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Rapid decay and polynomial growth for bicrossed products

PIERRE FIMA AND HUA WANG

Abstract

We study the rapid decay property and polynomial growth for duals of bicrossed products coming from a matched pair of a discrete group and a compact group.

1 Introduction

In the breakthrough paper [Ha78], Haagerup showed that the norm of the reduced C^* -algebra $C_r^*(\mathbb{F}_N)$ of the free group on N -generators \mathbb{F}_N , can be controlled by the Sobolev l^2 -norms associated to the word length function on \mathbb{F}_N . This is a striking phenomenon which actually occurs in many more cases. Jolissaint recognized this phenomenon, called Rapid Decay (or property (RD)), and studied it in a systematic way in [Jo90]. Property (RD) has now many applications. Let us mention the remarkable one concerning K -theory. Property (RD) allowed Jolissaint [Jo89] to show that the K -theory and $C_r^*(\Gamma)$ equals the K -theory of subalgebras of rapidly decreasing functions on Γ (Jolissaint did attribute this result to Connes). This result was then used by V. Lafforgue in his approach to the Baum-Connes conjecture via Banach KK -theory [La00, La02].

In this paper, we view discrete quantum groups as duals of compact quantum groups. The theory of compact quantum groups has been developed by Woronowicz [Wo87, Wo88, Wo98]. Property (RD) for discrete quantum groups has been introduced and studied by Vergnioux [Ve07]. Property (RD) has been refined later [BVZ14] in order to fit in the context of non-unimodular discrete quantum groups.

In this paper, we study the permanence of property (RD) under the bicrossed product construction. This construction was initiated by Kac [Ka68] in the context of finite quantum groups and was extensively studied later by many authors in different settings. The general construction, for locally compact quantum groups, was developed by Vaes-Vainerman [VV03]. In the context of compact quantum groups given by matched pairs of classical groups, an easier approach, that we will follow, was given by Fima-Mukherjee-Patri [FMP17].

Following [FMP17], the bicrossed product construction associates to a matched pair (Γ, G) of a discrete group Γ and a compact group G (see Section 2.2) a compact quantum group \mathbb{G} , called the bicrossed product. Given a length function l on the set of equivalence classes $\text{Irr}(\mathbb{G})$ of irreducible unitary representations of \mathbb{G} one can associate in a canonical way, as explained in Proposition 4.2, a pair of length functions (l_Γ, l_G) on Γ and $\text{Irr}(G)$ respectively. Such a pair satisfies some compatibility relations and every pair of length functions (l_Γ, l_G) on $(\Gamma, \text{Irr}(G))$ satisfying those compatibility relations will be called matched (see Definition 4.1). Any matched pair (l_Γ, l_G) on $(\Gamma, \text{Irr}(G))$ allows one to reconstruct a canonical length function on $\text{Irr}(\mathbb{G})$. The main result of the present paper is the following.

Theorem A. *Let (Γ, G) be a matched pair of a discrete group Γ and a compact group G . Denote by \mathbb{G} the bicrossed product. The following are equivalent.*

1. $\widehat{\mathbb{G}}$ has property (RD) .
2. There exists a matched pair of length function (l_Γ, l_G) on $(\Gamma, \text{Irr}(G))$ such that both (Γ, l_Γ) and (\widehat{G}, l_G) have (RD) .

For amenable discrete groups, property (RD) is equivalent to polynomial growth [Jo90] and the same occurs for discrete quantum groups [Ve07]. Hence, for the compact classical group G one has that (\widehat{G}, l_G) has (RD) if and only if it has polynomial growth. Note that a bicrossed product of a matched pair (Γ, G) is co-amenable if and only if Γ is amenable [FMP17]. The following theorem shows the permanence of polynomial growth under the bicrossed product construction.

Theorem B. *Let (Γ, G) be a matched pair of a discrete group Γ and a compact group G . Denote by \mathbb{G} the bicrossed product. The following are equivalent.*

1. $\widehat{\mathbb{G}}$ has polynomial growth.
2. There exists a matched pair of length function (l_Γ, l_G) on $(\Gamma, \text{Irr}(G))$ such that both (Γ, l_Γ) and (\widehat{G}, l_G) have polynomial growth.

The main ingredient to prove Theorem A and B is the classification of the irreducible unitary representation of a bicrossed product and the fusion rules.

The paper is organized as follows. Section 2 is a preliminary section in which we introduce our notations. In section 3 we classify the irreducible unitary representation of a bicrossed product and describe their fusion rules. Finally, in section 4, we prove Theorem A and Theorem B.

2 Preliminaries

2.1 Notations

For a Hilbert space H , we denote by $\mathcal{U}(H)$ its unitary group and by $\mathcal{B}(H)$ the C^* -algebra of bounded linear operators on H . When H is finite dimensional, we denote by Tr the unique trace on $\mathcal{B}(H)$ such that $\text{Tr}(1) = \dim(H)$. We use the same symbol \otimes for the tensor product of Hilbert spaces, unitary representations of compact quantum groups, minimal tensor product of C^* -algebras. For a compact quantum group G , we denote by $\text{Irr}(G)$ the set of equivalence classes of irreducible unitary representations and $\text{Rep}(G)$ the collection of finite dimensional unitary representations. We will often denote by $[u]$ the equivalence class of an irreducible unitary representation u . For $u \in \text{Rep}(G)$, we denote by $\chi(u)$ its character, i.e., viewing $u \in \mathcal{B}(H) \otimes C(G)$ for some finite dimensional Hilbert space H , one has $\chi(u) := (\text{Tr} \otimes \text{id})(u) \in C(G)$. We denote by $\text{Pol}(G)$ the unital C^* -algebra obtained by taking the Span of the coefficients of irreducible unitary representation, by $C_m(G)$ the enveloping C^* -algebra of $\text{Pol}(G)$ and by $C(G)$ the C^* -algebra generated by the GNS construction of the Haar state on $C_m(G)$. We also denote by $\varepsilon : C_m(G) \rightarrow \mathbb{C}$ the counit and we use the same symbol $\varepsilon \in \text{Irr}(G)$ to denote the trivial representation and its class in $\text{Irr}(G)$. In the entire paper, the word representation means a unitary and finite dimensional representation.

2.2 Compact bicrossed products

In this section, we follow the approach and the notations of [FMP17].

Let (Γ, G) be a pair of a countable discrete group Γ and a second countable compact group G with a left action $\alpha : \Gamma \rightarrow \text{Homeo}(G)$ of Γ on the compact space G by homeomorphisms and a right action $\beta : G \rightarrow S(\Gamma)$ of G on the discrete space Γ , where $S(\Gamma)$ is the Polish group of bijections of Γ , the topology being the one of pointwise convergence i.e., the smallest one for which the evaluation maps $S(\Gamma) \rightarrow \Gamma$, $\sigma \mapsto \sigma(\gamma)$ are continuous, for all $\gamma \in \Gamma$, where Γ has the discrete topology. Here, α is a group homomorphism and β is an antihomomorphism. The pair (Γ, G) is called a matched pair if $\Gamma \cap G = \{e\}$ with e being the common unit for both G and Γ , and if the actions α and β satisfy the following matched pair relations:

$$\forall g, h \in G, \gamma, \mu \in \Gamma, \quad \alpha_\gamma(gh) = \alpha_\gamma(g)\alpha_{\beta_g(\gamma)}(h), \quad \beta_g(\gamma\mu) = \beta_{\alpha_s(g)}(\gamma)\beta_g(\mu) \quad \text{and} \quad \alpha_\gamma(e) = \beta_g(e) = e. \quad (2.1)$$

We also write $\gamma \cdot g := \beta_g(\gamma)$. From now on, we assume (Γ, G) is matched. It is shown in [FMP17, Proposition 3.2] that β is automatically continuous. By continuity of β and compactness of G , every β orbit is finite. Moreover, the sets $G_{r,s} := \{g \in G : r \cdot g = s\}$ are clopen (see [FMP17, Section 2.1]). Let $v_{rs} = 1_{G_{r,s}} \in C(G)$ be the characteristic function of $G_{r,s}$. It is shown in [FMP17, Section 2.1] that, for all β -orbits $\gamma \cdot G \in \Gamma/G$, the unitary $v_{\gamma \cdot G} := \sum_{r,s \in \gamma \cdot G} e_{rs} \otimes v_{rs} \in \mathcal{B}(l^2(\gamma \cdot G)) \otimes C(G)$ is a unitary representation of G as well as a magic unitary, where $e_{rs} \in \mathcal{B}(l^2(\gamma \cdot G))$ are the canonical matrix units and the Haar probability measure ν on G is α -invariant.

It is shown in [FMP17, Theorem 3.4] that there exists a unique compact quantum group \mathbb{G} , called the bicrossed product of the matched pair (Γ, G) , such that $C(\mathbb{G}) = \Gamma_\alpha \rtimes C(G)$ is the reduced C^* -algebraic crossed product, generated by a copy of $C(G)$ and the unitaries u_γ , $\gamma \in \Gamma$ and $\Delta : C(\mathbb{G}) \rightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$ is the unique unital $*$ -homomorphism satisfying $\Delta|_{C(G)} = \Delta_G$ (the comultiplication on $C(G)$) and $\Delta(u_\gamma) = \sum_{r \in \gamma \cdot G} u_\gamma v_{\gamma r} \otimes u_r$ for all $\gamma \in \Gamma$. It is also shown that the Haar state on \mathbb{G} is a trace and is given by the formula $h(u_\gamma F) = \delta_{\gamma,1} \int_G F dv$ for all $\gamma \in \Gamma$ and $F \in C(G)$.

3 Representation theory of bicrossed products

3.1 Classification of irreducible representations

In this section we classify the irreducible representations of a bicrossed product. Let (Γ, G) be a matched pair of a discrete countable group Γ and a second countable compact group G with actions α, β .

For $\gamma \in \Gamma$ we denote by $G_\gamma := G_{\gamma, \gamma}$ the stabilizer of γ for the action $\beta : \Gamma \curvearrowright G$. Note that G_γ is an open (hence closed) subgroup of G , hence of finite index: its index is $|\gamma \cdot G|$. We view $C(G_\gamma) = v_{\gamma \cdot} C(G) \subset C(G)$ as a non-unital C^* -subalgebra. Let us denote by ν the Haar probability measure on G and note that $\nu(G_\gamma) = \frac{1}{|\gamma \cdot G|}$ so that the Haar probability measure ν_γ on G_γ is given by $\nu_\gamma(A) = |\gamma \cdot G| \nu(A)$ for all Borel subset A of G_γ .

For $\gamma \in \Gamma$ we fix a section, still denoted γ , $\gamma : \gamma \cdot G \rightarrow G$ of the canonical surjection $G \rightarrow \gamma \cdot G : g \mapsto \gamma \cdot g$. This means that $\gamma : \gamma \cdot G \rightarrow G$ is an injective map such that $\gamma \cdot \gamma(r) = r$ for all $r \in \gamma \cdot G$. We choose the section γ such that $\gamma(\gamma) = 1$, for all $\gamma \in \Gamma$. For $r, s \in \gamma \cdot G$, we denote by $\psi_{r,s}^\gamma$ the ν -preserving homeomorphism of G defined by $\psi_{r,s}^\gamma(g) = \gamma(r)g\gamma(s)^{-1}$. It follows from our choices that $\psi_{\gamma, \gamma}^\gamma = \text{id}$ for all $\gamma \in \Gamma$. Moreover, for all $g \in G$, one has $\psi_{r,s}^\gamma(g) \in G_\gamma$ if and only if $g \in G_{r,s}$. It follows that $\psi_{r,r}^\gamma$ is an isomorphism and an homeomorphism from G_r to G_γ intertwining the Haar probability measures.

Let $u : G_\gamma \rightarrow \mathcal{U}(H)$ be a unitary representation of G_γ and view u as a continuous function $G \rightarrow \mathcal{B}(H)$ which is zero outside G_γ i.e. a partial isometry in $\mathcal{B}(H) \otimes C(G)$ such that $uu^* = u^*u = \text{id}_H \otimes v_{\gamma \cdot}$. Define, for $r, s \in \gamma \cdot G$, the partial isometry $u_{r,s} := u \circ \psi_{r,s}^\gamma := (g \mapsto u(\psi_{r,s}^\gamma(g))) \in \mathcal{B}(H) \otimes C(G)$ and note that $u_{r,s}^* u_{r,s} = u_{r,s} u_{r,s}^* = \text{id}_H \otimes 1_{G_{r,s}}$. In the sequel we view $u_{r,s} \in \mathcal{B}(H) \otimes C(G) \subset \mathcal{B}(H) \otimes C(\mathbb{G})$ and we define:

$$\gamma(u) := \sum_{r,s \in \gamma \cdot G} e_{rs} \otimes (1 \otimes u_r v_{rs}) u_{r,s} \in \mathcal{B}(l^2(\gamma \cdot G)) \otimes \mathcal{B}(H) \otimes C(\mathbb{G}),$$

where we recall that e_{rs} , for $r, s \in \gamma \cdot G$, are the matrix units associated to the canonical orthonormal basis of $l^2(\gamma \cdot G)$.

