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†Submitted to Springer's RACSAM in December 2021

Abstract

We study the Radiative Transfer equations coupled with the time dependent temperature equation of a fluid: existence, uniqueness, a maximum principle are established. A short numerical section illustrates the pros and cons of the method.

Keywords: Radiative transfer, Temperature equation, Integral equation, Numerical analysis

Introduction

In fluid mechanics, *Radiative Transfer* is an important subfield of *Heat Transfer* with many applications to combustion, micro-wave ovens and climate models.

For the physics of radiative transfer for the atmosphere the reader is sent to Goody and Yung (1961), Bohren (2006), to the numerically oriented Zdunkowski and Trautmann (2003) and to the two mathematically oriented Chandrasekhar (1950) and Fowler (2011).

When Planck's theory of black bodies is used the radiations have a continuum of frequencies governed by the temperature of the emitting body.

Even when the interactions with the fluid medium are neglected, the radiative transfer equations have 5 spatial dimensions. Hence the problem is numerically quite difficult. The stratified approximation is used when the radiation source is far, typically, the Sun. It is a two dimensional model with one spatial and one angular dimension to which the authors have contributed recently: see Bardos and Pironneau (2021) for stratified radiative transfer alone and Pironneau (2021), Golse and Pironneau (2022), Golse and Pironneau (2021) for stratified radiative transfer coupled with the temperature equation in the stationary case.

In this article the complete 5 dimensional radiative transfer model is studied when coupled with the time dependent temperature equation in the fluid.

Existence and uniqueness of a solution is well known when the physically constants do not depend on the frequency of the radiating source, the so called grey model (see Bardos et al (1988)) and Golse (1987)). In the non-grey case, some results have been obtained by Mercier (1987), Golse and Perthame (1986), et al. The present article extends these studies done in the eighties.

The radiative transfer system coupled to the Navier-Stokes equations has been studied by Pomraning (1973) and Ghattassi et al (2020) at least. In the later an existence theorem is given when the coefficients depend on the spatial variables but not on the frequencies of the source.

The paper begins with a statement of the radiative transfer equations in Section 1. In Section 2 an existence result is given. In Section 3, uniqueness is shown. The proof is complex and relies on an argument given by Mercier (1987) and Crandall and Tartar (1980). A maximum principle is also shown. Finally in Section 4 a numerical example is given.

1 Fundamental equations and approximations

To find the temperature T in an incompressible fluid exposed to electromagnetic waves it is necessary to solve the Navier-Stokes equations coupled with the Radiative Transfer equations, as explained in Pomraning (1973). It is a complex partial differential system formulated in terms of a time dependent temperature field $T(\mathbf{x},t)$ function of the position $\mathbf{x}$ in the physical domain Ω and a light intensity field $I_\nu(\mathbf{x},\boldsymbol{\omega},t)$ of frequency ν in each direction $\boldsymbol{\omega}$:

Given $I_\nu, T, \mathbf{u}, \rho$ at time zero, find $I_\nu, T, \mathbf{u}, p, \rho$, such that for all $\{\mathbf{x},\boldsymbol{\omega},t,\nu\} \in \Omega\times\mathbb{S}_2\times(0,\bar{T})\times\mathbb{R}^+$,

$$\begin{aligned}
&\frac{1}{c}\partial_t I_\nu + \boldsymbol{\omega}\cdot\boldsymbol{\nabla} I_\nu + \rho\bar{\kappa}_\nu a_\nu\left[I_\nu - \frac{1}{4\pi}\int_{\mathbb{S}^2} p(\boldsymbol{\omega},\boldsymbol{\omega}')I_\nu(\boldsymbol{\omega}')d\omega'\right] \\
&\qquad\qquad\qquad\qquad = \rho\bar{\kappa}_\nu(1-a_\nu)[B_\nu(T)-I_\nu], \\
&\rho c_V(\partial_t T + \mathbf{u}\cdot\nabla T) - \nabla\cdot(\rho c_P \kappa_T \nabla T) \\
&\qquad + \frac{1}{c}\int_0^\infty\int_{\mathbb{S}^2} I_\nu d\mu d\nu + \boldsymbol{\nabla}\cdot\int_0^\infty\int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega}')\boldsymbol{\omega} d\omega d\nu = 0 \\
&\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \frac{\mu_F}{\rho}\Delta\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{g}, \quad \nabla\cdot\mathbf{u} = 0, \quad \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0,
\end{aligned} \tag{1}$$

where ∇, Δ are with respect to $\mathbf{x}$, $B_\nu(T) = \dfrac{2\hbar\nu^3}{c^2[\mathrm{e}^{\frac{\hbar\nu}{kT}} - 1]}$, is the Planck function, $\hbar, c, k$ are the Planck constant, the speed of light in the medium and the Boltzmann constant. The density of the medium is ρ, the pressure is p; c_P, c_V are the compressibility of the fluid at constant pressure or volume; in large area these may be altitude/depth dependent. The absorption coefficient $\kappa_\nu := \rho\bar{\kappa}_\nu$ comes from computations in atomic physics, but for our purpose it is seen as the percentage of light absorbed per unit length. The scattering albedo is $a_\nu \in (0,1)$ and $\frac{1}{4\pi}p(\boldsymbol{\omega},\boldsymbol{\omega}')$ is the probability that a ray in direction $\boldsymbol{\omega}'$ scatters in direction $\boldsymbol{\omega}$. The constants κ_T and μ_F are the thermal and molecular diffusions; $\mathbf{g}$ is the gravity.

Existence of solution for the fluid part of (1) has been established by Lions (1996).

1.1 The Mathematical Problem

Denote the angular average radiative intensity by $J_\nu(t,\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega})d\omega$. If $\frac{1}{c}$ is neglected in (1), the following holds:

$$\boldsymbol{\nabla}\cdot\int_0^\infty\int_{\mathbb{S}^2} I_\nu(\boldsymbol{\omega}')\boldsymbol{\omega}d\omega d\nu = 4\pi\int_0^\infty \rho\kappa_\nu(1-a_\nu)\left(B_\nu(T) - J_\nu\right)d\nu\,. \tag{2}$$

Consequently we are led to study the well-posedness of the following system for I_ν, J_ν, T:

$$\begin{cases} \boldsymbol{\omega}\cdot\nabla I_\nu + \kappa_\nu I_\nu = \kappa_\nu(1-a_\nu)B_\nu(T) + \kappa_\nu a_\nu J_\nu\,, \quad J_\nu := \displaystyle\int_{\mathbb{S}^2} I_\nu \frac{\mathrm{d}\boldsymbol{\omega}}{4\pi}\,, \\ \partial_t T + \mathbf{u}\cdot\nabla T - \lambda\Delta T = \displaystyle\int_0^\infty \kappa_\nu(1-a_\nu)(J_\nu - B_\nu(T))\mathrm{d}\nu\,, \\ I_\nu(\mathbf{x},\boldsymbol{\omega}) = Q_\nu(\mathbf{x},\boldsymbol{\omega})\,, \quad \boldsymbol{\omega}\cdot\mathbf{n} < 0\,,\ \mathbf{x}\in\partial\Omega\,, \qquad \dfrac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0\,, \\ T\big|_{t=0} = T_{in}\,. \end{cases} \tag{3}$$

Here Ω is assumed to be a bounded open subset of $\mathbf{R}^3$ with C^1 boundary, and we denote by $\mathbf{n}$ the outward unit normal field on $\partial\Omega$. We further assume that $\nu \mapsto \kappa_\nu$ and $\nu \mapsto a_\nu$ are measurable functions satisfying

$$0 \le \kappa_m \le \kappa_\nu \le \kappa_M\,, \qquad 0 \le a_\nu \le a_M < 1\,, \qquad \nu > 0\,,\ a.e.\,,$$

for some positive constants a_M and $\kappa_m < \kappa_M$. Finally, we assume that the fluid velocity field $(t,\mathbf{x}) \mapsto \mathbf{u}(t,\mathbf{x})$ is smooth on $[0,+\infty)\times\overline{\Omega}$ and satisfies

$$\nabla\cdot\mathbf{u}(t,\mathbf{x}) = 0 \text{ for } \mathbf{x}\in\Omega\,, \qquad \mathbf{u}(t,\mathbf{x}) = 0 \text{ for } \mathbf{x}\in\partial\Omega\,, \qquad t \ge 0\,.$$

