Computing the set of asymptotic critical values of polynomial mappings from smooth algebraic sets

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Abstract

Let $\boldsymbol{f} = (f_1, \ldots, f_p) \in \mathbb{Q}[z_1, \ldots, z_n]$ be a polynomial tuple. Define the polynomial mapping $\boldsymbol{f} : X \to \mathbb{C}^p$, where X is a smooth algebraic set defined by the simultaneous vanishing of the reduced regular sequence $g_1, \ldots, g_m$, with $m + p \leq n$. Let $d = \max(\deg f_1, \ldots, \deg f_p, \deg g_1, \ldots, \deg g_m)$, $\mathrm{d}\boldsymbol{f}$ be the differential of $\boldsymbol{f}$ and κ be the function measuring the distance of a linear operator to the set of singular linear operators from $\mathbb{C}^n$ to $\mathbb{C}^p$. We consider the problem of computing the set of asymptotic critical values of $\boldsymbol{f}$. This is the set of values c in the target space of $\boldsymbol{f}$ such that there exists a sequence of points $(\mathbf{x}_i)_{i\in\mathbb{N}}$ tending to ∞ for which $\boldsymbol{f}(\mathbf{x}_i)$ tends to c and $\|\mathbf{x}_i\|\kappa(\mathrm{d}\boldsymbol{f}(\mathbf{x}_i))$ tends to 0 when i tends to infinity.

The union of the classical and asymptotic critical values contains the so-called bifurcation set of a polynomial mapping. Thus, by computing both the critical values and the asymptotic critical values, one can utilise generalisations of Ehresmann's fibration theorem in non-proper settings for applications in polynomial optimisation and computational real algebraic geometry.

We design new efficient algorithms for computing the set of asymptotic critical values of a polynomial mapping restricted to a smooth algebraic set. By investigating the degree of the objects constructed in our algorithms, we give the first bound on the degree of this set of values of pD, where $D = d^{n-p-1} \sum_{i=0}^{p+1} \binom{n+p-1}{m+2p+i} d^i$. We also give the first complexity analysis of this problem, showing that it requires at most $O^\sim\left(p(p+1)D^{p+5} + (n+m+2p)^{d+3}D^{p+4}\right)$ operations in the base field. Moreover, in the special case $p = 1$, we give another complexity estimate of $O^\sim((n+m+2)^{d+3}D^5)$ arithmetic operations.

Additionally, we show how to apply these algorithms to polynomial optimisation problems and the problem of computing sample points per connected component of a semi-algebraic set defined by a single inequality/inequation.

We provide implementations of our algorithms and use them to test their practical capabilities. We show that our algorithms significantly outperform the current state-of-the-art algorithms by tackling previously out of reach benchmark examples.

Keywords: Asymptotic critical values, Polynomial optimisation, Gröbner bases

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1. Introduction

Definition of asymptotic critical values. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and let $\boldsymbol{f} = (f_1, \dots, f_p) \in \mathbb{K}[x_1, \dots, x_n]^p$ be a polynomial mapping. Let $\boldsymbol{g} = (g_1, \dots, g_m)$ be a reduced regular sequence such that the variety $X = \mathbf{V}(g_1, \dots, g_m) \subset \mathbb{C}^n$ is smooth. We consider the polynomial mapping

$$\boldsymbol{f} : \boldsymbol{x} = (x_1, \dots, x_n) \in X \mapsto (f_1(x_1, \dots, x_n), \dots, f_p(x_1, \dots, x_n)) \in \mathbb{K}^p.$$

We assume that $n \geq m + p$ and that this mapping is dominant, so that the image of $\boldsymbol{f}$ is dense in $\mathbb{C}^p$. For ease of notation, we shall denote $x_1, \dots, x_n$ by $\boldsymbol{x}$, $z_1, \dots, z_n$ by $\mathbf{z}$ and, for the value of the polynomial mapping $\boldsymbol{f}$, $c_1, \dots, c_p$ by $\mathbf{c}$. Denote by $\mathrm{d}\boldsymbol{f}$ the differential of the mapping $\boldsymbol{f}$ and, for a given point $\boldsymbol{x} \in X$, $\mathrm{d}\boldsymbol{f}(\boldsymbol{x})$ the differential of $\boldsymbol{f}$ at $\boldsymbol{x}$, a linear map from the tangent space $T_{\boldsymbol{x}}X$ of X at $\boldsymbol{x}$ to the tangent space $T_{\boldsymbol{f}(\boldsymbol{x})}\mathbb{K}^p$ of $\mathbb{K}^p$ at $\boldsymbol{f}(\boldsymbol{x})$.

Denote by $L(\mathbb{K}^n, \mathbb{K}^p)$ the space of linear mappings from $\mathbb{K}^n$ to $\mathbb{K}^p$ and by Σ the singular set of $L(\mathbb{K}^n, \mathbb{K}^p)$. First defined in [22], denote by ν the distance of an operator $A \in L(\mathbb{K}^n, \mathbb{K}^p)$ to the set of singular operators: [18, Proposition 2.2]

$$\nu(A) = \mathrm{dist}(A, \Sigma) = \inf_{B \in \Sigma} \|A - B\|.$$

Then, the set of asymptotic critical values of the polynomial mapping $\boldsymbol{f}$ restricted to the algebraic set X is defined as follows:

$$K_\infty(\boldsymbol{f}) = \{\mathbf{c} \in \mathbb{C}^p \mid \exists (\boldsymbol{x}_t)_{t \in \mathbb{N}} \subset X \text{ s.t.} \|\boldsymbol{x}_t\| \to \infty, \boldsymbol{f}(\boldsymbol{x}_t) \to c \text{ and } \|\boldsymbol{x}_t\| \nu(\mathrm{d}\boldsymbol{f}(\boldsymbol{x}_t)) \to 0\}$$

Let $\mathrm{jac}(\boldsymbol{f}, \boldsymbol{g})$ be the Jacobian matrix associated to the mapping $(f_1, \dots, f_p, g_1, \dots, g_m)$,

$$\mathrm{jac}(\boldsymbol{f}, \boldsymbol{g}) = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial f_p}{\partial z_1} & \cdots & \frac{\partial f_p}{\partial z_n} \\ \frac{\partial g_1}{\partial z_1} & \cdots & \frac{\partial g_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial z_1} & \cdots & \frac{\partial g_m}{\partial z_n} \end{bmatrix}.$$

For $1 \leq j \leq p$, denote by $\mathrm{jac}(\boldsymbol{f}, \boldsymbol{g})^{[j]}$ the submatrix of $\mathrm{jac}(\boldsymbol{f}, \boldsymbol{g})$ obtained by removing the jth row. Note that we only ever remove one of the first p rows. Denote by N_j the kernel of the matrix $\mathrm{jac}(\boldsymbol{f}, \boldsymbol{g})^{[j]}$. In the special case $(p, m) = (1, 0)$, the matrix has no entries, so by convention we say its kernel is $\mathbb{K}^n$. Let $w_j(z)$ be the restriction of the differential $\mathrm{d}f_j$ to the kernel N_j.

Following [15, Proposition 2.3], for a linear subspace $H \subset \mathbb{K}^n$ defined by vectors $B_1, \dots, B_m$, let $F \in L(H, \mathbb{K}^p)$ be a linear map represented by a matrix with rows $(A_1, \dots, A_p) \subset \mathbb{K}^n$. We consider the so-called Kuo distance defined by

$$\kappa(F) = \min_{1 \leq j \leq p} \mathrm{dist}\left(A_i, \mathrm{span}((A_j)_{i \neq j}, (B_k)_{1 \leq k \leq m})\right).$$

In particular, for $z \in X$ we have that

$$\kappa(\mathrm{d}\boldsymbol{f}(z)) = \min_{1 \leq j \leq p} ||w_j(z)||.$$

By [15, Corollary 2.1], the function ν is equivalent to the Kuo distance. Hence, an equivalent definition of the set of asymptotic critical values, the one that we shall primarily use, is the following:

$$K_\infty(\boldsymbol{f}) = \{\mathbf{c} \in \mathbb{C}^p \mid \exists (\boldsymbol{x}_t)_{t\in\mathbb{N}} \subset X \text{ s.t.} \|\boldsymbol{x}_t\| \to \infty, \boldsymbol{f}(\boldsymbol{x}_t) \to c \text{ and } \|\boldsymbol{x}_t\|\kappa(\mathrm{d}\boldsymbol{f}(\boldsymbol{x}_t)) \to 0\}.$$

Restriction to a proper algebraic subset of $\mathbb{C}^n$ can affect the asymptotic critical values of a polynomial mapping in subtle ways. For example, a path that leads to an asymptotic critical value in the unrestricted setting may not satisfy the Jacobian condition in the definition of $K_\infty(\boldsymbol{f})$. However, restricting $\boldsymbol{f}$ to an algebraic set that contains this path and thereby adding rows to said Jacobian, can result in this path now satisfying all the above conditions.

Example 1. *Let $f = z_1^2 + (z_1z_2 - 1)^2$. First we consider the global case, $f : \mathbb{C}^2 \to \mathbb{C}$. We shall show that $0 \in K_\infty(f)$. The gradient is equal to*

$$\mathrm{d}f = (2z_1 + 2z_2(z_1z_2 - 1), 2z_1(z_1z_2 - 1)).$$

Then, consider the path $z(t) = (t, 1/t - t)$ as $t \to 0$. We see that $\|z(t)\| \to \infty$ and $f(z(t)) = t^2 + t^4 \to 0$. Furthermore, we have that $\mathrm{d}f(z(t)) = (2t^3, -2t^3)$. Since $p = 1$ and $m = 0$, the Kuo distance κ can be simply replaced by the 2-norm. Hence,

$$\|z(t)\|^2\|\mathrm{d}f(z(t))\|^2 = 8t^6\left(t^2 + \left(\frac{1}{t} - t\right)^2\right) \to 0,$$

and so 0 is an asymptotic critical value of f.

Note that the path $y(t) = (t, 1/t)$ satisfies the first two conditions for a path towards the asymptotic critical value 0, $\|y(t)\| \to \infty$ and $f(y(t)) \to 0$ as $t \to 0$. However, $\mathrm{d}f(y(t)) = (2t, 0)$ and so

$$\|y(t)\|^2\|\mathrm{d}f(y(t))\|^2 = 4t^2\left(t^2 + \frac{1}{t^2}\right) = 4t^4 + 4 \to 4.$$

Now consider algebraic set $X = \mathbf{V}(g) = \mathbf{V}(z_1z_2 - 1)$ and the restricted polynomial map $f : X \to \mathbb{C}$. Then, consider the Jacobian

$$\mathrm{jac}(f, g) = \begin{bmatrix} 2z_1 + 2z_2(z_1z_2 - 1) & 2z_1(z_1z_2 - 1) \\ z_2 & z_1 \end{bmatrix}.$$

Let N_1 be the kernel of $\mathrm{d}g$, then $w_1 = \mathrm{d}f|_{N_j}$ and $\kappa(\mathrm{d}f) = \|w_1\|$. Choose a basis for N_1, $(-z_1, z_2)$. Then,

$$w_1 = -2z_1(2z_1 + 2z_2(z_1z_2 - 1)) + z_2(2z_1(z_1z_2 - 1)) = 2z_1z_2 - 2z_1^2z_2^2 - 4z_1^2.$$

Clearly, the path $y(t)$ is in the set $\mathbf{V}(z_1z_2 - 1)$ for all $t > 0$, so we have $\|y(t)\| \to \infty$ and $f(y(t)) \to 0$ as $t \to 0$ but now we also have

$$\|y(t)\|^2\kappa(\mathrm{d}f(y(t)))^2 = \|y(t)\|^2\|w_1(y(t))\|^2 = 16t^4\left(t^2 + \frac{1}{t^2}\right) \to 0.$$

Motivation. Denote by $K_0(\boldsymbol{f})$, the set of critical values of $\boldsymbol{f}$

$$K_0(\boldsymbol{f}) = \{\mathbf{c} \in \mathbb{C}^p \mid \exists \boldsymbol{x} \in X \text{ s.t. } \boldsymbol{f}(\boldsymbol{x}) = c \text{ and } \operatorname{rank}(\operatorname{jac}(\boldsymbol{f}, \boldsymbol{g})(\boldsymbol{x})) < m + p\}\,.$$

The set of generalised critical values is thus defined to be the union of the classical critical values and the asymptotic critical values, $K(\boldsymbol{f}) = K_0(\boldsymbol{f}) \cup K_\infty(\boldsymbol{f})$. In [22], the author proved that this set contains the so-called bifurcation set of $\boldsymbol{f}$. Essentially, this provides a generalisation of Ehresmann's fibration theorem to non-proper settings. Thus,

$$\boldsymbol{f} : X \setminus f^{-1}(K(\boldsymbol{f})) \to \mathbb{K}^p \setminus K(\boldsymbol{f})$$

is a locally trivial fibration which by definition, means that for all connected open sets $U \subset \mathbb{K}^p \setminus K(\boldsymbol{f})$, for all $y \in U$ there exists a diffeomorphism φ such that the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{f}^{-1}(y) \times U & \xrightarrow{\varphi} & \boldsymbol{f}^{-1}(U) \\ & \searrow^{\pi} & \downarrow \boldsymbol{f} \\ & & U \end{array}$$

where π is the projection map onto U [15, Theorem 3.1]. However, for this to be computationally meaningful, we require the set $K(\boldsymbol{f})$ not to be dense in $\mathbb{K}^p$. It is well known that by Bertini's algebraic version of Sard's theorem, the set $K_0(\boldsymbol{f})$ has codimension at least one in $\mathbb{K}^p$. Crucially, it has also been shown that the set of asymptotic critical values satisfies a generalised Sard's theorem [15, Theorem 3.3].

Therefore, the computation of the generalised critical values for effective uses in real algebraic geometry is appealing. Their fibration property has been capitalised upon in [13, 24] to design algorithms for

- exact polynomial optimisation (i.e. computing the minimal polynomial of the infimum of the map $x \to f(x)$ restricted to $X \cap \mathbb{R}^n$ and an isolating interval for this infimum),
- computing sample points for each connected component of a semi-algebraic set defined by a single inequality.

Prior works. Computing the set of critical values of a polynomial mapping restricted to an algebraic set is classical. By the Jacobian criterion under the assumption that X is smooth and $\boldsymbol{g}$ is a reduced regular sequence, one may consider the algebraic set defined by the intersection of X with the variety defined by the maximal minors of $\operatorname{jac}(\boldsymbol{f}, \boldsymbol{g})$ to find the critical points of $\boldsymbol{f}$. Then, the set $K_0(\boldsymbol{f})$ is equal to the set of values of $\boldsymbol{f}$ at these points [8, Corollary 16.20].

As far as we are aware, the first work towards the computation of the asymptotic critical values of a polynomial mapping was given in [18]. This is based on a geometric characterisation of $K_\infty(\boldsymbol{f})$ that allows one to construct an algebraic set of codimension at least one in $\mathbb{C}^p$ that contains the asymptotic critical values. Then, one can construct polynomials defining this algebraic set by using algorithms that compute elimination ideals in polynomial rings, such as Gröbner basis based algorithms. Note that the authors of this paper only consider the global setting. Later, the authors of [15] proposed an algorithm for computing the generalised critical values of a polynomial mapping restricted to an algebraic set. This follows a similar schematic of defining algebraic sets, considering their intersections with linear hyperspaces and projecting onto the target space. However, this algorithm constructs $(p(m+p))^{\binom{n}{m+p}}$ locally closed sets in $\mathbb{C}^{(n+1)\binom{n}{m+p}+p+n}$ before projecting onto $\mathbb{C}^p$ making the algorithm impractical. Furthermore, a complexity analysis for this algorithm is lacking.

