Wandering across the Weierstrass function, while revisiting its properties

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In honor of Jean-Yves Chemin

Abstract

The Weierstrass function is known as one of these so-called pathological mathematical objects, continuous everywhere, while nowhere differentiable. In the sequel, we have chosen, first, to concentrate on the unconventional history of this function, a function breaking with the mathematical canons of classical analysis of the XIX$^{\text{th}}$ century. We recall that it then took nearly a century for new mathematical properties of this function to be brought to light. It has since been the object of a renewed interest, mainly as regards the box-dimension of the related curve. We place ourselves in this vein, and, thanks to our result of 2018, which shows that this value can be obtained in a simple way, without calling for theoretical background in dynamic systems theory, we put forward the link between the non-differentiability and the value of the box-dimension of the curve.

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Introduction

The Weierstrass function, introduced in the the second part of the nineteenth century by Karl Weierstrass [KH16], [Wei75], is known as one of these so-called pathological mathematical objects, continuous everywhere, while nowhere differentiable; given $\lambda \in]0,1[$, and b such that $\lambda\, b > 1 + \dfrac{3\,\pi}{2}$, it is the sum of the uniformly convergent trigonometric series

$$x \in \mathbb{R} \mapsto \sum_{n=0}^{+\infty} \lambda^n \, \cos\left(\pi\, b^n\, x\right).$$

The story of this function, and its introduction, by Karl Weierstrass, is of interest. It has to be placed in both a mathematical and a historical context. On the mathematical point of view, of course, much better than done by Bernhard Riemann in 1861 [Dar75], because the proof of the non-differentiability was given to the whole community, it challenged all the existing theories that went back to André-Marie Ampère at the beginning of the century, and led a new impulse that aroused, in the community, the emergence of new functions bearing the same type of properties.

In the historical point of view, it coincides with the global upgrade, material, moral and conceptual, initiated by Prussia in the XIX$^{\text{th}}$ century, within the framework of German unity, upgrade which is certainly behind the appointment of Karl Weierstrass, a former high-school teacher, as Professor at the Friedrich-Wilhelms University of Berlin.

Karl Weierstrass had distinguished himself by his results on Abelian functions [Wei54], [Wei56]: the German University could not miss such a talent. This choice proved more than just right. The introduction of the Weierstrass function has made history. Its impact lasts since, even if it took a while before new properties came to light.

Actually, in addition to its nowhere differentiability, an interesting feature of the function is its self similarity properties. After the works of A. S. Besicovitch and H. D. Ursell [BU37], it is Benoît Mandelbrot [Man77] who particularly highlighted the fractal properties of the Weierstrass Curve. He also conjectured that the Hausdorff dimension of the graph is $D_{\mathcal{W}} = 2 + \dfrac{\ln \lambda}{\ln b}$.

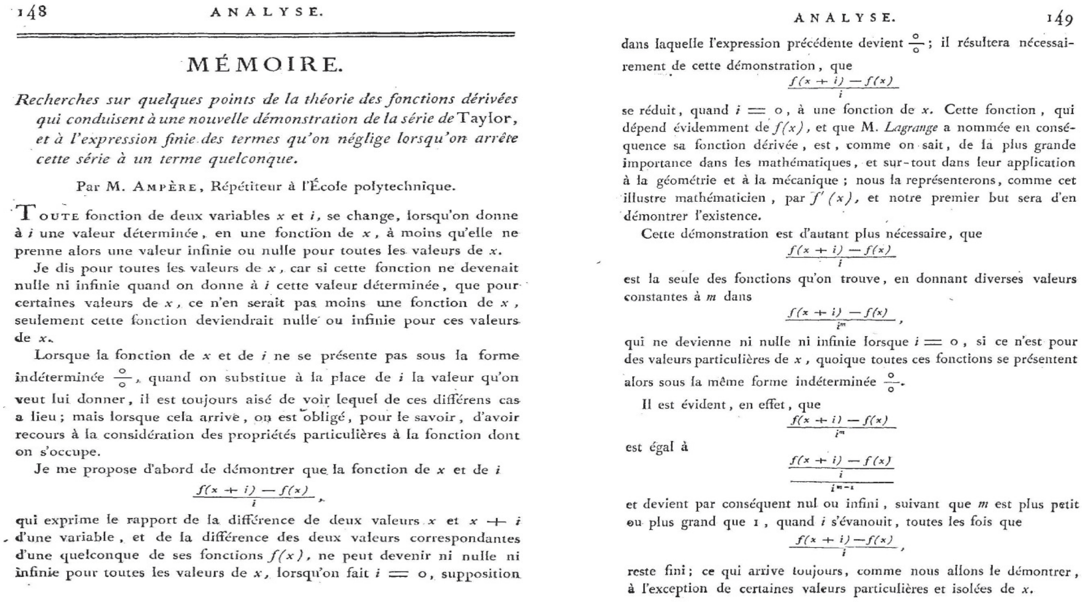
In the view of all that we have evoked, it seemed important to us to consider the Weierstrass function under the prism of an historical perspective, as we expose it in section 1, all the more as interesting discussions still occupy the mathematician community, and us in particular.

For instance, in [Dav18], we have showed that, in the case where $b = N_b$ is an integer, and contrarily to existing work on the subject, the box-counting dimension (or Minkowski dimension) of the Weierstrass curve, which happens to be equal to its Hausdorff dimension [KMPY84], [BBR14], can be obtained in a simple way, without calling for theoretical background in dynamic systems theory, as it is usually the case. At stake are prefractals, by means of a sequence of graphs, that converge towards the Weierstrass Curve. This sequence of graphs enables one to show nice geometric properties, since, for any natural integer m, the consecutive vertices of the m^{th}-order graph $\Gamma_{\mathcal{W}_m}$ are the vertices of simple not self-intersecting polygons with N_b sides, as it is exposed in section 2, polygons which play a part in the determination of the box-counting dimension of the curve.

Also, we improve or retrieve more classical results, and rather simply, as exposed in the sequel: in section 3, we put the light on the fact that our result concerning the box-dimension of the graph also gives an explicit lower bound, which is not given in existing works. Furthermore, we give a new proof of the non-differentiability of the Weierstrass function in the aforementioned case.

1 An historical overview: From Ampère and well-established beliefs, to the so-called pathological objects

In 1806, André-Marie Ampère [Amp06] gave what he considered as a "proof", according to which, for a given curve, it is always possible, except in a finite number of points, to calculate the slope. This "proof", that one can find in the Mathematics books of the time, served as a reference until the mid-nineteenth century.



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MÉMOIRE.

Recherches sur quelques points de la théorie des fonctions dérivées qui conduisent à une nouvelle démonstration de la série de Taylor, *et à l'expression finie des termes qu'on néglige lorsqu'on arrête cette série à un terme quelconque.*

Par M. Ampère, Répétiteur à l'École polytechnique.

Toute fonction de deux variables x et i, se change, lorsqu'on donne à i une valeur déterminée, en une fonction de x, à moins qu'elle ne prenne alors une valeur infinie ou nulle pour toutes les valeurs de x.

Je dis pour toutes les valeurs de x, car si cette fonction ne devenait nulle ni infinie quand on donne à i cette valeur déterminée, que pour certaines valeurs de x, ce n'en serait pas moins une fonction de x, seulement cette fonction deviendrait nulle ou infinie pour ces valeurs de x.

Lorsque la fonction de x et de i ne se présente pas sous la forme indéterminée $\frac{0}{0}$, quand on substitue à la place de i la valeur qu'on veut lui donner, il est toujours aisé de voir lequel de ces différens cas a lieu ; mais lorsque cela arrive, on est obligé, pour le savoir, d'avoir recours à la considération des propriétés particulières à la fonction dont on s'occupe.

Je me propose d'abord de démontrer que la fonction de x et de i

$$\frac{f(x+i)-f(x)}{i},$$

qui exprime le rapport de la différence de deux valeurs x et $x+i$ d'une variable, et de la différence des deux valeurs correspondantes d'une quelconque de ses fonctions $f(x)$, ne peut devenir ni nulle ni infinie pour toutes les valeurs de x, lorsqu'on fait $i=0$, supposition

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dans laquelle l'expression précédente devient $\frac{0}{0}$; il résultera nécessairement de cette démonstration, que

$$\frac{f(x+i)-f(x)}{i}$$

se réduit, quand $i=0$, à une fonction de x. Cette fonction, qui dépend évidemment de $f(x)$, et que M. *Lagrange* a nommée en conséquence sa fonction dérivée, est, comme on sait, de la plus grande importance dans les mathématiques, et sur-tout dans leur application à la géométrie et à la mécanique ; nous la représenterons, comme cet illustre mathématicien, par $f'(x)$, et notre premier but sera d'en démontrer l'existence.

Cette démonstration est d'autant plus nécessaire, que

$$\frac{f(x+i)-f(x)}{i}$$

est la seule des fonctions qu'on trouve, en donnant diverses valeurs constantes à m dans

$$\frac{f(x+i)-f(x)}{i^m},$$

qui ne devienne ni nulle ni infinie lorsque $i=0$, si ce n'est pour des valeurs particulières de x, quoique toutes ces fonctions se présentent alors sous la même forme indéterminée $\frac{0}{0}$.

Il est évident, en effet, que

$$\frac{f(x+i)-f(x)}{i^m}$$

est égal à

$$\frac{\frac{f(x+i)-f(x)}{i}}{i^{m-1}}$$

et devient par conséquent nul ou infini, suivant que m est plus petit ou plus grand que 1, quand i s'évanouit, toutes les fois que

$$\frac{f(x+i)-f(x)}{i},$$

reste fini ; ce qui arrive toujours, comme nous allons le démontrer, à l'exception de certaines valeurs particulières et isolées de x.

The beginning of the memoir of André-Marie Ampère [Amp06].

This lasted a certain time, until the 1860's to be exact ; let us quote the french mathematician Jean Gaston Darboux [Dar75]:

"Until the appearance of Riemann's memoir on trigonometric series, no doubt had been raised about the existence of the derivative of continuous functions. Excellent, illustrious geometers, among whom Ampère, had tried to give rigorous proofs of the existence of the derivative. These attempts were, no doubt, far from being satisfactory ; but, I repeat, no doubt had been expressed about the very existence of a derivative for continuous functions."

Gaston Darboux of course refers to the mention, in 1861, by Bernhard Riemann, then Professor at the University of Göttingen, of the existence of a continuous function that would not be nowhere differentiable:

$$x \mapsto \mathcal{R}(x) = \sum_{n=1}^{+\infty} \frac{\sin n^2 x}{n^2}$$

It is not clear wether Riemann gave a proof. If he did so, there is no mention of it in the literature of the time. And no one, at that time too, knew how to obtain it.

About two years later, during the winter 1863-1864, the former high school teacher (1842-1855) Karl Weierstrass, who had been appointed in 1856 Professor at what would then become the Friedrich-Wilhelm University of Berlin (the Königliches Gewerbeinstitut), gave a course on the theory of analytic functions. In this peculiar course took place the first evocation of a new function, continuous everywhere, and nowhere differentiable, which would then be called after him "Weierstrass function". How did this function come to Weierstrass's mind ? Some, like J.-P. Kahane [Kah64], suggest that it could be attributed to the Riemann function, for which he did not know how to prove the non-differentiable feature. Without taking sides, it may simply come from the fact that these questions, that were in the air, aroused interest in the mathematical community of the time. To use terminology currently in vogue, it is what historians today call "circulation of ideas".

It is interesting to note that the appointment of Karl Weierstrass as Professor coincides with the global upgrade, material, moral and conceptual, initiated by Prussia. Prussia wanted the German science to dominate the world. So, when whe Austrian Minister of Education, Leopold Graf von Thun und Hohenstein, proposed to Karl Weierstrass the creation of a chair, in the university of his choice, with an annual salary of 2000 gulden [KH16], Berlin immediately made a counter offer. This is the culmination of the regeneration Prussian process, launched in 1806, after the defeat of Iena against Napoleon.

In 1864, therefore, the Friedrich-Wilhelm University attributed a chair to Karl Weierstrass, at the exact moment when Bismarck began the German unification (War of Duchies). Everything was then connected: science, industry, prosperity, military and political power.

Beyond this configuration, what is of main interest to us is the specific story of the function, and, if one can say, its emergence in the mathematical communauty of the time. This of course leads one to consider the oldest known evidence, which can be found in a fac-similé of manuscript notes taken by Hermann Amandus Schwarz, then 20 years old, who attended the course (ABBAW, Nachlass Schwarz, Nr. 29, Archivs der Berlin-Brandenburgischen Akademie der Wissenschaften, [KH16]:

"It is not proved that such functions have derivatives. Proofs are erroneous if I show that there are such functions which are continuous in the above sense, but do not possess a derivative in any point."