2 Existence

Given a passive parameter t, consider the auxiliary problem

$$\begin{cases} \boldsymbol{\omega} \cdot \nabla I_\nu(t, \mathbf{x}, \boldsymbol{\omega}) = \kappa_\nu (S_\nu(t, \mathbf{x}) - I_\nu(t, \mathbf{x}, \boldsymbol{\omega})) \,, & \mathbf{x} \in \Omega \,,\ |\boldsymbol{\omega}| = 1 \,, \\ I_\nu(t, \mathbf{x}, \boldsymbol{\omega}) = Q_\nu(\mathbf{x}, \boldsymbol{\omega}) \,, & \boldsymbol{\omega} \cdot \mathbf{n} < 0 \,, \end{cases}$$

where the source S_ν is isotropic, i.e. not a function of $\boldsymbol{\omega}$. Define the exit time

$$\tau_{\mathbf{x},\boldsymbol{\omega}} = \sup\{s > 0 \text{ s.t. } \mathbf{x} - s\boldsymbol{\omega} \in \Omega\} \,.$$

By the method of characteristics

$$I_\nu(t, \mathbf{x}, \boldsymbol{\omega}) = \mathbf{1}_{\tau_{\mathbf{x},\boldsymbol{\omega}} < +\infty} Q_\nu(\mathbf{x} - \tau_{\mathbf{x},\boldsymbol{\omega}} \boldsymbol{\omega}) \mathrm{e}^{-\kappa_\nu \tau_{\mathbf{x},\boldsymbol{\omega}}} + \int_0^{\tau_{\mathbf{x},\boldsymbol{\omega}}} \mathrm{e}^{-\kappa_\nu s} \kappa_\nu S_\nu(t, \mathbf{x} - s\boldsymbol{\omega}) \mathrm{d}s \,.$$

Averaging in $\boldsymbol{\omega}$, one finds

$$\begin{aligned} J_\nu(t, \mathbf{x}) = \mathcal{J}[S_\nu](t, \mathbf{x}) := & \tfrac{1}{4\pi} \int_{\mathbb{S}^2} \mathbf{1}_{\tau_{\mathbf{x},\boldsymbol{\omega}} < +\infty} Q_\nu(\mathbf{x} - \tau_{\mathbf{x},\boldsymbol{\omega}} \boldsymbol{\omega}) \mathrm{e}^{-\kappa_\nu \tau_{\mathbf{x},\boldsymbol{\omega}}} \mathrm{d}\boldsymbol{\omega} \\ & + \tfrac{1}{4\pi} \int_{\mathbb{S}^2} \int_0^{\tau_{\mathbf{x},\boldsymbol{\omega}}} \mathrm{e}^{-\kappa_\nu s} \kappa_\nu S_\nu(t, \mathbf{x} - s\boldsymbol{\omega}) \mathrm{d}s \mathrm{d}\boldsymbol{\omega} \,. \end{aligned} \tag{4}$$

Since $\kappa_\nu > 0$, the functional $\mathcal{J}$ satisfies the following monotonicity property:

$$\begin{aligned} & S_\nu(t, \mathbf{x}) \le S'_\nu(t, \mathbf{x}) \text{ for a.e. } \mathbf{x} \in \Omega \text{ and } t > 0 \\ \implies & \mathcal{J}[S_\nu](t, \mathbf{x}) \le \mathcal{J}[S'_\nu](t, \mathbf{x}) \text{ for a.e. } \mathbf{x} \in \Omega \text{ and } t > 0 \,. \end{aligned}$$

In particular,

$$\begin{aligned} & 0 \le Q_\nu(\mathbf{x}, \boldsymbol{\omega}) \,,\ S_\nu(t, \mathbf{x}) \le B_\nu(T_M) \,, \quad x \in \Omega \,,\ |\boldsymbol{\omega}| = 1 \,,\ \nu, t > 0 \\ \implies & 0 \le \mathcal{J}[S_\nu](t, \mathbf{x}) \le B_\nu(T_M) \,, \quad x \in \Omega \,,\ \nu, t > 0 \,. \end{aligned}$$

That $\mathcal{J}[S_\nu] \ge 0$ is obvious. As for the upper bound, observe that

$$\begin{aligned} \mathcal{J}[B_\nu(T_M)] = & \tfrac{1}{4\pi} B_\nu(T_M) \int_{\mathbb{S}^2} \mathrm{e}^{-\kappa_\nu \tau_{\mathbf{x},\boldsymbol{\omega}}} \mathrm{d}\boldsymbol{\omega} + \tfrac{1}{4\pi} B_\nu(T_M) \int_{\mathbb{S}^2} \int_0^{\tau_{\mathbf{x},\boldsymbol{\omega}}} \mathrm{e}^{-\kappa_\nu s} \kappa_\nu \mathrm{d}s \mathrm{d}\boldsymbol{\omega} \\ = & \tfrac{1}{4\pi} B_\nu(T_M) \left[\int_{\mathbb{S}^2} \mathrm{e}^{-\kappa_\nu \tau_{\mathbf{x},\boldsymbol{\omega}}} \mathrm{d}\boldsymbol{\omega} + \int_{\mathbb{S}^2} (1 - \mathrm{e}^{-\kappa_\nu \tau_{\mathbf{x},\boldsymbol{\omega}}}) \mathrm{d}\boldsymbol{\omega} \right] = B_\nu(T_M) \,, \end{aligned}$$

so that the desired upper bound follows from the monotonicity of $\mathcal{J}$.

In order to solve the system (3), we consider the iterative scheme detailed in Algorithm 1, where we have assumed that

$$0 \le T_{in}(\mathbf{x}) \le T_M \,, \quad 0 \le Q_\nu(\mathbf{x}, \boldsymbol{\omega}) \le B_\nu(T_M) \,, \quad x \in \Omega \,,\ |\boldsymbol{\omega}| = 1 \,,\ \nu > 0 \,.$$

Algorithm 1 to solve (3).

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1. Start from $T^0 \equiv 0$ and $J_\nu^0 = \mathcal{J}[0]$;
2. for $n = 0, 1, \dots, N-1$
   (a) for all $\nu \in (0, \infty)$ and all $\tau \in (0, Z)$, by knowing $T^n \equiv T^n(t, \mathbf{x})$ and $J_\nu^n \equiv J_\nu^n(t, \mathbf{x})$, define with (4)
   $$J^{n+1} = \mathcal{J}[a_\nu J_\nu^n + (1 - a_\nu) B_\nu(T^n)]\,;$$
   (b) Define T^{n+1} to be the solution of the semilinear drift-diffusion equation
   $$\begin{cases} \partial_t T^{n+1} + \mathbf{u} \cdot \nabla T^{n+1} - \lambda \Delta T^{n+1} + \mathcal{B}(T^{n+1}) = \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^{n+1} \mathrm{d}\nu\,, \\ T^{n+1}\big|_{t=0} = T_{in}\,, \qquad \dfrac{\partial T^{n+1}}{\partial n}\Big|_{\partial\Omega} = 0\,, \quad \mathbf{x} \in \Omega\,,\ t > 0\,, \end{cases}$$
   where $\mathcal{B}(T) := \int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(\min(T_+, T_M)) \mathrm{d}\nu\,.$

