COMPARISON OF TWO EQUIVARIANT η-FORMS

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ABSTRACT. In this paper, we first define the equivariant infinitesimal η-form, then we compare it with the equivariant η-form, modulo exact forms, by a locally computable form. As a consequence, we obtain the singular behavior of the equivariant η-form, modulo exact forms, as a function on the acting Lie group. This result extends a result of Goette and it plays an important role in our recent work on the localization of η-invariants and on the differential K-theory.

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Date: June 16, 2022.

0. Introduction

In order to find a well-defined index for a first order elliptic differential operator over an even-dimensional compact manifold with nonempty boundary, Atiyah-Patodi-Singer [1] introduced a global boundary condition which is particularly significant for applications. In this final index formula, the contribution from the boundary is given by the Atiyah-Patodi-Singer (APS) η-invariant associated with the restriction of the operator on the boundary. Formally, the η-invariant is equal to the number of positive eigenvalues of the self-adjoint operator minus the number of its negative eigenvalues. If the manifold admits a compact Lie group action, in [31], extending the APS index theorem [1], Donnelly proved a Lefschetz type formula for manifolds with boundary. The contribution of the boundary is expressed as the equivariant η-invariant η_g.

Note that the η-invariant and the equivariant η-invariant are well-defined for any compact manifold. In [36, Theorem 0.5], Goette studied the singularity of η_g at $g = e$ the identity element, when the group action is locally free. He defined the equivariant infinitesimal η-invariant as a formal power series and express the singularity of η_g at $g = e$ as a locally computable term through the comparison of the equivariant infinitesimal η-invariant and the equivariant η-invariant.

In [19, 20], Bismut and Goette established the general comparison formulas for holomorphic analytic torsions and de Rham torsions. They used the analytic localization techniques developed by Bismut and Lebeau in [21] and developed new techniques to overcome the difficulty that the operators do not have lower bounds. In the holomorphic case [19, Theorem 0.1], besides the predictable Bott-Chern current, in the final formula, there is an exotic additive characteristic class of the normal bundle, which is closely related to the Gillet-Soulé R-genus [35] and Bismut's equivariant extension [10]. In the real case [20, Theorem 0.1], in the final formula, besides the predictable Chern-Simons current, they discovered an exotic locally computable diffeomorphism invariant of the fixed point set, the so-called V-invariant. The mysterious V-invariant should be understood as a finite dimensional analogue of the real analytic (de Rham) torsion.

On the other hand, extending the works of Bismut-Freed [17] and Cheeger [27] on the Witten's holonomy conjecture, Bismut and Cheeger [13] studied the adiabatic limit for a fibration of compact spin manifolds and found that under the invertible assumption of the fiberwise Dirac operator, the adiabatic limit of the η-invariant of the associated Dirac operators on the total space is expressible in terms of a canonically constructed differential form, $\tilde{\eta}$, so-called Bismut-Cheeger η-form, on the base space. Later, Dai [28] extended this result to the case when the kernels of the fiberwise Dirac operators form a vector bundle over the base manifold. The Bismut-Cheeger η-form, $\tilde{\eta}$, is the families version of the η-invariant and its 0-degree part is just the APS η-invariant. It appears naturally as the boundary contribution of the family index theorem for manifolds with boundary (cf. [14, 15, 49, 50]). We cite also [57] for a nice topological application of eta forms. As the holomorphic analytic torsion and its family version, Bismut-Köhler holomorphic torsion form [22] are the analytic counterpart to the direct image in Arakelov geometry [54], whose foundation was developed by Gillet-Soulé and Bismut in the 1980s, the Bismut-Cheeger η-form is also the analytic counterpart to the direct image in differential K-theory introduced by Freed-Hopkins [33] and developed further by [26], [34], [38], [53], etc.

When the fibration admits a fiberwise compact Lie group action, the Bismut-Cheeger η-form could be naturally extended to the equivariant η-form $\widetilde{\eta}_g$. Recently, the functoriality of equivariant η-forms with respect to the composition of two submersions was established in [39], which extends the previous work of Bunke-Ma [25] for usual η-forms for flat vector bundles with duality, cf. [5, 6, 23, 29, 45, 46, 47, 51] for related works on η-forms and holomorphic torsions.

In the same way as fixed-point formula has two equivariant versions, the Lefschetz fixed-point formula and Kirillov-like formula of Berline-Vergne [4], the same is true for equivariant η-forms. In this paper, we use the analytic techniques of Bismut-Goette in [19] to define the equivariant infinitesimal Bismut-Cheeger η-form and prove a general comparison formula between the equivariant infinitesimal Bismut-Cheeger η-form and the equivariant Bismut-Cheeger η-form which extend the work of Goette [36]. In particular, we express the singularity of $\widetilde{\eta}_g$ modulo exact forms, at any $g \in G$ as a locally computable differential form.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. We assume that G acts isometrically on an odd-dimensional compact oriented Riemannian manifold X and the G-action lifts on a Clifford module $\mathcal{E}$ over X. In general, the equivariant APS η-invariant η_g is not a continuous function on $g \in G$. In [36], Goette studied the singularity of the equivariant η-invariant η_g at $g = e$. He defined a formal power series $\eta_K \in \mathbb{C}[[\mathfrak{g}^*]]$ for $K \in \mathfrak{g}$, called the equivariant infinitesimal η-invariant and showed that if the Killing vector field K^X induced by K has no zeroes on X, for any $N \in \mathbb{N}$, as $0 \neq t \to 0$,

$$[\eta_{tK}]_N - \eta_{e^{tK}} = \mathcal{M}_{tK} + \mathcal{O}(t^N), \tag{0.1}$$

where $[\eta_{tK}]_N$ is the part of the formal power series η_{tK} with degree $\leq N$ and $\mathcal{M}_{tK}$ could be expressed precisely as a locally computable term. Moreover, there exist $c_j(K) \in \mathbb{C}$ such that when $t \to 0$,

$$\mathcal{M}_{tK} = \sum_{j=1}^{(\dim X+1)/2} c_j(K) t^{-j} + \mathcal{O}(t^0). \tag{0.2}$$

It means that if the Killing vector field K^X is nowhere vanishing, the singular behavior of $\eta_{e^{tK}}$ when $t \to 0$ could be computed as the integral of the local terms explicitly.

In this paper, we show first that η_{tK} is an analytic function on t for t small enough and for any $0 \neq K \in \mathfrak{g}$,

$$\eta_{tK} - \eta_{e^{tK}} = \mathcal{M}_{tK}, \quad \text{for } t \neq 0 \text{ small enough.} \tag{0.3}$$

In Theorem 0.2, we establish a general version of (0.3), in particular, its family version.

Let's explain in detail our result here. Let $\pi : W \to B$ be a smooth submersion of smooth compact manifolds with fiber X. Note that $n = \dim X$ can be even or odd. Let $TX = TM/B$ be the relative tangent bundle to the fiber X. We assume that TX is oriented and that the compact Lie group G acts fiberwise on W and as identity on B and preserves the orientation of TX.

Let g^{TX} be a G-invariant metric on TX. Let $(\mathcal{E}, h^{\mathcal{E}})$ be a Clifford module of TX to the fiber X and we assume that the G-action lifts on $(\mathcal{E}, h^{\mathcal{E}})$ and is compatible with the Clifford action. Let $\nabla^{\mathcal{E}}$ be a G-invariant Clifford connection on $(\mathcal{E}, h^{\mathcal{E}})$, i.e., $\nabla^{\mathcal{E}}$ is a G-invariant Hermitian connection on $(\mathcal{E}, h^{\mathcal{E}})$ and compatible with the Clifford action (see (1.19)). Let D be the fiberwise Dirac operator associated with $(g^{TX}, \nabla^{\mathcal{E}})$ (see (1.20)).

We assume that the kernels $\mathrm{Ker}(D)$ **form a vector bundle over** B**.** Then for any $g \in G$, the equivariant η-form $\tilde{\eta}_g$ is well-defined (see Definition 1.4)[1].

In the whole paper, if $n = \dim X$ is even, $\mathcal{E}$ is naturally $\mathbb{Z}_2$-graded by the chirality operator Γ defined in (1.15) and the supertrace for $A \in \mathrm{End}(\mathcal{E})$ is defined by $\mathrm{Tr}_s[A] := \mathrm{Tr}[\Gamma A]$; if $\dim X$ is odd, $\mathcal{E}$ is ungraded. For $\sigma = \alpha \otimes A$ with $\alpha \in \Lambda(T^*B)$, $A \in \mathrm{End}(\mathcal{E})$, we define $\mathrm{Tr}[\sigma] := \alpha \cdot \mathrm{Tr}[A]$. We denote by $\mathrm{Tr}^{\mathrm{odd}}[\sigma]$ the odd degree part of $\mathrm{Tr}[\sigma]$. Set

$$\widetilde{\mathrm{Tr}}[\sigma] = \begin{cases} \mathrm{Tr}_s[\sigma] & \text{if } n = \dim X \text{ is even;} \\ \mathrm{Tr}^{\mathrm{odd}}[\sigma] & \text{if } n = \dim X \text{ is odd.} \end{cases} \tag{0.4}$$

For $\alpha \in \Omega^j(\mathbb{R} \times B)$, the space of j-th differential forms on $\mathbb{R} \times B$, set

$$\psi_{\mathbb{R}\times B}(\alpha) = \begin{cases} (2i\pi)^{-\frac{j}{2}} \cdot \alpha & \text{if } j \text{ is even;} \\ \pi^{-\frac{1}{2}} (2i\pi)^{-\frac{j-1}{2}} \cdot \alpha & \text{if } j \text{ is odd.} \end{cases} \tag{0.5}$$

Let t be the coordinate of $\mathbb{R}$ in $\mathbb{R} \times B$. If $\alpha = \alpha_0 + dt \wedge \alpha_1$, with $\alpha_0, \alpha_1 \in \Lambda(T^*B)$, set

$$[\alpha]^{dt} := \alpha_1. \tag{0.6}$$

Let $\mathcal{L}_K$ be the infinitesimal action on $\mathscr{C}^\infty(W, \mathcal{E})$ induced by $K \in \mathfrak{g}$ (see (2.3)).

For $g \in G$, we denote by $Z(g) \subset G$ the centralizer subgroup of g with Lie algebra $\mathfrak{z}(g)$. Let $W^g = \{x \in W : gx = x\}$ be the fixed point set of g. Then the restriction of π on W^g, $\pi|_{W^g} : W^g \to B$ is a fibration with compact fiber X^g.

Let $\mathbb{B}_t$ be the rescaled Bismut superconnection defined in (1.23). Let d be the exterior differential operator.

[1]For even dimensional fiber, any family of Dirac operators could be deformed to another one which satisfies this assumption and has the same family index in $K^0(B)$ (see e.g., [3, §9.5]). But for odd dimensional fiber, some topological obstruction appears: if a family of Dirac operators D satisfies this assumption, the family index of D vanishes in $K^1(B)$ (this fact is implicitly contained in [2], a proof of which is presented in [32, Theorem 4.1]). Recently, for odd dimensional fiber case, Wittmann [56] defined an η-form under the assumption that the family of Dirac operators has one eigenvalue of multiplicity one crossing zero transversally. It is expected that many properties of Bismut-Cheeger η-form could be extended to this case.

Let $\widehat{\mathrm{A}}_{g,K}(\cdot)$ and $\mathrm{ch}_{g,K}(\cdot)$ be equivariant infinitesimal versions of the $\widehat{\mathrm{A}}$-form and the Chern character form (cf. (2.15) and (2.16)). The following result extends the equivariant infinitesimal η-invariant to the family case at any $g \in G$ (see Definition 2.3, (2.31), (2.32), (2.36) and (2.37)).

Theorem 0.1. *For any $g \in G$, there exists $\beta > 0$ such that if $K \in \mathfrak{z}(g)$ with $|K| < \beta$, the integral*

$$\widetilde{\eta}_{g,K} = -\int_0^{+\infty} \left\{ \psi_{\mathbb{R}\times B} \widetilde{\mathrm{Tr}} \left[g \exp\left(-\left(\mathbb{B}_t + \frac{c(K^X)}{4\sqrt{t}} + dt \wedge \frac{\partial}{\partial t} \right)^2 - \mathcal{L}_K \right) \right] \right\}^{dt} dt \tag{0.7}$$

is a well-defined differential form on B, and

$$d\tilde{\eta}_{g,K} = \begin{cases} \displaystyle\int_{X^g} \widehat{\mathrm{A}}_{g,K}(TX, \nabla^{TX})\, \mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) \\ \qquad\qquad - \mathrm{ch}_{ge^K}(\mathrm{Ker}(D), \nabla^{\mathrm{Ker}(D)}) & \text{if } n \text{ is even;} \\ \displaystyle\int_{X^g} \widehat{\mathrm{A}}_{g,K}(TX, \nabla^{TX})\, \mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) & \text{if } n \text{ is odd.} \end{cases} \tag{0.8}$$

Moreover, for fixed $K \in \mathfrak{z}(g)$, $\widetilde{\eta}_{g,zK}$ is an analytic function of $z \in \mathbb{C}$ for $|zK| < \beta$.

In the sequel, $\widetilde{\eta}_{g,K}$ is called the equivariant infinitesimal (Bismut-Cheeger) η-form.

Let $\vartheta_K \in T^*X$ be the 1-form which is dual to K^X by the metric g^{TX}. Now we state the main result of this paper.

Theorem 0.2. *For $g \in G$ and $K_0 \in \mathfrak{z}(g)$, there exists $\beta > 0$ such that for any $K = zK_0$, $K \neq 0$ and $-\beta < z < \beta$, modulo exact forms on B, we have*

$$\tilde{\eta}_{g,K} = \tilde{\eta}_{ge^K} + \mathcal{M}_{g,K}, \tag{0.9}$$

where $\mathcal{M}_{g,K}$ is a well-defined integral defined by

$$\mathcal{M}_{g,K} = -\int_0^{+\infty} \int_{X^g} \frac{\vartheta_K}{2i\pi v} \exp\left(\frac{d\vartheta_K - 2i\pi |K^X|^2}{2i\pi v} \right) \widehat{\mathrm{A}}_{g,K}(TX, \nabla^{TX})\, \mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) \frac{dv}{v}, \tag{0.10}$$

and $t^{\lfloor (\dim W^g+1)/2 \rfloor} \mathcal{M}_{g,tK}$ is real analytic on $t \in \mathbb{R}$, $|t| < 1$. Moreover, we have

$$d\mathcal{M}_{g,K} = \int_{X^g} \widehat{\mathrm{A}}_{g,K}(TX, \nabla^{TX})\, \mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) - \int_{X^{ge^K}} \widehat{\mathrm{A}}_{ge^K}(TX, \nabla^{TX})\, \mathrm{ch}_{ge^K}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}). \tag{0.11}$$

By Theorem 0.1, $\tilde{\eta}_{g,tK}$ is an analytic function of t near $t = 0$. Thus when $t \to 0$, modulo exact forms, the singularity of $\tilde{\eta}_{ge^{tK}}$ is the same as that of $-\mathcal{M}_{g,tK}$.

Note that the general comparison formula for the two versions of equivariant holomorphic analytic torsions is established in [19, Theorem 5.1], which is the model of our paper. The analytical tools in this paper are inspired by those of [19] with necessary modifications. For this problem on de Rham torsion forms, a comparison formula is stated in [20, Theorem 5.13].

Remark 0.3. Let G act on an odd dimensional compact Riemannian manifold (X, g^{TX}) and on a Clifford module $(\mathcal{E}, h^{\mathcal{E}}, \nabla^{\mathcal{E}})$ compatible with the Clifford action. Then for $g = e$ the identity element of G, (0.7) defines a complex number η_K for any $K \in \mathfrak{g}$, $|K| < \beta$. As formal power series on K, this η_K is just the equivariant infinitesimal η-invariant η_K in [36, Definition 0.4].

Let $P \to B$ be a G-principal bundle with connection and associated curvature Ω. Then we get naturally a fibration $P \times_G X \to B$ with fiber X. Let $\widetilde{\eta}$ be the associated Bismut-Cheeger η-form. For this fibration, by Bismut [8, §1d), §3b)], under the notation of (1.23), the term $c(T^H)$ in the Bismut superconnection is $c(\Omega)$, and $(\nabla^{\mathbb{E},u})^2 = \mathcal{L}_\Omega$, thus we get [36, Lemma 1.14],

$$\widetilde{\eta} = \eta_{\frac{i}{2\pi}\Omega}. \tag{0.12}$$

Thus we can understand the formal power series of η_K as a universal η-form.

Remark 0.4. Assume temporarily that $B = \mathrm{pt}$, $\dim X = n$ is odd, and X is the boundary of a G-equivariant Riemannian manifold Z, which has product structure near X. We also assume that $\mathcal{E}_Z = \mathcal{E}_Z^+ \oplus \mathcal{E}_Z^-$ is a G-equivariant Clifford module on Z such that $\mathcal{E}_Z^+\big|_X = \mathcal{E}$ and $\mathcal{E}_Z^\pm$ near X is the pull-back of $\mathcal{E}$ as Hermitian vector bundles with connections.

Let D_Z be the associated Dirac operator on $\mathcal{E}_Z$ over Z. Then the index of $D_Z^+ := D_Z|_{\mathscr{C}^\infty(Z,\mathcal{E}_Z^+)}$ with respect to the Atiyah-Patodi-Singer (APS) boundary condition is a virtual representation of G. For $g \in G$, its equivariant APS index $\mathrm{Ind}_{\mathrm{APS},g}(D_Z^+)$ can be computed by Donnelly's theorem [31],

$$\mathrm{Ind}_{\mathrm{APS},g}(D_Z^+) = \int_{Z^g} \widehat{\mathrm{A}}_g(TZ, \nabla^{TZ})\, \mathrm{ch}_g(\mathcal{E}_Z/\mathcal{S}_Z, \nabla^{\mathcal{E}_Z}) - \frac{1}{2}\left(\eta_g(D) + \mathrm{Tr}\,|_{\mathrm{Ker}(D)}[g]\right). \tag{0.13}$$

By combining (0.9), (0.11) (more precisely the Stokes formula [24, p. 775], (3.30) and (3.33)), and (0.13), for any $K \in \mathfrak{g}$, there exists $\beta > 0$ such that, for any $-\beta < t < \beta$, we have

$$\mathrm{Ind}_{\mathrm{APS},e^{tK}}(D_Z^+) = \int_Z \widehat{\mathrm{A}}_{tK}(TZ, \nabla^{TZ})\, \mathrm{ch}_{tK}(\mathcal{E}_Z/\mathcal{S}_Z, \nabla^{\mathcal{E}_Z}) - \frac{1}{2}\left(\eta_{tK}(D) + \mathrm{Tr}\,|_{\mathrm{Ker}(D)}[e^{tK}]\right). \tag{0.14}$$

Here $\widehat{\mathrm{A}}_{tK}(\cdot) := \widehat{\mathrm{A}}_{e,tK}(\cdot)$ and $\mathrm{ch}_{tK}(\cdot) := \mathrm{ch}_{e,tK}(\cdot)$.

The main result of this paper is announced in [41] and plays an important role in our recent work [42].

