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Abstract. We classify the group of birational automorphisms of Hilbert schemes of points on algebraic K3 surfaces of Picard rank one. We study whether these automorphisms are symplectic or non-symplectic and if there exists a hyperkähler birational model on which they become biregular. We also present new geometrical constructions of these automorphisms.

1. Introduction

Hilbert schemes of points on smooth complex K3 surfaces are examples of irreducible holomorphic symplectic (ihs) manifolds, i.e. compact simply connected Kähler manifolds with a unique, up to scalar, holomorphic two-form, which is everywhere non-degenerate. The second integral cohomology group of ihs manifolds admits a lattice structure, provided by the Beauville-Bogomolov-Fujiki (BBF) quadratic form. The global Torelli theorem for K3 surfaces has been generalized to ihs manifolds, albeit in a weaker form, making possible to investigate automorphisms of these manifolds by studying their action on the BBF lattice.

For an integer $t \geq 1$ consider a complex algebraic K3 surface S whose Picard group is generated by an ample line bundle H with $H^2 = 2t$, i.e. a very general element of the 19-dimensional space of $2t$-polarized K3 surfaces. A classification of the group of biregular automorphisms of the Hilbert scheme $S^{[n]} := \mathrm{Hilb}^n(S)$ has been given by Boissière, An. Cattaneo, Nieper-Wißkirchen and Sarti [5] in the case $n = 2$, and by the second author for all $n \geq 3$ [8]. In particular, $\mathrm{Aut}(S^{[n]})$ is either trivial or generated by an involution which is non-symplectic, i.e. it does not preserve the generator of $H^{2,0}(S^{[n]})$. If $t \neq 1$, the involution is not induced by an involution of the surface S (we say that it is non-natural). The results of [5] and [8] provide explicit numerical conditions on the parameters n and t for the existence of a non-trivial biregular automorphism on $S^{[n]}$. The first aim of the paper is to give a similar classification for the group $\mathrm{Bir}(S^{[n]})$ of birational automorphisms of the Hilbert scheme, thus generalizing the results of Debarre and Macrí [11, 10] for the case $n = 2$.

Section 2 briefly recalls the basic theory of Pells equations, which are the fundamental number theoretic tool in our study. Then, in Section 3 we combine the description of the movable cone of $S^{[n]}$ due to Bayer and Macrí [2] with Markman's results on monodromy operators [16] to obtain numerical necessary and sufficient conditions on n and t for the existence of birational automorphisms on $S^{[n]}$. For

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$\sigma \in \mathrm{Bir}(S^{[n]})$ let $\sigma^* \in O(H^2(S^{[n]},\mathbb{Z}))$ be its pullback action on the BBF lattice. We set $H^2(S^{[n]},\mathbb{Z})^{\sigma^*} := \left\{v \in H^2(S^{[n]},\mathbb{Z}) \mid \sigma^*(v) = v\right\}$ and $H^2(S^{[n]},\mathbb{Z})_{\sigma^*} := (H^2(S^{[n]},\mathbb{Z})^{\sigma^*})^{\perp}$.

Theorem 1.1. *Let S be an algebraic K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$ and $n \geq 2$.*

If $t \geq 2$, there exists a non-trivial birational automorphism $\sigma : S^{[n]} \dashrightarrow S^{[n]}$ if and only if $t(n-1)$ is not a square and the minimal solution $(X,Y) = (z,w)$ of Pell's equation $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$ satisfies $w \equiv 0 \pmod 2$ and $(z,z) \equiv (j,k) \in \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \times \frac{\mathbb{Z}}{2t\mathbb{Z}}$ with $(j,k) \in \{(1,1),(1,-1),(-1,-1)\}$. If so $\mathrm{Bir}(S^{[n]}) = \langle\sigma\rangle \cong \mathbb{Z}/2\mathbb{Z}$ and

- *if $(j,k) = (1,-1)$, then σ is non-symplectic with $H^2(S^{[n]},\mathbb{Z})^{\sigma^*} \cong \langle 2\rangle$;*
- *if $(j,k) = (-1,-1)$, then σ is non-symplectic with $H^2(S^{[n]},\mathbb{Z})^{\sigma^*} \cong \langle 2(n-1)\rangle$;*
- *if $(j,k) = (1,1)$, then σ is symplectic with $H^2(S^{[n]},\mathbb{Z})_{\sigma^*} \cong \langle -2(n-1)\rangle$.*

If $t = 1$, let $(X,Y) = (a,b)$ be the integer solution of $(n-1)X^2 - Y^2 = -1$ with smallest $a,b > 0$. If $n-1$ is a square or $b \equiv \pm 1 \pmod{n-1}$, then $\mathrm{Bir}(S^{[n]}) = \mathrm{Aut}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z}$. Otherwise $n \geq 9$, $\mathrm{Bir}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Aut}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z}$.

For more details on the action $\sigma^* \in O(H^2(S^{[n]},\mathbb{Z}))$, see Proposition 3.1. We remark that, by a classical result of Saint-Donat [26], the case $t = 1$ is the only one where the K3 surface S has a non-trivial automorphism (S is a double covering of $\mathbb{P}^2$ ramified over a sextic curve), hence $\mathrm{Aut}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by the corresponding natural (non-symplectic) involution for all $n \geq 2$.

It is interesting to notice that, differently from the biregular case, birational involutions of $S^{[n]}$ can be symplectic, for (infinitely many) suitable choices of n, t. In Proposition 3.5 we give examples of sequences of degrees of polarizations $t = t_k(n)$ which realize all cases in Theorem 1.1.

By [16, §5.2], the closed movable cone of a projective ihs manifold X has a wall-and-chamber decomposition

$$\overline{\mathrm{Mov}(X)} = \bigcup_g g^* \,\mathrm{Nef}(X') \tag{1}$$

where the union is taken over all non-isomorphic ihs birational models $g : X \dashrightarrow X'$. The chambers $g^*\,\mathrm{Nef}(X')$ are permuted by the action of any birational automorphism of X. By Markman's Hodge-theoretic global Torelli theorem [16, Theorem 1.3] biregular automorphisms are exactly those which map $\mathrm{Nef}(X)$ to itself. If X admits a birational automorphism σ, it is natural to ask whether there exists an ihs birational model $g : X \dashrightarrow X'$ such that $g \circ \sigma \circ g^{-1}$ is biregular on X'. We confront this problem for Hilbert schemes $S^{[n]}$ in Section 4, by using the explicit description of the walls between chambers coming from [2, Theorem 12.3]. For a fixed $n \geq 2$, we say that a value $t \geq 1$ is *n-irregular* if, for a very general $2t$-polarized K3 surface S, the group $\mathrm{Bir}(S^{[n]})$ contains an involution which is not biregular on any ihs birational model of $S^{[n]}$. We provide a numerical characterization of n-irregular values in Propositions 4.3 and 4.5. In particular, we verify that symplectic birational involutions remain strictly birational on all birational models. On the other hand, in the case of non-symplectic birational automorphisms there are only finitely many n-irregular t's for a fixed $n \geq 2$ (see Corollary 4.7).

Section 5 provides an in-depth analysis of the case $n = 3$. By studying solutions of two particular Pell's equations we show that, when $\mathrm{Bir}(S^{[3]})$ is not trivial, the number of chambers in the movable cone of $S^{[3]}$ (i.e. the number of non-isomorphic ihs birational models) is either one, two, three or five. For $n = 2$ it is known that the number of chambers in $\mathrm{Mov}(S^{[2]})$ is at most three. As n increases, computing an upper bound for the number of chambers becomes more and more difficult, since walls arise from the solutions of an increasing number of Pell's equations. The new results for $n = 3$ are summed up in Proposition 5.6.

In Section 6 we show how Theorem 1.1 can be generalized to study birational maps between Hilbert schemes of points on two distinct K3 surfaces of Picard rank one (which need to be Fourier-Mukai partners). In the literature, isomorphic Hilbert schemes of points on two non-isomorphic K3 surfaces are called (strongly) ambiguous. The main result of the section is Theorem 6.2, an improved version of the criterion [17, Theorem 2.2] for the determination of ambiguous pairs (up to isomorphism or birational equivalence).

In Section 7 we go back to the case of biregular involutions. For ihs manifolds of $K3^{[n]}$-type (i.e. deformation equivalent to Hilbert schemes of n points on a K3 surface) we compare moduli spaces of polarized manifolds and moduli spaces of manifolds with an involution whose action on the second cohomology has invariant lattice of rank one. Theorem 7.5 gives a modular interpretation of the classification of non-symplectic involutions for manifolds of $K3^{[n]}$-type (see for instance [7]), which for $n = 2$ has been highlighted in [4]. For $n \geq 3$ this requires some additional care, since we deal with moduli spaces which may be non-connected.

Finally, in Section 8 we give new geometrical examples of birational involutions as in Theorem 1.1, for Hilbert schemes of points on quartic surfaces. Finding similar constructions is an interesting problem, since the Torelli-like results used in the proof of Theorem 1.1 give no insight on the geometry. Moreover, we describe up to deformation birational involutions of $S^{[n]}$, when S is a general K3 surface of degree $2((n-1)k^2 + 1)$ for some integer k. These involutions are obtained by deforming a combination of Beauville involutions on the Hilbert scheme of n points of a specific K3 surface of Picard rank two, in such a way that the deformation path goes only through ihs manifolds which are still Hilbert schemes of points on K3's.

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2. Preliminaries on Pell's equations

A *generalized Pell's equation* is a quadratic diophantine equation

$$X^2 - rY^2 = m \tag{2}$$

in the unknowns $X, Y \in \mathbb{Z}$, for $r \in \mathbb{N}$ and $m \in \mathbb{Z} \setminus \{0\}$. If $m = 1$, the equation is called *standard*. If the equation (2) is solvable and r is not a square, the (infinite) set of solutions is divided into equivalence classes: two solutions (X, Y) and (X', Y')

are said to be equivalent if

$$\frac{XX' - rYY'}{m} \in \mathbb{Z}, \qquad \frac{XY' - X'Y}{m} \in \mathbb{Z}.$$

If the equation is standard, then it is solvable and all solutions are equivalent.

Inside any equivalence class of solutions, the *fundamental* solution (X, Y) is the one with smallest non-negative Y, if there is a single solution with this property in the class. Otherwise, the smallest non-negative value of Y is realized by two *conjugate* solutions (X, Y), $(-X, Y)$: in this case, the fundamental solution of the class will be (X, Y) with $X > 0$. If (X, Y) is a fundamental solution, all other solutions (X', Y') in the same equivalence class are of the form

$$\begin{cases} X' = aX + rbY \\ Y' = bX + aY \end{cases} \tag{3}$$

where (a, b) is a solution of the standard Pell's equation $a^2 - rb^2 = 1$.

A solution (X, Y) of (2) is called *positive* if $X > 0$, $Y > 0$. The *minimal* solution of the equation is the positive solution with smallest X. In particular, the minimal solution is one of the fundamental solutions. We also use the expression "minimal solution with a property P" to denote the positive solution with smallest X among those which satisfy the property P.

Let $(X, Y) = (z, w)$ be the minimal solution of the equation $X^2 - rY^2 = 1$. The half-open interval $[(\sqrt{m}, 0), (z\sqrt{m}, w\sqrt{m}))$ on the hyperbola $X^2 - rY^2 = m$ contains exactly one solution for each equivalence class. The solutions in this interval are all the solutions (X, Y) of (2) such that $X > 0$ and $0 \leq \frac{Y}{X} < \frac{w}{z}$. Moreover, if (X, Y) is a fundamental solution of (2) and $m > 0$, then:

$$0 < |X| \leq \sqrt{\frac{(z+1)m}{2}}, \qquad 0 \leq Y \leq w\sqrt{\frac{m}{2(z+1)}}. \tag{4}$$

We will often make use of the following lemma.

Lemma 2.1. *For $t \geq 1$ and $n \geq 2$ such that $t(n-1)$ is not a square, let (z, w) be the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$. If $w \equiv 0 \pmod 2$, then*

- (*i*) *$z \equiv 1 \pmod{2(n-1)}$ and $z \equiv 1 \pmod{2t}$ if and only if (z, w) is not the minimal solution of $X^2 - t(n-1)Y^2 = 1$.*
- (*ii*) *$z \equiv -1 \pmod{2(n-1)}$ and $z \equiv -1 \pmod{2t}$ if and only if the equation $X^2 - t(n-1)Y^2 = -1$ has integer solutions.*
- (*iii*) *if $z \equiv 1 \pmod{2(n-1)}$ and $z \equiv -1 \pmod{2t}$ then the equation $(n-1)X^2 - tY^2 = -1$ has integer solutions; if $t \geq 2$ the converse also holds.*
- (*iv*) *if $z \equiv -1 \pmod{2(n-1)}$ and $z \equiv 1 \pmod{2t}$ then the equation $(n-1)X^2 - tY^2 = 1$ has integer solutions; if $n \geq 3$ the converse also holds.*

Proof. Let $w = 2m$ for some $m \in \mathbb{N}$.

(*i*) – (*ii*) Write $z = 2(n-1)p \pm 1 = 2tq \pm 1$ for $p, q \in \mathbb{N}$. Then $p((n-1)p \pm 1) = tm^2$ and $(n-1)p = tq$, hence $r := \frac{p}{t} \in \mathbb{N}$. We have $r((n-1)rt \pm 1) = m^2$ and it follows that there exist $s, u \in \mathbb{N}$ such that $r = s^2$, $(n-1)tr \pm 1 = u^2$, $m = su$, hence $u^2 - t(n-1)s^2 = \pm 1$. If the sign is $+$, notice that $u < z$, therefore (z, w) is not the minimal solution of $X^2 - t(n-1)Y^2 = 1$. Conversely, assume first that $X^2 - t(n-1)Y^2 = -1$ has integer solutions and let (a, b) be the minimal one. Then, by [10, Lemma A.2] the minimal solution of

$X^2-t(n-1)Y^2=1$ is $(z,w)=(2t(n-1)b^2-1, 2ab)$, which satisfies $z\equiv -1 \pmod{2(n-1)}$ and $z\equiv -1 \pmod{2t}$. Similarly, if the minimal solution (u,s) of $X^2-t(n-1)Y^2=1$ does not satisfy $u\equiv \pm 1 \pmod{n-1}$, then by (3) we have $(z,w)=(2t(n-1)s^2+1, 2us)$, hence $z\equiv 1 \pmod{2(n-1)}$ and $z\equiv 1 \pmod{2t}$.