Several attempts to improve this algorithmic pattern have been made in the global case with $p = 1$. We mention [24] in which the author makes the connetion between generalised critical values and properties of polar varieties. This connection is exploited in [16] where the authors build rational arcs that reach all the generalised critical values of a polynomial. Moreover, in [17], the authors make a distinction between asymptotic critical values, detecting those that are found non-trivially, meaning away from the critical locus of the polynomial, something not covered in this paper.

Main results. By adapting the results of [18, Section 4], building a geometric characterisation of $K_\infty(\boldsymbol{f})$ using Lagrange multipliers, we develop efficient algorithms for computing asymptotic critical values under the restriction to a smooth algebraic set. We introduce an element of randomisation to avoid some combinatorial steps in the algorithm designed in [15]. Next, with a geometric result, we reduce the computation of $K_\infty(\boldsymbol{f})$ to intersecting the Zariski closure of some locally closed subset of $\mathbb{C}^{n+m+2p}$ with a linear affine subspace of codimension 2 such that the projection onto the target space of $\boldsymbol{f}$ of this intersection contains $K_\infty(\boldsymbol{f})$. Then, by taking advantage of the multi-homogeneous structure of the objects defined in this algorithm, we give a bound on the degree of the asymptotic critical values.

Theorem 2. *Let $\boldsymbol{f} = (f_1, \ldots, f_p) \in \mathbb{K}[\mathbf{z}]^p$ be a dominant polynomial mapping from a smooth algebraic set defined by a reduced regular sequence $\boldsymbol{g} = (g_1, \ldots, g_m)$. Let $d = \max(\deg f_1, \ldots, \deg f_p, \deg g_1, \ldots, \deg g_m)$. Then, the asymptotic critical values of $\boldsymbol{f}$ are contained in a hypersurface of degree at most*

$$pd^{n-p-1} \sum_{i=0}^{p+1} \binom{n+p-1}{m+2p-i} d^i.$$

We note that in many cases, the bound given in Theorem 2, combined with the bound on the degree of the critical values in [10, Corollary 2] in the $p = 1$ case, is less than the bound given on the degree of the generalised critical values in [15, Theorem 4.1]. However, for certain values of the parameters m, p and n, the latter bound is actually smaller. This is discussed in Section 9.

While in practice, and in our experiments, Gröbner bases are the tool of choice for performing the algebraic elimination routines necessary in our algorithms, we study their complexity by utilising the geometric resolution algorithm given in [12]. We recall the "soft-Oh" notation: $f(n) \in \tilde{O}(g(n))$ means that $f(n) \in g(n) \log^{O(1)}(3+g(n))$, see also [11, Chapter 25, Section 7].

We now give our first complexity result. The following is for the special case $p = 1$, which is of particular importance for many applications such as polynomial optimisation.

Theorem 3. *Let $f \in \mathbb{K}[\mathbf{z}]$ be a polynomial from a smooth algebraic set defined by a reduced regular sequence $\boldsymbol{g} = (g_1, \ldots, g_m)$. Let $d = \max(\deg f, \deg g_1, \ldots, \deg g_m)$ and*

$D = d^{n-2}\sum_{i=0}^{p+1}\binom{n}{m+2-i}d^i$. *There exists an algorithm which, on input* $f, \boldsymbol{g}$*, outputs a non-zero polynomial* $H \in \mathbb{K}[c]$ *such that* $K_\infty(\boldsymbol{f}) \subset \mathbf{V}(H)$ *using at most*

$$O^\sim((n+m+2)^{d+3}D^5)$$

arithmetic operations in $\mathbb{K}$.

The $p = 1$ case relies on resultant computation for a smaller complexity bound. However, in the $p > 1$ case, we must change our methodology. We use the FGLM algorithm [9] which has dominant complexity in our algorithm, to arrive at the following result.

Theorem 4. *Let* $\boldsymbol{f} = (f_1, \ldots, f_p) \in \mathbb{K}[\mathbf{z}]^p$ *be a dominant polynomial mapping from a smooth algebraic set defined by a reduced regular sequence* $\boldsymbol{g} = (g_1, \ldots, g_m)$*. Let* $d = \max(\deg f_1, \ldots, \deg f_p, \deg g_1, \ldots, \deg g_m)$*. Let* $D = d^{n-p-1}\sum_{i=0}^{p+1}\binom{n+p-1}{m+2p-i}d^i$*. There exists an algorithm which, on input* $\boldsymbol{f}$ *and* $\boldsymbol{g}$*, outputs* p *finite lists of non-zero polynomials* $G_i \subset \mathbb{K}[\mathbf{c}]$ *such that* $K_\infty(\boldsymbol{f}) \subset (\mathbf{V}(G_1) \cup \cdots \cup \mathbf{V}(G_p)) \subsetneq \mathbb{C}^p$ *using at most*

$$O^\sim\left(p(p+1)D^{p+5} + (n+m+2p)^{d+3}D^{p+4}\right)$$

arithmetic operations in $\mathbb{K}$.

Furthermore, we have implemented all the algorithms given in this paper in the Maple [19] computer algebra system. For the Gröbner basis computations, we rely on the Gröbner package in Maple. Testing these implementations for a wide range of benchmark examples, we illustrate that our algorithms significantly outperform the state-of-the-art.

Structure of the paper. In Section 2, we develop the geometric characterisation of the asymptotic critical values given in [18] to the setting of restrictions to smooth algebraic sets. Then, we explore an interpretation of this characterisation in terms of Lagrange multipliers that leads directly to an algorithm for computing the set of asymptotic critical values. In Section 3, we prove our main geometric result, upon which the efficiency of our algorithms relies. Then, in Section 4, we apply the results of the previous two sections to introduce two elements of randomisation in order to design new algorithms more efficient than the state-of-the-art. In Sections 5 and 6, we prove our main results by analysing our new algorithms. An additional algorithm, deriving from a different interpretation of the geometric characterisation of the asymptotic critical values is presented in Section 7. Finally, in Section 9, we compare all the algorithms given in this paper in terms of time. Furthermore, we compare our degree result to the bound given in [15, Theorem 4.1] and to the true number of asymptotic critical values for a set of benchmark examples.

2. Preliminaries

Let $\boldsymbol{f} : X \to \mathbb{C}^p$ be a dominant polynomial mapping defined from an algebraic set $X = \mathbf{V}(\boldsymbol{g})$ where X is smooth and is defined by $\boldsymbol{g}$, a reduced regular sequence. By [15, Theorem 3.3] the set of asymptotic critical values of $\boldsymbol{f}$ has codimension at least one in $\mathbb{C}^p$. The aim of this section is to define an algebraic set containing $K_\infty(\boldsymbol{f})$ that also has codimension at least one in $\mathbb{C}^p$.

To access the asymptotic behaviour algebraically, we utilise the following transformation that sends $z_s = 0$ to ∞:

$$\tau_s(z) = \left(\frac{z_1}{z_s}, \dots, \frac{z_{s-1}}{z_s}, \frac{1}{z_s}, \frac{z_{s+1}}{z_s}, \dots, \frac{z_n}{z_s}\right).$$

For each choice of $s = 1, \dots, n$, $j = 1, \dots, p$ and point $\boldsymbol{x} \in X$, let $W_s^j(\boldsymbol{x})$ be the graph of $x_s w_j(\boldsymbol{x})$, a point in the Grassmannian of linear subspaces of $\mathbb{C}^n \times \mathbb{C}$ that are of dimension $n-p-m+1$, denoted by $\mathbb{G}_{n-p-m+1}(\mathbb{C}^n \times \mathbb{C})$. We remark that this point is well defined for $\boldsymbol{x} \in X$ such that the kernel of $\mathrm{jac}(\boldsymbol{f}, \boldsymbol{g})^{[j]}$ has dimension $n-p-m+1$. Therefore, since X is a smooth affine variety and the mapping $\boldsymbol{f}$ is dominant, this is well-defined outside of a proper Zariski closed subset of X.

Then, define the rational mapping

$$\begin{aligned} M_s^j(\boldsymbol{f}) : X \setminus \{z_s = 0\} &\to \mathbb{C}^p \times \mathbb{G}_{n-p-m+1}(\mathbb{C}^n \times \mathbb{C}), \\ z &\mapsto (\boldsymbol{f}(\tau_s(z)), W_s^j(\tau_s(z))). \end{aligned}$$

Let $\Lambda = \mathbb{G}_{n-p-m+1}(\mathbb{C}^n \times 0)$. This is the set of $(n-p-m+1)$-dimensional graphs of linear maps from $\mathbb{C}^n$ to $\mathbb{C}$ that are identically the zero map.

$$L_s^j(\boldsymbol{f}) = \overline{\mathrm{graph}\, M_s^j(\boldsymbol{f})} \cap (\{z \in X | z_s = 0\} \times \mathbb{C}^p \times \Lambda).$$

Define $\pi : X \times \mathbb{C}^p \times \mathbb{G}_{n-k+1}(\mathbb{C}^n \times \mathbb{C}) \to \mathbb{C}^p$ to be the projection map and take $K_s^j(\boldsymbol{f}) = \pi(L_s^j(\boldsymbol{f}))$. We shall prove in this section that $L_s^j(\boldsymbol{f})$ is an algebraic set.

This framework suggests looking at each coordinate tending to infinity separately. Instead, we shall introduce a probabilistic element that allows one to investigate every coordinate tending to infinity at once.

Denote by $\mathrm{GL}_n(\mathbb{K})$ the group of $n \times n$ invertible matrices with entries in $\mathbb{K}$. For a group element $A \in \mathrm{GL}_n(\mathbb{K})$, consider the polynomial mapping $\boldsymbol{f}^A$ given by $\boldsymbol{f}^A(z) = \boldsymbol{f}(Az)$ restricted to the algebraic set $\mathbf{V}(\boldsymbol{g}^A)$ where $\boldsymbol{g}^A(z) = \boldsymbol{g}(Az)$. The following lemma shows that $K_\infty(\boldsymbol{f}^A) = K_\infty(\boldsymbol{f})$.

Lemma 5. *Let $\boldsymbol{f} : X \to \mathbb{C}^p$ be a polynomial mapping from an algebraic set X. Let $A \in \mathrm{GL}_n(\mathbb{K})$ be an invertible matrix and define the polynomial mapping*

$$\boldsymbol{f}^A : z \in A^{-1}X \to \boldsymbol{f}(Az) \in \mathbb{C}^p.$$

Then, $K_\infty(\boldsymbol{f}) = K_\infty(\boldsymbol{f}^A)$.

Proof. Let $\mathbf{c} \in K_\infty(\boldsymbol{f})$ be an asymptotic critical value with a path $z(t) \subset X$ such that $\|z(t)\| \to \infty, \boldsymbol{f}(z(t)) \to \mathbf{c}$ and $\|z(t)\|\nu(\mathrm{d}\boldsymbol{f}(z(t))) \to 0$ as $t \to \infty$. Then, for a given invertible matrix $A \in \mathrm{GL}_n(\mathbb{K})$, define the path $y(t) = A^{-1}z(t)$. Clearly, as $t \to \infty$, $\|y(t)\| \to \infty$ and $f^A(y(t)) \to \mathbf{c}$. Then, to prove that $\mathbf{c} \in K_\infty(\boldsymbol{f}^A)$, it remains to show that $\|y(t)\|\nu(\mathrm{d}\boldsymbol{f}^A(y(t))) \to 0$. Firstly, $\|y(t)\| \leq \|A^{-1}\|\|z(t)\|$. Moreover, by the chain rule,

$$\mathrm{d}\boldsymbol{f}^A(y(t)) = \mathrm{d}\boldsymbol{f}^A(A^{-1}z(t)) = \mathrm{d}\boldsymbol{f}(z(t))A.$$

Then, since A is an invertible matrix and since the Rabier distance is the distance to the set of singular operators [18, Proposition 2.2], we have that $\|z(t)\|\nu(\mathrm{d}\boldsymbol{f}(z(t))A) \to 0$ and hence $\mathbf{c} \in K_\infty(\boldsymbol{f}^A)$. The reverse direction holds with the same argument. □

The advantage of the transformation τ_s is that it allows us to give an algebraic description of the sets $L_s^j(\boldsymbol{f})$. By elimination of variables, we are then able to compute the Zariski-closure of the projection of $L_s^j(\boldsymbol{f})$ on the **c**-space. This gives, in general, a superset of the asymptotic critical values of codimension at least 1 in $\mathbb{C}^p$. Moreover, in the special case $p = 1$, of particular interest for many applications such as polynomial optimisation, this inclusion becomes equality.

First, we give a lemma that will allow a Lagrange multiplier interpretation of the Kuo distance.

Lemma 6. *Consider the linear maps $F : \mathbb{C}^n \to \mathbb{C}^m$, defined by vectors $(F_1, \ldots, F_m) \subset \mathbb{C}^n$, and $L : \mathbb{C}^n \to \mathbb{C}$ with $n \geq m$. defined, with a slight abuse of notation, by the vector L. Then,*

$$L \in \operatorname{span}(F_1, \ldots, F_m) \iff \ker F \subset \ker L.$$

Proof. We shall prove this by double inclusion. Firstly, suppose that $L \in \operatorname{span}(F_1, \ldots, F_m)$. Then, let $x \in \ker F$ so that $F_i \cdot x = 0$ for all i. Thus, there exists $y \in \mathbb{C}^m$ such that for all x we have $L \cdot x = \sum_{i=1}^m y_i(F_i \cdot x) = 0$. Hence, $x \in \ker L$ and $\ker F \subset \ker L$.

On the other hand, suppose that $\ker F \subset \ker L$. Consider a trivial extension of F, the linear map $G : \mathbb{C}^n \to \mathbb{C}^n$, defined by $(F_1, \ldots, F_n)$ where $F_{m+1} = \cdots = F_n \equiv 0$. Then, consider the induced isomorphism $\tilde{G} : \mathbb{C}^n/\ker F \to \operatorname{im} G$ and the linear map $\tilde{L} : \mathbb{C}^n/\ker F \to \mathbb{C}$. By extending a basis of $\operatorname{im} G \cong \mathbb{C}^m$ to a basis of $\mathbb{C}^n$, we may extend $\tilde{L}$ to $\hat{L} : \mathbb{C}^n \to \mathbb{C}$ so that

$$\hat{L} \circ G = \hat{L}|_{\operatorname{im} G} \circ G = \tilde{L} \circ \tilde{G}^{-1} \circ G = \tilde{L} \circ \pi = L,$$

where π is the projection map from $\mathbb{C}^n$ to $\mathbb{C}^n/\ker F$. Therefore, for any $x \in \mathbb{C}^n$ we have

$$L \cdot x = L(x) = \sum_{i=1}^{n} y_i(F_i \cdot x) = \sum_{i=1}^{m} y_i(F_i \cdot x)$$

for some $y \in \mathbb{C}^n$ and so $L \in \operatorname{span}(F_1, \ldots, F_m)$. □

In the algorithms presented in this paper, we shall derive polynomials whose simultaneous vanishing set is the Zariski closure of the graph of the set $M_s^j(\boldsymbol{f}^A)$. For this purpose, we introduce $m + p - 1$ new variables $\boldsymbol{\lambda}$, that will be Lagrange multipliers. Additionally, since we consider a graph we also have n new variables $\mathbf{u}$ and p new variables $\mathbf{c}$ that will correspond to the values of the the map $M_s^j(\boldsymbol{f}^A)$.

For a reduced rational function, φ/θ, we define the function numer by $\operatorname{numer}(\varphi/\theta) = \varphi$. For a vector of reduced rational functions, $(\varphi_1/\theta_1, \ldots, \varphi_m/\theta_m)$, we extend the function numer so that $\operatorname{numer}(\varphi_1/\theta_1, \ldots, \varphi_m/\theta_m) = (\varphi_1, \ldots, \varphi_m)$.