This paper is organized as follows. In Section 1, we recall the definition of the equivariant Bismut-Cheeger η-form. In Section 2, we state the family Kirillov formula and define the equivariant infinitesimal η-form, in particular, we establish Theorem 0.1 modulo some technical details. In Section 3, we prove that $\mathcal{M}_{g,tK}$ in (0.10) is well-defined and state our main result, Theorem 0.2. In Section 4, we state some intermediate results and prove Theorem 0.2. In Section 5, we give an analytic proof of the family Kirillov formula and the technical details to establish Theorem 0.1 following the lines of [19, §7]. For the convenience to compare the arguments here with those in [19, §7], especially how the extra terms for the families version appear, the structure of this section is formulated almost the same as in [19, §7]. In Section 6, we prove the intermediate results in Section 4 using the analytical techniques in [19, §8, §9].

From Remark 1.3, to simplify the presentation, in Sections 5, 6, we will **assume** that TX^g is oriented.

Notation: we use the Einstein summation convention in this paper: when an index variable appears twice in a single term and is not otherwise defined, it implies summation of that term over all the values of the index.

We denote by $\lfloor x \rfloor$ the maximal integer not larger than x.

We denote by d the exterior differential operator and d^B when we like to insist the base manifold B. Let $\Omega^{\mathrm{even/odd}}(B,\mathbb{C})$ be the space of even/odd degree complex valued differential forms on B. For a real vector bundle E, we denote by $\dim E$ the real rank of E.

If $\mathcal{A}$ is a $\mathbb{Z}_2$-graded algebra, and if $a, b \in \mathcal{A}$, then we will note $[a,b] := ab - (-1)^{\deg a \cdot \deg b} ba$ as the supercommutator of a, b. In the whole paper, if $\mathcal{A}$, $\mathcal{A}'$ are $\mathbb{Z}_2$-graded algebras we will note $\mathcal{A}\widehat{\otimes}\mathcal{A}'$ as the $\mathbb{Z}_2$-graded tensor product as in [3, §1.3]. If one of $\mathcal{A}, \mathcal{A}'$ is ungraded, we understand it as $\mathbb{Z}_2$-graded by taking its odd part as zero.

For the fiber bundle $\pi : W \to B$, we will often use the integration of the differential forms along the oriented fibers X in this paper. Since the fibers may be odd dimensional, we must make precisely our sign conventions: for $\alpha \in \Omega^\bullet(B)$ and $\beta \in \Omega^\bullet(W)$, then

$$\int_X (\pi^*\alpha) \wedge \beta = \alpha \wedge \int_X \beta. \tag{0.15}$$

Acknowledgments. B. L. is partially supported by Science and Technology Commission of Shanghai Municipality (STCSM), grant No.18dz2271000, Natural Science Foundation of Shanghai, grant No.20ZR1416700 and NSFC No.11931007. X. M. is partially supported by NSFC No.11528103, No.11829102, ANR-14-CE25-0012-01, and funded through the Institutional Strategy of the University of Cologne within the German Excellence Initiative. Part of this work was done while the authors were visiting University of Science and Technology in China and Wuhan University.

1. Equivariant η-forms

In this section, we recall the definition of the equivariant η-form in the language of Clifford modules. In Section 1.1, we recall the definition of the Clifford algebra. In Section 1.2, we explain the Bismut superconnection. In Section 1.3, we define the equivariant η-form for Clifford module.

1.1. **Clifford algebras.**

Let $(V, \langle , \rangle)$ be a Euclidean space, such that $\dim V = n$, with orthonormal basis $\{e_i\}_{i=1}^n$. Let $c(V)$ be the Clifford algebra of V defined by the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}. \tag{1.1}$$

To avoid ambiguity, we denote by $c(e_i)$ the element of $c(V)$ corresponding to e_i.

If $e \in V$, let $e^* \in V^*$ correspond to e by the scalar product $\langle , \rangle$ of V. The exterior algebra ΛV^* is a module of $c(V)$ defined by

$$c(e)\alpha = e^* \wedge \alpha - i_e \alpha \tag{1.2}$$

for any $\alpha \in \Lambda V^*$, where $\wedge$ is the exterior product and i is the contraction operator. The map $a \mapsto c(a) \cdot 1$, $a \in c(V)$, induces an isomorphism of vector spaces

$$\sigma : c(V) \to \Lambda V^*. \tag{1.3}$$

1.2. **Bismut superconnection.**

Let $\pi : W \to B$ be a smooth submersion of smooth compact manifolds with n-dimensional fibers X. Let $TX = TW/B$ be the relative tangent bundle to the fibers X.

Let G be a compact Lie group acting on W along the fibers X, that is, if $g \in G$, $\pi \circ g = \pi$. Then G acts on TW and on TX. Let $T^HW \subset TW$ be a G-invariant horizontal subbundle, so that

$$TW = T^HW \oplus TX. \tag{1.4}$$

Since G is compact, such T^HW always exists. Let $P^{TX} : TW \to TX$ be the projection associated with the splitting (1.4). Note that

$$T^HW \cong \pi^*TB. \tag{1.5}$$

Let g^{TX} be a G-invariant metric on TX. Let g^{TB} be a Riemannian metric on TB. We equip TW with the G-invariant metric via (1.4) and (1.5),

$$g^{TW} = \pi^*g^{TB} \oplus g^{TX}. \tag{1.6}$$

Let $\nabla^{TW,L}$ (resp. ∇^{TB}) be the Levi-Civita connection on (TW, g^{TW}) (resp. (TB, g^{TB})). Let ∇^{TX} be the connection on TX defined by

$$\nabla^{TX} = P^{TX}\nabla^{TW,L}P^{TX}. \tag{1.7}$$

It is G-invariant. Let ∇^{TW} be the G-invariant connection on TW, via (1.4) and (1.5),

$$\nabla^{TW} = \pi^*\nabla^{TB} \oplus \nabla^{TX}. \tag{1.8}$$

Put

$$S = \nabla^{TW,L} - \nabla^{TW}. \tag{1.9}$$

Then S is a 1-form on W with values in antisymmetric elements of $\mathrm{End}(TW)$. Let T be the torsion of ∇^{TW}. By [8, (1.28)], if $U, V, Z \in TW$,

$$\begin{aligned} &S(U)V - S(V)U + T(U, V) = 0,\\ &2\langle S(U)V, Z\rangle + \langle T(U, V), Z\rangle + \langle T(Z, U), V\rangle - \langle T(V, Z), U\rangle = 0. \end{aligned} \tag{1.10}$$

If U is a vector field on B, let U^H be its lift in T^HW and let $\mathcal{L}_{U^H}$ be the Lie derivative operator associated with the vector field U^H. Then $\mathcal{L}_{U^H}$ acts on the tensor algebra of TX. In particular, if $U \in TB, \left(g^{TX}\right)^{-1}\mathcal{L}_{U^H}g^{TX}$ defines a self-adjoint endomorphism of TX. If U, V are vector fields on B, from [11, Theorem 1.1],

$$T(U^H, V^H) = -P^{TX}[U^H, V^H], \tag{1.11}$$

and if $U \in TB$, $Z, Z' \in TX$,

$$T(U^H, Z) = \frac{1}{2}\left(g^{TX}\right)^{-1}\mathcal{L}_{U^H}g^{TX}Z, \quad T(Z, Z') = 0. \tag{1.12}$$

From (1.10) and (1.12), if $U \in TB$, $Z, Z' \in TX$, we have

$$\langle S(Z)Z', U^H\rangle = -\langle T(U^H, Z), Z'\rangle = -\langle T(U^H, Z'), Z\rangle. \tag{1.13}$$

We recall some properties in [11, §1.1].

Proposition 1.1. *1) The connection ∇^{TX} does not depend on g^{TB} and on each fiber X, it restricts to the Levi-Civita connection of (TX, g^{TX}).*

2) If $U \in TB$, then

$$\nabla^{TX}_{U^H} = \mathcal{L}_{U^H} + \frac{1}{2}\left(g^{TX}\right)^{-1}\mathcal{L}_{U^H} g^{TX}. \tag{1.14}$$

3) The tensors T and $\langle S(\cdot)\cdot, \cdot\rangle$ do not depend on g^{TB}.

Let $c(TX)$ be the Clifford algebra bundle of (TX, g^{TX}), whose fiber at $x \in W$ is the Clifford algebra $c(T_xX)$ of the Euclidean space (T_xX, g^{T_xX}). Let $\mathcal{E}$ be a Clifford module of $c(TX)$. It means that $\mathcal{E}$ is a complex vector bundle and restricted on a fiber, $\mathcal{E}_x$ is a representation of $c(T_xX)$. We assume that the G-action lifts on $\mathcal{E}$ and commutes with the Clifford action.

From now on, we assume that TX is G-equivariant oriented.

In the whole paper, if n is even, as in [3, Lemma 3.17], for a locally oriented orthonormal frame $e_1, \cdots, e_n$ of TX, we define the chirality operator by

$$\Gamma = i^{n/2} c(e_1)\cdots c(e_n). \tag{1.15}$$

Then Γ does not depend on the choice of the frame, commutes with the G-action and $\Gamma^2 = \mathrm{Id}$. Thus $\mathcal{E}$ is naturally $\mathbb{Z}_2$-graded by the chirality operator Γ. The supertrace for $A \in \mathrm{End}(\mathcal{E})$ is defined by

$$\mathrm{Tr}_s[A] := \mathrm{Tr}[\Gamma A]. \tag{1.16}$$

If n is odd, $\mathcal{E}$ is ungraded.

Let $h^{\mathcal{E}}$ be a G-invariant Hermitian metric on $\mathcal{E}$. For $b \in B$, let $\mathbb{E}_b$ be the set of smooth sections over $X_b = \pi^{-1}(b)$ of $\mathcal{E}|_{X_b}$. As in [8], we will regard $\mathbb{E}$ as an infinite dimensional vector bundle over B. Let $dv_X(x)$ be the Riemannian volume element of X_b. The bundle $\mathbb{E}_b$ is naturally endowed with the Hermitian product

$$\langle s, s'\rangle_0 = \int_{X_b} \langle s, s'\rangle(x) dv_X(x), \quad \text{for } s, s' \in \mathbb{E}. \tag{1.17}$$

Then G acts on $\mathbb{E}_b = \mathscr{C}^\infty(X_b, \mathcal{E}|_{X_b})$ as

$$(g.s)(x) = g(s(g^{-1}x)) \quad \text{for any } g \in G. \tag{1.18}$$

Let $\nabla^{\mathcal{E}}$ be a G-invariant Clifford connection on $\mathcal{E}$ (cf. [3, §10.2]), that is, $\nabla^{\mathcal{E}}$ is G-invariant, preserves $h^{\mathcal{E}}$ and for any $U \in TW$, $Z \in \mathscr{C}^\infty(W, TX)$,

$$\left[\nabla^{\mathcal{E}}_U, c(Z)\right] = c\left(\nabla^{TX}_U Z\right). \tag{1.19}$$

The fiberwise Dirac operator is defined by

$$D = \sum_{i=1}^n c(e_i)\nabla^{\mathcal{E}}_{e_i}, \tag{1.20}$$

which is independent of the choice of the orthonormal frame $\{e_i\}_{i=1}^n$.

Let $k \in (T^HW)^*$ such that for any $U \in TB$, $\mathcal{L}_{U^H} dv_X(x)/dv_X(x) = 2k(U^H)(x)$. The connection $\nabla^{\mathbb{E},u}$ on $\mathbb{E}$ defined by (cf. [16, Definition 1.3])

$$\nabla^{\mathbb{E},u}_U s := \nabla^{\mathcal{E}}_{U^H} s + k(U^H)s \quad \text{for } s \in \mathscr{C}^\infty(B, \mathbb{E}) = \mathscr{C}^\infty(W, \mathcal{E}), \tag{1.21}$$

is G-invariant and preserves the G-invariant L^2-product (1.17) (see e.g., [16, Proposition 1.4]).

Let $\{f_p\}$ be a local frame of TB and $\{f^p\}$ be its dual. Set

$$\nabla^{\mathbb{E},u} = f^p \wedge \nabla^{\mathbb{E},u}_{f_p}, \quad c(T^H) = \frac{1}{2} c\left(T(f_p^H, f_q^H)\right) f^p \wedge f^q \wedge . \tag{1.22}$$

Then $c(T^H)$ is a section of $\pi^*\Lambda^2(T^*B)\widehat{\otimes}\operatorname{End}(\mathcal{E})$.

By [8, (3.18)], the rescaled Bismut superconnection $\mathbb{B}_u$, $u>0$, is defined by

$$\mathbb{B}_u = \sqrt{u}D + \nabla^{\mathbb{E},u} - \frac{1}{4\sqrt{u}}c(T^H) : \mathscr{C}^\infty(B, \Lambda(T^*B)\widehat{\otimes}\mathbb{E}) \to \mathscr{C}^\infty(B, \Lambda(T^*B)\widehat{\otimes}\mathbb{E}). \tag{1.23}$$

Obviously, the Bismut superconnection $\mathbb{B}_u$ commutes with the G-action. Furthermore, $\mathbb{B}_u^2$ is a 2nd-order elliptic differential operator along the fiber X (cf. [8, (3.4)]) acting on $\Lambda(T^*B)\widehat{\otimes}\mathbb{E}$. Let $\exp(-\mathbb{B}_u^2)$ be the heat operators associated with the fiberwise elliptic operator $\mathbb{B}_u^2$.

1.3. **Equivariant η-forms.** Take $g \in G$ fixed and set $W^g = \{x \in W : gx = x\}$, the fixed point set of g. Then W^g is a submanifold of W and $\pi|_{W^g} : W^g \to B$ is a fibration with compact fiber X^g. Let $N_{W^g/W}$ denote the normal bundle of W^g in W, then

$$N_{W^g/W} := \frac{TW}{TW^g} = \frac{TX}{TX^g} =: N_{X^g/X}. \tag{1.24}$$

Let $\{X^g_\alpha\}_{\alpha\in\mathfrak{B}}$ be the connected components of X^g with

$$\dim X^g_\alpha = \ell_\alpha. \tag{1.25}$$

By an abuse of notation, we will often simply denote by all ℓ_α the same ℓ.

Assumption 1.2. We assume that the kernels $\operatorname{Ker}(D)$ form a vector bundle over B.

For $\sigma = \alpha\widehat{\otimes}A$ with $\alpha \in \Lambda(T^*B)$, $A \in \operatorname{End}(\mathcal{E})$, we define

$$\operatorname{Tr}[\sigma] = \alpha \cdot \operatorname{Tr}[A], \quad \operatorname{Tr}^{\mathrm{odd}}[\sigma] = \{\alpha\}^{\mathrm{odd}} \cdot \operatorname{Tr}[A], \quad \operatorname{Tr}^{\mathrm{even}}[\sigma] = \{\alpha\}^{\mathrm{even}} \cdot \operatorname{Tr}[A], \tag{1.26}$$

where $\{\alpha\}^{\mathrm{odd/even}}$ is the odd or even degree part of α. Set

$$\widetilde{\operatorname{Tr}}[\sigma] = \begin{cases} \operatorname{Tr}_s[\sigma] := \alpha \cdot \operatorname{Tr}[\Gamma A] & \text{if } n = \dim X \text{ is even;} \\ \operatorname{Tr}^{\mathrm{odd}}[\sigma] & \text{if } n = \dim X \text{ is odd.} \end{cases} \tag{1.27}$$

Let $\operatorname{End}_{c(TX)}(\mathcal{E})$ be the set of endomorphisms of $\mathcal{E}$ supercommuting with the Clifford action. It is a vector bundle over W. As in [3, Definition 3.28], we define the relative trace $\operatorname{Tr}^{\mathcal{E}/\mathcal{S}} : \operatorname{End}_{c(TX)}(\mathcal{E}) \to \mathbb{C}$ by: for any $A \in \operatorname{End}_{c(TX)}(\mathcal{E})$,

$$\operatorname{Tr}^{\mathcal{E}/\mathcal{S}}[A] = \begin{cases} 2^{-n/2}\operatorname{Tr}_s[\Gamma A] & \text{if } n = \dim X \text{ is even;} \\ 2^{-(n-1)/2}\operatorname{Tr}[A] & \text{if } n = \dim X \text{ is odd.} \end{cases} \tag{1.28}$$

Let $R^{TX} = \left(\nabla^{TX}\right)^2$, $R^{\mathcal{E}} = \left(\nabla^{\mathcal{E}}\right)^2$ be the curvatures of ∇^{TX}, $\nabla^{\mathcal{E}}$. Then

$$R^{\mathcal{E}/\mathcal{S}} := R^{\mathcal{E}} - \frac{1}{4}\langle R^{TX}e_i, e_j\rangle c(e_i)c(e_j) \in \mathscr{C}^\infty(W, \Lambda^2(T^*W) \otimes \operatorname{End}_{c(TX)}(\mathcal{E})) \tag{1.29}$$

is the twisting curvature of the Clifford module $\mathcal{E}$ as in [3, Proposition 3.43].

Note that if TX has a G-equivariant spin structure, then there exists a G-equivariant Hermitian vector bundle E such that $\mathcal{E} = \mathcal{S}_X \otimes E$, with $\mathcal{S}_X$ the spinor bundle of TX, $\nabla^{\mathcal{E}}$ is induced by ∇^{TX} and a G-invariant Hermitian connection ∇^E on E and

$$R^{\mathcal{E}/\mathcal{S}} = R^E = (\nabla^E)^2. \tag{1.30}$$

We denote the differential of g by dg which gives a bundle isometry $dg : N_{X^g/X} \to N_{X^g/X}$. As G is compact, we know that there is an orthonormal decomposition of real vector bundles over W^g,

$$TX|_{W^g} = TX^g \oplus N_{X^g/X} = TX^g \oplus \bigoplus_{0<\theta\leq\pi} N(\theta), \tag{1.31}$$

where $dg|_{N(\pi)} = -\mathrm{Id}$ and for each θ, $0 < \theta < \pi$, $N(\theta)$ is the underlying real vector bundle of a complex vector bundle N_θ over W^g on which dg acts by multiplication by $e^{i\theta}$. Since g preserves the metric and the orientation of TX, thus $\det(dg|_{N(\pi)}) = 1$, this means $\dim N(\pi)$ is even. So the normal bundle $N_{X^g/X}$ is even dimensional.

Since ∇^{TX} commutes with the group action, its restriction on W^g, $\nabla^{TX}|_{W^g}$, preserves the decomposition (1.31). Let ∇^{TX^g} and $\nabla^{N(\theta)}$ be the corresponding induced connections on TX^g and $N(\theta)$, with curvatures R^{TX^g} and $R^{N(\theta)}$.

Set

$$\begin{aligned}\widehat{\mathrm{A}}_g(TX, \nabla^{TX}) = {} & \det{}^{\frac{1}{2}}\left(\frac{\frac{i}{4\pi}R^{TX^g}}{\sinh\left(\frac{i}{4\pi}R^{TX^g}\right)}\right) \\ & \cdot \prod_{0<\theta\leq\pi}\left(i^{\frac{1}{2}\dim N(\theta)}\det{}^{\frac{1}{2}}\left(1 - g\exp\left(\frac{i}{2\pi}R^{N(\theta)}\right)\right)\right)^{-1} \in \Omega^{2\bullet}(W^g, \mathbb{C}).\end{aligned} \tag{1.32}$$

The sign convention in (1.32) is that the degree 0 part in $\prod_{0<\theta\leq\pi}$ is given by $\left(\frac{e^{i\theta/2}}{e^{i\theta}-1}\right)^{\frac{1}{2}\dim N(\theta)}$.

