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Abstract

We prove that certain sequences of Laurent polynomials, obtained from a fixed Laurent polynomial P by monomial substitutions, give rise to sequences of Mahler measures which converge to the Mahler measure of P. This generalizes previous work of Boyd and Lawton, who considered univariate monomial substitutions. We provide moreover an explicit upper bound for the error term in this convergence, generalizing work of Dimitrov and Habegger, and a full asymptotic expansion for a family of 2-variable polynomials, whose Mahler measures were studied independently by the third author.

1 Introduction

Let $P \in \mathbf{Z}[z_1]$ be a monic polynomial with integer coefficients, and write $P(z_1) = \prod_{j=1}^{d}(z_1 - \alpha_j)$ for its complex factorization. Then, for every $k \in \mathbf{N}$ the polynomials $P^{(k)}(z_1) := \prod_{j=1}^{d}(z_1 - \alpha_j^k)$ have integer coefficients, and one can experimentally see that if $p \in \mathbf{N}$ is a prime number, the integer $P^{(p)}(1) \in \mathbf{Z}$ is often prime itself. This observation, first made by Pierce [51], led Lehmer [42] to study the growth of the sequence $\{P^{(k)}(1)\}_{k=1}^{+\infty}$. In particular, he showed that $|P^{(k+1)}(1)/P^{(k)}(1)| \to \exp(m(P))$ as $k \to +\infty$, if no root α_j lies on the unit circle. Here, $m(P) \in \mathbf{R}$ stands for the *logarithmic Mahler measure*, which is defined as:

$$m(P) := \int_{[0,1]^n} \log|P(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})| \, dt_1 \cdots dt_n$$

for every non-zero Laurent polynomial $P \in \mathbf{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \setminus \{0\}$. In particular, $m(P) \geq 0$ whenever P has integer coefficients.

Lehmer's discovery lead him to wonder whether there exist polynomials $P \in \mathbf{Z}[z_1]$ with arbitrarily small, positive Mahler measure, which would lead to slowly increasing sequences $\{P^{(k)}(1)\}_{k=1}^{+\infty}$. This seemingly simple question is open to this day, and the Mahler measure of Lehmer's polynomial $z_1^{10} + z_1^9 - z_1^7 - z_1^6 - z_1^5 - z_1^4 - z_1^3 + z_1 + 1 \in \mathbf{Z}[z_1]$ is still the smallest, non-zero Mahler measure of a polynomial $P \in \mathbf{Z}[z_1] \setminus \{0\}$ which has been computed (see [59] for a survey).

One of the most interesting strategies to attack Lehmer's problem has been proposed by Boyd [7]. He observed that if the set:

$$\mathscr{M} := \bigcup_{n=1}^{+\infty} m(\mathbf{Z}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \setminus \{0\}) \subseteq \mathbf{R}_{\geq 0}$$

is closed, then indeed there exists $m_0 > 0$ such that for each $P \in \mathbf{Z}[t]$ either $m(P) = 0$ or $m(P) \geq m_0$. This observation follows from the fact that each Mahler measure $m(P)$ of a multivariate polynomial $P \in \mathbf{Z}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \setminus \{0\}$ is the limit of a sequence of Mahler measures of univariate polynomials. More precisely, Boyd shows that:

$$\lim_{a_1 \to +\infty} \cdots \lim_{a_n \to +\infty} m(P(t^{a_1}, \ldots, t^{a_n})) = m(P)$$

where each limit is taken independently. It seems natural to ask what kind of monomial substitutions in the variables of P give the same convergence. Indeed, for a Laurent polynomial $P \in \mathbf{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ and a matrix $A = (a_{i,j}) \in \mathbf{Z}^{m \times n}$, one can consider the polynomial P_A, in m variables, given by:

$$P_A(z_1, \ldots, z_m) = P(z_1^{a_{1,1}} \cdots z_m^{a_{m,1}}, \ldots, z_1^{a_{1,n}} \cdots z_m^{a_{m,n}}).$$

The substitutions appearing in the previous limit proven by Boyd are the special case of the row-matrix $A = (a_1, \ldots, a_n)$. In order to generalize Boyd's result, Lawton [40] considered the quantity $\rho(A)$ associated to the matrix A defined as the smallest ℓ^∞-norm of an integer vector in the kernel of A:

$$\rho(A) := \min\{\|v\|_\infty \colon v \in \mathbf{Z}^n \setminus \{0\}, \, A \cdot v = 0\}.$$

Lawton showed that if A_d is a sequence of row-matrices, with $\rho(A_d) \to \infty$, then we have:

$$\lim_{d \to \infty} m(P_{A_d}) = m(P).$$

Recently, Dimitrov and Habegger [20, Theorem A.1] have given an upper bound on the rate of convergence which is a negative power of $\rho(A)$. Strikingly, the exponent depends only on the number of non-vanishing coefficients in P. The constant involved depends also on the degree of P and the number n of variables.

Moreover, Smyth [60] used Lawton's result to show that the set $\mathscr{M}$ can be written as a nested ascending union of closed subsets of $\mathbf{R}$. In fact, Smyth proves more generally that for every Laurent polynomial $P \in \mathbf{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \setminus \{0\}$, the set:

$$\mathscr{M}(P) := \bigcup_{m=1}^{+\infty} \{m(P_A) \colon A \in \mathbf{Z}^{m\times n} \text{ such that } P_A \neq 0\} \tag{1}$$

is closed. Smyth shows moreover that $\mathscr{M}$ is the nested ascending union of the sets $\mathscr{M}\left(\sum_{j=1}^n z_{2j-1} - z_{2j}\right)$ for $n \to +\infty$ (see [60, Proposition 14]).

With this context in mind, it seems natural to understand sequences $m(P_{A_d})$ and their convergence when A_d is a sequence of $m \times n$-matrices, and not only of row-matrices. The present paper aims at initializing a systematic study of these sequences. To do so, first of all we devote Section 3 to the proof of the following theorem (see Theorem 3.1) which very naturally generalizes the theorems of Boyd and Lawton to the multivariate setting:

Theorem 1.1. *For every non-zero Laurent polynomial $P \in \mathbf{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \setminus \{0\}$, and every sequence of integer $m \times n$-matrices $\{A_d\}_{d \in \mathbf{N}} \subseteq \mathbf{Z}^{m\times n}$ such that $\lim_{d\to+\infty} \rho(A_d) = +\infty$, we have that $\lim_{d\to+\infty} m(P_{A_d}) = m(P)$.*

We then proceed in Section 4 to obtain an upper bound for the error term $|m(P_A) - m(P)|$, which generalizes the bound proved by Dimitrov and Habegger [20, Theorem A.1]. In fact, our proof of Theorem 4.1, which occupies the entirety of Section 4, follows a strategy similar to the one of Dimitrov and Habegger. More precisely, we proceed, as they do, by regularizing the function $\log|P|$, and we bound separately the error terms for the regularizations (see Corollary 4.14) and the integrals of the differences between $\log|P|$ and the regularized functions (see Proposition 4.10). However, our regularization proceeds by using the smooth functions $\frac{1}{2}\log(|P|^2 + \varepsilon)$, which extend holomorphically to a neighbourhood of the unit torus (see Proposition 4.11), whereas the regularization carried out in [20] uses functions which are not smooth in general. Let us point out as well that our proof of Proposition 4.10 relies on an estimate of Dobrowolski [22, Theorem 1.3] (see also Lemma 4.6), which is similar to [20, Lemma A.3] (see Remark 4.9 for a comparison). Along the proof, we show furthermore that if a polynomial P does not vanish on the unit torus $\mathbf{T}^n$, then $m(P_A)$ tends to $m(P)$ exponentially fast as $\rho(A) \to +\infty$ (see Corollary 4.4).

Section 5 gives some insight on which optimal rate of convergence and even on what kind of asymptotic expansion one can expect for the convergence of $m(P_A)$ towards $m(P)$. For the case of 1-variable Mahler measures converging to 2-variable ones, the question of asymptotic expansion has been studied by Condon [17]. His work, which we review in Section 5.1, shows that for a large class of 2-variable polynomials, the rate of convergence for the limit $m(P(z_1, z_1^d)) \to m(P)$ is an integer power of $1/d$, and a full asymptotic expansion for the error term can be obtained. Note that this rate of convergence is much better than the bounds provided by [20, Theorem A.1] and by Theorem 4.1. Moreover, Condon proceeds in giving experimental evidences for other polynomials, exhibiting what seems to be a rate of convergence comparable to a rational power of $1/\rho(A)$. The full description of the rate of convergence, even in this particular case, is still open.

However, we exhibit in Section 5.2 the example of the 4-variate polynomial $P_\infty(z_1, \dots, z_4) = (1-z_1)(1-z_2) - (1-z_3)(1-z_4)$ and a sequence A_d of 2×4 integer matrices, such that the polynomials P_{A_d} are intimately related to the sequence:

$$P_d(z_1, z_2) := \sum_{0 \le i+j \le d} z_1^i z_2^j$$

and in particular $m(P_{A_d}) = m(P_d)$. The sequence of Mahler measures $m(P_d)$ was thoroughly studied by the third named author of the present paper in [47], where she proved that $m(P_d) \to -18 \cdot \zeta'(-2)$ as $d \to +\infty$. We use Theorem 3.1 to give a new proof of this convergence, using the equality $-18\zeta'(-2) = m(P_\infty)$, which is due to D'Andrea and Lalín [18, Theorem 7]. We then provide a complete asymptotic expansion for the error term $|m(P_d) - m(P_\infty)|$ as $d \to +\infty$ in Theorem 5.1. In particular, we prove that:

$$|m(P_d) - m(P_\infty)| \sim_{d\to\infty} \frac{\log(\rho(A_d))}{\rho(A_d)^2}.$$

The logarithmic term represents a different behavior than what Condon studied and proved and is still much better than our general bound. So, this example shows how far we are from fully understanding the optimal rate of convergence and asymptotic expansion of $m(P_{A_d})$ to $m(P)$ in a general multivariate setting.

1.1 Historical remarks

We devote this subsection to a short historical overview of the existing results using and generalizing the work of Boyd [6, 7] and Lawton [40]. First of all, Boyd himself [6] used an earlier version of this theorem to characterize those Laurent polynomials $P \in \mathbf{Z}[\underline{z}^{\pm 1}] \setminus \{0\}$ such that $m(P) = 0$. Moreover, Lawton's result has been used by Schinzel [54] to provide an explicit bound on the Mahler measure of a polynomial, which generalizes a classical result of Gonçalves [30] (see also [28, Theorem 1.22]). Furthermore, the work of Boyd and Mossinghoff [9], later generalized by Otmani, Rhin and Sac-Épée [50], used Lawton's result as a starting point for an investigation of the genuine limit points in the set $\mathscr{M}$. On the other hand, Dobrowolski [21] used Lawton's limit formula to answer a question of Schinzel. Moving on, Dubickas and Jankauskas [25] used Lawton's theorem to construct many non-reciprocal univariate polynomials whose Mahler measures lie in the interval $[m(z_1^3 - z_1 - 1), m(1 + z_1 + \cdots + z_n)]$, whereas Dobrowolski and Smyth [23], as well as Akhtari and Vaaler [1], used the theorem of Lawton to study Mahler measures of polynomials with a bounded number of monomials. Finally, Dubickas [24] and Habegger [34] used Lawton's result in their investigations of sums of roots of unity, whereas, as we already mentioned, Smyth [60] used Lawton's limit formula to prove that the sets $\mathscr{M}(P)$ defined in (1) are closed.

Let us point out that Lawton's result has found applications also outside number theory. First of all, Lind, Schmidt and Ward [43] used it to provide a lower bound for the entropy of the dynamical system associated to a Laurent polynomial in terms of its Mahler measure. Moreover, the work of Silver and Williams [55, 56], later generalized by Raimbault [53] and Lê [41], applied Lawton's result to knot theory, in order study the convergence of Mahler measures of Alexander polynomials, and the growth of homology, under surgery operations. Staying in the realm of knot theory, the work of Champanerkar and Kofman [15, 16], later generalized by Cai and Todd [12], used Lawton's theorem to study Mahler measures of Jones polynomials. Moving to the world of von Neumann algebras, Deninger [19, Theorem 17] proved a continuity result for Fuglede-Kadison determinants on the space of marked groups, which implies Lawton's result under the strong assumption that P does not vanish on the torus $\mathbf{T}^n$. Note that Deninger's result is reminiscent of the classical theorem of Szegö [19, Theorem 1], which approximates univariate Mahler measures in terms of Toepliz determinants. A multivariate analogue of Szegö's result has been recently found by Hajli [35]. Finally, Lawton's result has been used by Lück in functional analysis, to study spectral density functions [45] and twists of L^2-invariants [46].

To conclude this subsection, let us mention some existing generalizations and improved versions of Lawton's result. First of all, Champanerkar and Kofman prove in [16, Lemma 3.3] that one can perform signed monomial substitutions. Moreover, Duke [26, Theorem 6] provides the first term in the asymptotic expansion of the difference $m(z_1^n + z_1^m + 1) - m(z_1 + z_2 + 1)$ as $(n,m) \to +\infty$. Furthermore, Lalín and Sinha [39] mention a generalization of Lawton's theorem [39, Theorem 30] to the multiple Mahler measure, introduced in previous work of Kurokawa, Lalín and Ochiai [38]. Such a generalization was rigorously proved by Issa and Lalín [36], who dealt also with the generalized Mahler measures defined by Gon and Oyanagi [29]. On the other hand, Carter, Lalín, Manes, Miller and Mocz [14, Proposition 1.3] recently proved a weak generalization of Lawton's result to dynamical Mahler measures (introduced in [14, Definition 1.1]).