Lemma 7. *Let $\boldsymbol{f} \in \mathbb{K}[\mathbf{z}]^p$ be a dominant polynomial mapping with domain a smooth algebraic set X defined by a reduced, regular sequence $(g_1, \ldots, g_m)$ and let $A \in \mathrm{GL}_n(\mathbb{K})$. Then, there exist polynomials $h_1, \ldots, h_{n+m+p} \in \mathbb{K}[\mathbf{z}, \mathbf{c}, \mathbf{u}, \boldsymbol{\lambda}]$ such that*

$$\overline{\operatorname{graph} M_s^j(\boldsymbol{f}^A)} = \overline{\mathbf{V}(h_1, \ldots, h_{n+m+p}) \setminus \mathbf{V}(z_s)},$$

$$L_s^j(\boldsymbol{f}^A) = \overline{\operatorname{graph} M_s^j(\boldsymbol{f}^A)} \cap \mathbf{V}(z_s, u_1, \ldots, u_n).$$

Proof. Firstly, since $A \in \mathrm{GL}_n(\mathbb{K})$, $\boldsymbol{f}$ being dominant implies that $\boldsymbol{f}^A$ is also dominant. Thus, $M_s^j(\boldsymbol{f}^A)$ is well-defined outside of a nowhere dense algebraic set.

Clearly, by the first p components of the map $M_s^j(\boldsymbol{f}^A)$ we take $h_1, \ldots, h_p$ to be $\mathrm{numer}(\boldsymbol{f}^A(\tau_s(z)) - \mathbf{c})$, where $\mathbf{c}$ are new indeterminates for the value of $\boldsymbol{f}^A$ at $\tau_s(z)$ and we take the numerators of these rational functions to get polynomials. We shall handle the denominators of these polynomials later by removing the algebraic set they define, thus ensuring these rational functions are always well-defined.

Then, by the restriction to the algebraic set $X = \mathbf{V}(g_1, \ldots, g_m)$, which after the transformation by A becomes $\mathbf{V}(g_1^A, \ldots, g_m^A)$, we set $h_{p+1}, \ldots, h_{p+m}$ to be $\mathrm{numer}(\boldsymbol{g}^A(\tau_s(z)))$. Now, we need an algebraic interpretation of $W_s^j(\tau_s(z))$ and $\Lambda = \mathbb{G}_{n-p-m+1}(\mathbb{C}^n \times 0)$.

Recall that $W_s^j(\tau_s(z))$ is an element of the Grassmannian $\mathbb{G}_{n-p+1}(\mathbb{C}^n \times \mathbb{C})$, since the map $M_s^j(\boldsymbol{f}^A)$ is well-defined outside of a nowhere dense algebraic set, and that $w_j(z)$ is the restriction of $z_s\, \mathrm{d}f_j^A$ to the kernel of the Jacobian matrix of $\boldsymbol{f}$ with the jth row removed. Then, the condition that $W_s^j(\tau_s(z)) \to W$, for some $W \in \Lambda$, along some path to an asymptotic critical value implies that the kernel of $\mathrm{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)^{[j]}$ tends to a subset of the kernel of $z_s\, \mathrm{d}f_j^A$. By Lemma 6, this is equivalent to the evaluation of $z_s\, \mathrm{d}f_j^A$ tending to vector in the span of the evaluation of $\mathrm{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)^{[j]}$ at $\tau_s(z)$. Thus, we may use Lagrange multipliers to represent $W_s^j(\tau_s(z))$ and its limit in Λ. Hence, we set $h_{m+p+1}, \ldots, h_{n+m+p}$ to be the numerators of the following polynomials at $\tau_s(z)$,

$$z_s\, \mathrm{d}f_j^A - \sum_{i=1}^{m+p-1} \lambda_i\, \mathrm{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)_i^{[j]} - \mathbf{u},$$

where $\mathbf{u}$ are n new indeterminates to represent the value of this Lagrangian function and $\boldsymbol{\lambda}$ are Lagrange multipliers.

Now, note that all the denominators of the rational functions we have defined are all powers of z_s. Thus, according to the definition of the map $M_s^j(\boldsymbol{f}^A)$, by removing the algebraic set $\mathbf{V}(z_s)$ from $\mathbf{V}(h_1, \ldots, h_{n+m+p})$, we get exactly the graph of $M_s^j(\boldsymbol{f}^A)$. Therefore, the algebraic closures give us the first equality

$$\overline{\mathrm{graph}\, M_s^j(\boldsymbol{f}^A)} = \overline{\mathbf{V}(h_1, \ldots, h_{n+m+p}) \setminus \mathbf{V}(z_s)}.$$

Secondly, to compute $L_s^j(\boldsymbol{f}^A)$ we intersect with the space $(\{z \in X | z_s = 0\} \times \mathbb{C}^p \times \Lambda)$. As discussed above, by Lemma 6, the intersection with Λ is achieved by setting the introduced $\mathbf{u}$ variables to 0. Then, the second equality is clear

$$L_s^j(\boldsymbol{f}^A) = \overline{\mathrm{graph}\, M_s^j(\boldsymbol{f}^A)} \cap \mathbf{V}(z_s, u_1, \ldots, u_n). \qquad \square$$

We now have an algebraic description of $L_s^j(\boldsymbol{f}^A)$ and hence of the np sets $K_s^j(\boldsymbol{f}^A)$. However, we shall see that by choosing a sufficiently generic A, it suffices to consider only p of these sets, for instance the sets $K_1^j(\boldsymbol{f}^A)$.

Lemma 8. *Let $\boldsymbol{f} \in \mathbb{K}[\mathbf{z}]^p$ be a dominant polynomial mapping with domain a smooth algebraic set X. There exists a non-empty Zariski open subset $\mathcal{O}_{\mathrm{GL}}$ of $\mathrm{GL}_n(\mathbb{K})$ such that for $A \in \mathcal{O}_{\mathrm{GL}}$ the following equality holds:*

$$K_\infty(\boldsymbol{f}) \subseteq \bigcup_{j=1}^{p} K_1^j(\boldsymbol{f}^A).$$

Proof. Suppose $c \in K_\infty(\boldsymbol{f})$. Then, there exists some sequence $(\boldsymbol{x}_t)_{t\in\mathbb{N}} \subset X$ such that as $t \to \infty$,

$$\|\boldsymbol{x}_t\| \to \infty, \boldsymbol{f}(\boldsymbol{x}_t) \to c \text{ and } \|\boldsymbol{x}_t\|\kappa(\mathrm{d}\boldsymbol{f}(\boldsymbol{x}_t)) \to 0.$$

By the isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, consider two discs in $\mathbb{R}^{2n}$ centred at c and 0 respectively. From the latter two limits, one defines a finite number of polynomials with real components that defines semi-algebraic sets again in $\mathbb{R}^{2n}$. By intersecting these sets with their respective discs, one can apply the curve selection lemma at infinity [18, Lemma 3.3], an extension of the classical curve selection lemma [5, Theorem 2.5.5] obtained by considering a semialgebraic compactification of $\mathbb{R}^{2n}$. Recall that such semi algebraic curves may be chosen to be Nash curves [5, Proposition 8.1.12].

Therefore, there exists a path $\gamma : (0, 1) \to X$ such that

$$\boldsymbol{f}(\gamma(t)) \to c, \|\gamma(t)\| \to \infty \text{ and } \|\gamma(t)\|\kappa(\mathrm{d}\boldsymbol{f}(\gamma(t))) \to 0 \text{ as } t \to 0,$$

where each component of γ is expressible as a Puiseux series in t. Denote this expression $z(t)$. Then, by the definition of Puiseux series, each component of $z(t)$ has finitely many terms with negative exponents. Let r be the least rational number such that t^r has a non-zero coefficient for some component of $z(t)$, or in other words, the exponent of the term that tends to infinity fastest as $t \to 0$. Then, for each $1 \leq i \leq n$ we have

$$z_i = \sum_{k\geq r} z_{ik}t^k.$$

Consider the group of $n \times n$ invertible matrices $\mathrm{GL}_n(\mathbb{K})$ with entries in $\mathbb{K}$. For $B = (b_{ik})_{1\leq i,k\leq n} \in \mathrm{GL}_n(\mathbb{K})$, let $y = Bz$ and set

$$y_1 = \sum_{k\geq r} y_{1k}t^k.$$

Consider the coefficient

$$y_{1r} = \sum_{k=1}^{n} b_{1k}z_{kr}.$$

Then, $y_{1r} = 0$ defines the Zariski-closed subset $\mathscr{C}$ of $\mathrm{GL}_n(\mathbb{K})$ such that $B \in \mathscr{C}$ implies that the first component of $Bz(t)$ is such that r is not the least exponent. By definition, some y_i is non-zero and so $\mathscr{C}$ is a proper subset. Therefore, there exists a non-empty Zariski-open subset $\mathcal{O}_{\mathrm{GL}}^{-1}$ of $\mathrm{GL}_n(\mathbb{K})$ such that for $B \in \mathcal{O}_{\mathrm{GL}}^{-1}$, $\|(Tz)_1(t)\|$ tends to infinity at the same speed as $\|z(t)\|$ as $t \to 0$. Let $\mathcal{O}_{\mathrm{GL}}$ be the non-empty Zariski closed subset of $\mathrm{GL}_n(\mathbb{K})$ defined by $A \in \mathcal{O}_{\mathrm{GL}} \iff A^{-1} \in \mathcal{O}_{\mathrm{GL}}^{-1}$.

Choose some $A \in \mathcal{O}_{\mathrm{GL}}$ and consider the polynomial mapping $\boldsymbol{f}^A = \boldsymbol{f}(Az)$ restricted to the algebraic set defined by $X^A = \mathbf{V}(\boldsymbol{g}^A) = \mathbf{V}(\boldsymbol{g}(Az))$ and the path $\Gamma(t) = A^{-1}z(t)$. As $t \to 0$, $\|\Gamma(t)\| \to \infty$ and $\boldsymbol{f}^A(\Gamma(t)) \to c$. Furthermore, by definition of $\mathcal{O}_{\mathrm{GL}}$, we have that $\|\Gamma_1(t)\| \to \infty$ as $t \to 0$. Recall that κ is equivalent to ν. Thus, since $A \in \mathrm{GL}_n(\mathbb{K})$, by [18, Corollary 2.1], we have $\|y(t)\|\nu(\mathrm{d}\boldsymbol{f}^A(y(t))) \to 0$ which implies that $\|\Gamma(t)\|\kappa(\mathrm{d}\boldsymbol{f}^A(\Gamma(t))) \to 0$. Choose j such that $\kappa(\mathrm{d}\boldsymbol{f}^A(\Gamma(t))) = \|w_j(\Gamma(t))\|$. Then, since the Grassmannian $\mathbb{G}_{n-k+1}(\mathbb{K}^n \times \mathbb{K})$ is compact, there is a limit W_1^j of graphs

$\Gamma_1(t)w_j(\Gamma(t))$ where $W_1^j \in \Lambda$ by [21, Lemma 5.1]. Therefore, we have in the limit $(0, c, W_1^j) \in L_1^j(\boldsymbol{f})$ and so $c \in K_1^j(\boldsymbol{f}^A)$. Thus,

$$K_\infty(\boldsymbol{f}) = K_\infty(\boldsymbol{f}^A) \subseteq \bigcup_{(s,j)=(1,1)}^{(n,p)} K_s^j(\boldsymbol{f}^A) = \bigcup_{j=1}^{p} K_1^j(\boldsymbol{f}^A). \qquad \square$$

3. Geometric result

In this section, we state our main geometric result that will form the basis of the proof of correctness of the probabilistic algorithms we give in Sections 4 and 7.

Proposition 9. *Let $W \subset \mathbb{C}^N$ be an algebraic set. Let Z be a hyperplane of $\mathbb{C}^N$ such that $\overline{W \setminus Z} = V_1 \cup \cdots \cup V_k$ for some positive k. Suppose that $V_1, \ldots, V_k$ have dimension m. Let π be the canonical projection map from W onto $\mathbb{C}^n$ so that π restricted to V_i is dominant for all i and let $\mathbb{G}_2(\mathbb{C}^n)$ be the Grassmannian of planes through the origin in $\mathbb{C}^n$. Then, there exists a Zariski-open dense subset $\mathcal{O}$ of $\mathbb{G}_2(\mathbb{C}^n)$ such that for all $E \in \mathcal{O}$,*

$$\overline{\pi^{-1}(E) \setminus Z} = \overline{W \setminus Z} \cap \pi^{-1}(E), \quad \dim \overline{\pi^{-1}(E) \setminus Z} = m - n + 2$$

Proof. Note that since there is a finite number of irreducible components of W that are not contained in Z, it suffices to consider the case $k = 1$ as a finite intersection of dense Zariski-open subsets is still a dense Zariski-open subset. Hence, let V be the n-dimensional irreducible component of W so that $\overline{W \setminus Z} = V$.

Let $V_H \in \mathbb{P}^N$ be the projectivisation of V. Then, the map π naturally extends to a projection map $\pi_H : V_H \to \mathbb{P}^n$. Note that π_H is a morphism of varieties since $\dim V = n$ and π is dominant. Thus, by [20, Theorem 1.1], the preimage of every line $L \in \mathbb{P}^n$, π_H^{-1} is connected and hence irreducible in the Zariski topology of V_H. This implies that the preimage of π of a generic line in $\mathbb{C}^n$, a hyperspace section of V of dimension 2, is irreducible. Let $\mathbb{C}[u_1, \ldots, u_n]$ be a coordinate ring of $\mathbb{C}^n$, then each of these lines may be parametrised by the equations

$$u_1 = a_1e_1 + b_1, \ldots, u_n = a_ne_1 + b_n,$$

where e_1 is a parameter and $\mathbf{a}, \mathbf{b}$ are vectors of $\mathbb{C}^n$ outside of some proper Zariski-closed subset C. Then, from such a line we get a plane E defined in two parameters and the equations

$$u_1 = a_1e_1 + b_1e_2, \ldots, u_n = a_ne_1 + b_ne_2.$$

Then, there exists a dense Zariski-open subset $\mathcal{O}_1$ of $\mathbb{G}_2(\mathbb{C}^n)$ so that for all $E \in \mathcal{O}_1$ the preimage $\pi^{-1}(E)$ is an irreducible two-dimensional section of V.

Consider $E \in \mathcal{O}_1$ and the parametrisation given by $u_1 = a_1e_1 + b_1e_2, \ldots, u_n = a_ne_1 + b_ne_2$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ be parameters and consider the ideal $I(\pi^{-1}(E))$. Since Z is a hyperplane of $\mathbb{C}^N$, there exists a linear form F such that $Z = \mathbf{V}(F)$. Then, the subset of E such that $\pi^{-1}(E) \subset Z$ is given by the normal form of F with respect to a Gröbner basis of $I(\pi^{-1}(E))$. Either the normal form is identically zero, or we obtain a polynomial whose coefficients are polynomials in the parameters $\mathbf{a}, \mathbf{b}$. Thus, this subset of E that we must avoid is a Zariski-closed subset. It remains to show that this Zariski-closed subset is not $\mathbb{G}_2(\mathbb{C}^n)$. To do so, take some $x \in V \setminus Z$. Since π is dominant, there exists some $E \in \mathbb{G}_2(\mathbb{C}^n)$ such that $x \in \pi^{-1}(E)$. Recall that $V \setminus Z$ is a dense Zariski-open subset of V. Hence, there exists a Zariski-open dense subset $\mathcal{O}_2$ of $\mathbb{G}_2(\mathbb{C}^n)$ such that for all $E \in \mathcal{O}_2$, $\pi^{-1}(E) \nsubseteq Z$.