By [3, Lemma 6.10], along W^g, the action of $g \in G$ on $\mathcal{E}$ may be identified with a section $g^{\mathcal{E}}$ of $c(N_{X^g/X}) \otimes \mathrm{End}_{c(TX)}(\mathcal{E})$. Under the isomorphism (1.3), $\sigma(g^{\mathcal{E}}) \in \mathscr{C}^\infty(W^g, \Lambda(N^*_{X^g/X}) \otimes \mathrm{End}_{c(TX)}(\mathcal{E}))$. Let $\sigma_{n-\ell}(g^{\mathcal{E}}) \in \mathscr{C}^\infty(W^g, \Lambda^{n-\ell}(N^*_{X^g/X}) \otimes \mathrm{End}_{c(TX)}(\mathcal{E}))$ be the highest degree part of $\sigma(g^{\mathcal{E}})$ in $\Lambda(N^*_{X^g/X})$. Then we define the localized relative Chern character $\mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}})$ as in [3, Definition 6.13]:

$$\begin{aligned}\mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) := {} & \frac{2^{(n-\ell)/2}}{\det^{1/2}(1 - g|_{N_{X^g/X}})}\mathrm{Tr}^{\mathcal{E}/\mathcal{S}}\left[\sigma_{n-\ell}(g^{\mathcal{E}})\exp\left(-\frac{R^{\mathcal{E}/\mathcal{S}}|_{W^g}}{2i\pi}\right)\right] \\ & \in \Omega^\bullet\big(W^g, \det N_{X^g/X}\big).\end{aligned} \tag{1.33}$$

Remark 1.3. In general, TX^g is not necessary oriented. The orientation of TX allows us to identify $\det N_{X^g/X}$ as the orientation line of X^g, thus the integral $\int_{X^g}$ of a form in $\Omega^\bullet\big(W^g, \det N_{X^g/X}\big)$ makes sense as in [3, Theorem 6.16]. Assume that TX^g is oriented, then the orientations of TX^g and TX induce canonically an orientation on $N_{X^g/X}$. By pairing with the volume form of $N_{X^g/X}$, we obtain

$$\mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) \in \Omega^\bullet(W^g, \mathbb{C}). \tag{1.34}$$

If TX has a G-equivariant spin^c structure, then TX^g is canonically oriented (cf. [3, Proposition 6.14], [44, Lemma 4.1]). If TX has a G-equivariant spin structure, $\mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}})$ under the above convention is just the usual equivariant Chern character (cf. (1.30))

$$\mathrm{ch}_g(E, \nabla^E) = \mathrm{Tr}^E\left[g\exp\left(-\frac{R^E|_{W^g}}{2i\pi}\right)\right]. \tag{1.35}$$

As in (0.5), for $\alpha \in \Omega^j(B)$, set

$$\psi_B(\alpha) = \begin{cases} (2i\pi)^{-\frac{j}{2}} \cdot \alpha & \text{if } j \text{ is even;} \\ \pi^{-\frac{1}{2}} (2i\pi)^{-\frac{j-1}{2}} \cdot \alpha & \text{if } j \text{ is odd.} \end{cases} \tag{1.36}$$

Then from the equivariant family local index theorem (see e.g., [8, Theorem 4.17], [17, Theorem 2.10], [40, Theorem 2.2], [43, Theorem 1.3]), for any $u > 0$, the differential form $\psi_B \widetilde{\mathrm{Tr}}[g \exp(-\mathbb{B}_u^2)] \in \Omega^\bullet(B, \mathbb{C})$ is closed, its cohomology class is independent of $u > 0$, and

$$\lim_{u \to 0} \psi_B \widetilde{\mathrm{Tr}}[g \exp(-\mathbb{B}_u^2)] = \int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX}) \ \mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}). \tag{1.37}$$

Let $P^{\mathrm{Ker}(D)} : \mathbb{E} \to \mathrm{Ker}(D)$ be the orthogonal projection with respect to (1.17). Let

$$\nabla^{\mathrm{Ker}(D)} = P^{\mathrm{Ker}(D)} \nabla^{\mathbb{E},u} P^{\mathrm{Ker}(D)} \tag{1.38}$$

and $R^{\mathrm{Ker}(D)}$ be the curvature of the connection $\nabla^{\mathrm{Ker}(D)}$ on $\mathrm{Ker}(D)$.

- If $n = \dim X$ is even, from the natural equivariant extension of [3, Theorem 9.19], we have

$$\lim_{u \to +\infty} \psi_B \,\mathrm{Tr}_s[g \exp(-\mathbb{B}_u^2)] = \mathrm{Tr}_s \left[g \exp \left(-\frac{R^{\mathrm{Ker}(D)}}{2i\pi} \right) \right] = \mathrm{ch}_g(\mathrm{Ker}(D), \nabla^{\mathrm{Ker}(D)}). \tag{1.39}$$

Since $\mathbb{B}_u$ is G-invariant, the equivariant version of [3, Theorem 9.17] shows that

$$\frac{\partial}{\partial u} \mathrm{Tr}_s \left[g \exp(-\mathbb{B}_u^2) \right] = -d^B \,\mathrm{Tr}_s \left[g \frac{\partial \mathbb{B}_u}{\partial u} \exp(-\mathbb{B}_u^2) \right]. \tag{1.40}$$

Thus for $0 < \varepsilon < T < +\infty$,

$$\mathrm{Tr}_s \left[g \exp(-\mathbb{B}_\varepsilon^2) \right] - \mathrm{Tr}_s \left[g \exp(-\mathbb{B}_T^2) \right] = d^B \int_\varepsilon^T \mathrm{Tr}_s \left[g \frac{\partial \mathbb{B}_u}{\partial u} \exp(-\mathbb{B}_u^2) \right] du. \tag{1.41}$$

The natural equivariant extension of [3, Theorems 9.23 and 10.32(1)] (cf. e.g., [39, (2.72) and (2.77)]) shows that

$$\begin{aligned} \mathrm{Tr}_s \left[g \frac{\partial \mathbb{B}_u}{\partial u} \exp(-\mathbb{B}_u^2) \right] &= \mathcal{O}(u^{-1/2}) \quad \text{as } u \to 0, \\ \mathrm{Tr}_s \left[g \frac{\partial \mathbb{B}_u}{\partial u} \exp(-\mathbb{B}_u^2) \right] &= \mathcal{O}(u^{-3/2}) \quad \text{as } u \to +\infty. \end{aligned} \tag{1.42}$$

In this case, by (1.36) and (1.42), the equivariant η-form is defined by

$$\tilde{\eta}_g = \int_0^{+\infty} \frac{1}{2i\sqrt{\pi}} \psi_B \,\mathrm{Tr}_s \left[g \ \frac{\partial \mathbb{B}_u}{\partial u} \ \exp(-\mathbb{B}_u^2) \right] du \in \Omega^{\mathrm{odd}}(B, \mathbb{C}). \tag{1.43}$$

By (1.37), (1.39), (1.41) and (1.43), we have

$$d^B \tilde{\eta}_g = \int_{X^g} \widehat{\mathrm{A}}_g(TX, \nabla^{TX}) \,\mathrm{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) - \mathrm{ch}_g(\mathrm{Ker}(D), \nabla^{\mathrm{Ker}(D)}). \tag{1.44}$$

- If n is odd, since the equivariant extension of [3, Theorem 9.19] also holds, we have

$$\lim_{u \to +\infty} \mathrm{Tr}^{\mathrm{odd}}[g \exp(-\mathbb{B}_u^2)] = \mathrm{Tr}^{\mathrm{odd}} \left[g \exp \left(-R^{\mathrm{Ker}(D)} \right) \right] = 0. \tag{1.45}$$

As an analogue of (1.41), for $0 < \varepsilon < T < +\infty$, we have

$$\mathrm{Tr}^{\mathrm{odd}} \left[g \exp(-\mathbb{B}_\varepsilon^2) \right] - \mathrm{Tr}^{\mathrm{odd}} \left[g \exp(-\mathbb{B}_T^2) \right] = d^B \int_\varepsilon^T \mathrm{Tr}^{\mathrm{even}} \left[g \frac{\partial \mathbb{B}_u}{\partial u} \exp(-\mathbb{B}_u^2) \right] du. \tag{1.46}$$

Following the same arguments in the proof of (1.42), we have

$$
\begin{aligned}
&\operatorname{Tr}^{\mathrm{even}}\left[g\frac{\partial \mathbb{B}_u}{\partial u}\exp(-\mathbb{B}_u^2)\right] = \mathcal{O}(u^{-1/2}) \quad \text{as } u \to 0,\\
&\operatorname{Tr}^{\mathrm{even}}\left[g\frac{\partial \mathbb{B}_u}{\partial u}\exp(-\mathbb{B}_u^2)\right] = \mathcal{O}(u^{-3/2}) \quad \text{as } u \to +\infty.
\end{aligned} \tag{1.47}
$$

In this case, by (1.36) and (1.47), the equivariant η-form is defined by

$$
\tilde{\eta}_g = \int_0^{+\infty} \frac{1}{\sqrt{\pi}}\psi_B \operatorname{Tr}^{\mathrm{even}}\left[g\,\frac{\partial \mathbb{B}_u}{\partial u}\exp(-\mathbb{B}_u^2)\right] du \in \Omega^{\mathrm{even}}(B,\mathbb{C}). \tag{1.48}
$$

From (1.37), (1.45), (1.46) and (1.48), we get

$$
d^B\tilde{\eta}_g = \int_{X^g} \widehat{\mathrm{A}}_g(TX,\nabla^{TX})\,\mathrm{ch}_g(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}}). \tag{1.49}
$$

We write the definition of the equivariant η-form (1.43) and (1.48) in a uniform way using the notation $\{\cdot\}^{du}$ as in (0.6).

Definition 1.4. [39, Definition 2.3] For $g \in G$ fixed, under Assumption 1.2, the equivariant Bismut-Cheeger η-form is defined by

$$
\tilde{\eta}_g := -\int_0^{+\infty} \left\{\psi_{\mathbb{R}\times B}\,\widetilde{\operatorname{Tr}}\left[g\exp\left(-\left(\mathbb{B}_u + du\wedge\frac{\partial}{\partial u}\right)^2\right)\right]\right\}^{du} du \in \Omega^{\bullet}(B,\mathbb{C}). \tag{1.50}
$$

If $g = e$ the identity element of G, (1.50) is exactly the Bismut-Cheeger η-form defined in [13]. If B is noncompact, (1.42) and (1.47) hold uniformly on any compact subset of B, thus Definition 1.4, (1.44) and (1.49) still hold.

2. Equivariant infinitesimal η-forms

In this section, we state the family Kirillov formula and define the equivariant infinitesimal η-form. In Section 2.1, we state the families version of the Kirillov formula. In Section 2.2, we define the equivariant infinitesimal η-form, and establish Theorem 0.1 modulo some technical details.

In this section, we use the same notations and assumptions in Section 1. Especially, TX is G-equivariant oriented and Assumption 1.2 holds in this section.

2.1. Moment maps and the family Kirillov formula.

Let $|\cdot|$ be a G-invariant norm on the Lie algebra $\mathfrak{g}$ of G. For $K \in \mathfrak{g}$, let

$$
K^X(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} e^{tK}\cdot x \quad \text{for } x \in W \tag{2.1}
$$

be the induced vector field on W. Since G acts fiberwise on W, $K^X \in \mathscr{C}^\infty(W,TX)$ and

$$
[K^X, K'^X] = -[K,K']^X \qquad \text{for any } K,K' \in \mathfrak{g}. \tag{2.2}
$$

For $K \in \mathfrak{g}$, let $\mathcal{L}_K$ be the corresponding Lie derivative given by

$$
\mathcal{L}_K s = \left.\frac{\partial}{\partial t}\right|_{t=0}\left(e^{-tK}.s\right), \tag{2.3}
$$

for $s \in \mathscr{C}^\infty(W, \mathcal{E})$ (cf. (1.18)). The associated moment maps $m^{TX}(\cdot)$, $m^{\mathcal{E}}(\cdot)$ are defined by [3, Definition 7.5] (see also [19, Definition 2.1]),

$$\begin{aligned} m^{TX}(K) &:= \nabla^{TX}_{K^X} - \mathcal{L}_K|_{TX} \in \mathscr{C}^\infty(W, \mathrm{End}(TX)), \\ m^{\mathcal{E}}(K) &:= \nabla^{\mathcal{E}}_{K^X} - \mathcal{L}_K|_{\mathcal{E}} \in \mathscr{C}^\infty(W, \mathrm{End}(\mathcal{E})). \end{aligned} \tag{2.4}$$

Since the vector field K^X is Killing and ∇^{TX}, $\nabla^{\mathcal{E}}$ preserve the corresponding metrics, $m^{TX}(K)$ and $m^{\mathcal{E}}(K)$ are skew-adjoint actions of $\mathrm{End}(TX)$ and $\mathrm{End}(\mathcal{E})$ respectively. By Proposition 1.1, the connection ∇^{TX} is the Levi-Civita connection of (TX, g^{TX}) when it is restricted on a fiber. Since the G-action is along the fiber, we have

$$m^{TX}(K) = \nabla^{TX}_{\cdot} K^X \in \mathscr{C}^\infty(W, \mathrm{End}(TX)). \tag{2.5}$$

Since the connection ∇^{TX} is G-invariant, from (2.4) (cf. [3, (7.4)] or [19, (2.8)]),

$$\nabla^{TX}_{\cdot} m^{TX}(K) + i_{K^X} R^{TX} = 0. \tag{2.6}$$

We denote by $m^{\mathcal{S}}(K) \in \mathrm{End}(\mathcal{E})$ by

$$m^{\mathcal{S}}(K) := \frac{1}{4}\langle m^{TX}(K)e_i, e_j\rangle c(e_i)c(e_j). \tag{2.7}$$

If TX is spin, $m^{\mathcal{S}}(K)$ is just the moment map of the spinor. Set

$$m^{\mathcal{E}/\mathcal{S}}(K) := m^{\mathcal{E}}(K) - m^{\mathcal{S}}(K). \tag{2.8}$$

From (1.29), we set (cf. [19, (2.30)])

$$R^{TX}_K = R^{TX} - 2i\pi\, m^{TX}(K), \quad R^{\mathcal{E}/\mathcal{S}}_K = R^{\mathcal{E}/\mathcal{S}} - 2i\pi\, m^{\mathcal{E}/\mathcal{S}}(K). \tag{2.9}$$

Then R^{TX}_K (resp. $R^{\mathcal{E}/\mathcal{S}}_K$) is called the equivariant curvature of TX (resp. equivariant twisted curvature of $\mathcal{E}$).

Let $Z(g) \subset G$ be the centralizer of $g \in G$ with Lie algebra $\mathfrak{z}(g)$. Then in the sense of the adjoint action,

$$\mathfrak{z}(g) = \{K \in \mathfrak{g} : g.K = K\}. \tag{2.10}$$

We fix $g \in G$ from now on. In the sequel, we always take $K \in \mathfrak{z}(g)$. Put

$$W^K = \{x \in W : K^X(x) = 0\}. \tag{2.11}$$

Then W^K, which is the fixed point set of the group generated by K, is a totally geodesic submanifold along each fiber X. Set

$$W^{g,K} = W^g \cap W^K. \tag{2.12}$$

Then $W^{g,K}$ is also a totally geodesic submanifold along each fiber X. Moreover, if $K_0 \in \mathfrak{z}(g)$ and $z \in \mathbb{R}$, for z small enough, we have

$$W^{g,zK_0} = W^{ge^{zK_0}}. \tag{2.13}$$

Since the G-action is trivial on B, $W^K \to B$ and $W^{g,K} \to B$ are fibrations with compact fiber X^K and $X^{g,K}$. As in (1.25), by an abuse of notation, we will often simply denote by

$$\dim X^{g,K} = \ell'. \tag{2.14}$$

Observe that $m^{TX}(K)|_{X^g}$ acts on TX^g and $N_{X^g/X}$. Also it preserves the splitting (1.31). Let $m^{TX^g}(K)$ and $m^{N(\theta)}(K)$ be the restrictions of $m^{TX}(K)|_{X^g}$ to TX^g and $N(\theta)$. We define the corresponding equivariant curvatures $R_K^{TX^g}$, $R_K^{N(\theta)}$ as in (2.9).

For $K \in \mathfrak{z}(g)$ with $|K|$ small enough, comparing with (1.32), set

$$\widehat{\mathrm{A}}_{g,K}(TX,\nabla^{TX}) = \det{}^{\frac{1}{2}}\left(\frac{\frac{i}{4\pi}R_K^{TX^g}}{\sinh\left(\frac{i}{4\pi}R_K^{TX^g}\right)}\right) \cdot \prod_{k>0}\left(i^{\frac{1}{2}\dim N(\theta)}\det{}^{\frac{1}{2}}\left(1-g\exp\left(\frac{i}{2\pi}R_K^{N(\theta)}\right)\right)\right)^{-1} \in \Omega^{2\bullet}(W^g,\mathbb{C}). \tag{2.15}$$

Note that W compact and $|K|$ small guarantee that the denominator in (2.15) is invertible. Comparing with (1.33), set

$$\mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}}) := \frac{2^{(n-\ell)/2}}{\det^{1/2}(1-g|_{N_{X^g/X}})}\mathrm{Tr}^{\mathcal{E}/\mathcal{S}}\left[\sigma_{n-\ell}(g^{\mathcal{E}})\exp\left(-\frac{R_K^{\mathcal{E}/\mathcal{S}}|_{W^g}}{2i\pi}\right)\right]. \tag{2.16}$$

As in (1.35), if TX has a G-equivariant spin structure, $\mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}})$ is just the equivariant infinitesimal Chern character in [19, Definition 2.7],

$$\mathrm{ch}_{g,K}(E,\nabla^E) = \mathrm{Tr}^E\left[g\exp\left(-\frac{R_K^E|_{W^g}}{2i\pi}\right)\right] \in \Omega^{2\bullet}(W^g,\mathbb{C}), \tag{2.17}$$

where $m^E(K) = \nabla^E_{K^X} - \mathcal{L}_K$, $R_K^E := R^E - 2i\pi m^E(K)$ as in (2.4) and (2.9).

Set

$$d_K = d - 2i\pi\, i_{K^X}. \tag{2.18}$$

Then by (2.6) (cf. [3, Theorem 7.7]),

$$d_K\widehat{\mathrm{A}}_{g,K}(TX,\nabla^{TX}) = 0, \quad d_K\,\mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}}) = 0. \tag{2.19}$$

Recall that $\mathbb{B}_t$ is the rescaled Bismut superconnection in (1.23). Set

$$\mathbb{B}_{K,t} = \mathbb{B}_t + \frac{c(K^X)}{4\sqrt{t}}. \tag{2.20}$$

Then $\mathbb{B}_{K,t}^2$ is a 2nd-order elliptic differential operator along the fiber X acting on $\Lambda(T^*B)\widehat{\otimes}\mathbb{E}$. If the base B is a point, then the operator $\mathbb{B}_{K,t}$ is $\sqrt{t}D + \frac{c(K^X)}{4\sqrt{t}}$, and it was introduced by Bismut [7] in his heat kernel proof of the Kirillov formula for the equivariant index. As observed by Bismut [8, §1d), §3b)] (cf. also [3, §10.7]), its square plus $\mathcal{L}_{K^X}$ is the square of the Bismut superconnection for a fibration with compact structure group, by replacing K^X by the curvature of the fibration. Thus we can roughly interpret $\mathbb{B}_{K,t}$ as the Bismut superconnection by extending our fibration by a fibration with compact structure group.