Applying Theorem 10.9 in Brezis (2011) (see also section 4.7.2 in Lions and Magenes (1972)) shows that, for each $q \in L^2(0, T;\ H^{-1}(\Omega))$, there exists a unique solution to the convection-diffusion problem

$$\begin{cases} \partial_t \theta + \mathbf{u} \cdot \nabla \theta - \lambda \Delta \theta = q\,, \quad \mathbf{x} \in \Omega\,,\ t > 0\,, \\ \theta\big|_{t=0} = T_{in}\,, \qquad \dfrac{\partial \theta}{\partial n}\Big|_{\partial\Omega} = 0\,, \end{cases}$$

of the form

$$\theta(t, \cdot) = \Sigma(t, 0) T_{in} + \int_0^t \Sigma(t, s) q(s, \cdot) \mathrm{d}s \in L^2(0, T;\ H^1(\Omega)) \cap C(0, T;\ L^2(\Omega))\,.$$

With

$$q := \int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^{n+1} \mathrm{d}\nu - \mathcal{B}(T^{n+1})\,,$$

we see that T^{n+1} is a fixed point of the map $\mathcal{F}$ defined by

$$\mathcal{F}(\theta)(t, \cdot) = \Sigma(t, 0) T_{in} + \int_0^t \Sigma(t, s) \left(\int_0^\infty \kappa_\nu (1 - a_\nu) J_\nu^{n+1}(s, \cdot) \mathrm{d}\nu - \mathcal{B}(\theta(s, \cdot)) \right) \mathrm{d}s\,.$$

Observe that $\mathcal{B}$ is Lipschitz continuous with

$$\mathrm{Lip}(\mathcal{B}) \le \kappa_M(1-a_M) \sup_{0\le\theta\le T_M} \int_0^\infty B'_\nu(\theta)d\nu = 4\kappa_M(1-a_M)T_M^3\,,$$

so that, arguing as in the proof of Theorem 1.2 in chapter 6 of Pazy (1983) shows that $\mathcal{F}$ has a unique fixed point. This defines a unique solution $T^{n+1} \in C([0,+\infty);\ L^2(\Omega)) \cap L^2_{loc}(0,\infty;\ H^1(\Omega))$.

Next, we seek to compare the solutions T and T' of

$$\begin{cases} \partial_t T + \mathbf{u}\cdot\nabla T - \lambda\Delta T + \mathcal{B}(T) = R\,, & \mathbf{x}\in\Omega\,,\ t>0\,,\\ T\big|_{t=0} = T_{in}\,, & \dfrac{\partial T}{\partial n}\Big|_{\partial\Omega} = 0\,, \end{cases}$$

and

$$\begin{cases} \partial_t T' + \mathbf{u}\cdot\nabla T' - \lambda\Delta T' + \mathcal{B}(T') = R'\,, & \mathbf{x}\in\Omega\,,\ t>0\,,\\ T'\big|_{t=0} = T'_{in}\,, & \dfrac{\partial T'}{\partial n}\Big|_{\partial\Omega} = 0\,, \end{cases}$$

under the assumption that $0 \le R \le R'$ on $(0,+\infty)\times\Omega$. Proceeding as in the proof of Lemma 6.2 of Golse and Pironneau (2021), we multiply both sides of the equality satisfied by the difference $T-T'$ by $(T-T')_+$:

$$\begin{aligned} \partial_t \tfrac{1}{2}(T-T')_+^2 + \mathbf{u}\cdot\nabla\tfrac{1}{2}(T-T')_+^2 - \lambda\Delta\tfrac{1}{2}(T-T')_+^2 + \lambda|\nabla(T-T')_+|^2 \\ +(\mathcal{B}(T)-\mathcal{B}(T'))(T-T')_+ = (R-R')(T-T')_+ \le 0\,. \end{aligned}$$

Integrating over Ω, and taking into account the boundary conditions satisfied by $\mathbf{u}$ and $(T-T')$ shows that

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}(T-T')_+^2(t,\mathbf{x})\mathrm{d}\mathbf{x} + \lambda\int_\Omega |\nabla(T-T')_+|^2(t,\mathbf{x})\mathrm{d}\mathbf{x} + \int_\Omega (\mathcal{B}(T)-\mathcal{B}(T'))(T-T')_+ \le 0\,,$$

since

$$\int_{\partial\Omega}\left(\mathbf{u}\cdot n(T-T')_+^2 - \lambda\frac{\partial}{\partial n}(T-T')_+^2\right)\mathrm{d}\sigma = 0\,.$$

Then, $T\mapsto\mathcal{B}(T)$ is nondecreasing on $\mathbf{R}$, since $\kappa_\nu(1-a_\nu)\ge 0$ and $T\mapsto B_\nu(\min(T_+,T_M))$ is nondecreasing on $\mathbf{R}$ for each $\nu>0$. Hence

$$(\mathcal{B}(T)-\mathcal{B}(T'))(T-T')_+ \ge 0$$

so that

$$\int_\Omega \tfrac{1}{2}(T-T')_+^2(t,\mathbf{x})\mathrm{d}\mathbf{x} \le \int_\Omega \tfrac{1}{2}(T_{in}-T'_{in})_+^2(\mathbf{x})\mathrm{d}\mathbf{x} = 0\,.$$

Therefore

$$T_{in}\le T'_{in} \text{ on } \Omega \text{ and } R\le R' \text{ on } (0,+\infty)\times\Omega \implies T\le T' \text{ on } (0,+\infty)\times\Omega\,.$$

This comparison argument shows that

$$0 \le J_\nu^{n+1}(t,\mathbf{x}) \le B_\nu(T_M) \text{ on } (0,+\infty)\times\Omega \implies 0 \le T^{n+1} \le T_M \text{ on } (0,+\infty)\times\Omega\,.$$

By the same token,

$$J_\nu^n \le J_\nu^{n+1} \text{ on } (0,+\infty)\times\Omega \implies T^n \le T^{n+1} \text{ on } (0,+\infty)\times\Omega\,.$$

On the other hand, the monotonicity property of the function $\mathcal{J}$ shows that

$$\begin{aligned} T^{n-1} \le T^n \text{ and } J_\nu^{n-1} \le J_\nu^n \implies J_\nu^n =& \mathcal{J}[a_\nu J_\nu^{n-1} + (1-a_\nu)B_\nu(T^{n-1})] \\ \le& \mathcal{J}[a_\nu J_\nu^n + (1-a_\nu)B_\nu(T^n)] = J_\nu^{n+1} \end{aligned}$$

on $(0,+\infty)\times\Omega$. Summarising, we have essentially proved the following result.

Theorem 1 *Assume that*

$$0 \le \kappa_m \le \kappa_\nu \le \kappa_M\,, \qquad 0 \le a_\nu \le a_M < 1\,, \qquad \nu > 0\,, \quad a.e.\,,$$

and pick boundary and initial data satisfying, for some T_M,

$$0 \le T_{in}(\mathbf{x}) \le T_M\,, \quad 0 \le Q_\nu(\mathbf{x},\boldsymbol{\omega}) \le B_\nu(T_M)\,, \quad x\in\Omega\,,\ |\boldsymbol{\omega}| = 1\,,\ \nu > 0\,.$$

Then

(1) the sequences T^n and J_ν^n satisfy

$$0 = T^0 \le T^1 \le \ldots \le T^n \le T^{n+1} \le \ldots \le T_M \qquad \text{on } (0,\infty)\times\Omega\,,$$

and

$$0 \le J_\nu^0 \le J_\nu^1 \le \ldots \le J_\nu^n \le J_\nu^{n+1} \le \ldots \le B_\nu(T_M) \quad \text{on } (0,\infty)\times\Omega, \text{ for all } \nu > 0.$$

(2) There exist a temperature field $T \in C([0,+\infty);\ L^2(\Omega)) \cap L^2_{loc}(0,\infty;\ H^1(\Omega))$ and a radiative intensity $I_\nu \in L^\infty((0,\infty)\times\Omega\times\mathbb{S}^2\times(0,+\infty))$ satisfying (3) in the sense of weak solutions.