But one had to wait until 1872, July 18^{th}, for the first official (oral) presentation of the aforementioned Weierstrass function, at the Berlin Academy of Sciences, by Karl Weierstrass himself.

As regards the first written reference, it occured in a letter written by Karl Weierstrass to Paul-Gustave du Bois-Reymond, in 1873 [Wei73]:

BRIEFE VON K. WEIERSTRASS AN PAUL DU BOIS-REYMOND.

Berlin, 23. November 1873.
Potsdamer Str. No. 40.

Verehrter Herr Kollege!

— —

In Ihrer neuesten, mir von Borchardt mitgeteilten Abhandlung[1] haben Sie meinen Beweis, daß die Funktion

$$\sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

unter den angegebenen Bedingungen an keiner Stelle einen bestimmten Differential-Koeffizienten besitze, aufgenommen. Damit bin ich völlig einverstanden,

Beginning of the letter written by Karl Weierstrass to P.-G. Du Bois-Reymond [Wei73].

The translation is the following:

"Dear Colleague,

In your last paper, published by Borchardt, you expose my proof showing that the function (...) was everywhere non-differentiable under the conditions I gave. I agree with everything."

One may then wonder what was Weierstrass's point of view, on the Riemann function ? He layed the emphazis upon, of course, the lack of proof, but, also, on the lack of precision: was the $\mathcal{R}$ function non-differentiable everywhere, or at certain points only:

Es wäre zunächst nach meiner Ansicht zweckmäßig ausdrücklich zu erwähnen, daß Riemann bereits im Jahre 1861 einigen seiner Zuhörer die durch die Reihe

$$\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$$

dargestellte Funktion als eine solche, die keine Ableitung besitze, bezeichnet, seinen Beweis dafür aber niemandem mitgeteilt, sondern nur gelegentlich geäußert habe, derselbe sei aus der Theorie der elliptischen Funktionen zu holen. Auch sei nichts darüber bekannt, ob Riemann behauptet habe, seine Funktion besitze an keiner Stelle einen bestimmten Differentialquotienten, — im Kreise von Riemanns Schülern schien man wenigstens von der Existenz solcher Funktionen nichts gewußt zu haben, wie aus einer Äußerung Hankels (Untersuchungen über

Second extract of the letter written by Karl Weierstrass to P.-G. Du Bois-Reymond [Wei73].

"It seems appropriate to recall that Riemann presented this function to his students in 1861. This function is not differentiable, yet, the proof has not been communicated to anyone, it has been said that this could be done with the theory of elliptic functions. It is also not known whether Riemann claimed that his function was non-differentiable everywhere, or at certain points only."

This remark is all the more interesting, since it was not until the 1970's that the differentiable character of the $\mathcal{R}$ function at specific rational multiples of π, of the form:

$$\frac{2\,p+1}{2\,q+1}\,\pi \quad , \quad p,\, q \text{ integers}$$

was proved, by Joseph Gerver [Ger70].

As concerns the first publication, it took place in 1875, in the Crelle Journal, through an article written by P.-G. du Bois-Reymond [BR75]:

Journal

für die

reine und angewandte Mathematik.

In zwanglosen Heften.

Als Fortsetzung des von

A. L. Crelle

gegründeten Journals

herausgegeben

unter Mitwirkung der Herren

Schellbach, Kummer, Kronecker, Weierstrass

von

C. W. Borchardt.

Mit thätiger Beförderung hoher Königlich-Preussischer Behörden.

Neunundsiebzigster Band.

In vier Heften.

Mit einer Figurentafel.

Berlin, 1875.

Druck und Verlag von Georg Reimer.

Ganz etwas Anderes scheinen mir aber die Functionen zu bedeuten, die Herr *Weierstrass* seinen Bekannten mittheilt, die in *keinem* Punkte einen Differentialquotienten besitzen, was noch von keiner der vorher angeführten Functionen nachgewiesen worden ist, und welche bei ihrer grossen Einfachheit und scheinbaren Unverfänglichkeit ahnen lassen, eine wie verbreitete Eigenschaft die Nichtdifferentiirbarkeit der Functionen sein mag. Hier sind nicht besondere Zahlenarten, die doch schliesslich immer isolirt auftreten, mit gewissen Singularitäten behaftet, sondern diese sind durch das ganze Grössengebiet des Arguments gleichförmig und gleichsam stetig vertheilt **).

Um meine Zweifel zu zerstreuen, hatte Herr *Weierstrass* die Güte, mir ein Beispiel einer solchen Function mitzutheilen, und ich glaube mir die Fachgenossen zu Dank zu verpflichten, wenn ich es hier, wo es als Beispiel einer durchweg stetigen Function, die nicht zur folgenden Classe gehört, an seinem Platze ist, wörtlich nach der Aufzeichnung des Verfassers abdrucken lasse:

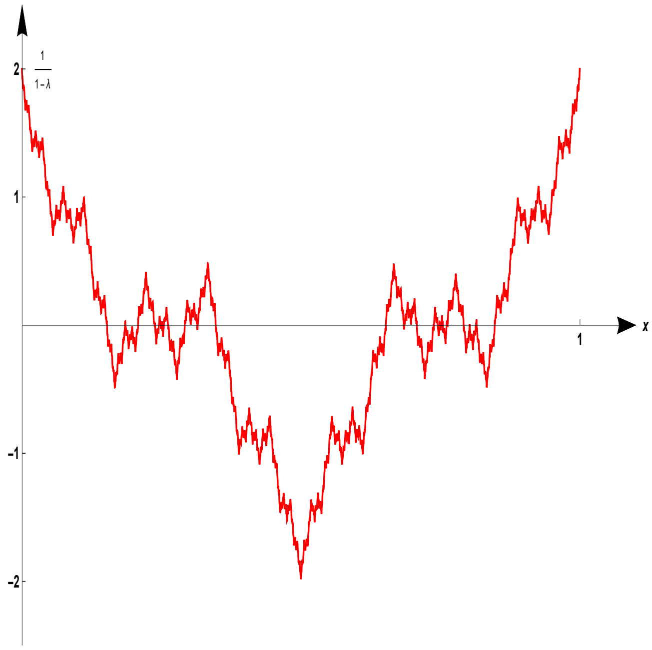
„Es sei x eine reelle Veränderliche, a eine ungerade ganze Zahl, b eine positive Constante, kleiner als Eins, und

$$f(x) = \sum_{0}^{\infty}{}_{n}(b^n \cos(a^n x)\pi);$$

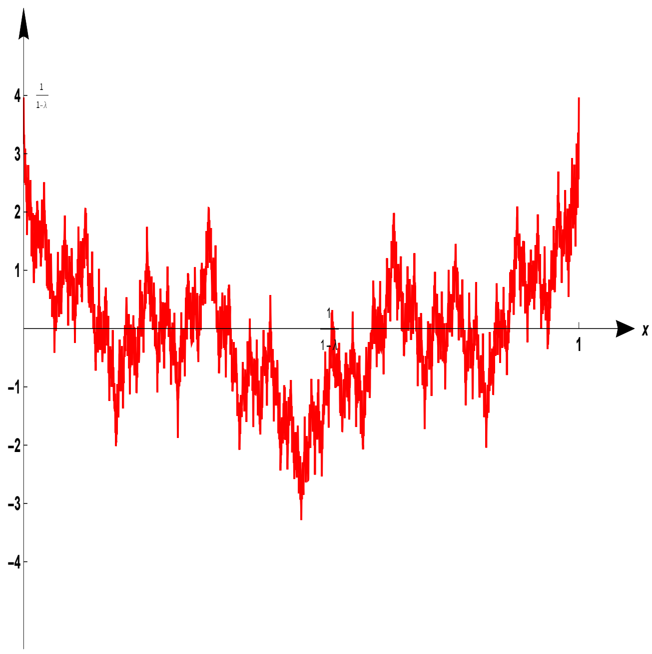
so ist $f(x)$ eine stetige Function, von der sich zeigen lässt, dass sie, sobald der Werth des Products ab eine gewisse Grenze übersteigt, *an keiner Stelle einen bestimmten Differentialquotienten hat.*

Extract of the article of P.-G. du Bois-Reymond in the Crelle Journal [BR75].

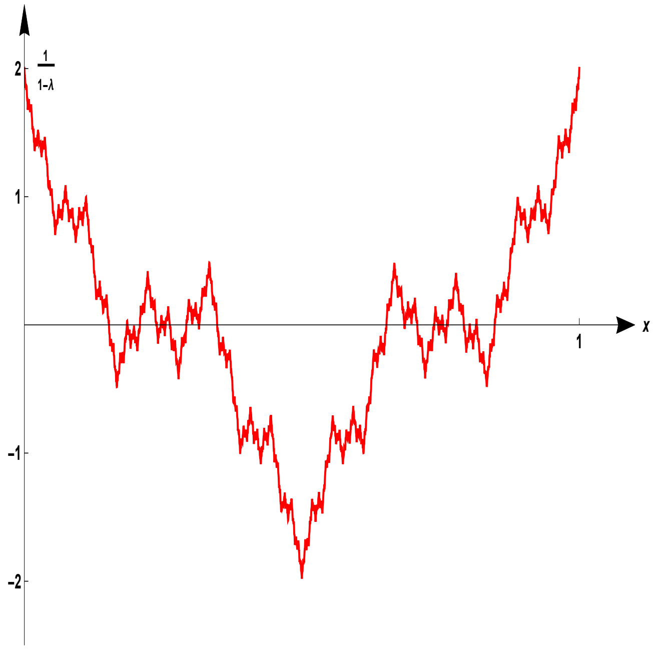
"The functions exposed by Mr. Weierstrass to his usual audience appear to me as being far different, since they possess nowhere a derivative ; this has never before been proved ; and despite an appearance of great simplicity, and as inconceivable as it may seem, they do not possess this expected property of differentiability. This does not concern isolated points, which could present singularities, but intervals evenly distributed throughout the field of study. To dissipate my doubts, Mr. Weierstrass was kind enough to give me an example of such a function, and I am very grateful to him ; it is an example of a function, continuous everywhere, which does not belong to the usual classes of functions. Listen how the author exposes it: "Given a real number x, a an odd integer, and b a positive constant, smaller than one (...) then $f(x)$ is a function continuous everywhere which, as soon as the product $a\,b$ exceeds a known value, is nowhere differentiable."



The Weierstrass Curve, in the case $N_b = 3$, $\lambda = \dfrac{1}{2}$.



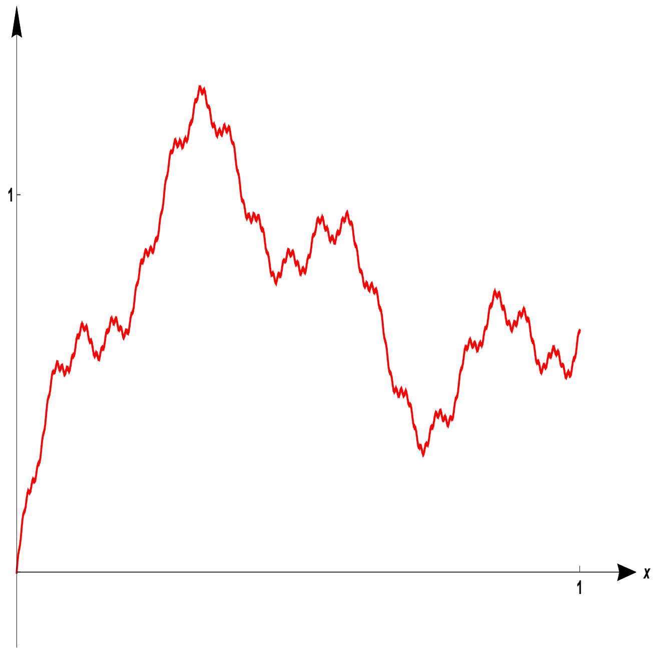
The Weierstrass Curve, in the case $N_b = 3$, $\lambda = \dfrac{3}{4}$.



The Weierstrass Curve, in the case $N_b = 7$, $\lambda = \dfrac{1}{2}$.

The impulse given by Weierstrass has led, from the 1870's, to the emergence of other functions of that type. One may quote, for instance, the one proposed by Jean Gaston Darboux [Dar75], [Dar79] :

$$x \mapsto \mathcal{D}arboux(x) = \sum_{n=1}^{+\infty} \frac{\sin\left((n+1)\,!\,x\right)}{n\,!}\,.$$



The Darboux Curve

Jean Gaston Darboux proves the non-differentiability of his function (see [Dar75], pages 107-108). The $(n+1)\,!$ instead of a $n\,!$ may intrigue. One has to look at the (non completely explicit) proof to understand that if a $n\,!$ had been substituted to the original $(n+1)\,!$, a $n+1$ factor crucial in the non-differentiable feature would have been reported missing.