Now we state the families version of the Kirillov formula and delayed a heat kernel proof of it to Section 5.

Theorem 2.1. *For any $K \in \mathfrak{z}(g)$ and $|K|$ small,*

- *if n is even, for $t > 0$, the differential form*

$$\psi_B\,\mathrm{Tr}_s\left[g\exp\left(-\mathbb{B}_{K,t}^2 - \mathcal{L}_K\right)\right] \in \Omega^{\mathrm{even}}(B,\mathbb{C})$$

is closed, the cohomology class defined by it is independent of t and

$$\lim_{t\to 0}\psi_B \operatorname{Tr}_s\left[g\exp\left(-\mathbb{B}_{K,t}^2-\mathcal{L}_K\right)\right]=\int_{X^g}\widehat{\mathrm{A}}_{g,K}(TX,\nabla^{TX})\,\mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}}). \tag{2.21}$$

- *if n is odd, for $t>0$, the differential form*

$$\psi_B \operatorname{Tr}^{\mathrm{odd}}\left[g\exp\left(-\mathbb{B}_{K,t}^2-\mathcal{L}_K\right)\right]\in\Omega^{\mathrm{odd}}(B,\mathbb{C})$$

is closed, the cohomology class defined by it is independent of t and

$$\lim_{t\to 0}\psi_B \operatorname{Tr}^{\mathrm{odd}}\left[g\exp\left(-\mathbb{B}_{K,t}^2-\mathcal{L}_K\right)\right]=\int_{X^g}\widehat{\mathrm{A}}_{g,K}(TX,\nabla^{TX})\,\mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}}). \tag{2.22}$$

If B is a point and $g=e$, this heat kernel proof of the Kirillov formula is given by Bismut in [7] (see also [3, Theorem 8.2]). If B is a point, (2.21) is established in [19]. For $g=e$, (2.21) is obtained in [55].

2.2. **Equivariant infinitesimal η-forms: Theorem 0.1.** For $t>0$, set

$$\mathcal{B}_{K,t}=\mathbb{B}_{K,t}+dt\wedge\frac{\partial}{\partial t}. \tag{2.23}$$

Then by (2.20),

$$\mathcal{B}_{K,t}^2=\mathbb{B}_{K,t}^2+dt\wedge\frac{\partial\mathbb{B}_{K,t}}{\partial t}=\left(\mathbb{B}_t+\frac{c(K^X)}{4\sqrt{t}}\right)^2+dt\wedge\frac{\partial}{\partial t}\left(\mathbb{B}_t+\frac{c(K^X)}{4\sqrt{t}}\right). \tag{2.24}$$

Theorem 2.2. *There exist $\beta>0$, $\delta,\delta'>0$, $C>0$, such that if $K\in\mathfrak{z}(g)$, $z\in\mathbb{C}$, $|zK|\le\beta$,*
a) for any $t\ge 1$,

$$\left|\left\{\widetilde{\mathrm{Tr}}\Big[g\exp\big(-\mathcal{B}_{zK,t}^2-z\mathcal{L}_K\big)\Big]\right\}^{dt}\right|\le\frac{C}{t^{1+\delta}}; \tag{2.25}$$

b) for any $0<t\le 1$,

$$\left|\left\{\widetilde{\mathrm{Tr}}\Big[g\exp\big(-\mathcal{B}_{zK,t}^2-z\mathcal{L}_K\big)\Big]\right\}^{dt}\right|\le C\,t^{\delta'-1}. \tag{2.26}$$

We delay the proof of Theorem 2.2 to Section 5.

- If $n=\dim X$ is even, then for $t>0$, as $\mathbb{B}_{K,t}$ commutes with g, $\mathcal{L}_K$, by [3, Lemma 9.15],

$$d^B\operatorname{Tr}_s\left[g\exp(-\mathbb{B}_{K,t}^2-\mathcal{L}_K)\right]=\operatorname{Tr}_s\Big[\left[\mathbb{B}_{K,t},g\exp(-\mathbb{B}_{K,t}^2-\mathcal{L}_K)\right]\Big]=0. \tag{2.27}$$

As in (1.39) (cf. [3, Proposition 8.11 and Theorem 9.19]), we have

$$\lim_{t\to+\infty}\psi_B\operatorname{Tr}_s\Big[g\exp\big(-\mathbb{B}_{K,t}^2-\mathcal{L}_K\big)\Big]=\mathrm{ch}_{ge^K}(\mathrm{Ker}(D),\nabla^{\mathrm{Ker}(D)}). \tag{2.28}$$

As in (1.40),

$$\begin{aligned}\frac{\partial}{\partial t}\operatorname{Tr}_s\left[g\exp(-\mathbb{B}_{K,t}^2-\mathcal{L}_K)\right]&=-d^B\operatorname{Tr}_s\left[g\frac{\partial\mathbb{B}_{K,t}}{\partial t}\exp(-\mathbb{B}_{K,t}^2-\mathcal{L}_K)\right]\\&=d^B\left\{\operatorname{Tr}_s\left[g\exp(-\mathcal{B}_{K,t}^2-\mathcal{L}_K)\right]\right\}^{dt}.\end{aligned} \tag{2.29}$$

Thus from (2.29), for $0 < \varepsilon < T < +\infty$,

$$\operatorname{Tr}_s\left[g\exp(-\mathbb{B}_{K,T}^2-\mathcal{L}_K)\right]-\operatorname{Tr}_s\left[g\exp(-\mathbb{B}_{K,\varepsilon}^2-\mathcal{L}_K)\right] = d^B\int_\varepsilon^T\left\{\operatorname{Tr}_s\left[g\exp(-\mathcal{B}_{K,t}^2-\mathcal{L}_K)\right]\right\}^{dt}dt. \tag{2.30}$$

In this case, for $|K|\leq\beta$, by Theorem 2.2, the equivariant infinitesimal η-form is defined by

$$\begin{aligned}\tilde{\eta}_{g,K} &= -\int_0^{+\infty}\frac{1}{2i\sqrt{\pi}}\psi_B\left\{\operatorname{Tr}_s\left[g\exp(-\mathcal{B}_{K,t}^2-\mathcal{L}_K)\right]\right\}^{dt}dt\\ &= \int_0^{+\infty}\frac{1}{2i\sqrt{\pi}}\psi_B\operatorname{Tr}_s\left[g\frac{\partial\mathbb{B}_{K,t}}{\partial t}\exp(-\mathbb{B}_{K,t}^2-\mathcal{L}_K)\right]dt\in\Omega^{\mathrm{odd}}(B,\mathbb{C}).\end{aligned} \tag{2.31}$$

By (2.21), (2.30) and (2.31), we have

$$d^B\tilde{\eta}_{g,K}=\int_{X^g}\widehat{\mathrm{A}}_{g,K}(TX,\nabla^{TX})\operatorname{ch}_{g,K}(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}})-\operatorname{ch}_{ge^K}(\operatorname{Ker}(D),\nabla^{\operatorname{Ker}(D)}). \tag{2.32}$$

• If n is odd, then for $t>0$, as $\mathbb{B}_{K,t}$ commutes with g, $\mathcal{L}_K$, again by the argument in [3, Lemma 9.15],

$$d^B\operatorname{Tr}^{\mathrm{odd}}\left[g\exp(-\mathbb{B}_{K,t}^2-\mathcal{L}_K)\right]=\operatorname{Tr}^{\mathrm{even}}\left[\left[\mathbb{B}_{K,t},g\exp(-\mathbb{B}_{K,t}^2-\mathcal{L}_K)\right]\right]=0. \tag{2.33}$$

As the same argument in (1.45),

$$\lim_{t\to+\infty}\operatorname{Tr}^{\mathrm{odd}}\left[g\exp\left(-\mathbb{B}_{K,t}^2-\mathcal{L}_K\right)\right]=0. \tag{2.34}$$

Comparing with (1.40) and (2.29), we have

$$\begin{aligned}\frac{\partial}{\partial t}\operatorname{Tr}^{\mathrm{odd}}\left[g\exp(-\mathbb{B}_{K,t}^2-\mathcal{L}_K)\right] &= -d^B\operatorname{Tr}^{\mathrm{even}}\left[g\frac{\partial\mathbb{B}_{K,t}}{\partial t}\exp(-\mathbb{B}_{K,t}^2-\mathcal{L}_K)\right]\\ &= d^B\left\{\operatorname{Tr}^{\mathrm{odd}}\left[g\exp(-\mathcal{B}_{K,t}^2-\mathcal{L}_K)\right]\right\}^{dt}.\end{aligned} \tag{2.35}$$

From Theorem 2.2, in this case, for $|K|\leq\beta$, the equivariant infinitesimal η-form is defined by

$$\begin{aligned}\tilde{\eta}_{g,K} &= -\int_0^{+\infty}\frac{1}{\sqrt{\pi}}\psi_B\left\{\operatorname{Tr}^{\mathrm{odd}}\left[g\exp(-\mathcal{B}_{K,t}^2-\mathcal{L}_K)\right]\right\}^{dt}dt\\ &= \int_0^{+\infty}\frac{1}{\sqrt{\pi}}\psi_B\operatorname{Tr}^{\mathrm{even}}\left[g\frac{\partial\mathbb{B}_{K,t}}{\partial t}\exp(-\mathbb{B}_{K,t}^2-\mathcal{L}_K)\right]dt\in\Omega^{\mathrm{even}}(B,\mathbb{C}).\end{aligned} \tag{2.36}$$

As in (1.49), by (2.22), (2.34), (2.35) and (2.36), we get

$$d^B\tilde{\eta}_{g,K}=\int_{X^g}\widehat{\mathrm{A}}_{g,K}(TX,\nabla^{TX})\operatorname{ch}_{g,K}(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}}). \tag{2.37}$$

Definition 2.3. For $K\in\mathfrak{z}(g)$, $|K|\leq\beta$, determined in Theorem 2.2, under Assumption 1.2, the equivariant infinitesimal Bismut-Cheeger η-form is defined by

$$\widetilde{\eta}_{g,K}=-\int_0^{+\infty}\left\{\psi_{\mathbb{R}\times B}\widetilde{\operatorname{Tr}}\Big[g\exp\left(-\mathcal{B}_{K,t}^2-\mathcal{L}_K\right)\Big]\right\}^{dt}dt. \tag{2.38}$$

By (0.5) and (1.36), (2.38) is a reformulation of (2.31) and (2.36). From (2.32) and (2.37), we establish the first part of Theorem 0.1.

Remark that the compactness of B guarantees the existence of the constant $\beta > 0$ in Definition 2.3.

From (2.31) and (2.36), it is obvious that if $K = 0$, $\widetilde{\eta}_{g,K} = \widetilde{\eta}_g$ in (1.50).

From the Duhamel's formula (cf. e.g., [3, Theorem 2.48]), we have

$$\frac{\partial}{\partial \bar{z}} \widetilde{\mathrm{Tr}}\Big[g \exp\big(-\mathcal{B}_{zK,t}^2 - z\mathcal{L}_K\big)\Big] = -\widetilde{\mathrm{Tr}}\left[g \frac{\partial(\mathcal{B}_{zK,t}^2 + z\mathcal{L}_K)}{\partial \bar{z}} \exp\big(-\mathcal{B}_{zK,t}^2 - z\mathcal{L}_K\big)\right] = 0. \tag{2.39}$$

Thus, $\widetilde{\mathrm{Tr}}\Big[g \exp\big(-\mathcal{B}_{zK,t}^2 - z\mathcal{L}_K\big)\Big]$ is $\mathscr{C}^\infty$ on $t > 0$ and holomorphic on $z \in \mathbb{C}$.

We fix $K \in \mathfrak{z}(g)$. Thus for $0 < \varepsilon < T < +\infty$, the function

$$\int_\varepsilon^T \Big\{\psi_{\mathbb{R}\times B} \widetilde{\mathrm{Tr}}\Big[g \exp\big(-\mathcal{B}_{zK,t}^2 - z\mathcal{L}_K\big)\Big]\Big\}^{dt} dt$$

is holomorphic on z. By Theorem 2.2 and the dominated convergence theorem, we have

$$\widetilde{\eta}_{g,zK} := -\int_0^{+\infty} \Big\{\psi_{\mathbb{R}\times B} \widetilde{\mathrm{Tr}}\Big[g \exp\big(-\mathcal{B}_{zK,t}^2 - z\mathcal{L}_K\big)\Big]\Big\}^{dt} dt \tag{2.40}$$

is holomorphic on $z \in \mathbb{C}$, $|zK| < \beta$. Thus we get the last part of Theorem 0.1.

The proof of Theorem 0.1 is completed.

3. Comparison of two equivariant η-forms

In this section, we state our main result. We use the same notations and assumptions in Sections 1 and 2.

Let $\vartheta_K \in T^*X$ be the 1-form which is dual to K^X by the metric g^{TX}, i.e., for any $U \in TX$,

$$\vartheta_K(U) = \langle K^X, U\rangle. \tag{3.1}$$

We identify ϑ_K to a vertical 1-form on W, i.e., to a 1-form which vanishes on T^HW. Then by (2.18) and (3.1), we have

$$d_K\vartheta_K = d\vartheta_K - 2i\pi\, |K^X|^2. \tag{3.2}$$

Let d^X be the exterior differential operator along the fiber X. By (2.5) and (3.1) (cf. [3, Lemma 7.15 (1)]), for $U, U' \in TX$, we have

$$d^X\vartheta_K(U, U') = 2\langle \nabla_U^{TX} K^X, U'\rangle = 2\langle m^{TX}(K)U, U'\rangle. \tag{3.3}$$

From (1.11) and (1.12), set

$$\widetilde{T} = 2T(f_p^H, e_i) f^p \wedge e^i \wedge + \frac{1}{2} T(f_p^H, f_q^H) f^p \wedge f^q \wedge . \tag{3.4}$$

From [3, Proposition 10.1] or [20, (3.61) and (3.94)],

$$d\vartheta_K = d^X\vartheta_K + \langle \widetilde{T}, K^X\rangle = d^X\vartheta_K + \vartheta_K(\widetilde{T}). \tag{3.5}$$

For $K \in \mathfrak{z}(g)$, $|K|$ small, $v > 0$, set

$$\begin{aligned} \alpha_K &= \widehat{\mathrm{A}}_{g,K}(TX, \nabla^{TX})\, \mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) \in \Omega^{2\bullet}\big(W^g, \det N_{X^g/X}\big), \\ \widetilde{e}_v &= -\int_{X^g} \frac{\vartheta_K}{8vi\pi} \exp\left(\frac{d_K\vartheta_K}{8vi\pi}\right) \alpha_K \in \Omega^\bullet(B, \mathbb{C}). \end{aligned} \tag{3.6}$$

Note that if $W^{g,K}=W^g$, as $\vartheta_K=0$ on $W^{g,K}$, we get $\widetilde{e}_v=0$.

Lemma 3.1. *If $W^{g,K}\subsetneq W^g$. Then $\widetilde{e}_v=\mathcal{O}(v^{-1})$ as $v\to+\infty$ and $\widetilde{e}_v=\mathcal{O}(v^{1/2})$ as $v\to 0$.*

Proof. By (3.2) and (3.6), we have

$$\widetilde{e}_v=-\sum_{j=0}^{\lfloor \dim W^g/2\rfloor}\frac{1}{j!}\left(\frac{1}{2i\pi}\right)^{j+1}\int_{X^g}\frac{\vartheta_K}{4v}\left(\frac{d\vartheta_K}{4v}\right)^j\exp\left(-\frac{|K^X|^2}{4v}\right)\cdot\alpha_K. \tag{3.7}$$

Thus when $v\to+\infty$, $\widetilde{e}_v=\mathcal{O}(v^{-1})$.

For $v\to 0$, we follow the argument in the proof of [9, Theorem 1.3]. For $x\in W^g$, if $K^X_x\neq 0$, when $v\to 0$, the integral term in (3.7) at x is of exponential decay. So the integral in (3.7) could be localized on a neighbourhood of $W^{g,K}$.

Let $N_{X^{g,K}/X^g}$ be the normal bundle of $W^{g,K}$ in W^g, and we identify it as the orthogonal complement of $TX^{g,K}=TX^g|_{W^{g,K}}\cap TX^K|_{W^{g,K}}$ in $TX^g|_{W^{g,K}}$. Recall that as K^X is a Killing vector field, for any $b\in B$, $X^{g,K}_b$ is totally geodesic in X^g_b, and as the same argument in Section 2.1, ∇^{TX^g}, $m^{TX}(K)$ preserve the splitting

$$TX^g=TX^{g,K}\oplus N_{X^{g,K}/X^g}\quad\text{ on } W^{g,K} \tag{3.8}$$

and $m^{TX}(K)=0$ on $TX^{g,K}$. In particular,

$$m^{N_{X^{g,K}/X^g}}(K)=m^{TX}(K)|_{N_{X^{g,K}/X^g}}\in\mathrm{End}(N_{X^{g,K}/X^g})\text{ is skew-adjoint and invertible.} \tag{3.9}$$

Combining with (2.6), it implies that $N_{X^{g,K}/X^g}$ is orientable, and we fix an orientation. Then the orientations on TX, $N_{X^{g,K}/X^g}$ induce the identifications over $W^{g,K}$,

$$\det(N_{X^g/X})\simeq\det(TX^g)\simeq\det(TX^{g,K}). \tag{3.10}$$

Given $\varepsilon>0$, let $\mathcal{U}''_\varepsilon$ be the ε-neighborhood of $W^{g,K}$ in $N_{X^{g,K}/X^g}$. There exists ε_0 such that for $0<\varepsilon\leq\varepsilon_0$, the fiberwise exponential map $(y,Z)\in N_{b,X^{g,K}/X^g}\to\exp^X_y(Z)\in X^g_b$ is a diffeomorphism from $\mathcal{U}''_\varepsilon$ into the tubular neighborhood $\mathcal{V}''_\varepsilon$ of $W^{g,K}$ in W^g. We denote $\widetilde{\mathcal{V}}''_\varepsilon$ the fiber of the fibration $\mathcal{V}''_\varepsilon\to B$. With this identification, let $\bar{k}(y,Z)$ be the function such that

$$dv_{X^g}(y,Z)=\bar{k}(y,Z)dv_{X^{g,K}}(y)dv_{N_{X^{g,K}/X^g}}(Z). \tag{3.11}$$

Here $dv_{X^g}\in\Lambda^{\max}(T^*X^g)\otimes\det(T^*X^g)$, $dv_{X^{g,K}}\in\Lambda^{\max}(T^*X^{g,K})\otimes\det(T^*X^{g,K})$ are the Riemannian volume forms of X^g, $X^{g,K}$ and $dv_{N_{X^{g,K}/X^g}}$ is the Euclidean volume form on $N_{X^{g,K}/X^g}$.

