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Abstract

We propose a definition of Lipschizian manifold that is more precise than the notion of Lipschitzian parameterization. It is modelled on the notion of differentiable manifold. We also give a notion of Lipschitzian submanifold and compare it with a notion devised by R.T. Rockafellar [30]. We endeavour to give a lucid view of the advantages and limitations of the different concepts. Among the examples we mention, the case of the graph of a maximally monotone operator and of the subjet of a convex function are the most notable.

Key words and phrases. Lipschitzian manifold; Lipschitzian parameterization; monotone operator; subdifferential; subjet

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Dedicated to R.T. Rockafellar on the occasion of his 85th birthday

Statements and Declarations

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in which notions of Lipschitz submanifolds of Euclidean spaces are studied; they coincide with the notions given in the draft of the present paper sent to D. Azé and several researchers. In the last paper a structure of measured space is introduced and integration is studied; more importantly, it is shown by an example that a Lipschitzian parameterization may differ from a Lipschitzian submanifold. In the first paper a Lipschitz Hodge Theory is presented on such manifolds. See also [33].

For many decades or even centuries, mathematicians worked on geometric objects we today call differentiable manifolds without defining them. As recalled by M. Berger in [8, p. 144], the famous geometer Elie Cartan wrote, in the late sixties for the second edition of his book,

"La notion générale de variété est assez difficile à définir avec précision."

(The general notion of manifold is quite difficult to define with precision).

The works of B. Riemann (1854), H. Weyl (1923), C. Chevalley (1939), H. Whitney (1936) led to the definition that is now widely accepted. In the nonsmooth case, Lipschitzian hypersurfaces of Euclidean spaces have been defined and used by P. Grisvard [15] and J. Nečas [23] in view of the study of partial differential equations on nonsmooth domains.

In this note, we endeavour to define a notion of Lipschitzian manifold, taking as a model the notion of differentiable manifold (see [8] or [20] for instance), replacing diffeomorphisms with *lipeomorphisms*, i.e., bijections that are Lipschitzian as well as their inverses. While in the differentiable case one can rely on the Inverse Mapping Theorem or the Embedding Theorem (see [28, Thm 5.22] for instance) to pass from embeddings to atlases, in the Lipschitz case such a tool is missing. Thus, one may expect that the notion of Lipschitzian parameterization cannot present the same features as the notion of Lipschitzian manifold. The recent developments of the geometric study of general metric spaces (see [7], [11], [16], [17], [18], [26] for instance) are incentives to explore such a notion. Many other incentives exist: for instance, one could explore the relationships between the tangent cone to the subjet of a convex function and the Alexandrov Theorem and the relationships between the geometry of the graph of a maximally monotone operator and the differentiability of its representative functions as defined in [28, section 9.4.5] for instance.

1 Definitions and examples

The definition we give is inspired by the notion of smooth or topological manifold (see [8], [20] for instance). As usual, we say that a family $(V_i)_{i\in I}$ of subsets of a set X is a *covering* if $\bigcup_{i\in I} V_i = X$.

Definition 1 *A set M is a Lipschitzian manifold if there is a covering $(V_i)_{i\in I}$ of M, a family $(E_i)_{i\in I}$ of normed spaces, a family $(U_i)_{i\in I}$ of sets such that U_i is an open subset of E_i and bijections $\varphi_i : U_i \to V_i$ such that for $(i,j) \in I^2$ the set $\varphi_i^{-1}(V_i \cap V_j)$ is open in U_i (or E_i), and the map $\varphi_j^{-1} \circ \varphi_i : \varphi_i^{-1}(V_i \cap V_j) \to \varphi_j^{-1}(V_i \cap V_j)$ is Lipschitzian. The maps φ_i are called charts and the collection of charts is called an atlas of M.*

Clearly, $\varphi_j^{-1} \circ \varphi_i$ is a lipeomorphism since its inverse is $\varphi_i^{-1} \circ \varphi_j$. Requiring that $\varphi_j^{-1} \circ \varphi_i$ is just locally Lipschitzian would not yield a more general notion since one could take restrictions of charts. Two atlases are said to be equivalent if their union is still an atlas. Strictly speaking, a Lipschitzian manifold is the datum of a set M and an equivalent class of atlases. When the atlas contains a single chart, we say that M is a Lipschitzian *monofold*. A Lipschitzian manifold is not necessarily embedded in some Euclidean space or normed space, on the contrary of compact differentiable manifolds (Nash's Theorem).