More precisely: by introducing a strictly positive integer N, Darboux uses a decomposition of his function of the form

$$\mathcal{D}arboux = \phi_N + \psi_N$$

where, for any real number x:

$$\phi_N(x) = \sum_{n=1}^{N-1} \frac{\sin((n+1)!\,x)}{n!} \quad , \quad \psi_N(x) = \sum_{n=N}^{+\infty} \frac{\sin((n+1)!\,x)}{n!} \,.$$

Given two strictly positive numbers h and ε such that:

$$N \times N! \times h = 2\,\varepsilon$$

and due to the second order Taylor expansion that the reader will have of course applied:

$$\phi_N(x+h) - \phi_N(x) \;=\; \sum_{n=1}^{N-1} \left\{ h\,(n+1)!\,\frac{\cos((n+1)!\,x)}{n!} - \frac{h^2}{2}\,((n+1)!)^2\,\frac{\sin((n+1)!\,x)}{n!} \right\} + o\left(h^2\right)$$

one "easily " (to use Darboux's terms) gets:

$$\begin{aligned}
\frac{\phi_N(x+h) - \phi_N(x)}{h} &= \sum_{n=1}^{N-1} \left\{ h\,(n+1)!\,\frac{\cos((n+1)!\,x)}{n!} - \frac{h^2}{2}\,((n+1)!)^2\,\frac{\sin((n+1)!\,x)}{n!} \right\} + o(h) \\
&= \sum_{n=1}^{N-1} \left\{ (n+1)!\,\frac{\cos((n+1)!\,x)}{n!} - \frac{h}{2}\,((n+1)!)^2\,\frac{\sin((n+1)!\,x)}{n!} \right\} + o(h) \\
&= \sum_{n=1}^{N-1} \left\{ (n+1)\,\cos((n+1)!\,x) - \frac{h}{2}\,(n+1)\,(n+1)!\,\sin((n+1)!\,x) \right\} + o(h) \\
&= \sum_{n=2}^{N} \left\{ n\,\cos(n!\,x) - \frac{h}{2}\,n\,n!\,\sin(n!\,x) \right\} + o(h)
\end{aligned}$$

Something is not clear in the original proof, because, instead of our previous expression, Darboux writes:

$$\frac{\phi_N(x+h) - \phi_N(x)}{h} = \sum_{n=???}^{N} n\,\cos(n!\,x) - \varepsilon\,\sin(N!\,x) + \omega(N,\varepsilon)$$

(we have written ??? for the lower bound in the sum, since the original text is not readable, one can hardly see if it is a "1", a "r" , a "x"), and where ω denotes a function such that, for a given ε:

$$\lim_{N \to +\infty} \omega(N,\varepsilon) = 0$$

So, with our current terminology, ω corresponds to a sum of "$o(\cdot)$", and details are reported missing.

The main point of the proof given by Darboux is in fact to point out that, for the values of the real number x such that

$$\lim_{N \to +\infty} \sin(N!\,x) = 0$$

the limit

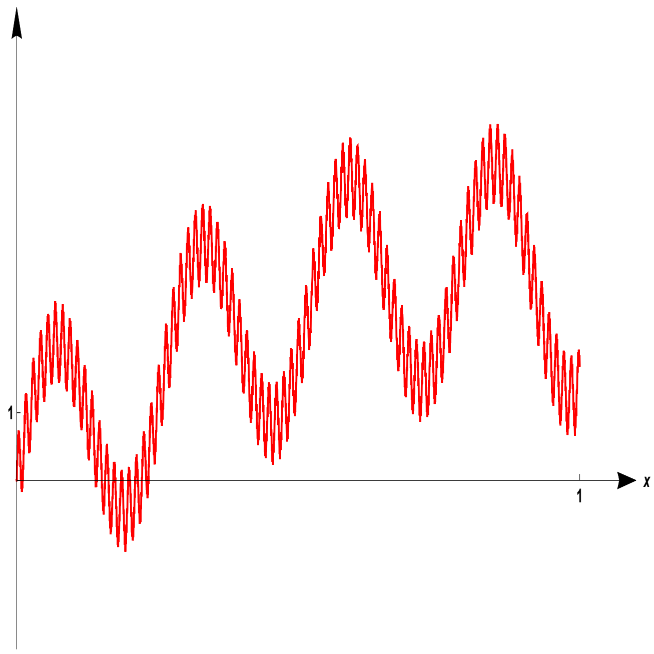
$$\lim_{N\to+\infty} \sum_{n=???}^{N} n\, \cos(n\,!\,x)$$

does not exist.

Very elegantly, Darboux quotes Riemann, Schwarz and some others, but not Weierstrass ...

One finds, after, another example given in 1877 by Ulisse Dini [Din77], [Din78]:

$$x \mapsto \mathcal{D}ini(x) = \sum_{n=1}^{+\infty} \frac{\alpha^n \cos(1\times 3\times 5\times \ldots \times (2\,n-1)\,x)}{1\times 3\times 5\times \ldots \times (2\,n-1)} \quad , \quad \alpha > 1 + \frac{3\,\pi}{2} \cdot$$



The Dini Curve, in the case $\alpha = \dfrac{3}{2} + \dfrac{3\,\pi}{2}$

As a result, the existence of these functions cast a chill on the mathematical community. Let us recall what wrote Charles Hermite, in one of his numerous letters to Thomas Stieltjes, in 1893 ([Cor05], letter 374):

"I turn away with fright and horror from this lamentable plague of continuous functions that have no derivatives."

As for Poincaré [Poi90], he stated that:

"Logic sometimes creates monsters. For half a century, one has seen the birth of strange functions, functions that look as little as possible as the honest ones, the useful ones. No more continuity, or continuity, but no derivatives, etc ... Even more, from the logical point of view, those strange functions appear as the most general ones, while those one may fall on by chance are relegated as special cases. They only have a tiny corner left."

Yet, and it is very important, contrary to the erroneous interpretations found in the literature ([JP15], page 4), Poincaré never described Weierstrass's work as offensive to common sense [Poi98]:

"To begin with, I shall quote a note read at the Berlin Academy on July 18, 1872, and where Weierstrass gave examples of continuous functions of a real argument which, for any value of this argument, do not possess a finite derivative. A hundred years ago, such a function would have been regarded as an outrage to common sense. A continuous function, one would have said, is in essence susceptible of being represented by a curve, and a curve obviously always has a tangent."

What Poincaré says about these functions was, nevertheless, rather hard [Poi99]:

"Formerly, when new functions arose, it was because they were devoted to some practical purpose ; today, they are invented expressly to put in default the reasoning of our fathers, and we will never get out of it."

Since then, the Weierstrass function has kept arousing interest. If this interest was initially due to its nowhere differentiability, its fractal properties, brought to light about ninety years later by B. Mandelbrot [Man82], pages 388-390, made the community consider it from a new angle. Mandelbrot was looking for an approximation of the Brownian motion, which accounts for its interest in the function introduced by Weierstrass.

By moving to a slightly more general frame, Mandelbrot thus chose to consider the related complex function defined, for any real number x, by:

$$\mathcal{W}_c(x) = \frac{1}{\sqrt{1-w^2}} \sum_{n=0}^{+\infty} w^n \, e^{2\,i\,\pi\,b^n\,x}$$

where

$$b > 1 \quad , \quad w = \frac{1}{b^H} = b^{D_{\mathcal{W}}-2} \quad , \quad 1 < 2 - H = D_{\mathcal{W}} < 2$$

After an introductory comparison with the Brownian motion, B. Mandelbrot placed himself on the point of view of physics, and, especially, to study the function's spectra: for each frequency f of the form $f = b^n$, $n \in \mathbb{N}^\star$ the spectral line of energy, i.e. the one that results from emission or absorption of light in a narrow range of frequencies, given by:

$$\frac{1}{1-w^2}\, w^{2\,n}$$

yields a cumulative energy in frequencies $f \geqslant b^n$ of:

$$\sum_{k=n}^{+\infty} \frac{1}{1-w^2}\, w^{2\,k} = \frac{1}{(1-w^2)^2}\, w^{2\,n} = \frac{1}{(1-w^2)^2}\, \frac{1}{b^{2\,n\,H}} = \frac{1}{(1-w^2)^2}\, \frac{1}{f^{2\,H}} \,.$$

B. Mandelbrot recalls then that, since "a function's derivative is obtained by multiplying its k^{th} Fourier coefficient by k", for physicists looking at the formal derivative of the complex Weierstrass function, the ${b^n}^{th}$ Fourier coefficient has an amplitude squared equal to:

$$\frac{1}{1-w^2}\, w^{2\,n}\, b^{2\,n} \,.$$

Thus, the cumulative energies for frequencies greater or equal than b^n are infinite, which enable physicists to obtain the non-differentiability of the $\mathcal{W}$ function as an "intuitively obvious" feature.

B. Mandelbrot then explains that, if "the total high frequency energy is infinite", it is thus "catastrophic for the theory", echoing the 1900's theory of Rayleigh and Jeans of blackbody radiation. By resuming his comparison with Brownian motion, and for the purpose of future applications, B. Mandelbrot thus proposes to take into account a modified version of the function, a one that would soon be called Weierstrass-Mandelbrot one, defined, for any real number x, by:

$$\mathcal{W}_{\mathcal{M}}(x) = \frac{1}{\sqrt{1-w^2}} \sum_{n=-\infty}^{+\infty} w^n \left\{ e^{2\,i\,\pi\,b^n\,x} - 1 \right\}$$

Better than the classical Weierstrass function, the $\mathcal{W}_{\mathcal{M}}$ function, still continuous everywhere, while nowhere differentiable, bears a scaling property and is self-affine:

$\forall m \in \mathbb{Z}, \forall x \in \mathbb{R}$:

$$\begin{aligned} \mathcal{W}_{\mathcal{M}}\left(b^m\,x\right) &= \frac{1}{\sqrt{1-w^2}} \sum_{n=-\infty}^{+\infty} w^n \left\{ e^{2\,i\,\pi\,b^{m+n}\,x} - 1 \right\} \\ &= \frac{1}{w^m}\, \mathcal{W}_{\mathcal{M}}(x) \\ &= b^{m\,H}\, \mathcal{W}_{\mathcal{M}}(x) \end{aligned}$$

To better stick real modelling, B. Mandelbrot then proposes to randomize the function, which enables one to approximate fractional Brown functions.

And as it has often been the case, B. Mandelbrot's intuition proved to be right: the Weierstrass-Mandelbrot function has practical applications. It was for instance shown in the 1990's that the function could be used in the modelling of turbulence [HSR92].

As for the classical Weierstrass function, it still occupies mathematicians. At stake is particularly the determination of the dimension of the Weierstrass Curve, whether one considers the box (or Minkowski-Bouligand) one, or the Hausdorff one. The value of the box-dimension, and how to obtain it, was first found in the works of J.-L. Kaplan et al. [KMPY84], or in the book of K. Falconer [Fal86] (example 11.3). Both box and Hausdorff dimensions are discussed in the paper of F. Przytycki and M. Urbański [PU89]. An intermediate discussion, by means of a new dimension index, is proposed in the one by T-Y. Hu and K-S. Lau [HL93]. As for the Hausdorff dimension, a proof is given by B. Hunt [Hun98] in 1998 in the case where arbitrary phases are included in each cosinusoidal term of the summation. Recently, K. Barańsky, B. Bárány and J. Romanowska [BBR14] proved that, for any value of the real number b, there exists a threshold value λ_b belonging to the interval $\left]\frac{1}{b},1\right[$ such that the aforementioned dimension is equal to $D_{\mathcal{W}}$ for every b in $]\lambda_b, 1[$. In [Kel17], G. Keller proposes what appears as a much simpler and very original proof. Results by W. Shen [She18] go further than the ones of [BBR14].

One may note that Weierstrass's work is not self-evident. It hits hard a whole academic tradition, mindful of order and classicism, resisting the challenge of what was considered obvious and acquired. Nearly a century will be necessary for the mathematical community to take seriously and start exploiting the very rich potential offered by the nowhere differentiability of the Weierstrass function. It is not a coincidence that the discovery of our Berlin professor meets a real and renewed interest when it is associated to the work on Brownian motion, thanks to Mandelbrot. The random, the erratic, the breaking of sense and direction definitely make their entry into the so-called "serious" science. This goes hand in hand with the extension of the notion of dimension. One might go further, and extend this constant to the whole of thought and knowledge, in the twentieth century, all disciplines combined, including arts and letters. Now, this movement of deciphering the irrational goes on. In the same vein, our contribution will now try to put forward the link between the non-differentiability and the value of the box-dimension of the curve.

2 Basic properties of the Weierstrass function - Towards the graph

In the sequel, we aim at describing some geometric properties of the Weierstrass Curve, properties which will be useful especially as regards theorem 3.1.

We place ourselves in the euclidian plane of dimension 2, referred to a direct orthonormal frame. The usual Cartesian coordinates are (x, y).