The irreducible unitary representations of \mathbb{G} are described as follows.

Theorem 3.1. *The following holds.*

1. For all $\gamma \in \Gamma$ and $u \in \text{Rep}(G_\gamma)$ one has $\gamma(u) \in \text{Rep}(\mathbb{G})$.
2. The character of $\gamma(u)$ is $\chi(\gamma(u)) = \sum_{r \in \gamma \cdot G} u_r v_{rr} \chi(u) \circ \psi_{r,r}^\gamma$.
3. For all $\gamma, \mu \in \Gamma$, $u \in \text{Rep}(G_\gamma)$ and $w \in \text{Rep}(G_\mu)$ one has

$$\dim(\text{Mor}_{\mathbb{G}}(\gamma(u), \mu(w))) = \delta_{\gamma, \mu} \dim(\text{Mor}_{G_\gamma}(u, w \circ \psi_{\gamma, \gamma}^\mu)).$$

4. For all $\gamma \in \Gamma$ and $u \in \text{Rep}(G_\gamma)$ one has $\overline{\gamma(u)} \simeq \gamma^{-1}(\overline{u} \circ \alpha_{\gamma^{-1}})$ (which makes sense since $\alpha_{\gamma^{-1}} : G_{\gamma^{-1}} \rightarrow G_\gamma$ is a group isomorphism and an homeomorphism).
5. $\gamma(u)$ is irreducible if and only if u is irreducible. Moreover, for any irreducible unitary representation u of \mathbb{G} there exists $\gamma \in \Gamma$ and v an irreducible representation of G_γ such that $u \simeq \gamma(v)$.

Proof. (1). Writing $\gamma(u) = \sum_{r,s} e_{r,s} \otimes V_{r,s}$, where $V_{r,s} := (1 \otimes u_r v_{rs})u_{r,s} \in \mathcal{B}(H) \otimes C(\mathbb{G})$, it suffices to check that, for all $r, s \in \gamma \cdot G$ one has $(\text{id} \otimes \Delta)(V_{r,s}) = \sum_{t \in \gamma \cdot G} (V_{r,t})_{12} (V_{t,s})_{13}$. We first claim that, for all $r, s \in \gamma \cdot G$, $(\text{id} \otimes \Delta)(u_{r,s}) = \sum_{t \in \gamma \cdot G} (u_{r,t})_{12} (u_{t,s})_{13}$. To check our claim, first recall that, for all $r, s \in \gamma \cdot G$ one has $\psi_{r,s}^\gamma(g) \in G_\gamma$ if and only if $r \cdot g = s$. Let $r, s \in \gamma \cdot G$ and $g, h \in G$. For $t = r \cdot g \in \gamma \cdot G$ one has :

$$u_{r,s}(gh) = u(\gamma(r)g\gamma(t)^{-1}\gamma(t)h\gamma(s)^{-1}) = u(\psi_{r,t}^\gamma(g)\psi_{t,s}^\gamma(h)) = \begin{cases} u_{r,t}(g)u_{t,s}(h) & \text{if } r \cdot gh = s, \\ 0 & \text{otherwise.} \end{cases}$$

Since we also have $u_{t,s}(h) = 0$ whenever $r \cdot gh \neq s$ we find, in both cases, that $u_{r,s}(gh) = u_{r,t}(g)u_{t,s}(h)$. Now, for $t \neq r \cdot g$ we have $u_{r,t}(g) = 0$ so the following formulae holds for any $r, s \in \gamma \cdot G$ and any $g, h \in G$:

$$v_{r,t}(g)u_{r,s}(gh) = u_{r,t}(g)u_{t,s}(h).$$

Hence, for all $r, s, t \in \gamma \cdot G$, $(1 \otimes v_{r,t} \otimes 1)(\text{id} \otimes \Delta)(u_{r,s}) = (u_{r,t})_{12} (u_{t,s})_{13}$. Using this we find:

$$\begin{aligned} \sum_{t \in \gamma \cdot G} (V_{r,t})_{12} (V_{t,s})_{13} &= \sum_t (1 \otimes u_r v_{rt} \otimes 1)(u_{r,t})_{12} (1 \otimes 1 \otimes u_t v_{ts})(u_{t,s})_{13} \\ &= \sum_t (1 \otimes u_r v_{rt} \otimes u_t v_{ts})(u_{r,t})_{12} (u_{t,s})_{13} = \left(1 \otimes \left(\sum_t u_r v_{rt} \otimes u_t v_{ts} \right) \right) (\text{id} \otimes \Delta)(u_{r,s}). \end{aligned}$$

Since v_γ is a unitary representation of G and a magic unitary we also have:

$$\Delta(u_r v_{rs}) = \sum_{t,t'} (u_r v_{rt} \otimes u_t)(v_{rt'} \otimes v_{t's}) = \sum_t u_r v_{rt} \otimes u_t v_{ts}.$$

This shows that $\gamma(u)$ is a representation of \mathbb{G} . We now check that $\gamma(u)$ is unitary. As before, since for all $r, s \in \gamma \cdot G$ one has $\psi_{r,s}^\gamma(g) \in G_\gamma$ if and only if $r \cdot g = s$ and because u is a unitary representation of G_γ , we have, for all $r, t \in \gamma \cdot G$, $(1 \otimes v_{rt})u_{r,t}u_{r,t}^* = 1 \otimes v_{rt}$. Hence,

$$\begin{aligned} \sum_{t \in \gamma \cdot G} V_{r,t} V_{s,t}^* &= \sum_t (1 \otimes u_r)(1 \otimes v_{rt})u_{r,t}u_{s,t}^*(1 \otimes v_{st})(1 \otimes u_s^*) \\ &= \delta_{r,s}(1 \otimes u_r) \left(\sum_t (1 \otimes v_{rt})u_{r,t}u_{r,t}^* \right) (1 \otimes u_r^*) = \delta_{r,s}(1 \otimes u_r) \left(\sum_t (1 \otimes v_{rt}) \right) (1 \otimes u_r^*) \\ &= \delta_{r,s}. \end{aligned}$$

A similar computations shows that $\sum_{t \in \gamma \cdot G} V_{t,r}^* V_{t,s} = \delta_{r,s}$.

(2). The character of $\gamma(u)$ is given by

$$\chi(\gamma(u)) = \sum_{r \in \gamma \cdot G} (\text{Tr} \otimes \text{id})(V_{r,r}) = \sum_r u_r v_{rr} (\text{Tr} \otimes \text{id})(u_{r,r}) = \sum_r u_r v_{rr} \chi(u) \circ \psi_{r,r}^\gamma.$$

(3). Let $\gamma, \mu \in \Gamma$ and u, w be representations of G_γ and G_μ respectively. Since the Haar measure on G is invariant under the action α and the homeomorphisms $\psi_{r,r}^\gamma$ and $\psi_{r,r}^\mu$, we find, by the character formulae in 2 and the crossed-product relations,

$$\begin{aligned} \dim(\text{Mor}(\gamma(u), \mu(w))) &= h(\chi(\gamma(u))\chi(\mu(w))^*) = \sum_{r \in \gamma \cdot G, s \in \mu \cdot G} h(u_{rs^{-1}} \alpha_s(v_{rr} v_{ss} \chi(u) \circ \psi_{r,r}^\gamma (\chi(w) \circ \psi_{s,s}^\mu)^*)) \\ &= \delta_{\gamma \cdot G, \mu \cdot G} \sum_{r \in \gamma \cdot G} \int_G \alpha_r(v_{rr} (\chi(u) \circ \psi_{r,r}^\gamma) (\overline{\chi(w) \circ \psi_{r,r}^\mu})) d\nu \\ &= \delta_{\gamma \cdot G, \mu \cdot G} \sum_{r \in \gamma \cdot G} \int_{G_r} (\chi(u) \circ \psi_{r,r}^\gamma) (\chi(\overline{w}) \circ \psi_{r,r}^\mu) d\nu \\ &= \delta_{\gamma \cdot G, \mu \cdot G} \sum_{r \in \gamma \cdot G} \int_{G_\mu} \chi(u) \circ (\psi_{r,\gamma}^\mu)^{-1} (\chi(\overline{w}) \circ \psi_{r,r}^\mu \circ (\psi_{r,r}^\gamma)^{-1} \circ (\psi_{r,\gamma}^\mu)^{-1}) d\nu \end{aligned}$$

Now, note that $\psi_{r,r}^\mu \circ (\psi_{r,r}^\gamma)^{-1} \circ (\psi_{\gamma,\gamma}^\mu)^{-1} = \text{Ad}(h)$, where $h = \mu(r)\gamma(r)^{-1}\mu(\gamma)^{-1}$. Moreover, $\mu \cdot h = \mu$ since:

$$\mu \cdot \mu(r)\gamma(r)^{-1}\mu(\gamma)^{-1} = r \cdot \gamma(r)^{-1}\mu(\gamma)^{-1} = \gamma \cdot \mu(\gamma)^{-1} = \mu.$$

Hence, $h \in G_\mu$. Since the characters of finite dimensional unitary representation of a group Λ are central functions i.e. invariant under $\text{Ad}(\lambda)$ for all $\lambda \in \Lambda$, we have $\chi(\bar{w}) \circ \psi_{r,r}^\mu \circ (\psi_{r,r}^\gamma)^{-1} \circ (\psi_{\gamma,\gamma}^\mu)^{-1} = \chi(\bar{w}) \circ \text{Ad}(h) = \chi(\bar{w})$. Hence:

$$\begin{aligned} \dim(\text{Mor}(\gamma(u), \mu(w))) &= \delta_{\gamma \cdot G, \mu \cdot G} \sum_{r \in \gamma \cdot G} \int_{G_\mu} \chi(u) \circ (\psi_{\gamma,\gamma}^\mu)^{-1} \chi(\bar{w}) d\nu = \delta_{\gamma \cdot G, \mu \cdot G} \int_{G_\mu} \chi(u) \circ (\psi_{\gamma,\gamma}^\mu)^{-1} \chi(\bar{w}) d\nu_\mu \\ &= \delta_{\gamma \cdot G, \mu \cdot G} \dim(\text{Mor}_{G_\mu}(u \circ (\psi_{\gamma,\gamma}^\mu)^{-1}, w)) = \delta_{\gamma \cdot G, \mu \cdot G} \int_{G_\gamma} \chi(u) \chi(\bar{w} \circ \psi_{\gamma,\gamma}^\mu) d\nu_\mu \\ &= \delta_{\gamma \cdot G, \mu \cdot G} \dim(\text{Mor}_{G_\gamma}(u, w \circ \psi_{\gamma,\gamma}^\mu)). \end{aligned}$$

(4). Note that, by the bicrossed product relations, we have, for all $\gamma \in \Gamma$ and $g \in G$, $(\gamma \cdot g)^{-1} = \gamma^{-1} \cdot \alpha_\gamma(g)$. Hence $v_{\gamma^{-1}\gamma^{-1}} \circ \alpha_\gamma = v_{\gamma\gamma}$ and $(\gamma \cdot G)^{-1} = \gamma^{-1} \cdot G$. In particular, $\alpha_\gamma : G_\gamma \rightarrow G_{\gamma^{-1}}$ is an homeomorphism and, by the bicrossed product relations, one has, for all $g \in G_\gamma$ and $h \in G$, $\alpha_\gamma(gh) = \alpha_\gamma(g)\alpha_{\gamma \cdot g}(h) = \alpha_\gamma(g)\alpha_\gamma(h)$ so that $\alpha_\gamma : G_\gamma \rightarrow G_{\gamma^{-1}}$ is also a group homomorphism.

For $r \in \gamma \cdot G$ one has $\gamma^{-1} \cdot \alpha_\gamma(\gamma(r)) = (\gamma \cdot \gamma(r))^{-1} = r^{-1} = \gamma^{-1} \cdot \gamma^{-1}(r^{-1})$. This implies that, for all $\gamma \in \Gamma$, there exists a map $\eta_\gamma : \gamma \cdot G \rightarrow G_{\gamma^{-1}}$ such that, for all $r \in \gamma \cdot G$, one has $\alpha_\gamma(\gamma(r)) = \eta_\gamma(r)\gamma^{-1}(r^{-1})$.