(iii) – (iv) Write $z=2(n-1)p\pm 1=2tq\mp 1$ for $p,q\in\mathbb{N}$. Then $p((n-1)p\pm 1)=tm^2$ and $(n-1)p\pm 1=tq$, hence $pq=m^2$. Since $\gcd(p,q)=1$, there exist $s,u\in\mathbb{N}$ such that $p=s^2$, $q=u^2$ and $m=su$, thus $(n-1)s^2-tu^2=\mp 1$. Vice versa, let (a,b) be the solution of $(n-1)X^2-tY^2=\pm 1$ with smallest $a>0$. By [10, Lemma A.2], the assumption $t\geq 2$ (if the sign is $-$) or $n\geq 3$ (if the sign is $+$) implies that the minimal solution of $X^2-t(n-1)Y^2=1$ is $(z,w)=(2(n-1)a^2\mp 1, 2ab)$, which satisfies $z\equiv \mp 1 \pmod{2(n-1)}$ and $z\equiv \pm 1 \pmod{2t}$. $\square$

3. Birational automorphisms of $S^{[n]}$

Let S be an algebraic K3 surface with $\mathrm{Pic}(S)=\mathbb{Z}H$, $H^2=2t$, $t\geq 1$. For $n\geq 2$, let $S^{[n]}$ be the Hilbert scheme of n points on S and $\{h,-\delta\}$ a basis for $\mathrm{NS}(S^{[n]})\subset H^2(S^{[n]},\mathbb{Z})$, where h is the class of the nef (not ample) line bundle induced by H on $S^{[n]}$ and 2δ is the class of the exceptional divisor of the Hilbert–Chow morphism $S^{[n]}\to S^{(n)}$. We consider $H^2(S^{[n]},\mathbb{Z})$ equipped with the (even) lattice structure given by the Beauville–Bogomolov–Fujiki integral quadratic form. In particular, we have $H^2(S^{[n]},\mathbb{Z})\cong H^2(S,\mathbb{Z})\oplus\mathbb{Z}\delta\cong U^{\oplus 3}\oplus E_8(-1)^{\oplus 2}\oplus\langle -2(n-1)\rangle$, where for an integer $d\neq 0$ we denote by $\langle d\rangle$ the rank one lattice generated by an element of square d.

3.1. **The action on cohomology.** By [24, Corollary 5.2] the group $\mathrm{Bir}(S^{[n]})$ is finite and the homomorphism $\mathrm{Bir}(S^{[n]})\to O(H^2(S^{[n]},\mathbb{Z})), \sigma\mapsto\sigma^*:=(\sigma^{-1})_*$ is injective by [3, Proposition 10]. Moreover, the kernel of $\Psi:\mathrm{Bir}(S^{[n]})\to O(\mathrm{NS}(S^{[n]}))$, $\sigma\mapsto \sigma^*|_{\mathrm{NS}(S^{[n]})}$, is the subgroup of natural automorphisms, which is isomorphic to $\mathrm{Aut}(S)$ (see [5, Lemma 2.4] and notice that $\ker(\Psi)\subset\mathrm{Aut}(S^{[n]})$ by the global Torelli theorem [16, Theorem 1.3]). By [26, §5], this implies that $\ker(\Psi)=\{\mathrm{id}\}$ if $t\geq 2$, while for $t=1$ we have $\ker(\Psi)=\langle\iota^{[n]}\rangle$, where ι is the involution which generates $\mathrm{Aut}(S)$ and $\iota^{[n]}$ is the natural involution induced by ι on $S^{[n]}$.

Let $\mathrm{Mov}(S^{[n]})\subset\mathrm{NS}(S^{[n]})_{\mathbb{R}}$ be the movable cone of $S^{[n]}$, i.e. the open cone generated by the classes of divisors whose base locus has codimension at least two. It contains the ample cone $\mathcal{A}_{S^{[n]}}$. If there exists $\sigma\in\mathrm{Bir}(S^{[n]})$ non-natural, then the action σ^* on $\mathrm{NS}(S^{[n]})$ is a non-trivial isometry and by [16, Lemma 6.22] it preserves the movable cone; more specifically, σ^* exchanges the two extremal rays of $\mathrm{Mov}(S^{[n]})$. By [2, Proposition 13.1], $t(n-1)$ is not a square, $(n-1)X^2-tY^2=1$ has no integer solutions (if $n\neq 2$) and $\overline{\mathrm{Mov}(S^{[n]})}=\langle h, zh-tw\delta\rangle_{\mathbb{R}_{\geq 0}}$, where (z,w) is the minimal solution of $z^2-t(n-1)w^2=1$ with $z\equiv\pm 1 \pmod{n-1}$; moreover, we have $z\equiv \pm 1 \pmod{2(n-1)}$ and $w\equiv 0 \pmod 2$.

By imposing the conditions $\sigma^*(h)=zh-tw\delta$, $\sigma^*(zh-tw\delta)=h$, one computes that the matrix which describes $\sigma^*\in O(\mathrm{NS}(S^{[n]}))$ with respect to the basis $\{h,-\delta\}$ is

$$\begin{pmatrix} z & -(n-1)w \\ tw & -z \end{pmatrix}. \tag{5}$$

This is an involution of $\mathrm{NS}(S^{[n]})$ thus from the description of $\ker(\Psi)$ that we recalled above we conclude that the group $\mathrm{Bir}(S^{[n]})$ is either $\{\mathrm{id}\}$, $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Assuming that there exists a non-natural birational involution $\sigma\in\mathrm{Bir}(S^{[n]})$, then if $t\geq 2$ we have $\mathrm{Bir}(S^{[n]})=\langle\sigma\rangle$, while $\mathrm{Bir}(S^{[n]})=\langle\iota^{[n]},\sigma\rangle$ if $t=1$. Notice that the isometry σ^* is the reflection of $\mathrm{NS}(S^{[n]})$ which fixes the line spanned by $(n-1)wh-(z-1)\delta$. Let $\nu=bh-a\delta\in\mathrm{NS}(S^{[n]})$ be the primitive generator of this line with $a,b>0$, $(a,b)=1$.

In the following, for any even lattice L with bilinear form $(-,-)$ we denote by $A_L:=L^\vee/L$ the discriminant group, which is a finite abelian group equipped with a finite quadratic form $q_L: A_L\to\mathbb{Q}/2\mathbb{Z}$ induced by the quadratic form on L. If A_L is cyclic of order m, we write $A_L\cong\frac{\mathbb{Z}}{m\mathbb{Z}}(\alpha)$ if q_L takes value $\alpha\in\mathbb{Q}/2\mathbb{Z}$ on a generator of A_L. For $g\in O(L)$ we denote by $\overline{g}$ the isometry induced by g on A_L. In the case of the lattice $H^2(S^{[n]},\mathbb{Z})$ we have $A_{H^2(S^{[n]},\mathbb{Z})}\cong\frac{\mathbb{Z}}{2(n-1)\mathbb{Z}}\left(-\frac{1}{2(n-1)}\right)$. For any $x\in H^2(S^{[n]},\mathbb{Z})$ let $\mathrm{div}(x)$ be the divisibility of x in $H^2(S^{[n]},\mathbb{Z})$, i.e. the positive generator of the ideal $(x,H^2(S^{[n]},\mathbb{Z}))\subset\mathbb{Z}$. The transcendental lattice of $S^{[n]}$ is $\mathrm{Tr}(S^{[n]})=\mathrm{NS}(S^{[n]})^\perp\subset H^2(S^{[n]},\mathbb{Z})$.

Proposition 3.1. *Let S be an algebraic K3 surface with $\mathrm{Pic}(S)=\mathbb{Z}H$, $H^2=2t$, $t\geq 1$. For $n\geq 2$, let $\sigma\in\mathrm{Bir}(S^{[n]})$ be a non-natural automorphism and $\nu=bh-a\delta\in\mathrm{NS}(S^{[n]})$ be the primitive generator of the line fixed by σ^* with $a,b>0$. Then one of the following holds:*

- $\sigma^*|_{\mathrm{Tr}(S^{[n]})}=-\mathrm{id}$ *and*
  - *either* $\overline{\sigma^*}=-\mathrm{id}$ *and* $\nu^2=2$, *i.e.* (a,b) *is an integer solution of* $(n-1)X^2-tY^2=-1$;
  - *or* $\overline{\sigma^*}=\mathrm{id}$, $\nu^2=2(n-1)$ *and* $\mathrm{div}(\nu)=n-1$, *i.e.* $(a,\frac{b}{n-1})$ *is an integer solution of* $X^2-t(n-1)Y^2=-1$.
- $n\geq 9$, $\sigma^*|_{\mathrm{Tr}(S^{[n]})}=\mathrm{id}$, $\overline{\sigma^*}=-\mathrm{id}$ *and* $\nu^2=2t$, *i.e. the minimal solution of* $X^2-t(n-1)Y^2=1$ *is* $(b,\frac{a}{t})$ *and* $b\not\equiv\pm1 \pmod{n-1}$.

Proof. Let (z,w) be the minimal solution of $X^2-t(n-1)Y^2=1$ with $z\equiv\pm1\pmod{n-1}$; then, as we recalled before, $z\equiv\pm1\pmod{2(n-1)}$ and $w\equiv 0\pmod 2$. By [16, Theorem 1.3], σ^* is a monodromy operator (see [16, Definition 1.1]) hence it acts on $A_{H^2(S^{[n]},\mathbb{Z})}$ as $\pm\mathrm{id}$ by [16, Lemma 9.2]. Moreover, since $\mathrm{Tr}(S^{[n]})\cong\mathrm{Tr}(S)$ has odd rank, $\sigma^*|_{\mathrm{Tr}(S^{[n]})}=\pm\mathrm{id}$ (because σ^* is a Hodge isometry; see [14, Corollary 3.3.5]). If $t\neq 1$, by imposing that $\sigma^*|_{\mathrm{Tr}(S^{[n]})}$ glues with the isometry $\sigma^*|_{\mathrm{NS}(S^{[n]})}$ of the form (5), we conclude (by [21, Corollary 1.5.2]) that $\sigma^*|_{\mathrm{Tr}(S^{[n]})}=\mathrm{id}$ if and only if $z\equiv 1\pmod{2t}$ and $\sigma^*|_{\mathrm{Tr}(S^{[n]})}=-\mathrm{id}$ if and only if $z\equiv-1\pmod{2t}$. On the other hand $\overline{\sigma^*}=\mathrm{id}$ if and only if $z\equiv-1\pmod{2(n-1)}$ and $\overline{\sigma^*}=-\mathrm{id}$ if and only if $z\equiv 1\pmod{2(n-1)}$, for all $t\geq 1$ (see [8, Remark 5.2]).

If σ^* acts as $-\mathrm{id}$ on $\mathrm{Tr}(S^{[n]})$ and $t\geq 2$, then the statement follows as in the proof of [8, Proposition 5.1] and by [8, Lemma 6.3].

Assume that $z\equiv 1\pmod{2t}$ (hence, $\sigma^*|_{\mathrm{Tr}(S^{[n]})}=\mathrm{id}$ or $t=1$). By Lemma 2.1 and by the hypothesis that $(n-1)X^2-tY^2=1$ has no integer solutions we have $z\equiv 1\pmod{2(n-1)}$, i.e. σ^* acts as $-\mathrm{id}$ on the discriminant group of $H^2(S^{[n]},\mathbb{Z})$. Moreover, $(z,w)=(2t(n-1)s^2+1,2us)$, where (u,s) is the minimal solution of $X^2-t(n-1)Y^2=1$. The axis of the reflection $\sigma^*|_{\mathrm{NS}(S^{[n]})}$ is spanned by

$$(n-1)wh-(z-1)\delta=2s(n-1)\left(uh-ts\delta\right).$$

Since $\gcd(u,ts)=1$, we conclude that the primitive generator of the axis of the reflection is $\nu = uh - ts\delta$, whose square is $2t$. By looking at quadratic residues modulo $n-1$, we see that the minimal solution (u,s) of $X^2 - t(n-1)Y^2 = 1$ can have $u \not\equiv \pm 1 \pmod{n-1}$ only if $n \geq 9$. □

Remark 3.2. Let ω be a generator of $H^{2,0}(S^{[n]})$. Then $\mathrm{NS}(S^{[n]}) = \omega^\perp \cap H^2(S^{[n]},\mathbb{Z})$, therefore an automorphism $\sigma \in \mathrm{Bir}(S^{[n]})$ is symplectic (i.e. $\sigma^*\omega = \omega$) if and only if $\sigma^*|_{\mathrm{Tr}(S^{[n]})} = \mathrm{id}$. As a consequence of [18, Lemma 3.5, Corollary 5.1] (see also [8, Proposition 4.1]) any symplectic non-trivial birational automorphism of $S^{[n]}$ is not biregular. We will return to this point in Section 4.

3.2. **Classification of birational automorphisms.** Let L be an even lattice and $l \in L$ such that $(l,l) \neq 0$. We define the reflection

$$R_l : L_\mathbb{R} \to L_\mathbb{R}, \quad m \mapsto m - 2\frac{(m,l)}{(l,l)}l.$$

This map restricts to a well-defined isometry $R_l \in O(L)$ if and only if the divisibility of l is either $|(l,l)|$ or $|(l,l)|/2$. Recall that the *real spinor norm* of L is the group homomorphism $\mathrm{sn}^L_\mathbb{R} : O(L) \to \mathbb{R}^*/\left(\mathbb{R}^*\right)^2 \cong \{\pm 1\}$ defined as

$$\mathrm{sn}^L_\mathbb{R}(g) = \left(-\frac{(v_1,v_1)}{2}\right)\cdots\left(-\frac{(v_r,v_r)}{2}\right) \pmod{(\mathbb{R}^*)^2}$$

where $g_\mathbb{R} = R_{v_1} \circ \ldots \circ R_{v_r}$ is the factorization of $g_\mathbb{R} \in O(L_\mathbb{R})$ with respect to reflections defined by elements $v_i \in L$. In particular, if the signature of L is (l_+, l_-), after diagonalizing the bilinear form of L over $\mathbb{R}$ it is immediate to check that $\mathrm{sn}^L_\mathbb{R}(-\mathrm{id}) = (-1)^{l_+}$.

Proof of Theorem 1.1 for $t \geq 2$. By Proposition 3.1, Lemma 2.1 and the previous discussion, the numerical conditions in the statement of the theorem are necessary for the existence of a birational automorphism of $S^{[n]}$ with non-trivial action on $\mathrm{NS}(S^{[n]})$ (i.e. a non-trivial automorphism, since $t \neq 1$). Assume now that these conditions hold. If $n \neq 2$ the equation $(n-1)X^2 - tY^2 = 1$ has no integer solutions, since $(j,k) \neq (-1,1)$ (Lemma 2.1). We therefore have $\overline{\mathrm{Mov}(S^{[n]})} = \langle h, zh - tw\delta\rangle_{\mathbb{R}_{\geq 0}}$ by [2, Proposition 13.1]. Using again Lemma 2.1, we give the following definitions depending on (j,k).