Let $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$. Then, $\mathcal{O}$ is a Zariski-open dense subset of $\mathbb{G}_2(\mathbb{C}^n)$. Fix some $E \in \mathcal{O}$. Then, since $\pi^{-1}(E)$ is irreducible and is not contained in Z we have that

$$\overline{\pi^{-1}(E) \setminus Z} = \pi^{-1}(E). \qquad \square$$

We aim to apply the results of Proposition 9 to reduce the dimension of the algebraic sets we consider in our algorithms. First, however, we give an algebraic condition that is sufficient to prove the required dominance of the projection from the graph of $M_1^j(\boldsymbol{f}^A)$ onto the $\mathbf{u}$-space. For that purpose, for given $\boldsymbol{f}, \boldsymbol{g}$ and $A \in \mathrm{GL}_n(\mathbb{K})$, define for each $1 \leq j \leq p$ the polynomial mappings $H_j = (\boldsymbol{g}^A, z_1\, \mathrm{d}f_j^A - \sum_{i=1}^{m+p-1} \lambda_i \,\mathrm{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)_i^{[j]})$.

Lemma 10. *Let $\boldsymbol{f} \in \mathbb{K}[\mathbf{z}]^p$ be a dominant polynomial mapping with domain a smooth algebraic set X defined by a reduced, regular sequence $(g_1, \dots, g_m)$ and let $A \in \mathrm{GL}_n(\mathbb{K})$ be such that the results of Lemma 8 hold. Let π be the projection map from $\overline{\mathrm{graph}\, M_s^j(\boldsymbol{f}^A)}$ onto the $\mathbf{u}$-space. If the Jacobian matrix $\mathrm{jac}(H_j)$ has full rank for all j, then π is a dominant map.*

Proof. Fix some $1 \leq j \leq p$. We aim to show that the set of points in $\mathbb{C}^m$ that are not in the image of π is a proper Zariski-closed subset. Thus, choose a generic point $(a_1, \dots, a_n)$. We shall show that there exists a point $(z_1, \dots, z_n, c_1, \dots, c_p, u_1, \dots, u_n, \lambda_1, \dots, \lambda_{m+p-1}) \in \overline{\mathrm{graph}\, M_s^j(\boldsymbol{f}^A)}$ where $(u_1, \dots, u_n) = (a_1, \dots, a_n)$.

By Lemma 7, we have that

$$\overline{\mathrm{graph}\, M_1^j(\boldsymbol{f}^A)} = \overline{\mathbf{V}(h_1, \dots, h_{n+m+p}) \setminus \mathbf{V}(z_1)},$$

where

$$\begin{aligned}
&\text{for } 1 \leq i \leq m \quad h_{p+i} = \mathrm{numer}(\boldsymbol{g}_i^A(\tau_1(z))),\\
&\text{for } 1 \leq i \leq n \quad h_{m+p+1} = \mathrm{d}f_j^A(\tau_1(z))_i - \sum_{k=1}^{m+p-1} \lambda_k \,\mathrm{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)_{k,i}^{[j]}(\tau_1(z)) - z_1 u_i).
\end{aligned}$$

Then, a generic point of graph $M_1^j(\boldsymbol{f}^A)$ is one such that $z_1 \neq 0$. Thus, τ_1 is invertible and we can divide by z_1. Hence, locally we have that $(h_{p+1}, \dots, h_{n+m+p}) = H_j$. Consider the system of equations

$$\begin{cases} v_1 & = g_1^A \\ & \vdots \\ v_m & = g_m^A \\ w_1 & = z_1 \frac{\partial f_j^A}{\partial z_1} - \sum_{i=1}^{m+p-1} \lambda_i \operatorname{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)_{i,1}^{[j]} \\ & \vdots \\ w_n & = z_1 \frac{\partial f_j^A}{\partial z_n} - \sum_{i=1}^{m+p-1} \lambda_i \operatorname{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)_{i,n}^{[j]} \\ x_1 & = b_1 \\ & \vdots \\ x_{p-1} & = b_{p-1} \end{cases}$$

for generic $b_1, \ldots, b_{p-1} \in \mathbb{C}^{p-1}$. Since the Jacobian of H_j has full rank, by the genericity of $\mathbf{b}$ we have that the Jacobian of the polynomials on the right hand side of these equations has full rank. Thus, by the inverse function theorem there exist equations, defined for $z_1 \neq 0$, $(\mathbf{z}, \boldsymbol{\lambda}) = (\phi_1(\mathbf{v}, \mathbf{w}, \boldsymbol{x}), \ldots, \phi_{n+m+p-1}(\mathbf{v}, \mathbf{w}, \boldsymbol{x}))$. Therefore, substituting $\mathbf{v}$ for 0, $\mathbf{w}$ for $\mathbf{a}$ and $\boldsymbol{x}$ for $\mathbf{b}$, we have constructed a point $(\mathbf{z}, \boldsymbol{f}^A(\mathbf{z}), \mathbf{a}, \boldsymbol{\lambda}) \in \operatorname{graph} M_1^j(\boldsymbol{f}^A)$. Hence, the image of π is a Zariski-dense subset of $\mathbb{C}^m$ and so π is dominant. □

4. Algorithms

4.1. Subroutines

The algorithms in this paper rely primarily on algebraic geometric operations. Through the ideal-variety correspondence, these shall be performed through ideal theoretic operations. We give three subroutines of this type that will be used in our algorithms and proofs.

Eliminate$(P, \mathbf{v}, \mathbf{w})$**:**

> **Input:** P, a finite basis of an ideal, I, of a polynomial ring (with base field $\mathbb{K}$ and two lists of indeterminates, $\mathbf{v}$ and $\mathbf{w}$) which we denote $\mathbb{K}[\mathbf{v}, \mathbf{w}]$.
>
> **Output:** E, a finite basis of the ideal $I \cap \mathbb{K}[\mathbf{w}]$.

Intersect$(P_1, \ldots, P_k)$**:**

> **Input:** $P_1, \ldots, P_k$, finite bases of ideals, $I_1, \ldots, I_k$, of a polynomial ring.
>
> **Output:** P, a finite basis of the ideal $\bigcap_{i=1}^k I_i$.

Saturate(P_1, P_2)**:**

> **Input:** P_1, P_2, finite bases of ideals, I_1, I_2, of a polynomial ring.
>
> **Output:** S, a finite basis of the ideal $I_1 : I_2^\infty$.

Remark 11. *These ideal theoretic operations can be computed algorithmically. For example, Gröbner bases can be computed to solve all the above problems. We refer to [7, Chapter 3, Section 1, Theorem 2], [3, Proposition 6.19] and [2, 8] for algorithms for computing a finite basis for respectively elimination ideals, intersection of ideals and the saturation of ideals. There exist algorithms for computing Gröbner bases that are correct and terminate [7, Chapter 2, Section 7, Theorem 2].*

4.2. Computing asymptotic critical values

Algorithm 1: acv1

Input: $\boldsymbol{g}$ a reduced regular sequence defining a smooth algebraic set X, $\boldsymbol{f} : X \to \mathbb{K}^p$ a dominant polynomial mapping with components in the ring $\mathbb{K}[\mathbf{z}]$ and the list $\mathbf{z}$.

Output: R, a finite list of polynomials whose zero set has codimension at least 1 in $\mathbb{C}^p$ and contains the set of asymptotic critical values of $\boldsymbol{f}$.

1 Generate a random change of variables $A \in \mathbb{K}^{n\times n}$.
2 Generate random numbers $\mathbf{a}, \mathbf{b} \in \mathbb{K}^n$ and set $\boldsymbol{f}^A \leftarrow \boldsymbol{f}(Az)$, $\boldsymbol{g}^A \leftarrow \boldsymbol{g}(Az)$.
3 **For** j **from** 1 **to** p **do**
4     $\mathbf{v}(z) \leftarrow z_1 \, \mathrm{d}f_j^A - \lambda_1 \, \mathrm{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)_1^{[j]} - \cdots - \lambda_{m+p-1} \, \mathrm{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)_{m+p-1}^{[j]} - \mathbf{a}e_1$.
5     $N(z) \leftarrow \{f_1^A - c_1, \ldots, f_p^A - c_p, g_1^A, \ldots, g_m^A, v_1 - b_1 e_2, \ldots, v_n - b_n e_2\}$.
6     $G \leftarrow \mathrm{numer}(N(\tau_1(z)))$.
7     $G_s \leftarrow \mathrm{Saturate}(G, z_1)$.
8     $L \leftarrow G_s \cup \{z_1, e_1, e_2\}$.
9     $V_j \leftarrow \mathrm{Eliminate}(L, \{\mathbf{z}, e_1, e_2, \lambda_1, \ldots, \lambda_{m+p-1}\}, \{\mathbf{c}\})$.
10 R $\leftarrow$ Intersect$(V_1, \ldots, V_p)$.
11 **Return** R.

Theorem 12. *Let $\boldsymbol{f} = (f_1, \ldots, f_p) \in \mathbb{K}[\mathbf{z}]^p$ be a dominant polynomial mapping from a smooth algebraic set defined by a reduced regular sequence $\boldsymbol{g} = (g_1, \ldots, g_m)$. Suppose that $A \in \mathrm{GL}_n(\mathbb{K})$ satisfies the genericity condition of Lemma 8 and suppose that $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ define a plane $E \subset \mathbb{C}^n$ that satisfies the genericity condition of Proposition 9. Suppose that $\mathrm{jac}(H_j)$ has full rank for all j. Then, Algorithm 1 terminates and returns as output a finite basis whose zero set has codimension at least 1 in $\mathbb{C}^p$ and contains the set of asymptotic critical values of $\boldsymbol{f}$.*

Proof. Firstly, Algorithm 1 relies on multivariate polynomial routines that are correct and terminate, see Remark 11. Hence, Algorithm 1 terminates in finitely many steps. By the choice of A, we may apply Lemma 8. Therefore, we aim to compute sets $K_1^j(\boldsymbol{f}^A)$ for $1 \le j \le p$. However, since these sets are semi-algebraic, we shall instead compute their closures in the Zariski topology which have the same dimension. We shall show that the algebraic sets defined by the list of polynomials V_j computed in step 9 contains $\overline{K_1^j(\boldsymbol{f}^A)}$ and has codimension at least 1 in $\mathbb{C}^p$. Then, the union of these algebraic sets, $\mathbf{V}(\mathrm{R})$ as computed in step 10, contains the asymptotic critical values of $\boldsymbol{f}$ and has codimension at least 1 in $\mathbb{C}^p$ by [7, Chapter 9, Section 4, Theorem 8].

Thus, fix some $1 \le j \le p$. By Lemma 7, there exists polynomials $h_1, \dots, h_{n+m+p} \in \mathbb{K}[\mathbf{z}, \mathbf{c}, \mathbf{u}, \boldsymbol{\lambda}]$ such that

$$\overline{\operatorname{graph} M_1^j(\boldsymbol{f}^A)} = \overline{\mathbf{V}(h_1, \dots, h_{n+m+p}) \setminus \mathbf{V}(z_1)},$$

$$L_1^j(\boldsymbol{f}^A) = \overline{\operatorname{graph} M_1^j(\boldsymbol{f}^A)} \cap \mathbf{V}(z_1, u_1, \dots, u_n).$$

Let E be the plane in the $\mathbf{u}$-space parametrised by the equations $u_i = a_i e_1 + b_i e_2$. Let $W = \mathbf{V}(h_1, \dots, h_{n+m+p})$. Then, $\overline{\operatorname{graph} M_1^j(\boldsymbol{f}^A)}$ is the union of the irreducible components of W that do not vanish on $\mathbf{V}(z_1)$. Then, $\overline{\operatorname{graph} M_1^j(\boldsymbol{f}^A)}$ is equidimensional of dimension $n + p - 1$ and, by Lemma 10, the projection map π from $\overline{\operatorname{graph} M_1^j(\boldsymbol{f}^A)}$ onto the $\mathbf{u}$-space is dominant. Then, by Proposition 9, by the choice of E we have that

$$\overline{\pi^{-1}(E) \setminus \mathbf{V}(z_1)} = \overline{W \setminus \mathbf{V}(z_1)} \cap \pi^{-1}(E).$$

Therefore, since E contains the origin of the $\mathbf{u}$ space,

$$\begin{aligned} L_1^j(\boldsymbol{f}^A) &= \overline{\operatorname{graph} M_1^j(\boldsymbol{f}^A)} \cap \mathbf{V}(z_1, u_1, \dots, u_n) \\ &= \overline{W \setminus \mathbf{V}(z_1)} \cap \mathbf{V}(z_1, u_1, \dots, u_n) \\ &= \overline{\pi^{-1}(E) \setminus \mathbf{V}(z_1)} \cap \mathbf{V}(z_1, u_1, \dots, u_n) \\ &= \overline{\pi^{-1}(E) \setminus \mathbf{V}(z_1)} \cap \mathbf{V}(z_1, e_1, e_2). \end{aligned}$$

Thus, we may replace u_i by $a_i e_1 + b_i e_2$, its value in the parametrisation of E. By the definition of $h_1, \dots, h_{n+m+p}$, the resulting polynomials are exactly those defined in step 6. Therefore, the algebraic set defined by L as defined in step 8 is $L_1^j(\boldsymbol{f}^A)$. Then, by [7, Chapter 4, Section 4, Theorem 4], eliminating all variables except $\mathbf{c}$ computes the closure of the projection onto the $\mathbf{c}$-space. The resulting algebraic set is exactly $\overline{K_1^j(\boldsymbol{f}^A)}$. Moreover, by Proposition 9, the hyperspace section $\pi^{-1}(E)$ has dimension $(n+p-1)-(n-2) = p+1$. By the dominance of the projection onto the $\mathbf{u}$-space, e_1, e_2 are not identically zero on $\pi^{-1}(E)$, hence $L_1^j(\boldsymbol{f}^A)$ has dimension at most $p-1$. Therefore, the projection $K_1^j(\boldsymbol{f}^A)$ onto the $\mathbf{c}$-space has codimension at least one in $\mathbb{C}^p$. □

Note that by step 8 of Algorithm 1, we find equations defining an algebraic set of dimension at most $p+1$. However, we then intersect with 3 hyperplanes but we only require the dimension to drop by 2. We take advantage of this behaviour in the following algorithm, which reduces the number of equations and variables by one each. This algorithm will subsequently allow us to obtain sharper degree bounds on the set of asymptotic critical values.

Algorithm 2: acv2

Input: $\boldsymbol{g}$ a reduced regular sequence defining a smooth algebraic set X,
$\boldsymbol{f} : X \to \mathbb{K}^p$ a dominant polynomial mapping with components in the ring $\mathbb{K}[\mathbf{z}]$ and the list $\mathbf{z}$.

Output: R, a finite list of polynomials whose zero set has codimension at least 1 in $\mathbb{C}^p$ and contains the set of asymptotic critical values of $\boldsymbol{f}$.

1 Generate a random change of variables $A \in \mathbb{K}^{n \times n}$.
2 Generate random numbers $\mathbf{a}, \mathbf{b} \in \mathbb{K}^n$ and set $\boldsymbol{f}^A \leftarrow \boldsymbol{f}(Az)$, $\boldsymbol{g}^A \leftarrow \boldsymbol{g}(Az)$.
3 **For** j **from** 1 **to** p **do**
4 $\quad \mathbf{v}(z) \leftarrow z_1 \, \mathrm{d}f_j^A - \lambda_1 \operatorname{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)_1^{[j]} - \cdots - \lambda_{m+p-1} \operatorname{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)_{m+p-1}^{[j]} - \mathbf{a}e_1$.
5 $\quad N'(z) \leftarrow \{f_1^A - c_1, \ldots, f_p^A - c_p, g_1^A, \ldots, g_m^A, b_2 v_1 - b_1 v_2, \ldots, b_n v_1 - b_1 v_n\}$.
6 $\quad G' \leftarrow \mathrm{numer}(N'(\tau_1(z)))$.
7 $\quad G'_s \leftarrow \mathrm{Saturate}(G', z_1)$.
8 $\quad L' \leftarrow G'_s \cup \{z_1, e_1\}$.
9 $\quad V'_j \leftarrow \mathrm{Eliminate}(L', \{\mathbf{z}, e_1, \boldsymbol{\lambda}\}, \{\mathbf{c}\})$.
10 $\mathrm{R}' \leftarrow \mathrm{Intersect}(V'_1, \ldots, V'_p)$.
11 **Return** R'.