Let now $r \in \gamma \cdot G$ and $g \in G_r$. One has, using the bicrossed product relations, that $e = \alpha_r(\gamma(r)\gamma(r)^{-1}) = \alpha_\gamma(\gamma(r))\alpha_r(\gamma(r)^{-1})$, hence

$$(\alpha_\gamma \circ \psi_{r,r}^\gamma)(g) = \alpha_\gamma(\gamma(r))\alpha_r(g)\alpha_r(\gamma(r)^{-1}) = \alpha_\gamma(\gamma(r))\alpha_r(g)(\alpha_\gamma(\gamma(r)))^{-1} = \eta_\gamma(r)(\psi_{r^{-1},r^{-1}}^{\gamma^{-1}} \circ \alpha_r)(g)(\eta_\gamma(r))^{-1}.$$

Hence, for all $\gamma \in \Gamma$, if $w \in \text{Rep}(G_{\gamma^{-1}})$, since $\chi(w) \in C(G_{\gamma^{-1}})$ is central we have

$$\chi(w) \circ \alpha_\gamma \circ \psi_{r,r}^\gamma(g) = \chi(w) \circ \psi_{r^{-1},r^{-1}}^{\gamma^{-1}} \circ \alpha_r(g) \quad \text{for all } r \in \gamma \cdot G, g \in G_r.$$

Since, as we seen above, $\gamma^{-1} \cdot G = (\gamma \cdot G)^{-1}$ and because $\chi(\bar{u} \circ \alpha_{\gamma^{-1}}) = \chi(\bar{u}) \circ \alpha_{\gamma^{-1}}$ we find, by the character formulae in 2, $\chi(\gamma^{-1}(\bar{u} \circ \alpha_{\gamma^{-1}})) = \sum_{r \in \gamma \cdot G} u_{r^{-1}} v_{r^{-1}r^{-1}} \chi(\bar{u}) \circ \alpha_{\gamma^{-1}} \circ \psi_{r^{-1},r^{-1}}^{\gamma^{-1}}$. It then follows from the crossed-product relations and the discussion above :

$$\begin{aligned} \chi(\gamma^{-1}(\bar{u} \circ \alpha_{\gamma^{-1}})) &= \sum_{r \in \gamma \cdot G} u_{r^{-1}} v_{r^{-1}r^{-1}} \chi(\bar{u}) \circ \alpha_{\gamma^{-1}} \circ \psi_{r^{-1},r^{-1}}^{\gamma^{-1}} \\ &= \sum_{r \in \gamma \cdot G} (\chi(\bar{u}) \circ \alpha_{\gamma^{-1}} \circ \psi_{r^{-1},r^{-1}}^{\gamma^{-1}} \circ \alpha_r)(v_{r^{-1}r^{-1}} \circ \alpha_r) u_{r^{-1}} \\ &= \sum_{r \in \gamma \cdot G} \chi(\bar{u}) \circ \psi_{r,r}^\gamma v_{rr} u_r^* = \sum_{r \in \gamma \cdot G} (\chi(u) \circ \psi_{r,r}^\gamma v_{rr})^* u_r^* \\ &= \chi(\gamma(u))^* \end{aligned}$$

(5). The statement on irreducibility following from 3, it suffices, by the general theory, to show that the linear span X of coefficients of representations of the form $\gamma(u)$, for $\gamma \in \Gamma$ and u an irreducible unitary representation of G_γ , is a dense subset of $C(\mathbb{G})$. Note that, for all $\gamma \in \Gamma$, the relation $1 = \sum_{r \in \gamma \cdot G} v_{\gamma r}$ implies that any function in $C(G)$ is a sum of continuous functions with support in $G_{\gamma,r} := \{g \in G : \gamma \cdot g = r\}$, for $r \in \gamma \cdot G$. Moreover, since $G_{\gamma,r} = (\psi_{\gamma,r}^\gamma)^{-1}(G_\gamma)$, any continuous function on G with support in $G_{\gamma,r}$ is of the form $F \circ \psi_{\gamma,r}^\gamma$, where $F \in C(G_\gamma)$. Since the linear span of coefficients of irreducible unitary representation of G_γ is dense in $C(G_\gamma)$, it suffices to show that, for any $\gamma \in \Gamma$, for any irreducible unitary representation of G_γ , $u : G_\gamma \rightarrow \mathcal{U}(H)$, any coefficient $u_{ij} \in C(G_\gamma) = v_{\gamma} C(G) \subset C(G)$ satisfies $u_\gamma u_{ij} \in X$. But this is obvious since one has

$$u_\gamma u_{ij} = u_\gamma v_{\gamma} u_{i,j} = u_\gamma v_{\gamma} u_{i,j} \circ \psi_{\gamma,\gamma}^\gamma = \gamma(u)_{\gamma,\gamma,i,j} \in X. \quad \square$$

Finally, the fusion rules are described as follows.

Let $\gamma, \mu \in \Gamma$, $u : G_\gamma \rightarrow \mathcal{U}(H_u)$, $v : G_\mu \rightarrow \mathcal{U}(H_v)$ by unitary representations of G_γ and G_μ respectively. For any $r \in (\gamma \cdot G)(\mu \cdot G)$, we define the r -twisted tensor product of u and v , denoted $u \otimes_r v$ as a unitary representation of G_r on $K_r \otimes H_u \otimes H_v$, where

$$K_r := \text{Span}(\{e_s \otimes e_t : s \in \gamma \cdot G \text{ and } t \in \mu \cdot G \text{ such that } st = r\}) \subset l^2(\gamma \cdot G) \otimes l^2(\mu \cdot G).$$

For $g \in G$, we define:

$$(u \otimes_r v)(g) = \sum_{\substack{s, s' \in \gamma \cdot G \\ t, t' \in \mu \cdot G \\ st = r = s't'}} e_{ss'} \otimes e_{tt'} \otimes v_{ss'}(\alpha_t(g))v_{tt'}(g)u(\psi_{s, s'}^\gamma(\alpha_t(g))) \otimes v(\psi_{t, t'}^\mu(g)) \in \mathcal{U}(K_r \otimes H_u \otimes H_v).$$

Theorem 3.2. *The following holds.*

1. For all $\gamma, \mu \in \Gamma$, all $r \in (\gamma \cdot G)(\mu \cdot G)$ and all u, v finite dimensional unitary representations of G_γ, G_μ respectively the element $u \otimes_r v$ is a unitary representation of G_r .
2. The character of $u \otimes_r w$ is $\chi(u \otimes_r v) = \sum_{s \in \gamma \cdot G, t \in \mu \cdot G, st=r} (v_{ss} \circ \alpha_t)v_{tt}(\chi(u) \circ \psi_{s, s}^\gamma \circ \alpha_t)(\chi(v) \circ \psi_{t, t}^\mu)$.
3. For all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ and all u, v, w unitary representations of $G_{\gamma_1}, G_{\gamma_2}$ and G_{γ_3} respectively, the number $\dim(\text{Mor}_{\mathbb{C}}(\gamma_1(u), \gamma_2(v) \otimes \gamma_3(w)))$ is equal to:

$$\begin{cases} \frac{1}{|\gamma_1 \cdot G|} \sum_{r \in \gamma_1 \cdot G \cap (\gamma_2 \cdot G)(\gamma_3 \cdot G)} \dim(\text{Mor}_{G_r}(u \circ \psi_{r, r}^{\gamma_1}, v \otimes_r w)) & \text{if } \gamma_1 \cdot G \cap (\gamma_2 \cdot G)(\gamma_3 \cdot G) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let us observe that, by the bicrossed product relations, we have, for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$,

$$\gamma_1 \cdot G \cap (\gamma_2 \cdot G)(\gamma_3 \cdot G) \neq \emptyset \Leftrightarrow \gamma_1 \cdot G \subset (\gamma_2 \cdot G)(\gamma_3 \cdot G).$$

Proof. (1). Put $w = u \otimes_r v$ and let $g, h \in G_r$. Then, $w(gh)$ is equal to:

$$\sum_{s, s' \in \gamma \cdot G, t, t' \in \mu \cdot G, st = s't' = r} e_{ss'} \otimes e_{tt'} \otimes v_{ss'}(\alpha_t(gh))v_{tt'}(gh)u(\psi_{s, s'}^\gamma(\alpha_t(gh))) \otimes v(\psi_{t, t'}^\mu(gh)).$$

Since $v_{ty}(g) \neq 0$ precisely when $t \cdot g = y$, the factor $v_{ss'}(\alpha_t(gh))v_{tt'}(gh)u(\psi_{s, s'}^\gamma(\alpha_t(gh))) \otimes v(\psi_{t, t'}^\mu(gh))$ is equal to:

$$\begin{aligned} & \sum_{x \in \gamma \cdot G, y \in \mu \cdot G} v_{sx}(\alpha_t(g))v_{xs'}(\alpha_{t \cdot g}(h))v_{ty}(g)v_{yt'}(h)u(\psi_{s, x}^\gamma(\alpha_t(g))u(\psi_{x, s'}^\gamma(\alpha_{t \cdot g}(h)))) \otimes v(\psi_{t, y}^\mu(g))v(\psi_{y, t'}^\mu(h)) \\ &= \sum_{x \in \gamma \cdot G, y \in \mu \cdot G} v_{sx}(\alpha_t(g))v_{xs'}(\alpha_y(h))v_{ty}(g)v_{yt'}(h)u(\psi_{s, x}^\gamma(\alpha_t(g))u(\psi_{x, s'}^\gamma(\alpha_y(h)))) \otimes v(\psi_{t, y}^\mu(g))v(\psi_{y, t'}^\mu(h)). \end{aligned}$$

Moreover, since for all $g \in G_r$ and all s, t such that $st = r$, one has, whenever $t \cdot g = y$ and $s \cdot \alpha_t(g) = x$, that $xy = (s \cdot \alpha_t(g))(t \cdot g) = (st) \cdot g = r \cdot g = r$, it follows that the only non-zero terms in the last sum are for $x \in \gamma \cdot G$ and $y \in \mu \cdot G$ such that $xy = r$. By the properties of the matrix units we see immediately that $w(gh) = w(g)w(h)$. To end the proof of (1), it suffices to check that $w(1) = 1$, which is clear, and that $w(g)^* = w(g^{-1})$ for all $g \in G_r$. So let $g \in G_r$. One has:

$$w(g)^* = \sum_{s, s' \in \gamma \cdot G, t, t' \in \mu \cdot G, st = r = s't'} e_{ss'} \otimes e_{tt'} \otimes v_{s's}(\alpha_{t'}(g))v_{t't}(g)u((\psi_{s', s}^\gamma(\alpha_{t'}(g)))^{-1}) \otimes v((\psi_{t', t}^\mu(g))^{-1}).$$

Note that for all $t, t' \in \Gamma$ and all $g \in G$, one has $v_{s's}(g) = v_{ss'}(g^{-1})$. Also, using the bicrossed product relations one finds that $\alpha_r(g)^{-1} = \alpha_{r \cdot g}(g^{-1})$ for all $r \in \Gamma$ and $g \in G$. In particular, $v_{s's}(\alpha_{t'}(g))v_{t't}(g) =$

$v_{ss'}(\alpha_t(g^{-1}))v_{tt'}(g^{-1})$ and, when $t' \cdot g = t$, one has $\psi_{s',s}^\gamma(\alpha_{t'}(g))^{-1} = \psi_{s,s'}^\gamma(\alpha_t(g^{-1}))$. It follows immediately that $w(g)^* = w(g^{-1})$.

(2). Is a direct computation.