Example 1. The boundary M of the unit square $[-1,1]^2$ of $\mathbb{R}^2$ is a Lipschitzian manifold having an atlas formed with two charts $\varphi_i : U_i :=]-3,3[\to M \cap V_i$, with $i \in I := \{-1,1\}$ $V_{-1} := M \cap (\mathbb{R}\times]-\infty,1[)$, $V_1 := M \cap (\mathbb{R}\times]-1,+\infty[)$ given by

$$\varphi_i(u) = (-1, iu-2) \text{ for } u \in]-3,-1], (i,u) \text{ for } u \in [-1,1], (1, 2i-iu) \text{ for } u \in [1,3[.$$

One easily check that the map $(\varphi_{-1})^{-1} \circ \varphi_1$ and its inverse are Lipschitzian on $]-3,-1[\cup]1,3[$.

Example 2. Let $\mathcal{C}$ be the set of bounded, closed, convex subsets of a normed space E. Then $\mathcal{C}$ is a Lipschitzian manifold, in fact a Lipschitzian monofold. This can be seen by using the atlas consisting of a single chart, the Hörmander map $h : \mathcal{H} \to \mathcal{C}$, where $\mathcal{H}$ is the space of continuous positively homogeneous functions on the unit ball B^* of the dual space E^* of E endowed with the norm of uniform convergence. It is given by $h(f) := \{e \in E : \langle e^*, e\rangle \leq f(e^*)\ \forall e^* \in B^*\}$ for $f \in \mathcal{H}$; its inverse is the map $h^{-1} : \mathcal{C} \to \mathcal{H}$ given by $C \mapsto f_C$ with $f_C(e^*) := \sup_{e\in C}\langle e^*, e\rangle$. Both h and h^{-1} are isometries when $\mathcal{C}$ is endowed with the Pompeiu-Hausdorff metric defined by $d(C,D) := \max(e(C,D), e(D,C))$, with $e(C,D)) := \sup_{c\in C} d(c,D)$, $d(c,D) := \inf_{d\in D} \|c-d\|$. Thus a segment $S := \{f_t := (1-t)f_0 + tf_1 : t \in [0,1]\}$ of $\mathcal{H}$ is transformed by h into the segment $\{C_t := (1-t)C_0 + tC_1, t \in [0,1]\}$ with $C_0 := h(f_0)$, $C_1 := h(f_1)$ and one can use differential calculus on $\mathcal{C}$. When E is a Hilbert space or a uniformly convex Banach space, another representation is given by the set $\mathcal{P}$ of nonexpansive convex projectors of E into itself. Here we say that $P : E \to E$ is a projector if $P \circ P = P$ and a projector P is a convex projector if

$$P((1-t)x_0 + tx_1) = (1-t)P(x_0) + tP(x_1) \qquad \forall t \in [0,1],\ x_0,\ x_1 \in E.$$

Then the parameterization is the map $p : \mathcal{P} \to \mathcal{C}$ given by $p(P) := C$, where $C := \{x \in E : P(x) = x\}$. However, $(\mathcal{P}, p)$ is not a parameterization in the sense given below.

A Lipschitzian manifold M can be endowed with a topology : a subset V of M is declared to be open if for all $i \in I$ the set $\varphi_i^{-1}(V \cap V_i)$ is open in U_i (or equivalently in E_i). Then φ_i is continuous; its inverse is also continuous since for any open subset U of U_i the set $\varphi_i(U)$ is open in M. Thus, Lipschitzian manifolds are topological manifolds; but it is known ([19]) that there are topological manifolds that cannot be given a differentiable structure.

A Lipschitzian manifold can be endowed with a metric or pseudo-metric in the following way. One first defines the length of an arc c i.e. a continuous map $c : T \to M$, where T is a compact interval of $\mathbb{R}$. In order to do so, one takes a finite covering $(V_i)_{i\in I_c}$ of $c(T)$ extracted from $(V_i)_{i\in I}$ and a subdivision (T_k) of T by subintervals such that $c(T_k)$ is contained in some $V_{i(k)}$ with $i(k) \in I_c$. The length of the restriction c_k of c to T_k is well defined since $V_{i(k)}$ can be endowed with the metric transported from the one in $U_{i(k)}$. Summing these lengths and taking the infimum, one gets the length of c. Then, given two points x, y of M one defines $d(x,y)$ as the infimum of the lengths of arcs joining x to y. One can check that the axioms of pseudo-metrics are satisfied.