To prove the correctness of this algorithm, we first prove a lemma.

Lemma 13. *Fix some $1 \leq j \leq p$ and let G and G' be the list of polynomials computed at step 6 of Algorithm 1 and at step 6 of Algorithm 2 respectively for the same sufficiently generic choice of $A, \mathbf{a}$ and $\mathbf{b}$. Then,*

$$\langle G' \rangle = \langle G \rangle \cap \mathbb{C}[\mathbf{z}, e_1, \boldsymbol{\lambda}, \mathbf{c}].$$

Proof. Firstly, the polynomials $f_1^A - c_1, \ldots, f_p^A - c_p, g_1^A, \ldots, g_m^A$ at $\tau_1(z)$ are elements of both lists G and G' which are contained in the polynomial ring $\mathbb{C}[\mathbf{z}, \mathbf{c}]$. Hence, we need only consider the remaining polynomials that are in the ring $\mathbb{C}[\mathbf{z}, e_1, e_2, \boldsymbol{\lambda}]$.

We shall prove this by double inclusion. Firstly, take some $\mathrm{numer}(b_i v_1(\tau_1(z)) - b_1 v_i(\tau_1(z))) \in G'$. We have that $b_i \,\mathrm{numer}(v_1(\tau_1(z)) - b_1 e_2) - b_1 \,\mathrm{numer}(v_i(\tau_1(z)) - b_i e_2) \in \langle G \rangle \cap \mathbb{C}[\mathbf{z}, e_1, \boldsymbol{\lambda}, \mathbf{c}]$. However, since v_1, v_2 have the same degree in $\mathbf{z}$,

$$\begin{aligned} b_i \,\mathrm{numer}(v_1(\tau_1(z)) - b_1 e_2) - b_1 \,\mathrm{numer}(v_i(\tau_1(z)) - b_i e_2) = \\ = b_i \,\mathrm{numer}(v_1(\tau_1(z))) - b_1 \,\mathrm{numer}(v_i(\tau_1(z))) \\ = \mathrm{numer}(b_i v_1(\tau_1(z)) - b_1 v_i(\tau_1(z))). \end{aligned}$$

On the other hand, let $h \in \langle G \rangle \cap \mathbb{C}[\mathbf{z}, e_1, \boldsymbol{\lambda}, \mathbf{c}]$. Let $G = \{h_1, \ldots, h_{n+m+p}\}$, then $h \in \langle G \rangle$ equals $\sum_{i=1}^{n+m+p} y_i h_i$ such that $y_i \in \mathbb{C}[\mathbf{z}, e_1, e_2, \boldsymbol{\lambda}, \mathbf{c}]$ and all e_2-terms are cancelled. Considering a monomial ordering such that e_2 is the largest monomial, $(y_1, \ldots, y_{n+m+p})$ is a syzygy on the leading terms of $h_1, \ldots, h_{n+m+p}$ that involve e_2. The S-polynomials generate the set of syzygies [7, Chapter 2, Section 10, Proposition 5] but by an elementary application of the exchange lemma, these S-polynomials are elements of $\langle G' \rangle$. Hence, $h \in \langle G' \rangle$. ☐

Thus, Algorithm 2 is the same as Algorithm 1 except that we eliminate e_2 before the saturation step 7.

Theorem 14. *Let $\boldsymbol{f} = (f_1, \dots, f_p) \in \mathbb{K}[\mathbf{z}]^p$ be a dominant polynomial mapping from a smooth algebraic set defined by a reduced regular sequence $\boldsymbol{g} = (g_1, \dots, g_m)$. Suppose that $A \in \mathrm{GL}_n(\mathbb{K})$ satisfies the genericity condition of Lemma 8 and suppose that $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ define a plane $E \subset \mathbb{C}^n$ that satisfies the genericity condition of Proposition 9. Suppose that $\mathrm{jac}(H_j)$ has full rank for all j. Then, Algorithm 2 terminates and returns as output a finite basis whose zero set has codimension at least 1 in $\mathbb{C}^p$ and contains the set of asymptotic critical values of $\boldsymbol{f}$.*

Proof. As in Algorithm 1, Algorithm 2 relies on multivariate polynomial routines that are correct and terminate, see Remark 11. Hence, Algorithm 2 terminates in finitely many steps.

Fix some $1 \leq j \leq p$ and let G and G' be the list of polynomials computed at step 6 of Algorithm 1 and at step 6 of Algorithm 2 respectively for the same sufficiently generic choice of $A, \mathbf{a}$ and $\mathbf{b}$. By Lemma 10 and Proposition 9, the projection from $\mathbf{V}(G)$ onto the (e_1, e_2)-space is dominant. By Lemma 13 and [7, Chapter 4, Section 4, Theorem 4], $\mathbf{V}(G')$ is the Zariski closure of the projection π_{e_2} of $\mathbf{V}(G)$ that eliminates e_2. Thus, $\mathbf{V}(G')$ remains two-dimensional and by Proposition 9, we have that

$$\overline{\mathbf{V}(G') \setminus \mathbf{V}(z_1)} = \overline{\overline{\pi_{e_2}(\mathbf{V}(G))} \setminus \mathbf{V}(z_1)} = \overline{\pi_{e_2}\left(\overline{\mathbf{V}(G) \setminus \mathbf{V}(z_1)}\right)}.$$

By [7, Chapter 4, Section 4, Theorem 10], this is equal to $\mathbf{V}(G'_s)$, where G'_s is the list computed at step 7. Therefore, there exist embeddings of the algebraic sets defined in Algorithm 2 in their counterparts defined in Algorithm 1. Thus, $\mathbf{V}(\mathrm{R}')$ contains $K_1^j(\boldsymbol{f}^A)$. It remains to show that $\mathbf{V}(\mathrm{R}')$ is contained in a proper Zariski-closed subset of $\mathbb{C}^p$.

By the dominance of the projection onto the e_1-axis, we have that e_1 is not identically zero over $\mathbf{V}(G'_s)$. Furthermore, by the saturation in step 7, z_1 is not identically zero either. Hence, $\mathbf{V}(L')$ has dimension at most $p-1$. Thus, $\mathbf{V}(V'_j)$ has codimension at least 1 in $\mathbb{C}^p$ and so does $\mathbf{V}(\mathrm{R}')$. ☐

5. Degree result

Theorem 2. *Let $\boldsymbol{f} = (f_1, \dots, f_p) \in \mathbb{K}[\mathbf{z}]^p$ be a dominant polynomial mapping from a smooth algebraic set defined by a reduced regular sequence $\boldsymbol{g} = (g_1, \dots, g_m)$. Let $d = \max(\deg f_1, \dots, \deg f_p, \deg g_1, \dots, \deg g_m)$. Then, the asymptotic critical values of $\boldsymbol{f}$ are contained in a hypersurface of degree at most*

$$pd^{n-p-1} \sum_{i=0}^{p+1} \binom{n+p-1}{m+2p-i} d^i.$$

Proof. Let $A \in \mathrm{GL}_n(\mathbb{K})$ and $\mathbf{a}, \mathbf{b} \in \mathbb{K}^n$ be such that the genericity assumptions of Theorem 14 hold. By Lemma 8, we have that

$$K_\infty(\boldsymbol{f}) \subseteq \bigcup_{j=1}^{p} K_1^j(\boldsymbol{f}^A).$$

By Theorem 14, the sets $K_1^1, \dots, K_1^p$ are contained in the algebraic sets $V_1, \dots, V_p$ returned by each pass of step 9 of Algorithm 2. Thus, we shall bound the degree of $K_\infty(\boldsymbol{f})$ by p times a bound on the degree of V_1, since by symmetry the bounds on the degree of each V_i will be equal. Then, let G, G_s, L and V_1 be the finite lists of polynomials as defined in the $j = 1$ pass of Algorithm 2 in steps 6, 7 8 and 9 respectively.

By [7, Chapter 4, Section 4, Theorem 10], the saturation

$$\langle G_s \rangle = \langle G \rangle : \langle z_1 \rangle^\infty$$

corresponds to the variety

$$\mathbf{V}(G_s) = \overline{\mathbf{V}(G) \setminus \mathbf{V}(z_1)}.$$

Thus, $\mathbf{V}(G_s)$ is the union of a subset of the irreducible components of $\mathbf{V}(G)$. Clearly, the degree of $\mathbf{V}(L)$ is then also bounded by the degree of $\mathbf{V}(G)$ and since projection cannot increase the degree either [14, Lemma 2], we have that the degree of $\mathbf{V}(G)$ is at least the degree of $\mathbf{V}(V_1)$.

Now, we shall bound the degree of $\mathbf{V}(G)$ by taking advantage of the multi-homogeneous structure. Firstly, we note that G consists of $n + m + p - 1$ polynomials in $n + m + 2p$ variables. We shall split the variables into $\mathbf{z}$ and $\mathbf{c}, e_1, \boldsymbol{\lambda}$. Note that G consists of m polynomials of degree at most d that depend only on $\mathbf{z}$ and $n + p - 1$ polynomials that have degree at most d in $\mathbf{z}$ and degree at most 1 in the remaining variables. Then, by the multi-homogeneous Bézout bound [27, Proposition 3], the p-equidimensional component of $\mathbf{V}(G)$ has degree at most the sum of the coefficients of the normal form of the polynomial $(dv_1 + v_2)^{n+p-1} d^m v_1^m$ with respect to the ideal $\langle v_1^{n+1}, v_2^{m+2p+1} \rangle$. Therefore, by binomial expansion, the degree of $\mathbf{V}(V_1)$ is at most

$$\begin{aligned} \deg \mathbf{V}(V_1) &\leq d^m \sum_{k=n-m-p-1}^{n-m} \binom{n+p-1}{k} d^k \\ &= d^{n-p-1} \sum_{i=0}^{p+1} \binom{n+p-1}{m+2p-i} d^i. \end{aligned}$$

Multiplication by p completes the proof of the bound in the statement. □

Note that for m, p, d fixed, the degree of the set of asymptotic critical values is in $O(n^{2p+m} d^{n-p-1})$.

6. Complexity result

In this subsection, we analyse the worst-case complexity of Algorithm 2. We focus on this algorithm in particular due to the fact it allows us to obtain the lowest degree bound, through the multi-homogeneous Bézout bound as in Section 5. Additionally, this algorithm handles the fewest variables out of the algorithms given in this paper. This is important as the dimension of the ambient space will be a factor in the complexity analysis.

Firstly, let $\mathsf{M}(d)$ be the number of base field operations required for multiplying two univariate polynomials of degree at most d. For example, using the Cantor–Kaltofen algorithm, we would have that $\mathsf{M}(n) = O(n \log n \log \log n)$ [6].

Algorithm 2 takes as input a polynomial mapping $\boldsymbol{f} : X \to \mathbb{C}^p$, $\boldsymbol{f} = (f_1(z), \ldots, f_p) \in \mathbb{K}[\mathbf{z}]$, where X is a smooth algebraic set defined by a reduced regular sequence $\boldsymbol{g} = (g_1, \ldots, g_m)$. Let $d = \max(\deg f_1, \ldots, \deg f_p, \deg g_1, \ldots, \deg g_m)$. The first steps of this algorithm, for each $1 \le j \le p$, is to construct a list of polynomials $h_1, \ldots, h_{n+m+p-1}$. In Section 5 it is proven that these polynomials define an algebraic set of degree at most

$$d^{n-p-1} \sum_{i=0}^{p+1} \binom{n+p-1}{m+2p-i} d^i.$$

This will be a key factor in our complexity result and so we denote this degree by D.

The remaining steps of Algorithm 2 involve applying some algebraic elimination subroutines with the list of polynomials $h_1, \ldots, h_{n+m+p-1}$ as the initial input. To obtain reasonable complexity results, we opt to use the geometric resolution algorithm given in [12]. However, since these polynomials define an algebraic set of dimension $p+1$, we must first specialise our system to obtain a zero-dimensional input for the geometric resolution algorithm. Then, we can apply the lifting algorithm of [28] to obtain a parametric system. Performing the final necessary intersections and projections of varieties is then done by resultant computation. In the end, we will obtain a polynomial whose solution set contains the set of asymptotic critical values.

Firstly, recall the representation given as the output of the geometric resolution algorithm. Consider polynomials $\varphi_1, \ldots, \varphi_\ell, \psi \in \mathbb{K}[x_1, \ldots, x_\ell]$. Suppose that $\varphi_1, \ldots, \varphi_\ell$ are a regular sequence so that the system S defined by $\varphi_1 = \cdots = \varphi_\ell = 0, \psi \neq 0$, is zero-dimensional of degree $\mathcal{D}$. Let T be a linear form in the variables $x_1, \ldots, x_\ell$. Then, with the system S as input, the geometric resolution algorithm returns a representation of the solution set of S as follows:

$$\begin{cases} Q(T) & = 0 \\ \frac{dQ}{dT}(T)\, x_1 & = V_1(T) \\ & \vdots \\ \frac{dQ}{dT}(T)\, x_\ell & = V_\ell(T), \end{cases}$$

where $Q, V_1, \ldots, V_\ell \in \mathbb{Q}[T]$ are univariate polynomials such that $\deg Q = \mathcal{D}, \deg V_i < \mathcal{D}$. Note that this representation is well-defined outside of the Zariski-closed subset $\mathbf{V}(\frac{dQ}{dT})$ of $\mathbb{C}^\ell$. We can now restate and prove our main complexity result.

We recall the complexity of the geometric resolution algorithm in the specialised context in which we shall use it [12, Theorem 1].

Lemma 15. *Let $\boldsymbol{f} = (f_1, \ldots, f_p) \in \mathbb{K}[\mathbf{z}]^p$ be a dominant polynomial mapping from a smooth algebraic set defined by a reduced regular sequence $\boldsymbol{g} = (g_1, \ldots, g_m)$. Let $d = \max(\deg f_1, \ldots, \deg f_p, \deg g_1, \ldots, \deg g_m)$ and $D = d^{n-p-1} \sum_{i=0}^{p+1} \binom{n+p-1}{m+2p-i} d^i$. Fix some $1 \le j \le p$ and define $(h_1, \ldots, h_{n+m+p-1})$ to be the output of step 6 of Algorithm 2. Let $L_1, \ldots, L_{p+1}$ be generic linear forms of the variables of the h_i. Let $y_1, \ldots, y_{p+1} \in \mathbb{K}$ be such that the following system S is zero-dimensional,*

$$h_1 = \cdots = h_{n+m+p-1} = 0,\ L_1 = y_1, \ldots, L_{p+1} = y_{p+1},\ z_1 \neq 0.$$

Then, a geometric resolution of this system can be computed within

$$O^{\sim}((n+m+2p)^{d+3} D^2)$$

arithmetic operations in the base field $\mathbb{K}$.

Proof. Let δ be the degree of the system

$$h_1 = \cdots = h_{n+m+p-1} = 0,\ L_1 = y_1, \ldots, L_p = y_p,\ z_1 \neq 0.$$

Then, since the final equation we include is $L_{p+1} = y_{p+1}$ which has degree 1, by [12, Theorem 1], computing a geometric resolution of the zero-dimensional system S requires at most

$$O((n+m+2p)((n+m+2p)P + (n+m+2p)^{\Omega})\mathsf{M}(\delta)^2)$$

arithmetic operations in $\mathbb{K}$, where P is the evaluation complexity. Since $L_1, \ldots, L_p$ are generic linear forms, we have that $\delta = D$. Moreover, assuming that $d \geq 2$ is fixed, we may bound the evaluation complexity by $(n+m+2p)\binom{n+m+2p+D}{n+m+2p} = O((n+m+2p)^{d+1})$. Thus, excluding logarithmic factors, this step has complexity in the class

$$O^{\sim}((n+m+2p)^{d+3}D^2). \qquad \square$$

A particular case of interest is $p = 1$. Indeed, the study of this case allows one to tackle applications such as exact polynomial optimisation and other problems in computational real algebraic geometry. Hence, we first give a complexity result in this special case.