In addition to the previously mentioned results, Dobrowolski [22] generalized a crucial estimate of Lawton [40, Theorem 1] on the measure of the set of points $\underline{z} \in \mathbf{T}^n$ where a polynomial is small. Similar bounds have been provided by Lück [45, Proposition 2.1], Habegger [34, Lemma A.4], and Dimitrov and Habegger [20, Lemma A.3] (see Remarks 4.8 and 4.9 for a comparison). Finally, Gu and Lalín [31, Proposition 8] have recently proved a multivariate convergence of Mahler measures, which can be obtained as a corollary of Theorem 1.1, for one particular family of polynomials (see for more details Remark 5.4).

2 Notation and conventions

For the reader's convenience, we collect in this section the notation that we most frequently use in the rest of the paper.

2.1 Generalities

We let $\mathbf{N} = \{0,1,\dots\}$ denote the natural numbers, $\mathbf{Z}$ denote the integers, $\mathbf{R}$ denote the real numbers, $\mathbf{C}$ denote the complex ones, and $\mathbf{C}^\times = \mathbf{C} \setminus \{0\}$. For any $n \in \mathbf{N}$, we denote by $\underline{z}_n = (z_1, \dots, z_n)$ the coordinates of $\mathbf{C}^n$, and by

$$\mathbf{T}^n := \{\underline{z}_n \in \mathbf{C}^n \colon |z_1| = \cdots = |z_n| = 1\}$$

the n-dimensional real-analytic unit torus. Moreover, for every $p \in \mathbf{R}_{\geq 1}$, we let $\|\cdot\|_p \colon \mathbf{C}^n \to \mathbf{R}_{\geq 0}$ denote the ℓ^p-norm, defined for every $\underline{z}_n \in \mathbf{C}^n$ by:

$$\|\underline{z}_n\|_p := \left(\sum_{j=1}^n |z_j|^p\right)^{1/p}$$

and we let $\|\cdot\|_\infty \colon \mathbf{C}^n \to \mathbf{R}_{\geq 0}$ denote the ℓ^∞-norm, defined by $\|\underline{z}_n\|_\infty := \max\{|z_1|, \dots, |z_n|\}$. Finally, for any natural number n and any real number $\delta > 0$, we define the *annulus*

$$\mathscr{C}_\delta := \left\{\underline{z}_n \in (\mathbf{C}^\times)^n \colon \sum_{i=1}^n |\log|z_i|| \leq \delta\right\} \tag{2}$$

which is a closed neighbourhood of the torus $\mathbf{T}^n$ in $(\mathbf{C}^\times)^n$.

2.2 Matrices

Fix a matrix $A = (a_{i,j}) \in \mathbf{Z}^{m \times n}$. We denote by $d(A)$ the dimension of the linear subspace $\ker(A) \subseteq \mathbf{R}^n$, and by $\kappa_A := \ker(A) \cap \mathbf{Z}^n$ the integer lattice within this subspace. Moreover, we define:

$$\rho(A) := \min\{\|v\|_\infty \colon v \in \kappa_A,\, v \neq 0\} \tag{3}$$

which is the first successive minimum of the lattice κ_A with respect to the ℓ^∞-norm. By convention, we put $\rho(A) = +\infty$ when $\kappa_A = \{0\}$.

Finally, we consider the monomial substitution:

$$\underline{z}_m^A := (z_1^{a_{1,1}} \cdots z_m^{a_{m,1}}, \dots, z_1^{a_{1,n}} \cdots z_m^{a_{m,n}}) \tag{4}$$

and for any Laurent polynomial $P \in \mathbf{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$, we define $P_A \in \mathbf{C}[z_1^{\pm 1}, \dots, z_m^{\pm 1}]$ by setting $P_A(z_1, \dots, z_m) := P(\underline{z}_m^A)$. In particular, we consider vectors $v = (v_1, \dots, v_n) \in \mathbf{Z}^n$ as column matrices, so that $\underline{z}_n^v = z_1^{v_1} \cdots z_n^{v_n}$ is a monomial.

2.3 Measure theory

For every $n \in \mathbf{N}$, we denote by $\mu_n := \frac{1}{(2\pi i)^n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ the probability Haar measure on $\mathbf{T}^n$. More generally, for every matrix $A = (a_{i,j}) \in \mathbf{Z}^{m \times n}$, we let μ_A be the probability measure on $\mathbf{T}^n$ defined as the push-forward of μ_m along the map $\mathbf{T}^m \to \mathbf{T}^n$ given by $\underline{z}_m \mapsto \underline{z}_m^A$. Note in particular that $\mu_{\mathrm{Id}_n} = \mu_n$. Finally, for every non-zero Laurent polynomial $P \in \mathbf{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$, we let:

$$m(P) := \int_{\mathbf{T}^n} \log|P(\underline{z}_n)| d\mu_n(\underline{z}_n) \in \mathbf{R} \tag{5}$$

denote the *logarithmic Mahler measure* of P.

2.4 Fourier coefficients

For every integrable function $f \colon \mathbf{T}^n \to \mathbf{C}$, and every vector $v \in \mathbf{Z}^n$, we denote by:

$$c_v(f) := \int_{\mathbf{T}^n} \frac{f(\underline{z}_n)}{\underline{z}_n^v} d\mu_n(\underline{z}_n)$$

the corresponding Fourier coefficient. In particular, if $P \in \mathbf{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ is a Laurent polynomial, then $P(\underline{z}_n) = \sum_{v \in \mathbf{Z}^n} c_v(P) \cdot \underline{z}_n^v$.

2.5 Polynomials

Fix a non-zero Laurent polynomial $P \in \mathbf{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. We denote by $N_P \subseteq \mathbf{R}^n$ its *Newton polytope*, which is the convex hull in $\mathbf{R}^n$ of the *support* set $\mathrm{supp}(P) := \{v \in \mathbf{Z}^n \colon c_v(P) \neq 0\}$, and by $k(P) := |\mathrm{supp}(P)|$ the number of non-zero monomials appearing in P. Moreover, we denote by $\mathrm{diam}(P)$ the *diameter* of P, which is the smallest $d \in \mathbf{N}$ such that N_P is contained inside a translate of $[0,d]^n$. We also write $L_1(P) := \sum_{v \in \mathbf{Z}^n} |c_v(P)|$ for the *length* of P, and $L_\infty(P) := \max_{v \in \mathbf{Z}^n} |c_v(P)|$ for the *modulus* of P.

Furthermore, we let $V_P \hookrightarrow \mathbf{G}_m^n$ be the hypersurface defined by P, so that $V_P(\mathbf{C}) := \{\underline{z}_n \in (\mathbf{C}^\times)^n \colon P(\underline{z}_n) = 0\}$. We also define the *conjugate reciprocal* of P by:

$$P^*(\underline{z}_n) := \overline{P}(\underline{z}_n^{-1}) = \sum_{v \in \mathbf{Z}^n} \overline{c_v(P)} \cdot \underline{z}_n^{-v}. \tag{6}$$

Finally, for every $t \geq 0$ we define the set $S(P,t) := \{\underline{z} \in \mathbf{T}^n \colon |P(\underline{z})| \leq t\} \subseteq \mathbf{T}^n$.

2.6 Constants

For every non-zero Laurent polynomial $P \in \mathbf{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$ we define a constant:

$$\rho_0(P) := \max\left\{\mathrm{diam}(P) + 1,\ 7\left(\frac{\mathrm{diam}(P)}{n}\right)^2,\ \exp(5 \cdot k(P) \cdot n^2)\right\} \in \mathbf{R}_{>0}, \tag{7}$$

and a further family of constants:

$$\delta_\varepsilon(P) := \min\left(\frac{\sqrt{\varepsilon}}{\mathrm{diam}(P) L_1(P)}, \frac{\log(4/3)}{\mathrm{diam}(P)}\right) \in \mathbf{R}_{>0} \tag{8}$$

depending on a positive real number $\varepsilon > 0$.

3 A higher dimensional analogue of Lawton's theorem

The aim of this section is to show that the Mahler measure $m(P) := \int_{\mathbf{T}^n} \log|P(\underline{z}_n)| d\mu_n(\underline{z}_n)$ of any non-zero Laurent polynomial $P(\underline{z}_n) \in \mathbf{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$ can be approximated by suitable sequences of "lower-dimensional" Mahler measures, as specified in the following theorem (see also Theorem 1.1).

Theorem 3.1. *Let $n \in \mathbf{N}$ be an integer, and $P(\underline{z}_n) \in \mathbf{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$ be a non-zero Laurent polynomial. Then, for every sequence of matrices $A_d \in \mathbf{Z}^{m_d \times n}$ such that $\lim_{d \to +\infty} \rho(A_d) = +\infty$, we have the convergence $\lim_{d \to +\infty} m(P_{A_d}) = m(P)$.*

3.1 Convergence of measures and integrals

In order to prove Theorem 3.1, we start by relating the growth of $\rho(A)$ to the weak convergence of the push-forward measures μ_A:

Lemma 3.2. *Fix $n \in \mathbf{Z}_{\geq 1}$, and let $A_d \in \mathbf{Z}^{m_d \times n}$ be a sequence of integral matrices, with fixed number of columns, such that $\rho(A_d) \to +\infty$ as $d \to +\infty$. Then the sequence of measures μ_{A_d} on $\mathbf{T}^n$ converges weakly to the measure μ_{Id_n}.*

Proof. This result is classical. We follow the lines of [6, Lemma 1], which treats the case when $m_d = 1$ for every d. By the definition of weak convergence and push-forward of measures, and by Weierstraß approximation, it is sufficient to prove that:

$$\lim_{d\to+\infty}\left(\int_{\mathbf{T}^{m_d}} Q(\underline{z}_{m_d}^{A_d})\, d\mu_{m_d}(\underline{z}_{m_d})\right) = \int_{\mathbf{T}^n} Q(\underline{z}_n)\, d\mu_n(\underline{z}_n) \tag{9}$$

for every Laurent polynomial $Q(\underline{z}_n) \in \mathbf{C}[\underline{z}_n^{\pm 1}]$. We see now immediately that for every $d \in \mathbf{N}$, the following identities hold true:

$$\int_{\mathbf{T}^{m_d}} Q(\underline{z}_{m_d}^{A_d})\, d\mu_{m_d}(\underline{z}_{m_d}) = \sum_{v\in\mathbf{Z}^n} c_v(Q)\cdot \int_{\mathbf{T}^{m_d}} \underline{z}_{m_d}^{A_d\cdot v}\, d\mu_{m_d}(\underline{z}_{m_d}) = \sum_{\substack{v\in\mathbf{Z}^n\\ A_d\cdot v=0}} c_v(Q) = \sum_{v\in\kappa_{A_d}} c_v(Q) \tag{10}$$

Now, set $R := \max\{\|v\|_\infty \colon v \in \mathrm{supp}(Q)\}$. If $\rho(A_j) > R$, then the only vector $v \in \kappa_{A_j}$ for which it may happen that $c_v(Q) \neq 0$ is the null vector $v = 0$. In this case, we have the identity:

$$\sum_{v\in\kappa_{A_d}} c_v(Q) = c_0(Q) = \int_{\mathbf{T}^n} Q(\underline{z}_n)\, d\mu_{\mathrm{Id}_n}(\underline{z}_n)$$

which, combined with (10), shows (9), because the sequence on the left is eventually constantly equal to the right hand side. □

Weak convergence of measures implies the convergence of integrals of any continuous function. Unfortunately, we would like a convergence of integrals of $\log|P|$, which is singular. However, uniform estimates on L^2-norms are enough to guarantee that the weak-convergence of measures implies convergence of integrals, as shown in the following general Lemma 3.3. In this lemma, we choose to work with continuous functions possibly having $+\infty$-values, for which the integral for any measure on the torus is naturally defined (possibly $+\infty$), as explained for instance in [52, Chapter 1].