When M has an atlas formed by a single chart, the preceding contruction can be simplified and one gets a metric.

When the atlas is countable, a notion of negligible set (or even of measure) can be introduced on a Lipschitzian manifold.

In order to complete the picture of the category of Lipschitzian manifolds one has to define morphisms. A map $f : M \to M'$ between two Lipschitzian manifolds is said to be Lipschitzian around $\overline{x} \in M$ if there are charts $\varphi : U \to V$, $\varphi' : U' \to V'$ of M and M' respectively with $\overline{x} \in V$, $f(\overline{x}) \in V'$ such that $\varphi'^{-1} \circ f \circ \varphi$ is Lipschitzian around $\varphi^{-1}(\overline{x})$. A map $f : M \to M'$ is locally Lipschitzian if it is Lipschitzian around any $\overline{x} \in M$. Note that the Lipschitz rate of f cannot be defined, unless M and M' have atlasses whose changes of charts are Lipschitzian with a prescribed rate. Obviously, the composition of two locally Lipschitzian maps is locally Lipschitzian.

A notion of Lipschitzian submanifold can be devised by analogy with the notion of differentiable submanifold.

Definition 2 *A subset M' of a Lipschitzian manifold M is said to be a Lipschitzian submanifold of M if for some atlas $(\varphi_i)_{i\in I} := (\varphi_i, U_i, V_i)_{i\in I}$ of M there is a subset I' of I such that $(M' \cap V_i)_{i\in I'}$ is a covering of M' and $U_i = U_i' \times U_i''$ where U_i' (resp. U_i'') is an open subset of some normed space E_i' (resp. E_i'') such that $E_i = E_i' \times E_i''$ and $\varphi_i(U_i' \times \{0\}) = M' \cap V_i$ for all $i \in I'$.*

Then one sees that M' is a Lipschitzian manifold for the (equivalent class) of charts given by the restrictions of φ_i to U_i', E_i' being identified with $E_i' \times \{0\}$ for $i \in I'$.

When M is a normed space E, two variants of this definition can be given. The first one is the notion of *Lipschitzian parameterization* and is somewhat looser than the preceding definition. It consists in a Lipschitzian bijection $\varphi : U \to M'$ whose inverse is also Lipschitzian, U being an open subset of a normed space. Then M' is a Lipschitzian monofold, but not necessarily a Lipschitzian submanifold of E. Clearly the graph G of a Lipschitzian mapping has a Lipschitzian parameterization. But, as an inverse mapping theorem is lacking, one cannot assert that a Lipschitzian parameterization is enough to prove that its image is a Lipschitzian submanifold. More research along the lines of [13, Cor 6.8], [24, Cor 6.8] and [25] is needed.

The second variant is tighter than the preceding definition and coincides with the definition introduced by R.T Rockafellar [30], which is as follows (when E is finite dimensional, an assumption we avoid here).

Definition 3 *A subset M of a normed space E is a Lipschitzian submanifold of E in the sense of Rockafellar (or a graphical Lipschitzian submanifold of E) if for any $\overline{x} \in M$ there is an open neighborhood V of $\overline{x}$ in E, a splitting $E := E' \times E''$ into a product of two linear subspaces and a C^1-diffeomorphism Φ*

of V onto an open subset $U := U' \times U''$ of $E' \times E''$ such that $\Phi(M \cap V) = G \cap U$, where G is the graph of a Lipschitzian map $g : U' \to U''$.

Example 2. The set M of Example 1 is a graphical submanifold of $\mathbb{R}^2$ as one can see by using the diffeomorphisms $\Phi : (u, v) \mapsto (u + v, u - v)$ and $\Psi : (u, v) \mapsto (u - v, u + v)$ around each vertex of the square.