Theorem 3. *Let $f \in \mathbb{K}[\mathbf{z}]$ be a polynomial from a smooth algebraic set defined by a reduced regular sequence $\boldsymbol{g} = (g_1, \ldots, g_m)$. Let $d = \max(\deg f, \deg g_1, \ldots, \deg g_m)$ and $D = d^{n-2}\sum_{i=0}^{2}\binom{n}{m+2-i}d^i$. There exists an algorithm which, on input $f, \boldsymbol{g}$, outputs a non-zero polynomial $H \in \mathbb{K}[c]$ such that $K_\infty(\boldsymbol{f}) \subset \mathbf{V}(H)$ using at most*

$$O^{\sim}((n+m+2)^{d+3}D^5)$$

arithmetic operations in $\mathbb{K}$.

Proof. To prove this result, we shall analyse the complexity of Algorithm 2. We aim to construct a non-zero polynomial $H \in \mathbb{K}[c]$ such that $K_\infty(\boldsymbol{f}) \subset \mathbf{V}(H)$. Thus, we begin choosing a sufficiently generic linear change of coordinate A and vectors $\mathbf{a}, \mathbf{b} \in \mathbb{K}^n$ so that the results of Lemma 8 and Proposition 9 hold. Then, with $j = 1$, let G' by the result of step 6 of Algorithm 2 so that

$$G' = (h_1, \ldots, h_{n+m}) \subset \mathbb{K}[\mathbf{z}, c, e_1, \boldsymbol{\lambda}].$$

We are interested in the irreducible components of the algebraic set defined by G' that are not contained in $\mathbf{V}(z_1)$. By Proposition 9, these components have dimension at most 2. However, the geometric resolution algorithm that we will rely upon requires the input system to be zero-dimensional. Thus, knowing that we can lift the result of a specialised computation, we introduce two generic linear forms of the variables $(\mathbf{z}, c, e_1, \boldsymbol{\lambda})$ that when specialised will reduce $\mathbf{V}(G')$ to a zero-dimensional algebraic set [12, 28]. Let L_1, L_2 be these linear forms. Then, with $y_1, y_2 \in \mathbb{K}$ generic, consider the zero-dimensional system:

$$h_1 = \cdots = h_{n+m} = 0,\ L_1 = y_1, L_2 = y_2,\ z_1 \neq 0.$$

Using the geometric resolution algorithm of [12], with T another linear form, we compute a representation

$$\begin{cases} q(T) & = 0 \\ \frac{dq}{dT}(T)c & = v_1(T) \\ \frac{dq}{dT}(T)z_1 & = v_2(T) \\ & \vdots \\ \frac{dq}{dT}(T)z_n & = v_{n+1}(T) \\ \frac{dq}{dT}(T)e_1 & = v_{n+2}(T) \\ \frac{dq}{dT}(T)\lambda_1 & = v_{n+3}(T) \\ & \vdots \\ \frac{dq}{dT}(T)\lambda_m & = v_{n+m+2}(T) \end{cases}$$

of this system where $q, v_1, \ldots, v_{n+m+2} \in \mathbb{K}[T]$ have degree at most D, the degree bound given in Theorem 2. By Lemma 15, this requires at most

$$O^{\sim}((n+m+2)^{d+3}D^2)$$

arithmetic operations in $\mathbb{K}$.

We can then consider a lifted representation with polynomials of degree at most D in L_1, L_2, T using the algorithm given in [28].

$$\begin{cases} Q(L_1, L_2, T) & = 0 \\ \frac{dQ}{dT}(L_1, L_2, T)c & = V_1(L_1, L_2, T) \\ \frac{dQ}{dT}(L_1, L_2, T)z_1 & = V_2(L_1, L_2, T) \\ & \vdots \\ \frac{dQ}{dT}(L_1, L_2, T)z_n & = V_{n+1}(L_1, L_2, T) \\ \frac{dQ}{dT}(L_1, L_2, T)e_1 & = V_{n+2}(L_1, L_2, T) \\ \frac{dQ}{dT}(L_1, L_2, T)\lambda_1 & = V_{n+3}(L_1, L_2, T) \\ & \vdots \\ \frac{dQ}{dT}(L_1, L_2, T)\lambda_m & = V_{n+m+2}(L_1, L_2, T) \end{cases}$$

Note that V_i, Q are indeed polynomials as L_1, L_2 are generic linear forms. Hence the system is in Noether position and the number of solutions is constant for all specialisations, counted with multiplicities. We aim to compute the intersection of this system with $\mathbf{V}(z_1, e_1)$, as in step 8 in Algorithm 2. To do so, we compute the the projection of this system onto the (c, z_1, e_1)-space, and then will set z_1 and e_1 to zero. We accomplish this using evaluation-interpolation techniques. Specialising the L_i variables, eliminating T, and then interpolating the result. Therefore, we can in fact skip the lifting step and instead considering many different geometric resolutions by choosing different generic y_1, y_2. However, the existence of the lifted system will inform us on the degree of the polynomials we must interpolate.

Consider the first 2 equations of the specialised system, and eliminate the variable T by computing the resultant in T, $W = \mathrm{Res}_T(q, \frac{dq}{dT}c - v_1)$, a univariate polynomial in c. By [11, Corollary 11.21], we can compute this bivariate resultant within $O^{\sim}(D^2)$ arithmetic operations in $\mathbb{K}$. On the other hand, in the lifted system, we may express the polynomials V_1, V_2, V_{n+2}, Q as univariate polynomials in T by a Kronkecker substitution. Since L_1, L_2, T appear with degree at most D, the Kronecker substituted polynomials will have degree in the order of $O(D^3)$ in T [11, Chapter 8.4].

Therefore, we must specialise the system in $O(D^3)$ points in y_1, y_2 and compute the same number of geometric resolutions and resultants. We then interpolate the resulting polynomials to find a polynomials $F \in \mathbb{K}[c, z_1, e_1]$. By [11, Chapter 10.2], this can be accomplished within $O^{\sim}(D^3)$ operations.

Then, define $H(c) = F(c, 0, 0)$. We have that $\mathbf{V}(H)$ contains the algebraic set defined by the result of step 9 in Algorithm 2. By Theorem 14, the algebraic set has codimension at least 1 in $\mathbb{C}$ and so H is non-zero. Hence, the overall complexity is dominated by computing $O(D^3)$ geometric resolutions and is in the class

$$O^{\sim}((n+m+2)^{d+3}D^5). \qquad \square$$

In the case $p > 1$, we are no longer able to use a single resultant to eliminate the linear form T from the $p+1$ equations in the parametric representation we obtain from the geometric resolution algorithm. Thus, we opt for the FGLM algorithm to compute a representation where T is the greatest variable and so can be eliminated [9].

Theorem 4. *Let $\boldsymbol{f} = (f_1, \ldots, f_p) \in \mathbb{K}[\mathbf{z}]^p$ be a dominant polynomial mapping from a smooth algebraic set defined by a reduced regular sequence $\boldsymbol{g} = (g_1, \ldots, g_m)$. Let $d = \max(\deg f_1, \ldots, \deg f_p, \deg g_1, \ldots, \deg g_m)$. Let $D = d^{n-p-1} \sum_{i=0}^{p+1} \binom{n+p-1}{m+2p-i} d^i$. There exists an algorithm which, on input $\boldsymbol{f}$ and $\boldsymbol{g}$, outputs p finite lists of non-zero polynomials $G_i \subset \mathbb{K}[\mathbf{c}]$ such that $K_\infty(\boldsymbol{f}) \subset (\mathbf{V}(G_1) \cup \cdots \cup \mathbf{V}(G_p)) \subsetneq \mathbb{C}^p$ using at most*

$$O^{\sim}\left(p(p+1)D^{p+5} + (n+m+2p)^{d+3}D^{p+4}\right)$$

arithmetic operations in $\mathbb{K}$.

Proof. As in the proof of Theorem 3, we shall analyse the complexity of Algorithm 2 and so for each $1 \le j \le p$, we begin with the list of polynomials

$$G' = (h_1, \ldots, h_{n+m+p-1}) \subset \mathbb{K}[\mathbf{z}, \mathbf{c}, e_1, \boldsymbol{\lambda}].$$

By the proof of Theorem 2, the degree of $\mathbf{V}(G')$ is at most D. Moreover, by Proposition 9, the system

$$h_1 = \cdots = h_{n+m+p-1} = 0, \; z_1 \neq 0$$

has dimension $p+1$. Hence, we introduce $p+1$ generic linear forms, $L_1, \ldots, L_{p+1}$, of the variables $(\mathbf{z}, \mathbf{c}, e_1, \boldsymbol{\lambda})$ and specialise them to generic $y_1, \ldots, y_{p+1} \in \mathbb{K}$ respectively to reduce to a zero-dimensional algebraic set. Consider a parametric representation of the system

$$h_1 = \cdots = h_{n+m+p-1} = 0, \; L_1 = y_1, \ldots, L_{p+1} = y_{p+1}, \; z_1 \neq 0,$$

with T another linear form,

$$\begin{cases} q(T) & = 0 \\ \frac{dq}{dT}(T)c_1 & = v_1(T) \\ & \vdots \\ \frac{dq}{dT}(T)c_p & = v_p(T) \\ \frac{dq}{dT}(T)z_1 & = v_{p+1}(T) \\ & \vdots \\ \frac{dq}{dT}(T)z_n & = v_{n+p}(T) \\ \frac{dq}{dT}(T)e_1 & = v_{n+p+1}(T) \\ \frac{dq}{dT}(T)\lambda_1 & = v_{n+p+2}(T) \\ & \vdots \\ \frac{dq}{dT}(T)\lambda_{m+p-1} & = v_{n+m+2p}(T) \end{cases}$$

where $q, v_1, \ldots, v_{n+m+2p} \in \mathbb{K}[T]$ have degree at most D, the degree bound given in Theorem 2. By Lemma 15, such a representation can be computed using the geometric resolution algorithm within

$$O^{\sim}((n+m+2p)^{d+3}D^2)$$

arithmetic operations in the base field $\mathbb{K}$.

Consider the first $p+1$ equations of this rational parameterisation. Note that these form a Gröbner basis with respect to a lexicographic ordering with T as the least variable. Using the FGLM algorithm [9], we can compute a Gröbner basis defining the same ideal but with respect to a term ordering where T is the greatest variable, thereby eliminating T. Thus, let $\prec$ be the lexicographic monomial ordering $T > c_p > \cdots > c_1$. Let G_2 be the Gröbner basis with respect to the ordering $\prec$ returned by the FGLM algorithm with input basis $(q, \frac{dq}{dT}c_1 - v_1, \ldots, \frac{dq}{dT}c_p - v_p)$. Since the FGLM algorithm returns a reduced Gröbner basis and since the input polynomials system has degree D, we have that G_2 contains at most $(p+1)D$ polynomials. By [9, Theorem 5.1], this requires at most $O((p+1)D^3)$ arithmetic operations in $\mathbb{K}$.

We aim to compute the intersection of the system obtained from the FGLM algorithm with $\mathbf{V}(z_1, e_1)$. To do so, we shall interpolate polynomials in $\mathbf{c}, z_1, e_1$ from different systems obtained by many specialisations. As in Theorem 3, by [28], there exists a lifted representation with polynomials, since we have Noether position, of degree at most D in $L_1, \ldots, L_{p+1}, T$.

$$\begin{cases} Q(L_1,\dots,L_{p+1},T) & = 0 \\ \frac{dQ}{dT}(L_1,\dots,L_{p+1},T)c_1 & = V_1(L_1,\dots,L_{p+1},T) \\ & \vdots \\ \frac{dQ}{dT}(L_1,\dots,L_{p+1},T)c_p & = V_p(L_1,\dots,L_{p+1},T) \\ \frac{dQ}{dT}(L_1,\dots,L_{p+1},T)z_1 & = V_{p+1}(L_1,\dots,L_{p+1},T) \\ & \vdots \\ \frac{dQ}{dT}(L_1,\dots,L_{p+1},T)z_n & = V_{n+p}(L_1,\dots,L_{p+1},T) \\ \frac{dQ}{dT}(L_1,\dots,L_{p+1},T)e_1 & = V_{n+p+1}(L_1,\dots,L_{p+1},T) \\ \frac{dQ}{dT}(L_1,\dots,L_{p+1},T)\lambda_1 & = V_{n+p+2}(L_1,\dots,L_{p+1},T) \\ & \vdots \\ \frac{dQ}{dT}(L_1,\dots,L_{p+1},T)\lambda_{m+p-1} & = V_{n+m+2p}(L_1,\dots,L_{p+1},T). \end{cases}$$

We may express the corresponding polynomials $V_1,\dots,V_{p+1},V_{n+p+1},Q$ as univariate polynomials in T by a Kronecker substitution. Since $L_1,\dots,L_{p+1},T$ appear with degree at most D, the Kronecker substituted polynomials will have degree in the order of $O(D^{p+2})$ in T [11, Chapter 8.4]. Therefore, the polynomials we wish to interpolate have at most the same degree and so we must specialise the system in $O(D^{p+2})$ points in $y_1,\dots,y_{p+1}$. We then interpolate the resulting polynomials to find polynomials $F_1,\dots,F_{(p+1)D} \in \mathbb{K}[\mathbf{c},z_1,e_1]$. By [11, Chapter 10.2], this can be accomplished within $O^{\sim}(D^{p+2})$ operations. Define $G_i(\mathbf{c}) = F_i(c_1,\dots,c_p,0,0)$. Then, the polynomials $(G_1,\dots,G_{(p+1)D})$ define an algebraic set containing the algebraic set defined by the result of step 9 in Algorithm 2.

Therefore, for each $1 \le j \le p$, we output a seperate list of these G_i. Hence, the overall complexity is given by calling the geometric resolution and FGLM algorithms $O(D^{p+2})$ times and so is in the class

$$O^{\sim}\left(p(p+1)D^{p+5} + (n+m+2p)^{d+3}D^{p+4}\right). \qquad \square$$

7. Alternate description of the Jacobian condition

In this section, we develop a different interpretation of the geometric characterisation of the asymptotic critical values given in Section 2. Instead of a Lagrange multiplier based approach, we construct a basis of the kernel of $\mathrm{jac}(\boldsymbol{f},\boldsymbol{g})^{[j]}$ by introducing a matrix of new variables. Thus, define the set of variables $\mathbf{u} = \{u_{i,k} : 1 \le i \le n, 1 \le k \le n-m-p+1\}$ and the variable matrix

$$M_U = \begin{bmatrix} u_{1,1} & \cdots & u_{1,n-m-p+1} \\ \vdots & \ddots & \vdots \\ u_{n,1} & \cdots & u_{n,n-m-p+1} \end{bmatrix}.$$

Firstly, we introduce the equations

$$\mathrm{jac}(\boldsymbol{f},\boldsymbol{g})^{[j]} \cdot M_U = 0,$$

so that the columns of the matrix M_U are elements of the kernel of $\mathrm{jac}(\boldsymbol{f}, \boldsymbol{g})^{[j]}$. Then, we ensure that the matrix M_U has full rank by introducing a matrix of sufficiently generic scalars $T_U \in \mathbb{K}^{(n-m-p+1)\times n}$ and the equations

$$T_U M_U = \mathrm{Id}_{n-m-p+1},$$

where $\mathrm{Id}_{n-m-p+1}$ is the identity matrix of size $n-m-p+1$.

Lemma 16. *There exists a proper Zariski open subset $\mathcal{O}_M$ of $\mathbb{K}^{(n-m-p+1)\times n}$ such that if $T_U \in \mathcal{O}_M$ then $T_U M_U = \mathrm{Id}_{n-m-p+1}$ implies that* $\mathrm{rank}(M_U) = n-m-p+1$.