Lemma 3.3. *Let ν_k be a sequence of probability measures on $\mathbf{T}^n$, which converges weakly to some probability measure ν_∞. Let $f\colon \mathbf{T}^n \to \mathbf{R}\cup\{+\infty\}$ be a continuous function, which is uniformly L^2 for the family $\{\nu_k \colon k \in \mathbf{N}\cup\{\infty\}\}$. Then we have the convergence $\int_{\mathbf{T}^n} f d\nu_k \to \int_{\mathbf{T}^n} f d\nu_\infty$ as $k \to +\infty$.*

Proof. By assumption, there exists a positive real number $C \in \mathbf{R}_{>0}$ such that $\int_{\mathbf{T}^n} |f|^2 d\nu_k \leq C$ for every $k \in \mathbf{N}\cup\{\infty\}$. Fix $\varepsilon > 0$ and let $\lambda = \frac{C}{\varepsilon}$. Define the set $S_\lambda = \{t \in \mathbf{T}^n \colon |f(t)| > \lambda\}$. The L^2-bounds yield, for any $k \in \mathbf{N}\cup\{\infty\}$:

$$\left|\int_{S_\lambda} (f-\lambda) d\nu_k\right| \leq \left|\int_{S_\lambda} f d\nu_k\right| \leq \int_{S_\lambda} |f|\frac{|f|}{\lambda} d\nu_k \leq \frac{1}{\lambda}\int_{\mathbf{T}^n} |f|^2 d\nu_k \leq \frac{C}{\lambda} = \varepsilon. \tag{11}$$

Now, let $\tilde{f}$ be the continuous function $\min(f,\lambda)$, which is bounded from above by λ. For every $\underline{z} \in \mathbf{T}^n$, we have the equality $f(\underline{z}) = \tilde{f}(\underline{z}) + (f(\underline{z}) - \lambda)\cdot\chi_{S_\lambda}(\underline{z})$, where χ_{S_λ} denotes the characteristic function of S_λ. Hence, for all $k \in \mathbf{N}$ we have the bound:

$$\left|\int_{\mathbf{T}^n} f d\nu_k - \int_{\mathbf{T}^n} f d\nu_\infty\right| \leq \left|\int_{\mathbf{T}^n} \tilde{f} d\nu_k - \int_{\mathbf{T}^n} \tilde{f} d\nu_\infty\right| + \left|\int_{S_\lambda} (f-\lambda) d\nu_k\right| + \left|\int_{S_\lambda} (f-\lambda) d\nu_\infty\right|. \tag{12}$$

The last two terms on the right hand side of (12) are bounded by ε, thanks to (11). For k big enough, by the convergence $\nu_k \to \nu_\infty$, the first one is less than ε. So we have proven that, for k big enough, we have:

$$\left|\int_{\mathbf{T}^n} f d\nu_k - \int_{\mathbf{T}^n} f d\nu_\infty\right| \leq 3\varepsilon$$

which shows that $\int_{\mathbf{T}^n} f d\nu_k \to \int_{\mathbf{T}^n} f d\nu_\infty$ as $k \to +\infty$. □

3.2 Uniform L^2-bounds and convergence of Mahler measures

Our goal is to prove that $m(P_A) = \int_{\mathbf{T}^n} \log|P| d\mu_A$ converges to $m(P) = \int_{\mathbf{T}^n} \log|P| d\mu_A$. From the previous results we know that an uniform L^2-bound for this functions would grant the convergence. The following estimate is essentially obtained by Dimitrov and Habegger in [20, Appendix A], where they deal with Lawton theorem and improves the rate of convergence, see also [34].

Proposition 3.4 (Dimitrov & Habegger)**.** *Let $n,k \in \mathbf{N}$ be two integers. Then, there exists a constant $C > 0$ such that, for every non-zero Laurent polynomial $P(\underline{z}_n) \in \mathbf{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$ with $k(P) = k$ and $L_\infty(P) = 1$, and every matrix $A \in \mathbf{Z}^{m\times n}$ with $\rho(A) > \operatorname{diam}(P)$ and $m \leq n$, the following holds:*

$$\|\log|P|\|_{2,\mu_A}^2 := \int_{\mathbf{T}^l} \left|\log|P(\underline{z}_m^A)|\right|^2 d\mu_A \leq C \quad \textit{and} \quad \|\log|P|\|_{2,\mu_n}^2 \leq C.$$

Proof. A direct computation, similar to the one carried out in (10), proves that, for any matrix $A \in \mathbf{Z}^{m\times n}$, we have:

$$P(\underline{z}_m^A) = \sum_{v\in\mathbf{Z}^n} c_v(P)\underline{z}_m^{A\cdot v} = \sum_{w\in\mathbf{Z}^m}\left(\sum_{\substack{v\in\mathbf{Z}^n\\ A\cdot v = w}} c_v(P)\right)\underline{z}_m^w. \tag{13}$$

Two vectors $v, v' \in \mathbf{Z}^n$ contribute non-trivially to the same monomial in the above sum if and only if $c_v(P) \neq 0$, $c_{v'}(P) \neq 0$ and $A\cdot v = A \cdot v'$, or equivalently $v - v' \in \kappa_A$. By definition of $\operatorname{diam}(P)$ (see Section 2.5), in this case, we have $\|v - v'\|_\infty \leq \operatorname{diam}(P)$. We see that if $\rho(A) > \operatorname{diam}(P)$ the only possibility is $v - v' = 0$. In other terms, each monomial of P_A comes from a single monomial of $P(\underline{z}_m^A)$ with the same coefficient and no compensations.

So, for any A with $\rho(A) > \operatorname{diam}(P)$, the polynomial $P_A = P(\underline{z}_m^A)$ has $m \leq n$ variables, $k(P_A) = k(P) = k$ non-vanishing coefficients, and $L_\infty(P_A) = L_\infty(P) = 1$. Our proposition comes then directly from the estimates of Dimitrov and Habegger. They show in [20, Lemma A.3] [1] that for every $l,k \in \mathbf{Z}_{\geq 1}$, there exists a constant $C_{l,k} > 0$ such that for any Laurent polynomial $Q \in \mathbf{C}[\underline{z}_l^{\pm 1}]$, with $k(Q) = k$ and $L_\infty(Q) = 1$, we have:

$$\int_{\mathbf{T}^l} (\log|Q|)^2 d\mu_l \leq C_{l,k}.$$

Thanks to the considerations in the previous paragraph, this bound applies both to $Q = P$ and to $Q = P_A$, hence we can take $C := \max\{C_{m,k} \colon m \leq n\}$. □

Theorem 3.1 is now an easy consequence of the other results proved in this section:

Proof of Theorem 3.1. We first make an easy reduction: up to multiplying P by a constant a, we may and will assume that $L_\infty(P) = 1$. Indeed, we have, for all $a \in \mathbf{C}^*$, both $m(aP) = \log|a| + m(P)$ and $m(aP_{A_j}) = \log|a| + m(P_{A_j})$. So the problem of convergence is equivalently solved for P or aP. Observe moreover that, for every $d \in \mathbf{N}$, we have the following identities:

$$m(P_{A_d}) = \int_{\mathbf{T}^{m_d}} \log|P(\underline{z}_{m_d}^{A_d})| d\mu_{m_d} = \int_{\mathbf{T}^n} \log|P| d\mu_{A_d}.$$

Let $d_0 \in \mathbf{N}$ be such that, for $d \geq d_0$, we have $\rho(A_d) \geq \operatorname{diam}(P)$. From Proposition 3.4, we know that the function $\log|P|$ is uniformly L^2 for the family $\{\mu_{A_d}, d \geq d_0\} \cup \{\mu_n\}$. Moreover, we know from Lemma 3.2 that the family μ_{A_d} converges weakly to μ_n as $d \to +\infty$. Thus, we have:

$$\lim_{d\to+\infty} m(P(\underline{z}_{m_d}^{A_d})) = \lim_{d\to+\infty} \int_{\mathbf{T}^n} \log|P| d\mu_{A_d} = \int_{\mathbf{T}^n} \log|P| d\mu_n = m(P)$$

thanks to Lemma 3.3. □

We will see an example of application of Theorem 3.1 in Section 5.2. Meanwhile, we analyze more carefully in the next section the convergence, to obtain an upper bound on its rate.

4 An error term in the convergence

The aim of this section is to improve Theorem 3.1 by providing an explicit upper bound for the error term $|m(P_A) - m(P)|$, where $P \in \mathbf{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$ is a non-zero Laurent polynomial, and $A \in \mathbf{Z}^{m\times n}$ is an integral matrix. We will assume without loss of generality that P is not a monomial (*i.e.* $k(P) > 1$), and that $n \geq 2$, because otherwise $m(P_A) = m(P)$ for every non-zero integral matrix $A \in \mathbf{Z}^{m\times n}$. Then, we obtain the following result, which generalizes [20, Theorem A.1] to higher dimensions:

Theorem 4.1. *Fix two integers $k,n \in \mathbf{Z}_{\geq 2}$. Let $P \in \mathbf{C}[\underline{z}_n^{\pm 1}]$ be a Laurent polynomial with $k(P) = k$ non-zero coefficients, and let $\rho_0(P)$ be the constant defined in (7). Then, for every $m \in \mathbf{Z}_{\geq 1}$ and every matrix $A \in \mathbf{Z}^{m\times n}$ such that $\rho(A) \geq \rho_0(P)$, the following inequality holds:*

$$|m(P_A) - m(P)| \leq 7 \cdot k^2 \cdot (3^n n) \cdot \log(\rho(A)) \cdot \left(\frac{\operatorname{diam}(P)}{\rho(A)}\right)^{\frac{1}{n(k-1)}}. \tag{14}$$

[1]The first version of the paper [20] contains a slight error in the proof of this Lemma, that will be corrected in a forthcoming second version

4.1 An explicit exponential convergence for polynomials without toric points

The aim of this section is to show that, for a Laurent polynomial $P \in \mathbf{C}[\underline{z}_n^{\pm 1}]$ which does not vanish on the unit torus $\mathbf{T}^n$, the convergence $m(P_A) \to m(P)$ as $\rho(A) \to +\infty$ is exponentially fast, and its speed can be explicitly bounded, as we will see in Corollary 4.4. This result follows easily from the more general Theorem 4.2 (which we will use later on in the case P is arbitrary). Its proof uses crucially the standard fact that the Fourier coefficients of a holomorphic function on a neighborhood of the torus $\mathbf{T}^n$ decay exponentially. Recall from Section 2.2 that for any matrix $A \in \mathbf{Z}^{m\times n}$, we set $d(A) := \dim(\ker(A))$.

Theorem 4.2. *Fix two natural numbers $n,m \geq 1$, an open $U \subseteq \mathbf{C}^n$ containing $\mathbf{T}^n$, and a holomorphic function $f\colon U \to \mathbf{C}$. Then, for every real number $\delta > 0$ such that U contains the annulus $\mathscr{C}_\delta$ defined in (2), and every matrix $A \in \mathbf{Z}^{m\times n}$ such that $\rho(A) \geq \frac{2d(A)}{3\delta}$, the following estimate holds:*

$$\left| \int_{\mathbf{T}^n} f(\underline{z}) d\mu_A(\underline{z}) - \int_{\mathbf{T}^n} f(\underline{z}) d\mu_n(\underline{z}) \right| \leq (d(A)+1)3^{d(A)} \cdot \frac{\max_{\mathscr{C}_\delta}|f|}{\exp(\delta\rho(A))}. \tag{15}$$

Proof. Let $r = e^\delta$ and $\kappa_A = \ker(A) \cap \mathbf{Z}^n$, as in Section 2.2. For any $\nu \in \mathbf{Z}^n$, write $c_\nu(f) := \int_{\mathbf{T}^n} f(\underline{z})\underline{z}^{-\nu} d\mu_n(\underline{z})$ for the ν-th Fourier coefficient of f, as in Section 2.4. Since f is holomorphic on U, the Fourier series $\sum_{\nu\in\mathbf{Z}^n} c_\nu(f)\underline{z}^\nu$ converges normally to f on $\mathbf{T}^n$, and the dominated convergence theorem gives

$$\int_{\mathbf{T}^n} f(\underline{z}) d\mu_A(\underline{z}) - \int_{\mathbf{T}^n} f(\underline{z}) d\mu_n(\underline{z}) = \left(\sum_{\nu\in\mathbf{Z}^n} c_\nu(f) \int_{\mathbf{T}^n} \underline{z}^\nu d\mu_A(\underline{z}) \right) - c_0(f) = \left(\sum_{\nu\in\kappa_A} c_\nu(f) \right) - c_0(f) = \sum_{\nu\in\kappa_A\setminus\{0\}} c_\nu(f). \tag{16}$$

To bound the Fourier coefficients $c_\nu(f)$, we use the holomorphicity of f on U. To be more precise, let us associate to every vector $\underline{h} = (h_1,\dots,h_n) \in \mathbf{R}^n$ the torus $\mathbf{T}_{\underline{h}} := \{\underline{z} \in \mathbf{C}^n\colon |z_j| = e^{h_j},\ \forall j \in \{1,\dots,n\}\}$. Then, for every $\nu \in \mathbf{Z}^n\setminus\{0\}$ and every $\underline{h} \in \mathbf{R}^n$ such that $\|\underline{h}\|_1 \leq \delta$, the homotopy invariance of integrals of holomorphic functions implies

$$c_\nu(f) = \int_{\mathbf{T}_{\underline{h}}} f(\underline{z})\underline{z}^{-\nu} d\mu_n(\underline{z}) \tag{17}$$

because $\mathbf{T}_{\underline{h}} \subseteq \mathscr{C}_\delta \subseteq U$ by assumption. Now let $j_0 \in \{1,\dots,n\}$ be any integer such that $\|\nu\|_\infty = |\nu_{j_0}|$, and take $\underline{h} \in \mathbf{R}^n$ to be the vector with $h_j := 0$ for any $j \in \{1,\dots,n\} \setminus \{j_0\}$, and $h_{j_0} := \delta \cdot \nu_{j_0}/|\nu_{j_0}|$. Then, using (17) we see that:

$$|c_\nu(f)| \leq \max_{\mathscr{C}_\delta}|f| \cdot r^{-\|\nu\|_\infty}. \tag{18}$$

Combining (16) and (18), we get

$$\left| \int_{\mathbf{T}^n} f(\underline{z}) d\mu_A(\underline{z}) - \int_{\mathbf{T}^n} f(\underline{z}) d\mu_n(\underline{z}) \right| \leq \sum_{\nu\in\kappa_A\setminus\{0\}} |c_\nu(f)| \leq \max_{\mathscr{C}_\delta}|f| \sum_{\nu\in\kappa_A\setminus\{0\}} r^{-\|\nu\|_\infty}. \tag{19}$$