Let $e^1,\cdots,e^\ell$ be a locally orthonormal frame of T^*X^g. For $\beta\in\Omega^\bullet\big(W^g,\det(N_{X^g/X})\big)$, let $[\beta]^{\max}$ be the coefficient of $e^1\wedge\cdots\wedge e^\ell\otimes\widehat{e^1\wedge\cdots\wedge e^\ell}$ of β, here $\widehat{e^1\wedge\cdots\wedge e^\ell}$ means the local frame of $\det(N_{X^g/X})$ induced by $e^1\wedge\cdots\wedge e^\ell$ via (3.10). Consider the dilation δ_v, $v>0$, of

$N_{X^{g,K}/X^g}$ by $\delta_v(y,Z)=(y,\sqrt{v}Z)$. We have

$$\begin{aligned}(3.12)\quad &\int_{\widetilde{\mathcal{V}}''_\varepsilon}\frac{\vartheta_K}{4v}\left(\frac{d\vartheta_K}{4v}\right)^j\exp\left(-\frac{|K^X|^2}{4v}\right)\alpha_K\\
&=\int_{X^{g,K}}\int_{Z\in N_{X^{g,K}/X^g},|Z|<\varepsilon}\left[\frac{\vartheta_K|_{(y,Z)}}{4v}\left(\frac{d\vartheta_K|_{(y,Z)}}{4v}\right)^j\exp\left(-\frac{|K^X(y,Z)|^2}{4v}\right)\alpha_K(y,Z)\right]^{\max}\\
&\qquad\qquad\cdot\bar{k}(y,Z)dv_{X^{g,K}}(y)dv_{N_{X^{g,K}/X^g}}(Z)\\
&=\int_{X^{g,K}}\int_{Z\in N_{X^{g,K}/X^g},|Z|<\varepsilon/\sqrt{v}}\left[\frac{\delta_v^*\vartheta_K|_{(y,Z)}}{4v}\left(\frac{d\delta_v^*\vartheta_K|_{(y,Z)}}{4v}\right)^j\exp\left(-\frac{|K^X(y,\sqrt{v}Z)|^2}{4v}\right)\right.\\
&\qquad\qquad\left.\cdot\,\delta_v^*\alpha_K|_{(y,Z)}\right]^{\max}\cdot\bar{k}(y,\sqrt{v}Z)dv_{X^{g,K}}(y)dv_{N_{X^{g,K}/X^g}}(Z).\end{aligned}$$

Let $\nabla^{N_{X^{g,K}/X^g}}$ be the connection on $N_{X^{g,K}/X^g}$ induced by ∇^{TX} as explained after (1.31). Let $\pi_N:N_{X^{g,K}/X^g}\to W^{g,K}$ be the obvious projection. With respect to $\nabla^{N_{X^{g,K}/X^g}}$, we have the canonical splitting of bundles over $N_{X^{g,K}/X^g}$,

$$(3.13)\qquad TN_{X^{g,K}/X^g}=T^HN_{X^{g,K}/X^g}\oplus\pi_N^*N_{X^{g,K}/X^g}.$$

By (1.4) and (3.13), we have

$$(3.14)\qquad T^HN_{X^{g,K}/X^g}\simeq\pi_N^*TW^{g,K}\simeq\pi_N^*(T^HW\oplus TX^{g,K}).$$

On $N_{X^{g,K}/X^g}$, by (3.13) and (3.14), we have

$$\begin{aligned}(3.15)\quad \Lambda(T^*N_{X^{g,K}/X^g})&=\Lambda(T^{H*}N_{X^{g,K}/X^g})\widehat{\otimes}\pi_N^*\Lambda(N^*_{X^{g,K}/X^g})\\
&\simeq\pi_N^*\left(\Lambda(T^*W^{g,K})\widehat{\otimes}\Lambda(N^*_{X^{g,K}/X^g})\right).\end{aligned}$$

For $y\in W^{g,K}$ fixed, we take $Y_1,Y_1'\in T_yW^{g,K}$, $Y^V,Y'^V\in N_{X^{g,K}/X^g,y}$, then $Y=Y_1+Y^V$, $Y'=Y_1'+Y'^V$ are sections of $TN_{X^{g,K}/X^g}$ along $N_{X^{g,K}/X^g,y}$ under our identification (3.13), i.e.,

$$(3.16)\qquad Y_{(y,Z)}=Y_1^H(y,Z)+Y^V,\quad Y'_{(y,Z)}=Y_1'^H(y,Z)+Y'^V.$$

Here $Y_1^H,Y_1'^H\in T^HN_{X^{g,K}/X^g}$ are the lifts of Y_1,Y_1'.

Let θ_0 be the one form on total space $\mathcal{N}$ of $N_{X^{g,K}/X^g}=N_{W^{g,K}/W^g}$ given by

$$(3.17)\quad \theta_0(Y)_{(y,Z)}=\langle m^{TX}(K)Z,Y^V\rangle_y\quad\text{for } Y=Y_1^H+Y^V\in T^HN_{X^{g,K}/X^g}\oplus(\pi_N^*N_{X^{g,K}/X^g}).$$

By [3, Lemma 7.15 (2)], we have

$$(3.18)\qquad \frac{1}{v}\delta_v^*\vartheta_K=\theta_0+\mathcal{O}(v^{1/2}).$$

From (3.18), we get

$$(3.19)\qquad \frac{1}{v}\delta_v^*d\vartheta_K=\frac{1}{v}d\delta_v^*\vartheta_K=d\theta_0+\mathcal{O}(v^{1/2}).$$

As in the argument before [3, p218, Lemma 7.16], by (3.8), we calculate that for $(y,Z)\in N_{X^{g,K}/X^g}$,

$$(3.20)\qquad d\theta_0(Y,Y')_{(y,Z)}=2\langle m^{TX}(K)Y^V,Y'^V\rangle_y-\langle R^{TX}(Y_1^H,Y_1'^H)(m^{TX}(K)Z),Z\rangle_y.$$

By (2.5) and (2.12), for $y \in W^{g,K}$,

$$\frac{1}{v}|K^X(y,\sqrt{v}Z)|^2 = |m^{TX}(K)Z|^2 + \mathcal{O}(v^{1/2}). \tag{3.21}$$

From (3.12), (3.18), (3.19) and (3.21), for any $\alpha \in \Omega^\bullet\big(W^g, \det(N_{X^g/X})\big)$, as $v \to 0$,

$$\begin{aligned}\int_{\widetilde{\mathcal{V}}''_\varepsilon} \frac{\vartheta_K}{4v}\left(\frac{d\vartheta_K}{4v}\right)^j \exp\left(-\frac{|K^X|^2}{4v}\right)\alpha \\ = \int_{X^{g,K}} \alpha_y \int_{Z\in N_{X^{g,K}/X^g}} \frac{\theta_0}{4}\left(\frac{d\theta_0}{4}\right)^j \exp\left(-\frac{|m^{TX}(K)Z|^2}{4}\right) + \mathcal{O}(v^{1/2}).\end{aligned} \tag{3.22}$$

From (3.20), $d\theta_0$ is an even polynomial in Z. However from (3.17), θ_0 is linear in Z. Thus the last integral in (3.22) is zero. Therefore, as $v \to 0$, we have

$$\int_{X^g} \frac{\vartheta_K}{4v}\left(\frac{d\vartheta_K}{4v}\right)^j \exp\left(-\frac{|K^X|^2}{4v}\right)\alpha_K = \mathcal{O}(v^{1/2}). \tag{3.23}$$

The proof of Lemma 3.1 is completed. □

Remark that when B is a point, for $g = e$, Lemma 3.1 is proved in [37, Proposition 2.2].

From Lemma 3.1 and (3.6), the following integral is well-defined,

$$\mathcal{M}_{g,K} := \int_0^{+\infty} \widetilde{e}_v \frac{dv}{v}. \tag{3.24}$$

Proposition 3.2. *For any $K_0 \in \mathfrak{z}(g)$, there exists $\beta > 0$ such that for $K = zK_0$, $-\beta < z < \beta$, we have*

$$\begin{aligned} d^B\mathcal{M}_{g,K} = \int_{X^g} \widehat{\mathrm{A}}_{g,K}(TX,\nabla^{TX})\,\mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}}) \\ - \int_{X^{ge^K}} \widehat{\mathrm{A}}_{ge^K}(TX,\nabla^{TX})\,\mathrm{ch}_{ge^K}(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}}).\end{aligned} \tag{3.25}$$

And there exist $c_j(K) \in \Omega^\bullet(B,\mathbb{C})$ for $1 \le j \le \lfloor(\dim W^g+1)/2\rfloor$ such that $\mathcal{M}_{g,tK}$ is smooth on $|t|<1$, $t \neq 0$ and as $t \to 0$, we have

$$\mathcal{M}_{g,tK} = \sum_{j=1}^{\lfloor(\dim W^g+1)/2\rfloor} c_j(K)t^{-j} + \mathcal{O}(t^0). \tag{3.26}$$

Moreover, $t^{\lfloor(\dim W^g+1)/2\rfloor}\mathcal{M}_{g,tK}$ is real analytic in t for $|t|<1$.

Proof. By (2.18), $d_K^2 = -2i\pi\mathcal{L}_K$, and ϑ_K is K-invariant. We know

$$\frac{\partial}{\partial v}\left(\exp\left(\frac{d_K\vartheta_K}{2vi\pi}\right)\right) = -\frac{1}{v^2}d_K\left(\frac{\vartheta_K}{2i\pi}\exp\left(\frac{d_K\vartheta_K}{2vi\pi}\right)\right). \tag{3.27}$$

We define the corresponding equivariant curvature $R_K^{N_{X^{g,K}/X^g}}$ as in (2.9) via (3.8). By the proof of (3.23) and [9, Theorem 1.3], we know that there exists $C > 0$, such that for any

$v \in (0,1]$, $\alpha \in \Omega^\bullet(W^g, \det(N_{X^g/X})) = \Omega^\bullet(W^g, o(TX^g))$,

$$\begin{aligned}(3.28)\qquad &\left| \int_{X^g} \exp\left(\frac{d_K \vartheta_K}{2vi\pi}\right)\alpha - \int_{X^{g,K}} \frac{i^{-(\ell-\ell')/2}\alpha}{\det^{1/2}\left(R_K^{N_{X^{g,K}/X^g}}/(2i\pi)\right)} \right| \leq C\sqrt{v}\|\alpha\|_{\mathscr{C}^1(W^g)},\\ &\left| \int_{X^g} \frac{\vartheta_K}{2vi\pi}\exp\left(\frac{d_K \vartheta_K}{2vi\pi}\right)\alpha \right| \leq C\sqrt{v}\|\alpha\|_{\mathscr{C}^1(W^g)}.\end{aligned}$$

Let Q_K be the current on W^g such that if $\alpha \in \Omega^\bullet(W^g, \det(N_{X^g/X}))$, then

$$(3.29)\qquad \int_{X^g} Q_K \alpha = -\int_0^{+\infty} \int_{X^g} \frac{\vartheta_K}{2vi\pi}\exp\left(\frac{d_K \vartheta_K}{2vi\pi}\right)\alpha \frac{dv}{v}.$$

From (3.7), the second equation of (3.28), we know (3.29) is well-defined. From (3.27)-(3.29), the following equality of currents on W^g holds (cf. [12, Theorem 1.8]):

$$(3.30)\qquad d_K Q_K = 1 - \frac{i^{-(\ell-\ell')/2}\delta_{W^{g,K}}}{\det^{1/2}\left(R_K^{N_{X^{g,K}/X^g}}/(2i\pi)\right)},$$

where $\delta_{W^{g,K}}$ is the current of integration on $W^{g,K}$. From (3.6), (3.24) and (3.29), we get

$$\begin{aligned}(3.31)\quad \mathcal{M}_{g,K} &= \int_{X^g} Q_K \alpha_K\\ &= -\int_0^{+\infty} \int_{X^g} \frac{\vartheta_K}{2vi\pi}\exp\left(\frac{d_K \vartheta_K}{2vi\pi}\right)\widehat{\mathrm{A}}_{g,K}(TX, \nabla^{TX})\,\mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}})\frac{dv}{v}.\end{aligned}$$

For $x \in W^g$, $K \in \mathfrak{z}(g)$, we have $K^X(x) \in T_x X^g$. From [3, (1.7)], for $\sigma \in \Omega^\bullet(W^g, o(TX^g))$, using the sign convention in (0.15), we have

$$(3.32)\qquad d^B \int_{X^g} \sigma = \int_{X^g} d\sigma = \int_{X^g} d_K \sigma.$$

From (2.12), proceeding as the same calculation in the proof of [3, Theorem 8.2], we get, as elements in $\Omega^\bullet(W^{g,K}, \det(N_{X^g/X}))$,

$$(3.33)\qquad \frac{\widehat{\mathrm{A}}_{g,K}(TX, \nabla^{TX})\,\mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}})}{i^{(\ell-\ell')/2}\det^{1/2}\left(R_K^{N_{X^{g,K}/X^g}}/(2i\pi)\right)} = \widehat{\mathrm{A}}_{ge^K}(TX, \nabla^{TX})\,\mathrm{ch}_{ge^K}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}).$$

As α_K is d_K-closed, by (2.13) and (3.29)-(3.33), we get (3.25).

For $t \neq 0$, by (3.31) and changing the variables $v \mapsto vt^2$, we have

$$(3.34)\qquad \mathcal{M}_{g,tK} = -\int_0^{+\infty} \int_{X^g} \frac{\vartheta_K}{2vi\pi t} \sum_{k=0}^{\lfloor(\dim W^g - 1)/2\rfloor} \left(\frac{(d\vartheta_K)^k}{(2vi\pi t)^k k!}\right)\exp\left(-\frac{|K^X|^2}{v}\right)\alpha_{tK}\frac{dv}{v}.$$

From the arguments in the proof of (3.23), we get (3.26). From (2.15), (2.16) and (3.6), we see that α_{tK} is real analytic on t for $|t| < 1$. Following the proof of (3.23),

$$\int_0^{+\infty} \int_{X^g} \frac{\vartheta_K}{v}\left(\frac{d\vartheta_K}{v}\right)^k \exp\left(-\frac{|K^X|^2}{v}\right)\alpha_{tK}\frac{dv}{v}$$

is uniformly absolutely integrable on v for $|t| < 1$. Thus $t^{\lfloor(\dim W^g + 1)/2\rfloor}\mathcal{M}_{g,tK}$ is real analytic on t for $|t| < 1$.

The proof of Proposition 3.2 is completed. $\square$

From Proposition 3.2, we could state our main result, Theorem 0.2 as follows.

Theorem 3.3. *For any $g \in G$, $K_0 \in \mathfrak{z}(g)$, there exists $\beta > 0$ such that for $K = zK_0$, $-\beta < z < \beta$, $K \neq 0$, we have*

$$\tilde{\eta}_{g,K} = \tilde{\eta}_{ge^K} + \mathcal{M}_{g,K} \in \Omega^\bullet(B,\mathbb{C})/d\Omega^\bullet(B,\mathbb{C}). \tag{3.35}$$

Observe that by (2.40), $\tilde{\eta}_{g,tK}$ is analytic on t for t small. By (3.35), when $t \to 0$, modulo exact forms, the singularity of $\tilde{\eta}_{ge^{tK}}$ is the same as that of $-\mathcal{M}_{g,tK}$ in (3.26).

Note that Theorem 3.3 is compatible with (1.44), (1.49), (2.32), (2.37) and (3.25).

Remark 3.4. For $K \in \mathfrak{z}(g)$, $M = \lfloor (\dim W^g - 1)/2 \rfloor$, on $W^g \setminus \{K^X = 0\}$, we have

$$\begin{aligned} Q_K &= -\sum_{j=0}^{M} \frac{1}{j!}\left(\frac{1}{2i\pi}\right)^{j+1} \int_0^{+\infty} \frac{\vartheta_K}{v}\left(\frac{d\vartheta_K}{v}\right)^j \exp\left(-\frac{|K^X|^2}{v}\right)\frac{dv}{v} \\ &= -\sum_{j=0}^{M} \frac{1}{j!}\left(\frac{1}{2i\pi}\right)^{j+1} \frac{\vartheta_K}{|K^X|^2}\frac{(d\vartheta_K)^j}{|K^X|^{2j}} \int_0^{+\infty} v^j e^{-v} dv \\ &= -\sum_{j=0}^{M} \frac{\vartheta_K (d\vartheta_K)^j}{(2i\pi)^{j+1}|K^X|^{2j+2}} = -\frac{\vartheta_K}{2i\pi|K^X|^2}\left(1 - \frac{d\vartheta_K}{2i\pi|K^X|^2}\right)^{-1}. \end{aligned} \tag{3.36}$$

From (3.18)-(3.21), we know that there exists $C > 0$ such that

$$\left|K^X(y,Z)\right|^2 \geq C|Z|^2, \tag{3.37}$$

and for $Y_1 \in T_y W^{g,K}$,

$$i_{Y_1^H}\vartheta_K = \mathcal{O}(|Z|^3), \quad i_{Y_1^H} d\vartheta_K = \mathcal{O}(|Z|^2). \tag{3.38}$$

From (3.36)-(3.38) and the rank $\ell - \ell'$ of $N_{X^{g,K}/X^g}$ is even, we know that near $W^{g,K}$,

$$Q_K(y,Z) = \mathcal{O}(|Z|^{1-(\ell-\ell')}). \tag{3.39}$$

Thus as a current over W^g, Q_K is in fact locally integrable over W^g and given by (3.36). For $g = e$, and $B = \mathrm{pt}$, this is exactly [37, Proposition 2.2].

Assume now K^X has no zeros, for $t \neq 0$ small enough, by (3.6), (3.24), (3.35) and (3.36), we have

$$\widetilde{\eta}_{g,tK} = \widetilde{\eta}_{ge^{tK}} - \int_{X^g} \frac{\vartheta_K}{2i\pi t|K^X|^2}\left(1 - \frac{d\vartheta_K}{2i\pi t|K^X|^2}\right)^{-1} \alpha_{tK} \in \Omega^\bullet(B,\mathbb{C})/d\Omega^\bullet(B,\mathbb{C}). \tag{3.40}$$

In particular, for $g = e$ and $B = \mathrm{pt}$, (3.40) as Taylor expansion at $t = 0$ is [36, Theorem 0.5].

4. A proof of Theorem 3.3

In this section, we state some intermediate results and prove Theorem 3.3. The proofs of the intermediate results are delayed to Section 6.