This notion is more demanding than the notion of Lipschitzian submanifold we introduced since Φ and Φ^{-1} are required to be bijections of class C^1 and since E is a normed vector space (it could also be a differentiable manifold). These additional requirements allow to define a tangent cone to M (and even tangent cones of various nature) at each of its points. Requiring more regularity would permit to study higher order notions. However, such requirements are not possible for an arbitrary submanifold of a Lipschitzian manifold. The next proposition clarifies the relationship between the two notions of Lipschitzian submanifold.

Proposition 4 *A graphical Lipschitzian submanifold of a normed space E is a Lipschitzian submanifold of E.*

The next proposition justifies this statement.

Proposition 5 *Given two normed spaces X, Y and an open subset W of X, the graph G of a map $g : W \to Y$ is a monofold, i.e. a Lipschitzian manifold with an atlas consisting in a single chart ψ. Moreover, if g is Lipschitzian, G is a Lipschitzian submanifold of $X \times Y$.*

Proof. The map $\psi : w \to (w, g(w))$ is clearly a bijection of W onto G. We call ψ the canonical chart for G; thus G is a monofold. Let $\varphi : W \times Y \to W \times Y$ be given by $\varphi(w, y) = (w, y + g(w))$. When g is Lipschitzian, φ is a Lipschitzian bijection with Lipschitzian inverse φ^{-1} given by $\varphi^{-1}(w, z) = (w, z - g(w))$. Then for any $w \in W$ one has $\varphi(w, 0) = (w, g(w)) \in G$ and $\varphi^{-1}(G) \subset W \times \{0\}$. Thus $\varphi(W \times \{0\}) = G$ and G is a Lipschitzian submanifold of $W \times Y$. $\square$

Lipschitzian manifolds abound, as show the preceding proposition and the simple examples presented in the following propositions. Moreover, a deep result of Sullivan [33] asserts that any topological manifold of dimension $k \neq 4$ has a structure of Lipschitzian manifold. Our motivation is however drawn from more concrete problems. Among the questions we have in mind are the relationships between the structure of Lipschitzian manifold of a maximally monotone operator M with its representative functions f_M and p_M as defined in [28, Section 9.4.5] and its references.

One may wonder whether the unit sphere S of a Banach space E is a Lipschitzian submanifold of E. When the norm $n(\cdot)$ is Fréchet differentiable off 0, S is a smooth manifold of class C^1 since then $n(\cdot)$ is of class C^1 off 0 by [28, Cor 6.8] and is a submersion since $n'(x)x = 1$ for all $x \in S$. Let us give a similar result under a weaker differentiability assumption.

Proposition 6 *If $j : E \to \mathbb{R}_+$ is a sublinear function satisfying $j^{-1}(0) = \{0\}$, then $S := j^{-1}(1)$ is a Lipschitzian submanifold of E.*

If j is of class C^1 the set S is a graphical Lipschitzian submanifold (and a differentiable manifold).

Proof. Given $e \in S$, using the Hahn-Banach theorem we can pick $h \in \partial j(e)$ such that $h(e) = 1 = j'(e)e$ and let $H := h^{-1}(0)$, $U := H \times]-1, +\infty[$, $V := h^{-1}(\mathbb{P})$ with $\mathbb{P} :=]0, +\infty[$. We define $\varphi : U \to E$ by

$$\varphi(z, r) := (r+1)\frac{z + (r+1)e}{j(z + (r+1)e)}, \qquad (z, r) \in U.$$

Clearly, φ is well defined since by convexity

$$j(z+(r+1)e) = (r+1)j((r+1)^{-1}z+e) \geq (r+1)(j(e)+h((r+1)^{-1}z)) = r+1 > 0 \qquad (z,r) \in U$$

and $\varphi(z, r) \in V$ for $(z, r) \in U$ since

$$h(\varphi(z, r)) = \frac{r+1}{j(z + (r+1)e)}(r+1)h(e) = \frac{(r+1)^2}{j(z + (r+1)e)} > 0.$$

Given $v \in V$, setting $r := j(v) - 1$, $s := j(v)^2/h(v)$, $z := j(v)v/h(v) - j(v)e$, we see that $\varphi(z, r) = v$ and the inverse of φ is given by

$$\varphi^{-1}(v) = (\frac{j(v)}{h(v)}v - j(v)e, j(v) - 1).$$

Since $\varphi(H \times \{0\}) = S$ and since φ and φ^{-1} are locally Lipschitzian we get that S is a Lipschizian submanifold of E. □

Proposition 2.11 of [31] enables us to transfer the example of Proposition 5 to the class $\mathcal{M}$ of maximal monotone multimaps from a Hilbert space X into itself. For the sake of completeness we present this proposition (with a slight change) and its proof using a map of common use in symplectic geometry.