Notation. In the following, λ and b are two real numbers such that:

$$0 < \lambda < 1 \quad , \quad b = N_b \in \mathbb{N} \quad \text{and} \quad \lambda\, N_b > 1.$$

We will consider the Weierstrass function $\mathcal{W}$, defined, for any real number x, by:

$$\mathcal{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) \cdot$$

Definition 2.1. Weierstrass Curve

We will call *Weierstrass Curve* the restriction to $[0,1[\times\mathbb{R}$, of the graph of the Weierstrass function, and denote it by $\Gamma_{\mathcal{W}}$.

Property 2.1. ***Periodic properties of the Weierstrass function***
For any real number x:

$$\mathcal{W}(x+1) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x + 2\,\pi\, N_b^n\right) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) = \mathcal{W}(x).$$

The study of the Weierstrass function can be restricted to the interval $[0,1[$.

The restriction $\Gamma_{\mathcal{W}}$ to $[0,1[\times\mathbb{R}$, of the Weierstrass Curve, is approximated by prefractals (sequence of graphs, built through an iterative process).

To this purpose, we introduce the iterated function system of the family of C^∞ maps from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$\{T_0, \ldots, T_{N_b-1}\}$$

where, for any integer i belonging to $\{0, \ldots, N_b-1\}$, and any (x, y) of $\mathbb{R}^2$:

$$T_i(x,y) = \left(\frac{x+i}{N_b}, \lambda\, y + \cos\left(2\,\pi\left(\frac{x+i}{N_b}\right)\right)\right).$$

Remark 2.1. For any i of $\{0, \ldots, N_b-1\}$, the map T_i is not a contraction.

The **Gluing Lemma [BD85]** does not apply, but:

Lemma 2.2. *For any integer i belonging to $\{0,\ldots,N_b-1\}$, the map T_i is a bijection of the graph of the Weierstrass function on $\mathbb{R}$.*

Proof. Let us consider $i \in \{0,\ldots,N_b-1\}$.

Consider a point $(y,\mathcal{W}(y))$ of $\Gamma_{\mathcal{W}}$, and let us look for a real number x of $[0,1]$ such that:

$$T_i\left(x,\mathcal{W}(x)\right) = \left(y,\mathcal{W}(y)\right).$$

One has:

$$y = \frac{x+i}{N_b}.$$

Then:

$$x = N_b\, y - i \cdot$$

This enables one to obtain:

$$\mathcal{W}(x) = \mathcal{W}(N_b\, y - i) = \sum_{n=0}^{+\infty} \lambda^n \, \cos\left(2\,\pi\, N_b^{n+1}\, y - 2\,\pi\, N_b^n\, i\right) = \sum_{n=0}^{+\infty} \lambda^n \, \cos\left(2\,\pi\, N_b^{n+1}\, y\right)$$

and:

$$\begin{array}{rcl} T_i\left(x,\mathcal{W}(x)\right) & = & \left(\dfrac{x+i}{N_b}, \lambda\, \mathcal{W}(x) + \cos\left(2\,\pi\,\left(\dfrac{x+i}{N_b}\right)\right)\right) \\ & = & \left(y, \displaystyle\sum_{n=0}^{+\infty} \lambda^{n+1}\, \cos\left(2\,\pi\, N_b^{n+1}\, y\right) + \cos\left(2\,\pi\, y\right)\right) \\ & = & \left(y, \displaystyle\sum_{n=0}^{+\infty} \lambda^{n}\, \cos\left(2\,\pi\, N_b^{n}\, y\right)\right) \\ & = & \left(y,\mathcal{W}(y)\right). \end{array}$$

There exists thus a unique real number x such that:

$$T_i\left(x,\mathcal{W}(x)\right) = \left(y,\mathcal{W}(y)\right).$$

□

Property 2.3.

$$\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}}).$$

Proof. This immediately comes from Lemma 2.2.

□

Definition 2.2. Word, on the graph $\Gamma_{\mathcal{W}}$

Let m be a strictly positive integer. We will call **number-letter** any integer $\mathcal{M}_i$ of $\{0, \ldots, N_b - 1\}$, and **word of length** $|\mathcal{M}| = m$, on the graph $\Gamma_{\mathcal{W}}$, any set of number-letters of the form:

$$\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_m) .$$

We will write:

$$T_{\mathcal{M}} = T_{\mathcal{M}_1} \circ \ldots \circ T_{\mathcal{M}_m} .$$

Definition 2.3. For any integer i belonging to $\{0, ..., N_b - 1\}$, let us denote by:

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1 - \lambda} \cos \left(\frac{2\,\pi\, i}{N_b - 1} \right) \right)$$

the fixed point of the map T_i.

We will denote by V_0 the ordered set (according to increasing abscissa), of the points:

$$\{P_0, ..., P_{N_b - 1}\}$$

since, for any i of $\{0, ..., N_b - 2\}$:

$$x_i \leqslant x_{i+1} .$$

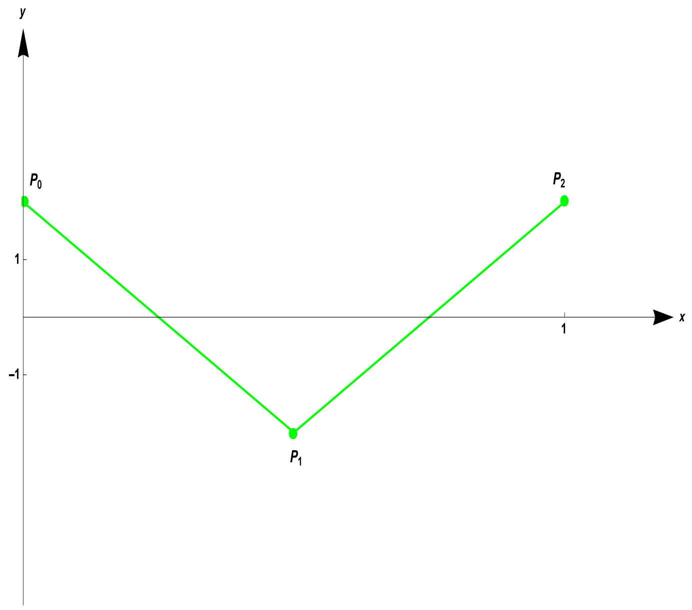
The set of points V_0, where, for any i of $\{0, ..., N_b - 2\}$, the point P_i is linked to the point P_{i+1}, constitutes an oriented graph (according to increasing abscissa), that we will denote by $\Gamma_{\mathcal{W}_0}$. V_0 is called the set of vertices of the graph $\Gamma_{\mathcal{W}_0}$.

For any natural integer m, we set:

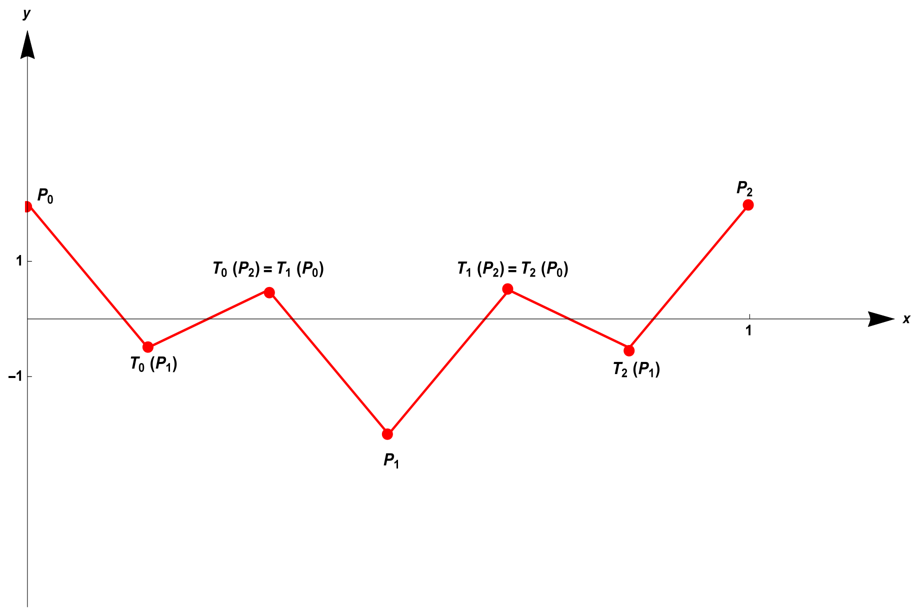
$$V_m = \bigcup_{i=0}^{N_b - 1} T_i\,(V_{m-1}) .$$

The set of points V_m, where two consecutive points are linked, is an oriented graph (according to increasing abscissa), which we will denote by $\Gamma_{\mathcal{W}_m}$. V_m is called the set of vertices of the graph $\Gamma_{\mathcal{W}_m}$. We will denote, in the following, by $\mathcal{N}_m^{\mathcal{S}}$ the number of vertices of the graph $\Gamma_{\mathcal{W}_m}$, and we will write:

$$V_m = \left\{ \mathcal{S}_0^m, \mathcal{S}_1^m, \ldots, \mathcal{S}_{\mathcal{N}_m^{\mathcal{S}} - 1}^m \right\} .$$



The fixed points P_0, P_1, P_2, and the graph $\Gamma_{\mathcal{W}_0}$, in the case where $\lambda = \dfrac{1}{2}$, and $N_b = 3$.



The graph $\Gamma_{\mathcal{W}_1}$, in the case where $\lambda = \dfrac{1}{2}$, and $N_b = 3$. $T_0(P_2) = T_1(P_0)$ et $T_1(P_2) = T_2(P_1)$.

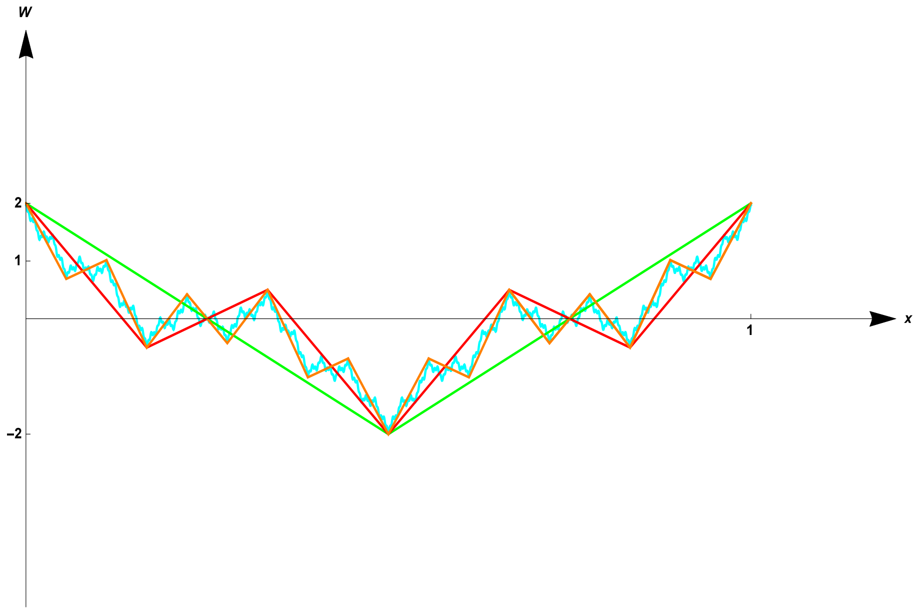
Property 2.4. *For any natural integer m:*

$$V_m \subset V_{m+1}.$$

Property 2.5. *For any integer i belonging to $\{0, ..., N_b - 2\}$:*

$$T_i\left(P_{N_b-1}\right) = T_{i+1}\left(P_0\right).$$

Proof. Since:



The graphs $\Gamma_{\mathcal{W}_0}$ *(in green),* $\Gamma_{\mathcal{W}_1}$ *(in red),* $\Gamma_{\mathcal{W}_2}$ *(in orange),* $\Gamma_{\mathcal{W}}$ *(in cyan), in the case where* $\lambda = \dfrac{1}{2}$, *and* $N_b = 3$.

$$P_0 = \left(0, \frac{1}{1-\lambda}\right) \quad , \quad P_{N_b-1} = \left(\frac{N_b-1}{N_b-1}, \frac{1}{1-\lambda}\cos\left(\frac{2\pi\,(N_b-1)}{N_b-1}\right)\right) = \left(1, \frac{1}{1-\lambda}\right)$$

one has :

$$\begin{cases} T_i\left(P_{N_b-1}\right) & = & \left(\dfrac{1+i}{N_b}, \dfrac{\lambda}{1-\lambda} + \cos\left(2\pi\left(\dfrac{1+i}{N_b}\right)\right)\right) \\ T_{i+1}\left(P_0\right) & = & \left(\dfrac{i+1}{N_b}, \dfrac{\lambda}{1-\lambda} + \cos\left(2\pi\left(\dfrac{i+1}{N_b}\right)\right)\right) \end{cases}$$

□

Property 2.6. *The sequence* $\left(\mathcal{N}_m^{\mathcal{S}}\right)_{m\in\mathbb{N}}$ *is an arithmetico-geometric one, with* $\mathcal{N}_0^{\mathcal{S}} = N_b$ *as first term:*

$$\forall\, m \in \mathbb{N} : \quad \mathcal{N}_{m+1}^{\mathcal{S}} = N_b\,\mathcal{N}_m^{\mathcal{S}} - (N_b - 2)$$

Proof. This results comes from the fact that each graph $\Gamma_{\mathcal{W}_m}$, $m \in \mathbb{N}^\star$, is built from its predecessor $\Gamma_{\mathcal{W}_{m-1}}$ by applying the N_b maps T_i, $0 \leqslant i \leqslant N_b - 1$, to the vertices of $\Gamma_{\mathcal{W}_{m-1}}$. Since, for any i of $\{0, ..., N_b - 2\}$:

$$T_i\left(P_{N_b-1}\right) = T_{i+1}\left(P_0\right)$$

the, $N_b - 2$ points appear twice if one takes into account the images of the $\mathcal{N}_{m-1}$ vertices of $\Gamma_{\mathcal{W}_{m-1}}$ by the whole set of maps T_i, $0 \leqslant i \leqslant N_b - 1$.