4.1. **Some intermediate results.** For $t>0$, $v>0$, set

$$\mathcal{C}_{v,t}=\mathbb{B}_t+\frac{\sqrt{t}c(K^X)}{4}\left(\frac{1}{t}-\frac{1}{v}\right)+dt\wedge\frac{\partial}{\partial t}+dv\wedge\frac{\partial}{\partial v}. \tag{4.1}$$

Then $\mathcal{C}_{v,t}$ is a superconnection associated with the fibration $(\mathbb{R}_+^*)^2\times W\to(\mathbb{R}_+^*)^2\times B$. From the argument in the proof of [3, Theorem 9.17], we have

$$d^{\mathbb{R}^2\times B}\widetilde{\mathrm{Tr}}[g\exp(-\mathcal{C}_{v,t}^2-\mathcal{L}_K)]=0. \tag{4.2}$$

For $\alpha\in\Lambda(T^*(\mathbb{R}^2\times B))$,

$$\alpha=\alpha_0+dv\wedge\alpha_1+dt\wedge\alpha_2+dv\wedge dt\wedge\alpha_3,\quad \alpha_i\in\Lambda(T^*B), i=0,1,2,3, \tag{4.3}$$

as in (0.6), set

$$[\alpha]^{dv}:=\alpha_1,\quad [\alpha]^{dt}:=\alpha_2,\quad [\alpha]^{dv\wedge dt}:=\alpha_3. \tag{4.4}$$

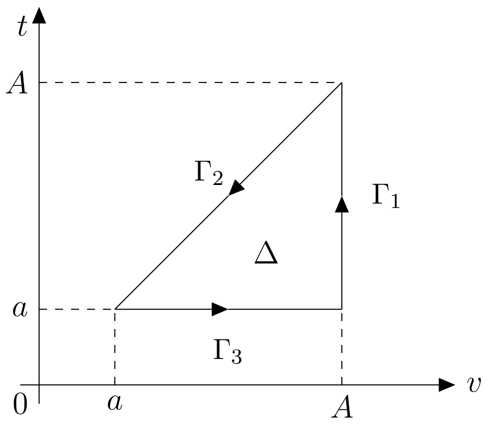
Definition 4.1. We define $\beta_{g,K}$ to be the part of $-\psi_{\mathbb{R}^2\times B}\widetilde{\mathrm{Tr}}[g\exp(-\mathcal{C}_{v,t}^2-\mathcal{L}_K)]$ of degree one with respect to the coordinates (v,t). We denote by

$$\alpha_{g,K}=-\left\{\psi_{\mathbb{R}^2\times B}\widetilde{\mathrm{Tr}}[g\exp(-\mathcal{C}_{v,t}^2-\mathcal{L}_K)]\right\}^{dv\wedge dt}. \tag{4.5}$$

From comparing the coefficient of $dv\wedge dt$ part of (4.2), we have

$$\left(dv\wedge\frac{\partial}{\partial v}+dt\wedge\frac{\partial}{\partial t}\right)\beta_{g,K}=-dv\wedge dt\wedge d^B\alpha_{g,K}. \tag{4.6}$$

Take a, A, $0<a\le 1\le A<+\infty$. Let $\Gamma=\Gamma_{a,A}$ be the oriented contour in $\mathbb{R}_{+,v}\times\mathbb{R}_{+,t}$:



The contour Γ is made of three oriented pieces $\Gamma_1,\Gamma_2,\Gamma_3$ indicated in the above picture. For $1\le k\le 3$, set $I_k^0=\int_{\Gamma_k}\beta_{g,K}$. Also Γ bounds an oriented triangular domain Δ.

By Stocks' formula and (4.6),

$$\sum_{k=1}^3 I_k^0=\int_{\partial\Delta}\beta_{g,K}=\int_\Delta\left(dv\wedge\frac{\partial}{\partial v}+dt\wedge\frac{\partial}{\partial t}\right)\beta_{g,K}=-d^B\left(\int_\Delta\alpha_{g,K}dv\wedge dt\right). \tag{4.7}$$

The proof of the following theorem is left to Section 5.11.

Theorem 4.2. *For $K\in\mathfrak{z}(g)$, $|K|$ small enough, there exist $\delta>0$, $C>0$ such that for any $t\ge 1$, $v\ge t$, we have*

$$\left|[\beta_{g,K}(v,t)]^{dt}\right|\le\frac{C}{t^{1+\delta}}. \tag{4.8}$$

For $\alpha \in \Omega^j(B,\mathbb{C})$, we define

$$\phi(\alpha) := \{\psi_{\mathbb{R}\times B}(dv\wedge\alpha)\}^{dv} = \begin{cases} \pi^{-\frac{1}{2}}\,(2i\pi)^{-\frac{j}{2}}\cdot\alpha & \text{if } j \text{ is even;}\\ (2i\pi)^{-\frac{j+1}{2}}\cdot\alpha & \text{if } j \text{ is odd.}\end{cases} \tag{4.9}$$

Comparing with (1.27), we set

$$\widetilde{\mathrm{Tr}}' = \begin{cases} \mathrm{Tr}_s & \text{if } n \text{ is even;}\\ \mathrm{Tr}^{\mathrm{even}} & \text{if } n \text{ is odd.}\end{cases} \tag{4.10}$$

For $0 < t \le v$, set

$$\mathcal{B}_{K,t,v} = \left(\mathbb{B}_t + \frac{\sqrt{t}c(K^X)}{4}\left(\frac{1}{t}-\frac{1}{v}\right)\right)^2 + \mathcal{L}_K. \tag{4.11}$$

Then by Definition 4.1, (4.1) and (4.11), we have

$$\begin{aligned}
&[\beta_{g,K}(v,t)]^{dt}\\
&\quad = -\left\{\psi_{\mathbb{R}_t\times B}\widetilde{\mathrm{Tr}}\left[g\exp\left(-\mathcal{B}_{K,t,v} - dt\wedge\frac{\partial}{\partial t}\left(\mathbb{B}_t + \frac{\sqrt{t}c(K^X)}{4}\left(\frac{1}{t}-\frac{1}{v}\right)\right)\right)\right]\right\}^{dt}\\
&\quad = \phi\widetilde{\mathrm{Tr}}'\left[g\frac{\partial}{\partial t}\left(\mathbb{B}_t + \frac{\sqrt{t}c(K^X)}{4}\left(\frac{1}{t}-\frac{1}{v}\right)\right)\exp\left(-\mathcal{B}_{K,t,v}\right)\right],\\
&[\beta_{g,K}(v,t)]^{dv} = -\left\{\psi_{\mathbb{R}_v\times B}\widetilde{\mathrm{Tr}}\left[g\exp\left(-\mathcal{B}_{K,t,v} - dv\frac{\sqrt{t}c(K^X)}{4v^2}\right)\right]\right\}^{dv}\\
&\quad = \phi\widetilde{\mathrm{Tr}}'\left[g\frac{\sqrt{t}c(K^X)}{4v^2}\exp\left(-\mathcal{B}_{K,t,v}\right)\right].
\end{aligned} \tag{4.12}$$

Thus as $\mathcal{B}_{K,t,t} = \mathbb{B}_t^2 + \mathcal{L}_K$, by (4.12), on Γ_2, we have

$$\begin{aligned}
\beta_{g,K}(v,t) &= dt\wedge\phi\widetilde{\mathrm{Tr}}'\left[g\frac{\partial\mathbb{B}_t}{\partial t}\exp\left(-\mathbb{B}_t^2 - \mathcal{L}_K\right)\right]\\
&= -dt\wedge\left\{\psi_{\mathbb{R}\times B}\,\widetilde{\mathrm{Tr}}\left[g\exp\left(-\left(\mathbb{B}_t + dt\wedge\frac{\partial}{\partial t}\right)^2 - \mathcal{L}_K\right)\right]\right\}^{dt}.
\end{aligned} \tag{4.13}$$

In the remainder of this section, we use Theorem 4.2 and the following estimates to prove Theorem 3.3. The proofs of these estimates are delayed to Section 6. Recall that $\widetilde{e}_v$ is defined in (3.6).

Theorem 4.3. *For $K_0 \in \mathfrak{z}(g)$, there exists $\beta > 0$ such that for $K = zK_0$, $-\beta < z < \beta$, $K \neq 0$,*
a) when $t \to 0$,

$$\phi\widetilde{\mathrm{Tr}}'\left[g\frac{\sqrt{t}c(K^X)}{4v}\exp\left(-\mathcal{B}_{K,t,v}\right)\right] \to -\widetilde{e}_v; \tag{4.14}$$

b) there exist $C > 0$, $\delta \in (0,1]$, such that for $t \in (0,1]$, $v \in [t,1]$,

$$\left|\phi\widetilde{\mathrm{Tr}}'\left[g\frac{\sqrt{t}c(K^X)}{4v}\exp\left(-\mathcal{B}_{K,t,v}\right)\right] + \widetilde{e}_v\right| \le C\left(\frac{t}{v}\right)^{\delta}; \tag{4.15}$$

c) there exists $C>0$ such that for $t\in(0,1]$, $v\geq 1$,

$$\left|\widetilde{\mathrm{Tr}}'\left[g\frac{\sqrt{t}c(K^X)}{4v}\exp\left(-\mathcal{B}_{K,t,v}\right)\right]\right|\leq\frac{C}{v};\tag{4.16}$$

d) for $v\geq 1$,

$$\lim_{t\to0}\widetilde{\mathrm{Tr}}'\left[g\frac{c(K^X)}{4\sqrt{tv}}\exp\left(-\mathcal{B}_{K,t,tv}\right)\right]=0.\tag{4.17}$$

4.2. **A proof of Theorem 3.3.** We now finish the proof of Theorem 3.3 by using Theorems 4.2 and 4.3. By (4.7), we know that $I_1^0+I_2^0+I_3^0$ is an exact form on B. We take the limits $A\to+\infty$ and then $a\to0$ in the indicated order. We claim that the limit of the part $I_j^0(A,a)$ as $A\to+\infty$ exists, denoted by $I_j^1(a)$, and the limit of $I_j^1(a)$ as $a\to0$ exists, denoted by I_j^2 for $j=1,2,3$.

i) By (4.11) and (4.12), $[\beta_{g,K}(v,t)]^{dt}$ is uniformly bounded for $v\geq1$, $t\in I$, a compact interval $I\subset(0,+\infty)$, and

$$\lim_{v\to+\infty}[\beta_{g,K}(v,t)]^{dt}=[\beta_{g,K}(+\infty,t)]^{dt}.\tag{4.18}$$

From Theorem 4.2, (2.24), (4.12) and the dominated convergence theorem, we see that

$$I_1^1(a)=\lim_{A\to+\infty}\int_a^A[\beta_{g,K}(A,t)]^{dt}dt=-\int_a^{+\infty}\left\{\psi_{\mathbb{R}\times B}\widetilde{\mathrm{Tr}}\Big[g\exp\left(-\mathcal{B}_{K,t}^2-\mathcal{L}_K\right)\Big]\right\}^{dt}dt.\tag{4.19}$$

Thus by Theorem 2.2 and Definition 2.3, we have

$$I_1^2=-\int_0^{+\infty}\left\{\psi_{\mathbb{R}\times B}\widetilde{\mathrm{Tr}}\Big[g\exp\left(-\mathcal{B}_{K,t}^2-\mathcal{L}_K\right)\Big]\right\}^{dt}dt=\widetilde{\eta}_{g,K}.\tag{4.20}$$

ii) From Definition 1.4, (2.3) and (4.13), we have

$$I_2^2=\int_0^{+\infty}\left\{\psi_{\mathbb{R}\times B}\,\widetilde{\mathrm{Tr}}\left[g\exp\left(-\left(\mathbb{B}_t+dt\wedge\frac{\partial}{\partial t}\right)^2-\mathcal{L}_K\right)\right]\right\}^{dt}dt=-\widetilde{\eta}_{ge^K}.\tag{4.21}$$

iii) For the term $I_3^0(A,a)$, set

$$\begin{aligned}J_1&=-\int_a^1\widetilde{e}_v\frac{dv}{v},\\ J_2&=\int_1^{+\infty}\phi\widetilde{\mathrm{Tr}}'\left[g\frac{\sqrt{a}c(K^X)}{4v}\exp\left(-\mathcal{B}_{K,a,v}\right)\right]\frac{dv}{v},\\ J_3&=\int_1^{1/a}\left(\phi\widetilde{\mathrm{Tr}}'\left[g\frac{c(K^X)}{4\sqrt{av}}\exp\left(-\mathcal{B}_{K,a,av}\right)\right]+\widetilde{e}_{av}\right)\frac{dv}{v}.\end{aligned}\tag{4.22}$$

Clearly, by Theorem 4.3 c) and (4.12), we have

$$I_3^1(a)=J_1+J_2+J_3.\tag{4.23}$$

By (4.14), (4.16) and (4.22), from the dominated convergence theorem, we find that as $a\to0$,

$$J_2\to J_2^1=-\int_1^{+\infty}\widetilde{e}_v\frac{dv}{v}.\tag{4.24}$$

By (4.15), there exist $C > 0$, $\delta \in (0, 1]$ such that for $a \in (0, 1]$, $1 \le v \le 1/a$,

$$\left| \phi \widetilde{\mathrm{Tr}}' \left[g \frac{c(K^X)}{4\sqrt{av}} \exp\left(-\mathcal{B}_{K,a,av}\right) \right] + \widetilde{e}_{av} \right| \le \frac{C}{v^\delta}. \tag{4.25}$$

Using Lemma 3.1, (4.17), (4.22), (4.25), and the dominated convergence theorem, as $a \to 0$,

$$J_3 \to J_3^1 = 0. \tag{4.26}$$

By (3.24), (4.22)-(4.24) and (4.26), we have

$$I_3^2 = -\int_0^{+\infty} \widetilde{e}_v \frac{dv}{v} = -\mathcal{M}_{g,K}. \tag{4.27}$$

By [30, §22, Theorem 17], $d\Omega^\bullet(B, \mathbb{C})$ is closed under the uniformly convergence. Thus, by (4.7),

$$\sum_{j=1}^{3} I_j^2 \equiv 0 \bmod d\Omega^\bullet(B, \mathbb{C}). \tag{4.28}$$

By (4.20), (4.21), (4.27) and (4.28), the proof of Theorem 3.3 is completed.

5. Construction of the equivariant infinitesimal η-forms

In this section, we prove Theorems 2.2 and 4.2 following the lines of [19, §7] and give a heat kernel proof of the family Kirillov formula, Theorem 2.1. For the convenience to compare the arguments in this section with those in [19], especially how the extra terms for the family version appear, the structure of this section is formulated almost the same as in [19, §7].

This section is organized as follows. In Section 5.1, we prove Theorem 2.2 a). In Sections 5.2-5.10, we give proofs of Theorems 2.1 and 2.2 b). In Section 5.11, we prove Theorem 4.2.

5.1. The behaviour of the trace as $t \to +\infty$.

Set

$$C_{K,t} = \mathbb{B}_t + \frac{c(K^X)}{4\sqrt{t}} + t \cdot dt \wedge \frac{\partial}{\partial t}. \tag{5.1}$$

For $z \in \mathbb{C}$, we denote by

$$\mathcal{A}_{zK,t} := C_{zK,t}^2 + z\mathcal{L}_K. \tag{5.2}$$

Then Theorem 2.2 a) is implied by the following estimate.

Theorem 5.1. *For $\beta > 0$ fixed, there exist $C > 0$, $\delta > 0$ such that if $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \le \beta$, $t \ge 1$,*

$$\left| \left\{ \widetilde{\mathrm{Tr}}[g \exp(-\mathcal{A}_{zK,t})] \right\}^{dt} \right| \le \frac{C}{t^\delta}. \tag{5.3}$$

Proof. This subsection is devoted to the proof of Theorem 5.1. □

In this subsection, we fix $\beta > 0$. The constants in this subsection may depend on β.

For $b \in B$, recall that $\mathbb{E}_b$ is the vector space of the smooth sections of $\mathcal{E}$ on X_b. For $\mu \in \mathbb{R}$, let $\mathbb{E}_b^\mu$ be the Sobolev spaces of the order μ of sections of $\mathcal{E}$ on X_b. We equip $\mathbb{E}_b^0$ by the Hermitian product $\langle\ ,\ \rangle_0$ in (1.17). Let $\|\cdot\|_0$ be the corresponding norm of $\mathbb{E}_b^0$. For $\mu \in \mathbb{Z}$, let $\|\cdot\|_\mu$ be the Sobolev norm of $\mathbb{E}_b^\mu$ induced by ∇^{TX} and $\nabla^{\mathcal{E}}$.

Recall that we assume that the kernels $\mathrm{Ker}(D)$ form a vector bundle over B. We denote by P the orthogonal projection from $\mathbb{E}^0$ to $\mathrm{Ker}(D)$ and let $P^\perp = 1 - P$.

Recall that $P^{TX} : TW = T^H W \oplus TX \to TX$ is the projection defined by (1.4). For $s, s' \in \mathbb{E}$, $t \geq 1$, we set

$$
\begin{aligned}
|s|^2_{t,0} &: = \|s\|^2_0\,, \\
|s|^2_{t,1} &: = \|Ps\|^2_0 + t\|P^\perp s\|^2_0 + t\|\nabla^{\mathcal{E}}_{P^{TX}\cdot} P^\perp s\|^2_0\,.
\end{aligned}
\tag{5.4}
$$

Set

$$
|s|_{t,-1} = \sup_{0\neq s'\in\mathbb{E}^1} \frac{|\langle s, s'\rangle_0|}{|s'|_{t,1}}.
\tag{5.5}
$$

Then (5.4) and (5.5) define Sobolev norms on $\mathbb{E}^1$ and $\mathbb{E}^{-1}$. Since $\nabla^{\mathcal{E}}_{P^{TX}\cdot} P$ is an operator along the fiber X with smooth kernel, we know that $|\cdot|_{t,1}$ (resp. $|\cdot|_{t,-1}$) is equivalent to $\|\cdot\|_1$ (resp. $\|\cdot\|_{-1}$) on $\mathbb{E}^1$ (resp. $\mathbb{E}^{-1}$).

Let $\mathcal{A}^{(0)}_{zK,t}$ be the piece of $\mathcal{A}_{zK,t}$ which has degree 0 in $\Lambda(T^*(\mathbb{R}\times B))$.

Lemma 5.2. *There exist $c_1, c_2, c_3, c_4 > 0$, such that for any $t \geq 1$, $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $s, s' \in \mathbb{E}$,*

$$
\begin{aligned}
\mathrm{Re}\left\langle \mathcal{A}^{(0)}_{zK,t} s, s\right\rangle_0 &\geq c_1|s|^2_{t,1} - c_2|s|^2_{t,0}, \\
\left|\mathrm{Im}\left\langle \mathcal{A}^{(0)}_{zK,t} s, s\right\rangle_0\right| &\leq c_3|s|_{t,1}|s|_{t,0}, \\
\left|\left\langle \mathcal{A}^{(0)}_{zK,t} s, s'\right\rangle_0\right| &\leq c_4|s|_{t,1}|s'|_{t,1}.
\end{aligned}
\tag{5.6}
$$

Proof. From (1.23), (5.1) and (5.2), we have

$$
\mathcal{A}^{(0)}_{zK,t} = tD^2 + \frac{z}{4}\left[D, c(K^X)\right] - z^2\frac{|K^X|^2}{16t} + z\mathcal{L}_K.
\tag{5.7}
$$

So we have

$$
\begin{aligned}
\mathrm{Re}\langle \mathcal{A}^{(0)}_{zK,t} s, s\rangle_0 &= \left\langle \left(tD^2 + \mathrm{Im}(z)i\left(\frac{1}{4}\left[D, c(K^X)\right] + \mathcal{L}_K\right) - \mathrm{Re}(z^2)\frac{|K^X|^2}{16t}\right) s, s\right\rangle_0, \\
\mathrm{Im}\langle \mathcal{A}^{(0)}_{zK,t} s, s\rangle_0 &= \left\langle \left(-\mathrm{Re}(z)i\left(\frac{1}{4}\left[D, c(K^X)\right] + \mathcal{L}_K\right) - \mathrm{Im}(z^2)\frac{|K^X|^2}{16t}\right) s, s\right\rangle_0.
\end{aligned}
\tag{5.8}
$$

From (5.4), there exist $c_1', c_2', c_3', c_4' > 0$ such that for any $t \geq 1$, $|zK| \leq \beta$, $\epsilon > 0$,

$$
\begin{aligned}
\left\langle \left(tD^2 - \mathrm{Re}(z^2)\frac{|K^X|^2}{16t}\right) s, s\right\rangle_0 &\geq c_1'|s|^2_{t,1} - c_2'|s|^2_{t,0}, \\
\left|\left\langle \frac{\mathrm{Im}(z)}{4}\left[D, c(K^X)\right] s, s\right\rangle_0\right| &\leq c_3'|s|_{t,1}|s|_{t,0} \leq c_3'\epsilon|s|^2_{t,1} + \frac{c_3'}{4\epsilon}|s|^2_{t,0}, \\
|\langle |z|\mathcal{L}_K s, s\rangle_0| &\leq c_4'|s|_{t,1}|s|_{t,0} \leq c_4'\epsilon|s|^2_{t,1} + \frac{c_4'}{4\epsilon}|s|^2_{t,0}.
\end{aligned}
\tag{5.9}
$$

By taking $\epsilon = \min\{c_1'/(4c_3'), c_1'/(4c_4')\}$, from (5.8), we get the first estimate of (5.6).