The only remaining step to prove the theorem is to bound the sum appearing in the right-hand-side of (19). It is an independent estimate, which we state separately in Lemma 4.3. Note that we fulfill its assumptions: κ_A is a lattice of full rank inside the vector space $\ker(A)$ of dimension $d(A)$. Moreover, its first successive minimum with respect to the ℓ^∞-norm is by definition $\rho(A)$. Eventually, we have by assumption $\rho(A)\log(r) = \rho(A)\delta \geq 2d(A)/3$, so we can use the bound (21) of the following Lemma 4.3. □

Lemma 4.3. *Fix a real vector space V of finite dimension $d \in \mathbb{Z}_{\geq 1}$, and a norm $\|\cdot\|\colon V \to \mathbb{R}_{\geq 0}$. Let $\Lambda \subseteq V$ be a lattice of full rank, and denote by $\rho := \lambda_1(\Lambda, \|\cdot\|) := \min\{\|\lambda\|\colon \lambda \in \Lambda \setminus \{0\}\}$ its first successive minimum with respect to the norm $\|\cdot\|$. Then, we have the following estimate:*

$$\sum_{\nu\in\Lambda} r^{-\|\nu\|} \leq \frac{3^d d!}{r^\rho} \sum_{k=0}^{d} \frac{1}{(d-k)!} \left(\frac{2}{3\rho\log(r)} \right)^k \qquad (r > 1). \tag{20}$$

In particular, if $\rho\log(r) \geq 2d/3$, we have

$$\sum_{\nu\in\Lambda} r^{-\|\nu\|} \leq \frac{(d+1)3^d}{r^\rho} \qquad (r > 1). \tag{21}$$

Proof. First of all, set $B(x,q) := \{y \in V\colon \|y - x\| < q\}$ and $N_q := |B(0,q) \cap \Lambda|$ for every $x \in V$ and $q \geq 0$. Now, observe that:

$$\begin{aligned} \sum_{\nu\in\Lambda} r^{-\|\nu\|} &= \sum_{q=1}^{+\infty} |\{\nu \in \Lambda\colon \|\nu\| = q\}| \cdot r^{-q} = \sum_{q=1}^{+\infty} (N_q - N_{q-1}) \cdot r^{-q} = \\ &= -\frac{N_{\rho-1}}{r^{\rho-1}} + \log(r) \int_{\rho-1}^{+\infty} \frac{N_{\lfloor t\rfloor}}{r^t} dt = -\frac{1}{r^\rho} + \log(r) \int_{\rho}^{+\infty} \frac{N_{\lfloor t\rfloor}}{r^t} dt \end{aligned} \tag{22}$$

as follows from Abel's summation formula (see [2, Theorem 4.2]), since $N_q = 1$ if $q \leq \rho - 1$. Moreover, note that the inclusion

$$\bigsqcup_{x \in B(0,q)\cap\Lambda} B\left(x, \frac{\rho}{2}\right) \subseteq B(0, q+\rho/2) \qquad (q > 0)$$

provides the bound

$$N_q \leq \frac{\mathrm{vol}\left(B\left(0, q+\frac{\rho}{2}\right)\right)}{\mathrm{vol}\left(B\left(0, \frac{\rho}{2}\right)\right)} = \left(\frac{2q}{\rho} + 1\right)^d \qquad (q \geq 0) \tag{23}$$

(see [4, Theorem 2.1] for more details). Applying the bound (23) to (22), we get:

$$\sum_{v \in \Lambda} r^{-\|v\|} \leq -\frac{1}{r^\rho} + \frac{\log(r)}{\rho^d} \int_\rho^{+\infty} \frac{(2t+\rho)^d}{r^t} dt = -\frac{1}{r^\rho} + \frac{2^d r^{\rho/2}}{(\rho \log(r))^d} \int_{3\rho\log(r)/2}^{+\infty} u^d e^{-u} du, \tag{24}$$

where the last equality follows from the change of variables $2u = (2t+\rho)\log(r)$. Putting $x = 3\rho\log(r)/2$, we recognize the incomplete gamma function [27, § 9.2.1]:

$$\Gamma(d+1, x) = \int_x^{+\infty} u^d e^{-u} du = d!\, e^{-x} \sum_{k=0}^{d} \frac{x^k}{k!}.$$

The inequality (20) in the Lemma follows. Now under the assumption $x \geq d$, the right hand side of (20) is bounded by

$$\frac{3^d d!}{r^\rho} \sum_{k=0}^{d} \frac{1}{(d-k)!\, d^k} \leq \frac{3^d d!}{r^\rho} \sum_{k=0}^{d} \frac{1}{d!} \leq \frac{(d+1)3^d}{r^\rho}.$$

□

To conclude this section, let us see how to deduce the exponential convergence of $m(P_A) \to m(P)$ as $\rho(A) \to +\infty$ from the previous Theorem 4.2. Note that if a polynomial P does not vanish on the torus $\mathbf{T}^n$, then there is some $\delta > 0$ such that it does not vanish on the annulus $\mathscr{C}_\delta$ defined in (2).

Corollary 4.4. *Let $P \in \mathbf{C}[\underline{z}_n^{\pm 1}]$ be a Laurent polynomial that does not vanish on the torus $\mathbf{T}^n$. Then there exist $r > 1$ and $C > 0$ such that for every matrix $A \in \mathbf{Z}^{m \times n}$ with the property that $\rho(A) \geq 2d(A)/3\log(r)$, we have the following estimate:*

$$|m(P_A) - m(P)| \leq \frac{C}{r^{\rho(A)}}. \tag{25}$$

More precisely, one can take $r = e^\delta$, where $\delta > 0$ is any real number such that P does not vanish on the annulus $\mathscr{C}_\delta$.

Proof. Let P^* be the conjugate reciprocal of P, introduced in Section 2.5. Note that for $\underline{z} = (z_1, \ldots, z_n) \in \mathbf{T}^n$, we have

$$|P(\underline{z})|^2 = P(z_1, \ldots, z_n)\overline{P(z_1, \ldots, z_n)} = P(z_1, \ldots, z_n)\overline{P}(\overline{z_1}, \ldots, \overline{z_n}) = P(z_1, \ldots, z_n)\overline{P}\left(\frac{1}{z_1}, \ldots, \frac{1}{z_n}\right) = PP^*(\underline{z}).$$

This shows that $PP^* = |P|^2$ on the torus $\mathbf{T}^n$. Fix $\delta > 0$ such that P does not vanish on the annulus $\mathscr{C}_\delta$. Since the involution $(z_1, \ldots, z_n) \mapsto (\overline{z}_1^{-1}, \ldots, \overline{z}_n^{-1})$ preserves $\mathscr{C}_\delta$, the polynomial P^* also does not vanish on $\mathscr{C}_\delta$. So, the differential form $\omega := \frac{1}{2} d\log(PP^*) = d(PP^*)/(2PP^*)$ is holomorphic on some open $U \subseteq \mathbf{C}^n$ containing $\mathscr{C}_\delta$. Moreover, the restriction of ω to the torus is equal to $d\log|P|$, hence is exact.

Now, for $U \supseteq \mathscr{C}_\delta$ small enough, each loop $\gamma \subseteq U$ is homologous to a loop $\gamma' \subseteq \mathbf{T}^n$. This implies that $\int_\gamma \omega = \int_{\gamma'} d\log|P| = 0$. Thus, de Rham's comparison theorem shows that there exists a unique holomorphic function $f \colon U \to \mathbf{C}$ such that $\omega = df$ on U and $f = \log|P|$ on $\mathbf{T}^n$. Hence $|m(P_A) - m(P)| = |\int_{\mathbf{T}^n} f\, d\mu_A - \int_{\mathbf{T}^n} f\, d\mu_n|$, and we can apply Theorem 4.2 because f is holomorphic on $U \supseteq \mathscr{C}_\delta$. This yields the bound (25), where we set $C := (n+1)3^n \cdot \max_{\mathscr{C}_\delta}|f|$. □

Remark 4.5. For a given Laurent polynomial $P \in \mathbf{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$ which does not vanish on $\mathbf{T}^n$, one can find an explicit $\delta > 0$, depending on $\min_{\mathbf{T}^n}|P|$, such that P does not vanish on $\mathscr{C}_\delta$. We will carry out this computation for a specific type of polynomial in Proposition 4.11.

4.2 An explicit error term in the general case

Let $P \in \mathbf{C}[\underline{z}_n^{\pm 1}]$ be a Laurent polynomial in n variables, which is not a monomial. Given a matrix $A \in \mathbf{Z}^{m \times n}$, we wish to prove Theorem 4.1, which gives a precise estimate for the error $|m(P_A) - m(P)|$. In order to do so, we approximate the function:

$$\begin{aligned} f \colon \mathbf{T}^n &\to \mathbb{R} \cup \{-\infty\} \\ \underline{z} &\mapsto \log|P(\underline{z})| \end{aligned}$$

which is singular when P vanishes on $\mathbf{T}^n$, with the smooth functions $f_\varepsilon(\underline{z}) = \frac{1}{2}\log(|P(\underline{z})|^2 + \varepsilon)$. Then, one has that:

$$\begin{aligned} |m(P_A) - m(P)| \leq & \left| \int_{\mathbf{T}^n} \frac{1}{2}\log(|P|^2+\varepsilon) - \log|P| d\mu_A \right| + \left| \int_{\mathbf{T}^n} \frac{1}{2}\log(|P|^2+\varepsilon) - \log|P| d\mu_n \right| \\ & + \left| \int_{\mathbf{T}^n} f_\varepsilon(\underline{z}) d\mu_A - \int_{\mathbf{T}^n} f_\varepsilon(\underline{z}) d\mu_n(\underline{z}) \right| \end{aligned} \tag{26}$$

and we proceed by bounding each integral separately. We will show in Proposition 4.11 that f_ε is holomorphic on a neighborhood of the torus, so the results of the previous section apply to the last integral. However, two phenomena are competing here. On the one hand, you need to take ε small enough to make the first two integrals small. On the other hand, the annulus $\mathscr{C}_\delta$ on which f_ε is holomorphic becomes smaller and smaller when $\varepsilon \to 0$, which weakens the bound for the third integral given by Theorem 4.2. Thus, the proof of Theorem 4.1 will consist in choosing a suitable value of ε, depending on the quantity $\rho(A)$, which balances these two phenomena.

In order to bound the first two integrals, we rely on an explicit estimate for the measure of the set of points $\underline{z} \in \mathbf{T}^n$ where $|P(\underline{z})|$ is small. This estimate is expressed in the following Lemma 4.6, which is an immediate consequence of a result of Dobrowolski [22, Theorem 1.3].

Lemma 4.6 (Dobrowolski)**.** *For any Laurent polynomial $P \in \mathbf{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$ with $k = k(P) \geq 2$ non-vanishing coefficients, the following bound:*

$$\mu_n(S(P,t)) \leq 6 \cdot (k-1) \cdot n \cdot \left(\frac{t}{L_\infty(P)} \right)^{\frac{1}{n(k-1)}} \tag{27}$$

holds for every $t \geq 0$, where $L_\infty(P) := \max_{\nu \in \mathbf{Z}^n} |c_\nu(P)|$ and $S(P,t) := \{\underline{z} \in \mathbf{T}^n \colon |P(\underline{z})| \leq t\}$, as we defined in Section 2.5.

Proof. Replacing (P,t) by $(P/L_\infty(P), t/L_\infty(P))$ if necessary, we may assume without loss of generality that $L_\infty(P) = 1$. Under this assumption, we can further assume that $t \leq 1$, since $\mu_n(S(P,t)) \leq 1$ for every $t \geq 0$. Now, for every integer $j \in \{1, \dots, n\}$, let $k_j = k_j(P)$ be the number of non-zero terms of P seen as a polynomial in z_j. Then, Dobrowolski's work [22, Theorem 1.3] gives us the bound:

$$\mu_n(\{\underline{z} \in \mathbf{T}^n \colon |P(\underline{z})| \leq t\}) \leq (\mathfrak{C}(k_1) + \dots + \mathfrak{C}(k_n)) \cdot t^{\frac{1}{\sum_{j=1}^n (k_j - 1)}}$$

where $\mathfrak{C}(x) := (x-1)\left(\frac{12\sqrt{2}}{\pi}\right)^{\frac{x-2}{x-1}}$ for $x > 1$, and $\mathfrak{C}(1) = 0$. Since clearly $k_j \leq k$, and $t \leq 1$ by assumption, we see that $t^{\frac{1}{\sum_{j=1}^n (k_j-1)}} \leq t^{\frac{1}{n(k-1)}}$ and $\mathfrak{C}(k_1) + \dots + \mathfrak{C}(k_n) \leq n \cdot \mathfrak{C}(k)$. To conclude, it suffices to observe that $12\sqrt{2} \leq 6\pi$, which implies that $\mathfrak{C}(x) \leq 6(x-1)$ for every $x > 1$. □

Remark 4.7. Note that the estimate $\sum_{j=1}^n (k_j(P) - 1) \leq n(k(P) - 1)$, used in the proof of Lemma 4.6, is quite rough. In fact, for a Laurent polynomial $P \in \mathbf{C}[\underline{z}_n^{\pm 1}]$ which is generic (in a suitable sense), the quantity $k(P)$ is rather comparable to the product $k_1(P) \cdots k_n(P)$. However, for every $A \in \mathbf{Z}^{m \times n}$ with $m \leq n$ we have that $m(k(P_A) - 1) \leq n(k(P) - 1)$, whereas it is not clear in general how to compare $\sum_{j=1}^m (k_j(P_A) - 1)$ and $\sum_{j=1}^n (k_j(P) - 1)$.