Proposition 7 *Given a Hilbert space X and $c \in \mathbb{P} :=]\mathbf{0}, +\infty[$, the Lipschitzian map $\chi_c : X^2 \to X^2$ given by*

$$\chi_c(u, v) = 2^{-1/2}c(u + v, u - v)$$

with inverse given by

$$\chi_c^{-1}(w, z) = 2^{-1/2}c^{-1}(w + z, w - z)$$

transforms the graph of any nonexpansive map $g : D_g \to X$ defined on some subset D_g of X onto the graph of a monotone operator $M : X \rightrightarrows X$ and conversely, χ_c^{-1} transforms the graph of any monotone operator M onto the graph of a nonexpansive map g defined on some subset D of X. For $c = 1$, χ_c is an isometry and $\chi_c^{-1} = \chi_c$.

If M is maximally monotone one can require that $D = X$.

We use the parameter c because, while the choice $c = 1$ has the advantage of producing an isometry, the choices $c = 2^{1/2}$ or $c = 2^{-1/2}$ have the advantage of simplicity in the writing.

Proof. The relations $\chi_c \circ \chi_c^{-1} = I_{X^2}$ and $\chi_c^{-1} \circ \chi_c = I_{X^2}$ are immediate as is the fact that χ_c and χ_c^{-1} are Lipschitzian and, for $c = 1$, monometries (i.e. preserve distances), hence isometries. Now, for (u, v), $(u', v') \in X^2$ and $(w, z) := \chi_c(u, v)$, $(w', z') := \chi_c(u', v')$ one has

$$\begin{aligned} \|u - u'\|^2 - \|v - v'\|^2 &= \langle (u - u') - (v - v'), (u - u') + (v - v') \rangle \\ &= \langle (u - v) - (u' - v'), (u + v) - (u' + v') \rangle \\ &= 2^{-1} c^{-2} \langle z - z', w - w' \rangle \end{aligned}$$

so that, if (u, v), $(u', v') \in G := \operatorname{gph} g$, with g nonexpansive, one sees that $\langle z - z', w - w' \rangle \geq 0$ and $M := \chi_c(G)$ is monotone. Conversely, if M is (the graph of) a monotone multimap, if $(u, v) = \chi_c^{-1}(w, z)$, $(u', v') = \chi_c^{-1}(w', z')$ with (w, z), $(w', z') \in M$, when $u' = u$ we see that $v' = v$, so that $G_c := \chi_c^{-1}(M)$ is the graph of a map g_c and g_c is nonexpansive since $\|v - v'\| \leq \|u - u'\|$ in view of the preceding string of equalities.

In order to prove the last assertion one can follow the argument of the proof of [31, Thm 9.58] about the extension of a nonexpansive map, using the weak topology of X instead of the topology associated with the Euclidean metric. □

Thus the graph of a monotone operator $M : X \rightrightarrows X$ is a graphical Lipschitzian submanifold, hence is a Lipschitzian submanifold of $X \times X$. In [31, Thm 12.15] the *Minty parameterization* (of the graph) of a maximally monotone operator M on $\mathbb{R}^n$ is defined. Let us describe it. It is known ([5, Thm 3.5.8], [22], [28, Thm 9.28, Cor. 9.10], [31, Thm 12.12]) that given $w \in X$, there is a unique pair $(u, v) \in M$ (identified with the graph of M) such that $u + v = w$. One sets $u = P(w)$, $v = Q(w)$.The maps P, Q are described in [31, Thm 12.15] by

$$P := (I + M)^{-1} \qquad Q := (I + M^{-1})^{-1}.$$

They are single-valued, maximally monotone, nonexpansive maps satisfying $Q \subset M \circ P$ and $P + Q = I$, the identity map on X. The *Minty parameterization* of M is the lipeomorphism $w \mapsto (P(w), Q(w))$ from X to M. It makes M a Lipschitzian manifold (and even monofold) but it does not show that M is a Lipschitzian submanifold of $X \times X$. Our aim is now to show that.