- *(i)* If $(j,k) = (1,-1)$, then $(n-1)X^2 - tY^2 = -1$ has integer solutions. Let (a,b) be the solution with smallest $a > 0$ and set $\nu = bh - a\delta$. By Lemma 2.1 we have $(z,w) = (2(n-1)a^2+1, 2ab)$, since we are assuming $t \geq 2$. As $\nu^2 = 2$, we can define $\phi = -R_\nu \in O(H^2(S^{[n]},\mathbb{Z}))$. Then $\overline{\phi} = -\mathrm{id} \in O(A_{H^2(S^{[n]},\mathbb{Z})})$ by [12, Proposition 3.1].
- *(ii)* If $(j,k) = (-1,-1)$, then $X^2 - t(n-1)Y^2 = -1$ has integer solutions. Let (a,b) be the minimal solution and set $\nu = (n-1)bh - a\delta$. Then $(z,w) = (2t(n-1)b^2 - 1, 2ab)$. The class ν has square $2(n-1)$ and divisibility $n-1$ (see [8, Lemma 6.3]), so we can define $\phi = -R_\nu \in O(H^2(S^{[n]},\mathbb{Z}))$ and $\overline{\phi} = \mathrm{id}$ by generalizing [12, Corollary 3.4].
- *(iii)* If $(j,k) = (1,1)$, let (b,a) be the minimal solution of $X^2 - t(n-1)Y^2 = 1$ and set $\nu = bh - ta\delta$. Then $(z,w) = (2t(n-1)a^2+1, 2ab)$. The primitive element $\gamma = (n-1)ah - b\delta$ satisfies $\gamma^2 = -2(n-1)$, while the divisibility of γ in $H^2(S^{[n]},\mathbb{Z})$ is either $n-1$ or $2(n-1)$. This implies that $\phi = R_\gamma \in O(H^2(S^{[n]},\mathbb{Z}))$ and moreover $\overline{\phi} = -\mathrm{id}$ by [16, Proposition 9.12].

Notice that there is no ambiguity in the definitions of ν and ϕ when $n = 2$. By construction, $\phi|_{\mathrm{Tr}(S^{[n]})} = -\mathrm{id}$ in cases (i), (ii) and $\phi|_{\mathrm{Tr}(S^{[n]})} = \mathrm{id}$ in case (iii). Thus ϕ extends to a Hodge isometry of $H^2(S^{[n]}, \mathbb{C})$. Moreover, ϕ is orientation-preserving. Indeed, let $\mathrm{sn}_{\mathbb{R}} := \mathrm{sn}_{\mathbb{R}}^{H^2(S^{[n]},\mathbb{Z})}$; then if $\phi = -R_\nu$ we have

$$\mathrm{sn}_{\mathbb{R}}(\phi) = \mathrm{sn}_{\mathbb{R}}(-\mathrm{id})\mathrm{sn}_{\mathbb{R}}(R_\nu) = -\mathrm{sn}_{\mathbb{R}}(R_\nu) = \mathrm{sgn}(\nu^2) = +1$$

where we used the fact that $H^2(S^{[n]}, \mathbb{Z})$ has signature $(3, 20)$. If $\phi = R_\gamma$, then $\mathrm{sn}_{\mathbb{R}}(\phi) = -\mathrm{sgn}(\gamma^2) = +1$. We conclude, by [16, Lemma 9.2], that $\phi \in \mathrm{Mon}^2_{\mathrm{Hdg}}(S^{[n]})$ (in the case $\phi = R_\gamma$, also see [16, Proposition 9.12]).

From the relations between (a, b) and (z, w) that we remarked for each of the three pairs (j, k), it is immediate to check that $\nu \in \mathrm{Mov}(S^{[n]})$ and that $\phi|_{\mathrm{NS}(S^{[n]})}$ is the isometry (5) fixing the line spanned by ν, hence $\phi(\mathrm{Mov}(S^{[n]})) = \mathrm{Mov}(S^{[n]})$. Then, by [16, Lemma 6.22 and Lemma 6.23] $\phi \in \mathrm{Mon}^2_{\mathrm{Bir}}(S^{[n]})$, i.e. there exists a birational automorphism $\sigma \in \mathrm{Bir}(S^{[n]})$ such that $\sigma^* = \phi$. $\square$

We are left to discuss the case $t = 1$. For all $n \geq 2$ the group $\mathrm{Aut}(S^{[n]})$ contains the natural involution induced by the generator of $\mathrm{Aut}(S)$. Here, the equation $(n-1)X^2 - tY^2 = -1$ has always integer solutions, while $X^2 - t(n-1)Y^2 = -1$ has no integer solutions if we assume that $(n-1)X^2 - tY^2 = 1$ has none.

Proof of Theorem 1.1 for $t = 1$. Let (z, w) be the minimal solution of $z^2 - (n-1)w^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$. If there exists a non-natural automorphism then $w \equiv 0 \pmod 2$, which implies $z \equiv 1 \pmod{2t}$. Thus, by the proof of Proposition 3.1, (z, w) is not the minimal solution of the equation, i.e. $b \not\equiv \pm 1 \pmod{n-1}$. On the other hand, if we assume $b \not\equiv \pm 1 \pmod{n-1}$ then $(n-1)X^2 - Y^2 = 1$ has no integer solutions by [10, Lemma A.2]. The existence of a symplectic automorphism follows as in the proof for $t \geq 2$, case (iii). By [8, Proposition 1.1], this automorphism is not biregular. $\square$

If S is a 2-polarized K3 surface of Picard rank one, the smallest n such that $\mathrm{Bir}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is $n = 9$.

Remark 3.3. Notice that $t = 1$ is the only value of t such that the isometry (5) of $\mathrm{NS}(S^{[n]})$ can be glued to both $+\mathrm{id}, -\mathrm{id} \in O(\mathrm{Tr}(S^{[n]}))$. This follows from [21, Corollary 1.5.2], since the discriminant group of $\mathrm{Tr}(S^{[n]})$ is $\frac{\mathbb{Z}}{2t\mathbb{Z}}$. If $t = 1$, let $\nu = bh - a\delta \in \mathrm{NS}(S^{[n]})$ be as in the proof of Theorem 1.1. Then $(-R_\nu) \oplus (-\mathrm{id}) \in O(\mathrm{NS}(S^{[n]}) \oplus \mathrm{Tr}(S^{[n]}))$ extends to $-R_\nu \in O(H^2(S^{[n]}, \mathbb{Z}))$, while $(-R_\nu) \oplus \mathrm{id}$ extends to R_γ, with $\gamma = (n-1)ah - b\delta$. If $n - 1$ is not a square and $b \not\equiv \pm 1 \pmod{n-1}$, both isometries R_γ and $-R_\nu$ of the cohomology lattice are Hodge monodromies which lift to (distinct) non-natural birational automorphisms of $S^{[n]}$. The lift of R_γ is symplectic while the lift of $-R_\nu$ is non-symplectic. Each automorphism can be recovered by composing the other with $\iota^{[n]}$, where ι is the covering involution which generates $\mathrm{Aut}(S)$. Indeed, $(\iota^{[n]})^*|_{\mathrm{NS}(S^{[n]})} = \mathrm{id}$ and $(\iota^{[n]})^*|_{\mathrm{Tr}(S^{[n]})} = -\mathrm{id}$ (since $\iota^{[n]}$ is a natural involution).

Remark 3.4. If for $n \geq 2$, $t \geq 2$ there exists a non-trivial birational automorphism on $S^{[n]}$, it is biregular if and only if n, t satisfy condition (iii) of [8, Theorem 6.4], which guarantees that $\mathcal{A}_{S^{[n]}} = \mathrm{Mov}(S^{[n]})$. If $t \leq 2n - 3$ the automorphism is not biregular by [8, Proposition 1.1].

The following proposition provides examples of polarization degrees for the K3 surface S so that $S^{[n]}$ has a non-natural birational automorphism, for each of the three different actions on cohomology of Proposition 3.1.

Proposition 3.5. *Fix $n \geq 2$. For $t \geq 1$, we denote by S an algebraic K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$.*

- (i) *There exist infinitely many t's such that $\mathrm{Bir}(S^{[n]})$ contains a non-natural, non-symplectic automorphism σ with $H^2(S^{[n]}, \mathbb{Z})^{\sigma^*} \cong \langle 2 \rangle$, e.g. $t = (n-1)k^2 + 1$ for $k \geq 1$.*
- (ii) *There exists t such that $\mathrm{Bir}(S^{[n]})$ contains a non-natural, non-symplectic automorphism σ with $H^2(S^{[n]}, \mathbb{Z})^{\sigma^*} \cong \langle 2(n-1) \rangle$ if and only if -1 is a quadratic residue modulo $n-1$. If so, this happens for infinitely many t's, e.g. $t = (n-1)k^2 + 2qk + \frac{q^2+1}{n-1}$ for $k, q \geq 1$ and $q^2 \equiv -1 \pmod{n-1}$.*
- (iii) *There exists t such that $\mathrm{Bir}(S^{[n]})$ contains a non-natural, symplectic automorphism σ if and only if $n - 1 = \frac{q^2-1}{h}$ for some $q \geq 3, h \not\equiv 0 \pmod{q \pm 1}$. If so, $H^2(S^{[n]}, \mathbb{Z})_{\sigma^*} \cong \langle -2(n-1) \rangle$ and this happens for infinitely many t's, e.g. $t = (n-1)k^2 + 2qk + h$ for $k \geq 1$.*

Proof. As before, we denote by (z, w) the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$.

- (i) If $t = (n-1)k^2 + 1$ and $k \geq 1$, then $t(n-1)$ is not a square and $(z, w) = (2t-1, 2k) = (2(n-1)k^2 + 1, 2k)$ by [10, Lemma A.2], since $(k, 1)$ is the solution of $(n-1)X^2 - tY^2 = -1$ with smallest $X > 0$. We conclude with Theorem 1.1.
- (ii) If there exists t such that $\mathrm{Bir}(S^{[n]})$ contains a non-natural, non-symplectic automorphism σ with $H^2(S^{[n]}, \mathbb{Z})^{\sigma^*} \cong \langle 2(n-1) \rangle$, then by Theorem 1.1 and Lemma 2.1 the equation $X^2 - t(n-1)Y^2 = -1$ is solvable, hence -1 is a quadratic residue modulo $n-1$. On the other hand, let $t = (n-1)k^2 + 2qk + \frac{q^2+1}{n-1}$ for $k, q \geq 1$ and $q^2 \equiv -1 \pmod{n-1}$. The minimal solution of $X^2 - t(n-1)Y^2 = -1$ is $(q + (n-1)k, 1)$, therefore $t(n-1)$ is not a square and $(z, w) = (-1 + 2t(n-1), 2q + 2(n-1)k)$, again by [10, Lemma A.2]. The existence of the automorphism and its action on cohomology follow then from Theorem 1.1.
- (iii) If $\mathrm{Bir}(S^{[n]})$ contains a symplectic automorphism σ, then by Theorem 1.1 and Lemma 2.1 the minimal solution (u, s) of $X^2 - t(n-1)Y^2 = 1$ has $u \not\equiv \pm 1 \pmod{n-1}$. Hence $n - 1 = \frac{u^2-1}{ts^2} \geq 8$ and $u \pm 1 \nmid ts^2$. Let now $t = (n-1)k^2 + 2qk + h$ for k, q, h as in the statement. The minimal solution of $X^2 - t(n-1)Y^2 = 1$ is $(u, s) = (q + (n-1)k, 1)$. Since $h \not\equiv 0 \pmod{q \pm 1}$, we have $q \not\equiv \pm 1 \pmod{n-1}$, therefore $(z, w) = (1 + 2t(n-1), 2q + 2(n-1)k)$, which allows us to conclude by using Theorem 1.1. $\square$

4. Decomposition of the movable cone and automorphisms of birational models

We consider the wall-and-chamber decomposition (1) of the closed movable cone $\overline{\mathrm{Mov}(S^{[n]})} \subset \mathrm{NS}(S^{[n]})_{\mathbb{R}} = \mathbb{R}h \oplus \mathbb{R}\delta$. As shown in [8, Lemma 2.5] by using [2, Theorem 12.1], the walls are spanned by the classes of the form $Xh - 2tY\delta$ which lie in the movable cone, for (X, Y) a positive solution of one of Pell's equations

$X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ with $X \equiv \pm\alpha \pmod{2(n-1)}$, where the possible values of ρ and α are:

$$\begin{cases} \rho = -1 \text{ and } 1 \leq \alpha \leq n-1; \\ \rho = 0 \text{ and } 3 \leq \alpha \leq n-1; \\ 1 \leq \rho < \frac{n-1}{4} \text{ and } 4\rho + 1 \leq \alpha \leq n-1. \end{cases} \tag{6}$$

Definition 4.1. For $n \geq 2$, let S be an algebraic K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 1$. The value t is said to be *n-irregular* if there exists a birational automorphism $\sigma \in \mathrm{Bir}(S^{[n]})$ such that for all ihs birational models $g : S^{[n]} \dashrightarrow X$ the involution $g \circ \sigma \circ g^{-1} \in \mathrm{Bir}(X)$ is not biregular.

Remark 4.2. If the decomposition of $\overline{\mathrm{Mov}(S^{[n]})}$ has an odd number of chambers, and there exists a non-natural automorphism $\sigma \in \mathrm{Bir}(S^{[n]})$, then the generator ν of the axis of the reflection $\sigma^*|_{\mathrm{NS}(S^{[n]})}$ is in the interior of one of the chambers. This follows from the fact that the isometry σ^* acts on the set of chambers by [16, Lemma 5.12], hence if their number is odd, one of them is preserved by σ^*, and therefore it contains ν. As a consequence, there exists an ihs birational model (corresponding to the preserved chamber) on which the involution becomes biregular, by [16, Theorem 1.3]. Notice that $\overline{\mathrm{Mov}(S^{[n]})}$ has only one chamber if and only if $\sigma \in \mathrm{Aut}(S^{[n]})$. On the other hand, if there is an even number of chambers in the decomposition of $\overline{\mathrm{Mov}(S^{[n]})}$, then ν lies on one of the walls in the interior of the cone, and therefore σ is not biregular on any of the ihs birational models of $S^{[n]}$.

This implies that t is n-irregular if and only if $\mathrm{Bir}(S^{[n]})$ contains a non-natural birational automorphism and the wall-and-chamber decomposition of $\mathrm{Mov}(S^{[n]})$ has an even number of chambers.

Proposition 4.3. *For $n \geq 2$ and $t \geq 1$, assume that there exists a non-natural birational automorphism $\sigma \in \mathrm{Bir}(S^{[n]})$ and either $t = 1$ or $\sigma^*|_{\mathrm{Tr}(S^{[n]})} = \mathrm{id}$. Then t is n-irregular.*

Proof. By the proof of Proposition 3.1, if $t = 1$ or $\sigma^*|_{\mathrm{Tr}(S^{[n]})} = \mathrm{id}$ the axis of the reflection $\sigma^*|_{\mathrm{NS}(S^{[n]})}$ is spanned by the class $\nu = bh - ta\delta$, where (b, a) is the minimal solution of $X^2 - t(n-1)Y^2 = 1$. Moreover, $n \geq 9$ and $\overline{\mathrm{Mov}(S^{[n]})} = \langle h, zh - tw\delta \rangle_{\mathbb{R}_{\geq 0}}$ with $(z, w) = (2t(n-1)a^2 + 1, 2ab)$. Since $(b+1)(b-1) = t(n-1)a^2$, we have $c := \max\{\gcd(n-1, b-1), \gcd(n-1, b+1)\} \geq 2$. We define $\alpha := \max\{4, \frac{2(n-1)}{c}\}$, which is an even integer such that $4 \leq \alpha \leq n-1$. We consider Pell's equation $X^2 - 4t(n-1)Y^2 = \alpha^2$. The pair $(X, Y) = (b\alpha, a\frac{\alpha}{2})$ is a solution with the property $\frac{2Y}{X} = \frac{a}{b}$. We also have $X = b\alpha \equiv \pm\alpha \pmod{2(n-1)}$ by construction, hence ν lies on the wall (in the interior of the movable cone) spanned by the class $Xh - 2tY\delta$. $\square$

By the previous proposition, a non-natural birational automorphism of $S^{[n]}$ which acts on $\mathrm{Tr}(S^{[n]})$ as id (or as $\pm$id if $t = 1$) is not biregular on any ihs birational model of $S^{[n]}$, which is stronger than what we stated in Remark 3.2. By Proposition 3.5, for n fixed the number of n-irregular values t for which this happens is either zero or infinite.