Proof Statement (1) is a consequence of the monotonicity properties established above. It implies statement (2) by the following elementary observations: first, one can pass to the limit by monotone convergence in the expression

$$J_\nu^{n+1} = \mathcal{J}[a_\nu J_\nu^n + (1-a_\nu)B_\nu(T^n)]$$

to find that

$$J_\nu = \mathcal{J}[[a_\nu J_\nu + (1-a_\nu)B_\nu(T)]\,,$$

since B_ν is an increasing function for each $\nu > 0$. By the method of characteristics, the formula

$$\begin{aligned} I_\nu(t,\mathbf{x},\boldsymbol{\omega}) =& \mathbf{1}_{\tau_{\mathbf{x},\boldsymbol{\omega}}<+\infty} Q_\nu(\mathbf{x} - \tau_{\mathbf{x},\boldsymbol{\omega}}\boldsymbol{\omega})\mathrm{e}^{-\kappa_\nu \tau_{\mathbf{x},\boldsymbol{\omega}}} \\ &+ \int_0^{\tau_{\mathbf{x},\boldsymbol{\omega}}} \mathrm{e}^{-\kappa_\nu s}\kappa_\nu(a_\nu J_\nu(t,\mathbf{x}-s\boldsymbol{\omega}) + (1-a_\nu)B_\nu(T(t,\mathbf{x}-s\boldsymbol{\omega})))\mathrm{d}s \end{aligned}$$

defines a solution of the transfer equation in (3).

Finally, for each $\phi \in C_c([0,\infty);\ H^1(\Omega))$ such that $\partial_t \phi \in L^2((0,\infty)\times\Omega)$, one has

$$\int_0^\infty \int_\Omega (\lambda \nabla T^n \cdot \nabla \phi - T^n(\partial_t \phi + \mathbf{u}\cdot\nabla\phi))\mathrm{d}\mathbf{x}\mathrm{d}t = \int_\Omega T_{in}(\mathbf{x})\phi(0,\mathbf{x})\mathrm{d}\mathbf{x}$$
$$\int_0^\infty \int_0^\infty \int_\Omega \kappa_\nu(1-a_\nu)(J_\nu^n - B_\nu(T^n))\phi \mathrm{d}\mathbf{x}\mathrm{d}\nu\mathrm{d}t\,.$$

One can pass to the limit by dominated convergence in all the terms except the one involving ∇T^n. This last term is mastered by the energy balance for the convection-diffusion equation:

$$\tfrac12 \int_\Omega T^n(t,\mathbf{x})^2 \mathrm{d}x + \lambda \int_0^t \int_\Omega |\nabla T^n(t,\mathbf{x})|^2 \mathrm{d}\mathbf{x} = \tfrac12 \int_\Omega T_{in}(\mathbf{x})^2 \mathrm{d}x$$
$$+ \int_0^t \int_0^\infty \int_\Omega \kappa_\nu(1-a_\nu)(J_\nu^n - B_\nu(T^n))T^n \mathrm{d}\mathbf{x}\mathrm{d}\nu\mathrm{d}s \le t|\Omega|\mathcal{B}(T_M)\,,$$

which implies that T^n is bounded in $L^2_{loc}(0,\infty;\ H^1(\Omega))$. Since we already know that $T^n \to T$ in $L^p((0,\tau)\times\Omega)$ for all $p\in[1,\infty)$ as $n\to\infty$, we conclude that $T^n \to T$ weakly in $L^2_{loc}(0,\infty;\ H^1(\Omega))$. With this last piece of information, we pass to the limit in the weak formulation of the convection-diffusion equation and conclude that

$$\int_0^\infty \int_\Omega (\lambda \nabla T \cdot \nabla \phi - T(\partial_t \phi + \mathbf{u}\cdot\nabla\phi))\mathrm{d}\mathbf{x}\mathrm{d}t = \int_\Omega T_{in}(\mathbf{x})\phi(0,\mathbf{x})\mathrm{d}\mathbf{x}$$
$$\int_0^\infty \int_0^\infty \int_\Omega \kappa_\nu(1-a_\nu)(J_\nu - B_\nu(T))\phi \mathrm{d}\mathbf{x}\mathrm{d}\nu\mathrm{d}t\,.$$

In other words, T is a weak solution of the second equation in (3). This concludes the proof.

□

3 Uniqueness and maximum principle

For each $\epsilon > 0$ let $s_\epsilon \in C^\infty(\mathbf{R})$ be such that

$$s_\epsilon(z) = 0 \text{ for } z \le 0\,, \quad s_\epsilon(z) = 1 \text{ for } z \ge \epsilon\,, \quad 0 \le \epsilon s'_\epsilon(z) \le 2 \text{ for } z \in \mathbf{R}\,,$$

and let $s_+(z) = \mathbf{1}_{z>0}$. Set

$$S_\epsilon(y) = \int_0^y s_\epsilon(z)dz\,.$$

Henceforth, we use the notation

$$\langle\!\langle \phi \rangle\!\rangle := \tfrac{1}{4\pi} \int_0^\infty \int_{\mathbb{S}^2} \phi(\boldsymbol{\omega},\nu)\mathrm{d}\boldsymbol{\omega}\mathrm{d}\nu$$

Let (I_ν, T) and (I'_ν, T') be two solutions of the system above; then

$$\nabla \cdot \langle\!\langle \boldsymbol{\omega}(I_\nu - I'_\nu)_+ \rangle\!\rangle + D_1 + D_2 = 0\,,$$

where

$$D_1 := \langle\!\langle \kappa_\nu(1-a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))s_+(I_\nu - I'_\nu)\rangle\!\rangle\,,$$

and

$$D_2 := \langle\!\langle \kappa_\nu a_\nu((I_\nu - I'_\nu) - (J_\nu - J'_\nu))s_+(I_\nu - I'_\nu)\rangle\!\rangle\,.$$

Since

$$\int_{\mathbb{S}^2}((I_\nu - I'_\nu) - (J_\nu - J'_\nu))\mathrm{d}\boldsymbol{\omega} = 0,$$

one has

$$D_2 := \langle\!\langle \kappa_\nu a_\nu((I_\nu - I'_\nu) - (J_\nu - J'_\nu))(s_+(I_\nu - I'_\nu) - s_+(J_\nu - J'_\nu))\rangle\!\rangle \geq 0\,,$$

since $z \mapsto s_+(z)$ is nondecreasing, so that

$$((I_\nu - I'_\nu) - (J_\nu - J'_\nu))(s_+(I_\nu - I'_\nu) - s_+(J_\nu - J'_\nu)) \geq 0\,.$$

On the other hand

$$D_1 = D_3^\epsilon + s_\epsilon(T-T')\langle\!\langle \kappa_\nu(1-a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))\rangle\!\rangle$$

where

$$D_3^\epsilon = \langle\!\langle \kappa_\nu(1-a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))(s_+(I_\nu - I'_\nu) - s_\epsilon(T-T'))\rangle\!\rangle\,,$$

while

$$\begin{aligned}\partial_t S_\epsilon(T-T') + \mathbf{u}\cdot\nabla S_\epsilon(T-T') - \lambda\Delta(T-T')s_\epsilon(T-T')\\ = s_\epsilon(T-T')\int_0^\infty \kappa_\nu(1-a_\nu)((J_\nu - J'_\nu) - (B_\nu(T) - B_\nu(T')))\mathrm{d}\nu\\ = 4\pi s_\epsilon(T-T')\langle\!\langle \kappa_\nu(1-a_\nu)((J_\nu - J'_\nu) - (B_\nu(T) - B_\nu(T')))\rangle\!\rangle\,.\end{aligned}$$