Remark 4.8. We note that Lück [45, Proposition 2.1] provided another estimate for the measure $\mu_n(S(P,t))$. However, this bound depends on the *width* $\mathrm{wd}(P)$, which is a quantity defined in [45, § 1.2] that turns out to be comparable with $\mathrm{diam}(P)$. In particular $\mathrm{wd}(P_A) \to +\infty$ as $\rho(A) \to +\infty$, which makes Lück's bound not adapted to our purposes. More precisely, if we used Lück's bound in the proof of Proposition 4.10, we would get an estimate for the first two integrals appearing in (26) which would diverge as $\rho(A) \to +\infty$.

Remark 4.9. We note that Habegger [34, Lemma A.4] and Dimitrov and Habegger [20, Lemma A.3] stated another estimate for $\mu_n(S(P,t))$, where the exponent $1/(n(k-1))$ appearing in (27) is replaced with $1/(2(k-1))$. However, there is a slight mistake in the proof of [34, Lemma A.4] that will be corrected in the second version (to appear) of [20, Lemma A.3]. We choose in this paper to work with Dobrowolski's bound in order to obtain estimates of the constants appearing. However, if no explicit estimates of the constants are needed, our method with Dimitrov-Habegger's bound readily gives an exponent $\frac{1}{2(k-1)}$ in Theorem 4.1, as discussed in Remark 4.15. The quality of this last exponent in the bound (14) is better than the one obtained in [20, Theorem A.1] for $m = 1$, noting that the authors explicitly remark that they do not strive to get optimal exponent.

The crucial property of the bound provided by (27) is that the constants involved remain bounded if we replace P by P_A, for any matrix $A \in \mathbf{Z}^{m \times n}$. Under the additional assumptions that $L_\infty(P) = 1$ and $m \leq n$, this suffices to bound the first two integrals appearing in (26), as we show in the following Proposition 4.10. This proposition follows from Lemma 4.6 by a *Tauberian estimate*, similar in spirit to the ones considered in [61]. Note that the aforementioned assumptions are harmless, as we will explain at the beginning of the proof of Theorem 4.1.

Proposition 4.10. *Fix two natural numbers $k \in \mathbf{Z}_{\geq 2}$ and $n \in \mathbf{Z}_{\geq 1}$. Let $P \in \mathbf{C}[\underline{z}_n^{\pm 1}]$ be a Laurent polynomial such that $k(P) = k$ and $L_\infty(P) = 1$. Let A be a matrix in $\mathbf{Z}^{m \times n}$ such that $\rho(A) > \mathrm{diam}(P)$ and $m \leq n$. Then, for every $\varepsilon > 0$ the following inequalities hold:*

$$0 \leq \int_{\mathbf{T}^n} \frac{1}{2}\log(|P|^2+\varepsilon) - \log|P| d\mu_A \leq 12 \cdot (k \cdot n)^2 \cdot \varepsilon^{\frac{1}{2n(k-1)}}.$$

Proof. Let ν be the measure on $\mathbf{R}_{\geq 0}$ defined as the push-forward of μ_A along the (measurable) function $|P|\colon \mathbf{T}^n \to \mathbf{R}_{\geq 0}$. Moreover, let us define the functions $\phi(t) := \frac{1}{2}\log\left(1+\frac{\varepsilon}{t^2}\right)$ and $\psi(s) := \sqrt{\frac{\varepsilon}{e^{2s}-1}}$, so that $\psi(\phi(x)) = \phi(\psi(x)) = x$ for every $x > 0$. Writing χ_S for the characteristic function of a subset $S \subseteq \mathbf{R}$, we have the following identities:

$$\begin{aligned}\int_{\mathbf{T}^n} \frac{1}{2}\log(|P|^2+\varepsilon) - \log|P| d\mu_A &= \int_0^{+\infty} \phi(t) d\nu(t) = \int_0^{+\infty}\left(\int_0^{+\infty} \chi_{[0,\phi(t)]}(s)\, ds\right) d\nu(t) \\ &= \int_0^{+\infty}\left(\int_0^{+\infty} \chi_{[0,\psi(s)]}(t)\, d\nu(t)\right) ds = \int_0^{+\infty} \nu([0,\psi(s)]) ds.\end{aligned} \tag{28}$$

Now, thanks to Lemma 4.6, we have that:

$$\nu([0,t]) \leq 6\cdot k\cdot n\cdot t^{\frac{1}{n(k-1)}} \tag{29}$$

for every $t \in \mathbf{R}_{>0}$. Indeed, this estimate clearly holds if $t \geq 1$, because $\nu([0,t]) \leq 1$ for every $t \in \mathbf{R}_{>0}$. Moreover, the assumption $\rho(A) > \operatorname{diam}(P)$ implies that $k(P_A) = k(P) = k$ and $L_\infty(P_A) = L_\infty(P) = 1$, as we explained at the beginning of the proof of Proposition 3.4. Hence, for $0 < t < 1$ we see from Lemma 4.6 that:

$$\nu([0,t]) := \mu_A(\{\underline{z} \in \mathbf{T}^n \colon |P(\underline{z})| < t\}) = \mu_m(\{\underline{z} \in \mathbf{T}^m \colon |P_A(\underline{z})| < t\}) \leq 6\cdot(k(P_A)-1)\cdot m\cdot t^{\frac{1}{m(k(P_A)-1)}} \leq 6\cdot k\cdot n\cdot t^{\frac{1}{n(k-1)}}.$$

Combining (29) with the identities provided by (28), we get the bound:

$$\int_{\mathbf{T}^n} \frac{1}{2}\log(|P|^2+\varepsilon) - \log|P| d\mu_A = \int_0^{+\infty} \nu([0,\psi(s)]) ds \leq \left(6kn\cdot\int_0^{+\infty} \left(e^{2u}-1\right)^{-\frac{1}{2n(k-1)}} du\right)\cdot \varepsilon^{\frac{1}{2n(k-1)}}.$$

The last integral can be computed by substituting $t = e^{2u}-1$ and by using Cauchy's residue theorem (see [13, p. 107]):

$$\int_0^{+\infty} \left(e^{2u}-1\right)^{-\frac{1}{2n(k-1)}} du = \frac{1}{2}\int_0^{+\infty} \frac{t^{-\frac{1}{2n(k-1)}}}{(t+1)} dt = \frac{\pi}{2\cdot\sin\left(\frac{\pi}{2n(k-1)}\right)}.$$

To conclude, we observe that $2\sin(x) \geq x$ when $0 \leq x \leq \pi/2$. □

We now tackle the last term in (26). We wish to bound it using Theorem 4.2. To this end, we must show that the function $\frac{1}{2}\log(|P|^2+\varepsilon)$, defined on the torus $\mathbf{T}^n$, can be extended to a holomorphic function on a neighborhood of $\mathbf{T}^n$. Following the idea used to prove Corollary 4.4, we use the Laurent polynomial PP^*, which is equal to $|P|^2$ on $\mathbf{T}^n$, and we consider the function $\frac{1}{2}\log(PP^*+\varepsilon)$. The following proposition shows that this function is well-defined and holomorphic on an explicit annulus containing $\mathbf{T}^n$.

Proposition 4.11. *Fix $n \in \mathbf{N}$, a Laurent polynomial $P \in \mathbf{C}[\underline{z}_n^{\pm 1}] \setminus \{0\}$, and $\varepsilon > 0$. Let P^* be the conjugate reciprocal of P, introduced in Section 2.5. Let $\delta := \delta_\varepsilon(P)$ be the constant defined in (8). Then the function $f_\varepsilon := \frac{1}{2}\log(PP^*+\varepsilon)$, defined using the principal branch of the logarithm, is holomorphic on an open neighborhood of the annulus $\mathscr{C}_\delta$ defined in (2).*

Proof. Denote by Q the Laurent polynomial PP^*. Since $Q = |P|^2$ on the torus, the image $Q(\mathbf{T}^n)$ is a segment contained in $\mathbf{R}_{\geq 0}$. We will show that $Q(\mathscr{C}_\delta)$ is contained in the half-plane $\{w \in \mathbf{C} : \operatorname{Re}(w) > -\varepsilon\}$, so that we may consider the principal branch of the logarithm of $Q+\varepsilon$ on $\mathscr{C}_\delta$. Our strategy is to fix a point $\underline{u} = (u_1,\dots,u_n)$ on $\mathbf{T}^n$, and to bound from below the real part of Q near $\underline{u}$ in radial directions. We therefore write $\underline{z} = (e^{h_1}u_1,\dots,e^{h_n}u_n)$, and consider the function $g_{\underline{u}} : \mathbf{R}^n \to \mathbf{C}$ defined by

$$g_{\underline{u}}(\underline{h}) = Q(e^{h_1}u_1,\dots,e^{h_n}u_n).$$

Note in particular that $g_{\underline{u}}(\mathbf{0}) = Q(\underline{u}) \in \mathbf{R}_{\geq 0}$. Let us apply the multivariable Taylor theorem to $g_{\underline{u}}$ at the origin:

$$g_{\underline{u}}(\underline{h}) = g_{\underline{u}}(\mathbf{0}) + \sum_{i=1}^n \frac{\partial g_{\underline{u}}}{\partial h_i}(\mathbf{0})\cdot h_i + R_{\underline{u}}(\underline{h})$$

where the remainder is given in Lagrange's form by

$$R_{\underline{u}}(\underline{h}) = \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 g_{\underline{u}}}{\partial h_i \partial h_j}(\alpha_{\underline{h}}\underline{h})\cdot h_i h_j \qquad (0 < \alpha_{\underline{h}} < 1).$$

Lemma 4.12. *The partial derivatives $\partial g_{\underline{u}}/\partial h_i(\mathbf{0})$, $1 \leq i \leq n$, are purely imaginary.*

Proof. From the definition of Q, we have $\overline{Q(z_1,\dots,z_n)} = Q(1/\overline{z_1},\dots,1/\overline{z_n})$ for every $\underline{z} \in (\mathbf{C}^\times)^n$. Substituting $\underline{z} = (e^{h_1}u_1,\dots,e^{h_n}u_n)$ gives $\overline{g_{\underline{u}}(\underline{h})} = g_{\underline{u}}(-\underline{h})$. Differentiating with respect to h_i at $\mathbf{0}$, we get the result. □

Lemma 4.12 ensures that the real part of Q behaves *quadratically* in $\underline{h}$ near the torus. More precisely, we have

$$\mathrm{Re}(Q(\underline{z})) = \mathrm{Re}(g_{\underline{u}}(\underline{h})) = g_{\underline{u}}(\mathbf{0}) + \mathrm{Re}(R_{\underline{u}}(\underline{h})) \geq \mathrm{Re}(R_{\underline{u}}(\underline{h})) \geq -|R_{\underline{u}}(\underline{h})|. \tag{30}$$

Now, let us introduce the differential operators $D_k = z_k(\partial/\partial z_k)$ for every $k \in \{1,\dots,n\}$. Notice that $\partial g_{\underline{u}}/\partial h_k(\underline{h}) = (D_k Q)(\underline{z})$, and similarly for the higher order derivatives. So it suffices to bound $D_i D_j Q(\underline{z})$. Expanding this polynomial, we get

$$D_i D_j Q(\underline{z}) = D_i D_j \left(\sum_{v,w \in \mathbf{Z}^n} c_v(P)\overline{c_w(P)}\underline{z}^{v-w} \right) = \sum_{v,w \in \mathbf{Z}^n} (v_i - w_i)(v_j - w_j) c_v(P)\overline{c_w(P)}\underline{z}^{v-w}$$

which gives the bound

$$|D_i D_j Q(\underline{z})| \leq \sum_{v,w \in \mathbf{Z}^n} |v_i - w_i||v_j - w_j||c_v(P)||c_w(P)| \prod_{k=1}^n e^{|h_k||v_k - w_k|} \leq \mathrm{diam}(P)^2 L_1(P)^2 e^{\mathrm{diam}(P)\|\underline{h}\|_1}. \tag{31}$$

Combining (30) and (31), we obtain

$$\mathrm{Re}(Q(\underline{z})) \geq -|R_{\underline{u}}(\underline{h})| \geq -\frac{1}{2}\sum_{i,j=1}^n \left| \frac{\partial^2 g_{\underline{u}}}{\partial h_i \partial h_j}(\alpha_{\underline{h}}\,\underline{h}) \right| \cdot |h_i||h_j| \geq -\frac{1}{2}\mathrm{diam}(P)^2 L_1(P)^2 e^{\mathrm{diam}(P)\|\underline{h}\|_1} \cdot \|\underline{h}\|_1^2. \tag{32}$$

Finally, if $\underline{z} \in \mathscr{C}_\delta$ then $\|\underline{h}\|_1 = \sum_{i=1}^n |\log|z_i|| \leq \delta$, and the definition (8) of δ implies that the right-hand side of (32) is $\geq -\frac{2}{3}\varepsilon$. We thus have $\mathrm{Re}(Q+\varepsilon) > 0$ on an open neighbourhood of $\mathscr{C}_\delta$, as we wanted to show. □

Thanks to Proposition 4.11, we may apply Theorem 4.2 to the functions f_ε. However, we also need to bound f_ε on the domain $\mathscr{C}_{\delta_\varepsilon(P)}$. This is the content of the following lemma.