Example 4.4. For any odd $k \geq 5$ with $k \not\equiv \pm 1 \pmod 8$, define $t = \frac{k^2-1}{8} \in \mathbb{N}$. We can readily check that $S^{[9]}$ admits a symplectic non-natural involution by Theorem 1.1, hence t is 9-irregular by Proposition 4.3.

For non-symplectic automorphisms when $t \neq 1$, the behaviour is different. We now show that, for a fixed $n \geq 2$, there is only a finite number of n-irregular values t for which the non-natural birational automorphism of $S^{[n]}$ acts as $-\operatorname{id}$ on $\operatorname{Tr}(S^{[n]})$, and we provide an algorithm to compute them.

Proposition 4.5. *Let $t \geq 2$ and $n \geq 2$, such that $t(n-1)$ is not a square and $(n-1)X^2 - tY^2 = 1$ has no integer solutions (if $n \neq 2$). Assume that one between $(n-1)X^2 - tY^2 = -1$ and $X^2 - t(n-1)Y^2 = -1$ admits integer solutions; define $\ell = 1$ if the first equation is solvable or $\ell = n-1$ if the second one is, and let (a, b) be the solution with smallest $a > 0$. Then t is n-irregular if and only if there exists a pair (α, ρ) as in (6) and a positive solution (X, Y) of Pell's equation $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ with $X \equiv \pm\alpha \pmod{2(n-1)}$ such that*

$$4t\ell Y^2 = (\alpha^2 - 4\rho(n-1))a^2.$$

Proof. Let $\nu = \ell bh - a\delta \in \operatorname{Mov}(S^{[n]})$ be the class of square 2ℓ as in the proof of Theorem 1.1. Then ν lies on one of the walls in the interior of $\operatorname{Mov}(S^{[n]})$ if and only if $\frac{a}{\ell b} = 2t\frac{Y}{X}$, where (X, Y) is a positive solution of one of the equations $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ with $X \equiv \pm\alpha \pmod{2(n-1)}$ and (α, ρ) as in (6). We write the condition on the slopes as:

$$\frac{Y}{X} \cdot \frac{\ell b}{2(n-1)a} = \frac{1}{4t(n-1)}$$

and we observe that $(2(n-1)a)^2 - 4t(n-1)(\ell b)^2 = -4\ell(n-1)$. Hence this becomes:

$$\sqrt{\frac{1}{4t(n-1)} - \frac{\alpha^2 - 4\rho(n-1)}{4t(n-1)X^2}}\sqrt{\frac{1}{4t(n-1)} + \frac{4\ell(n-1)}{4t(n-1)(2(n-1)a)^2}} = \frac{1}{4t(n-1)}.$$

If we rearrange the equation, we obtain:

$$\ell X^2 - (n-1)(\alpha^2 - 4\rho(n-1))a^2 = \ell(\alpha^2 - 4\rho(n-1)) \tag{7}$$

i.e. $4t\ell Y^2 = (\alpha^2 - 4\rho(n-1))a^2$. □

Remark 4.6.

- Notice that, by (7) and $(n-1)a^2 - t(\ell b)^2 = -\ell$, we have $X^2 = (\alpha^2 - 4\rho(n-1))t\ell b^2$. However, $t\ell$ has to divide $\alpha^2 - 4\rho(n-1)$ (because $t\ell$ is coprime with a), hence $\alpha^2 - 4\rho(n-1) = t\ell r^2$ for some $r \in \mathbb{N}$. This gives an easy way to compute a (finite) list of candidates for the n-irregular values t, among the divisors of $\alpha^2 - 4\rho(n-1)$ for (α, ρ) as in (6). In particular, $t \leq (n-1)(n+3)$.
- In order to check if a value $t \geq 2$ is n-irregular in Proposition 4.5, it is enough to consider the pairs (α, ρ) such that $\alpha^2 - 4\rho(n-1) > 0$ and the positive solutions (X, Y) with smallest X in each equivalence class of solutions of $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$, otherwise the wall spanned by $Xh - 2tY\delta$ is not in the interior of the movable cone.

Corollary 4.7. *For a fixed $n \geq 2$, the number of n-irregular values t for which $\operatorname{Bir}(S^{[n]})$ contains a non-symplectic birational automorphism is finite. In particular, for $n \leq 8$ there is only a finite number of n-irregular values t.*

Proof. By Proposition 3.1, assuming $t \geq 2$ a non-natural automorphism $\sigma \in \operatorname{Bir}(S^{[n]})$ satisfies $\sigma^*|_{\operatorname{Tr}(S^{[n]})} = -\operatorname{id}$ if and only if the minimal solution of $X^2 - t(n-1)Y^2 = 1$ has $X \equiv \pm 1 \pmod{n-1}$. In particular, this always holds for $n \leq 8$.

Then, by Theorem 1.1, either $(n-1)X^2 - tY^2 = 1$ or $X^2 - t(n-1)Y^2 = 1$ is solvable. The statement follows from Proposition 4.5 and Remark 4.6. □

In Table 1, for $n \leq 14$ we list all n-irregular values t as in Corollary 4.7. We separate the values of t for which the middle wall of the movable cone is spanned by a primitive class ν of square 2 (i.e. $\ell = 1$), from those where the generator has square $2(n-1)$ and divisibility $n-1$ (i.e. $\ell = n-1$). For $n \neq 9, 12$, the values t in the table are all the n-irregular values, 9 and 12 being the only $n \leq 14$ which satisfy Proposition 3.5 (*iii*). We observe that, for $n \leq 5$, all n-irregular values are of the form $t = n$ or $t = 4n - 3$.

TABLE 1. n-irregular values t for $n \leq 14$ as in Corollary 4.7.

n	n-irregular t's s.t. $\nu^2 = 2$	n-irregular t's s.t. $\nu^2 = 2(n-1)$
2	5	
3	$3, 9$	/
4	$4, 13$	/
5	$5, 17$	/
6	$6, 9, 21$	/
7	$7, 25, 49$	/
8	$2, 4, 8, 11, 16, 29, 37$	/
9	$1, 9, 33, 57$	/
10	$10, 13, 37, 61, 85$	/
11	$11, 19, 41, 49, 121$	/
12	$3, 4, 5, 12, 15, 25, 27, 45, 125$	/
13	$1, 13, 49$	/
14	$14, 17, 22, 38, 49, 53, 77, 121, 133$	5

Lemma 4.8. *For all $n \geq 3$, the value $t = n$ is n-irregular. For all $n \geq 2$, the value $t = 4n - 3$ is n-irregular.*

Proof. Notice that $t = n$ and $t = 4n-3$ are values of the form $t = (n-1)k^2 + 1$, for $k = 1, 2$ respectively. By Proposition 3.5 the Hilbert scheme $S^{[n]}$ has a non-natural birational automorphism which acts as $-\mathrm{id}$ on $\mathrm{Tr}(S^{[n]})$. The minimal solution of $(n-1)X^2 - tY^2 = -1$ is $(a, b) = (k, 1)$.

- If $t = n$ and $n \geq 3$, consider the equation $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ for $\rho = -1$ and $\alpha = 2$. Its minimal solution is $(X, Y) = (2n, 1)$, which satisfies $X \equiv \alpha \pmod{2(n-1)}$.
- If $t = 4n - 3$ and $n \geq 2$, consider the equation $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ for $\rho = -1$ and $\alpha = 1$. Its minimal solution is $(X, Y) = (4n-3, 1)$, which satisfies $X \equiv \alpha \pmod{2(n-1)}$.

In both cases the relation $4tY^2 = (\alpha^2 - 4\rho(n-1))a^2$ holds, hence the statement follows from Proposition 4.5. □

In the case $t = n$, it is known that the automorphism of $S^{[n]}$ which acts as the reflection in the (only) wall contained in the interior of $\mathrm{Mov}(S^{[n]})$ is Beauville's involution [3, §6], which is biregular if and only if $n = 2$.

In Proposition 3.5 we showed that for $t = (n-1)k^2 + 1$, $k \geq 1$ the Hilbert scheme $S^{[n]}$ has a non-symplectic birational involution. We observed in Lemma 4.8

that the involution is not biregular for $k = 1, 2$, while by [8, Proposition 6.7] it is biregular (i.e. $\mathcal{A}_{S^{[n]}} = \mathrm{Mov}(S^{[n]})$) whenever $k \geq \frac{n+3}{2}$. It seems however that this lower bound on k is far from optimal.

Conjecture 4.9. *Let $n \geq 2$ and $t = (n-1)k^2 + 1$, for $k \in \mathbb{N}$. Let S be an algebraic K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$. If $k \geq 3$, then the non-natural non-symplectic involution which generates $\mathrm{Bir}(S^{[n]})$ is biregular.*

We checked computationally that the conjecture holds for all $n \leq 14$.

The results of [8, Proposition 6.7] can be adapted also to the other family of polarizations in Proposition 3.5 giving rise to non-symplectic birational involutions.

Lemma 4.10. *Let $n \geq 3$ and $q \in \mathbb{N}$ be such that $q^2 \equiv -1 \pmod{n-1}$. Let $t = (n-1)k^2 + 2qk + \frac{q^2+1}{n-1}$ for $k \geq 1$ and S an algebraic K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$. If $k \geq \frac{n+3}{2}$, then the non-natural non-symplectic involution which generates $\mathrm{Bir}(S^{[n]})$ is biregular.*

Proof. By Proposition 3.5 we just need to prove that $\mathcal{A}_{S^{[n]}} = \mathrm{Mov}(S^{[n]})$. Define $Q = (n-1)k + q$. For $k \geq \frac{n+3}{2}$ we have $4t(n-1) > (n+3)^2(n-1)^2 \geq \big(\alpha^2 - 4\rho(n-1)\big)^2$ for all (α, ρ) as in (6), hence the solutions of $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ are encoded in the convergents of the continued fraction of $\sqrt{4t(n-1)} = 2\sqrt{Q^2+1} = [2Q; \overline{Q, 4Q}]$ (for details see [9, Chapter XXXIII, §16]). The proof then follows as for [8, Proposition 6.7]. □

5. Kähler-type chambers and automorphisms for $n = 3$

It is known that for a K3 surface S of Picard rank one the number of Kähler-type chambers in the decomposition of $\overline{\mathrm{Mov}(S^{[2]})}$ is $d \in \{1, 2, 3\}$ (see for instance [10, Example 3.18]). We now detail the computation of the number of chambers for $n = 3$ in the cases where $\overline{\mathrm{Mov}(S^{[n]})} = \langle h, zh - tw\delta \rangle_{\mathbb{R}_{\geq 0}}$, with (z, w) the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$. In particular, this holds whenever $\mathrm{Bir}(S^{[3]}) \neq \{\mathrm{id}\}$.

For an integer $t \geq 2$ such that $2t$ is not a square, consider the following two generalized Pell's equations:

$$P_{8t}(9) : X^2 - 8tY^2 = 9$$
$$P_{8t}(12) : X^2 - 8tY^2 = 12.$$

Remark 5.1.
- If we write the two equations as $X^2 - 8tY^2 = 8 + \alpha^2$, with $\alpha \in \{1, 2\}$, then it is clear that all solutions (X, Y) satisfy $X \equiv \pm\alpha \pmod 4$.
- If (X, Y) is a solution of $P_{8t}(9)$ and (X', Y') is a solution of $P_{8t}(12)$, then $\frac{Y}{X} \neq \frac{Y'}{X'}$ since $\sqrt{\frac{12}{9}}$ is not rational.
- Assume that $P_{8t}(12)$ is solvable and let (X, Y) be a solution. Then $X = 2Z$ for some $Z \in \mathbb{Z}$ such that (Z, Y) is a solution of $P_{2t}(3) : Z^2 - 2tY^2 = 3$. Notice that $Z \equiv 1 \pmod 2$, $Y \equiv 1 \pmod 2$ and $t \equiv 1 \pmod 2$, since 3 is not a quadratic residue modulo 4. By [20, Theorem 110], if $P_{2t}(3)$ is solvable it has one equivalence class of solutions if $t \equiv 0 \pmod 3$, otherwise it has two classes of solutions, which are conjugate. Then one can easily see that the same holds for $P_{8t}(12)$.

The next two lemmas give bounds for the number of equivalence classes of solutions of $P_{8t}(9)$ and $P_{8t}(12)$, depending on t.

Lemma 5.2. *If* $t \equiv 3 \pmod{18}$*, then* $P_{8t}(12)$ *is either not solvable or it has one class of solutions. If* $t \equiv 5, 11, 17 \pmod{18}$*, then* $P_{8t}(12)$ *is either not solvable or it has two classes of solutions. In all other cases,* $P_{8t}(12)$ *is not solvable.*

Proof. By Remark 5.1 the equation $P_{8t}(12)$, if it is solvable, has one class of solutions when $t \equiv 0 \pmod 3$, two classes otherwise. We also remarked that solutions (X, Y) of $P_{8t}(12)$ are of the form $(2Z, Y)$ for (Z, Y) solution of $P_{2t}(3)$, and that $P_{8t}(12)$ is not solvable if $t \equiv 0 \pmod 2$.

If $t \equiv 1 \pmod 3$, write $t = 3q+1$ for $q \in \mathbb{N}_0$ and assume that (Z, Y) is a solution of $P_{2t}(3)$. Then $Z^2 - (6q + 2)Y^2 = 3$, i.e. $Z^2 \equiv 2Y^2 \pmod 3$. This implies that $Z \equiv Y \equiv 0 \pmod 3$, which gives a contradiction. Hence, $P_{8t}(12)$ has no solutions.

If $t \equiv 0 \pmod 9$, write $t = 9q$ for $q \in \mathbb{N}$. If (X, Y) is a solution of $P_{8t}(12)$, then $X = 6X'$ for some $X' \in \mathbb{N}$ and $3(X')^2 - 6qY^2 = 1$, which is impossible. Hence, $P_{8t}(12)$ has no solutions.

If $t \equiv 6 \pmod 9$, write $t = 9q + 6$ for $q \in \mathbb{N}_0$. If (X, Y) is a solution of $P_{8t}(12)$, then $X = 3X'$ for some $X' \in \mathbb{N}$ and $3(X')^2 - 8(3q + 2)Y^2 = 4$. This implies $Y^2 \equiv -1 \pmod 3$, which cannot be. Hence, $P_{8t}(12)$ has no solutions. □

Lemma 5.3. *If* $t \equiv 1 \pmod 3$ *or* $t \equiv 3, 6 \pmod 9$*, then all solutions of* $P_{8t}(9)$ *are equivalent to* $(X, Y) = (3, 0)$*. If* $t \equiv 0 \pmod 9$*, then* $P_{8t}(9)$ *has either one, two or three classes of solutions. If* $t \equiv 2 \pmod 3$*, then* $P_{8t}(9)$ *has either one or three classes of solutions.*

Proof. If $t = 3q+1$, for $q \in \mathbb{N}_0$, then $X^2 \equiv 2Y^2 \pmod 3$, hence all solutions (X, Y) are of the form $(3X', 3Y')$, with $(X')^2 - 8t(Y')^2 = 1$. This is now a standard Pell's equation, which has only one class of solutions, thus the same holds for $P_{8t}(9)$.