Thus

$$\begin{aligned}4\pi\nabla\cdot\langle\!\langle \boldsymbol{\omega}(I_\nu - I'_\nu)_+\rangle\!\rangle + \partial_t S_\epsilon(T-T') + \mathbf{u}\cdot\nabla S_\epsilon(T-T') - \lambda\Delta(T-T')s_\epsilon(T-T')\\ +4\pi(D_3^\epsilon + D_2) = 0\end{aligned}$$

Then we integrate both sides of this equality on Ω:

$$\begin{aligned}\frac{d}{dt}\int_\Omega S_\epsilon(T-T')d\mathbf{x} + 4\pi\int_{\partial\Omega}\langle\!\langle \boldsymbol{\omega}\cdot\mathbf{n}(I_\nu - I'_\nu)_+\rangle\!\rangle\mathrm{d}\sigma(\mathbf{x}) + \int_{\partial\Omega} S_\epsilon(T-T')\mathbf{u}\cdot\mathbf{n}\mathrm{d}\sigma(\mathbf{x})\\ +\lambda\int_\Omega|\nabla(T-T')|^2 s'_\epsilon(T-T')\mathrm{d}\mathbf{x} - \lambda\int_{\partial\Omega} s_\epsilon(T-T')\frac{\partial(T-T')}{\partial n}\mathrm{d}\sigma(\mathbf{x})\\ = -4\pi\int_\Omega(D_3^\epsilon + D_2)\mathrm{d}\mathbf{x}\,.\end{aligned}$$

Using the boundary conditions, specifically that

$$I_\nu(\mathbf{x},\boldsymbol{\omega}) = Q_\nu(\mathbf{x},\boldsymbol{\omega}) \text{ and } I'_\nu(\mathbf{x},\boldsymbol{\omega}) = Q'_\nu(\mathbf{x},\boldsymbol{\omega}), \quad \boldsymbol{\omega}\cdot\mathbf{n} < 0,$$

with

$$(Q_\nu - Q'_\nu)(\mathbf{x},\boldsymbol{\omega}) \leq 0, \quad \boldsymbol{\omega}\cdot\mathbf{n} < 0,$$

implies

$$\int_{\partial\Omega} \langle\!\langle \boldsymbol{\omega}\cdot\mathbf{n}(I_\nu - I'_\nu)_+ \rangle\!\rangle \mathrm{d}\sigma(\mathbf{x}) = \int_{\partial\Omega} \langle\!\langle (\boldsymbol{\omega}\cdot\mathbf{n})_+ (I_\nu - I'_\nu)_+ \rangle\!\rangle \mathrm{d}\sigma(\mathbf{x}) \geq 0,$$
$$\int_{\partial\Omega} S_\epsilon(T - T')\mathbf{u}\cdot\mathbf{n}\mathrm{d}\sigma(\mathbf{x}) = \int_{\partial\Omega} s_\epsilon(T - T')\frac{\partial(T - T')}{\partial n}\mathrm{d}\sigma(\mathbf{x}) = 0.$$

Hence

$$\begin{aligned} &\int_\Omega S_\epsilon(T - T')(t,\mathbf{x})d\mathbf{x} + 4\pi\int_0^t\int_{\partial\Omega} \langle\!\langle (\boldsymbol{\omega}\cdot\mathbf{n})_+ (I_\nu - I'_\nu)_+ \rangle\!\rangle(\tau,\mathbf{x})\mathrm{d}\sigma(\mathbf{x})\mathrm{d}\tau \\ &+\lambda\int_0^t\int_\Omega |\nabla(T - T')(\tau,\mathbf{x})|^2 s'_\epsilon(T - T')(\tau,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}\tau + 4\pi\int_0^t\int_\Omega (D_3^\epsilon + D_2)(\tau,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}\tau \\ &= \int_\Omega S_\epsilon(T - T')(0,\mathbf{x})d\mathbf{x} = 0 \end{aligned}$$

under the assumption that

$$T\Big|_{t=0} = T_{in} \quad \text{and } T'\Big|_{t=0} = T'_{in} \quad \text{with } T_{in} \leq T'_{in}.$$

Assume that

$$\kappa_\nu(1 - a_\nu)(I_\nu + I'_\nu + B_\nu(T) + B_\nu(T')) \in L^1([0,t]\times\Omega\times\mathbb{S}^2\times(0,+\infty));$$

by dominated convergence

$$\int_0^t\int_\Omega D_3^\epsilon(\tau,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}\tau \to \int_0^t\int_\Omega D_3(\tau,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}\tau$$

where

$$D_3 = \langle\!\langle \kappa_\nu(1 - a_\nu)((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))(s_+(I_\nu - I'_\nu) - s_+(T - T'))\rangle\!\rangle \geq 0$$

since $z \mapsto s_+(z)$ is nondecreasing and

$$s_+(T - T') = s_+(B_\nu(T) - b_\nu(T'))$$

because B_ν is increasing for each $\nu > 0$, so that

$$((I_\nu - I'_\nu) - (B_\nu(T) - B_\nu(T')))(s_+(I_\nu - I'_\nu) - s_+(T - T')) \geq 0.$$

By Fatou's lemma

$$\int_\Omega S_\epsilon(T-T')(t,\mathbf{x})d\mathbf{x} \to \int_\Omega (T-T')_+(t,\mathbf{x})d\mathbf{x}$$

for a.e. $t \geq 0$, so that

$$\int_\Omega (T-T')_+(t,\mathbf{x})d\mathbf{x} + 4\pi\int_0^t\int_{\partial\Omega} \langle\!\langle(\boldsymbol{\omega}\cdot\mathbf{n})_+(I_\nu - I'_\nu)_+\rangle\!\rangle(\tau,\mathbf{x})\mathrm{d}\sigma(\mathbf{x})\mathrm{d}\tau \\ +4\pi\int_0^t\int_\Omega (D_3+D_2)(\tau,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}\tau \leq 0\,,$$

since

$$\varliminf_{\epsilon\to 0}\int_0^t\int_\Omega |\nabla(T-T')(\tau,\mathbf{x})|^2 s'_\epsilon(T-T')(\tau,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}\tau \geq 0\,.$$

Since all the terms on the left hand side of the previous equality are nonnegative, one has

$$\begin{aligned}\int_\Omega (T-T')_+(t,\mathbf{x})d\mathbf{x} =& 4\pi\int_0^t\int_{\partial\Omega} \langle\!\langle(\boldsymbol{\omega}\cdot\mathbf{n})_+(I_\nu - I'_\nu)_+\rangle\!\rangle(\tau,\mathbf{x})\mathrm{d}\sigma(\mathbf{x})\mathrm{d}\tau \\ =& 4\pi\int_0^t\int_\Omega (D_3+D_2)(\tau,\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}\tau = 0 \quad \text{for a.e. } t>0\,.\end{aligned}$$

Once it is known that $T \leq T'$ a.e. on $(0,+\infty)\times\Omega$, one has

$$\int_\Omega \langle\!\langle\kappa_\nu(1-a_\nu)(I_\nu - I'_\nu)_+\rangle\!\rangle\mathrm{d}\mathbf{x} + \int_{\partial\Omega}\langle\!\langle(\boldsymbol{\omega}\cdot\mathbf{n})_+(I_\nu - I'_\nu)_+\rangle\!\rangle(\tau,\mathbf{x})\mathrm{d}\sigma(\mathbf{x}) + \int_\Omega D_2\mathrm{d}\mathbf{x} \\ = \int_\Omega \langle\!\langle\kappa_\nu(1-a_\nu)(B_\nu(T)-B_\nu(T'))(t,\mathbf{x})s_+(I_\nu - I'_\nu)\rangle\!\rangle\mathrm{d}\mathbf{x} \leq 0$$

since $B_\nu(T) - B_\nu(T') \leq 0$ while $s_+(I_\nu - I'_\nu) \geq 0$, so that $I_\nu \leq I'_\nu$ a.e. on $(0,+\infty)\times\Omega\times\mathbb{S}^2\times(0,+\infty)$.