For this purpose, let us introduce the *Minty chart* of such a graph. It is inspired by the illuminating Figure 12.4 of [31], but in fact it has been defined in [30].

Theorem 8 *Let $M : X \rightrightarrows X$ be a maximally monotone multimap. Then there exists a Lipschitzian map $\mu : X \times X \rightrightarrows X \times X$ whose inverse is also Lipschitzian such that $M = \mu(X \times \{0\})$. Thus M is a Lipschitzian submanifold of X^2.*

We call μ the *Minty chart* and we observe that $w \mapsto \mu(w, 0)$ is the Minty parameterization defined in [31, Thm 12.15].

Proof. The idea of the proof is simple. Introducing the diffeomorphism $\chi := \chi_c$ for $c = 2^{1/2}$ given by $\chi(u,v) = (u+v, u-v)$, one defines μ by

$$\mu := \chi \circ \varphi, \qquad \chi := \chi_c$$

where φ is canonical chart for the nonexpansive map $g := g_c$ associated with M. Its inverse is given by $\mu^{-1} = \varphi^{-1} \circ \chi^{-1}$. Restricting μ to $X \times \{0\}$, we see that $\mu(X \times \{0\}) = \chi(G) = M$:

$$X \times \{0\} \overset{\varphi}{\to} G \overset{\chi}{\to} M.$$

In order to describe g or φ one notes that $\varphi = \chi^{-1} \circ \mu$ and that for $w \in X$ one has $(w, g(w)) = \varphi(w, 0) = \chi^{-1}(\mu(w, 0))$, so that, setting $(u, v) = \mu(w, 0)$ one has $(w, g(w)) = \chi^{-1}(u, v) = (u + v, u - v)$. Requiring that $w \mapsto \mu(w, 0)$ be the Minty embedding $w \mapsto (P(w), Q(w))$ or $u = P(w)$, $v = Q(w)$, one obtains that $g(w) = P(w) - Q(w)$ and

$$\begin{aligned} \varphi(w,z) &= (w, g(w) + z) = (w, P(w) - Q(w) + z), \\ \mu(w,z) &= (w + P(w) - Q(w) + z, w - z - P(w) + Q(w)). \end{aligned}$$

□

Let us observe that the Minty chart μ enables to transfer the phase curve $(x(\cdot), x'(\cdot))$ of a solution to the differential inclusion $-x'(t) \in M(x(t))$ into a curve in the image of M by μ^{-1} and that may be convenient as it is a curve in $X \times \{0\}$ which can be studied with classical means.

Now let us turn to subjets of convex functions. The *subjet* (or characteristic manifold) J_f of a closed proper convex function $f : X \to \mathbb{R}_\infty$ on a Hilbert space X (identified with its dual) is the set

$$J_f := \{(x, y, s) \in X \times X \times \mathbb{R} : y \in \partial f(x),\ s = f(x)\},$$

where $\partial f(x)$ is the subdifferential of f at x. In [32, Prop. 6.5] a locally Lipschitzian parameterization of this set is introduced. Here we show that it is a Lipschitzian submanifold of $X \times X \times \mathbb{R}$. For such a purpose, we consider the map $\omega : X \times X \times \mathbb{R} \to X \times X \times \mathbb{R}$ given by

$$\omega(w, z, t) = (\mu(w, z), f(P(w)) + \frac{1}{2}\left\|Q(w)\right\|^2 + t)$$

where $P := (I + \partial f)^{-1}$, $Q := I - P$. It is a locally Lipschitzian (in fact boundedly Lipschitzian, i.e. Lipschitzian on bounded subsets) map because P and Q are Lipschitzian and μ is Lipschizian, $f \circ P + \frac{1}{2}\left\|Q(\cdot)\right\|^2$ is the Moreau 1-regularized function f_1 of f, given by

$$f_1(w) = \inf_{u \in X} (f(u) + \frac{1}{2}\left\|w - u\right\|^2)$$

It is Lipschitzian on the ball $B(0, r)$: given w, $z \in B(0, r)$, setting $u := P(w)$, $v = P(z)$, one has