If $t = 3q$ for $q \in \mathbb{N}$ and $(q, 3) = 1$, then $X = 3X'$ for some $X' \in \mathbb{Z}$ such that $3(X')^2 - 8qY^2 = 3$. Since $3 \nmid q$ we need $(X, Y) = (3X', 3Y')$ with $(X')^2 - 24q(Y')^2 = 1$. There exists only one class of solutions (X', Y') for this standard Pell's equation, thus also the solutions of $P_{8t}(9)$ form a single class.

Assume now that $t = 9q$ for $q \in \mathbb{N}$. Then a solution (X, Y) of $P_{8t}(9)$ is of the form $(X, Y) = (3X', Y)$ with $(X')^2 - 8qY^2 = 1$. Let (z, w) be the minimal solution of $P_{8t}(1) : z^2 - 8tw^2 = 1$. Notice that the solutions of $P_{8t}(1)$ are the pairs $(X', \frac{Y}{3})$ for all solutions (X', Y) of $(X')^2 - 8qY^2 = 1$ such that $Y \equiv 0 \pmod 3$. Let (a, b) be the minimal solution of $(X')^2 - 8qY^2 = 1$. Since this is a standard Pell's equation, by (3) its next two solutions (for increasing values of X') are $(a^2 + 8qb^2, 2ab)$ and $(a^3 + 24qab^2, 8qb^3 + 3a^2b)$. We observe that one among these first three positive solutions (X', Y) has $Y \equiv 0 \pmod 3$, since either a^2 (hence, a) or $8qb^2 = a^2 - 1$ is divisible by 3. The positive solution (X', Y) with this property and smallest X' is therefore equal to $(z, 3w)$, thus the corresponding solution $(X, Y) = (3X', Y)$ of $P_{8t}(9)$ satisfies $\frac{Y}{X} = \frac{w}{z}$, i.e. it is the first positive solution in the same equivalence class of $(3, 0)$. We conclude that $P_{8t}(9)$ has either one, two or three classes of solutions.

Finally, assume that $t \equiv 2 \pmod 3$. Let (z, w) be the minimal solution of $P_{8t}(1)$ and let (u_1, v_1), (u_2, v_2) be two positive solutions of $P_{8t}(9)$ such that, for $i = 1, 2$:

$$0 < u_i \leq 3\sqrt{\frac{z+1}{2}}, \quad 0 < v_i \leq \frac{3w}{\sqrt{2(z+1)}}. \tag{8}$$

By (4), this is equivalent to asking that either (u_i, v_i) or $(-u_i, v_i)$ is a fundamental solution of $P_{8t}(9)$, different from $(3, 0)$. Thus u_1, v_1, u_2, v_2 are not divisible by three. From $u_1^2 - 8tv_1^2 = 9$ and $u_2^2 - 8tv_2^2 = 9$ we get

$$u_1 v_2 \equiv \pm u_2 v_1 \pmod 9, \qquad u_1 u_2 \equiv \pm 8t v_1 v_2 \pmod 9 \tag{9}$$

where the signs in the two congruences coincide. If we now multiply $u_1^2 - 8tv_1^2 = 9$ and $u_2^2 - 8tv_2^2 = 9$ member by member, we obtain

$$\left(\frac{u_1 u_2 \mp 8t v_1 v_2}{9}\right)^2 - 8t \left(\frac{u_1 v_2 \mp u_2 v_1}{9}\right)^2 = 1 \tag{10}$$

where by (9) the two squares in the LHS term are integers. If we assume that (u_1, v_1) and (u_2, v_2) are distinct, then $u_1 v_2 \mp u_2 v_1 \neq 0$. Since (z, w) is the minimal solution of $P_{8t}(1)$, from (10) we have $|u_1 v_2 \mp u_2 v_1| \geq 9w$. However, from (8) we compute $|u_1 v_2 \mp u_2 v_1| < 9w$, which is a contradiction. We conclude $u_1 = u_2$, $v_1 = v_2$. Thus, there are at most three classes of solutions for $P_{8t}(9)$: the class of $(3, 0)$ and possibly the classes of (u_1, v_1) and $(-u_1, v_1)$. Notice that the latter two classes are always distinct: in order for them to coincide we would need

$$\frac{u_1^2 + 8tv_1^2}{9} \in \mathbb{Z}, \qquad \frac{2u_1 v_1}{9} \in \mathbb{Z}$$

which does not happen, since $(u_1, 3) = (v_1, 3) = 1$. Hence, the equation has either one or three classes of solutions. $\square$

We remark that all cases in the statements of Lemma 5.2 and Lemma 5.3 occur, for suitable values of t.

Let S be an algebraic K3 surface such that $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 2$. As explained in Section 4, if $\mathrm{Bir}(S^{[3]}) \neq \{\mathrm{id}\}$ then $2t$ is not a square, $2X^2 - tY^2 = 1$ has no integer solutions and the minimal solution of $X^2 - 2tY^2 = 1$ has $Y \equiv 0 \pmod 2$. As a consequence, $\overline{\mathrm{Mov}(S^{[3]})} = \langle h, zh - 2tw\delta \rangle_{\mathbb{R}_{\geq 0}}$, where (z, w) is the minimal solution of $P_{8t}(1) : z^2 - 8tw^2 = 1$. The walls in the interior of the movable cone are the rays through $Xh - 2tY\delta$, for (X, Y) positive solution of $X^2 - 8tY^2 = 8 + \alpha^2$ such that $\alpha \in \{1, 2\}$, $X \equiv \pm\alpha \pmod 4$ and $0 < \frac{Y}{X} < \frac{w}{z}$. By Remark 5.1, this implies that the number of chambers in the interior of the cone coincides with the number of combined equivalence classes of solutions for $P_{8t}(9)$ and $P_{8t}(12)$ (the class of the solution $(3, 0)$ of $P_{8t}(9)$ determines the two extremal rays of the movable cone). Hence, Lemma 5.2 and Lemma 5.3 give the following result.

Proposition 5.4. *Let $t \geq 2$ such that $2t$ is not a square, $2X^2 - tY^2 = 1$ has no integer solutions and the minimal solution of $X^2 - 2tY^2 = 1$ has $Y \equiv 0 \pmod 2$. The following table provides the numbers of classes of solutions for the equations $P_{8t}(9)$ and $P_{8t}(12)$ and the number of chambers in the movable cone of $S^{[3]}$, for S an algebraic K3 surface such that $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$.*

$t \mod 18$	0	1	2	3	4	5	6	7	8
# eq. classes $P_{8t}(9)$	1, 2, 3	1	1, 3	1	1	1, 3	1	1	1, 3
# eq. classes $P_{8t}(12)$	0	0	0	0, 1	0	0, 2	0	0	0
# chambers	1, 2, 3	1	1, 3	1, 2	1	1, 3, 5	1	1	1, 3

$t \mod 18$	9	10	11	12	13	14	15	16	17
# eq. classes $P_{8t}(9)$	$1,2,3$	1	$1,3$	1	1	$1,3$	1	1	$1,3$
# eq. classes $P_{8t}(12)$	0	0	$0,2$	0	0	0	0	0	$0,2$
# chambers	$1,2,3$	1	$1,3,5$	1	1	$1,3$	1	1	$1,3,5$

In all cases where $\mathrm{Mov}(S^{[3]})$ has no walls in its interior (i.e. there is only one chamber), then any birational automorphism is biregular by the global Torelli theorem for ihs manifolds.

Corollary 5.5. *Let S be an algebraic K3 surface such that $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 1$. If $t \equiv 1, 4, 6, 7, 10, 12, 13, 15, 16 \pmod{18}$, then $\mathrm{Bir}(S^{[3]}) = \mathrm{Aut}(S^{[3]})$.*

From Theorem 1.1, Corollary 4.7 and the results of this section we conclude the following.

Proposition 5.6. *Let S be an algebraic K3 surface such that $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$. If $t = 1$, then $\mathrm{Bir}(S^{[3]}) = \mathrm{Aut}(S^{[3]}) \cong \mathbb{Z}/2\mathbb{Z}$. If $t \geq 2$, then $\mathrm{Bir}(S^{[3]}) \neq \{\mathrm{id}\}$ if and only if:*

- *$2t$ is not a square;*
- *$2X^2 - tY^2 = 1$ has no integer solutions;*
- *either $2X^2 - tY^2 = -1$ or $X^2 - 2tY^2 = -1$ has integer solutions.*

If $\mathrm{Bir}(S^{[3]}) \neq \{\mathrm{id}\}$, let d be the number of chambers in the decomposition of $\overline{\mathrm{Mov}(S^{[3]})}$. Then $d \in \{1, 2, 3, 5\}$ and one of the following holds:

- *$d = 1$ and $\mathrm{Bir}(S^{[3]}) = \mathrm{Aut}(S^{[3]}) \cong \mathbb{Z}/2\mathbb{Z}$;*
- *$d = 2$, $t = 3$ or $t = 9$, $\mathrm{Aut}(S^{[3]}) = \{\mathrm{id}\}$ and $\mathrm{Bir}(S^{[3]}) \cong \mathbb{Z}/2\mathbb{Z}$;*
- *$d = 3, 5$, $\mathrm{Aut}(S^{[3]}) = \{\mathrm{id}\}$ and $\mathrm{Bir}(S^{[3]}) \cong \mathbb{Z}/2\mathbb{Z}$*

If $t \neq 3, 9$ and $\sigma \in \mathrm{Bir}(S^{[3]})$, there exists an ihs sixfold X and a birational map $g : S^{[3]} \dashrightarrow X$ such that $g \circ \sigma \circ g^{-1} \in \mathrm{Aut}(X)$.

Table 2 lists the number d of chambers in the decomposition of $\overline{\mathrm{Mov}(S^{[3]})}$ and the structure of the groups $\mathrm{Aut}(S^{[3]})$, $\mathrm{Bir}(S^{[3]})$, for an algebraic K3 surface S with $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$, $1 \leq t \leq 30$ when $\mathrm{Bir}(S^{[3]}) \neq \{\mathrm{id}\}$.

TABLE 2. Chambers and automorphisms for $n = 3$.

t	d	$\mathrm{Aut}(S^{[3]})$	$\mathrm{Bir}(S^{[3]})$
1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	2	$\{\mathrm{id}\}$	$\mathbb{Z}/2\mathbb{Z}$
5	3	$\{\mathrm{id}\}$	$\mathbb{Z}/2\mathbb{Z}$
9	2	$\{\mathrm{id}\}$	$\mathbb{Z}/2\mathbb{Z}$
11	5	$\{\mathrm{id}\}$	$\mathbb{Z}/2\mathbb{Z}$
13	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
19	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
25	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
27	3	$\{\mathrm{id}\}$	$\mathbb{Z}/2\mathbb{Z}$
29	3	$\{\mathrm{id}\}$	$\mathbb{Z}/2\mathbb{Z}$

6. Ambiguous Hilbert schemes and birational models

Having classified the group of birational automorphisms of $S^{[n]}$, for a K3 surface S with Picard rank one, we explain in this section how the same approach can be used to study the more general problem of whether there exists a K3 surface Σ, again of Picard rank one, and a birational map $\phi : S^{[n]} \dashrightarrow \Sigma^{[n]}$ which do not come from an isomorphism $S \to \Sigma$. This is related to the notion of *(strong) ambiguity* for Hilbert schemes of points on K3 surfaces, which for $n = 2$ (and ϕ biregular) has been investigated in [11] and [27]. Some of the results of this section overlap (even though they are proved differently) with those of [17], where birationality of derived equivalent Hilbert schemes of K3 surfaces is studied.

It is known that, if $S^{[n]}$ and $\Sigma^{[n]}$ are birational, then S and Σ are Fourier–Mukai partners and $S^{[n]}$ and $\Sigma^{[n]}$ are derived equivalent (see [25, Proposition 10]). If $\mathrm{Pic}(S) = \mathbb{Z}H$ with $H^2 = 2t$, $t \geq 1$, then by [23, Proposition 1.10] the number of non-isomorphic FM partners of S is $2^{\rho(t)-1}$, where $\rho(t)$ denotes the number of prime divisors of t (and $\rho(1) = 1$).

We can classify the Fourier–Mukai partners Σ of S as follows (for details see for instance [23, Section 4]). The overlattice $L = H^2(\Sigma, \mathbb{Z})$ of $\mathrm{Tr}(S) \oplus \mathrm{NS}(S)$ (with integral Hodge structure defined by setting $L^{2,0} = \mathrm{Tr}(S)^{2,0}$) corresponds to an isotropic subgroup $I_L \subset A_{\mathrm{Tr}(S)} \oplus A_{\mathrm{NS}(S)} = \frac{\mathbb{Z}}{2t\mathbb{Z}}(-\frac{1}{2t}) \oplus \frac{\mathbb{Z}}{2t\mathbb{Z}}(\frac{1}{2t})$ (as in [21, Section 1.4]). Since $\mathrm{NS}(S)$ and $\mathrm{Tr}(S)$ are primitive in L, the group I_L is of the form $I_L = I_a := \langle \epsilon + a\eta \rangle$ for some $a \in (\mathbb{Z}/2t\mathbb{Z})^\times$, $a^2 \equiv 1 \pmod{4t}$, where ϵ (respectively, η) is a generator of $A_{\mathrm{Tr}(S)}$ (respectively, $A_{\mathrm{NS}(S)}$) on which the finite quadratic form takes value $-\frac{1}{2t}$ (respectively, $+\frac{1}{2t}$). For each $a \in (\mathbb{Z}/2t\mathbb{Z})^\times$, $a^2 \equiv 1 \pmod{4t}$, there exists a K3 surface Σ_a (unique up to isomorphism) such that $H^2(\Sigma_a, \mathbb{Z})$ is Hodge-isometric to the overlattice L_a of $\mathrm{Tr}(S) \oplus \mathrm{NS}(S)$ defined by I_a. Moreover, $\Sigma_a \cong \Sigma_b$ if and only if $b \equiv \pm a \pmod{2t}$. Indeed, $2^{\rho(t)-1}$ (which is the number of non-isomorphic FM partners of S) is the cardinality of $\left\{ a \in (\mathbb{Z}/2t\mathbb{Z})^\times, a^2 \equiv 1 \pmod{4t} \right\} / \pm \mathrm{id}$.