Summarising, we have proved the following

Theorem 2 *Let (I_ν, T) and (I'_ν, T') be two solutions of (3) such that*

$$\kappa_\nu(1-a_\nu)(I_\nu + I'_\nu + B_\nu(T) + B_\nu(T')) \in L^1([0,t]\times\Omega\times\mathbb{S}^2\times(0,+\infty))$$

for all $t>0$. Assume that

$$T\Big|_{t=0} = T_{in} \quad \text{and } T'\Big|_{t=0} = T'_{in} \quad \text{with } T_{in} \leq T'_{in}\,,$$

and that, when $\mathbf{x}\in\partial\Omega$,

$$I'_\nu(\mathbf{x},\boldsymbol{\omega}) = Q'_\nu(\mathbf{x},\boldsymbol{\omega}) \text{ and } I_\nu(\mathbf{x},\boldsymbol{\omega}) = Q_\nu(\mathbf{x},\boldsymbol{\omega}) \leq Q'_\nu(\mathbf{x},\boldsymbol{\omega})\,, \quad \boldsymbol{\omega}\cdot\mathbf{n}<0\,.$$

Then

$$I_\nu \leq I'_\nu \text{ and } T \leq T'\,.$$

If $T'_{in} = T_{in}$ and $Q_\nu = Q'_\nu$, exchanging the roles of (I_ν, T) and (I'_ν, T') in the theorem above leads to the following uniqueness result.

Corollary 1 There is at most one solution (I_ν, T) of (3) such that

$$\kappa_\nu(1-a_\nu)(I_\nu+B_\nu(T)) \in L^1([0,t]\times\Omega\times\mathbb{S}^2\times(0,+\infty)) \quad \text{for all } t>0.$$

In the case where (I'_ν, T) is a Planck equilibrium, i.e. $I'_\nu = B_\nu(T')$ with T' =constant, one obtains Mercier's maximum principle:

Corollary 2 If $0 \le Q_\nu \le B_\nu(T_M)$ and $0 \le T_{in} \le T_M$ and Ω has finite volume, the solution (I_ν, T) of (3) such that

$$\kappa_\nu(1-a_\nu)(I_\nu+B_\nu(T)) \in L^1([0,t]\times\Omega\times\mathbb{S}^2\times(0,+\infty))$$

satisfies

$$0 \le I_\nu(t,\mathbf{x},\boldsymbol{\omega}) \le B_\nu(T_M) \quad \text{and} \quad 0 \le T(t,\mathbf{x}) \le T_M$$

for a.e. $(t,\mathbf{x},\boldsymbol{\omega},\nu)$ in $(0,+\infty)\times\Omega\times\mathbb{S}^2\times(0,+\infty)$.

4 A Numerical Scheme

We begin with an important observation for the numerical implementation:

Proposition 3 *Equation* (4) *can be written as*

$$\tilde{J}_\nu(\mathbf{x}) = Y_{\kappa_\nu}(\mathbf{x}) \star \tilde{S}_\nu(\mathbf{x}) + \tilde{S}^E_\nu(\mathbf{x}), \ \text{with } Y_{\kappa_\nu}(\mathbf{x}) = \kappa_\nu \frac{e^{-\kappa_\nu|\mathbf{x}|}}{\pi|2\mathbf{x}|^{d-1}}, \ d=2,3. \tag{5}$$

where $\star$ *denotes a convolution and tildes indicate an extension by zero outside* Ω *and*

$$S^E_\nu(\mathbf{x}) = \frac{1}{2^{d-1}\pi}\int_{|\boldsymbol{\omega}|=1} \mathbf{1}_{\{\tau_{\mathbf{x},\boldsymbol{\omega}}<+\infty\}} Q_\nu(\mathbf{x}-s\boldsymbol{\omega})e^{-\kappa_\nu\tau_{\mathbf{x},\boldsymbol{\omega}}} d\omega \tag{6}$$

Proof This is because, by integration in spherical coordinates with $|\mathbf{x}| = s$,

$$\int_{|\boldsymbol{\omega}|=1}\int_0^\infty \kappa_\nu \tilde{S}_\nu(\mathbf{x}-s\boldsymbol{\omega})e^{-\kappa_\nu s} ds d\omega = \int_{\mathbb{R}^d} \kappa_\nu \tilde{S}_\nu(\mathbf{x}-\mathbf{x}')e^{-\kappa_\nu|\mathbf{x}'|}\frac{dx'}{|x'|^{d-1}}$$

□

Notice that the Fourier transform of Y_{κ_ν} satisfies

$$\mathcal{F}Y_{\kappa_\nu}(\xi) = \mathcal{F}Y_1\left(\frac{\xi}{\kappa_\nu}\right) = \frac{|\xi|}{2\pi\kappa_\nu}\arctan\frac{|\xi|}{\kappa_\nu}.$$

The numerical implementation is detailed in Algorithm 2.

Algorithm 2 To solve (5) with $S_\nu = a_\nu J_\nu + (1 - a_\nu) B_\nu(T)$

for each $\nu > 0$,

1. Compute $\mathbf{x} \mapsto \tilde{S}^E_\nu(\mathbf{x})$ by (6) and $\mathcal{F}Y_{\kappa_\nu} = \frac{\kappa_\nu}{2\pi|\xi|} \arctan(\kappa_\nu|\xi|)$.
2. **for** n=0,1,..,N
   (a) Compute the Fourier transforms $\mathcal{F}\tilde{S}_\nu$.
   (b) Compute $Y_{\kappa_\nu} \star S_\nu = \mathcal{F}^{-1}(\mathcal{F}Y_{\kappa_\nu} \cdot \mathcal{F}\tilde{S}_\nu)$.
   (c) Set $J^{n+1}_\nu(\mathbf{x}) = S^E_\nu(\mathbf{x}) + \mathcal{F}^{-1}(\mathcal{F}Y_{\kappa_\nu} \cdot \mathcal{F}S_\nu)$.
   (d) Compute T^{n+1} solution of

$$\int_0^\infty \kappa_\nu (1 - a_\nu) B_\nu(T) d\nu = \int_0^\infty \kappa_\nu (1 - a_\nu) J^{n+1}_\nu d\nu$$

.

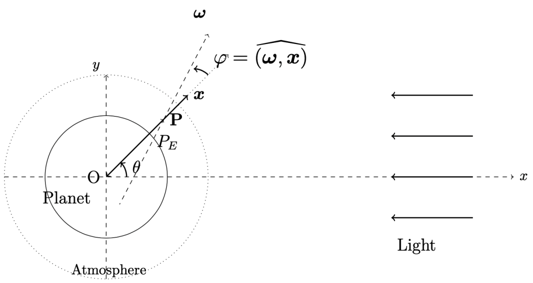

Fig. 1 *The light source, in the far right, sends lightwaves to the planet; it is assumed that the light is unaffected by the atmosphere. Hence point P in the atmosphere receives only the radiations emitted by the planet . A cross section of the planet and its atmosphere is shown in the plane defined by the axis Ox and the point P. The light intensity in the direction $\boldsymbol{\omega}$ is a function of the light intensity at P_E, the intersection of $\boldsymbol{\omega}$ and the circle $|\mathbf{x}| = R$.*

4.1 A Bidimensional Example

The geometry of the problem is shown on Figure 1 and the data are:

$$\Omega = \{x : \ |x| \in (R, R + H)\}, \quad Q_\nu(\mathbf{x}_E, \boldsymbol{\omega}) = Q^0 B_\nu(T_s) \frac{x^+_E}{R}.$$

These data are used with $\mathbf{x}_E = (x_E, y_E)$, the intersection of the line $\{\mathbf{x} - t\boldsymbol{\omega}\}_{t>0}$ with the circle $\{\mathbf{x} : |\mathbf{x}| = R\}$. As $|\mathbf{x} - t\boldsymbol{\omega}| = R$ requires $t^2 - 2t\mathbf{x}\cdot\boldsymbol{\omega} + |\mathbf{x}|^2 - R^2 = 0$, we have

$$\tau_{\mathbf{x},\boldsymbol{\omega}} = \mathbf{x} \cdot \boldsymbol{\omega} - \sqrt{(\mathbf{x} \cdot \boldsymbol{\omega})^2 - |\mathbf{x}|^2 + R^2}.$$

As explained in Bardos and Pironneau (2021), for numerical convenience the problem can be rescaled so that the Planck function is $B_\nu(T) = \nu^3/(\mathrm{e}^{\frac{\nu}{T}} - 1)$ with T in Kelvin divided by 4780. The Stefan Boltzmann formula becomes $\int_0^\infty B_\nu(T) \mathrm{d}\nu = \sigma T^4$ with $\sigma = \frac{\pi^4}{15}$. All cases are without scattering $a = 0$.