(3). One has $\dim(\text{Mor}_{\mathbb{C}}(\gamma_1(u), \gamma_2(v) \otimes \gamma_3(w))) = h(\chi(\gamma_1(u))^* \chi(\gamma_2(v)) \chi(\gamma_3(w)))$ which is equal to:

$$\begin{aligned}
& \sum_{r \in \gamma_1 \cdot G, s \in \gamma_2 \cdot G, t \in \gamma_3 \cdot G} h(\chi(\bar{u}) \circ \psi_{r,r}^{\gamma_1} v_{rr} u_r^* u_s v_{ss} \chi(v) \circ \psi_{s,s}^{\gamma_2} u_t v_{tt} \chi(w) \circ \psi_{t,t}^{\gamma_3}) \\
&= \sum_{r,s,t} h(u_{r-1st} \alpha_{t-1s-1r}(\chi(\bar{u}) \circ \psi_{r,r}^{\gamma_1} v_{rr}) \alpha_{t-1}(v_{ss} \chi(v) \circ \psi_{s,s}^{\gamma_2}) v_{tt} \chi(w) \circ \psi_{t,t}^{\gamma_3}) \\
&= \sum_{r \in \gamma_1 \cdot G} \sum_{s \in \gamma_2 \cdot G, t \in \gamma_3 \cdot G, st=r} \int_G \chi(\bar{u}) \circ \psi_{r,r}^{\gamma_1} v_{rr} \alpha_{t-1}(v_{ss} \chi(v) \circ \psi_{s,s}^{\gamma_2}) v_{tt} \chi(w) \circ \psi_{t,t}^{\gamma_3} d\nu \\
&= \sum_{r \in \gamma_1 \cdot G \cap (\gamma_2 \cdot G)(\gamma_3 \cdot G)} \frac{1}{|r \cdot G|} \int_{G_r} \chi(\bar{u}) \circ \psi_{r,r}^{\gamma_1} \chi(v \otimes w) d\nu_r \\
&= \frac{1}{|\gamma_1 \cdot G|} \sum_{r \in \gamma_1 \cdot G \cap (\gamma_2 \cdot G)(\gamma_3 \cdot G)} \dim(\text{Mor}_{G_r}(u \circ \psi_{r,r}^{\gamma_1}, v \otimes w)).
\end{aligned}$$

Note that, whenever $\gamma_1 \cdot G \cap ((\gamma_2 \cdot G)(\gamma_3 \cdot G)) = \emptyset$, there is no non-zero terms in the sum above. \square

3.2 The induced representation

In this section, we explain how the induced representation maybe viewed as a particular twisted tensor product.

For $\gamma \in \Gamma$ and $u : G_\gamma \rightarrow \mathcal{U}(H)$ is a unitary representation of G_γ we define the induced representation:

$$\text{Ind}_\gamma^G(u) := \varepsilon_{G_{\gamma^{-1}}} \otimes_1 u : G \rightarrow \mathcal{U}(l^2(\gamma \cdot G) \otimes H); g \mapsto \sum_{r,s \in \gamma \cdot G} e_{rs} \otimes v_{rs}(g) u(\psi_{rs}^\gamma(g)).$$

It follows from Theorem 3.2 that $\text{Ind}_\gamma^G(u)$ is indeed a unitary representation of G . We collect some elementary and well known facts about this representation in the following Proposition. Note that, in property 3, we use the symbol $\text{Res}_{G_\gamma}^G(u)$ for $u \in \text{Rep}(G)$ to denote the restriction of u to a representation of G_γ . Hence, property 3 motivates the name induced representation for the representation $\text{Ind}_\gamma^G(u)$.

Proposition 3.3. *The following holds.*

1. For all $\gamma \in \Gamma$ and all $u \in \text{Rep}(G_\gamma)$ one has $\chi(\text{Ind}_\gamma^G(u))(g) = \sum_{r \in \gamma \cdot G} v_{rr}(g) \chi(u)(\psi_{rr}^\gamma(g))$ for all $g \in G$.
2. For all $\gamma \in \Gamma$ and all $u, v \in \text{Rep}(G_\gamma)$ one has $u \simeq v \implies \text{Ind}_\gamma^G(u) \simeq \text{Ind}_\gamma^G(v)$.
3. For all $\gamma \in \Gamma, u \in \text{Rep}(G)$ and $v \in \text{Rep}(G_\gamma)$ one has $\dim(\text{Mor}_G(u, \text{Ind}_\gamma^G(v))) = \dim(\text{Mor}_{G_\gamma}(\text{Res}_{G_\gamma}^G(u), v))$.

Proof. (1). It is obvious, by definition of $\text{Ind}_\gamma^G(u)$.

(2). If $u \simeq v$ then $\chi(u) = \chi(v)$. Hence, $\chi(\text{Ind}_\gamma^G(u)) = \chi(\text{Ind}_\gamma^G(v))$ by (1). So $\text{Ind}_\gamma^G(u) \simeq \text{Ind}_\gamma^G(v)$.

(3). Let $\gamma \in \Gamma, u \in \text{Rep}(G)$ and $v \in \text{Rep}(G_\gamma)$. One has,

$$\dim(\text{Mor}_G(u, \text{Ind}_\gamma^G(v))) = \sum_{r \in \gamma \cdot G} \int_G \chi(\bar{u}) v_{rr} \chi(v) \circ \psi_{rr}^\gamma d\nu = \frac{1}{|\gamma \cdot G|} \sum_{r \in \gamma \cdot G} \int_{G_r} \chi(\bar{u}) \chi(v) \circ \psi_{rr}^\gamma d\nu_r.$$

Since $\psi_{rr}^\gamma : G_r \rightarrow G_\gamma$ is a Haar probability preserving homeomorphism we obtain

$$\dim(\text{Mor}_G(u, \text{Ind}_\gamma^G(v))) = \frac{1}{|\gamma \cdot G|} \sum_{r \in \gamma \cdot G} \int_{G_\gamma} \chi(\bar{u}) \circ (\psi_{rr}^\gamma)^{-1} \chi(v) d\nu_\gamma.$$

Finally, since, for all $g \in G$, $\chi(\bar{u}) \circ (\psi_{rr}^\gamma)^{-1}(g) = \chi(\bar{u})(g)$ (because $\chi(\bar{u})$ is a central function on G) it follows that:

$$\dim(\text{Mor}_G(u, \text{Ind}_\gamma^G(v))) = \frac{1}{|\gamma \cdot G|} \sum_{r \in \gamma \cdot G} \int_{G_\gamma} \chi(\bar{u}) \chi(v) d\nu_\gamma = \dim(\text{Mor}_{G_\gamma}(\text{Res}_{G_\gamma}^G(u), v)). \quad \square$$

4 Length functions

Recall that given a compact quantum group \mathbb{H} , a function $l : \text{Irr}(\mathbb{H}) \rightarrow [0, \infty)$ is called a *length function on $\text{Irr}(\mathbb{H})$* if $l([\epsilon]) = 0$, $l(\bar{x}) = l(x)$ and that $l(x) \leq l(y) + l(z)$ whenever $x \subset y \otimes z$. A length function on a discrete group Λ is a function $l : \Lambda \rightarrow [0, \infty)$ such that $l(1) = 0$, $l(r) = l(r^{-1})$ and $l(rs) \leq l(r) + l(s)$ for all $r, s \in \Lambda$.

Let (Γ, G) be a matched pair with bicrossed product \mathbb{G} . In view of the description of the irreducible representations of \mathbb{G} , the fusion rules and the contragredient representation, it is clear that to get a length function on $\text{Irr}(\mathbb{G})$, we need a family of maps $l_\gamma : \text{Irr}(G_\gamma) \rightarrow [0, +\infty[$, for $\gamma \in \Gamma$, satisfying the hypothesis of the following definition.

Definition 4.1. Let (Γ, G) be a matched pair, $l : \text{Irr}(G) \rightarrow [0, +\infty[$ and $l_\Gamma : \Gamma \rightarrow [0, +\infty[$ be length functions. The pair (l, l_Γ) is *matched* if, for all $\gamma \in \Gamma$, there exists a function $l_\gamma : \text{Irr}(G_\gamma) \rightarrow [0, +\infty[$ such that

- (i) $l_1 = l$ and $l_\gamma(\varepsilon_{G_\gamma}) = l_\Gamma(\gamma)$.
- (ii) For any $\gamma \in \Gamma$, $r \in \gamma \cdot G$, and $x \in \text{Irr}(G_\gamma)$, we have $l_\gamma(x) = l_r([u^x \circ \psi_{r,r}^\gamma])$.
- (iii) For any $\gamma \in \Gamma$, $x \in \text{Irr}(G_\gamma)$, we have $l_\gamma(x) = l_{\gamma^{-1}}([\bar{u}^x \circ \alpha_{\gamma^{-1}}])$.
- (iv) For any $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$, $x \in \text{Irr}(G_{\gamma_1})$, $y \in \text{Irr}(G_{\gamma_2})$, $z \in \text{Irr}(G_{\gamma_3})$, if $\gamma_3 \in (\gamma_1 \cdot G)(\gamma_2 \cdot G)$, and

$$\dim \text{Mor}_{G_r}(u^z \circ \psi_{r,r}^{\gamma_3}, u^x \otimes_r u^y) \neq 0 \quad (4.1)$$

for some $r \in \gamma_3 \cdot G$, then

$$l_{\gamma_3}(z) \leq l_{\gamma_1}(x) + l_{\gamma_2}(y). \quad (4.2)$$

The next Proposition shows that our notion of matched pair for length functions is the good one, as expected.

Proposition 4.2. *Let (Γ, G) be a matched pair with bicrossed product \mathbb{G} .*

1. *If l is a length function on $\text{Irr}(\mathbb{G})$ then the maps $l_G : \text{Irr}(G) = \text{Irr}(G_1) \rightarrow [0, +\infty[$, $x \mapsto l([1(x)])$ and $l_\Gamma : \Gamma \rightarrow [0, +\infty[$, $\gamma \mapsto l([\gamma(\varepsilon_{G_\gamma})])$ are length functions and the pair (l_Γ, l_G) is matched.*
2. *If l_Γ is any β -invariant length function on Γ then the map $l' : \text{Irr}(\mathbb{G}) \rightarrow [0, +\infty[$, $[\gamma(u^x)] \mapsto l_\Gamma(\gamma)$ is a well defined length function on $\text{Irr}(\mathbb{G})$.*
3. *If (l_Γ, l_G) is a matched pair of length functions on $(\Gamma, \text{Irr}(G))$ then l_Γ is β -invariant and the maps $l, \tilde{l} : \text{Irr}(\mathbb{G}) \rightarrow [0, +\infty[$, $l([\gamma(u^x)]) := l_\gamma(x)$ and $\tilde{l}([\gamma(u^x)]) := l_\gamma(x) + l_\Gamma(\gamma)$ are well-defined length functions.*

Proof. (1). Since $1(\varepsilon_G)$ is the trivial representation of \mathbb{G} one has $l_\Gamma(1) = 0$. Let $\gamma, \mu \in \Gamma$ and note that $\gamma\mu \in (\gamma \cdot G)(\mu \cdot G)$. Moreover,

$$\dim(\text{Mor}(\varepsilon_{G_{\gamma\mu}}, \varepsilon_{G_\gamma} \otimes_{\gamma\mu} \varepsilon_{G_\mu})) = \int_{G_{\gamma\mu}} \chi(\varepsilon_{G_\gamma} \otimes_{\gamma\mu} \varepsilon_{G_\mu}) d\nu_{G_{\gamma\mu}} = |\gamma\mu \cdot G| \sum_{s \in \gamma \cdot G, t \in \mu \cdot G, st = \gamma\mu} \int_{G_{\gamma\mu}} (v_{ss} \circ \alpha_t) v_{tt} d\nu$$

$$\begin{aligned}
&= |\gamma\mu \cdot G| \sum_{s \in \gamma \cdot G, t \in \mu \cdot G, st = \gamma\mu} \nu(\alpha_{t^{-1}}(G_s) \cap G_t \cap G_{\gamma\mu}) \\
&\geq \nu(\alpha_{\mu^{-1}}(G_\gamma) \cap G_\mu \cap G_{\gamma\mu}).
\end{aligned}$$

Hence, since $\alpha_{\mu^{-1}}(G_\gamma) \cap G_\mu \cap G_{\gamma\mu}$ is open and non empty (it contains 1) we deduce that

$$\dim(\text{Mor}(\varepsilon_{G_{\gamma\mu}}, \varepsilon_{G_\gamma} \otimes_{\gamma\mu} \varepsilon_{G_\mu})) > 0.$$

So $\varepsilon_{G_{\gamma\mu}} \subset \varepsilon_{G_\gamma} \otimes_{\gamma\mu} \varepsilon_{G_\mu}$ and, by the fusion rules of \mathbb{G} in Theorem 3.2, $(\gamma\mu)(\varepsilon_{G_{\gamma\mu}}) \subset \gamma(\varepsilon_{G_\gamma}) \otimes \mu(\varepsilon_{G_\mu})$. Hence, since l is a length function, $l_\Gamma(\gamma\mu) = l([\gamma\mu(\varepsilon_{G_{\gamma\mu}})]) \leq l([\gamma(\varepsilon_{G_\gamma})]) + l([\mu(\varepsilon_{G_\mu})]) = l_\Gamma(\gamma) + l_\Gamma(\mu)$. Finally, note that, by point 4 of Theorem 3.1, for all $\gamma \in \Gamma$, one has $\gamma^{-1}(\varepsilon_{G_{\gamma^{-1}}}) \simeq \overline{\gamma(\varepsilon_G)}$. Hence,

$$l_\Gamma(\gamma^{-1}) = l([\gamma^{-1}(\varepsilon_{G_{\gamma^{-1}}})]) = l([\overline{\gamma(\varepsilon_G)}]) = l([\gamma(\varepsilon_G)]) = l_\Gamma(\gamma).$$