Remark 6.1. If there exists a birational map $\phi : S^{[n]} \dashrightarrow \Sigma^{[n]}$ as above, then $\phi^*(\mathrm{Mov}(\Sigma^{[n]})) = \mathrm{Mov}(S^{[n]})$ ([16, Corollary 5.7]). Let $\{h_S, \delta_S\}$ and $\{h_\Sigma, \delta_\Sigma\}$ be the canonical bases for $\mathrm{NS}(S^{[n]})$ and $\mathrm{NS}(\Sigma^{[n]})$ respectively, as in Section 3.1. Then by [6, Theorem 1] the map ϕ is induced by an isomorphism of the underlying K3 surfaces if and only if $\phi^*(\delta_\Sigma) = \delta_S$. As a consequence, if ϕ does not come from an isomorphism $S \to \Sigma$ then $\phi^*(h_\Sigma)$ is the primitive generator of the extremal ray of $\mathrm{Mov}(S^{[n]})$ not spanned by h_S. This implies that Σ is uniquely defined, up to isomorphism: in fact, if we also have $\phi' : S^{[n]} \dashrightarrow (\Sigma')^{[n]}$ which does not come from an isomorphism $S \to \Sigma'$, then $\phi' \circ \phi^{-1} : \Sigma^{[n]} \dashrightarrow (\Sigma')^{[n]}$ is induced by an isomorphism $\Sigma \to \Sigma'$, since its pullback maps $h_{\Sigma'}$ to h_Σ (hence, also $\delta_{\Sigma'}$ to δ_Σ).

Theorem 6.2. *Let S be an algebraic K3 surface such that $\mathrm{Pic}(S) = \mathbb{Z}H$, with $H^2 = 2t$, $t \geq 1$. For $n \geq 2$, let (z, w) be the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$. There exists a K3 surface Σ and a birational map $\phi : S^{[n]} \dashrightarrow \Sigma^{[n]}$ which is not induced by an isomorphism $S \to \Sigma$ if and only if:*

- *$t(n-1)$ is not a square;*
- *if $n \neq 2$, $(n-1)X^2 - tY^2 = 1$ has no integer solutions;*
- *$z \equiv \pm 1 \pmod{2(n-1)}$ and $w \equiv 0 \pmod 2$.*

If so, the K3 surfaces S and Σ are isomorphic if and only if $z \equiv \pm 1 \pmod{2t}$. Moreover, ϕ is biregular if and only if, for all integers ρ, α as follows:

- $\rho = -1$ *and* $1 \leq \alpha \leq n-1$*, or*
- $\rho = 0$ *and* $3 \leq \alpha \leq n-1$*, or*
- $1 \leq \rho < \frac{n-1}{4}$ *and* $\max\left\{4\rho+1, \left\lceil 2\sqrt{\rho(n-1)}\right\rceil\right\} \leq \alpha \leq n-1$

if Pell's equation

$$X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$$

is solvable, the minimal solution (X, Y) *with* $X \equiv \pm\alpha \pmod{2(n-1)}$ *satisfies* $\frac{Y}{X} \geq \frac{w}{2z}$.

Proof. If there exists a birational map $\phi : S^{[n]} \dashrightarrow \Sigma^{[n]}$ which does not come from an isomorphism $S \to \Sigma$ then both extremal rays of $\mathrm{Mov}(S^{[n]})$ correspond to Hilbert–Chow contractions (Remark 6.1). By [2, Theorem 5.7], $t(n-1)$ is not a square, $(n-1)X^2 - tY^2 = 1$ has no integer solutions (if $n \neq 2$) and $z \equiv \pm1 \pmod{2(n-1)}$, $w \equiv 0 \pmod 2$. On the other hand, assume that these conditions are satisfied. Let $a \in (\mathbb{Z}/2t\mathbb{Z})^\times$, $a^2 \equiv 1 \pmod{4t}$, such that $S \cong \Sigma_a$. Since $z^2 - t(n-1)w^2 = 1$ and w is even, we can consider the FM partner Σ_{za} of S. As $\mathrm{Tr}(S^{[n]}) \cong \mathrm{Tr}(S)$ and $\mathrm{NS}(S^{[n]}) \cong \mathrm{NS}(S) \oplus \langle -2(n-1)\rangle$, the groups I_a and I_{za} (defined at the beginning of the section) can also be seen as isotropic subgroups of $A_{\mathrm{Tr}(S^{[n]})} \oplus A_{\mathrm{NS}(S^{[n]})}$ (in particular, we assume $2t\eta = h \in \mathrm{NS}(S^{[n]})$). It is then immediate to check that the overlattices of $\mathrm{Tr}(S^{[n]}) \oplus \mathrm{NS}(S^{[n]})$ defined by these two subgroups are $H^2(\Sigma_a^{[n]}, \mathbb{Z})$ and $H^2(\Sigma_{za}^{[n]}, \mathbb{Z})$ respectively. Let $\mu \in O(\mathrm{NS}(S^{[n]}))$ be the isometry (5). We have that $\mathrm{id} \oplus \mu \in O(\mathrm{Tr}(S^{[n]}) \oplus \mathrm{NS}(S^{[n]}))$ extends to a Hodge isometry $\psi : H^2(\Sigma_a^{[n]}, \mathbb{Z}) \to H^2(\Sigma_{za}^{[n]}, \mathbb{Z})$, because $\overline{\mu}(a\eta) = za\eta \in A_{\mathrm{NS}(S^{[n]})}$ (here we use the fact that w is even). Notice that the discriminant group of $H^2(S^{[n]}, \mathbb{Z})$ is generated by the class of $\frac{\delta}{2(n-1)}$ and $\overline{\mu}\left(\frac{\delta}{2(n-1)}\right) = -\frac{(n-1)wh}{2(n-1)} - \frac{z\delta}{2(n-1)} = \pm\frac{\delta}{2(n-1)}$. By [21, Corollary 1.5.2], ψ extends to a Hodge isometry $H^*(\Sigma_a, \mathbb{Z}) \to H^*(\Sigma_{za}, \mathbb{Z})$ between the Mukai lattices of the two K3 surfaces Σ_a, Σ_{za}. We conclude that $\Sigma_a^{[n]}$ and $\Sigma_{za}^{[n]}$ are birationally equivalent, by [16, Corollary 9.9].

As stated before, Σ_a and Σ_{za} are isomorphic if and only if $a \equiv \pm za \pmod{2t}$, i.e. $z \equiv \pm1 \pmod{2t}$. The isomorphism $\phi : S^{[n]} \dashrightarrow \Sigma^{[n]}$ is biregular if and only if the Hodge isometry $\phi^* : H^2(\Sigma^{[n]}, \mathbb{Z}) \to H^2(S^{[n]}, \mathbb{Z})$ is effective, that is if it maps ample classes to ample classes. Since ϕ^* permutes the two extremal rays of the movable cones, this is equivalent to asking that the movable cone of $S^{[n]}$ coincides with the ample cone, i.e. that there is only one chamber in the decomposition of $\overline{\mathrm{Mov}(S^{[n]})}$. The last part of the statement follows then as in the proof of [8, Theorem 6.4]. □

Remark 6.3. If $S \cong \Sigma$ the existence of a birational map $S^{[n]} \dashrightarrow \Sigma^{[n]}$ which does not come from an isomorphism of the K3 surfaces is equivalent to the existence of a non-natural birational automorphism in $\mathrm{Bir}(S^{[n]})$. Indeed, in this case Theorem 6.2 gives the same conditions of Theorem 1.1.

Remark 6.4. It can be readily checked that the conditions in the first part of Theorem 6.2 are equivalent to those of [17, Theorem 2.2]. Write $z = 2(n-1)k + \epsilon$ and $w = 2h$ for $k, h \in \mathbb{N}$ and $\epsilon \in \{\pm1\}$. Then, $z^2 - t(n-1)w^2 = 1$ implies $k(k(n-1)+\epsilon) = th^2$. Since the two factors in the LHS term are coprime, there exist $p, q, r, s \in \mathbb{N}$ such that $k = sp^2$, $k(n-1)+\epsilon = rq^2$, $h = pq$, $t = rs$. In particular, $(n-1)sp^2 - rq^2 = \pm1$, which is what is requested in [17, Theorem 2.2]. The FM partner Σ of S such that there exists a birational map $S^{[n]} \dashrightarrow \Sigma^{[n]}$ not coming from an isomorphism $S \to \Sigma$ is the moduli space $M_S(p^2s, pqH, q^2r) \cong M_S(s, H, r)$. Notice that $z \equiv \pm 1 \pmod{2t}$ (i.e. $\Sigma \cong S$) if and only if $\{r, s\} = \{1, t\}$.

Corollary 6.5. *Let S be an algebraic surface with $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$ and $n \geq 2$ an integer such that $S^{[n]}$ admits an involution which is not biregular. Assume that t is not n-irregular, hence there exists an ihs birational model X of $S^{[n]}$ with a biregular involution. Then X is not isomorphic to the Hilbert scheme of n points on a K3 surface.*

Proof. Let (z, w) be the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$. By contradiction, assume that $X \cong \Sigma^{[n]}$ for some K3 surface Σ. The groups $\mathrm{Aut}(\Sigma^{[n]})$ and $\mathrm{Aut}(S^{[n]})$ are different so S is not isomorphic to Σ, while $\Sigma^{[n]}$ and $S^{[n]}$ are birational. So $z \not\equiv \pm 1 \pmod{2t}$ by Theorem 6.2, which is in contrast with Theorem 1.1. □

By [8, Proposition 1.1], Corollary 6.5 is applicable for the values $t \leq 2n-3$ which satisfy Theorem 1.1 and which are not n-irregular (see Table 1), e.g. $(n, t) = (6, 2)$.

7. Moduli spaces of polarized manifolds of $K3^{[n]}$-type

We say that an ihs manifold is of $K3^{[n]}$-type if it is deformation equivalent to the Hilbert scheme of n points on a K3 surface. Let $\Lambda_n = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle$, which is isometric to the second cohomology lattice of any ihs manifold of $K3^{[n]}$-type (with the BBF quadratic form). A *polarization type* is the choice of a $O(\Lambda_n)$-orbit $[k]$, with $k \in \Lambda_n$ primitive and $(k, k) > 0$. We denote by $\mathcal{M}_{[k]}$ the moduli space of polarized manifolds (X, D), with X of $K3^{[n]}$-type and D a primitive ample class on X such that there exists an isomorphism $\eta : H^2(X, \mathbb{Z}) \to \Lambda_n$ (called a marking) such that $\eta(D) = k$. More generally, we denote by $\mathcal{M}^n_{2d,\gamma}$ the moduli space which parametrizes manifolds of $K3^{[n]}$-type with a polarization of square $2d$ and divisibility γ, for $d, \gamma \in \mathbb{N}$ (for more details on these moduli spaces, we refer to [13] and [1]).

From [16, §7.1], two polarized manifolds (X_1, D_1), $(X_2, D_2) \in \mathcal{M}^n_{2d,\gamma}$ are in the same connected component of the moduli space if and only if there exists a parallel transport operator $g : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ such that $g(D_1) = D_2$. Let $\iota_{X_i} : H^2(X_i, \mathbb{Z}) \hookrightarrow \Lambda := U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$ be in the canonical $O(\Lambda)$-orbit of primitive embeddings in the Mukai lattice (see [16, Corollary 9.5]). We denote by $T(X_i, D_i)$ the rank-two positive definite lattice which arises as saturation, inside Λ, of the sublattice generated by $\iota_{X_i}(D_i)$ and $\iota_{X_i}(H^2(X_i, \mathbb{Z}))^\perp$. Then there exists a polarized parallel transport operator between (X_1, D_1) and (X_2, D_2) if and only if $(T(X_1, D_1), \iota_{X_1}(D_1))$ and $(T(X_2, D_2), \iota_{X_2}(D_2))$ are isomorphic as lattices with a distinguished vector ([1, Proposition 1.6]).

Let X be a manifold of $K3^{[n]}$-type with a non-symplectic biregular involution $i \in \mathrm{Aut}(X)$ such that $H^2(X, \mathbb{Z})^{i^*}$ has rank one. By [7, Proposition 4.3] there exists an i^*-invariant primitive class $D \in \mathcal{A}_X$ such that (X, D) belongs to one of the following moduli spaces: $\mathcal{M}^n_{2,1}$; $\mathcal{M}^n_{2,2}$ (non-empty only if $n \equiv 0 \pmod 4$); $\mathcal{M}^n_{2(n-1),n-1}$ (non-empty only if $\left(\frac{-1}{n-1}\right) = 1$, using Legendre symbol).

Proposition 7.1. [1, Corollary 2.4 and Proposition 3.1], [13, Examples 3.8, 3.10] *Let $n \geq 2$ and for $r \in \mathbb{N}$ denote by $\rho(r)$ the number of distinct prime divisors of r.*

- *The moduli space $\mathcal{M}^n_{2,1}$ is connected; the polarization type is unique.*

- *If* $n \equiv 0 \pmod 4$, *then* $\mathcal{M}^n_{2,2}$ *is connected; the polarization type is unique.*
- *If* $n \equiv 0 \pmod 2$ *and* $\left(\frac{-1}{n-1}\right) = 1$, *then* $\mathcal{M}^n_{2(n-1),n-1}$ *has* $2^{\rho(n-1)-1}$ *connected components; the polarization type is unique.*
- *If* $n \equiv 1 \pmod 4$ *and* $\left(\frac{-1}{n-1}\right) = 1$, *then* $\mathcal{M}^n_{2(n-1),n-1}$ *has* $2^{\rho(\frac{n-1}{4})}$ *connected components.*
- *If* $n \equiv 3 \pmod 4$ *and* $\left(\frac{-1}{n-1}\right) = 1$, *then* $\mathcal{M}^n_{2(n-1),n-1}$ *has* $2^{\rho(\frac{n-1}{2})-1}$ *connected components.*

Given $n, t \geq 2$ we define the property

$(*)$: for an algebraic K3 surface S with $\mathrm{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$, the Hilbert scheme $S^{[n]}$ is equipped with a biregular non-natural automorphism.

Equivalently, this means that n, t satisfy the numerical conditions given in [8, Theorem 6.4].

Take n, t which satisfy $(*)$. Let (a, b) (resp. $(a, \frac{b}{n-1})$) be the integer solution of $(n-1)X^2 - tY^2 = -1$ (resp. $X^2 - t(n-1)Y^2 = -1$) with minimal $X > 0$, depending on which of the two equations is solvable (see Proposition 3.1). We consider the moduli space $\mathcal{K}_{2t}$ of $2t$-polarized K3 surfaces and $U \subset \mathcal{K}_{2t}$ the subset of elements (Σ, H) such that $D := bh - a\delta \in \mathrm{NS}(\Sigma^{[n]})$ is ample, where as usual $h \in \mathrm{NS}(\Sigma^{[n]})$ is the class of the line bundle induced by H and 2δ is the class of the exceptional divisor of the Hilbert–Chow contraction $\Sigma^{[n]} \to \Sigma^{(n)}$. In particular, by Proposition 3.1 U contains all $2t$-polarized $K3$ surfaces of Picard rank one. For $(\Sigma, H) \in U$, the polarized manifold $(\Sigma^{[n]}, D)$ is in one of the moduli spaces $\mathcal{M}^n_{2,1}$, $\mathcal{M}^n_{2,2}$ if $(n-1)X^2 - tY^2 = -1$ is solvable, otherwise $(\Sigma^{[n]}, D) \in \mathcal{M}^n_{2(n-1),n-1}$.