In the numerical tests $Q^0 = 5.74 \cdot 10^{-5}$, $T_{sun} = 1.209$, $R = 0.4$, $H = 0.3$.

4.1.1 The Grey Case

In the grey case (κ_ν independent of ν), the upper bar, as in $\bar{J}$, denotes the mean in ν. Then it is easy to see that we need to solve iteratively the integral equation:

$$\bar{J}(\mathbf{x}) = S^E(\mathbf{x}) + \sigma Y_\kappa \star \tilde{T}^4, \; S^E(\mathbf{x}) = \frac{Q^0 \sigma T_s^4}{2\pi} \int_0^{2\pi} (x - \tau_{\mathbf{x},\boldsymbol{\omega}} \cos\theta)^+ \mathrm{e}^{-\kappa \tau_{\mathbf{x},\boldsymbol{\omega}}} d\theta \quad (7)$$

with $Y_\kappa = \dfrac{\kappa}{2\pi|\mathbf{x}|} \mathrm{e}^{-\kappa|\mathbf{x}|}$. In absence of thermal diffusion, the temperature field is given by

$$\kappa \sigma T^4(\mathbf{x}) = \kappa \bar{J}(\mathbf{x}), \quad \mathbf{x} \in \Omega. \quad (8)$$

Figure 2 shows a numerical result obtained with $\kappa = 0.5$, $N = 10$ iterations, starting from $T^0 = 0.01$. The monotone behaviour of $\bar{J}^n$ is clearly seen (but not displayed here).

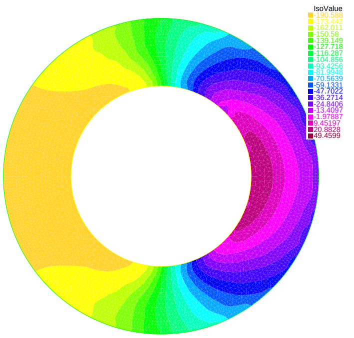

Fig. 2 Temperature map in the atmosphere of the planet which receives light from the right.

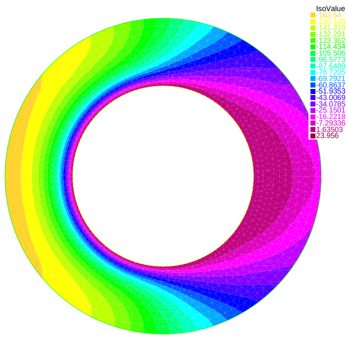

Fig. 3 Temperature map in the atmosphere of the planet which receives light from the right and has thermal diffusion.

The grid used for the FFT is 64×64. The mesh for the ring is 36×120 approximately uniform in polar coordinates. For S^E there are 60 integration points in θ. The computing time is 1 second per iteration on a core i9 MacBook 2020; convergence is reached after 5 iterations.

4.1.2 Non-Zero Thermal Diffusion

Let κ_T be the thermal diffusion and let T_E be the temperature of the planet. Then (8) must be replaced by

$$-\kappa_T \Delta T + \sigma T^4(\mathbf{x}) = \bar{J}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad T_{\partial\Omega} = T_E. \quad (9)$$

It is discretized with triangular finite elements of degree 1 and solved iteratively by a fixed point method whereby T^4 is replaced by $T_m^3 T_{m+1}$. Figure 3 shows a result with the same data used for Figure 2 and $\kappa_T = 0.01\sigma$. The temperature on the planet is fixed at 0.06, i.e. 13.8 C^o.

4.1.3 The Frequency Dependent Case

When κ_ν is not constant the problem is numerically expensive because one Fourier transform is needed at each integration point in the integrals in ν.

Recall that, when $a_\nu = 0$, we have to solve

$$\int_0^\infty \kappa_\nu B_\nu(T(\mathbf{x}))d\nu = \int_0^\infty \kappa_\nu Y_{\kappa_\nu} \star \tilde{B}_\nu(T)d\nu + \bar{S}^E(\mathbf{x}) \tag{10}$$

with

$$\bar{S}^E(\mathbf{x}) = \frac{Q^0}{2\pi}\int_0^{2\pi}\left((x - \tau_{\mathbf{x},\boldsymbol{\omega}}\cos\theta)^+ \int_0^\infty B_\nu(T_s)\kappa_\nu \mathrm{e}^{-\kappa_\nu \tau_{\mathbf{x},\boldsymbol{\omega}}} d\nu\right) d\theta \tag{11}$$

Extracting $\mathbf{x} \mapsto T(\mathbf{x})$ from (10), with a known right hand side, with a $\nu \mapsto \kappa_\nu$ given by values, is doable but expensive (see Golse and Pironneau (2021)). For a simple numerical example we may expand κ_ν in powers of ν:

$$\kappa_\nu \approx \kappa_0 + \kappa_1\nu + \kappa_2\nu^2 + \kappa_3\nu^3 + \kappa_4\nu^4 + \cdots \implies \int_0^\infty \kappa_\nu B_\nu(T) = \sigma\kappa_0 T^4$$
$$+ 24.886\kappa_1 T^5 + 122.081\kappa_2 T^6 + 726.012\kappa_3 T^7 + 5060.55\kappa_4 T^8 + \ldots \tag{12}$$

These numerical values are evaluations of polynomials of π and ζ function numbers computed with Maple.

For the numerical test we chose $\kappa_\nu = \kappa_0 + \kappa_1\nu := 0.5 \pm 0.03\nu$, $\nu \in (0, 15)$. Then we have to solve iteratively

$$\sigma\kappa_0 T^4(\mathbf{x}) + 24.886\kappa_1 T^5(\mathbf{x}) = Y_1 \star \mathcal{B}_\nu|_{\mathbf{x}} + S^E(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{13}$$

Figures 3, 4 and 5 illustrate the influence of a varying κ_ν on the temperature. There were 60 points for the integrations in ν, 60 points for the integrations in θ and 64×64 for the Discrete Fourier Transforms.

All programs were written with the high level PDE solver `freefem++` (see Hecht (2012)). The program for a non constant κ is evidently much slower and took 580 seconds per case.

5 Conclusion

By using a technique developed in Golse and Pironneau (2021) for the stratified radiative transfer problem, we have proved existence, uniqueness and a

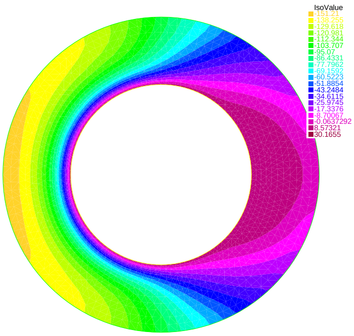

Fig. 4 *Same as above but with* $\kappa = 0.5 + 0.03\nu$, $\nu \in (0.01, 15)$.