□

Definition 2.4. Vertices of the graph $\Gamma_{\mathcal{W}}$

Two points X and Y of $\Gamma_{\mathcal{W}}$ will be called ***vertices*** of the graph $\Gamma_{\mathcal{W}}$ if there exists a natural integer m such that:

$$(X,Y) \in V_m^2$$

Definition 2.5. Consecutive vertices on the graph $\Gamma_{\mathcal{W}}$

Two points X and Y of $\Gamma_{\mathcal{W}}$ will be called ***consecutive vertices*** of the graph $\Gamma_{\mathcal{W}}$ if there exist a natural integer m, and an integer j of $\{0,...,N_b-2\}$, such that:

$$X = (T_{i_1} \circ \ldots \circ T_{i_m})(P_j) \quad \text{and} \quad Y = (T_{i_1} \circ \ldots \circ T_{i_m})(P_{j+1}) \qquad \{i_1,\ldots,i_m\} \in \{0,...,N_b-1\}^m$$

or:

$$X = (T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m})(P_{N_b-1}) \quad \text{and} \quad Y = (T_{i_1+1} \circ T_{i_2} \ldots \circ T_{i_m})(P_0).$$

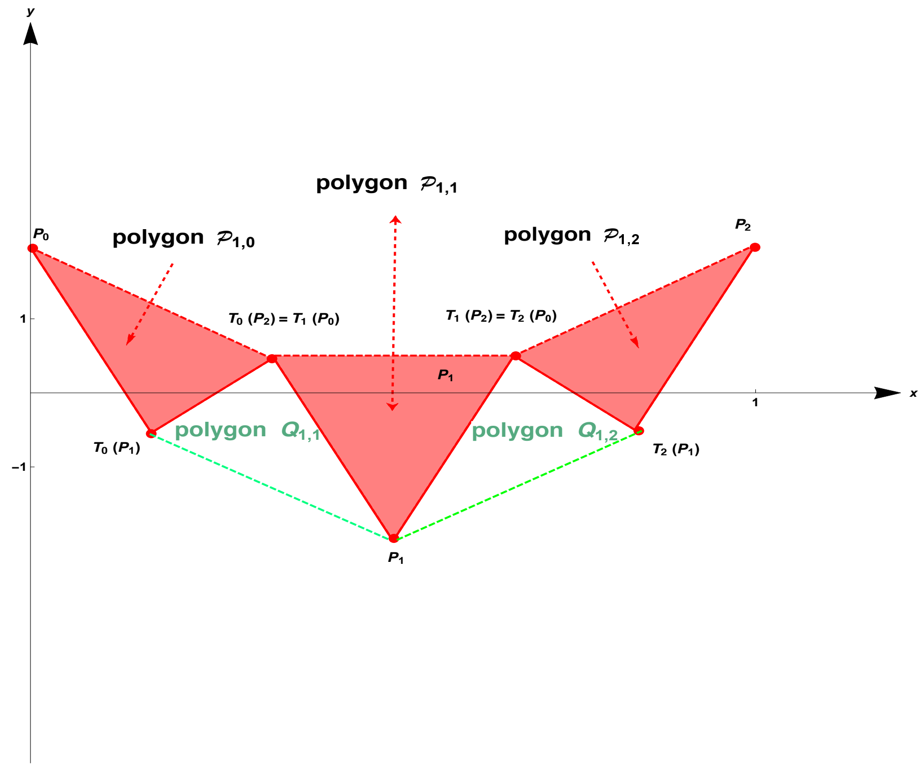
Remark 2.2. It is important to note that X and Y cannot be in the same time the images of P_j and P_{j+1}, $0 \leqslant j \leqslant N_{b-2}$, by $T_{i_1} \circ \ldots \circ T_{i_m}$, $(i_1,\ldots,i_m) \in \{0,...,N_b-2\}$, and of P_k and P_{k+1}, $0 \leqslant k \leqslant N_{b-2}$,
by $T_{p_1} \circ \ldots \circ T_{p_m}$, $(p_1,\ldots,p_m) \in \{0,...,N_b-2\}$. This result can be proved by induction, since, for any pair of integers (j,k) of $\{0,...,N_b-2\}^2$, for any i_m of $\{0,...,N_b-2\}$, and any p_m of $\{0,...,N_b-2\}$:

$$(i_m \neq p_m \quad \text{and} \quad j \neq k) \Longrightarrow (T_{i_m}(P_j) \neq T_{j_m}(P_k) \quad \text{and} \quad T_{i_m}(P_j) \neq T_{j_m}(P_k)).$$

Each map T_i, $0 \leqslant i \leqslant N_b-1$ is indeed injective.
Since the vertices of the initial graph $\Gamma_{\mathcal{W}_0}$ are distinct, one gets the expected result.

Property 2.7. *For any natural integer m, the $\mathcal{N}_m^{\mathcal{S}}$ consecutive vertices of the graph $\Gamma_{\mathcal{W}_m}$ are, also, the vertices of N_b^m simple polygons $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m-1$, with N_b sides (see Figure 3). For any integer j such that $0 \leqslant j \leqslant N_b^m-1$, one obtains each polygon $\mathcal{P}_{m,j}$ by linking the point number j to the point number $j+1$ if $j = i \bmod N_b$, $0 \leqslant i \leqslant N_b-2$, and the point number j to the point number $j-N_b+1$ if $j = -1 \bmod N_b$.*
In the same way, the $\mathcal{N}_m^{\mathcal{S}}-2$ consecutive vertices of the graph $\Gamma_{\mathcal{W}_m}$, distinct of P_0 and P_{N_b-1}, are the vertices of N_b^m-1 simple polygons $\mathcal{Q}_{m,j}$, $1 \leqslant j \leqslant N_b^m-2$, with N_b sides. For any integer j such that $1 \leqslant j \leqslant N_b^m-2$, one obtains each polygon $\mathcal{Q}_{m,j}$ by linking the point number j to the point number $j+1$ if $j = i \bmod N_b$, $1 \leqslant i \leqslant N_b-1$, and the point number j to the point number $j-N_b+1$ if $j = 0 \bmod N_b$.

These polygons generate a Borel set of $\mathbb{R}^2$.



The polygons $\mathcal{P}_{1,0}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$,$\mathcal{Q}_{1,1}$, $\mathcal{Q}_{1,2}$ *in the case where* $\lambda = \frac{1}{2}$, *and* $N_b = 3$.

Property 2.8. *For any natural integer* m, *and any integer* $j \in \{0, \ldots, N_b^m - 1\}$, *there exists a word* $\mathcal{M}_{m,j}^{\mathcal{P}}$ *of length* m *such that the set of consecutive vertices of each* N_b*-gon* $\mathcal{P}_{m,j}$ *is of the form:*

$$\left\{T_{\mathcal{M}_{m,j}^{\mathcal{P}}}\,(P_k)\right\}_{0 \leqslant k \leqslant N_b - 1}$$

In the same way, for any natural integer m, *and any integer* $j \in \{1, \ldots, N_b^m - 2\}$, *there exists a word* $\mathcal{M}_{m,j}^{\mathcal{Q}}$ *of lentgh* m *such that the set of consecutive vertices of each* N_b*-gon* $\mathcal{Q}_{m,j}$ *is of the form:*

$$\left\{T_{\mathcal{M}_{m,j}^{\mathcal{Q}}}\,(P_{k+1})\right\}_{0 \leqslant k \leqslant N_b - 1}$$

Proof. The above result is obtained by induction.
It is obvious that, for $m = 1$, the consecutive vertices of the N_b-gons $\mathcal{P}_{1,0}$, $\mathcal{P}_{1,1}$, ..., $\mathcal{P}_{1,N_b-1}$ are the respective images $T_0\,(P_0)$, $T_0\,(P_1)$, ..., $T_0\,(P_{N_b-1})$, ..., $T_{N_b-1}\,(P_0)$, $T_{N_b-1}\,(P_1)$, ..., $T_{N_b-1}\,(P_{N_b-1})$.

Now, given a natural integer m, let us assume that, for any integer $j \in \{1, \ldots, N_b^m - 2\}$, there exists a word $\mathcal{M}_{m,j}^{\mathcal{Q}}$ of lentgh m such that the set of consecutive vertices of each N_b-gon $\mathcal{P}_{m,j}$ is of the form:

$$\left\{T_{\mathcal{M}_{m,j}^{\mathcal{P}}}\,(P_k)\right\}_{0 \leqslant k \leqslant N_b - 1}.$$

Since:

$$V_{m+1} = \bigcup_{i=0}^{N_b-1} T_i\,(V_m)$$

the set of consecutive vertices of each N_b-gon $\mathcal{P}_{m+1,j}$ is thus of the form:

$$\left\{T_0 \circ T_{\mathcal{M}_{m,j}^{\mathcal{P}}}\,(P_k)\,, \ldots, T_{N_b-1} \circ T_{\mathcal{M}_{m,j}^{\mathcal{P}}}\,(P_k)\right\}_{0 \leqslant k \leqslant N_b - 1}$$

which naturally yields the searched result at the step $m+1$:

$$\begin{array}{ccc} T_{\mathcal{M}^{\mathcal{P}}_{m+1,0}} & = & T_0 \circ T_{\mathcal{M}^{\mathcal{P}}_{m,0}} \\ & \vdots & \\ T_{\mathcal{M}^{\mathcal{P}}_{m+1,N_b-1}} & = & T_{N_b-1} \circ T_{\mathcal{M}^{\mathcal{P}}_{m,0}} \\ & \vdots & \\ T_{\mathcal{M}^{\mathcal{P}}_{m+1,N_b^{m+1}-N_b}} & = & T_0 \circ T_{\mathcal{M}^{\mathcal{P}}_{m,N_b-1}} \\ & \vdots & \\ T_{\mathcal{M}^{\mathcal{P}}_{m+1,N_b^{m+1}-1}} & = & T_{N_b-1} \circ T_{\mathcal{M}^{\mathcal{P}}_{m,N_b-1}}. \end{array}$$

The second part of the property can be proved similarly.

□

Notation. For any natural integer m, we will respectively denote by

$$\left\{\mathcal{M}^{\mathcal{P}}_{m,j}\right\}_{0\leqslant j\leqslant N_b^m-1} \quad , \quad \left\{\mathcal{M}^{\mathcal{Q}}_{m,j}\right\}_{0\leqslant j\leqslant N_b^m-1}$$

the ordered sets of the words of length m related to the sets of N_b-gons $\mathcal{P}_{m,j}$, $0\leqslant j\leqslant N_b^m-1$ and $\mathcal{Q}_{m,j}$, $1\leqslant j\leqslant N_b^m-2$ as given in Property 2.8.

Property 2.9. *The set* $\bigcup_{m\in\mathbb{N}} V_m$ *is dense in* $\Gamma_{\mathcal{W}}$.

Proof. Since the function $\mathcal{W}$ is continuous, it suffices to remark that the set of the abscissae of the vertices is dense in $[0,1]$. Given a natural integer i, let us denote by A_i the set of the abscissae of V_i. The set A_i is transformed into $\dfrac{A_i}{N_b}$ by the map T_0, then, this set is shifted by T_1, …, T_{N_b-1}, and this produces a new set of points, the distance between two consecutive new points having been divided by N_b.
Formally, as exposed in the above, for any natural integer m, and any integer $j\in\{0,\ldots,N_b^m-1\}$, there exists a word $\mathcal{M}^{\mathcal{P}}_{m,j}$ of length m such that the set of consecutive vertices of each N_b-gon $\mathcal{P}_{m,j}$ is of the form:

$$\left\{T_{\mathcal{M}^{\mathcal{P}}_{m,j}}\left(P_k\right)\right\}_{0\leqslant k\leqslant N_b-1}$$

Let us write $T_{\mathcal{M}^{\mathcal{P}}_{m,j}}$ under the form:

$$T_{\mathcal{M}^{\mathcal{P}}_{m,j}} = T_{i_m}\circ T_{i_{m-1}}\circ\ldots\circ T_{i_1}$$

where $(i_1,\ldots,i_m)\in\{0,\ldots,N_b-1\}^m$.