$$f_1(w) = f(u) + \frac{1}{2}\left\|w - u\right\|^2 \leq f(v) + \frac{1}{2}\left\|w - v\right\|^2 = f_1(z) - \frac{1}{2}\left\|z - v\right\|^2 + \frac{1}{2}\left\|w - v\right\|^2,$$

hence

$$f_1(w) - f_1(z) \leq \langle w - v + z - v, w - z \rangle \leq 2(r + k) \|w - z\|,$$

where k is such that $P(B(0,r)) \subset B(0,k)$. The existence of k stems from the fact that there are a and b in $\mathbb{R}_+$ such that $f(v) \geq b - a \|v\|$ for all $v \in X$, so that, taking d in the domain of f, one has

$$b - a \|v\| + \frac{1}{2} \|z - v\|^2 \leq f(d) + \frac{1}{2} \|z - d\|^2.$$

Since μ is surjective, ω is seen to be onto. In fact, it has an inverse given by

$$\omega^{-1}(w, z, s) = (\mu^{-1}(w, z), s - f(P(w)) - \frac{1}{2} \|Q(w)\|^2)$$

This inverse also is boundedly Lipschitzian, i.e. Lipschitzian on any bounded subset.

2 Tangent cone

If M is a Lipschitzian submanifold of a normed space E, one can define the tangent cone to M and the normal cone to M at an arbitrary point $x \in M$, as in the case of an arbitrary subset of E and this can be done in different ways, according to the definitions of Bouligand or Clarke, for instance. But one cannot use an atlas to characterize it. On the contrary, when M is a graphical Lipschitzian submanifold of E this tangent cone corresponds to the tangent cone to some graph via the derivative of a chart at x. The invariance of tangent cones under diffeomorphisms shows that this cone does not depend on the choice of a chart Φ.

For an abstract Lipschitzian manifold the construction of a substitute to a tangent cone is not so clear. A possible way may consists in selecting a class of curves replacing straight lines or half-lines or differentiable curves. If M is a Lipschitzian manifold with atlas $(\varphi_i)_{i \in I}$, two curves $c_1 : T_1 \to M$ and $c_2 : T_2 \to M$ (where $T_1 := [0, t_1]$, $T_2 := [0, t_2]$) are said to be tangent at $x := c_1(0) = c_2(0)$ if for some chart φ_i one has $\lim_{t \to 0_+} (1/t) \left\|\varphi_i^{-1}(c_1(t)) - \varphi_i^{-1}(c_2(t))\right\| = 0$. This condition is independent of the choice of the chart φ_i and defines an equivalence condition. However, the set of such equivalence classes is too large to represent a tangent cone to M.

Thus, we turn to means to detect a smaller set. Such means have been given in any metric space M either assuming M is locally compact ([16]) or without making this assumption ([26, Def. 2.1]). Assuming M is endowed with a metric d defining its topology, in order to define a tangent cone $T_x M$ to M at some point $x \in M$, two conditions on a curve c in M issued from x are required in [26, Def. 2.1]:

c is *rhythmed* in the sense that the limit of $d(c(t), c(0))/t$ exists,

c is a *cadence* in the sense that it is rhythmed and for any $a \in \mathbb{P} :=]0, \infty[$ the limit of $d(c(st), c(t))/t$ as $(s,t) \to (a, 0_+)$ exists and is equal to $|a - 1| \lim_{t \to 0_+} d(c(t), c(0))/t$.

Then a kind of tangent cone to M at x is defined as the set of equivalence classes of cadences issued from x; in [26] it is called the set $V(M,x)$ of *velocities* of M at x. If $f : M \to M'$ is a locally Lipschitzian map with values in another Lipschitzian manifold endowed with a metric d', a substitute to the tangent map of f can be introduced when for any velocity v and any representant c of v the curve $f \circ c$ is a cadence.