So l_Γ is a length function on Γ . It is obvious that l_G is a length function on $\text{Irr}(G)$. Let us prove that the pair (l_Γ, l_G) is matched. Indeed, defining $l_\gamma : \text{Irr}(G_\gamma) \rightarrow [0, +\infty[$ by $l_\gamma(x) = l([\gamma(u^x)])$, point (i) of Definition 4.1 is clear while point (ii) follows from point 3 of Theorem 3.1, since it implies $[\gamma(u^x)] = [r(u^x \circ \psi_{\gamma, \gamma}^r)]$, thus

$$l_\gamma(x) = l([\gamma(u^x)]) = l([r(u^x \circ \psi_{\gamma, \gamma}^r)]) = l_r([u^x \circ \psi_{\gamma, \gamma}^r]).$$

Next, by point 4 of Theorem 3.1, we have $\overline{[\gamma(u^x)]} = [\gamma^{-1}(\overline{u^x}) \circ \alpha_{\gamma^{-1}}]$ thus,

$$l_\gamma(x) = l([\overline{[\gamma(u^x)]}]) = l([\gamma^{-1}(\overline{u^x}) \circ \alpha_{\gamma^{-1}}]) = l_{\gamma^{-1}}([\overline{u^x} \circ \alpha_{\gamma^{-1}}]),$$

which proves point (ii) of Definition 4.1. Finally, for point (iv), the fusion rules in Theorem 3.2 imply

$$\dim \text{Mor}(\gamma_3(u^z), \gamma_1(u^x) \otimes \gamma_2(u^y)) = \frac{1}{|\gamma \cdot G|} \sum_{r \in \gamma_3 \cdot G} \dim \text{Mor}_{G_r}(u^z \circ \psi_{r, r}^{\gamma_3}, u^x \otimes_r u^y). \quad (4.3)$$

If $\dim \text{Mor}_{G_r}(u^z \circ \psi_{r, r}^{\gamma_3}, u^x \otimes_r u^y) \neq 0$ for some $r \in \gamma_3 \cdot G$, then (4.3) is also nonzero, which means, by irreducibility of $\gamma_3(u^z)$ that $[\gamma_3(u^z)] \subseteq [\gamma_1(u^x)] \otimes [\gamma_2(u^y)]$. Hence, since l is a length function on $\text{Irr}(\mathbb{G})$,

$$l_{\gamma_3}(z) = l([\gamma_3(u^z)]) \leq l([\gamma_1(u^x)]) + l([\gamma_2(u^y)]) = l_{\gamma_1}(x) + l_{\gamma_2}(y).$$

(2). Since l_Γ is β -invariant, the map l' is well defined by Theorem 3.1. It is clear that $l'(\varepsilon_G) = 0$ and, by point 4 (and 5) of Theorem 3.1 and since l' is a length function we also have that $l'(z) = l'(\overline{z})$ for all $z \in \text{Irr}(\mathbb{G})$. Let now $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$, $x \in \text{Irr}(G_{\gamma_1})$, $y \in \text{Irr}(G_{\gamma_2})$ and $z \in \text{Irr}(G_{\gamma_3})$ be such that $\gamma_1(u^x) \subset \gamma_2(u^y) \otimes \gamma_3(u^z)$ then, by point 3 in Theorem 3.2, there exists $r \in \gamma_1 \cdot G$, $s \in \gamma_2 \cdot G$ and $t \in \gamma_3 \cdot G$ such that $r = st$ (and $u^x \circ \psi_{r, r}^{\gamma_1} \subset u^y \otimes u^z$). Then,

$$l'([\gamma_1(u^x)]) = l_\Gamma(\gamma_1) = l_\Gamma(r) \leq l_\Gamma(s) + l_\Gamma(t) = l_\Gamma(\gamma_2) + l_\Gamma(\gamma_3) = l'([\gamma_2(u^y)]) + l'([\gamma_3(u^z)]).$$

(3). Let (l_Γ, l_G) be a matched pair of length functions. By points 1 and 2 of Definition 4.1 we have, for all $\gamma \in \Gamma$ and all $r \in \gamma \cdot G$, $l_\Gamma(\gamma) = l_\Gamma(\varepsilon_{G_\gamma}) = l_r([\varepsilon_{G_\gamma} \circ \psi_{r, r}^\gamma]) = l_r(\varepsilon_{G_r}) = l_\Gamma(r)$. Hence, l_Γ is β -invariant. By assertion (2) we just proved above, we get a length function l' on $\text{Irr}(\mathbb{G})$. Now, it is clear from Definition 4.1, the fusion rules and the adjoint representation of a bicrossed product (point 3 of Theorem 3.2 and point 4 of Theorem 3.1) that $l : [\gamma(u^x)] \mapsto l_\gamma(x)$ is a length function on $\text{Irr}(\mathbb{G})$. Since $\tilde{l} = l + l'$, \tilde{l} is also a length function on $\text{Irr}(\mathbb{G})$. \square

5 Rapid decay and polynomial growth

In this section we study property (RD) and polynomial growth for bicrossed-products.

5.1 Generalities

We use the notion of property *(RD)* developed by Vergnioux in [Ve07] (see also [BVZ14]) and recall the definition below. Since we are only dealing with Kac algebras, we recall the definition of the Fourier transform and rapid decay only for Kac algebras.

Let \mathbb{H} be a compact quantum group. We use the notation $l^\infty(\widehat{\mathbb{H}}) := \bigoplus_{x \in \text{Irr}(\mathbb{H})} \mathcal{B}(H_x)$ to denote the l^∞ direct sum. The c_0 direct sum is denoted by $c_0(\widehat{\mathbb{H}}) \subset l^\infty(\widehat{\mathbb{H}})$ and the algebraic direct sum is denoted by $c_c(\widehat{\mathbb{H}}) \subset c_0(\widehat{\mathbb{H}})$. An element $a \in c_c(\widehat{\mathbb{H}})$ is said to have finite support and its finite support is denoted by $\text{Supp}(a) := \{x \in \text{Irr}(\mathbb{H}) : ap_x \neq 0\}$, where p_x , for $x \in \text{Irr}(\mathbb{H})$ denotes the central minimal projection of $l^\infty(\widehat{\mathbb{H}})$ corresponding to the block $\mathcal{B}(H_x)$.

For a compact quantum group \mathbb{H} which is always supposed to be of Kac type, and $a \in C_c(\widehat{\mathbb{H}})$ we define its Fourier transform as:

$$\mathcal{F}_{\mathbb{H}}(a) = \sum_{x \in \text{Irr}(\mathbb{H})} \dim(x) (\text{Tr}_x \otimes \text{id})(u^x(ap_x \otimes 1)) \in \text{Pol}(\mathbb{H}),$$

and its ‘‘Sobolev 0-norm’’ by $\|a\|_{\mathbb{H},0}^2 = \sum_{x \in \text{Irr}(\mathbb{H})} \dim(x) \text{Tr}_x((a^*a)p_x)$.

Given a length function $l : \text{Irr}(\mathbb{H}) \rightarrow [0, \infty)$, consider the element $L = \sum_{x \in \text{Irr}(\mathbb{H})} l(x)p_x$ which is affiliated to $c_0(\widehat{\mathbb{H}})$. Let q_n denote the spectral projections of L associated to the interval $[n, n+1)$.

The pair $(\widehat{\mathbb{H}}, l)$ is said to have:

- *Polynomial growth* if there exists a polynomial $P \in \mathbb{R}[X]$ such that for every $k \in \mathbb{N}$ one has

$$\sum_{x \in \text{Irr}(\mathbb{H}), k \leq l(x) < k+1} \dim(x)^2 \leq P(k)$$

- *Property (RD)* if there exists a polynomial $P \in \mathbb{R}[X]$ such that for every $k \in \mathbb{N}$ and $a \in q_k c_c(\widehat{\mathbb{H}})$, we have $\|\mathcal{F}(a)\|_{C(\mathbb{H})} \leq P(k) \|a\|_{\mathbb{H},0}$.

Finally, $\widehat{\mathbb{H}}$ is said to have *polynomial growth (resp. property (RD))* if there exists a length function l on $\text{Irr}(\mathbb{H})$ such that $(\widehat{\mathbb{H}}, l)$ has polynomial growth (resp. property (RD)).

It is known from [Ve07] that if $(\widehat{\mathbb{H}}, l)$ has polynomial growth then $(\widehat{\mathbb{H}}, l)$ has rapid decay and the converse also holds when we assume \mathbb{H} to be co-amenable. Moreover, it is shown also shown in [Ve07] that duals of compact connected real Lie groups have polynomial growth. The fact that polynomial growth implies *(RD)* can easily be deduced from the following lemma.

Lemma 5.1. *Let \mathbb{H} be a CQG, $F \subset \text{Irr}(\mathbb{H})$ a finite subset and $a \in l^\infty(\widehat{\mathbb{H}})$ with $ap_x = 0$ for all $x \notin F$. Then,*

$$\|\mathcal{F}_{\mathbb{H}}(a)\| \leq 2 \sqrt{\sum_{x \in F} \dim(x)^2} \|a\|_{\mathbb{H},0}.$$

Proof. One can copy the proof of Proposition 4.2, assertion (a), in [BVZ14] or the proof of Proposition 4.4, assertion (ii), in [Ve07]. \square

5.2 Technicalities

Let (Γ, G) be a matched pair with actions (α, β) and denote by \mathbb{G} the bicrossed product.

Recall that $\text{Irr}(\mathbb{G}) = \sqcup_{\gamma \in I} \text{Irr}(G_\gamma)$, where $I \subset \Gamma$ is a complete set of representatives for Γ/G . For $\gamma \in I$ and $x \in \text{Irr}(G_\gamma)$, we denote by $\gamma(x)$ the corresponding element in $\text{Irr}(\mathbb{G})$. If a complete set of representatives of $\text{Irr}(G_\gamma)$, $x \in \text{Irr}(G_\gamma)$ is given by $u^x \in \mathcal{B}(H_x) \otimes C(G_\gamma)$ then a representative for $\gamma(x)$ is given by

$$u^{\gamma(x)} := \sum_{r,s \in \gamma \cdot G} e_{rs} \otimes (1 \otimes u_r v_{rs}) u^x \circ \psi_{r,s} \in \mathcal{B}(l^2(\gamma \cdot G)) \otimes C(\mathbb{G}).$$

The lemma below gives a way of obtaining an element $\tilde{a} \in c_c(\widehat{G})$ from an $a \in c_c(\widehat{G}_\gamma)$ in a suitable way so that they are compatible with the Fourier transforms.

Lemma 5.2. *Let $\gamma \in \Gamma$ and $a \in c_c(\widehat{G}_\gamma)$. Define $\tilde{a} \in c_c(\widehat{G})$ by:*

$$\tilde{a}p_y = \sum_{x \in \text{supp}(a) \text{ and } y \subset \text{Ind}_\gamma^G(x)} \frac{\dim(x)}{\dim(y)} \sum_{i=1}^{\dim(\text{Mor}_G(y, \text{Ind}_\gamma^G(x)))} (S_i^y)^*(e_{\gamma\gamma} \otimes ap_x)S_i^y,$$

where $S_i^y \in \text{Mor}(y, \text{Ind}_\gamma^G(x))$ is a basis of isometries with pairwise orthogonal images. The following holds.

1. If (l_Γ, l) is a matched pair of length functions on $(\Gamma, \text{Irr}(G))$ then, for all $y \in \text{supp}(\tilde{a})$ one has

$$l(y) \leq \max(\{l_\gamma(x) : x \in \text{supp}(a)\}) + l_\Gamma(\gamma),$$

where $(l_\gamma)_{\gamma \in \Gamma}$ is any family of maps realizing the compatibility relations of Definition 4.1.

2. $\mathcal{F}_{G_\gamma}(a) = v_{\gamma\gamma}\mathcal{F}_G(\tilde{a})$.

3. $\|\tilde{a}\|_{G,0} \leq \|a\|_{G_\gamma,0}$.

Proof. (1). Since any $y \in \text{supp}(\tilde{a})$ is such that $y \subset \text{Ind}_\gamma^G(x) = \varepsilon_{G_{\gamma^{-1}}} \otimes_1 x$ for some $x \in \text{supp}(a)$, it follows that any $y \in \text{supp}(\tilde{a})$ satisfies $l(y) = l_1(y) \leq l_{\gamma^{-1}}(\varepsilon_{G_{\gamma^{-1}}}) + l_\gamma(x) = l_\Gamma(\gamma^{-1}) + l_\gamma(x) = l_\Gamma(\gamma) + l_\gamma(x)$ for some $x \in \text{supp}(a)$.