Lemma 4.13. *Let $P \in \mathbf{C}[\underline{z}_n^{\pm 1}]$ be a non-zero Laurent polynomial, and fix $\varepsilon > 0$. Let $\delta := \delta_\varepsilon(P) \in \mathbf{R}_{>0}$ be defined as in (8), and let f_ε be the function defined in Proposition 4.11. Then, we have that:*

$$|f_\varepsilon(\underline{z}_n)| \leq |\log\varepsilon| + 2|\log L_1(P)| + 3$$

for every $\underline{z}_n \in \mathscr{C}_\delta$.

Proof. Let $Q = PP^*$. We have $f_\varepsilon = \log(Q+\varepsilon) = \log|Q+\varepsilon| + i\arg(Q+\varepsilon)$, and by the proof of Proposition 4.11, the argument of $Q+\varepsilon$ stays in $]-\pi/2,\pi/2[$ on the domain $\mathscr{C}_\delta$. It remains to bound from below and from above the modulus of $Q+\varepsilon$.

The lower bound follows from (32), since $|Q(\underline{z})+\varepsilon| \geq \mathrm{Re}(Q(\underline{z}))+\varepsilon \geq \varepsilon/3$ for $\underline{z} \in \mathscr{C}_\delta$ as seen at the end of the previous proof. For the upper bound, let us write $\underline{z} = (e^{h_1}u_1,\dots,e^{h_n}u_n)$, where $\underline{u} = (u_1,\dots,u_n) \in \mathbf{T}^n$ and $\underline{h} = (h_1,\dots,h_n) \in \mathbf{R}^n$. Then, a simple application of the triangle inequality yields:

$$|Q(\underline{z})| \leq \sum_{v,w \in \mathbf{Z}^n} |c_v(P)||c_w(P)| \prod_{j=1}^n e^{|h_j||v_j - w_j|} \leq L_1(P)^2 e^{\delta_\varepsilon(P)\,\mathrm{diam}(P)} \leq \frac{4}{3}L_1(P)^2.$$

We deduce that:

$$|\log(Q(\underline{z})+\varepsilon)| \leq |\log|Q(\underline{z})+\varepsilon|| + \frac{\pi}{2} \leq \max\left(\left|\log\frac{\varepsilon}{3}\right|, \left|\log\left(\frac{4}{3}L_1(P)^2+\varepsilon\right)\right| \right) + \frac{\pi}{2}.$$

We conclude using the inequality $|\log(x+y)| \leq |\log(x)| + |\log(y)| + \log 2$, valid for any $x,y > 0$, which is easily proved by distinguishing the cases $x \leq y$ and $x \geq y$. □

Using Theorem 4.2, together with Proposition 4.11 and the bound of Lemma 4.13, we get:

Corollary 4.14. *Let $m,n \geq 1$ be integers, and $P(\underline{z}_n) \in \mathbf{C}[\underline{z}_n^{\pm 1}]$ be a non-zero Laurent polynomial. For $\varepsilon > 0$, let $f_\varepsilon = \frac{1}{2}\log(|P|^2 + \varepsilon)$, and let $\delta_\varepsilon(P) \in \mathbf{R}_{>0}$ be defined as in (8). Then, for every matrix $A \in \mathbf{Z}^{m\times n}$ such that $\rho(A)\delta_\varepsilon(P) \geq 2d(A)/3$, we have:*

$$\left| \int_{\mathbf{T}^n} f_\varepsilon d\mu_A - \int_{\mathbf{T}^n} f_\varepsilon d\mu_n \right| \leq (d(A)+1)3^{d(A)} \cdot \frac{|\log\varepsilon| + 2|\log L_1(P)| + 3}{\exp(\delta_\varepsilon(P)\cdot\rho(A))}.$$

We are finally ready to prove Theorem 4.1, by choosing a suitable value of ε.

Proof of Theorem 4.1. We fix the Laurent polynomial P and a matrix A in $\mathbf{Z}^{m\times n}$ verifying the assumption of Theorem 4.1, that is $\rho(A) \geq \rho_0(P)$. For this proof, we will simplify a bit the notations, by denoting $k := k(P)$, $\rho_0 := \rho_0(P)$, $\delta_\varepsilon := \delta_\varepsilon(P)$, $\rho := \rho(A)$ and $d = d(A)$. Note that $d \leq n$ by definition.

Since the quantity $|m(P) - m(P_A)|$ we want to bound does not change when multiplying P by a non-zero constant, we will assume without loss of generality that $L_\infty(P) = 1$. Finally, we may assume without loss of generality that $m \leq n$ (compare with [60, Theorem 4]). Indeed, let $U \in \mathbf{Z}^{m\times m}$ be a matrix with non-zero determinant such that $U \cdot A$ is in row-echelon form, and write $B \in \mathbf{Z}^{m'\times n}$ for the matrix obtained from $U \cdot A$ by deleting all the zero rows. Then, $m' = \mathrm{rk}(B) \leq n$ by construction, and one has that $\kappa_A = \kappa_B$, which implies that $\rho(A) = \rho(B)$. Moreover, one sees that $m(P_A) = m((P_A)_U) = m(P_{U\cdot A}) = m(P_B)$ by combining [58, Lemma 7] and [60, Lemma 6]. Hence, upon replacing A with B, we can and will assume without loss of generality that $m \leq n$, as we claimed.

Now, let us come back to the bound provided by (26):

$$|m(P_A) - m(P)| \leq \left| \int_{\mathbf{T}^n} \frac{1}{2}\log(|P|^2+\varepsilon) - \log|P| d\mu_A \right| + \left| \int_{\mathbf{T}^n} \frac{1}{2}\log(|P|^2+\varepsilon) - \log|P| d\mu_n \right| + \left| \int_{\mathbf{T}^n} f_\varepsilon(\underline{z}) d\mu_A - \int_{\mathbf{T}^n} f_\varepsilon(\underline{z}) d\mu_n(\underline{z}) \right|$$

which holds for any $\varepsilon > 0$. The first two terms are bounded by Proposition 4.10 under some conditions, whereas the third term is bounded by Corollary 4.14, under other assumptions. The strategy is to choose the value of ε such that we can indeed apply these two results and, moreover, that the two upper bounds become comparable. To that end, let us fix the quantity:

$$\varepsilon := \left(\frac{\operatorname{diam}(P) \cdot L_1(P)}{n \cdot (k-1)} \cdot \frac{\log(\rho)}{\rho} \right)^2 \tag{33}$$

First of all, let us observe that the assumptions of Proposition 4.10 are fulfilled. Indeed, $\rho \geq \rho_0 > \operatorname{diam}(P)$ by assumption. Moreover, we also assumed without loss of generality that $L_\infty(P) = 1$ and $m \leq n$.

To verify the assumptions of Corollary 4.14, we have to check that $\delta_\varepsilon \rho \geq 2d/3$. To see this, note first of all that the inequalities $\rho \geq \rho_0 \geq 7\left(\frac{\operatorname{diam}(P)}{n}\right)^2$ hold by assumption. Plugging this bound in (33), and using the elementary inequality $\frac{\log(\rho)}{\rho} \leq \frac{3}{4\sqrt{\rho}}$, we see that $\sqrt{\varepsilon} \leq \log(4/3) L_1(P)$. Combining this with the definition of δ_ε, given in (8), we get the equality:

$$\delta_\varepsilon = \frac{\sqrt{\varepsilon}}{\operatorname{diam}(P) \cdot L_1(P)} = \frac{1}{n \cdot (k-1)} \cdot \frac{\log(\rho)}{\rho}.$$

Then, the desired condition $\delta_\varepsilon \rho \geq 2d/3$ follows from the lower bound $\rho \geq \rho_0 \geq \exp(5kn^2) \geq \exp(n(k-1)d)$.

Eventually, to use efficiently Corollary 4.14, we need an upper bound on the quantity $|\log(\varepsilon)| + 2|\log L_1(P)| + 3$. We begin by noting that our assumption that $\rho \geq \rho_0 \geq 7\left(\frac{\operatorname{diam}(P)}{n}\right)^2$, and the remark $1 = L_\infty(P) \leq L_1(P) \leq kL_\infty(P) = k$, imply that

$$\varepsilon \leq \left(\frac{\operatorname{diam}(P) \cdot L_1(P)}{n \cdot (k-1)} \right)^2 \cdot \frac{9}{16\rho} \leq \left(\frac{k}{k-1} \right)^2 \cdot \frac{9}{16 \times 7} \leq 1.$$

So we can further use the bound $\rho \geq \rho_0 \geq \exp(5 \cdot k \cdot n^2) \geq \exp\left(\frac{k \cdot e^{3/2} \cdot n}{\operatorname{diam}(P)}\right)$, which holds by assumption, to obtain:

$$|\log(\varepsilon)| + 2\log(L_1(P)) + 3 = 2\log\left(\frac{ne^{3/2}(k-1)}{\operatorname{diam}(P)} \cdot \frac{\rho}{\log(\rho)} \right) \leq 2\log(\rho).$$

Applying Proposition 4.10 and Corollary 4.14 together with the last estimate, we get the following bound:

$$|m(P_A) - m(P)| \leq 24 \cdot (kn)^2 \cdot \varepsilon^{\frac{1}{2n(k-1)}} + (d+1)3^d \cdot \frac{2\log(\rho)}{\exp(\delta_\varepsilon \rho)}. \tag{34}$$

We can now use our definition (33) of ε, together with the bounds $L_1(P) \leq k$ and $d \leq n$, to obtain:

$$\begin{aligned} |m(P_A) - m(P)| &\leq 24 \cdot (kn)^2 \cdot \left(\frac{\operatorname{diam}(P) \cdot L_1(P)}{n \cdot (k-1)} \cdot \frac{\log(\rho)}{\rho} \right)^{\frac{1}{n(k-1)}} + (d+1)3^d \cdot \frac{2\log(\rho)}{\rho^{\frac{1}{n(k-1)}}} \\ &\leq \left(24 \cdot (kn)^2 \cdot \left(\frac{\operatorname{diam}(P) \cdot L_1(P)}{n \cdot (k-1)} \right)^{\frac{1}{n(k-1)}} + 2(d+1)3^d \right) \frac{\log(\rho)}{\rho^{\frac{1}{n(k-1)}}} \\ &\leq \left(24 \cdot (kn)^2 \cdot \left(\frac{k}{n(k-1)} \right)^{\frac{1}{n(k-1)}} + 2(n+1)3^n \right) \cdot \log(\rho) \cdot \left(\frac{\operatorname{diam}(P)}{\rho} \right)^{\frac{1}{n(k-1)}} \\ &\leq 7k^2(3^n n) \cdot \log(\rho) \cdot \left(\frac{\operatorname{diam}(P)}{\rho} \right)^{\frac{1}{n(k-1)}} \end{aligned}$$

where the last inequality follows from the bounds $\frac{k}{n(k-1)} \leq 1$ and $24(kn)^2 + 2(n+1)3^n \leq 7k^2(3^n n)$ for $n, k \geq 2$. □

Remark 4.15. As is clear from the proof of Theorem 4.1, the quality of the error term depends essentially only on the quality of the bound (27) provided by Lemma 4.6. To see this, fix a Laurent polynomial $P \in \mathbf{C}[\underline{z}_n^{\pm 1}]$ with $k(P) \geq 2$, and a matrix $A \in \mathbf{Z}^{m \times n}$ such that $m \leq n$ and $\rho(A) > \operatorname{diam}(P) + 1$, so that $k(P) = k(P_A)$ and $L_\infty(P) = L_\infty(P_A)$. Suppose moreover that there exist $a, C \in \mathbf{R}_{>0}$ such that the bounds:

$$\mu_n(S(P,t)) \leq C \cdot (t/L_\infty(P))^a \quad \text{and} \quad \mu_m(S(P_A,t)) \leq C \cdot (t/L_\infty(P_A))^a \tag{35}$$

hold true for every $t \in \mathbf{R}_{>0}$. Then, going through the proof of Theorem 4.1 one sees that the following bound holds:

$$|m(P_A) - m(P)| \leq 2 \cdot \max\left(C \cdot \frac{(k(P) \cdot a)^a}{a}, (d(A)+1) \cdot 3^{d(A)}\right) \cdot \log(\rho(A)) \cdot \left(\frac{\mathrm{diam}(P)}{\rho(A)}\right)^a \tag{36}$$

under the additional hypothesis that $\rho(A) \geq \max(\mathrm{diam}(P)+1,\ 7(\mathrm{diam}(P) \cdot (k(P)-1) \cdot a)^2,\ \exp(5n/a))$.

More precisely, the first two terms of (26) are bounded by $(2C/a) \cdot \varepsilon^{a/2}$, as follows from Proposition 4.10. Note in particular that the exponent appearing in this bound is halved with respect to the ones featured in (35), as a consequence of the square root appearing in the function $\psi(s) := \sqrt{\varepsilon/(e^{2s}-1)}$. However, Lemma 4.12 allows us to have a final bound (36) featuring the same exponent appearing in (35). More precisely, Lemma 4.12 allows us to define $\delta_\varepsilon(P)$ using $\sqrt{\varepsilon}$ instead of ε. This in turn allows us to set $\varepsilon := (\mathrm{diam}(P) \cdot L_1(P) \cdot a \cdot \log(\rho(A))/\rho(A))^2$. Then, we can bound the last term appearing in (26) by applying Corollary 4.14, and the presence of the square in the definition of ε implies that the exponent appearing in (36) can be taken to be the same as the one featured in (35).