Proof. Consider an $(n-m-p+1) \times n$ variable matrix. Then, the list of maximal minors of this matrix defines a proper Zariski closed subset of $\mathbb{K}^{(n-m-p+1)\times n}$ where the specialisations of T_U do not have full rank. Let $\mathcal{O}_M$ be the complement of this Zariski closed subset. Suppose $T_M \in \mathcal{O}_M$, then T_M has full rank and so $T_U M_U = \mathrm{Id}_{n-m-p+1}$ implies that $\mathrm{rank}(M_U) = n-m-p+1$. ☐

With the equations defined by Lemma 16, the columns of the matrix M_U are defined to be linearly independent. Therefore, since the kernel of $\mathrm{jac}(\boldsymbol{f}, \boldsymbol{g})^{[j]}$ has dimension $n-m-p+1$ outside of a proper Zariski-closed subset of $\mathbb{C}^n$, they form a basis of the kernel. Thus, we give Algorithm 3, an alternative version to Algorithm 2 which, as we shall prove, terminates and returns the same output.

Algorithm 3: acv3

Input: $\boldsymbol{g}$ a reduced regular sequence defining a smooth algebraic set X, $\boldsymbol{f} : X \to \mathbb{K}^p$ a dominant polynomial mapping with components in the ring $\mathbb{K}[\mathbf{z}]$, a variable matrix M_U of size $n \times (n-m-p+1)$ with entries in the variable list $\mathbf{u}$ and the list $\mathbf{z}$.

Output: R, a finite list of polynomials whose zero set has codimension at least 1 in $\mathbb{C}^p$ and contains the set of asymptotic critical values of $\boldsymbol{f}$.

Generate a random scalar matrix T_U with entries $t_{1,1}, \dots, t_{n-m-p+1,n} \in \mathbb{K}$.
Generate a random change of variables $A \in \mathbb{K}^{n\times n}$.
Generate random numbers $\mathbf{a}, \mathbf{b} \in \mathbb{K}^{n-m-p+1}$ and set $\boldsymbol{f}^A \leftarrow \boldsymbol{f}(Az)$, $\boldsymbol{g}^A \leftarrow \boldsymbol{g}(Az)$.
For j **from** 1 **to** p **do**
  $R_U \leftarrow$ List of polynomials $T_U M_U - \mathrm{Id}_{n-m-p+1}$.
  $J_U \leftarrow$ List of equations of $\mathrm{jac}(\boldsymbol{f}^A, \boldsymbol{g}^A)^{[j]} M_U$.
  $(v_1(z), \dots, v_{n-p-m+1}(z)) \leftarrow \mathrm{d}f_j^A M_U - \mathbf{a}e_1$.
  $N'(z) \leftarrow \{f_1^A - c_1, \dots, f_p^A - c_p, g_1^A, \dots, g_m^A, b_2 v_1 - b_1 v_2, \dots, b_{n-m-p+1} v_1 - b_1 v_{n-m-p+1}\} \cup J_U$.
  $G' \leftarrow \mathrm{numer}(N'(\tau_1(z))) \cup R_U$.
  $G'_s \leftarrow \mathrm{Saturate}(G', z_1)$.
  $L' \leftarrow G'_s \cup \{z_1, e_1\}$.
  $V'_j \leftarrow \mathrm{Eliminate}(L', \{\mathbf{z}, e_1, \mathbf{u}\}, \{\mathbf{c}\})$.
$\mathrm{R}' \leftarrow \mathrm{Intersect}(V'_1, \dots, V'_p)$.
Return R'.

Theorem 17. *Let $\boldsymbol{f} = (f_1, \ldots, f_p) \in \mathbb{K}[\mathbf{z}]^p$ be a dominant polynomial mapping from a smooth algebraic set defined by a reduced regular sequence $\boldsymbol{g} = (g_1, \ldots, g_m)$. Suppose that $A \in \mathrm{GL}_n(\mathbb{K})$ satisfies the genericity condition of Lemma 8 and suppose that $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$ define a plane $E \subset \mathbb{C}^n$ that satisfies the genericity condition of Proposition 9. Suppose that $\mathrm{jac}(H_j)$ has full rank for all j. Suppose that the projection map π from $\overline{\mathrm{graph}\, M_s^j(\boldsymbol{f}^A)}$ to $\mathbb{C}^n$ is dominant. Then, Algorithm 3 terminates and returns as output a finite basis whose zero set has codimension at least 1 in $\mathbb{C}^p$ and contains the set of asymptotic critical values of $\boldsymbol{f}$.*

Proof. Since the genericity condition of Lemma 16 holds, the equations $R_U \cup J_U$ give conditions for the columns of the matrix M_U to define a basis for the kernel of $\mathrm{jac}((\boldsymbol{f}^A, \boldsymbol{g}^A)^{[j]})$. Thus, the equations $f_j^A M_U$ define w_j, the restriction of the differential $\mathrm{d}f_j$ to the kernel.

The remainder of the algorithm follows exactly Algorithm 2, with only one exception. The projection onto the **c**-space now requires the elimination of the newly introduced variables $\mathbf{u}$ instead of $\boldsymbol{\lambda}$. Thus, by Theorem 14, Algorithm 3 terminates and returns as output a finite basis whose zero set has codimension at least 1 in $\mathbb{C}^p$ and contains the set of asymptotic critical values of $\boldsymbol{f}$. □

Remark 18. *One can perform a similar analysis of the degree of the objects computed in, and the complexity of, Algorithm 3 as is done for Algorithm 2 in Sections 5 and 6. Indeed, one can take advantage of both the multi-homogeneous structure of the polynomials constructed in step 9 as well as the number of linear forms from Lemma 16. Thus, using the multi-homogeneous Bézout bound, one can arrive at the following formula [27, Proposition 3].*

$$pd^m \sum_{i=0}^{n-m} \sum_{j=n-m-p-1-i}^{n-m-i} \binom{n-m}{i} \binom{(m+p-1)(n-m-p+1)}{j} d^i (d-1)^j.$$

Moreover, we saw that the number of variables is an important factor in the complexity of the geometric resolution algorithm used in Section 6 to perform the ideal theoretic operations [12]. Algorithm 3 works within a polynomial ring with $n(n-m-p+1)-m-p$ more variables than Algorithm 2. Hence, this leads to a worse bound on the degree and the worst-case complexity as $n \to \infty$ whenever $m + p < n$.

Note that Algorithm 3 introduces many more variables than Algorithm 2 which leads to a worse arithmetic complexity. However, as we will see in Section 9, there are some problems for which Algorithm 3 is faster.

8. Applications

8.1. Solving Polynomial Optimisation Problems

In this subsection we present how to use the algorithms detailed in this paper to solve polynomial optimisation problems without inequalities.

Firstly, we review the problem we wish to solve. Consider a polynomial $f \in \mathbb{Q}[\mathbf{z}]$. We aim to compute the infimum of this polynomial over a smooth algebraic set X defined by a reduced regular sequence $\boldsymbol{g}$, $\inf_{\boldsymbol{x} \in X} f(\boldsymbol{x}) = f^* \in \mathbb{R} \cup \{-\infty\}$. We can solve this problem exactly by computing the generalised critical values of f restricted to X.

There are three cases:

- f^* is reached. Then, f^* is a critical value of f;
- f^* is reached only at infinity, meaning that there is no minimiser $\boldsymbol{x} \in X$ but instead a path $\boldsymbol{x}_t \in \mathbb{R}^n$ that approaches the infimum as $\|\boldsymbol{x}_t\| \to \infty$. Then, f^* is an asymptotic critical value of f;
- $f^* = -\infty$.

Note that this methodology allows for the consideration of a non-compact domain X. The procedure is as follows: We first compute an algebraic representation of the generalised critical values of f restricted to X. One method to accomplish this is to compute asymptotic critical values and classical critical values separately. Firstly, we compute a polynomial whose roots contain the asymptotic critical values by using, for example Algorithm 2. Then, by the Jacobian criterion [8, Corollary 16.20], one can compute a geometric resolution of the system comprised of the polynomials $f - c$, $\boldsymbol{g}$ and the maximal minors of the Jacobian of f and $\boldsymbol{g}$, to obtain a polynomial representation of the critical values of f.

There are algebraic elimination algorithms that compute such polynomials with rational coefficients, for example Gröbner bases [7, Chapter 2] or the geometric resolution algorithm designed in [12], since we assumed that $\boldsymbol{f} \in \mathbb{Q}[\mathbf{z}]$. See Section 6 for a discussion on implementing Algorithm 2 using the geometric resolution algorithm. Thus, after finding a common denominator, we may assume these polynomials have integer coefficients. Then, we may use a real root isolation algorithm such as in [23], based on Descartes' rule of sign [1, Theorem 2.44], to compute isolating intervals with rational endpoints for all real roots of these polynomials.

Let $C = \{c_1, \ldots, c_k\} \subset \mathbb{R}$ be the finite set of real algebraic numbers that are the real roots of the above polynomials. Then, the set C contains the generalised critical values of f. By [15, Theorem 3.1], the polynomial f with restricted domain $f : X \setminus f^{-1}(K(f)) \to \mathbb{R} \setminus K(f)$ is a locally trivial fibration over each connected component of $\mathbb{R} \setminus K(f)$. Therefore, since C is finite, the restriction $f : X \setminus f^{-1}(C) \to \mathbb{R} \setminus C$ is also a locally trivial fibration. Hence, to decide the emptiness of each connected component of $\mathbb{R} \setminus C$, it is sufficient to decide the emptiness of one fibre for each connected component.

After computing the isolating intervals for the elements of C, we may now choose rational numbers $r_1, \ldots, r_k$ so that

$$r_1 < c_1 < r_2 < \cdots < r_k < c_k.$$

We must assess the emptiness of the fibres of these values. We do so using the algorithm designed in [26]. We consider, for $1 \le i \le k$, the ideal $\langle f - r_i \rangle$. This algorithm requires a radical ideal such that $\mathbf{V}(f - r_i)$ is smooth and equidimensional. Since r_i is outside of these isolating intervals, we have that $\mathbf{V}(f - r_i)$ is smooth and equidimensional. Furthermore, while $\langle f - r_i \rangle$ may not be radical, we have that $\mathbf{V}(\sqrt{\langle f - r_i \rangle}) = \mathbf{V}(f - r_i)$ and so we consider $\sqrt{\langle f - r_i \rangle}$ to decide the emptiness of $\mathbf{V}_{\mathbb{R}}(f - r_i) = \mathbf{V}(f - r_i) \cap \mathbb{R}^n$.

Firstly, if $\mathbf{V}_{\mathbb{R}}(f - r_1)$ is non-empty then we must be in the third case and so $f^* = -\infty$. For the remaining two cases, let c_j be the least critical value and let i be the least index such that $\mathbf{V}_{\mathbb{R}}(f - r_i)$ is non-empty, if such an index or critical value exist. If $r_i > c_j$, which one may decide from the isolating intervals, then c_j is the minimum of f. Otherwise, $r_i \le c_j$ and c_{i-1} is an asymptotic critical value and is the infimum of f. Finally, if such an index does not exist, then c_j is the minimum of f and if f also does not have any critical values, then the infimum is c_k.

We consider the complexity of the algorithm for polynomial optimisation described above. For a polynomial $f \in \mathbb{Q}[\mathbf{z}]$ and reduced regular sequence $\boldsymbol{g}$ of degree at most d, we first compute a polynomial representation of $K(f)$. With

$$D = \binom{n}{m+2} d^{n-2} + \binom{n}{m+1} d^{n-1} + \binom{n}{m} d^n,$$

by Theorem 3, one can compute a polynomial representation of the asymptotic critical values with complexity

$$O^{\sim}\left((n+m+2)^{d+3} D^5\right).$$

By [10, Corollary 2], the set of critical values has degree at most $d^m(d-1)^{n-m}\binom{n}{m}$. Hence, with the geometric resolution algorithm, one can compute a polynomial representation of the critical values within

$$O^{\sim}\left((n+1)^{d+3} d^{2m} (d-1)^{2(n-m)} \binom{n}{m}^2\right)$$

arithmetic operations in $\mathbb{Q}$ [12]. By 2 and [10, Corollary 2], the product of these polynomials has at most

$$\Delta = \left(\binom{n}{m+2} d^{n-2} + \binom{n}{m+1} d^{n-1} + \binom{n}{m} d^n\right) + d^m (d-1)^{n-m} \binom{n}{m}$$

roots and hence f has at most this many generalised critical values. With β bounding the bit-size of the input polynomial, isolating the real roots with the algorithm designed in [23] requires $O(\beta\Delta^4)$ operations. We must then choose at most $d^n + 1$ points in $\mathbb{Q}$, the $r_1, \ldots, r_\Delta$ as above, and decide the emptiness of each $\mathbf{V}_{\mathbb{R}}(f - r_i)$. This requires the use of the algorithm designed in [26] at most Δ times with each computation requiring $O(n^7\Delta^3)$ operations. Thus, one can compute an isolating interval for the infimum of a polynomial $f \in \mathbb{Q}[\mathbf{z}]$ restricted to an algebraic set defined by a reduced regular sequence with degrees at most d in approximately $O^{\sim}(n^7\Delta^4 + (n+m+2)^{d+3}D^5)$ arithmetic operations in $\mathbb{Q}$.

Example 19. *Consider the polynomial $f = z_1^2 z_2^2 + 2z_1 z_2^3 + z_2^4 + z_1^2 + 3z_1 z_2 + 2z_2^2$. First, we compute the set of generalised critical values. Note that in this simple example it is possible to find exactly the real algebraic numbers that contain the generalised critical values because the degrees of the polynomials we compute in our algorithms are small. We find that $K_0(f) = \{0\}$ and using Algorithm 2 we find $K_\infty(f) \subset \{-\frac{1}{4}\}$. Now, to show that $f^* = -\frac{1}{4}$ one must first show that f is bounded from below. To do so, decide the emptiness of the real variety $\mathbf{V}_{\mathbb{R}}(f - r)$ for some rational number $r < -\frac{1}{4}$. For example, we can choose $r = -1$ and find that this variety is indeed empty. Finally, one must show that $-\frac{1}{4}$ truly is an asymptotic critical value as Algorithm 2 computes a superset of the asymptotic critical values. Thus, one shows that f takes values less than 0 by once again deciding the emptiness of a fibre. So, consider the variety $\mathbf{V}_{\mathbb{R}}(f + \frac{1}{8})$ and find that it is not empty. This shows that f takes values less than 0 and since*

$$f|_{\mathbb{R}^2 \setminus f^{-1}\{0, -\frac{1}{4}\}} \to \mathbb{R} \setminus \{0, -\frac{1}{4}\}$$

is a locally trivial fibration [15, Theorem 3.1], we conclude that the infimum of f is $-\frac{1}{4}$.

Example 20. *Consider the polynomial* $f = z_1^3 + z_1^2 z_2^2 - 2z_1 z_2 + 1$*. We find that* $K_0(f) = \{1\}$ *and* $K_\infty(f) \subset \{0\}$*. We first test the third case. Take a value less than* 0*, for example* -1*, and decide the emptiness of* $\mathbf{V}_\mathbb{R}(f+1)$*. We find that this fibre is not empty and so by [15, Theorem 3.1], we conclude that* $f^* = -\infty$*.*

For more information on solving polynomial optimisation problems, we refer to [13, 25, 29].