One has then:

$$x\left(T_{\mathcal{M}^{\mathcal{P}}_{m,j}}\left(P_k\right)\right) = \frac{x_k}{N_b^m}+\sum_{p=1}^{m}\frac{i_p}{N_b^p} \quad , \quad x\left(T_{\mathcal{M}^{\mathcal{P}}_{m,j}}\left(P_{k+1}\right)\right) = \frac{x_{k+1}}{N_b^m}+\sum_{p=1}^{m}\frac{i_p}{N_b^p}$$

and thus:

$$x\left(T_{\mathcal{M}^{\mathcal{P}}_{m,j}}\left(P_{k+1}\right)\right)-x\left(T_{\mathcal{M}}\left(P_k\right)\right)=\frac{1}{\left(N_b-1\right)N_b^m}$$

One deduces then:

$$[0,1]=\bigcup_{k=0}^{\left(N_b-1\right)\left(N_b^m-1\right)}\left[\frac{k}{\left(N_b-1\right)N_b^m},\frac{k+1}{\left(N_b-1\right)N_b^m}\right]=\bigcup_{0\leqslant j\leqslant N_b^m-1,\,0\leqslant k\leqslant N_b-1}\left[x\left(T_{\mathcal{M}^{\mathcal{P}}_{m,j}}\left(P_k\right)\right),x\left(T_{\mathcal{M}^{\mathcal{P}}_{m,j}}\left(P_{k+1}\right)\right)\right]$$

Let us now consider a point $X=(x,\mathcal{W}(x))$ of $\Gamma_{\mathcal{W}}$, and a strictly positive number ε. Due to the continuity of the Weierstrass function, there exists a natural integer m_0 such that, for any $m\geqslant m_0$:

$$\forall\, x'\in[0,1]:\quad |x-x'|\leqslant\frac{1}{\left(N_b-1\right)N_b^m}\Longrightarrow\left|\mathcal{W}(x)-\mathcal{W}\left(x'\right)\right|\leqslant\varepsilon$$

By using our preliminary results, one deduces the existence of a natural integer $m_1\geqslant m_0$ such that, for any $m\geqslant m_1$, the real number x belongs to an interval of the form:

$$\left[\frac{k}{\left(N_b-1\right)N_b^m},\frac{k+1}{\left(N_b-1\right)N_b^m}\right]\quad,\quad 0\leqslant k\leqslant\left(N_b-1\right)\left(N_b^m-1\right)$$

or, equivalently:

$$\left[x\left(T_{\mathcal{M}^{\mathcal{P}}_{m,j}}\left(P_k\right)\right),x\left(T_{\mathcal{M}^{\mathcal{P}}_{m,j}}\left(P_{k+1}\right)\right)\right]\quad,\quad 0\leqslant j\leqslant N_b^m-1,\;0\leqslant k\leqslant N_b-1$$

Thus:

$$\left|\mathcal{W}(x)-\mathcal{W}\left(x\left(T_{\mathcal{M}^{\mathcal{P}}_{m,j}}\left(P_k\right)\right)\right)\right|\leqslant\varepsilon$$

which yields the expected density result. □

Definition 2.6. Polygonal domain delimited by the graph $\Gamma_{\mathcal{W}_m}$, $m\in\mathbb{N}$

For any natural integer m, well call **polygonal domain delimited by the graph** $\Gamma_{\mathcal{W}_m}$, and denote by $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$, the reunion of the N_b^m polygons $\mathcal{P}_{m,j}$, $0\leqslant j\leqslant N_b^m-1$ and $\mathcal{Q}_{m,j}$, $1\leqslant j\leqslant N_b^m-2$.

Remark 2.3. The introduction of this polygonal domain arises naturally as one builds the Weierstrass curve. In the literature, one can already find approximating polygons, for instance in the case of the Peano curve, as introduced by W. Wunderlich [Wun73]. Such a notion was then adopted by H. Sagan [Sag86], [Sag94]. As showed by H. Sagan, among other advantages, such polygons enable to obtain the exact coordinates of nodal points, which is of course also the case for the Weierstrass curve. The term "approximating" is justified in so far as the polygons approximate the considered curve uniformly. In our case, we have choosen a slightly different, whatever equivalent, definition of convergence.

Definition 2.7. Convergence of the sequence of polygonal domains $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$

We will say that the sequence of polygonal domains $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ converges towards the graph $\Gamma_{\mathcal{W}}$ if, when the integer m tends towards infinity, the Lebesgue measure of all polygons $\mathcal{P}_{m,j}$, $0\leqslant j\leqslant N_b^m-1$ and $\mathcal{Q}_{m,j}$, $1\leqslant j\leqslant N_b^m-2$, tends towards zero.

Property 2.10. *For any natural integer m, the vertices of the N_b-gons $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$, are not self-intersecting.*

Proof. Let us prove, by induction, that the vertices of the N_b−gons $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$ are not self-intersecting.
For any integer i belonging to $\{0, ..., N_b - 1\}$:

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1-\lambda} \cos\left(\frac{2\pi i}{N_b - 1} \right) \right)$$

Thus, for any integer i belonging to $\{0, ..., N_b - 2\}$:

$$\begin{aligned} y_{i+1} - y_i &= \frac{1}{1-\lambda} \left\{ \cos\left(\frac{2\pi(i+1)}{N_b - 1} \right) - \cos\left(\frac{2\pi i}{N_b - 1} \right) \right\} \\ &= -\frac{2}{1-\lambda} \sin\left(\frac{2\pi(i+1+i)}{2N_b - 1} \right) \sin\left(\frac{2\pi(i+1-i)}{2N_b - 1} \right) \\ &= -\frac{2}{1-\lambda} \sin\left(\frac{\pi(2i+1)}{N_b - 1} \right) \sin\left(\frac{\pi}{N_b - 1} \right) \end{aligned}$$

For the values of the integer i such that:

$$\frac{\pi(2i+1)}{N_b - 1} \leqslant \pi + 2p\pi \qquad p \in \left\{ 0, 1, \ldots, \left\lfloor \frac{N_b}{2(N_b - 1)} \right\rfloor \right\}$$

i.e.:

$$i \leqslant \left\lfloor \frac{N_b - 2}{2} + p(N_b - 1) \right\rfloor \qquad p \in \left\{ 0, 1, \ldots, \left\lfloor \frac{N_b}{2(N_b - 1)} \right\rfloor \right\}$$

one gets:

$$y_{i+1} - y_i \leqslant 0$$

To this point, one may note that the compatibility condition:

$$\frac{N_b - 2}{2} + p(N_b - 1) \leqslant N_b - 1$$

leads to:

$$p \leqslant \frac{N_b}{2(N_b - 1)}$$

The sole entire admissible value for the integer p is thus: 0.

In the same way, one shows that, for the values of the integer i such that:

$$\pi \leqslant \frac{\pi(2i+1)}{N_b - 1} \leqslant 2\pi$$

i.e.:

$$\left\lfloor \frac{N_b - 2}{2} \right\rfloor \leqslant i \leqslant \left\lfloor \frac{2N_b - 3}{2} \right\rfloor$$

one gets:

$$y_{i+1} - y_i \geqslant 0$$

This proves that the set $\{P_0, P_1, \ldots, P_{N_b-1}\}$ belongs to a non-self-intersecting continuous closed loop in the plane.
One may also note that since the sequences $(y_i)_{0\leqslant i\leqslant \left\lfloor \frac{N_b-2}{2}\right\rfloor}$ and $(y_i)_{\left\lfloor \frac{N_b-2}{2}\right\rfloor\leqslant i\leqslant \left\lfloor \frac{2\,N_b-3}{2}\right\rfloor}$ are respectively non-increasing and non-decreasing, the polygon $P_0 P_1 \ldots P_{N_b-1}$ is convex. One has then just to use the self-similarity of the graph, and reason by induction; for any strictly positive integer m:

$$V_m = \bigcup_{0\leqslant i\leqslant N_b-1} T_i\left(V_{m-1}\right)$$

By assuming that the points of V_{m-1} belong to a non-self-intersecting continuous closed loop in the plane, it is also the case of their images $T_i\left(V_{m-1}\right)$ by each map T_i, $0\leqslant i\leqslant N_b-1$. For any integer i belonging to $\{1,\ldots,N_b-2\}$, $T_i\left(V_{m-1}\right)$ and $T_{i+1}\left(V_{m-1}\right)$ have exactly one common vertex, which happens to be the last point of $T_i\left(V_{m-1}\right)$, and the first one of $T_{i+1}\left(V_{m-1}\right)$. Moreover, $T_i\left(V_{m-1}\right)$ and $T_{i+1}\left(V_{m-1}\right)$ are ordered sets, according to increasing abscissae.

The proof is done in a similar way for the vertices of the N_b-gons $\mathcal{Q}_{m,j}$, $1\leqslant j\leqslant N_b^m-2$.

□

Definition 2.8. Edge relation, on the graph $\Gamma_{\mathcal{W}}$

Given a natural integer m, two points X and Y of $\Gamma_{\mathcal{W}_m}$ will be called ***adjacent*** if and only if X and Y are two consecutive vertices of $\Gamma_{\mathcal{W}_m}$. We will write:

$$X \underset{m}{\sim} Y$$

This edge relation ensures the existence of a word $\mathcal{M}=(\mathcal{M}_1,\ldots,\mathcal{M}_m)$ of length m, such that X and Y both belong to the iterate:

$$T_{\mathcal{M}}\,V_0 = \left(T_{\mathcal{M}_1}\circ\ldots\circ T_{\mathcal{M}_m}\right)\,V_0$$

Given two points X and Y of the graph $\Gamma_{\mathcal{W}}$, we will say that X and Y are ***adjacent*** if and only if there exists a natural integer m such that:

$$X \underset{m}{\sim} Y$$

Proposition 2.11. ***Adresses, on the Weierstrass Curve***

Given a strictly positive integer m, and a word $\mathcal{M}=(\mathcal{M}_1,\ldots,\mathcal{M}_m)$ of length $m\in\mathbb{N}^\star$, on the graph $\Gamma_{\mathcal{W}_m}$, for any integer j of $\{1,\ldots,N_b-2\}$, any $X=T_{\mathcal{M}}(P_j)$ of $V_m\setminus V_0$, i.e. distinct from one of the N_b fixed point P_i, $0\leqslant i\leqslant N_b-1$, has exactly two adjacent vertices, given by:

$$T_{\mathcal{M}}(P_{j+1}) \quad \textit{and} \quad T_{\mathcal{M}}(P_{j-1})$$

where:

$$T_{\mathcal{M}} = T_{\mathcal{M}_1}\circ\ldots\circ T_{\mathcal{M}_m}$$

By convention, the adjacent vertices of $T_{\mathcal{M}}(P_0)$ are $T_{\mathcal{M}}(P_1)$ and $T_{\mathcal{M}}(P_{N_b-1})$, those of $T_{\mathcal{M}}(P_{N_b-1})$, $T_{\mathcal{M}}(P_{N_b-2})$ and $T_{\mathcal{M}}(P_0)$.

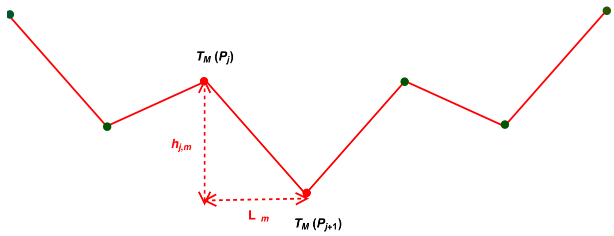
3 From the box-counting dimension to the non-differentiability

Notation.
For any integer j belonging to $\{0, \ldots, N_b - 1\}$, any natural integer m, and any word $\mathcal{M}$ of length m, we set:

$$T_{\mathcal{M}}(P_j) = (x(T_{\mathcal{M}}(P_j)), y(T_{\mathcal{M}}(P_j)))$$

$$L_m = x(T_{\mathcal{M}}(P_{j+1})) - x(T_{\mathcal{M}}(P_j)) = \frac{1}{(N_b - 1)\, N_b^m}$$

$$h_{j,m} = y(T_{\mathcal{M}}(P_{j+1})) - y(T_{\mathcal{M}}(P_j)) .$$



3.1 Box-counting dimension

Notation. We will denote by:

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b}$$

the Hausdorff dimension of $\Gamma_{\mathcal{W}}$ (see [BBR14], [Kel17]).