The other estimates in (5.6) follow directly from (5.4) and (5.8).

The proof of Lemma 5.2 is completed. □

By using Lemma 5.2 and exactly the same argument in [21, Theorem 11.27], we get

Lemma 5.3. *There exist $c, C > 0$, such that if $t \geq 1$, $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \leq \beta$,*

$$\lambda \in U_c := \left\{ \lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq \frac{\mathrm{Im}(\lambda)^2}{4c^2} - c^2 \right\}, \tag{5.10}$$

the resolvent $(\lambda - \mathcal{A}^{(0)}_{zK,t})^{-1}$ exists, and moreover for any $s \in \mathbb{E}$,

$$\begin{aligned} |(\lambda - \mathcal{A}^{(0)}_{zK,t})^{-1} s|_{t,0} &\leq C|s|_{t,0}, \\ |(\lambda - \mathcal{A}^{(0)}_{zK,t})^{-1} s|_{t,1} &\leq C(1+|\lambda|)^2 |s|_{t,-1}. \end{aligned} \tag{5.11}$$

The following lemma is the analogue of [11, Theorem 9.15].

Lemma 5.4. *There exist $C > 0, k \in \mathbb{N}$, such that for $t \geq 1$, $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $\lambda \in U_c$, with c in Lemma 5.3, the resolvent $(\lambda - \mathcal{A}_{zK,t})^{-1}$ exists, extends to a continuous linear operator from $\Lambda(T^*(\mathbb{R} \times B)) \otimes \mathbb{E}^{-1}$ into $\Lambda(T^*(\mathbb{R} \times B)) \otimes \mathbb{E}^1$, and moreover for $s \in \mathbb{E}$,*

$$|(\lambda - \mathcal{A}_{zK,t})^{-1} s|_{t,1} \leq C(1+|\lambda|)^k |s|_{t,-1}. \tag{5.12}$$

Proof. From (1.1), (1.23), (5.1) and (5.2),

$$\begin{aligned} \mathcal{A}_{zK,t} - \mathcal{A}^{(0)}_{zK,t} = \sqrt{t}\Big([D, \nabla^{\mathbb{E},u}] + \frac{1}{2} dt \wedge D \Big) + \big(\nabla^{\mathbb{E},u}\big)^2 - \frac{1}{4}[D, c(T^H)] \\ + \frac{1}{8\sqrt{t}} \Big(2[\nabla^{\mathbb{E},u}, zc(K^X) - c(T^H)] - dt \wedge (zc(K^X) - c(T^H)) \Big) \\ + \frac{1}{16t}\Big(2z\langle K^X, T^H\rangle + c(T^H)^2 \Big). \end{aligned} \tag{5.13}$$

By [8, Theorem 2.5], $[D, \nabla^{\mathbb{E},u}]$ and $\big(\nabla^{\mathbb{E},u}\big)^2$ are first order differential operators along the fiber. From $P[D, \nabla^{\mathbb{E},u}]P = 0$, we get

$$\Big| \langle \sqrt{t}[D, \nabla^{\mathbb{E},u}]s, s'\rangle_0 \Big| \leq C(|s|_{t,0}|s'|_{t,1} + |s|_{t,1}|s'|_{t,0}). \tag{5.14}$$

By (5.13) and (5.14), there exists $C' > 0$ such that for any $t \geq 1$, we have

$$|(\mathcal{A}_{zK,t} - \mathcal{A}^{(0)}_{zK,t})s|_{t,-1} \leq C'|s|_{t,1}. \tag{5.15}$$

Take $\lambda \in U_c$. Then since $\mathcal{A}_{zK,t} - \mathcal{A}^{(0)}_{zK,t}$ has positive degree in $\Lambda(T^*(\mathbb{R} \times B))$, we have

$$(\lambda - \mathcal{A}_{zK,t})^{-1} = \sum_{m=0}^{1+\dim B} (\lambda - \mathcal{A}^{(0)}_{zK,t})^{-1} \Big((\mathcal{A}_{zK,t} - \mathcal{A}^{(0)}_{zK,t})(\lambda - \mathcal{A}^{(0)}_{zK,t})^{-1} \Big)^m. \tag{5.16}$$

Therefore, by (5.11), (5.15) and (5.16), we obtain (5.12).

The proof of Lemma 5.4 is completed. □

Proposition 5.5. *There exists $C > 0$, such that for $t \geq 1$, $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $s \in \mathbb{E}$,*

$$\Big\| \Big(\exp(-\mathcal{A}_{zK,t}) - \exp(-\mathbb{B}^2_{zK,t} - z\mathcal{L}_K) \Big) s \Big\|_0 \leq \frac{C}{\sqrt{t}} \|s\|_0. \tag{5.17}$$

Proof. From (5.4) and (5.5), we know for $s \in \mathbb{E}$,

$$|P^\perp s|_{t,-1} = \sup_{0 \neq s' \in \mathbb{E}^1,\, Ps'=0} \frac{|\langle P^\perp s, s'\rangle_0|}{|s'|_{t,1}} = \frac{1}{\sqrt{t}} \|P^\perp s\|_{-1} \leq \frac{1}{\sqrt{t}} \|P^\perp s\|_0. \tag{5.18}$$

Note that from (2.20), (5.1) and (5.2), we have

$$\mathcal{A}_{zK,t} = \mathbb{B}^2_{zK,t} + z\mathcal{L}_K + dt \wedge \left(\frac{1}{2}\sqrt{t}D - \frac{1}{8\sqrt{t}}\big(zc(K^X) - c(T^H)\big) \right). \tag{5.19}$$

Thus $\mathbb{B}^2_{zK,t} + z\mathcal{L}_K$ has the same spectrum as $\mathcal{A}_{zK,t}$ and by omitting dt part, we know Lemma 5.4 holds for $\mathbb{B}^2_{zK,t} + z\mathcal{L}_K$. Thus from (5.12) and (5.18), for $\lambda \in U_c$, we have

$$\begin{aligned} \left\| (\lambda - \mathcal{A}_{zK,t})^{-1}\sqrt{t}D\Big(\lambda - (\mathbb{B}^2_{zK,t} + z\mathcal{L}_K)\Big)^{-1} s \right\|_0 &\leq \frac{C}{\sqrt{t}}(1+|\lambda|)^k \left\| \sqrt{t}DP^{\perp}\Big(\lambda - (\mathbb{B}^2_{zK,t} + z\mathcal{L}_K)\Big)^{-1} s \right\|_0 \\ &\leq \frac{C}{\sqrt{t}}(1+|\lambda|)^k \left| \Big(\lambda - (\mathbb{B}^2_{zK,t} + z\mathcal{L}_K)\Big)^{-1} s \right|_{t,1} \\ &\leq \frac{C^2}{\sqrt{t}}(1+|\lambda|)^{2k}|s|_{t,-1} \leq \frac{C^2}{\sqrt{t}}(1+|\lambda|)^{2k}\|s\|_0. \end{aligned} \tag{5.20}$$

Note that

$$\exp(-\mathcal{A}_{zK,t}) = \frac{1}{2i\pi}\int_{\partial U_c} e^{-\lambda}(\lambda - \mathcal{A}_{zK,t})^{-1} d\lambda, \tag{5.21}$$

and (5.21) also holds for $\mathbb{B}^2_{zK,t} + z\mathcal{L}_K$. From (5.19),

$$\begin{aligned} &(\lambda - \mathcal{A}_{zK,t})^{-1} - (\lambda - (\mathbb{B}^2_{zK,t} + z\mathcal{L}_K))^{-1} \\ &= (\lambda - \mathcal{A}_{zK,t})^{-1} \cdot \left(dt \wedge \left(\frac{1}{2}\sqrt{t}D - \frac{1}{8\sqrt{t}}\big(zc(K^X) - c(T^H)\big) \right) \right) \cdot (\lambda - (\mathbb{B}^2_{zK,t} + z\mathcal{L}_K))^{-1}. \end{aligned} \tag{5.22}$$

Now from (5.20)-(5.22), we get (5.17). The proof of Proposition 5.5 is completed. □

Since B is compact, there exists a family of smooth sections of TX, $U_1, \cdots, U_m$ such that for any $x \in W$, $U_1(x), \cdots, U_m(x)$ spans T_xX.

Let $\mathcal{D}$ be a family of operators on $\mathbb{E}$,

$$\mathcal{D} = \left\{ P^{\perp}\, \nabla^{\mathcal{E}}_{U_i} P^{\perp} \right\}. \tag{5.23}$$

From (5.7) and (5.13), by the same argument as the proof of [21, Proposition 11.29] (see also e.g., [11, Theorem 9.17], [39, Lemma 5.17]), we get the following lemma.

Lemma 5.6. *For any $k \in \mathbb{N}$ fixed, there exists $C_k > 0$ such that for $t \geq 1$, $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $Q_1, \cdots, Q_k \in \mathcal{D}$ and $s, s' \in \mathbb{E}$, we have*

$$|\langle [Q_1, [Q_2, \cdots [Q_k, \mathcal{A}_{zK,t}], \cdots]] s, s' \rangle_0| \leq C_k |s|_{t,1} |s'|_{t,1}. \tag{5.24}$$

For $k \in \mathbb{N}$, let $\mathcal{D}^k$ be the family of operators Q which can be written in the form

$$Q = Q_1 \cdots Q_k, \quad Q_i \in \mathcal{D}. \tag{5.25}$$

If $k \in \mathbb{N}$, we define the Hilbert norm $\|\cdot\|'_k$ by

$$\|s\|'^2_k = \sum_{\ell=0}^{k} \sum_{Q \in \mathcal{D}^\ell} \|Qs\|_0^2. \tag{5.26}$$

Since $P\nabla^{\mathcal{E}}_{PTX\cdot}$ and $\nabla^{\mathcal{E}}_{PTX\cdot}P$ are operators along the fiber with smooth kernels, the Sobolev norm $\|\cdot\|'_k$ is equivalent to the Sobolev norm $\|\cdot\|_k$. Thus, we also denote the Sobolev space with respect to $\|\cdot\|'_k$ by $\mathbb{E}^k$.

By using Lemma 5.6, as the proof of [21, Theorem 11.30], we get

Lemma 5.7. *For any $m \in \mathbb{N}$, there exist $p_m \in \mathbb{N}$ and $C_m > 0$ such that for $t \geq 1$, $\lambda \in U_c$, $s \in \mathbb{E}$,*

$$\|(\lambda - \mathcal{A}_{zK,t})^{-1}s\|'_{m+1} \leq C_m(1+|\lambda|)^{p_m}\|s\|'_m. \tag{5.27}$$

Let $\exp(-\mathcal{A}_{zK,t})(x,x')$, $\exp(-\mathbb{B}^2_{zK,t} - z\mathcal{L}_K)(x,x')$ be the smooth kernels of the operators $\exp(-\mathcal{A}_{zK,t})$, $\exp(-\mathbb{B}^2_{zK,t} - z\mathcal{L}_K)$ associated with $dv_X(x')$. By using Lemma 5.7, following the same progress as in the proof of [21, Theorem 11.31], we get

Proposition 5.8. *For $m \in \mathbb{N}$, there exists $C > 0$, such that for $b \in B$, $x, x' \in X_b$, $t \geq 1$, $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \leq \beta$,*

$$\sup_{|\alpha|,|\alpha'|\leq m}\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial x^{\alpha}\partial x'^{\alpha'}}\exp(-\mathcal{A}_{zK,t})(x,x')\right| \leq C. \tag{5.28}$$

By omitting dt part, we know Proposition 5.8 holds for $\exp(-\mathbb{B}^2_{zK,t} - z\mathcal{L}_K)(x,x')$. From Propositions 5.5, 5.8 and (5.19), by the arguments in [21, §11 p)], there exist $C > 0$, $\delta > 0$, such that for $t \geq 1$, $K \in \mathfrak{g}$, $|zK| \leq \beta$,

$$\left|\exp\left(-\mathcal{A}_{zK,t}\right)(x,x') - \exp(-\mathbb{B}^2_{zK,t} - z\mathcal{L}_K)(x,x')\right| \leq \frac{C}{t^{\delta}}. \tag{5.29}$$

From (5.19),

$$dt \wedge \left\{\widetilde{\mathrm{Tr}}\left[g\exp\left(-\mathcal{A}_{zK,t}\right)\right]\right\}^{dt} = \widetilde{\mathrm{Tr}}\left[g\Big(\exp\left(-\mathcal{A}_{zK,t}\right) - \exp(-\mathbb{B}^2_{zK,t} - z\mathcal{L}_K)\Big)\right]. \tag{5.30}$$

From (5.28) and (5.30), we get Theorem 5.1.

5.2. **A proof of Theorems 2.1 and 2.2 b).** Section 5.3 is devoted to the proof of the following theorem.

Theorem 5.9. *There exist $\beta > 0$, $C > 0$, $0 < \delta \leq 1$ such that if $K \in \mathfrak{z}(g)$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $0 < t \leq 1$,*

$$\left|\psi_{\mathbb{R}\times B}\widetilde{\mathrm{Tr}}\left[g\exp\left(-\mathcal{A}_{zK,t}\right)\right] - \int_{X^g}\widehat{\mathrm{A}}_{g,zK}(TX,\nabla^{TX})\ \mathrm{ch}_{g,zK}(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}})\right| \leq Ct^{\delta}. \tag{5.31}$$

Since $\int_{X^g}\widehat{\mathrm{A}}_{g,zK}(TX,\nabla^{TX})\ \mathrm{ch}_{g,zK}(\mathcal{E}/\mathcal{S},\nabla^{\mathcal{E}})$ does not have the dt term, we get Theorem 2.2 b) from Theorem 5.9, which we reformulate as follows.

Theorem 5.10. *There exist $\beta > 0$, $C > 0$, $\delta > 0$, such that if $K \in \mathfrak{z}(g)$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $0 < t \leq 1$,*

$$\left|\left\{\widetilde{\mathrm{Tr}}\Big[g\exp\big(-\mathcal{A}_{zK,t}\big)\Big]\right\}^{dt}\right| \leq Ct^{\delta}. \tag{5.32}$$

Proof of Theorem 2.1. If we omit the dt term in (5.31) and take $z = 1$, it follows that

$$\begin{aligned} &\Bigg|\psi_B \widetilde{\mathrm{Tr}}\left[g\exp\left(-\left(\mathbb{B}_t + \frac{c(K^X)}{4\sqrt{t}}\right)^2 - \mathcal{L}_K\right)\right] \\ &\qquad - \int_{X^g} \widehat{\mathrm{A}}_{g,K}(TX, \nabla^{TX})\, \mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}})\Bigg| \leq Ct^{\delta}. \end{aligned} \tag{5.33}$$

From (5.33), we get (2.21) and (2.22).

From (2.27), (2.29), (2.33) and (2.35), we get other parts of Theorem 2.1.

The proof of Theorem 2.1 is completed. □

For simplicity, we will assume in the remainder of this section that $n = \dim X$ is even. The functional analysis part is exactly the same for even and odd dimensional. We only explain in Remark 5.22 how to use the argument in the proof of [17, Theorem 2.10] to compute the local index in odd dimensional case.

5.3. **Finite propagation speed and localization.** The proof of the following lemma is the same as Lemma 5.2.

Lemma 5.11. *Given $\beta > 0$, there exist $C_1, C_2, C_2'(\beta), C_3(\beta), C_3'(\beta), C_4, C_5(\beta) > 0$ such that if $K \in \mathfrak{g}$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $s, s' \in \mathbb{E}$, $0 < t \leq 1$,*

$$\begin{aligned} \mathrm{Re}\langle t\mathcal{A}^{(0)}_{zK,t}s, s\rangle_0 &\geq C_1 t^2\|s\|_1^2 - (C_2t^2 + C_2'(\beta))\|s\|_0^2, \\ |\mathrm{Im}\langle t\mathcal{A}^{(0)}_{zK,t}s, s\rangle_0| &\leq C_3(\beta)t\|s\|_1\|s\|_0 + C_3'(\beta)\|s\|_0^2, \\ |\langle t\mathcal{A}^{(0)}_{zK,t}s, s'\rangle_0| &\leq C_4(t\|s\|_1 + C_5(\beta)\|s\|_0)(t\|s'\|_1 + C_5(\beta)\|s'\|_0). \end{aligned} \tag{5.34}$$

Moreover, as $\beta \to 0$, $C_2'(\beta)$, $C_3(\beta)$, $C_3'(\beta)$, $C_5(\beta) \to 0$.

In the sequel, we take $\beta > 0$ and always assume that $K \in \mathfrak{g}$, $|zK| \leq \beta$.

For $c > 0$, put

$$\begin{aligned} V_c &= \left\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq \frac{\mathrm{Im}(\lambda)^2}{4c^2} - c^2\right\}, \\ \Gamma_c &= \left\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) = \frac{\mathrm{Im}(\lambda)^2}{4c^2} - c^2\right\}. \end{aligned} \tag{5.35}$$

Note that U_c, V_c, Γ_c are the images of $\{\lambda \in \mathbb{C} : |\mathrm{Im}(\lambda)| \geq c\}$, $\{\lambda \in \mathbb{C} : |\mathrm{Im}(\lambda)| \leq c\}$, $\{\lambda \in \mathbb{C} : |\mathrm{Im}(\lambda)| = c\}$ by the map $\lambda \mapsto \lambda^2$.

The following lemma is an analogue of [19, Theorem 7.12].

Lemma 5.12. *There exists $C > 0$ such that given $c \in (0, 1]$, for $\beta > 0$ and $t \in (0, 1]$ small enough, if $\lambda \in U_c$, $|zK| \leq \beta$, the resolvent $(\lambda - t\mathcal{A}^{(0)}_{zK,t})^{-1}$ exists, extends to a continuous operator from $\mathbb{E}^{-1}$ into $\mathbb{E}^1$, and moreover, for $s \in \mathbb{E}$,*

$$\begin{aligned} \|(\lambda - t\mathcal{A}^{(0)}_{zK,t})^{-1}s\|_0 &\leq \frac{2}{c^2}\|s\|_0, \\ \|(\lambda - t\mathcal{A}^{(0)}_{zK,t})^{-1}s\|_1 &\leq \frac{C}{c^2t^4}(1 + |\lambda|)^2\|s\|_{-1}. \end{aligned} \tag{5.36}$$

Proof. From the same arguments in [19, (7.47)-(7.49)], by Lemma 5.11, if $\lambda \in \mathbb{R}$, $\lambda \leq -(C_2t^2 + C_2'(\beta))$, the resolvent $(\lambda - t\mathcal{A}^{(0)}_{zK,t})^{-1}$ exists.