The existence of such a set of velocities at x is ensured when there exists a chart φ that is *almost isometric at* $\varphi^{-1}(x)$ in the sense that for any $\varepsilon > 0$ there exists a neighborhood $U_\varepsilon \subset U$ of $\varphi^{-1}(x)$ such that for any $u, u' \in U_\varepsilon$ one has

$$(1+\varepsilon)^{-1} \|u - u'\| \le d(\varphi(u), \varphi(u')) \le (1+\varepsilon) \|u - u'\| .$$

It is easy to see that, when φ is such a chart, for any $e \in E$ the image c by φ of the segment $t \mapsto \varphi^{-1}(x) + te$ with $t \in [0, \theta]$ for some $\theta > 0$ fulfils the two conditions defining a cadence. Then the set of velocities of M contains the class of c and the set of velocities of M at x is not reduced to 0, the class of the constant curve, as it contains a subset in bijection with the linear space E.

Moreover, it is shown in [26] and easily seen, that when M is a subset of a normed space and $c : [0, \theta] \to M$ is a curve issued from $x \in M$ such that the right derivative $c'_+(0)$ of c at 0 exists, then c is a cadence. Then the set of velocities of M contains the classical tangent cone also called the incident cone $T^i(M,x)$ ([27]) or the adjacent cone ([5]). But it is not related to the contingent cone or the Clarke tangent cone, unless M is a graphical Lipschitzian submanifold.

When a Lipschitzian manifold M has a countable atlas $\{\varphi_i : i \in I\}$ whose charts have finite dimensional sources, one can define negligible subsets of M : $N \subset M$ is negligible if for all $i \in I$ the set $\varphi_i^{-1}(N \cap V_i)$ has measure zero. Using the Rademacher Theorem, one sees that the set N of $x \in M$ such that for all i, $j \in I$ with $x \in V_i \cap V_j$ the map $\varphi_j \circ \varphi_i^{-1}$ is differentiable at $\varphi_i(x)$ is negligible. Then, for all $x \in M \backslash N$ the tangent cone to M at x can be defined as the set of families $(e_i)_{i \in I}$ with $e_i \in E_i$ such that $e_j = D(\varphi_j \circ \varphi_i^{-1})(\varphi_i(x))(e_i)$ for all i, $j \in I$ with $x \in V_i \cap V_j$. It is clearly a finite dimensional vector space.

Locally Lipschitzian maps between Lipschitzian manifolds can be introduced and studied. When these manifolds are graphically Lipschitzian submanifolds of some normed spaces, the tools of nonsmooth analysis can be used since they are invariant under diffeomorphisms of class C^1. In our views, that is the main advantage of graphically Lipschitzian manifolds over general Lipschitzian manifolds or Lipschitzian parameterizations.

3 Conclusion and Open Questions

Inspired by an analogy with the notion of smooth manifold, we have introduced an abstract, but natural notion of Lipschitzian manifold and a notion of Lipschitzian submanifold of a Lipschitzian manifold, in particular of a normed space. We have shown that the graphs of maximally monotone multimaps on Hilbert spaces enjoy such a double structure. From the analysis we conduct the benefits of such a structure over the notion of Lipschitzian parameterization are put in full light and the comparison with the notion of graphical Lipschitzian manifold due to R.T. Rockafellar is made clear: the later notion is suited to an extension of the concepts and methods of nonsmooth analysis. Answers to the following questions and the use of generalized forms of the implicit function theorem as in [13], [24], and [25] may play this role.

Question 1. We note that at each point of differentiability of a chart of a submanifold M of some normed space E one can define a tangent space which is a closed linear subspace. Does this space coincides with the tangent cone defined in [26] using the metric structure of M?

Question 2. Does the structure of Lipschitzian manifold for a set M helps for regularizing M?

Question 3. Can one use some nonsmooth version of the implicit function theorem (as in [6], [13], [24] for instance) to pass from a Lipschitzian embedding to a structure of Lipschitzian submanifold?

Question 4. Curvature notions could be obtained for Lipschizian submanifolds of a finite dimensional Hilbert space using the Alexandrof Theorem.

Question 5. Is it possible (and useful) to extend our study of the Lipschizian manifold structure of a maximally monotone multimap to the class of hyper-accretive or hyper-dissipative multimaps on some appropriate class of Banach spaces?

Question 6. What is the interaction of the structure of Lipschizian manifold of a maximally monotone operator M with its representative functions f_M and p_M as defined in [14], [28, Section 9.4.5] (or any other representative function)?

Question 7. Does the structure of Lipschitzian manifold given to the subjet of a convex function shed a new light on the $\mathcal{U}$-Lagrangian described in [21]?

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