Definition 3.1 (Box-counting dimension). By definition of the box-counting dimension $D_{\mathcal{W}}$ (we refer, for instance, to [Fal86]), one has:

$$D_{\mathcal{W}} = - \lim_{\delta \to 0^+} \frac{\ln N_\delta(\Gamma_{\mathcal{W}})}{\ln \delta}$$

where $N_\delta(\Gamma_{\mathcal{W}})$ is any of the following:

i. the smallest number of sets of diameter at most δ that cover $\Gamma_{\mathcal{W}}$ on $[0, 1[$;

ii. the smallest number of closed balls of radius δ that cover $\Gamma_{\mathcal{W}}$ on $[0, 1[$;

iii. the smallest number of cubes of side δ that cover $\Gamma_{\mathcal{W}}$ on $[0, 1[$;

iv. the number of δ-mesh cubes that intersect $\Gamma_{\mathcal{W}}$ on $[0, 1[$;

v. the largest number of disjoint balls of radius δ with centers in $\Gamma_{\mathcal{W}}$ on $[0, 1[$.

Theorem 3.1. ***An upper bound and a lower bound, for the box-dimension of the Weierstrass Curve*** *[Dav18]*

For any integer j belonging to $\{0,1,\ldots,N_b-2\}$, each natural integer m, and each word $\mathcal{M}$ of length m, let us consider the rectangle, whose sides are parallel to the horizontal and vertical axes, of width:

$$L_m = x\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) - x\left(T_{\mathcal{M}}\left(P_j\right)\right) = \frac{1}{\left(N_b-1\right)N_b^m}$$

and height $|h_{j,m}|$, such that the points $T_{\mathcal{M}}\left(P_j\right)$ and $T_{\mathcal{M}}\left(P_{j+1}\right)$ are two vertices of this rectangle.

We set:

$$\eta_{\mathcal{W}} = 2\,\pi^2\,\left\{\frac{(2\,N_b-1)\,\lambda\,(N_b^2-1)}{(N_b-1)^2\,(1-\lambda)\,(\lambda\,N_b^2-1)} + \frac{2\,N_b}{(\lambda\,N_b^2-1)\,(\lambda\,N_b^3-1)}\right\}.$$

$$C_1(N_b) = \left\{\begin{array}{ll} (N_b-1)^{2-D_{\mathcal{W}}}\left\{\frac{2}{1-\lambda}\,\sin\left(\frac{\pi}{N_b-1}\right)\min\limits_{0\leqslant j\leqslant N_b-1}\left|\sin\left(\frac{\pi\,(2\,j+1)}{N_b-1}\right)\right| - \frac{2\,\pi}{N_b\,(N_b-1)}\,\frac{1}{\lambda\,N_b-1}\right\} & \textit{if} \quad N_b \textit{ is odd} \\ (N_b-1)^{2-D_{\mathcal{W}}}\max\left\{\frac{2}{1-\lambda}\,\sin\left(\frac{\pi}{N_b-1}\right)\min\limits_{0\leqslant j\leqslant N_b-1}\left|\sin\left(\frac{\pi\,(2\,j+1)}{N_b-1}\right)\right| - \frac{2\,\pi}{N_b\,(N_b-1)}\,\frac{1}{\lambda\,N_b-1}, \frac{4}{N_b^2}\,\frac{1-N_b^{-2}}{N_b^2-1}\right\} & \textit{if} \quad N_b \textit{ is even}\end{array}\right.$$

and:

$$C_2(N_b) = \eta_{\mathcal{W}}(N_b-1)^{2-D_{\mathcal{W}}}\,.$$

Then:

$$C_1(N_b)\,L_m^{2-D_{\mathcal{W}}} \leqslant |h_{j,m}| \leqslant C_2(N_b)\,L_m^{2-D_{\mathcal{W}}}.$$

Proof. **Sketch of proof** (for the detailed proof, we refer to [Dav18]

The proof is based on the fact that, given a strictly positive integer m, and two points X and Y of V_m such that:

$$X \underset{m}{\sim} Y$$

there exists a word $\mathcal{M}$ of length $|\mathcal{M}| = m$, on the graph $\Gamma_{\mathcal{W}}$, and an integer j of $\{0,\ldots,N_b-2\}^2$, such that:

$$X = T_{\mathcal{M}}\left(P_j\right) \quad , \quad Y = T_{\mathcal{M}}\left(P_{j+1}\right).$$

By writing $T_{\mathcal{M}}$ under the form:

$$T_{\mathcal{M}} = T_{i_m} \circ T_{i_{m-1}} \circ \ldots \circ T_{i_1}$$

where $(i_1,\ldots,i_m) \in \{0,\ldots,N_b-1\}^m$, one gets:

$$x\left(T_{\mathcal{M}}\left(P_j\right)\right) = \frac{x_j}{N_b^m} + \sum_{k=1}^{m}\frac{i_k}{N_b^k} \quad , \quad x\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) = \frac{x_{j+1}}{N_b^m} + \sum_{k=1}^{m}\frac{i_k}{N_b^k}$$

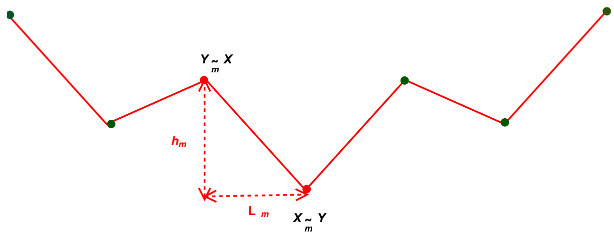
and:

$$\left\{\begin{array}{lcl} y\left(T_{\mathcal{M}}\left(P_j\right)\right) & = & \lambda^m\,y_j + \sum\limits_{k=1}^{m}\lambda^{m-k}\,\cos\left(2\,\pi\,\left(\frac{x_j}{N_b^k} + \sum\limits_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right) \\ y\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) & = & \lambda^m\,y_{j+1} + \sum\limits_{k=1}^{m}\lambda^{m-k}\,\cos\left(2\,\pi\,\left(\frac{x_{j+1}}{N_b^k} + \sum\limits_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right)\end{array}\right..$$

□

Notation. Given a natural integer m, we set:

$$h_m = L_m^{2-D_{\mathcal{W}}} = \frac{N_b^{(D_{\mathcal{W}}-2)\,m}}{(N_b-1)^{2-D_{\mathcal{W}}}}$$



$$|h_{jm}| \lesssim h_m$$

Remark 3.1. Comparison with previous results: explicit lower and upper bounds

It is worth noting that our result gives explicit lower and upper bounds for the quantity $|h_{j,m}|$, which enables one to obtain then the value of the box-counting dimension of the graph $\Gamma_{\mathcal{W}}$. Especially concerning the lower bound, such a result does not appear in the existing literature on the subject. Even if the result of G. Hardy [Har11], [Har16], at first destined to show the non-differentiability of the Weierstrass function, is not referenced among the ones related to the calculation of the box-dimension, one may note that he is the first to give a (non-explicit) upper bound and prove that, for any value of the real number x, and $\eta \to 0^+$:

$$\mathcal{W}(x+\eta) - \mathcal{W}(x) = \mathcal{O}\left(|\eta|^{2-D_{\mathcal{W}}}\right)$$

(see cite [Har11], Theorem 1.3.2 page 303).

Later, in [KMPY84], the authors rely on non-explicit lower-bound estimates. In [Hun98], as concerns the lower bound, the author calls for strictly positive constants K and K' which, again, are not given explicitely (see section 3., page 798). In [She18], also on the Hausdorff dimension of the graph, the estimates are so scattered that it is extremely difficult to reconstruct explicit ones. In [Kel17], again, there isn't any explicit lower bound, but general constants K_1 and $K(K_1)$.

Corollary 3.2. *The box-counting dimension of the graph $\Gamma_{\mathcal{W}}$ is exactly $D_{\mathcal{W}}$.*

Proof. i. Given a strictly positive integer m, let us first consider the subdivision of the interval $[0,1[$ into:

$$N_m = \frac{1}{L_m} = (N_b - 1)\, N_b^m$$

sub-intervals of length L_m. One has to determine a natural integer $\tilde{N}_m$ such that the graph of $\Gamma_{\mathcal{W}}$ on $[0,1[$ can be covered by $N_m \times \tilde{N}_m$ squares of side length L_m.

The difficulty is indeed to cover not only the approached m^{th}-order graph $\Gamma_{\mathcal{W}m}$, but any $(m+p)^{th}$-order graph $\Gamma_{\mathcal{W}m+p}$, $p \in \mathbb{N}$, and, thus, $\Gamma_{\mathcal{W}}$.

This is achieved thanks to the Hölder condition satisfied by the Weierstrass function [Zyg02]:

$$\forall\,(x,y) \in [0,1]^2 : \quad |\mathcal{W}(x) - \mathcal{W}(y)| \lesssim |x-y|^{2-D_{\mathcal{W}}}$$

Thus, given two adjacent vertices X and Y of the m^{th}-order graph $\Gamma_{\mathcal{W}m}$, all the points of the Weierstrass Curve that are between X and Y belong to a rectangle of height equal to $h_m = L_m^{2-D_{\mathcal{W}}}$, and of width L_m. A convenient cover of the Weierstrass Curve between X and Y requires at most:

$$\frac{h_m}{L_m} \quad \text{squares of side length } L_m$$

To cover the Weierstrass Curve on the semi-opened interval $[0,1[$ thus requires at most:

$$N_m \frac{h_m}{L_m} = \frac{1}{L_m}\frac{h_m}{L_m} = \frac{h_m}{L_m^2} \leqslant \frac{C_2\,L_m^{2-D_{\mathcal{W}}}}{L_m^2} = C_2\,L_m^{-D_{\mathcal{W}}}\,.$$

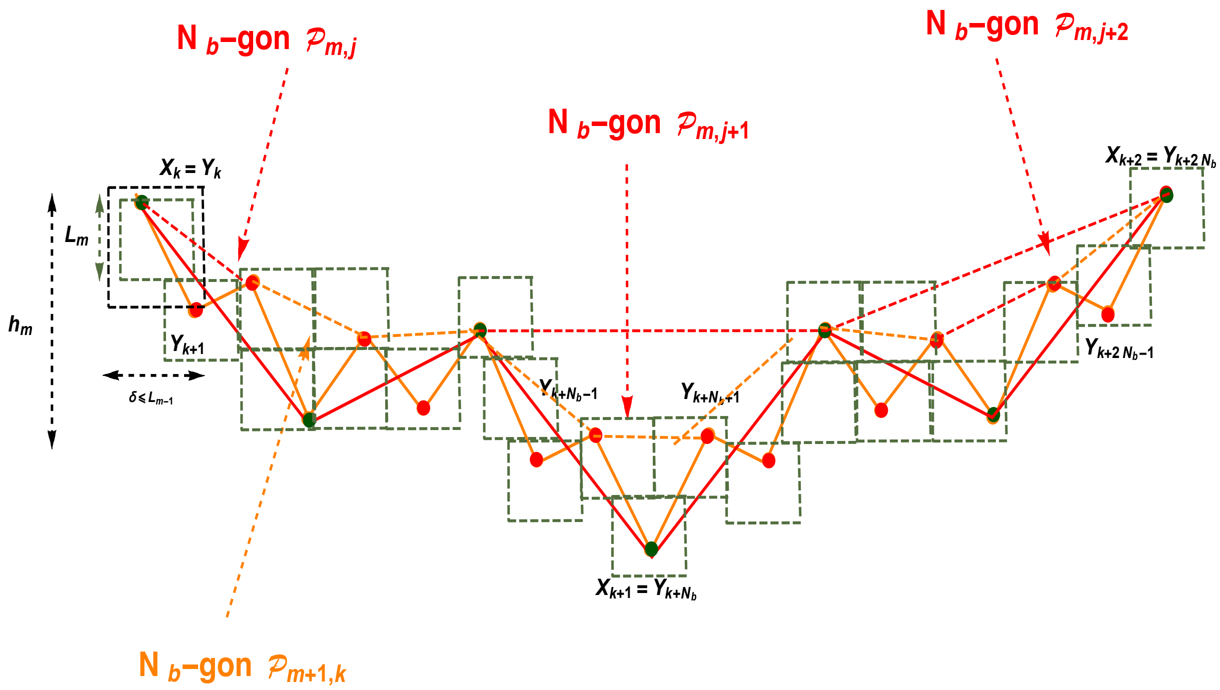
Let us now consider a strictly positive real number δ such that:

$$L_m < \delta \leqslant L_{m-1}$$

then, a δ-cover of $\Gamma_{\mathcal{W}}$ on $[0,1[$ is at most constituted of:

$$C_2\,L_m^{-D_{\mathcal{W}}}$$

squares of side L_m.