(2). One has:

$$\begin{aligned} v_{\gamma\gamma}\mathcal{F}_G(\tilde{a}) &= v_{\gamma\gamma} \sum_y \dim(y)(\text{Tr}_y \otimes \text{id})(u^y \tilde{a}p_y \otimes 1) \\ &= v_{\gamma\gamma} \sum_{x \in \text{supp}(a), y \subset \text{Ind}_\gamma^G(x)} \sum_{i=1}^{\dim(\text{Mor}(y, \text{Ind}_\gamma^G(x)))} \dim(x)(\text{Tr}_y \otimes \text{id})(u^y((S_i^y)^*(e_{\gamma\gamma} \otimes ap_x)S_i^y) \otimes 1) \\ &= v_{\gamma\gamma} \sum_{x,y,i} \dim(x)(\text{Tr}_y \otimes \text{id})(((S_i^y)^* \otimes 1)\text{Ind}_\gamma^G(u^x)(e_{\gamma\gamma} \otimes ap_x \otimes 1)(S_i^y \otimes 1)) \\ &= v_{\gamma\gamma} \sum_{x,y,i} \dim(x)(\text{Tr}_{l^2(\gamma \cdot G) \otimes H_x} \otimes \text{id})(\text{Ind}_\gamma^G(u^x)(e_{\gamma\gamma} \otimes ap_x \otimes 1)(S_i^y(S_i^y)^* \otimes 1)) \\ &= v_{\gamma\gamma} \sum_{x \in \text{supp}(a)} \dim(x)(\text{Tr}_{l^2(\gamma \cdot G) \otimes H_x} \otimes \text{id})(\text{Ind}_\gamma^G(u^x)(e_{\gamma\gamma} \otimes ap_x \otimes 1)) \\ &= v_{\gamma\gamma} \sum_{x \in \text{supp}(a)} \dim(x)(\text{Tr}_x \otimes \text{id})(u^x ap_x \otimes 1) = \mathcal{F}_{G_\gamma}(a), \end{aligned}$$

where, in the 3rd equation we use the fact that $(S_i^y)^* \in \text{Mor}(\text{Ind}_\gamma^G(x), y)$ and, in the last equation we use the definition of the representation $\text{Ind}_\gamma^G(u^x)$.

(3). One has:

$$\begin{aligned} \|\tilde{a}\|_{G,0}^2 &= \sum_y \dim(y)\text{Tr}_y(\tilde{a}^* \tilde{a}p_y) \\ &= \sum_{x \in \text{supp}(a), y \subset \text{Ind}_\gamma^G(x)} \sum_{i,j=1}^{\dim(\text{Mor}(y, \text{Ind}_\gamma^G(x)))} \dim(y) \frac{\dim(x)^2}{\dim(y)^2} \text{Tr}_y((S_i^y)^*(e_{\gamma\gamma} \otimes a^*p_x)S_i^y(S_j^y)^*(e_{\gamma\gamma} \otimes ap_x)S_j^y) \\ &= \sum_{x,y,i} \dim(x) \left(\frac{\dim(x)}{\dim(y)} \right) \text{Tr}_y((S_i^y)^*(e_{\gamma\gamma} \otimes a^*p_x)S_i^y(S_i^y)^*(e_{\gamma\gamma} \otimes ap_x)S_i^y) \end{aligned}$$

Since, for all y, i , $S_i^y(S_i^y)^*$ is a projection, one has $S_i^y(S_i^y)^* \leq 1$ hence,

$$\mathrm{Tr}_y((S_i^y)^*(e_{\gamma\gamma} \otimes a^* p_x) S_i^y (S_i^y)^*(e_{\gamma\gamma} \otimes a p_x) S_i^y) \leq \mathrm{Tr}_y((S_i^y)^*(e_{\gamma\gamma} \otimes a^* a p_x) S_i^y).$$

Moreover, by Proposition 3.3, one has $y \subset \mathrm{Ind}_\gamma^G(x)$ if and only if

$$\dim(\mathrm{Mor}_{G_\gamma}(\mathrm{Res}_{G_\gamma}^G(y), x)) = \dim(\mathrm{Mor}_G(y, \mathrm{Ind}_\gamma^G(x))) \neq 0.$$

Since x is irreducible, we find that $y \subset \mathrm{Ind}_\gamma^G(x) \Leftrightarrow x \subset \mathrm{Res}_{G_\gamma}^G(y)$. In particular, for any $y \subset \mathrm{Ind}_\gamma^G(x)$ one has $\dim(x) \leq \dim(y)$. Hence,

$$\begin{aligned} \|\tilde{a}\|_{G,0}^2 &\leq \sum_{x,y,i} \dim(x) \mathrm{Tr}_y((S_i^y)^*(e_{\gamma\gamma} \otimes a^* a p_x) S_i^y) = \sum_{x,y,i} \dim(x) \mathrm{Tr}_{l^2(\gamma \cdot G) \otimes H_x}(e_{\gamma\gamma} \otimes a^* a p_x (S_i^y)^* S_i^y) \\ &= \sum_{x \in \mathrm{supp}(a)} \dim(x) \mathrm{Tr}_{l^2(\gamma \cdot G) \otimes H_x}(e_{\gamma\gamma} \otimes a^* a p_x) = \sum_{x \in \mathrm{supp}(a)} \dim(x) \mathrm{Tr}_x(a^* a p_x) = \|a\|_{G,\gamma,0}^2. \quad \square \end{aligned}$$

Lemma 5.3. *Let (l_Γ, l) be a matched pair of length functions on $(\Gamma, \mathrm{Irr}(G))$. If (\widehat{G}, l) has polynomial growth then, there exists $C > 0$ and $N \in \mathbb{N}$ such that:*

- $\|\mathcal{F}_G(a)\| \leq C(k+1)^N \|a\|_{G,0}$ for all $a \in c_c(\widehat{G})$ with $\mathrm{supp}(a) \subset \{x \in \mathrm{Irr}(G) : l(x) < k+1\}$.
- $|\gamma \cdot G| \dim(x) \leq C(l_\Gamma(\gamma) + l_\gamma(x) + 1)^N$ for all $\gamma \in \Gamma, x \in \mathrm{Irr}(G_\gamma)$.
- For all $\gamma \in \Gamma, \sum_{x \in \mathrm{Irr}(G_\gamma), l_\gamma(x) < k+1} \dim(x)^2 \leq C^2(k + l_\Gamma(\gamma) + 1)^{2N}$.

Proof. Let $P \in \mathbb{R}[X]$ be such that $\sum_{x \in \mathrm{Irr}(G), k \leq l(x) < k+1} \dim(x)^2 \leq P(k)$ for all $k \in \mathbb{N}$ and let $C_1 > 0$ and $N_1 \in \mathbb{N}$ be such that $P(k) \leq C_1(k+1)^{N_1}$ for all $k \in \mathbb{N}$. By Lemma 5.1 one has, for all $a \in c_c(\widehat{G})$, with $\mathrm{supp}(a) \subset \{x \in \mathrm{Irr}(G) : k \leq l(x) < k+1\}$, $\|\mathcal{F}_G(a)\| \leq 2\sqrt{P(k)} \|a\|_{G,0} \leq \sqrt{C_1}(k+1)^{\frac{N_1}{2}} \|a\|_{G,0}$. Now, suppose that $\mathrm{supp}(a) \subset \{x \in \mathrm{Irr}(G) : l(x) < k+1\}$ so that $a \in q_k c_c(\widehat{G})$, where $q_k = \sum_{j=0}^k p_j$ and $p_j = \sum_{x \in \mathrm{Irr}(G), k \leq l(x) < k+1}$. One has,

$$\|\mathcal{F}_G(a)\| = \sum_{j=0}^k \|\mathcal{F}_G(ap_j)\| \leq \sum_{j=0}^k \sqrt{C_1}(j+1)^{\frac{N_1}{2}} \|a\|_{G,0} \leq \sqrt{C_1}(k+1)^{\frac{N_1}{2}+1} \|a\|_{G,0}. \quad (5.1)$$

Now, let $\gamma \in \Gamma$ and $x \in \mathrm{Irr}(G_\gamma)$. By Proposition 3.3 one has:

$$\begin{aligned} |\gamma \cdot G| \dim(x) &= \dim(\mathrm{Ind}_\gamma^G(x)) = \sum_{y \in \mathrm{Irr}(G)} \dim(\mathrm{Mor}_G(y, \mathrm{Ind}_\gamma^G(x))) \dim(y) \\ &= \sum_{y \in \mathrm{Irr}(G), y \subset \mathrm{Ind}_\gamma^G(x)} \dim(\mathrm{Mor}_{G_\gamma}(\mathrm{Res}_{G_\gamma}^G(y), x)) \dim(y). \end{aligned}$$

Note that $\dim(\mathrm{Mor}_{G_\gamma}(\mathrm{Res}_{G_\gamma}^G(y), x)) \leq \dim(y)$ for all x, y . Moreover, since $\mathrm{Ind}_\gamma^G(x) \simeq \varepsilon_{G_{\gamma^{-1}}} \otimes x$ and the pair (l_Γ, l) is matched, one has $\{y \in \mathrm{Irr}(G), y \subset \mathrm{Ind}_\gamma^G(x)\} \subset \{y \in \mathrm{Irr}(G) : l(y) \leq l_\Gamma(\gamma) + l_\gamma(x)\}$. Hence,

$$\begin{aligned} |\gamma \cdot G| \dim(x) &\leq \sum_{y \in \mathrm{Irr}(G), l(y) < l_\Gamma(\gamma) + l_\gamma(x) + 1} \dim(y)^2 = \sum_{j=0}^{l_\Gamma(\gamma) + l_\gamma(x)} \sum_{y \in \mathrm{Irr}(G), j \leq l(y) < j+1} \dim(y)^2 \\ &\leq \sum_{j=0}^{l_\Gamma(\gamma) + l_\gamma(x)} P(j) \leq C_1 \sum_{j=0}^{l_\Gamma(\gamma) + l_\gamma(x)} (j+1)^{N_1} \leq C_1(l_\Gamma(\gamma) + l_\gamma(x) + 1)^{N_1+1}. \quad (5.2) \end{aligned}$$

It follows from Equations (5.1) and (5.2) that $C := \mathrm{Max}(C_1, \sqrt{C_1})$ and $N := N_1 + 1$ do the job.

Let us show the last point. Fix $\gamma \in \Gamma$ and let $F \subset \text{Irr}(G_\gamma)$ a finite subset. Define $p_F \in c_c(\widehat{G}_\gamma)$ by $p_F = \sum_{x \in F} p_x$ and note that $\mathcal{F}_{G_\gamma}(p_F) = \sum_{x \in F} \dim(x)\chi(x)$ and $\|a\|_{G_\gamma,0}^2 = \sum_{x \in F} \dim(x)^2$. Suppose that $F \subset \{x \in \text{Irr}(G_\gamma) : l_\gamma(x) < k + 1\}$. Using Lemma 5.2 and the first part of the proof we find, since $\widetilde{p}_F \in c_c(\widehat{G})$ with $\text{supp}(\widetilde{p}_F) \subset \{x \in \text{Irr}(G) : l(x) < l_\Gamma(\gamma) + k + 1\}$,

$$\begin{aligned} \left\| \sum_{x \in F} \dim(x)\chi(x) \right\|^2 &= \|\mathcal{F}_{G_\gamma}(p_F)\|^2 = \|v_{\gamma\gamma}\mathcal{F}_G(\widetilde{p}_F)\|^2 \leq \|\mathcal{F}_G(\widetilde{p}_F)\|^2 \leq C^2(k + l_\Gamma(\gamma) + 1)^{2N} \|\widetilde{p}_F\|_{G,0}^2 \\ &\leq C^2(k + l_\Gamma(\gamma) + 1)^{2N} \|p_F\|_{G_\gamma,0}^2 = C^2(k + l_\Gamma(\gamma) + 1)^{2N} \sum_{x \in F} \dim(x)^2. \end{aligned}$$

It follows that:

$$\left(\sum_{x \in F} \dim(x)^2 \right)^2 = \left(\sum_{x \in F} \dim(x)\chi(x)(1) \right)^2 \leq \left\| \sum_{x \in F} \dim(x)\chi(x) \right\|_{C(G)}^2 \leq C^2(k + l_\Gamma(\gamma) + 1)^{2N} \sum_{x \in F} \dim(x)^2.$$

Hence, for all non empty finite subsets $F \subset \{x \in \text{Irr}(G_\gamma) : l_\gamma(x) < k + 1\}$ one has $\sum_{x \in F} \dim(x)^2 \leq C^2(k + l_\Gamma(\gamma) + 1)^{2N}$. The last assertion follows. \square

5.3 Polynomial growth for bicrossed product

We start with the following result.