Assume that $D^2 = 2$. One can readily check that the divisibility of D is $\gcd(b, 2(n-1))$, hence $(\Sigma^{[n]}, D) \in \mathcal{M}^n_{2,1}$ if b is odd while $(\Sigma^{[n]}, D) \in \mathcal{M}^n_{2,2}$ if b is even.

If $D^2 = 2(n-1)$, we are interested in determining the connected component of $\mathcal{M}^n_{2(n-1),n-1}$ which contains $(\Sigma^{[n]}, D)$. The canonical embedding $\iota_{\Sigma^{[n]}}$ in the Mukai lattice satisfies (see [16, Example 9.6])

$$\begin{aligned} \iota_{\Sigma^{[n]}} : H^2(\Sigma^{[n]}, \mathbb{Z}) &\hookrightarrow \Lambda \\ h &\mapsto (0, -H, 0) \\ -\delta &\mapsto (1, 0, n-1), \end{aligned}$$

where we identify Λ with $H^*(\Sigma, \mathbb{Z}) = H^0(\Sigma, \mathbb{Z}) \oplus H^2(\Sigma, \mathbb{Z}) \oplus H^4(\Sigma, \mathbb{Z})$. The image of $\iota_{\Sigma^{[n]}}$ is the orthogonal complement of $v = (1, 0, 1-n)$. Let $\beta = \frac{b}{n-1} \in \mathbb{N}$. The primitive sublattice $T^t_n := T(\Sigma^{[n]}, D) \subset \Lambda$ is generated by v and $w = \frac{1}{n-1}(av - \iota_{\Sigma^{[n]}}(D)) = (0, \beta H, -2a)$. With respect to the basis $\{v, w\}$, the polarization is $\iota_{\Sigma^{[n]}}(D) = av - (n-1)w$ and the bilinear form of the lattice is given by the matrix

$$\begin{pmatrix} 2(n-1) & 2a \\ 2a & 2t\beta^2 \end{pmatrix}.$$

In the following we assume that n, t satisfy $(*)$ and we denote by $\mathcal{M}(n, t)$ the connected component (of one the moduli spaces $\mathcal{M}^n_{2,1}$, $\mathcal{M}^n_{2,2}$, $\mathcal{M}^n_{2(n-1),n-1}$) which contains the polarized manifolds $(\Sigma^{[n]}, bh - a\delta)$ as above, for $(\Sigma, H) \in U$.

Lemma 7.2. *For* $n \geq 2$, *let* $t_1, t_2 \geq 2$ *be such that* $X^2 - t_i(n-1)Y^2 = -1$ *is solvable, and let* (a_i, β_i) *be the minimal solution. Assuming that both* t_1, t_2 *satisfy*

$(*)$, the connected components $\mathcal{M}(n,t_1)$, $\mathcal{M}(n,t_2)$ of $\mathcal{M}^n_{2(n-1),n-1}$ coincide if and only if $a_1 \equiv \pm a_2 \pmod{n-1}$.

Proof. By the previous discussion, the connected components coincide if and only if there exists an isometry $\lambda : T_n^{t_1} \to T_n^{t_2}$ such that $\lambda(a_1v_1 - (n-1)w_1) = a_2v_2 - (n-1)w_2$. In particular, $\lambda(w_1) = \pm w_2$ and $\lambda((n-1)w_1) = (n-1)w_2 - (a_2 \pm a_1)v_2$. Such isometry exists if and only if $a_1 \equiv \pm a_2 \pmod{n-1}$. □

We now consider the map ${}^n\phi_{2t} : U \to \mathcal{M}(n,t), (\Sigma, H) \mapsto (\Sigma^{[n]}, bh - a\delta)$. Clearly the image of ${}^n\phi_{2t}$ is contained in the Noether–Lefschetz locus of $\mathcal{M}(n,t)$, that is the subset of polarized manifolds whose Picard group has rank at least two. Let $\mathcal{C}_{2t} \subset \mathcal{M}(n,t)$ be the locus which parametrizes polarized manifolds (X,D) whose Picard group contains a primitive hyperbolic rank-two sublattice $K \ni D$ and such that $K^\perp \subset H^2(X,\mathbb{Z})$ has discriminant $-2t$. The proof of [11, Proposition 7.1] can be generalized to obtain the following result.

Proposition 7.3. *Given n,t which satisfy $(*)$, the subset $U \subset \mathcal{K}_{2t}$ is open and the rational map ${}^n\phi_{2t} : \mathcal{K}_{2t} \dashrightarrow \mathcal{M}(n,t)$ is birational onto an irreducible component of $\mathcal{C}_{2t}$.*

For $n \geq 2$, let $\mathcal{M}_{\Lambda_n}$ be the moduli space of marked ihs manifolds of $K3^{[n]}$-type and fix a connected component $\mathcal{M}^0_{\Lambda_n}$ (see [16, §1.1] for details). Let $\rho \in O(\Lambda_n)$ be an involution such that $(\Lambda_n)^\rho = \langle k \rangle$, with k primitive of square $2d > 0$ and divisibility γ for some $(d,\gamma) \in \{(1,1),(1,2),(n-1,n-1)\}$. Notice that $\rho = -R_k$. Following [15] and [7], we can consider the moduli space $\mathcal{M}^n_{\langle 2d\rangle,\rho} \subset \mathcal{M}^0_{\Lambda_n}$ of $(\rho, \langle 2d\rangle)$-polarized manifolds of $K3^{[n]}$-type. An element of this moduli space is a marked manifold (X,η) which admits a non-symplectic involution $i \in \mathrm{Aut}(X)$ such that $\rho = \eta \circ i^* \circ \eta^{-1}$. Notice that i is uniquely determined, by the injectivity of $\mathrm{Aut}(X) \to O(H^2(X,\mathbb{Z})), i \mapsto i^*$. By [7, Theorem 3.3], the moduli space $\mathcal{M}^n_{\langle 2d\rangle,\rho}$ is non-empty if and only if the induced isometry $\overline{\rho} \in O(A_{\Lambda_n})$ is $\pm\,\mathrm{id}_{A_{\Lambda_n}}$.

Let (X,η), (X',η') be two $(\rho,\langle 2d\rangle)$-polarized manifolds, with involutions $i \in \mathrm{Aut}(X)$ and $i' \in \mathrm{Aut}(X')$ such that $\rho = \eta \circ i^* \circ \eta^{-1} = \eta' \circ (i')^* \circ (\eta')^{-1}$. Let $T := (\Lambda_n)^\rho$ and $\mathrm{Mon}^2(\Lambda_n, T) := \left\{g \in \mathrm{Mon}^2(\Lambda_n) \mid g(T) = T\right\}$, where $\mathrm{Mon}^2(\Lambda_n) := \eta \,\mathrm{Mon}^2(X) \eta^{-1} = \eta' \,\mathrm{Mon}^2(X')(\eta')^{-1} \subset O(\Lambda_n)$. We say that the pairs (X,i), (X',i') are isomorphic if there exists an isomorphism $f : X \to X'$ such that $i' \circ f = f \circ i$. In this case we have $\eta \circ f^* \circ (\eta')^{-1} \in \mathrm{Mon}^2(\Lambda_n,T)$; moreover, $\eta \circ f^* \circ (\eta')^{-1} = \mathrm{id}_{\Lambda_n}$ if and only if (X,η) and (X',η') are equivalent in $\mathcal{M}^n_{\langle 2d\rangle,\rho}$.

Proposition 7.4. *The quotient $\mathcal{M}_\rho := \mathcal{M}^n_{\langle 2d\rangle,\rho} / \mathrm{Mon}^2(\Lambda_n,T)$ is the coarse moduli space of isomorphism classes of pairs (X,i), for $(X,\eta) \in \mathcal{M}^n_{\langle 2d\rangle,\rho}$ and $i \in \mathrm{Aut}(X)$ the involution such that $\rho = \eta \circ i^* \circ \eta^{-1}$.*

Proof. This follows from [15, Theorem 10.5]: since $(\Lambda_n)^\rho$ has rank one, $\mathcal{M}_\rho$ is irreducible by [15, Theorem 9.11], i.e. all $(\rho,\langle 2d\rangle)$-polarized manifolds of $K3^{[n]}$-type deform in the same family. For the same reason every pair $(X,i) \in \mathcal{M}_\rho$ is simple in the sense of [15, Definition 10.3]. □

The following statement generalizes [4, Corollary 4.1] for $n = 2$.

Theorem 7.5. *For $(d,\gamma) \in \{(1,1),(1,2),(n-1,n-1)\}$, let $\rho \in O(\Lambda_n)$ be an involution such that $\overline{\rho} = \pm\,\mathrm{id}_{A_{\Lambda_n}}$ and $(\Lambda_n)^\rho$ is generated by a primitive element k*

of square $2d$ and divisibility γ. Then $\mathcal{M}_\rho$ is isomorphic to a connected component of $\mathcal{M}_{[k]} \subset \mathcal{M}^n_{2d,\gamma}$.

Proof. Let (X,η) be $(\rho,\langle 2d\rangle)$-polarized, with $i \in \mathrm{Aut}(X)$ the non-symplectic involution such that $\rho = \eta \circ i^* \circ \eta^{-1}$. If D is the primitive ample divisor which generates $H^2(X,\mathbb{Z})^{i^*} = \eta^{-1}((\Lambda_n)^\rho)$, then $(X,D) \in \mathcal{M}^n_{2d,\gamma}$. The morphism $\alpha : \mathcal{M}_\rho \to \mathcal{M}^n_{2d,\gamma}, (X,i) \mapsto (X,D)$ is clearly well-defined at the level of moduli spaces and its image is contained in a connected component $\mathcal{M} \subset \mathcal{M}^n_{2d,\gamma}$. Indeed, if (X_1,η_1,i_1), (X_2,η_2,i_2) are $(\rho,\langle 2d\rangle)$-polarized then $g = \eta_2^{-1} \circ \eta_1$ is a parallel transport operator (because (X_1,η_1) and (X_2,η_2) lie in the same connected component of $\mathcal{M}_{\Lambda_n}$) such that $g(D_1) = D_2$. The connected component $\mathcal{M}$ is determined by the explicit choice of the isometry ρ. By construction, $\mathcal{M} \subset \mathcal{M}_{[k]}$ with $(\Lambda_n)^\rho = \langle k\rangle$. Let now (X,D) be a polarized manifold in $\mathcal{M}$. By [8, Proposition 5.3] there is a (unique) non-symplectic involution $i \in \mathrm{Aut}(X)$ such that $i^* = -R_D$. Since (X,D) is in the same connected component of any $\alpha(X',i')$ and $\rho = -R_k$ there exists a marking $\eta : H^2(X,\mathbb{Z}) \to \Lambda_n$ such that $(X,\eta) \in \mathcal{M}^0_{\Lambda_n}$ and $\rho = \eta \circ i^* \circ \eta^{-1}$, hence we have a well-defined map $\mathcal{M} \to \mathcal{M}_\rho, (X,D) \to (X,i)$ which is the inverse of α. □

Remark 7.6. Let $(d,\gamma) \in \{(1,1),(1,2),(n-1,n-1)\}$ and fix a connected component $\mathcal{M} \subset \mathcal{M}^n_{2d,\gamma}$. For $(X,D) \in \mathcal{M}$ let $\eta : H^2(X,\mathbb{Z}) \to \Lambda_n$ be a marking such that $(X,\eta) \in \mathcal{M}^0_{\Lambda_n}$ and define $\rho = -R_{\eta(D)} \in O(\Lambda_n)$. Then it is clear from the proof of the previous proposition that $\mathcal{M}$ is the connected component of $\mathcal{M}^n_{2d,\gamma}$ which is isomorphic to $\mathcal{M}_\rho$ via the map α.

8. Geometrical constructions

Let S be a $2t$-polarized K3 surface of Picard rank one and $n \geq 2$. Theorem 1.1 allows us to determine the values n,t for which $\mathrm{Bir}(S^{[n]}) \neq \{\mathrm{id}\}$, however it does not provide any indication on how to construct these automorphisms geometrically. If $t = n$, then $\mathrm{Bir}(S^{[n]})$ is generated by Beauville's (non-symplectic) involution [3, §6]. For $2 \leq n \leq 10$, $2 \leq t \leq 7$, $t \neq n$, the pairs (n,t) such that $\mathrm{Bir}(S^{[n]}) \neq \{\mathrm{id}\}$ are the following:

$$(n,t) = (2,5),(3,5),(4,7),(6,2),(8,2),(8,4),(9,3),(9,5).$$

The involution which generates $\mathrm{Bir}(S^{[n]})$ is symplectic for $(n,t) = (9,3),(9,5)$, non-symplectic in the other cases.

The general K3 surface S of degree $2t = 10$ is a transverse intersection $\mathrm{Gr}(2,5) \cap \Gamma \cap Q \subset \mathbb{P}^9$, where $\Gamma \cong \mathbb{P}^6$ and Q is a quadric. The birational involution of $S^{[2]}$ was constructed by O'Grady in [22, Section 4.3], while a geometric description for the involution of $S^{[3]}$ has been provided in [10, Example 4.12].

We give new constructions of the non-symplectic involutions for $(n,t) = (6,2)$, $(8,2)$. Let $\mathcal{K}_4$ be the moduli space of 4-polarized K3 surfaces. We denote by $D_{x,y} \subset \mathcal{K}_4$ the Noether–Lefschetz divisor corresponding to polarized K3 surfaces (S,H) for which there exists $B \in \mathrm{Pic}(S)$ with $(H,B) = x$, $B^2 = y$ and such that the sublattice of $\mathrm{Pic}(S)$ generated by B,H is primitive. A general K3 surface S of degree 4 can be embedded as a smooth quartic surface in $\mathbb{P}^3$ and we consider the polarization $H \in \mathrm{Pic}(S)$ given by a hyperplane section.

Example 8.1 ($n = 6$, $t = 2$)**.** Let $S \subset \mathbb{P}^3$ be a smooth quartic surface which does not contain any twisted cubic curve, e.g. $(S,H) \notin D_{3,-2}$. If $p_1,\ldots,p_6 \in S$ are in general linear position there exists a single rational normal curve (i.e. a twisted cubic) passing through them, which we denote by C_3. Since C_3 does not lie on S, the intersection $S \cap C_3$ consists of twelve points. By associating the p_i's with the six residual points of intersection, we obtain a birational involution of $S^{[6]}$.

For $n = 6$, $t = 2$ the equation $X^2 - t(n-1)Y^2 = -1$ is solvable, with minimal solution $(a, b) = (3, 1)$. Thus for $(S, H) \in \mathcal{K}_4$ with $\mathrm{Pic}(S) = \mathbb{Z}H$, the birational non-symplectic involution σ which generates $\mathrm{Bir}(S^{[6]})$ satisfies $H^2(S^{[6]}, \mathbb{Z})^{\sigma^*} = \mathbb{Z}\nu \cong \langle 10 \rangle$ with $\nu = 5h - 3\delta$.