5 Discussion of the speed of convergence

In this section, we study several situations where more can be said about the error term $m(P_A) - m(P)$, compared to the bounds given in Corollary 4.4 and Theorem 4.1. In particular, we devote Section 5.1 to an experimental study of the differences $m(P(z_1, z_1^d)) - m(P)$ for some two-variable polynomials $P \in \mathbf{Z}[\underline{z}_2]$. A full asymptotic expansion for these sequences, under a technical assumption on P, was provided by Condon [17], and our experiments are compatible with this result. Finally, we devote Section 5.2 to the study of a multivariate example where not only an equivalent of the error term can be obtained, but also a full asymptotic expansion. This example goes beyond Condon's framework, both with respect to the number of variables involved (as we study a family of 2-variable polynomials whose Mahler measures converge to the Mahler measure of a 4-variable one) and the type of expansion that we get, where a logarithmic term appears.

5.1 Asymptotic expansions in the presence of toric points

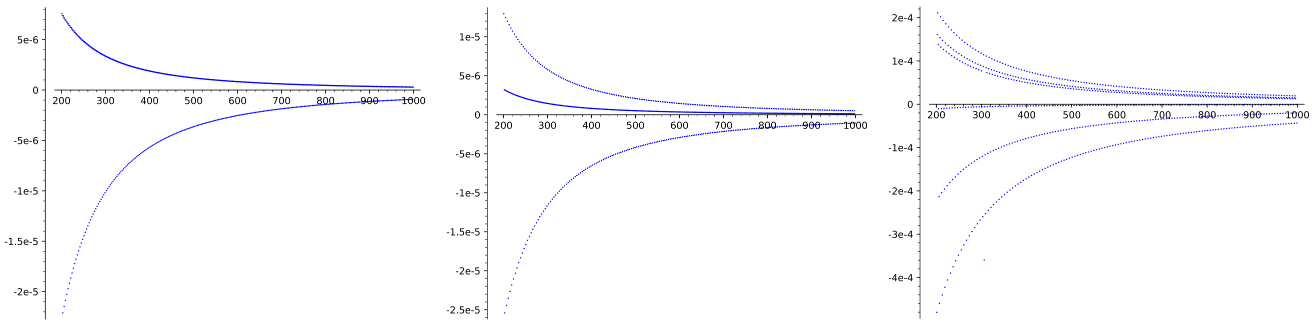
When a polynomial $P(\underline{z}_n) \in \mathbf{C}[\underline{z}_n^{\pm 1}]$ vanishes on the torus $\mathbf{T}^n$, one cannot hope that the error term $|m(P_A) - m(P)|$ decays exponentially fast as $\rho(A) \to +\infty$. This is already evident when $n = 2$, and we take the sequence of matrices $A_d := (1, d) \in \mathbf{Z}^{1\times 2}$, which results in the sequence of polynomials $P_{A_d}(z_1) := P(z_1, z_1^d)$. For $P(z_1, z_2) = z_1 + z_2 + 1$, the resulting sequence of polynomials $\{P_{A_d}(z_1) = z_1 + z_1^d + 1\}_{d=1}^{+\infty}$ was already studied by Boyd [6, Appendix 2], who proved that:

$$m(z_1 + z_1^d + 1) - m(z_1 + z_2 + 1) = \frac{c(d)}{d^2} + O\left(\frac{1}{d^3}\right)$$

where $c\colon \mathbf{Z} \to \mathbf{R}$ is a 3-periodic function. More precisely, $c(d) := -\sqrt{3}\pi/6$ if $d \equiv 2(3)$, and $c(d) := \sqrt{3}\pi/18$ otherwise. This is reflected by the fact that the plot of $m(P_{A_d}) - m(P)$, depicted in Figure 1a, consists of two branches. This is by no means an isolated phenomenon: we include in Figure 1 two other examples, taken from [8, Equation (1-7)] and [8, Table 1] respectively, of polynomials P for which the error term $m(P_{A_d}) - m(P)$ appears to be divided into a finite number of smooth branches.

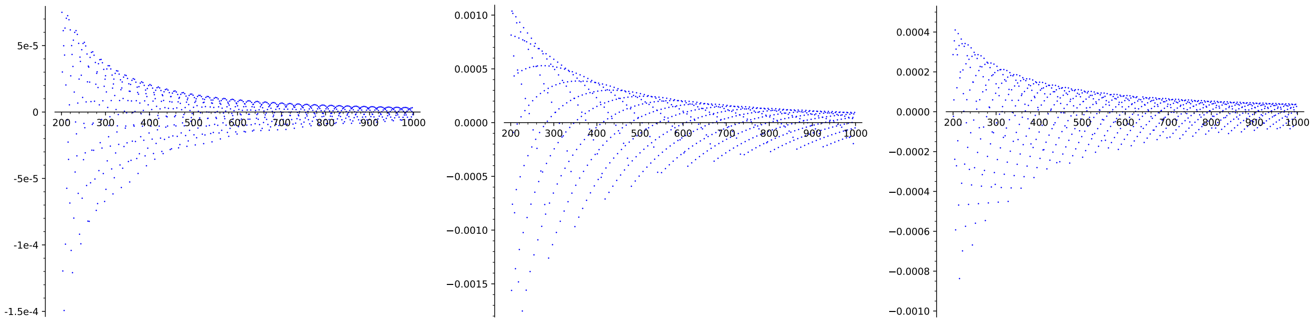
Note however that not all polynomials $P(z_1, z_2)$ give rise to an error term $m(P_{A_d}) - m(P)$ with this kind of behavior. This is depicted in Figures 2a to 2c, which display polynomials taken from [10, Table 1], [44, Example 4.8] and [48, Equation 1] respectively.

These different types of phenomena have been partially explained by Condon's work [17], which provides an asymptotic expansion for the error term $m(P_{A_d}) - m(P)$ of an irreducible polynomial $P \in \mathbf{C}[\underline{z}_2^{\pm 1}]$ such that P and $\partial P/\partial z_2$ do not have a



(a) $P(z_1, z_2) = z_2 + (z_1 + 1)$ (b) $P(z_1, z_2) = (z_1+1)z_2 + (z_1 - 1)$ (c) $P(z_1, z_2) = z_1 z_2^2 + (z_1^2 + z_1 + 1)z_2 + z_1$

Figure 1: Plots of $m(P_{A_d}) - m(P)$, for $A_d = (1, d)$, which seem to lie on finitely many smooth branches. The SageMath code we used to produce these plots is available online [32].



(a) $P = (z_1+1)^4 z_2 - (z_1^2+1)(z_1^2-z_1+1)$ (b) $P = (z_1^8+z_1^6+z_1^4+z_1^2+1)(z_2^2+1)+(2z_1^8-37z_1^6+5z_1^5+70z_1^4+5z_1^3-37z_1^2+2)z_2$ (c) $P = (z_1+1)z_2^2+(z_1^2+z_1+1)z_2+z_1^2+z_1$

Figure 2: Plots of $m(P_{A_d})-m(P)$, for $A_d=(1,d)$, which seem to lie on infinitely many smooth branches. The SAGEMATH code we used to produce these plots is available online [32].

common root on $\mathbf{T}^2$. To be more precise, we need to recall some terminology introduced by [17]. First of all, a function $c\colon \mathbf{R}\to\mathbf{R}$ is said to be *quasi-periodic* if it is the sum of finitely many continuous periodic functions. Then, for every collection of quasi-periodic functions $\{c_j\colon \mathbf{R}\to\mathbf{R}\}_{j\in\mathbf{N}}$ and every function $f\colon \mathbf{N}\to\mathbf{R}$ we will use the notation $f(d)\approx\sum_{j\in\mathbf{N}}\frac{c_j(d)}{d^j}$ if, for every $J\in\mathbf{N}$, there exists a constant $C_{f,J}>0$ such that the following bound:

$$\left| f(d) - \sum_{j=0}^{J-1}\frac{c_j(d)}{d^j}\right| \leq \frac{C_{f,J}}{d^J}$$

holds for every $d\geq 1$. This notation generalizes the usual notion of asymptotic series (see for example [5, Definition 1.3.1]), where the coefficients c_j are assumed to be constant. In particular, [17, Proposition 1] shows that, as in the classical case, any given function $f\colon \mathbf{N}\to\mathbf{R}$ has at most one asymptotic expansion of this kind. Then, Condon proves in [17, Theorem 1] that for any Laurent polynomial $P\in\mathbf{C}[z_2^{\pm 1}]$ such that P and $\partial P/\partial z_2$ do not have common zeros on $\mathbf{T}^2$, one has an asymptotic expansion:

$$m(P(z_1,z_1^d)) - m(P) \approx \sum_{j=2}^{+\infty}\frac{c_j(d)}{d^j} \tag{37}$$

where each $c_j\colon \mathbf{R}\to\mathbf{R}$ is an explicit quasi-periodic function, given by a linear combination of the periodic functions:

$$\{t\mapsto \mathfrak{B}_k(\langle\theta - t\varphi\rangle)\colon k\in\{2,\dots,j\},\ (e^{2\pi i\theta},e^{2\pi i\varphi})\in V_P(\mathbf{C})\cap\mathbf{T}^2\}$$

where $\mathfrak{B}_k(x)$ denotes the k-th Bernoulli polynomial, and $\langle x\rangle := x-\lfloor x\rfloor$ denotes the fractional part of a real number $x\in\mathbf{R}$. In particular, if $V_P(\mathbf{C})\cap\mathbf{T}^2\subseteq\mu_N\times\mu_N$, where $\mu_N\subseteq\mathbf{T}^1$ denotes the set of N-th roots of unity, each function c_j is N-periodic. This is precisely what happens for the two polynomials displayed in Figures 1a and 1b, and hence this periodicity of the coefficients c_j explains why each point $(d,m(P_{A_d})-m(P))\in\mathbf{R}^2$ seems to lie on a finite union of graphs of smooth functions. On the other hand, one can show that each of the polynomials displayed in Figure 2 has toric points whose coordinates are not roots of unity, and this gives rise to the depicted behavior, where the points $(d,m(P_{A_d})-m(P))\in\mathbf{R}^2$ seem to lie on an infinite union of graphs of smooth functions. Note finally that Figures 1c, 2b and 2c do not fall strictly within the framework of [17, Theorem 1], because in these cases P and $\partial P/\partial z_2$ have common roots on $\mathbf{T}^2$. This is related to the fact that $m(P_{A_d})-m(P)$ seems to decay more slowly than $1/d^2$ in some cases. For instance, extensive computational evidence (already mentioned in [17, § 8.3]), shows that for the polynomial P appearing in Figure 1c one might expect that $m(P_{A_d})-m(P)\sim c(d)/d^{3/2}$, where $c\colon\mathbf{Z}\to\mathbf{R}$ is 6-periodic.

5.2 An asymptotic expansion with a logarithmic term

This section is dedicated to the sequence of polynomials $P_d(z_1,z_2) := \sum_{0\leq i+j\leq d} z_1^i z_2^j\in\mathbf{C}[z_1,z_2]$, whose Mahler measure was widely studied in [47] by the third author of this paper. In particular, she proved that

$$\lim_{d\to+\infty} m(P_d) = \frac{9}{2\pi^2}\zeta(3) = -18\cdot\zeta'(-2) \tag{38}$$

where $\zeta(s)$ denotes Riemann's zeta function. This convergence is illustrated in Figure 3 and exhibits a much simpler behavior than the examples discussed before.

We can give a new proof of (38) using Theorem 3.1. More precisely, we can write:

$$P_d(\underline{z}) = \frac{1}{(1-z_1)(1-z_2)} - \left(\frac{z_1}{(1-z_1)(z_1-z_2)}\right)z_1^{d+1} - \left(\frac{z_2}{(1-z_2)(z_2-z_1)}\right)z_2^{d+1}$$

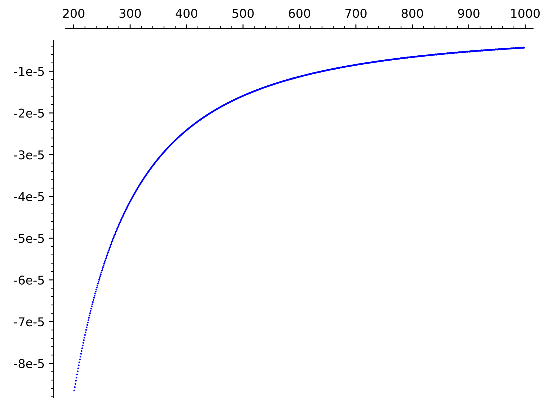


Figure 3: Plot of $m(P_d) - m(P_\infty)$. The SAGEMATH code we used to produce this plot is available online [32].

using the geometric series. Thus, we see that:

$$P_d(\underline{z}_2)(1-z_1)(1-z_2)(z_1-z_2) = z_1^{d+2}(z_2-1) + z_2^{d+2}(1-z_1) + (z_1-z_2) = (1-z_1^{d+2})(1-z_2) - (1-z_1)(1-z_2^{d+2})$$

which implies that $m(P_d) = m(P_\infty(\underline{z}_2^{M_d}))$, where $P_\infty(\underline{z}_4) := (1-z_1)(1-z_2) - (1-z_3)(1-z_4)$ and $M_d := \left(\begin{smallmatrix} d+2 & 0 & 1 & 0 \\ 0 & 1 & 0 & d+2 \end{smallmatrix}\right) \in \mathbf{Z}^{2\times 4}$. To apply Theorem 3.1, we need to compute $\rho(M_d)$. This is elementary:

$$\{v \in \mathbf{Z}^4 \mid M_d \cdot v = \mathbf{0}\} = \left\langle \left(\begin{smallmatrix} -1 \\ 0 \\ d+2 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ d+2 \\ 0 \\ -1 \end{smallmatrix}\right) \right\rangle_{\mathbf{Z}}$$

so that $\rho(M_d) = d+2$ for every $d \in \mathbf{N}$. Thus, Theorem 3.1 shows that:

$$\lim_{d\to+\infty} m(P_d) = \lim_{d\to+\infty} m(P_\infty(\underline{z}_2^{M_d})) = m(P_\infty).$$

Finally, D'Andrea and Lalín [18, Theorem 7] have proved that $m(P_\infty) = -18 \cdot \zeta'(-2)$, which yields back the convergence (38) proved by Mehrabdollahei in [47].