Given three consecutive vertices of $\Gamma_{\mathcal{W}_m}$, X_k, X_{k+1}, X_{k+2}, where k denotes a generic natural integer, Y_{k+1}, …, Y_{k+N_b-1} are the points of $V_{m+1}\setminus V_m$ such that: Y_{k+1}, …, Y_{k+N_b-1} are between X_k and X_{k+1}, and by Y_{k+N_b+1}, …, $Y_{k+2\,N_b-1}$, the points of $V_{m+1}\setminus V_m$ such that: Y_{k+N_b+1}, …, $Y_{k+2\,N_b-1}$ are between X_{k+1} and X_{k+2}. In magenta, one can see the δ-cover of squares of side L_m, in green, the δ-cover of squares of side δ.

Hence:

$$N_\delta\left(\Gamma_{\mathcal{W}}\right) \leqslant \frac{C_2(N_b)\, L_m^{-D_{\mathcal{W}}}}{N_b^{1-D_{\mathcal{W}}}}$$

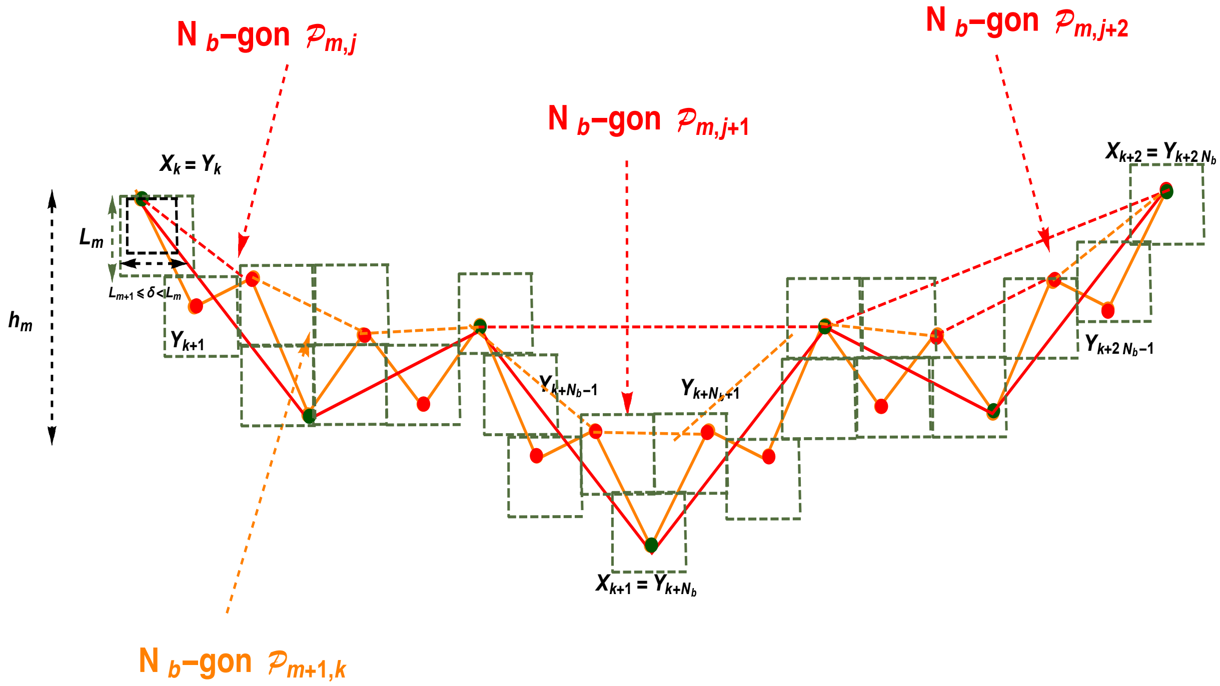
which yields:

$$-\limsup_{\delta\to 0^+} \frac{\ln N_\delta\left(\Gamma_{\mathcal{W}}\right)}{\ln \delta} \leqslant -\limsup_{m\to+\infty} \frac{\ln \dfrac{C_2(N_b)\, L_m^{-D_{\mathcal{W}}}}{N_b^{1-D_{\mathcal{W}}}}}{\ln L_m} = -\limsup_{m\to+\infty} \frac{\ln L_m^{-D_{\mathcal{W}}}}{\ln L_m} = D_{\mathcal{W}}$$

ii. Conversely, given a strictly positive real number δ such that:

$$L_{m+1} \leqslant \delta < L_m \quad , \quad m \in \mathbb{N}^\star$$

any square of side δ intersects at most N_b polygons $\mathcal{P}_{m+1,j}$, $0 \leqslant j \leqslant N_b^{m+1}-1$ that occur at step $m+1$ in the construction of $\Gamma_{\mathcal{W}}$ on $[0,1[$. Due, again, to the Hölder condition satisfied by the Weierstrass function [Zyg02], a N_b-gon $\mathcal{P}_{m+1,j}$, $0 \leqslant j \leqslant N_b^{m+1}-1$, can be inscribed in a rectangle of heigth at most equal to h_{m+1}, and of width L_{m+1}, which contains all the points of the curve that are between the extreme vertices of $\mathcal{P}_{m+1,j}$.



In green, a square of side $\delta \in [L_{m+1}, L_m[$ ***intersecting polygons*** $\mathcal{P}_{m+1,j}$***,*** $0 \leqslant j \leqslant N_b^{m+1}-1$***.***

There are N_b^{m+1} such polygons. One has to consider the vertical amplitude, taking account that the N_b^{m+1} polygons $\mathcal{P}_{m+1,j}$, $0 \leqslant j \leqslant N_b^{m+1}-1$, with N_b sides, are related to the elementary height h_{m+1}. This brings in a required number (related to the vertical amplitude) at least of:

$$N_b\,\frac{h_{m+1}}{\delta}$$

Thus:

$$N_\delta\left(\Gamma_{\mathcal{W}}\right) \geqslant \frac{1}{\delta} \times N_b\,\frac{h_{m+1}}{\delta} \geqslant \frac{N_b\, h_{m+1}}{L_m^2}$$

i.e.:

$$N_\delta\left(\Gamma_{\mathcal{W}}\right) \geqslant \frac{N_b\, h_{m+1}}{L_m^2} \geqslant \frac{N_b\, C_1(N_b)\, L_{m+1}^{2-D_{\mathcal{W}}}}{L_m^2}$$

i.e.:

$$N_\delta\left(\Gamma_{\mathcal{W}}\right) \geqslant \frac{N_b\, C_1(N_b)\, L_m^{2-D_{\mathcal{W}}}}{L_m^2\, N_b^{2-D_{\mathcal{W}}}}$$

i.e.:

$$N_\delta\left(\Gamma_{\mathcal{W}}\right) \geqslant \frac{C_1(N_b)\, L_m^{-D_{\mathcal{W}}}}{N_b^{1-D_{\mathcal{W}}}}$$

So, at least

$$\frac{C_1(N_b)\, L_m^{-D_{\mathcal{W}}}}{N_b^{1-D_{\mathcal{W}}}}$$

squares of side δ are required to cover $\Gamma_{\mathcal{W}}$ on $[0,1[$.
Hence:

$$N_\delta\left(\Gamma_{\mathcal{W}}\right) \geqslant \frac{C_1(N_b)\, L_m^{-D_{\mathcal{W}}}}{N_b^{1-D_{\mathcal{W}}}}$$

which yields:

$$-\liminf_{\delta\to 0^+} \frac{\ln N_\delta\left(\Gamma_{\mathcal{W}}\right)}{\ln \delta} \geqslant -\liminf_{m\to+\infty} \frac{\ln \dfrac{C_1(N_b)\, L_m^{-D_{\mathcal{W}}}}{N_b^{1-D_{\mathcal{W}}}}}{\ln L_m} = -\liminf_{m\to+\infty} \frac{\ln L_m^{-D_{\mathcal{W}}}}{\ln L_m} = D_{\mathcal{W}}$$

□

Corollary 3.3. *The sequence of polygonal domains* $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ *converges towards* $\Gamma_{\mathcal{W}}$*.*

Proof. For any natural integer m,, the aforementioned squares, the side length of which is at most equal to L_m, that can cover the graph $\Gamma_{\mathcal{W}_m}$ on $[0,1[$, also cover the polygonal domain $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$. Since

$$\lim_{m\to+\infty} L_m = 0$$

the convergence is obvious.

□

3.2 Non-differentiability of the Weierstrass function

The original proof of the non-differentiability of the Weierstrass function was given by K. Weierstrass in the case where $\lambda\, N_b > 1 + \frac{3\,\pi}{2}$, N_b being an odd positive integer (see [Tit39], pages 351-354). It is rather technical (two pages), and consists in proving that the $\mathcal{W}$ function has no finite derivative for any value of $x \in \mathbb{R}$, since the quantity:

$$\left| \frac{\mathcal{W}(x+h) - \mathcal{W}(x)}{h} \right|$$

takes arbitrary large values when $h \to 0^+$.

A slight improvement was given by T. J. Bromwich [Bro08], in the case where

$$\lambda\, N_b > 1 + \frac{3\,\pi}{2}\,(1-\lambda)$$

T. J. Bromwich seemed very proud of his result, and did not hesitate to qualify the seminal condition of Weierstrass of “unnecessarily narrow” ...

In [Har16], G. H. Hardy showed that all those conditions were artificial ones, which “arised in consequence of the methods employed”, and, as it could have been expected, did not correspond to “any essential feature of the function”. G. H. Hardy proved that in the general case, i.e. not depending on the fact that b was or was not an integer, and under the condition:

$$\lambda\, b > 1$$

the $\mathcal{W}$ function is not differentiable. Again, it is very technical, the aim being to obtain estimates that enable to get the expected result. Following the above remark of Hardy himself, one may say that, at the times, one did not have enough appropriate tools.

To get the expected limit, one simply requires a lower bound for the absolute value of the average rate of change

$$\left| \frac{\mathcal{W}(x+h) - \mathcal{W}(x)}{h} \right|$$

where h denotes a positive real number that tends to 0, which happens to be given by Theorem 3.1.

Corollary 3.4. *(of Theorem 3.1)*

In the case where

$$0 < \lambda < 1 \quad , \quad b = N_b \in \mathbb{N} \quad \text{and} \quad \lambda\, N_b > 1$$

the $\mathcal{W}$ function is non-differentiable.

Proof. Given a point $X = (x, \mathcal{W}(x))$ of $\Gamma_{\mathcal{W}}$, and a natural integer m, one may note that, for:

$$k_0 = \sup \left\{ k \in \{0, \ldots, \leqslant N_b - 1\}\,,\, x\left(T_{\mathcal{M}_{m,j}}\left(P_k\right)\right) \leqslant x \right\}$$

where $\mathcal{M}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$ denotes a word of length m, one has:

$$x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0}\right)\right) \leqslant x \leqslant x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0+1}\right)\right)$$

In the same time:

$$\left|x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0}\right)\right)-x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0+1}\right)\right)\right|=\frac{1}{\left(N_b-1\right)N_b^m}=L_m\underset{m\to+\infty}{\longrightarrow}0$$

Thus:

$$\begin{array}{rcl}\left|\mathcal{W}\left(x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0}\right)\right)\right)-\mathcal{W}\left(x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0+1}\right)\right)\right)\right| & \geqslant & C_1(N_b)\,L_m^{2-D_{\mathcal{W}}} \\ & = & C_1(N_b)\left|x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0}\right)\right)-x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0+1}\right)\right)\right|^{2-D_{\mathcal{W}}}\end{array}$$

which leads to:

$$\begin{array}{rcl}\left|\frac{\mathcal{W}\left(x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0}\right)\right)\right)-\mathcal{W}\left(x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0+1}\right)\right)\right)}{\underbrace{x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0}\right)\right)-x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0+1}\right)\right)}_{L_m}}\right| & \geqslant & C_1(N_b)\left|x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0}\right)\right)-x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0+1}\right)\right)\right|^{1-D_{\mathcal{W}}} \\ & = & C_1(N_b)\,L_m^{1-D_{\mathcal{W}}}\end{array}$$

where

$$1-D_{\mathcal{W}}=-1-\frac{\ln\lambda}{\ln N_b}=-\frac{\ln\left(\lambda\,N_b\right)}{\ln N_b}<0$$

By passing to the limit when the integer m tends towards infinity, one gets the non-differentiability expected result:

$$\lim_{m\to+\infty}\left|\frac{\mathcal{W}\left(x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0}\right)\right)\right)-\mathcal{W}\left(x\left(T_{\mathcal{M}_{m,j}}\left(P_{k_0+1}\right)\right)\right)}{L_m}\right|=+\infty\cdot$$

The key point of this proof is that the points of the m^{th} order prefractal graph approximation, in particular, $T_{\mathcal{M}_{m,j}}\left(P_{k_0}\right)$ and $T_{\mathcal{M}_{m,j}}\left(P_{k_0+1}\right)$, are **also** on the Weierstrass Curve. One thus naturally falls on the limit position of the secant.

□

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