Theorem 5.4. *Suppose that (l_G, l_Γ) is a matched pair of length functions on (Γ, G) . If both (Γ, l_Γ) and (\widehat{G}, l_G) has polynomial growth then (\mathbb{G}, \tilde{l}) have polynomial growth.*

Proof. Let $I \subset \Gamma$ be a complete set of representatives for Γ/G so that $\text{Irr}(\mathbb{G}) = \sqcup_{\gamma \in I} \text{Irr}(G_\gamma)$. Let $k \geq 1$ and define

$$F_k := \{z \in \text{Irr}(\mathbb{G}) : \tilde{l}(z) < k\} \subset \sqcup_{\gamma \in I_k} T_{\gamma,k},$$

where $I_k := \{\gamma \in \Gamma : l_\Gamma(\gamma) < k + 1\} \cap I$ and $T_{\gamma,k} := \{x \in \text{Irr}(G_\gamma) : l_\gamma(x) < k + 1\}$. Since (Γ, l_Γ) has polynomial growth, there exists a polynomial P_1 such that, for all $k \in \mathbb{N}$, $|I_k| \leq P_1(k)$. Moreover, since (\widehat{G}, l_G) has polynomial growth, we can apply Lemma 5.3 to get $C > 0$ and $N \in \mathbb{N}$ such that, for all $k \in \mathbb{N}$ and all $\gamma \in I_k$, one has $\sum_{x \in T_{\gamma,k}} \dim(x)^2 \leq C^2(2k + 2)^{2N}$ and, $|\gamma \cdot G| = |\gamma \cdot G| \dim(\varepsilon_G) \leq C(2k + 3)^N$. Hence, for all $k \geq 1$,

$$\begin{aligned} \sum_{z \in F_k} \dim(z)^2 &= \sum_{\gamma \in I_k} |\gamma \cdot G|^2 \sum_{x \in T_{\gamma,k}} \dim(x)^2 \leq C^2(2k + 2)^{2N} \sum_{\gamma \in I_k} |\gamma \cdot G|^2 \leq C^4(2k + 2)^{2N} (2k + 3)^{2N} |I_k| \\ &\leq C^4(2k + 2)^{2N} (2k + 3)^{2N} P_1(k). \end{aligned} \quad \square$$

To complete the proof of Theorem B, we need the following Proposition.

Proposition 5.5. *Assume that there exists a length function l on $\text{Irr}(\mathbb{G})$ such that $(\widehat{\mathbb{G}}, l)$ has polynomial growth and consider the matched pair of length functions (l_Γ, l_G) associated to l given in Proposition 4.2. Then (Γ, l_Γ) and (\widehat{G}, l_G) both have polynomial growth.*

Proof. Assume that $(\widehat{\mathbb{G}}, l)$ has polynomial growth. Since the map $\text{Irr}(G) \rightarrow \text{Irr}(\mathbb{G})$, $x \mapsto 1(x)$ is injective, dimension preserving and length preserving (by definition of l_G), it is clear that (\widehat{G}, l_G) has polynomial growth. Let us show that (Γ, l_Γ) also has polynomial growth. Let P be a polynomial witnessing (RD) for $(\widehat{\mathbb{G}}, l)$ and $k \in \mathbb{N}$. Define $F_k := \{\gamma \in \Gamma : k \leq l_\Gamma(\gamma) < k + 1\}$. One has, for all $k \in \mathbb{N}$,

$$\begin{aligned} |F_k| &= \sum_{k \leq l([\gamma(\varepsilon_G)]) < k+1} 1 \leq \sum_{k \leq l([\gamma(\varepsilon_G)]) < k+1} |\gamma \cdot G|^2 = \sum_{k \leq l([\gamma(\varepsilon_G)]) < k+1} \dim([\gamma(\varepsilon_G)])^2 \\ &\leq \sum_{z \in \text{Irr}(\mathbb{G}), k \leq l(z) < k+1} \dim(z)^2 \leq P(k). \end{aligned} \quad \square$$

5.4 Rapid decay for bicrossed product

Recall that $l^\infty(\widehat{\mathbb{G}}) = \bigoplus_{\gamma \cdot G \in \Gamma/G} \bigoplus_{x \in \text{Irr}(G_\gamma)} \mathcal{B}(l^2(\gamma \cdot G) \otimes H_x)$. Let us denote by $p_{\gamma(x)}$ the central projection of $l^\infty(\widehat{\mathbb{G}})$ corresponding to the block $\mathcal{B}(l^2(\gamma \cdot G) \otimes H_x)$ and define, for $\gamma \cdot G \in \Gamma/G$, the central projection :

$$p_\gamma := \sum_{x \in \text{Irr}(G_\gamma)} p_{\gamma(x)} \in l^\infty(\widehat{\mathbb{G}}).$$

Note that $p_\gamma l^\infty(\widehat{\mathbb{G}}) = \bigoplus_{x \in \text{Irr}(G_\gamma)} \mathcal{B}(l^2(\gamma \cdot G) \otimes H_x) \simeq \mathcal{B}(l^2(\gamma \cdot G)) \otimes L(G_\gamma)$, where $L(G_\gamma) = \bigoplus_{x \in \text{Irr}(G_\gamma)} \mathcal{B}(H_x)$ is the group von-Neumann algebra of G_γ (which is also the multiplier C^* -algebra of $C_r^*(G_\gamma) = \bigoplus_{x \in \text{Irr}(G_\gamma)}^{c_0} \mathcal{B}(H_x)$). Using this identification, we define $\pi_\gamma : c_0(\widehat{\mathbb{G}}) \rightarrow \mathcal{B}(l^2(\gamma \cdot G)) \otimes C_r^*(G_\gamma) \subset c_0(\widehat{\mathbb{G}})$ to be the $*$ -homomorphism given by $\pi_\gamma(a) = ap_\gamma$, for all $a \in c_0(\widehat{\mathbb{G}})$. We also write, for $a \in c_0(\widehat{\mathbb{G}})$, $\pi_\gamma(a) = \sum_{r,s \in \gamma \cdot G} e_{rs} \otimes \pi_{r,s}^\gamma(a)$, where we recall that (e_{rs}) are the matrix units associated to the canonical orthonormal basis $(e_r)_{r \in \gamma \cdot G}$ of $l^2(\gamma \cdot G)$ and $\pi_{r,s}^\gamma : c_0(\widehat{\mathbb{G}}) \rightarrow C_r^*(G_\gamma)$ is the completely bounded map defined by $\pi_{r,s}^\gamma := (\omega_{e_s, e_r} \otimes \text{id}) \circ \pi_\gamma$ and $\omega_{e_s, e_r} \in \mathcal{B}(l^2(\gamma \cdot G))$, $\omega_{e_s, e_r}(T) = \langle T e_s, e_r \rangle$.

We start with the following result.

Theorem 5.6. *Let (l_Γ, l_G) be a matched pair of length functions on $(\Gamma, \text{Irr}(G))$. Suppose that (\widehat{G}, l_G) has polynomial growth and (Γ, l_Γ) has (RD). Then $(\widehat{\mathbb{G}}, \widehat{l})$ has (RD).*

Proof. Let $a \in c_c(\widehat{\mathbb{G}})$ and write $a = \sum_{\gamma \in S} \sum_{x \in T_\gamma} ap_{\gamma(x)}$, where $S \subset I$ and $T_\gamma \subset \text{Irr}(G_\gamma)$ are finite subsets.

Claim. *The following holds.*

1. $\mathcal{F}_{\mathbb{G}}(a) = \sum_{\gamma \in S} |\gamma \cdot G| \left(\sum_{r,s \in \gamma \cdot G} u_r v_{rs} \mathcal{F}_{G_\gamma}(\pi_{s,r}^\gamma(a)) \circ \psi_{r,s}^\gamma \right)$.
2. $\|a\|_{\mathbb{G},0}^2 = \sum_{\gamma \in S} |\gamma \cdot G| \left(\sum_{r,s \in \gamma \cdot G} \|\pi_{r,s}^\gamma(a)\|_{G_\gamma,0}^2 \right)$.

Proof of the Claim.(1). A direct computation gives:

$$\begin{aligned} \mathcal{F}_{\mathbb{G}}(a) &= \sum_{\gamma \in S, x \in T_\gamma} |\gamma \cdot G| \dim(x) (\text{Tr}_{l^2(\gamma \cdot G) \otimes H_x}(\gamma(u^x) ap_{\gamma(x)} \otimes 1)) \\ &= \sum_{\gamma \in S, x \in T_\gamma} |\gamma \cdot G| \dim(x) \sum_{r,s \in \gamma \cdot G} u_r v_{rs} (\text{Tr}_x \otimes \text{id})(u^x \circ \psi_{r,s}^\gamma \pi_{s,r}^\gamma(a) p_x \otimes 1) \\ &= \sum_{\gamma \in S} |\gamma \cdot G| \sum_{r,s \in \gamma \cdot G} u_r v_{rs} \mathcal{F}_{G_\gamma}(\pi_{s,r}^\gamma(a)) \circ \psi_{r,s}^\gamma. \end{aligned}$$

(2). Since π_γ is a $*$ -homomorphism, we have $\pi_{r,s}^\gamma(a^* a) = \sum_{t \in \gamma \cdot G} \pi_{t,r}^\gamma(a)^* \pi_{t,s}^\gamma(a)$ hence,

$$\begin{aligned} \|a\|_{\mathbb{G},0}^2 &= \sum_{\gamma \in S, x \in T_\gamma} |\gamma \cdot G| \dim(x) \sum_{r,s \in \gamma \cdot G} (\text{Tr}_x \otimes \text{id})(\pi_{s,r}^\gamma(a)^* \pi_{r,s}^\gamma(a)) \\ &= \sum_{\gamma \in S} |\gamma \cdot G| \sum_{r,s \in \gamma \cdot G} \|\pi_{r,s}^\gamma(a)\|_{G_\gamma,0}^2. \end{aligned}$$

Let us now prove the theorem. Let $b = \sum_{\gamma \in S'} \sum_{t,t' \in \gamma \cdot G} u_t v_{tt'} F_\gamma \circ \psi_{t,t'}^\gamma \in C(\mathbb{G})$, where $F_\gamma \in C(G_\gamma)$ and $S' \subset I$ is a finite subset. For all $r \in \Gamma$, we denote by γ_r the unique element in I such that $\gamma_r \cdot G = r \cdot G$. We may re-order the sums and write:

$$\mathcal{F}_{\mathbb{G}}(a) = \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G| \left(\sum_{s \in r \cdot G} u_r v_{rs} \mathcal{F}_{G_{\gamma_r}}(\pi_{s,r}^{\gamma_r}(a)) \circ \psi_{r,s}^{\gamma_r} \right) \text{ and } b = \sum_{t \in \Gamma} u_t 1_{S' \cdot G}(t) \left(\sum_{t' \in t \cdot G} v_{tt'} F_{\gamma_t} \circ \psi_{t,t'}^{\gamma_t} \right).$$