The proof of (38) provided in [47] proceeds along very different lines. More precisely, Mehrabdollahei uses crucially the fact that P_d is always an exact polynomial (see [47, Definition 2.2]), which allows her to write:

$$m(P_d) = \frac{3}{d+1} \sum_{1\leq k\leq d+1} \frac{(d+2-2k)}{2\pi} \cdot D\left(e^{\frac{2\pi i k}{d+2}}\right) - \frac{3}{d+2} \sum_{1\leq k\leq d} \frac{(d+1-2k)}{2\pi} \cdot D\left(e^{\frac{2\pi i k}{d+1}}\right) \tag{39}$$

where $D(z) := \arg(1-z)\log|z| - \operatorname{Im}\left(\int_0^z \log(1-t)\frac{dt}{t}\right)$ denotes the Bloch-Wigner dilogarithm (see [47, Theorem p. 2]). Even if our proof of (38) does not use (39), the latter allows us to obtain a full asymptotic expansion for the error term $m(P_d) - m(P_\infty)$, which is the content of the following theorem.

Theorem 5.1. *We have the following asymptotic expansion of $m(P_d) - m(P_\infty)$ as $d \to +\infty$:*

$$m(P_d) - m(P_\infty) \approx \frac{1}{(d+1)(d+2)} \left[-\frac{\log(d)}{2} + \sum_{k=0}^{+\infty} \frac{\alpha_k}{d^k} \right] \tag{40}$$

where the coefficients $\alpha_k \in \mathbf{R}$ are defined as:

$$\alpha_0 := 6\big(\zeta'(-1) - \zeta'(-2)\big) + \frac{\log(2\pi)}{2} - 1$$
$$\alpha_k := \frac{12\cdot(-1)^k}{k(k+1)} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k+1}{2j} \cdot \frac{(2^{k+1-2j}-1)(2j-1)}{(2j+1)(2j+2)} \cdot B_{2j+2} \cdot \zeta(2j) \qquad (k \geq 1)$$

where B_n denotes the n-th Bernoulli number.

In particular, we have that $m(P_d) - m(P_\infty) \sim -\dfrac{\log(\rho(M_d))}{2\rho(M_d)^2}$ as $d \to +\infty$.

Proof. First of all, we observe that (39) can be rewritten as:

$$m(P_d) - m(P_\infty) = \frac{3}{(d+1)(d+2)} \left[-2\zeta'(-2) + (d+1)^3 E_{d+1}(f) - (d+2)^3 E_{d+2}(f) \right] \tag{41}$$

where $f(x) := \frac{1-2x}{2\pi} D(e^{2\pi i x})$ and $E_d(F) := \int_0^1 F(x)dx - \frac{1}{d}\left(\sum_{j=1}^d F\left(\frac{j}{d}\right)\right)$ for each integrable function $F\colon [0,1] \to \mathbf{R}$.

Now, write $h(x) := (2x-1)x\log(x)$ and observe that the function

$$g(x) := f(x) - \log(2\pi)(1-2x)^2 - h(x) - h(1-x)$$

is smooth on the closed interval $[0,1]$. Indeed, $f(x)$ and $g(x)$ are smooth on the open interval $(0,1)$, and $g(x) = g(1-x)$. Thus, to see that $g(x)$ is smooth on $[0,1]$ it is sufficient to compute the Maclaurin series:

$$g(x) = (1-2x)\left[-\log(2\pi) + (\log(2\pi)+2)x + \sum_{k=2}^{+\infty} \frac{1}{(1-k)k} \cdot x^k + \sum_{m=1}^{+\infty} \frac{\zeta(2m)}{m(2m+1)} \cdot x^{2m+1}\right] \tag{42}$$

which follows from the identity $\frac{d^2}{d\theta^2} D(e^{2i\theta}) = -2\cot(\theta)$. Moreover, (42) allows one to write the asymptotic expansion:

$$E_d(g) \approx \frac{3\log(2\pi)+2}{6} \cdot \frac{1}{d^2} + \sum_{m=2}^{+\infty} \frac{B_{2m}[2m+1+2(2m-3)\zeta(2m-2)]}{m(2m-1)(2m-2)(2m-3)} \cdot \frac{1}{d^{2m}}$$

using the classical Euler-Maclaurin summation formula [49, Equation 1].

This formula was extended by Navot to functions with a logarithmic singularity at one endpoint of the integration interval [49, Equation 7]. Applying this generalization to $h(x)$ we see that:

$$E_d(h) \approx \frac{1}{12} \cdot \frac{\log(d)}{d^2} - \left(\zeta'(-1) + \frac{1}{12}\right) \cdot \frac{1}{d^2} + 2\zeta'(-2) \cdot \frac{1}{d^3} - \sum_{m=2}^{+\infty} \frac{B_{2m}(2m+1)}{(2m)(2m-1)(2m-2)(2m-3)} \cdot \frac{1}{d^{2m}}$$

as follows from the Taylor expansion $h(x) = (x-1) + \frac{5}{2}(x-1)^2 + \sum_{k=3}^{+\infty} \frac{(-1)^{k+1}(k+2)}{k(k-1)(k-2)} \cdot (x-1)^k$.

Hence, observing that $E_d((1-2x)^2) = -\frac{2}{3} \cdot \frac{1}{d^2}$ and $E_d(h(x)) = E_d(h(1-x))$, we get:

$$E_d(f) \approx \frac{1}{6} \cdot \frac{\log(d)}{d^2} + \left(\frac{1-\log(2\pi)}{6} - 2\zeta'(-1)\right) \cdot \frac{1}{d^2} + 4\zeta'(-2) \cdot \frac{1}{d^3} + 4\sum_{m=2}^{+\infty} a_{2m} \cdot \frac{1}{d^{2m}} \tag{43}$$

where we set $a_k := \frac{B_k \zeta(k-2)}{k(k-1)(k-2)} \in \mathbf{Q} \cdot \pi^{k-2}$ for every integer $k \geq 4$.

Now, combining the identities:

$$(d+1)\log(d+1) - (d+2)\log(d+2) \approx -\log(d) - 1 + \sum_{k=1}^{+\infty} \frac{(-1)^k(2^{k+1}-1)}{k(k+1)} \cdot \frac{1}{d^k}$$

$$\frac{1}{(d+1)^{2m-3}} - \frac{1}{(d+2)^{2m-3}} \approx (2m-3)\sum_{j=1}^{+\infty} \binom{j+2m-4}{2m-3} \frac{(-1)^{j+1}(2^j-1)}{j} \cdot \frac{1}{d^{j+2m-3}}$$

with (41) and (43) we get:

$$\begin{aligned}(d+1)(d+2)(m(P_d) - m(P_\infty)) \approx & -\frac{\log(d)}{2} + \left(6(\zeta'(-1) - \zeta'(-2)) + \frac{\log(2\pi)}{2} - 1\right) + \\ & + \sum_{k=1}^{+\infty} \frac{(-1)^k(2^{k+1}-1)}{2k(k+1)} \cdot \frac{1}{d^k} \\ & + 12\sum_{k=2}^{+\infty} (-1)^k \cdot \left(\sum_{j=1}^{k-1} \binom{k-1}{k-j} \frac{(2^j-1)(k-j)}{j} a_{k-j+3}\right) \frac{1}{d^k}\end{aligned}$$

which after some rearrangement, gives us (40). □

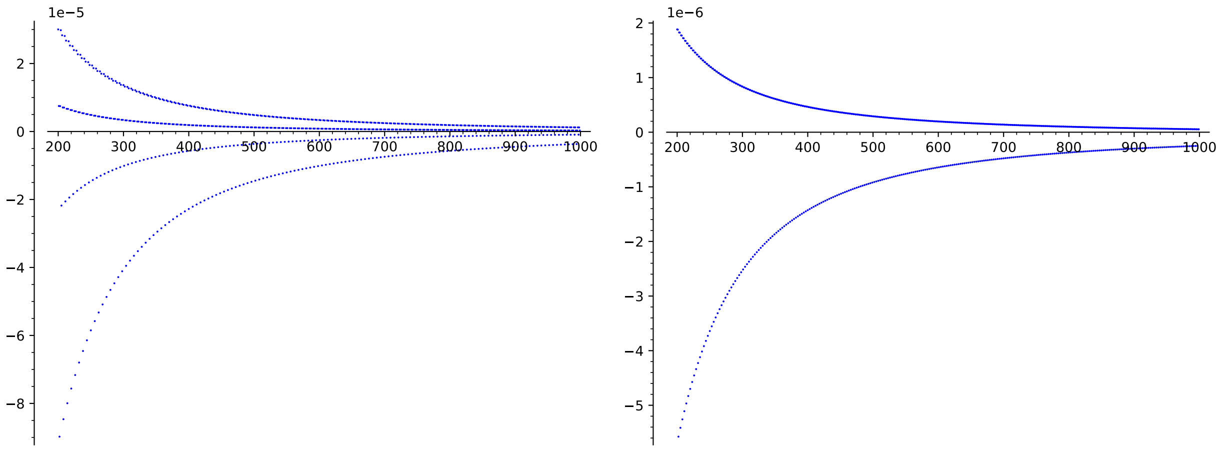
Remark 5.2. The asymptotic expansion (40) has been checked numerically using the PARI/GP program `Asympraw` available at [3].

Remark 5.3. We note that in the asymptotic expansion (40), the coefficients α_k do not depend on d, which is in contrast to what happened for the examples described in Section 5.1. Moreover, if $k \geq 1$ we see that α_k is a $\mathbf{Q}$-linear combination of $1, \pi^2, \pi^4, \ldots, \pi^{2\lfloor k/2 \rfloor}$, whereas α_0 and π are most likely algebraically independent.

Remark 5.4. Note that [31, Proposition 8] provides another family of polynomials, in three variables, whose Mahler measures converge to $m(P_\infty)$. They correspond to the monomial substitutions provided by the matrices:

$$A_{a,b} := \begin{pmatrix} b & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbf{Z}^{3\times 4}$$

taken as either $a \to +\infty$ or $b \to +\infty$, where $a, b \in \mathbf{N}$ are coprime. Since $\ker(A_{a,b}) \cap \mathbf{Z}^4 = \mathbf{Z} \cdot (-a, 0, 0, b)^t$, we see that $\rho(A_{a,b}) = \max(a,b)$, and so [31, Proposition 8] can be seen as a special case of Theorem 3.1. On the other hand, the proof provided by Gu and Lalín uses an explicit formula (see [31, Theorem 1]) for the Mahler measures of the three-variable polynomials $(P_\infty)_{A_{a,b}}$, which is similar to the formula (39) proved in [47] by the third named author of this paper.



(a) $A_d = (1, d)$

(b) $A_d = (d, 1)$

Figure 4: Plots of $m(P_{A_d}) - m(P)$, for $P = z_1^2 + z_2 + 1$.

5.3 Perspectives

We hope that the previous Sections 5.1 and 5.2 managed to convey to the reader our impression that understanding the rate of convergence (and, even more, the asymptotic expansions) of the difference $m(P_A) - m(P)$, remains a difficult and interesting challenge. In particular, the bound provided by Theorem 4.1 seems far from optimal, even for a general polynomial. Moreover, the actual rate of convergence, for a fixed polynomial P, seems to depend on the geometry of the real algebraic set $V_P(\mathbf{C}) \cap \mathbf{T}^n$, which can be quite complicated on its own (see [33, Example 5.2.5]). Furthermore, one should study as well the geometries of the intersections of this real algebraic set with the sub-tori cut out by the matrices A. We also lack a rationale explaining the logarithmic term appearing in the asymptotic expansion provided in Theorem 5.1.

To conclude, we note that the invariant $\rho(A)$, whose divergence is sufficient to guarantee the convergence $m(P_A) \to m(P)$ (as we showed in Theorem 3.1), will not suffice to express even the first term in the asymptotic expansion of $m(P_A) - m(P)$. More precisely, let $P = z_1^2 + z_2 + 1$, and consider the two sequences of matrices $A_d = (1, d)$ and $A_d = (d, 1)$. Then $\rho(A_d) = d$ in both cases, but the convergence patterns for $m(P_{A_d}) \to m(P)$, portrayed in Figure 4, are quite different, which can be rigorously proved using Condon's formula (37).

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