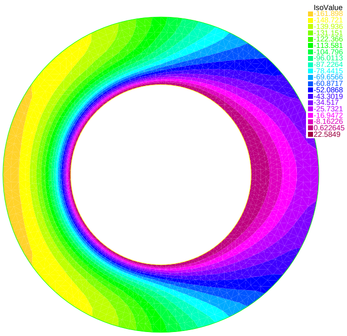

Fig. 5 *Same as above but with* $\kappa = 0.5 - 0.03\nu$, $\nu \in (0.01, 15)$.

maximum principle for a rather general form of the multidimensional Radiative Transfer system coupled with the time dependent temperature equation with drift.

The proofs are constructive and yield a robust and fairly fast numerical numerical algorithm, at least in the grey case, which encapsulate the exact solution between a lower and a higher numerical one obtained by starting from a guessed temperature field below (resp. above) the exact temperature field.

The 5 dimensional PDE is thus replaced by an iteration involving a three dimensional integral and a convolution integral easily computed with an FFT and which constitutes a tremendous gain in computing time over more classical finite element discretization as in LeHardy et al (2017).

Most remarkable is that there are essentially no constraint, besides positivity, on the absorption κ_ν and the scattering a_ν. If these depend on $\mathbf{x}$, a change of variable needs to be applied to return to the case κ_ν independent of $\mathbf{x}$. However if κ_ν depends on T the method does not work, except by adding an iteration loop, sending this dependency on the right hand side of the equation of I_ν.

Acknowledgments. The authors would like to thank Prof. Claude Bardos for the numerous discussions and references given.

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6 Appendix: Code documentation

The following may not appear in the published paper.

The following `freefem++` script `RT2Dfull2.edp` works for $\kappa_\nu = \kappa_0 + \kappa_1 \nu$, $\nu \in (\nu_{min}, \nu_{max})$. It recognizes the case κ constant (i.e. $\kappa_1 = 0$) and by avoiding the integrals in ν is then much faster in that case.

The data are:

```
// RT2Dfull2.edp. radiative transfer with no approximation
load "dfft"

int n=1, Niter=5,
  nx=n*32,ny=n*32,NN=nx*ny;
real Q0=2*sqrt(2.)*2.03e-5, Tsun=1.209, sigma=(pi^4)/15;

mesh Th=square(nx-1,ny-1,[-1+2*(nx-1)*x/nx,-1+2.*(ny-1)*y/ny]);
// warning the numbering of vertices (x,y) i  i = x/nx + nx*y/ny

real R= 0.4, H=0.3;
real dtheta=pi/30; // controls the integration on the unit circle
real kappa0=0.5, kappa1= 0.,
   kappaT=0.01; // if zero no heat equation
real numin= 0.01, numax = 15, dnu = (numax-numin)/100;

real source = Q0, R2=R*R; // auxiliaries
```

We need two domains, the square for the dFFT and the ring for the physics:

```
mesh Th=square(nx-1,ny-1,[-1+2*(nx-1)*x/nx,-1+2.*(ny-1)*y/ny]);
// warning the numbering of vertices (x,y) i  i = x/nx + nx*y/ny
border R1(t=0,2*pi){x=R*cos(t); y=R*sin(t);}
border RH(t=0,2*pi){x=(R+H)*cos(t); y=(R+H)*sin(t);}
mesh Rh= buildmesh(RH(120)+R1(-120));
```

The finite element spaces and the functions are defined by

```
fespace Vh(Th,P1);
fespace Wh(Rh,P1);

Vh<complex> u,v,w, F; // u,v,w for FFT and JJ for J(x,y)
Vh Jsource; // for S^E(x,y)
Wh Tc=0.01, Tch, Tca; // Tc is T(x,y) and Tch and Tca are auxiliaries
```

We need a function to define κ_ν, one to define the Planck function $B_\nu(T)$ and one to compute $\tau_{\mathbf{x},\boldsymbol{\omega}}$. The parameter `scal` in `twx` is here to save time and prevent recomputing the scalar product in `getSe`.

```
func real kappa(real nu) {return kappa0+kappa1*nu;}
func real Bnu(real T,real nu){ return sqr(nu)*nu/(exp(nu/T)-1);}
func real txw(real X,real Y, real scal){
  real aux =  sqr(scal) + R2 -sqr(X)-sqr(Y);
  if(aux>=0)
    if(scal>0) return scal - sqrt(aux);
      else return scal + sqrt(aux);
  else return -1;
}
```

Now `getSe` implements (11) or (7) when applicable.

```
func real getSe(real X, real Y){
  real aux = X*X+Y*Y;
  real Jxy=0;
  if(aux>R2 && aux<(R+H)*(R+H))
    for(real theta=0; theta<2*pi;theta+=dtheta){
    real wx=cos(theta),wy=sin(theta);
    real scal  = X*wx+Y*wy;
    real t = txw(X,Y,scal);
    real Bke = 0;
    if(t>0) {
      if(kappa1==0) Bke = kappa0*exp(-t*kappa0)*sigma*Tsun^4;
```

```
      else  for(real  nu=numin;nu<numax;nu+=dnu)
          Bke += Bnu(Tsun,nu)*kappa(nu)*exp(-kappa(nu)*t)*dnu;
      Jxy=Jxy+dtheta*max(X-t*wx,0.)*Bke;
    }
   }
  return  Jxy;
}
Jsource  =  getSe(x,y)*source/(2*pi);
```

Now let us compute Y_1 and its Fourier transform v. We could have use its analytical values but then would have had to struggle with the correspondance between the Fourier modes and the grid points. To avoid the singularity at $\mathbf{x} = 0$ we truncate it at $|\mathbf{x}| > R/4$. The FEM function u is needed to build an array of values at the grid points.

```
func  real  Yxy(real  X,real  Y,real  kappa){
  real  aux  =  sqrt(X*X+Y*Y);
  if(  aux>R/4  )  return  kappa*exp(-aux*kappa)/(2*pi*aux);
  else  return  0.;
}
```

The computation of the right hand side of (13) is done as follows

```
  F=0;
  if(kappa1==0){
     if(niter==0){u = Yxy(x,y,kappa0);  v[]=dfft(u[],ny,-1);}
     u = kappa0*sigma*Tc^4;
     w[]=dfft(u[],ny,-1);
     F=v*w*kappa0/sqr(NN);
  } else
  for(real  nu=numin;nu<numax;nu+=dnu){
    u= Yxy(x,y,kappa(nu));
    v[]=dfft(u[],ny,-1);
    u = Bnu(Tc(x,y),nu);
    w[]=dfft(u[],ny,-1);
    u=v*w/sqr(NN);
    F=F+u*kappa(nu)*dnu;
  }
  u[]=dfft(F[],ny,1);
  u= u + Jsource;
```

Finally the temperature is computed and converted into Celsius degree by the last formula.

```
  Tca=sqrt(sqrt(real(u) / (sigma*kappa0 + 24.886*kappa1*Tc) ));
  heat;
  u = sqr(Tc*Tc)*(sigma*kappa0 + 24.886*kappa1*Tc); // mean light intensity
  Tca=Tc*4780-273; // temperature in Celcius
  plot(Tca,ps="planettempdifffull2.ps", value=1,fill=1);
```

where `heat` is finite element solver for the temperature equation implemented as (notice how T is prescribed on the planet at 0.06, which is 13.8 Celsius.

```
problem   heat(Tc,Tch) = int2d(Rh)(kappaT*(dx(Tc)*dx(Tch)
      +dy(Tc)*dy(Tch)) +Tc*Tch) - int2d(Rh)(Tca*Tch)+on(R1,Tc=0.06);
```

These next to last 2 blocks must be encapsulated into a iteration loop

```
for(int  niter=0;niter<Niter;  niter++){
// the blocks here
}
```