8.2. Deciding the emptiness of semi-algebraic sets defined by a single inequality

In this subsection, we continue to explore the applications of algorithms computing generalised critical values. Let $f \in \mathbb{Q}[\mathbf{z}]$ be a polynomial with degree d and consider the semi-algebraic set S defined by the single inequality $f > 0$. The goal is to test the emptiness of the set S and in the case that S is not empty to compute at least one point in each connected component. There exists $e \in \mathbb{Q}^+$ small enough such that the problem is reduced to computing at least one point in each connected component of the real algebraic set $\mathbf{V}_\mathbb{R}(f - e)$. Such an e is small enough in this sense if it is less than the least positive generalised critical value of the map $z \in \mathbb{R}^n \to f(z) \in \mathbb{R}$, we refer to [24, Theorem 5.1]. To decide when this is the case, one computes isolating intervals for the generalised critical values by [1, Algorithm 10.63]. Once an appropriate e has been chosen, it remains to compute at least one point in each connected component of $\mathbf{V}_\mathbb{R}(f - e)$. This may be accomplished using the algorithm designed in [26]. To apply this algorithm, we require that $\langle f - e\rangle$ is radical and $\mathbf{V}(f - e)$ is equidimensional and smooth. Since e is away from any generalised critical values we have that $\mathbf{V}(f - e)$ is equidimensional and smooth. Moreover, if $\langle f - e\rangle$ is not radical, we may simply take the square-free part of $f - e$ instead as $\mathbf{V}(\sqrt{\langle f - e\rangle}) = \mathbf{V}(f - e)$.

As in the previous application, the complexity of computing isolating intervals for all real generalised critical values is in the class $O^\sim(n^7 d^{4n})$. After choosing an appropriate rational number e, it remains to apply the algorithm designed in [26]. This requires $O(n^7 d^3 n)$ operations. Therefore, the overall complexity of deciding the emptiness of the semi-algebraic set defined by $f > 0$ is in the class $O^\sim(n^7 d^{4n})$. Moreover, in the case where this set is not empty, at least one point in each connected component is computed.

Example 21. *Consider the polynomial* $f = z_1^2(1 - z_2) - (z_1 z_2^2 - 1)^2$*. Again, in this simple example we obtain polynomials of degree at most* 2 *from our algorithms and so we can give explicitly the set containing the generalised critical values. The polynomial giving the asymptotic critical values is* c *while for the critical values it is* $229c^2 - 202c - 27$*. Hence, we find that* $K(f) \subset \{0, 1, \frac{-27}{229}\}$*. We note that the value* 1 *is a critical value, hence we may decide immediately that the semialgebraic set defined by* $f > 0$ *is non-empty. Now, to compute at least one sample point in each connected component of this set, we must choose a suitable fibre to investigate. Thus, we choose a rational value greater than* 0 *and less than the least critical value, such as* $\frac{1}{2}$*, and use the algorithm in [26] to compute sample points for each connected component of* $\mathbf{V}_\mathbb{R}(f - \frac{1}{2})$*. We may do so because* $\langle f - \frac{1}{2}\rangle$ *is a radical ideal. Let* α *be a real root of* $x^4 + x - 1$*. Then,*

$$(z_1, z_2) = \left(\frac{3}{4}(\alpha^3 + \alpha^2 + 1), \alpha\right)$$

is a sample point.

9. Experiments

The three algorithms given in this paper have been implemented in the MAPLE computer algebra system [19]. For our timing results, we use the Groebner package implemented in MAPLE to perform the algebraic eliminations. Alternatively, to obtain our degree results, we use MSOLVE [4], implemented in C, for the Gröbner basis computations. We present the experimental results of these implementations with computations performed on a computing server with 1536 GB of memory and an Intel Xeon E7-4820 v4 2GHz processor. To closer analyse our algebraic complexity result, all computations were performed over finite fields so as to avoid additional computation time due to coefficient growth. We choose the finite field $\mathbb{F}_{2147483647} = \mathbb{F}_{2^{31}-1}$ so that the probability of choosing bad random values in our algorithms is low. All computations that could not be completed within two days have been given the entry ∞ in Tables 1 and 2 and the entry N/A in Tables 3 and 4.

With the intention of comparison, we have attempted to implement the algorithm by Kurdyka and Jelonek, given in [15, Section 5.1], that computes the set of generalised critical values of a polynomial mapping whose domain is restricted to an algebraic set. However, we see that this algorithm fails for some examples, such as $f = x^2 + (xy - 1)^2$ restricted to $\mathbf{V}(xy - 1)$. In Example 1, we saw that $0 \in \mathbb{K}_\infty(\boldsymbol{f})$, found along the path $(x, y) = (t, 1/t)$ as $t \to 0$. However, our implementation finds no values. Moreover, we understand that there may be some typos in the presentation of the algorithm. Based on our reading of this paper and the results obtained, we attempted another implementation fixing these mistakes. However, for the same example, we still fail to find the asymptotic critical value. On the other hand, in the global setting, one can infer an algorithm from the results of [18, Section 4] that is similar to a version of Algorithm 1 where we do not apply Proposition 9. This means we consider the polynomials directly as given in Lemma 6. Hence, an implementation of this algorithm, under the name acv0, will be compared to the algorithms designed in this paper. As we will see, this will illustrate the efficiency that Algorithms 1, 2 and 3 get from applying Proposition 9.

To further aid comparison, we implement versions of all these algorithms with and without a generic linear change of coordinates. This means not applying Lemma 8 and so we must compute np sets instead of p. However, while our complexity and degree results rely on a generic linear change of coordinates, for some problems this change can have a negative effect on the efficiency of the algorithm. This is to be expected for some sparse problems as such a generic change of coordinates destroys all structure in the input and means we perform operations on polynomials with dense support.

For our implementations of the algorithms given in this paper, and of the algorithm presented in [15, Section 5.1], see the webpage `https://www-polsys.lip6.fr/~ferguson/acv_algorithms.html`.

For the purpose of comparing the algorithms we develop, we introduce a number of families of polynomial mappings that have asymptotic critical values. Firstly, in the global setting, we give three families of polynomials. For $n \geq 2$, let

$$f_n = z_1^2 + \sum_{i=2}^{n} (z_1 z_i - 1)^2, \quad g_n = \sum_{i=1}^{n} \frac{\prod_{j=1}^{n} z_j^2}{z_i^2}, \quad h_n = \sum_{i=1}^{n} \prod_{j=1}^{i} z_j^{2^{i-j}}.$$

For $n \geq 2$, each of these polynomials has an asymptotic critical value at 0. For $n \geq 3$,

	with A				without A			
	acv0	acv1	acv2	acv3	acv0	acv1	acv2	acv3
System	time (s)							
f_{20}	∞	3.3	2.4	220	650	3.0	1.5	230
f_{40}	∞	150	130	∞	∞	29	18	∞
f_{60}	∞	2300	1600	∞	∞	120	84	∞
g_4	∞	8.4	0.028	0.3	2700	6.7	0.044	0.86
g_6	∞	∞	19	1300	∞	∞	5.1	21000
g_8	∞	∞	83000	∞	∞	∞	1500	∞
h_3	0.46	0.21	0.020	0.070	0.068	0.20	0.017	0.20
h_4	∞	230	0.47	21000	∞	∞	0.59	∞
h_5	∞	∞	120	∞	∞	∞	4200	∞
d_2n_{20}	21	0.10	0.15	2.9	450	0.35	0.35	83
d_2n_{100}	∞	160	160	∞	∞	20	26	∞
d_3n_5	∞	13000	0.075	0.14	∞	63000	0.21	0.33
d_3n_7	∞	∞	0.42	1.6	∞	∞	1.1	8.3
d_4n_4	∞	0.13	0.38	1.2	∞	∞	1.1	0.71
d_4n_6	∞	∞	3.7	22	∞	∞	18	120

Table 1: Timings for global systems given to 2 significant figures.

f_n also has an asymptotic critical value at n. Additionally, we consider two families of polynomial mappings restricted to algebraic sets. For $n \geq 2$, let

$$\alpha_n : \mathbf{V}(z_1z_2 - 1, \dots, z_1z_n - 1) \to \mathbb{C}, \quad \alpha_n(z) = z_1^2 + (z_1z_2 - 1)^2 + \cdots + (z_1z_n - 1)^2,$$
$$\beta_n : \mathbf{V}(z_1^3z_2 \cdots z_n - 1) \to \mathbb{C}, \quad \beta_n(z) = z_1 \cdots z_n.$$

For $n \geq 3$, the map β_n has an asymptotic critical value at 0. The polynomial mapping α_n, extended from Example 1, also has an asymptotic critical value at 0 for all $n \geq 2$. We note that the critical locus of α_n, β_n is empty, so these asymptotic critical values are non-trivial. The system α_n has a fixed degree of 4 for all n is restricted to an algebraic set defined by $n-1$ polynomials each of degree 2. This allows us to test how our algorithms behave as we greatly increase the number of variables and the number of constraints. On the other hand, β_n has linear degree in n and has one restraint of degree $n+2$. Additionally, we compare these algorithms with random dense polynomials in both the global setting and under the restriction to a hypersurface defined by a random dense polynomial of the same degree. We denote this type of system, with degrees d in k variables, by d_sn_k.

9.1. Timing experiments

In Table 1 and 2, we see that for structured systems like α_n and β_n, the generic linear change of coordinates increases the computation time. This can be explained by two factors: Firstly, the change of coordinates destroying the sparsity in the polynomials. Secondly, when there are many variables, the application of the linear change of variables A becomes more time consuming. For example, to solve the examples d_2n_{20}

	with A				without A			
	acv0	acv1	acv2	acv3	acv0	acv1	acv2	acv3
System	time (s)							
α_{10}	0.82	0.15	0.075	0.039	0.66	0.21	0.20	0.092
α_{20}	53	1.3	1.2	0.61	23	2.1	2.3	1.0
α_{30}	720	9.5	10	6.8	240	9.3	9.6	4.8
α_{40}	5200	42	39	36	1600	28	29	16
α_{50}	∞	110	110	86	5300	73	75	46
α_{60}	∞	280	280	210	∞	160	150	110
β_4	5.1	0.33	0.25	0.26	0.19	0.25	0.18	0.34
β_5	300	2.0	0.67	4.0	0.75	0.97	0.67	2.6
β_6	∞	7.5	3.1	7.2	3.9	2.7	2.1	5.5
β_7	∞	41	9.9	120	21	7.2	4.2	41
β_8	∞	190	38	420	130	14	13	55
β_9	∞	1000	240	∞	1100	35	25	370
d_2n_4	18	0.37	0.026	0.072	69	1.4	0.079	0.29
d_2n_6	∞	7.2	0.10	0.35	∞	41	0.30	2.0
d_3n_3	21000	220	0.21	280	59000	670	0.59	820
d_4n_2	2.1	2.2	0.19	0.013	3.9	5.1	0.33	0.020
d_4n_4	∞	∞	5300	∞	∞	∞	22000	∞
d_6n_2	660	770	1.2	0.050	1400	1500	2.3	0.082

Table 2: Timings for restricted systems given to 2 significant figures.

and d_2n_{100} applying A takes almost all computation time at around 0.1 and 160 seconds respectively. Similarly for the families f_n and α_n, applying the linear change of variables takes around half the time due to the large number of variables. Moreover, for generic systems, the change of coordinates effectively does not change the system. Hence, excluding the time spent applying A, the change of coordinates decreases computation time by approximately a factor of n, the number of variables, due to the algorithm computing one set instead of n sets.

Note that by considering the symmetry in the problem, one could improve the efficiency of our algorithms further. For example, for α_n and β_n, there is only one special variable, z_1. All other variables are symmetric and so the asymptotic critical values computed without a generic linear change of variables in the second to the nth set are the same. Therefore, one only needs to compute two sets, instead of n. Such symmetry reductions resulting in more efficient algorithms are a topic of future study.

From the timings presented in Table 2, the benefit of applying Proposition 9 is clear. Algorithms 1 and 2, which rely on this geometric result, are in general significantly faster than acv0. We note the special case $n = 2$, where Algorithm 1 can be slower than acv0. This is because in this setting we do not decrease the dimension of the algebraic sets we consider when we apply Proposition 9. However, we find that Algorithm 2 is in general faster than both acv0 and Algorithm 1. We also observe that the different formulations of this result, Algorithms 2 and 3, can have different behaviours depending on the problem. For example, Algorithm 2 computes the asymptotic critical values of β_n faster but Algorithm 3 is better at handling α_n as we increase the number of variables.

	Degree bound				True degree	
Sys.	acv2	Theorem 2	Crit. Values	[15, Theorem 4]	$K_\infty(\boldsymbol{f})$	$K(\boldsymbol{f})$
f_{20}	4	1.97×10^{13}	3.49×10^{9}	1.10×10^{12}	3	3
f_{40}	4	7.22×10^{25}	1.21×10^{19}	1.21×10^{24}	3	3
f_{60}	4	1.68×10^{38}	4.24×10^{28}	1.33×10^{36}	3	3
g_4	42	2 376	625	1 296	1	1
g_6	162	1 750 000	531 441	1 000 000	1	1
g_8	420	2 529 924 096	815 730 721	1 475 789 056	1	1
h_4	124	65 475	38 416	50 625	1	1
h_5	N/A	33 544 666	24 300 000	28 629 151	1	1
h_6	N/A	68 714 415 882	56 800 235 584	62 523 502 209	1	1
d_2n_{20}	3	61 341 696	1	1 048 576	0	1
d_2n_{100}	3	1.63×10^{33}	1	1.27×10^{30}	0	1
d_3n_5	64	918	32	243	0	32
d_3n_7	256	12 393	128	2 187	0	128
d_4n_4	135	608	81	256	0	81
d_4n_6	1215	14 080	729	4 096	0	729

Table 3: Degree of asymptotic/generalised critical values in the unrestricted case.

9.2. Degree experiments

We consider the degree of the algebraic set defined by the list of polynomials constructed in step 6 which is the basis of Theorem 2. Then, we give the bound of Theorem 2 as well as a bound on the number of critical values given in [10, Corollary 2] and compare this to the bound on the generalised critical values given in [15, Theorem 4].

In Table 3, we see that for unrestricted systems, the bound of [15, Theorem 4] is better. However, in Table 4 our degree bound is significantly smaller outside of a few cases where the parameters n and d are small. We note that the polynomial systems we compute in Algorithm 2 do not reach the bound of Theorem 2. Moreover, we are unaware of any examples of polynomial systems with a large number of asymptotic critical values, since generic systems contain no such values.

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	Degree bound				True degree	
Sys.	acv2	Theorem 2	Crit. Values	[15, Theorem 4]	$K_\infty(\boldsymbol{f})$	$K(\boldsymbol{f})$
α_{10}	2	30 146 560	787 320	161 414 428	1	1
α_{20}	2	$1.53\ \times 10^{14}$	$9.30\ \times 10^{10}$	$4.56\ \times 10^{16}$	1	1
α_{30}	2	$4.53\ \times 10^{20}$	$8.23\ \times 10^{15}$	$1.29\ \times 10^{25}$	1	1
α_{40}	2	$1.03\ \times 10^{27}$	$6.48\ \times 10^{20}$	$3.64\ \times 10^{33}$	1	1
α_{50}	2	$2.00\ \times 10^{33}$	$4.79\ \times 10^{25}$	$1.03\ \times 10^{42}$	1	1
α_{60}	2	$3.51\ \times 10^{39}$	$3.39\ \times 10^{30}$	$2.90\ \times 10^{50}$	1	1
β_4	21	6 624	3 000	7 986	1	1
β_5	25	111 475	45 360	199 927	1	1
β_6	29	2 146 304	806 736	6 075 000	1	1
β_7	33	46 707 759	16 515 072	217 238 121	1	1
β_8	37	1 136 000 000	382 637 520	8 938 717 390	1	1
β_9	41	30 575 371 299	9 900 000 000	416 051 452 971	1	1
d_2n_4	36	128	8	54	0	8
d_2n_6	64	1 184	12	486	0	12
d_3n_3	75	111	36	75	0	36
d_4n_2	32	36	24	28	0	24
d_4n_4	792	1 472	432	1 372	0	432
d_6n_2	72	78	60	66	0	60
d_6n_6	N/A	422 496	112 500	966 306	0	112 500

Table 4: Degree of asymptotic/generalised critical values in the restricted case.

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