Also, $\|a\|_{\mathbb{G},0}^2 = \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G| \left(\sum_{s \in r \cdot G} \|\pi_{r,s}^{\gamma_r}(a)\|_{G_{\gamma_r},0}^2 \right)$. Then, $\|\mathcal{F}_{\mathbb{G}}(a)b\|_{2,h_{\mathbb{G}}}$ is equal to :

$$\begin{aligned}
& \left\| \sum_{r,t \in \Gamma} u_{rt} 1_{S \cdot G}(r) 1_{S' \cdot G}(t) |r \cdot G| \left(\sum_{s \in r \cdot G, t' \in t \cdot G} v_{rs} \circ \alpha_t \mathcal{F}_{G_{\gamma_r}}(\pi_{s,r}^{\gamma_r}(a)) \circ \psi_{r,s}^{\gamma_r} \circ \alpha_t v_{tt'} F_{\gamma_t} \circ \psi_{t,t'}^{\gamma_t} \right) \right\|_{2, h_G}^2 \\
&= \sum_{x \in \Gamma} \left\| \sum_{\substack{r,t \in \Gamma \\ rt=x}} 1_{S \cdot G}(r) 1_{S' \cdot G}(t) |r \cdot G| \left(\sum_{s \in r \cdot G, t' \in t \cdot G} v_{rs} \circ \alpha_t \mathcal{F}_{G_{\gamma_r}}(\pi_{s,r}^{\gamma_r}(a)) \circ \psi_{r,s}^{\gamma_r} \circ \alpha_t v_{tt'} F_{\gamma_t} \circ \psi_{t,t'}^{\gamma_t} \right) \right\|_2^2 \\
&= \sum_{x \in \Gamma} \left\| \sum_{\substack{r,t \in \Gamma \\ rt=x}} 1_{S \cdot G}(r) 1_{S' \cdot G}(t) |r \cdot G| \left(\sum_{s \in r \cdot G} v_{rs} \circ \alpha_t \mathcal{F}_{G_{\gamma_r}}(\pi_{s,r}^{\gamma_r}(a)) \circ \psi_{r,s}^{\gamma_r} \circ \alpha_t \right) \left(\sum_{t' \in t \cdot G} v_{tt'} F_{\gamma_t} \circ \psi_{t,t'}^{\gamma_t} \right) \right\|_2^2 \\
&\leq \sum_x \left(\sum_{\substack{r,t \in \Gamma \\ rt=x}} 1_{S \cdot G}(r) 1_{S' \cdot G}(t) |r \cdot G| \left\| \sum_{s \in r \cdot G} v_{rs} \circ \alpha_t \mathcal{F}_{G_{\gamma_r}}(\pi_{s,r}^{\gamma_r}(a)) \circ \psi_{r,s}^{\gamma_r} \circ \alpha_t \right\|_\infty \left\| \sum_{t' \in t \cdot G} v_{tt'} F_{\gamma_t} \circ \psi_{t,t'}^{\gamma_t} \right\|_2 \right)^2 \\
&= \sum_x \left(\sum_{\substack{r,t \in \Gamma \\ rt=x}} \left(1_{S \cdot G}(r) |r \cdot G| \left\| \sum_{s \in r \cdot G} v_{rs} \mathcal{F}_{G_{\gamma_r}}(\pi_{s,r}^{\gamma_r}(a)) \circ \psi_{r,s}^{\gamma_r} \right\|_\infty \right) \left(1_{S' \cdot G}(t) \left\| \sum_{t' \in t \cdot G} v_{tt'} F_{\gamma_t} \circ \psi_{t,t'}^{\gamma_t} \right\|_2 \right) \right)^2 \\
&= \|\psi * \phi\|_{l^2(\Gamma)}^2,
\end{aligned}$$

where $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote respectively the L^2 -norm and the supremum norm on $C(G)$ and $\psi, \phi : \Gamma \rightarrow \mathbb{R}_+$ are finitely supported functions defined by :

$$\psi(r) := 1_{S \cdot G}(r) |r \cdot G| \left\| \sum_{s \in r \cdot G} v_{rs} \mathcal{F}_{G_{\gamma_r}}(\pi_{s,r}^{\gamma_r}(a)) \circ \psi_{r,s}^{\gamma_r} \right\|_\infty \quad \text{and} \quad \phi(t) := 1_{S' \cdot G}(t) \left\| \sum_{t' \in t \cdot G} v_{tt'} F_{\gamma_t} \circ \psi_{t,t'}^{\gamma_t} \right\|_2,$$

Note that $\|\phi\|_{l^2(\Gamma)}^2 = \|b\|_{2, h_G}^2$. Moreover, one has, since $\psi_{r,s}^\gamma : G_{r,s} \rightarrow G_\gamma$ is an homeomorphism,

$$\begin{aligned}
\|\psi\|_{l^2(\Gamma)}^2 &= \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^2 \left\| \sum_{s \in r \cdot G} v_{rs} \mathcal{F}_{G_{\gamma_r}}(\pi_{s,r}^{\gamma_r}(a)) \circ \psi_{r,s}^{\gamma_r} \right\|_\infty^2 \\
&\leq \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \sum_{s \in r \cdot G} \|v_{rs} \mathcal{F}_{G_{\gamma_r}}(\pi_{s,r}^{\gamma_r}(a)) \circ \psi_{r,s}^{\gamma_r}\|_\infty^2 \\
&= \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \sum_{s \in r \cdot G} \|\mathcal{F}_{G_{\gamma_r}}(\pi_{s,r}^{\gamma_r}(a))\|_{C(G_{\gamma_r})}^2.
\end{aligned}$$

For $k \in \mathbb{N}$ let $p_k = \sum_{\gamma \in I, x \in \text{Irr}(G_\gamma) : k \leq l(\gamma(x)) < k+1} p_{\gamma(x)} \in l^\infty(\widehat{\mathbb{G}})$, $p_k^{G_\gamma} = \sum_{x \in \text{Irr}(G_\gamma) : k \leq l_{G_\gamma}(x) < k+1} p_x \in l^\infty(\widehat{G}_\gamma)$ and suppose from now on that $a \in p_k c_c(\widehat{\mathbb{G}})$. Hence, we must have $S \subset \{\gamma \in \Gamma : l_\Gamma(\gamma) < k+1\}$ and, for all $\gamma \in S$, $T_\gamma \subset \{x \in \text{Irr}(G_\gamma) : l_{G_\gamma}(x) < k+1\}$. Hence, for all $\gamma \in S$ and all $r, s \in \gamma \cdot G$ one has $\pi_{r,s}^\gamma(a) \in q_k^\gamma c_c(\widehat{G}_\gamma)$, where $q_k^\gamma = \sum_{j=0}^k p_j^{G_\gamma}$.

Since (\widehat{G}, l_G) has polynomial growth, there exists $C > 0$ and $N \in \mathbb{N}$ satisfying the properties of Lemma 5.3. In particular, one has, for all $\gamma \in \Gamma$, $|\gamma \cdot G| \leq C(2l_\Gamma(\gamma) + 1)^N$. Moreover, since $S \subset \{g \in \Gamma : l_\Gamma(g) < k+1\}$ and l_Γ is β -invariant, it follows that $S \cdot G \subset \{g \in \Gamma : l_\Gamma(g) < k+1\}$. By Lemma 5.2 (and Lemma 5.3) we deduce that:

$$\begin{aligned}
\|\psi\|_{l^2(\Gamma)}^2 &\leq \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \sum_{s \in r \cdot G} \left\| v_{\gamma_r, \gamma_r} \widetilde{\mathcal{F}_G(\pi_{s,r}^{\gamma_r}(a))} \right\|^2 \leq \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \sum_{s \in r \cdot G} \left\| \widetilde{\mathcal{F}_G(\pi_{s,r}^{\gamma_r}(a))} \right\|^2 \\
&\leq \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \sum_{s \in r \cdot G} C^2(k + l_\Gamma(\gamma_r) + 1)^{2N} \left\| \widetilde{\pi_{s,r}^{\gamma_r}(a)} \right\|_{G,0}^2
\end{aligned}$$

$$\begin{aligned}
&\leq C^2(2k+2)^{2N} \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G|^3 \sum_{s \in r \cdot G} \|\pi_{s,r}^{\gamma_r}(a)\|_{G_{\gamma_r},0}^2 \\
&\leq C^4(2k+3)^{4N} \sum_{r \in \Gamma} 1_{S \cdot G}(r) |r \cdot G| \sum_{s \in r \cdot G} \|\pi_{s,r}^{\gamma_r}(a)\|_{G_{\gamma_r},0}^2 = C^4(2k+3)^{4N} \|a\|_{\mathbb{G},0}^2.
\end{aligned}$$

Since (Γ, l_Γ) has (RD) , let $C_2 > 0$ and $N_2 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, for all function ξ on Γ supported on $\{g \in \Gamma : l_\Gamma(g) < k+1\}$, we have $\|\xi * \eta\|_{l^2(\Gamma)} \leq C_2(k+1)^{N_2} \|\xi\|_{l^2(\Gamma)} \|\eta\|_{l^2(\Gamma)}$. Note that ψ is supported on $S \cdot G$ and $S \cdot G \subset \{g \in \Gamma : l_\Gamma(g) < k+1\}$. Hence, it follows from the preceding computations that:

$$\begin{aligned}
\|\mathcal{F}_{\mathbb{G}}(a)b\|_{2,h_{\mathbb{G}}}^2 &\leq \|\psi * \phi\|_{l^2(\Gamma)}^2 \leq C_2^2(k+1)^{2N_2} \|\psi\|_{l^2(\Gamma)} \|\phi\|_{l^2(\Gamma)} \leq C^4(2k+3)^{4N} C_2^2(k+1)^{2N_2} \|a\|_{\mathbb{G},0}^2 \|b\|_{2,h_{\mathbb{G}}}^2 \\
&= (P(k) \|a\|_{\mathbb{G},0}^2 \|b\|_{2,h_{\mathbb{G}}}^2)^2.
\end{aligned}$$

where $P(X) = C^2 C_2^2 (2X+3)^{2N} (X+1)^{N_2}$. It concludes the proof. \square

To complete the proof of Theorem A, we need the following Proposition.

Proposition 5.7. *Assume that there exists a length function l on $\text{Irr}(\mathbb{G})$ such that $(\widehat{\mathbb{G}}, l)$ has (RD) and consider the matched pair of length functions (l_Γ, l_G) associated to l given in Proposition 4.2. Then (Γ, l_Γ) has (RD) and (\widehat{G}, l_G) has polynomial growth.*

Proof. Suppose that $(\widehat{\mathbb{G}}, l)$ has (RD) . The fact that (\widehat{G}, l_G) has (RD) follows from the general theory (since $C(G) \subset C(\mathbb{G})$ intertwines the comultiplication and the associated injection $\text{Irr}(G) \rightarrow \text{Irr}(\mathbb{G})$, actually given by $(x \mapsto 1(x))$, preserves the length functions). Let us show that (Γ, l_Γ) has (RD) . Let $k \in \mathbb{N}$ and $\xi : \Gamma \rightarrow \mathbb{C}$ be a finitely supported function with support in $\{\gamma \in \Gamma : k \leq l_\Gamma(\gamma) < k+1\}$. Define $\tilde{\xi} \in c_c(\widehat{\mathbb{G}})$ by $\tilde{\xi} = \sum_{\gamma \in I} \frac{1}{|\gamma \cdot G|} \left(\sum_{r \in \gamma \cdot G} \xi(r) e_{rr} \right) p_{\gamma(1)}$, where we recall $e_{rs} \in \mathcal{B}(l^2(\gamma \cdot G))$ for $r, s \in \gamma \cdot G$ are the matrix units associated to the canonical orthonormal basis. Then,

$$\mathcal{F}_{\mathbb{G}}(\tilde{\xi}) = \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r) (\text{Tr}_{l^2(\gamma \cdot G)} \otimes \text{id})(u^{\gamma(1)}(e_{rr} \otimes 1)) = \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_{rr} \quad \text{also,}$$

$$\|\tilde{\xi}\|_{\mathbb{G},0}^2 = \sum_{\gamma \in I} |\gamma \cdot G| \text{Tr}_{l^2(\gamma \cdot G)} \left(\sum_{r \in \gamma \cdot G} \frac{|\xi(r)|^2}{|\gamma \cdot G|^2} e_{rr} \right) = \sum_{\gamma \in I} \frac{1}{|\gamma \cdot G|} \sum_{r \in \gamma \cdot G} |\xi(r)|^2 \leq \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} |\xi(r)|^2 = \|\xi\|_2^2.$$

Since ξ is supported in $\{\gamma \in \Gamma : k \leq l_\Gamma(\gamma) < k+1\}$ and l_Γ is β -invariant, it follows that $\text{supp}(\tilde{\xi}) \subset \{z \in \text{Irr}(\mathbb{G}) : k \leq l(z) < k+1\}$. Hence, denoting by P a polynomial witnessing (RD) for $(\widehat{\mathbb{G}}, l)$, we have:

$$\left\| \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_{rr} \right\| \leq P(k) \|\xi\|_2.$$

Denote by Ψ the unital $*$ -morphism $\Psi : C(\mathbb{G}) = \Gamma \rtimes C(G) \rightarrow C_r^*(\Gamma)$ such that $\Psi(u_\gamma F) = \lambda_\gamma F(1)$ for all $\gamma \in \Gamma$ and $F \in C(G)$. Since Ψ has norm one, denoting by $\lambda(\xi) \in C_r^*(\Gamma)$ the convolution operator by ξ , we have

$$\|\lambda(\xi)\| = \left\| \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r) \lambda_r \right\| = \left\| \Psi \left(\sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_{rr} \right) \right\| \leq \left\| \sum_{\gamma \in I} \sum_{r \in \gamma \cdot G} \xi(r) u_r v_{rr} \right\| \leq P(k) \|\xi\|_2.$$

This concludes the proof. \square

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