Example 8.2 ($n = 8$, $t = 2$)**.** Let $S \subset \mathbb{P}^3$ be a smooth quartic surface such that, if a curve $C \subset \mathbb{P}^3$ of degree four is contained in two different quadric surfaces, then $C \not\subset S$. This holds for instance outside the divisors $D_{4,0}, D_{4,2}$ (notice that we do not have to exclude $D_{4,-2}$, since a rational quartic curve in $\mathbb{P}^3$ is contained in a unique quadric surface). Consider $p_1, \ldots, p_8 \in S$ in general linear position. The linear system of quadric surfaces in $\mathbb{P}^3$ passing through $p_1, \ldots, p_8$ is a pencil. Its base locus is a quartic curve C_4, given by the intersection of two different divisors in the pencil. By the initial assumption C_4 is not contained in S, hence C_4 intersects S along sixteen points, eight of which are the $p_i's$. This gives rise to a birational involution of $S^{[8]}$.

For $n = 8, t = 2$ the equation $(n-1)X^2 - tY^2 = -1$ is solvable, with minimal solution $(a, b) = (1, 2)$. Thus for $(S, H) \in \mathcal{K}_4$ with $\mathrm{Pic}(S) = \mathbb{Z}H$, the birational non-symplectic involution σ which generates $\mathrm{Bir}(S^{[8]})$ satisfies $H^2(S^{[8]}, \mathbb{Z})^{\sigma^*} = \mathbb{Z}\nu \cong \langle 2 \rangle$ with $\nu = 2h - \delta$.

Take n, t which satisfy $(*)$. If $(n-1)X^2 - tY^2 = -1$ admits solutions let (a, b) be the minimal one, otherwise we take (a, b) such that $(a, \frac{b}{n-1})$ is the minimal solution of $X^2 - t(n-1)Y^2 = -1$. Consider the rational map ${}^n\phi_{2t} : \mathcal{K}_{2t} \dashrightarrow \mathcal{M}(n, t)$ and $U \subset \mathcal{K}_{2t}$ as in Proposition 7.3. In the following, for a K3 surface S and $L \in \mathrm{Pic}(S)$ we denote by $\widetilde{L} \in \mathrm{Pic}(S^{[n]})$ the line bundle induced by L on $S^{[n]}$ (and, with a small abuse of notation, its class in $\mathrm{NS}(S^{[n]})$).

Proposition 8.3. *For $n \geq 2$, $t \geq 2$ which satisfy $(*)$ and $(\Omega, \Theta) \in \mathcal{K}_{2t}$, assume that there is $j \in \mathrm{Bir}(\Omega^{[n]})$ which acts on $H^2(\Omega^{[n]}, \mathbb{Z})$ as $-R_D$, for $D = b\widetilde{\Theta} - a\delta$. Then for every $(S, H) \in U$ there exists a deformation family of ihs manifolds $\pi : \mathcal{X} \to B$ over an analytic connected base B, a line bundle $\mathcal{L}$ on $\mathcal{X}$ and points $0, p \in B$ such that*

- $(\pi^{-1}(0), \mathcal{L}_{|_{\pi^{-1}(0)}}) \cong (\Omega^{[n]}, b\widetilde{\Theta} - a\delta)$ *and* $(\pi^{-1}(p), \mathcal{L}_{|_{\pi^{-1}(p)}}) \cong (S^{[n]}, b\widetilde{H} - a\delta)$*;*
- *for every $b \in B \setminus \{0\}$ the pair $(\pi^{-1}(b), \mathcal{L}_{|_{\pi^{-1}(b)}})$ is of the form $(\Sigma^{[n]}, b\widetilde{L} - a\delta)$ for some $(\Sigma, L) \in U$.*

Proof. Since U is path-connected, for any (S_1, H_1), $(S_2, H_2) \in U$ there exists a polarized deformation of K3 surfaces $\pi' : (\mathcal{X}', \mathcal{H}) \to B'$ between (S_1, H_1) and (S_2, H_2) with $B' \subset U$. Passing to the Hilbert schemes, this gives a polarized deformation $\pi : (\mathcal{X}, \mathcal{F}) \to B := {}^n\phi_{2t}(B')$ between $(S_1^{[n]}, b\widetilde{H_1} - a\delta)$ and $(S_2^{[n]}, b\widetilde{H_2} - a\delta)$, whose fibers are all of the form $(\Sigma^{[n]}, b\widetilde{L} - a\delta)$ for $(\Sigma, L) \in U$.

Consider now (Ω, Θ) as in the statement. If D is ample then $(\Omega, \Theta) \in U$, hence the previous argument allows us to conclude. On the other hand, if j is not biregular then D is movable. However, since U is open, we can still find a family of ihs manifolds with a line bundle $\pi : (\mathcal{X}, \mathcal{L}) \to B$ such that the fiber over a point $0 \in B$ is $(\Omega^{[n]}, D)$, while for all $b \in B$, $b \neq 0$ the fiber $\pi^{-1}(b)$ is of the form $(\Sigma^{[n]}, b\widetilde{L} - a\delta)$ for $(\Sigma, L) \in U$. $\square$

Combined with the results of Section 7, the proposition gives a description of a deformation path of ihs manifolds with an involution, from $(\Omega^{[n]}, j)$ to $(S^{[n]}, \sigma)$, where σ is the non-symplectic involution of the Hilbert scheme of n points on a very general $2t$-polarized K3 surface S. This deformation path has the advantage to be explicit, in the sense that it is induced by a deformation of K3 surfaces from Ω to S. Hence it lies in a 19-dimensional subspace of the 20-dimensional moduli space of $(\langle 2d\rangle, \rho)$-polarized ihs manifolds of $K3^{[n]}$-type with the prescribed action on cohomology. This allows us to keep track of the action of the involution on length n subschemes of $K3$ surfaces, along the deformation path.

As an application of Proposition 8.3, we give a geometrical description (up to deformation) of the biregular involutions of Hilbert schemes of very general K3 surfaces of degree $2t = 2((n-1)k^2+1)$. This extends the results of [4, §6] for $n = 2$.

For $n, k \geq 2$ we consider the hyperbolic rank two lattice $M_{n,k}$ whose Gram matrix, with respect to a suitable basis, is

$$\begin{pmatrix} 2n & 2n+k-1 \\ 2n+k-1 & 2n \end{pmatrix}.$$

By [19, Corollary 2.9] there exists a K3 surface Σ whose Picard group is isomorphic to $M_{n,k}$. Let $d_1, d_2 \in \mathrm{Pic}(\Sigma)$ be primitive generators such that $(d_1, d_1) = (d_2, d_2) = 2n$, $(d_1, d_2) = 2n + k - 1$.

Lemma 8.4. *For $n \geq 2$ and $k \geq 3$, there exists $H \in \mathrm{Pic}(\Sigma)$ ample with $H^2 = 2((n-1)k^2+1)$. The Hilbert scheme $\Sigma^{[n]}$ has a birational involution whose action on $H^2(\Sigma^{[n]}, \mathbb{Z})$ is $-R_{(\widetilde{H}-k\delta)}$.*

Proof. Let $L = (k+1)d_1 - d_2 \in \mathrm{Pic}(\Sigma)$, which satisfies $L^2 = 2((n-1)k^2+1)$. Consider the chamber decomposition of $\{x \in \mathrm{Pic}(\Sigma)_{\mathbb{R}} : (x,x) > 0\}$ where the walls are orthogonal to (-2)-classes. We show that d_1, d_2, L are in the interior of the same chamber. Any (-2)-class in $\mathrm{Pic}(\Sigma)$ is of the form $N = (\frac{-(2n+k-1)y \pm \sqrt{\Delta}}{2n})d_1 + yd_2$, with $y \in \mathbb{Z}$ and $\Delta = y^2(k-1)(4n+k-1) - 4n$. A direct computation shows that $(N, d_2) \cdot (N, L) > 0$, hence d_2 and L are in the interior of the same chamber. By convexity, the same holds for $d_1 = \frac{d_2+L}{k+1}$. By [14, Corollary 8.2.9] there exists an isometry $\varrho \in H^2(\Sigma, \mathbb{Z})$ which maps d_1, d_2, L to the ample cone. We denote $D_i = \varrho(d_i)$ and $H = \varrho(L)$.

For $i = 1, 2$, we observe that D_i is very-ample by [26, §8], since there is no $B \in \mathrm{Pic}(\Sigma)$ such that $(B, B) = 0$ and $(B, D_i) \in \{1, 2\}$. As a consequence, we have two distinct embeddings $\varphi_{|D_i|} : \Sigma \hookrightarrow \mathbb{P}^{n+1}$ whose image is a surface of degree $(D_i, D_i) = 2n$. Each of these embeddings gives rise to a Beauville involution $\iota_i \in \mathrm{Bir}(\Sigma^{[n]})$, which acts on $H^2(\Sigma^{[n]}, \mathbb{Z})$ as the reflection fixing the line spanned by $\widetilde{D_i} - \delta$. Now we take $j = \iota_1\iota_2\iota_1 \in \mathrm{Bir}(\Sigma^{[n]})$: its invariant lattice is generated by $\iota_1^*(\widetilde{D_2} - \delta) = \widetilde{H} - k\delta$. $\square$

Recall that for $t = (n-1)k^2+1$ the minimal solution of $(n-1)X^2 - tY^2 = -1$ is $(k, 1)$. Thus, from the lemma and Proposition 8.3 we obtain the following geometric description.

Proposition 8.5. *For $n \geq 2$ and $k \geq \frac{n+3}{2}$, the non-natural, non-symplectic involution of the Hilbert scheme of n points on a K3 surface of Picard rank one in $\mathcal{K}_{2((n-1)k^2+1)}$ can be described, up to a deformation as in Proposition 8.3, as the* *conjugation of a birational Beauville involution with respect to another birational Beauville involution.*

Remark 8.6.
- The proposition holds for all $k \geq 3$ such that the birational involution which generates $\mathrm{Bir}(S^{[n]})$, for S a $2((n-1)k^2+1)$-polarized K3 surface of Picard rank one, is biregular (cfr. Conjecture 4.9).
- For $k = 2$ (i.e. $t = 4n-3$) the wall orthogonal to the (-2)-class $d_1 - d_2$ separates d_1 and d_2, therefore the construction of Lemma 8.4 cannot be performed. We already know that this value of t is n-irregular (Lemma 4.8), hence Proposition 8.3 is not applicable.

For $n = 2$ both Beauville involutions in Proposition 8.5 are biregular, as one can show that there are no lines on the quartic surfaces $\varphi_{|D_i|}(\Sigma) \subset \mathbb{P}^3$. For k odd, this construction is [4, Theorem 6.1].

References

1. A. Apostolov, *Moduli spaces of polarized irreducible symplectic manifolds are not necessarily connected*, Ann. Inst. Fourier (Grenoble) **64** (2014), no. 1, 189–202. MR 3330546
2. A. Bayer and E. Macrì, *MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations*, Invent. Math. **198** (2014), no. 3, 505–590. MR 3279532
3. A. Beauville, *Some remarks on Kähler manifolds with* $c_1 = 0$, Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39, Birkhäuser Boston, Boston, MA, 1983, pp. 1–26. MR 728605
4. S. Boissière, An. Cattaneo, D. G. Markushevich, and A. Sarti, *On the antisymplectic involutions of the Hilbert square of a K3 surface*, Izv. Ross. Akad. Nauk Ser. Mat. **83** (2019), no. 4, 86–99. MR 3985691
5. S. Boissière, An. Cattaneo, M. Nieper-Wisskirchen, and A. Sarti, *The automorphism group of the Hilbert scheme of two points on a generic projective K3 surface*, K3 surfaces and their moduli, Progr. Math., vol. 315, Birkhäuser/Springer, [Cham], 2016, pp. 1–15. MR 3524162
6. S. Boissière and A. Sarti, *A note on automorphisms and birational transformations of holomorphic symplectic manifolds*, Proc. Amer. Math. Soc. **140** (2012), no. 12, 4053–4062. MR 2957195
7. C. Camere, Al. Cattaneo, and An. Cattaneo, *Non-symplectic involutions of manifolds of* $K3^{[n]}$*-type*, Nagoya Math. J.; doi:10.1017/nmj.2019.43 (2020).
8. Al. Cattaneo, *Automorphisms of Hilbert schemes of points on a generic projective K3 surface*, Math. Nachr. **292** (2019), no. 10, 2137–2152.
9. G. Chrystal, *Algebra: An elementary text-book for the higher classes of secondary schools and for colleges, Part II*, Chelsea Publishing Co., New York, 1961. MR 0121327
10. O. Debarre, *Hyperkähler manifolds*, arXiv:1810.02087 (2018).
11. O. Debarre and E. Macrì, *On the period map for polarized hyperkähler fourfolds*, Int. Math. Res. Not. IMRN (2019), no. 22, 6887–6923. MR 4032178
12. V. Gritsenko, K. Hulek, and G. K. Sankaran, *The Kodaira dimension of the moduli of K3 surfaces*, Invent. Math. **169** (2007), no. 3, 519–567. MR 2336040
13. ———, *Moduli spaces of irreducible symplectic manifolds*, Compos. Math. **146** (2010), no. 2, 404–434. MR 2601632
14. D. Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016. MR 3586372
15. M. Joumaah, *Non-symplectic involutions of irreducible symplectic manifolds of* $K3^{[n]}$*-type*, Math. Z. **283** (2016), no. 3-4, 761–790. MR 3519981
16. E. Markman, *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, Complex and differential geometry, Springer Proc. Math., vol. 8, Springer, Heidelberg, 2011, pp. 257–322. MR 2964480
17. C. Meachan, G. Mongardi, and K. Yoshioka, *Derived equivalent Hilbert schemes of points on K3 surfaces which are not birational*, Math. Z. **294** (2020), no. 3-4, 871–880. MR 4074026
18. G. Mongardi, *Towards a classification of symplectic automorphisms on manifolds of* $K3^{[n]}$ *type*, Math. Z. **282** (2016), no. 3-4, 651–662. MR 3473636

19. D. R. Morrison, *On $K3$ surfaces with large Picard number*, Invent. Math. **75** (1984), no. 1, 105–121. MR 728142
20. T. Nagell, *Introduction to Number Theory*, John Wiley & Sons, Inc., New York; Almqvist & Wiksell, Stockholm, 1951. MR 0043111
21. V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238. MR 525944
22. K. G. O'Grady, *Involutions and linear systems on holomorphic symplectic manifolds*, Geom. Funct. Anal. **15** (2005), no. 6, 1223–1274. MR 2221247
23. K. Oguiso, *K3 surfaces via almost-primes*, Math. Res. Lett. **9** (2002), no. 1, 47–63. MR 1892313
24. ———, *Automorphism groups of Calabi-Yau manifolds of Picard number 2*, J. Algebraic Geom. **23** (2014), no. 4, 775–795. MR 3263669
25. D. Ploog, *Equivariant autoequivalences for finite group actions*, Adv. Math. **216** (2007), no. 1, 62–74. MR 2353249
26. B. Saint-Donat, *Projective models of $K3$ surfaces*, Amer. J. Math. **96** (1974), 602–639. MR 0364263
27. R. Zuffetti, *Strongly ambiguous Hilbert squares of projective K3 surfaces with Picard number one*, Rendiconti Sem. Mat. Univ. Pol. Torino **77** (2019), no. 1, 113–130.

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