Now we take $\lambda = a + ib$, $a, b \in \mathbb{R}$. By (5.34),

$$|\langle (t\mathcal{A}^{(0)}_{zK,t} - \lambda)s, s\rangle| \geq \sup \Big\{ C_1t^2\|s\|_1^2 - (C_2t^2 + C_2'(\beta) + a)\|s\|_0^2, \\ - C_3(\beta)t\|s\|_1\|s\|_0 + (|b| - C_3'(\beta))\|s\|_0^2 \Big\}. \tag{5.37}$$

Set

$$C(\lambda, t) = \inf_{u\geq 1} \sup \Big\{ C_1(tu)^2 - (C_2t^2 + C_2'(\beta) + a), -C_3(\beta)tu - C_3'(\beta) + |b| \Big\}. \tag{5.38}$$

Since $\|s\|_0 \leq \|s\|_1$, using (5.37), (5.38), we get

$$|\langle (t\mathcal{A}^{(0)}_{zK,t} - \lambda)s, s\rangle| \geq C(\lambda, t)\|s\|_0^2. \tag{5.39}$$

Now we fix $c \in (0, 1]$. Suppose that $\lambda \in U_c$, i.e.,

$$a \leq \frac{b^2}{4c^2} - c^2. \tag{5.40}$$

Assume that $u \in \mathbb{R}$ is such that

$$|b| - C_3(\beta)tu - C_3'(\beta) \leq c^2. \tag{5.41}$$

Then by (5.40) and (5.41),

$$C_1(tu)^2 - (C_2t^2 + C_2'(\beta) + a) \geq C_1(tu)^2 - \frac{b^2}{4c^2} + c^2 - C_2t^2 - C_2'(\beta) \\ \geq \left(C_1 - \frac{C_3(\beta)^2}{4c^2} \right)(tu)^2 - \frac{(c^2 + C_3'(\beta))C_3(\beta)}{2c^2}tu + c^2 - \frac{(c^2 + C_3'(\beta))^2}{4c^2} - C_2t^2 - C_2'(\beta). \tag{5.42}$$

The discriminant Δ of the polynomial in the variable tu in the right-hand side of (5.42) is given by

$$\Delta = -3c^2C_1 + 2C_1(C_3'(\beta) + 2C_2t^2 + 2C_2'(\beta)) + C_3(\beta)^2 \\ + \frac{1}{c^2}(C_1C_3'(\beta)^2 - C_2C_3(\beta)^2t^2 - C_2'(\beta)C_3(\beta)^2). \tag{5.43}$$

Clearly, for β, t small enough,

$$\Delta \leq -2c^2C_1, \quad C_1 - \frac{C_3^2(\beta)}{4c^2} > 0. \tag{5.44}$$

From (5.42)-(5.44), we get

$$C_1(t^2u)^2 - (C_2t^4 + C_2'(\beta) + a) \geq -\frac{\Delta}{4(C_1 - C_3^2(\beta)/(4c^2))} \geq \frac{c^2}{2}. \tag{5.45}$$

Ultimately, by (5.38)-(5.45), we find that for $\beta > 0$, $t \in (0, 1]$ small enough, if $\lambda \in U_c$,

$$C(\lambda, t) \geq \frac{c^2}{2}. \tag{5.46}$$

From (5.37), (5.38) and (5.46), we get the first equation of (5.36). Then combining with (5.34) and the argument in [19, (7.64)-(7.68)], we get the other part of Lemma 5.12.

The proof of Lemma 5.12 is completed. $\square$

As in (5.15), from (5.13), there exists $C > 0$, such that for any $0 < t \leq 1$, $s \in \mathbb{E}^1$,

$$\|(t\mathcal{A}_{zK,t} - t\mathcal{A}^{(0)}_{zK,t})s\|_{-1} \leq C\|s\|_1. \tag{5.47}$$

From Lemma 5.12, following the same process as the proof of (5.12), we get the following lemma.

Lemma 5.13. *There exist $k, m \in \mathbb{N}$, $C > 0$, such that given $c \in (0,1]$, for $\beta > 0$ and $t \in (0,1]$ small enough, if $\lambda \in U_c$, $|zK| \leq \beta$, the resolvent $(\lambda - t\mathcal{A}_{zK,t})^{-1}$ exists, extends to a continuous operator from $\Lambda(T^*(\mathbb{R} \times B)) \otimes \mathbb{E}^{-1}$ into $\Lambda(T^*(\mathbb{R} \times B)) \otimes \mathbb{E}^1$, and moreover, for $s \in \mathbb{E}$,*

$$\|(\lambda - t\mathcal{A}_{zK,t})^{-1}s\|_1 \leq \frac{C}{c^k t^k}(1 + |\lambda|)^m \|s\|_{-1}. \tag{5.48}$$

Definition 5.14. If H, H' are separable Hilbert spaces, if $1 \leq p < +\infty$, set

$$\mathscr{L}_p(H, H') = \{A \in \mathscr{L}(H, H') : \mathrm{Tr}[(A^*A)^{p/2}] < +\infty\}. \tag{5.49}$$

If $A \in \mathscr{L}_p(H, H')$, set

$$\|A\|_{(p)} := \left(\mathrm{Tr}[(A^*A)^{p/2}]\right)^{1/p}. \tag{5.50}$$

Then by [52, Chapter IX Proposition 6], $\|\cdot\|_{(p)}$ is a norm on $\mathscr{L}_p(H, H')$. Similarly, if $A \in \mathscr{L}(H, H')$, let $\|A\|_{(\infty)}$ be the usual operator norm of A.

In the sequel, the norms $\|\cdot\|_{(p)}$, $\|\cdot\|_{(\infty)}$ will always be calculated with respect to the Sobolev spaces $\mathbb{E}^0$.

From Lemma 5.13, we get the following lemma with the same proof as in [19, Theorem 7.13].

Lemma 5.15. *Given $q \geq 2\dim X + 1$, there exist $C > 0$, $k, m \in \mathbb{N}$, such that given $c \in (0,1]$, for $\beta > 0$ and $t \in (0,1]$ small enough, if $\lambda \in U_c$, $|zK| \leq \beta$,*

$$\begin{aligned} \|(\lambda - t\mathcal{A}_{zK,t})^{-1}\|_{(q)} &\leq \frac{C}{c^k t^k}(1 + |\lambda|)^m, \\ \|(\lambda - t\mathcal{A}_{zK,t})^{-q}\|_{(1)} &\leq \frac{C^q}{(c^k t^k)^q}(1 + |\lambda|)^{mq}. \end{aligned} \tag{5.51}$$

Let a_X be the inf of the injectivity radius of the fibers X. Let $\alpha \in (0, a_X/8]$. The precise value of α will be fixed later. The constants $C > 0$, $C' > 0$... may depend on the choice of α.

Let $f : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$f(u) = \begin{cases} 1 & \text{for } |u| \leq \alpha/2; \\ 0 & \text{for } |u| \geq \alpha. \end{cases} \tag{5.52}$$

Set

$$g(u) = 1 - f(u). \tag{5.53}$$

For $t > 0$, $a \in \mathbb{C}$, put

$$\begin{cases} F_t(a) = \displaystyle\int_{-\infty}^{+\infty} \exp(\sqrt{2}iua) \exp\left(-\frac{u^2}{2}\right) f(\sqrt{t}u) \frac{du}{\sqrt{2\pi}}, \\ G_t(a) = \displaystyle\int_{-\infty}^{+\infty} \exp(\sqrt{2}iua) \exp\left(-\frac{u^2}{2}\right) g(\sqrt{t}u) \frac{du}{\sqrt{2\pi}}. \end{cases} \tag{5.54}$$

Then $F_t(a)$, $G_t(a)$ are even holomorphic functions of a such that

$$\exp(-a^2) = F_t(a) + G_t(a). \tag{5.55}$$

Moreover when restricted on $\mathbb{R}$, F_t and G_t both lie in the Schwartz space $\mathcal{S}(\mathbb{R})$. Put

$$I_t(a) = G_t(a/\sqrt{t}). \tag{5.56}$$

Clearly, there exist uniquely defined holomorphic functions $\widetilde{F}_t(a)$, $\widetilde{G}_t(a)$, $\widetilde{I}_t(a)$ such that

$$F_t(a) = \widetilde{F}_t(a^2), \quad G_t(a) = \widetilde{G}_t(a^2), \quad I_t(a) = \widetilde{I}_t(a^2). \tag{5.57}$$

By (5.55) and (5.56), we have

$$\exp(-a) = \widetilde{F}_t(a) + \widetilde{G}_t(a), \quad \widetilde{I}_t(a) = \widetilde{G}_t(a/t). \tag{5.58}$$

From (5.58),

$$\exp(-\mathcal{A}_{zK,t}) = \widetilde{F}_t(\mathcal{A}_{zK,t}) + \widetilde{I}_t(t\mathcal{A}_{zK,t}). \tag{5.59}$$

From Lemma 5.15, the proof of the following lemma is the same as that of [19, Theorem 7.15].

Lemma 5.16. *There exist $\beta > 0$, $C > 0$, $C' > 0$ such that if $t \in (0,1]$, $K \in \mathfrak{g}$, $|zK| \leq \beta$,*

$$\|\widetilde{I}_t(t\mathcal{A}_{zK,t})\|_{(1)} \leq C\exp(-C'/t). \tag{5.60}$$

By (5.59) and (5.60), we find that to establish (5.31), we may as well replace $\exp(-\mathcal{A}_{zK,t})$ by $\widetilde{F}_t(\mathcal{A}_{zK,t})$.

Let $\widetilde{F}_t(\mathcal{A}_{zK,t})(x,x')$, $(x,x' \in X_b, b \in B)$ be the smooth kernel associated with the operator $\widetilde{F}_t(\mathcal{A}_{zK,t})$ with respect to $dv_X(x')$. Clearly the kernel of $g\widetilde{F}_t(\mathcal{A}_{zK,t})$ is given by $g\widetilde{F}_t(\mathcal{A}_{zK,t})(g^{-1}x,x')$. Then,

$$\mathrm{Tr}_s[g\widetilde{F}_t(\mathcal{A}_{zK,t})] = \int_X \mathrm{Tr}_s[g\widetilde{F}_t(\mathcal{A}_{zK,t})(g^{-1}x,x)]dv_X(x). \tag{5.61}$$

For $\varepsilon > 0$, $x \in X_b$, $b \in B$, let $B^X(x,\varepsilon)$ be the open ball in X_b with centre x and radius ε. Using finite propagation speed for solutions of hyperbolic equations (cf. [48, Appendix D.2]), we find that given $x \in X_b$, $\widetilde{F}_t(\mathcal{A}_{zK,t})(x,\cdot)$ vanishes on the complement of $B^X(x,\alpha)$ in X_b, and depends only on the restriction of the operator $\mathcal{A}_{zK,t}$ to the ball $B^X(x,\alpha)$. Therefore, we have shown that the proof of (5.31) can be made local on X_b. Therefore, we may and we will assume that TX_b is spin and

$$\mathcal{E} = \mathcal{S}_X \otimes E \tag{5.62}$$

over X_b, where $\mathcal{S}_X$ is the spinor of TX and E is a complex vector bundle, and the metric and the connection on $\mathcal{E}$ are induced from those on TX and E.

By the above, it follows that $g\widetilde{F}_t(\mathcal{A}_{zK,t})(g^{-1}x,x)$, $x \in X_b$ vanishes if $d^{X_b}(g^{-1}x,x) \geq \alpha$. Here d^{X_b} is the distance in $(X_b,\, g^{TX})$.

Now we explain our choice of α. Recall that $N_{X^g/X}$ is identified with the orthogonal bundle to TX^g in $TX|_{X^g}$. Given $\varepsilon > 0$, let $\mathcal{U}_\varepsilon$ be the ε-neighborhood of X^g_b in $N_{X^g/X}$. There exists $\varepsilon_0 \in (0, a_X/32]$ such that if $\varepsilon \in (0, 16\varepsilon_0]$, the fiberwise exponential map $(x,Z) \in N_{X^g/X} \to \exp^X_x(Z)$ is a diffeomorphism of $\mathcal{U}_\varepsilon$ on the tubular neighborhood $\mathcal{V}_\varepsilon$ of X^g in X. In the sequel, we identify $\mathcal{U}_\varepsilon$ and $\mathcal{V}_\varepsilon$. This identification is g-equivariant. We will assume that $\alpha \in (0, \varepsilon_0]$ is small enough so that for any $b \in B$, if $x \in X_b$, $d^{X_b}(g^{-1}x, x) \leq \alpha$, then $x \in \mathcal{V}_{\varepsilon_0}$.

By (5.60), (5.61) and the above considerations, it follows that for $\beta > 0$ small enough, the problem is localized on the ε_0-neighborhood $\mathcal{V}_{\varepsilon_0}$ of X^g.

As in (3.11), let $k(x, Z)$ be the smooth function on $\mathcal{U}_{\varepsilon_0}$ such that

$$dv_X(x, Z) = k(x, Z) dv_{X^g}(x) dv_{N_{X^g/X}}(Z). \tag{5.63}$$

In particular, $k|_{X^g} = 1$.

For $\omega \in \Lambda(T^*\mathbb{R}) \widehat{\otimes} \Lambda(T^*W^g)$, via (1.4) and (1.5), we will write

$$\omega = \sum_{1 \leq i_1 < \cdots < i_p \leq \ell} \omega_{i_1, \cdots, i_p} \wedge e^{i_1} \wedge \cdots \wedge e^{i_p}, \quad \text{for } \omega_{i_1, \cdots, i_p} \in \Lambda(T^*\mathbb{R}) \widehat{\otimes} \pi^* \Lambda(T^*B).$$

We denote by

$$\omega^{\max} := \omega_{1, \cdots, \ell} \in \Lambda(T^*\mathbb{R}) \widehat{\otimes} \pi^* \Lambda(T^*B). \tag{5.64}$$

Note that if the fiber is odd dimensional, our sign convention here is compatible with that in (0.15).

Theorem 5.17. *There exist $\beta > 0$, $\delta \in (0, 1]$ such that if $K \in \mathfrak{z}(g)$, $z \in \mathbb{C}$, $|zK| \leq \beta$, $t \in (0, 1]$, $x \in X^g$,*

$$\begin{aligned} \Bigg| t^{\frac{1}{2} \dim N_{X^g/X}} \int_{Z \in N_{X^g/X}, |Z| \leq \varepsilon_0/\sqrt{t}} & \psi_{\mathbb{R} \times B} \operatorname{Tr}_s[g \widetilde{F}_t(\mathcal{A}_{zK,t})(g^{-1}(x, \sqrt{t}Z), (x, \sqrt{t}Z))] \\ & \cdot k(x, \sqrt{t}Z) dv_{N_{X^g/X}}(Z) - \left\{ \widehat{\mathrm{A}}_{g,zK}(TX, \nabla^{TX}) \operatorname{ch}_{g,zK}(E, \nabla^E) \right\}^{\max} \Bigg| \leq C t^\delta. \end{aligned} \tag{5.65}$$

Proof. Sections 5.4-5.10 are devoted to the proof of this theorem. ☐

Proof of Theorem 5.9. By (5.61) and the finite propagation speed argument above, we have

$$\begin{aligned} \int_X \operatorname{Tr}_s[g \widetilde{F}_t(\mathcal{A}_{zK,t})(g^{-1}x, x)] dv_X(x) &= \int_{\mathcal{U}_{\varepsilon_0}} \operatorname{Tr}_s[g \widetilde{F}_t(\mathcal{A}_{zK,t})(g^{-1}x, x)] dv_X(x) \\ &= \int_{(x,Z) \in \mathcal{U}_{\varepsilon_0/\sqrt{t}}} t^{\frac{1}{2} \dim N_{X^g/X}} \operatorname{Tr}_s[g \widetilde{F}_t(\mathcal{A}_{zK,t})(g^{-1}(x, \sqrt{t}Z), (x, \sqrt{t}Z))] \\ &\qquad \times k(x, \sqrt{t}Z) dv_{X^g}(x) dv_{N_{X^g/X}}(Z). \end{aligned} \tag{5.66}$$

By Lemma 5.16, Theorem 5.17 and (5.66), there exist $\beta > 0$, $\delta \in (0, 1]$ such that for $K \in \mathfrak{z}(g)$, $|zK| \leq \beta$, $t \in (0, 1]$,

$$\left| \psi_{\mathbb{R} \times B} \operatorname{Tr}_s \left[g \exp\left(-\mathcal{A}_{zK,t}\right) \right] - \int_{X^g} \widehat{\mathrm{A}}_{g,zK}(TX, \nabla^{TX}) \operatorname{ch}_{g,zK}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}}) \right| \leq C t^\delta. \tag{5.67}$$

So we obtain Theorem 5.9. ☐

5.4. **A Lichnerowicz formula.** Let $e_1, \cdots, e_n$ be a locally defined smooth orthonormal frame of TX. Let (F, ∇^F) be a vector bundle with connection on X. We use the notation

$$(\nabla^F_{e_i})^2 = \sum_{i=1}^n (\nabla^F_{e_i})^2 - \nabla^F_{\sum_{i=1}^n \nabla^{TX}_{e_i} e_i}. \tag{5.68}$$

Let H be the scalar curvature of X. The following result is a combination of [7, Theorem 1.6], [19, Proposition 7.18] (for the term involved K^X and base $B = \mathrm{pt}$), [8, Theorem 3.5] (for Bismut's Lichnerowicz formula for Bismut superconnection) and [18, Theorem 2.10] (for the term involved dt).

Proposition 5.18. *The following identity holds,*

$$\begin{aligned}\mathcal{A}_{zK,t} = -t\Bigg(\nabla^{\mathcal{E}}_{e_i} &+ \frac{1}{2\sqrt{t}}\langle S(e_i)e_j, f^H_p\rangle c(e_j) f^p \wedge \\ &+\frac{1}{4t}\langle S(e_i) f^H_p, f^H_q\rangle f^p \wedge f^q \wedge - \frac{z\langle K^X, e_i\rangle}{4t} - dt \wedge \frac{c(e_i)}{4\sqrt{t}}\Bigg)^2 \\ &+ \frac{t}{4}H + \frac{t}{2}R^{\mathcal{E}/\mathcal{S}}(e_i, e_j)c(e_i)c(e_j) + \sqrt{t}R^{\mathcal{E}/\mathcal{S}}(e_i, f^H_p)c(e_i)f^p \wedge \\ &+ \frac{1}{2}R^{\mathcal{E}/\mathcal{S}}(f^H_p, f^H_q) f^p \wedge f^q \wedge - z m^{\mathcal{E}/\mathcal{S}}(K).\end{aligned} \tag{5.69}$$

Proof. From Bismut's Lichnerowicz formula (cf. [8, Theorem 3.5]),

$$\begin{aligned}\mathbb{B}_t^2 = -t\left(\nabla^{\mathcal{E}}_{e_i} + \frac{1}{2\sqrt{t}}\langle S(e_i)e_j, f^H_p\rangle c(e_j) f^p \wedge + \frac{1}{4t}\langle S(e_i) f^H_p, f^H_q\rangle f^p \wedge f^q \wedge\right)^2 \\ + \frac{t}{4}H + \frac{t}{2}R^{\mathcal{E}/\mathcal{S}}(e_i, e_j)c(e_i)c(e_j) + \sqrt{t}R^{\mathcal{E}/\mathcal{S}}(e_i, f^H_p)c(e_i)f^p \wedge + \frac{1}{2}R^{\mathcal{E}/\mathcal{S}}(f^H_p, f^H_q) f^p \wedge f^q \wedge .\end{aligned} \tag{5.70}$$

From (1.19), (2.5) and (2.7),

$$\frac{1}{4}[D, c(K^X)] = \frac{1}{4}c(e_k)c\left(\nabla^{TX}_{e_k} K^X\right) - \frac{1}{2}\langle K^X, e_j\rangle \nabla^{\mathcal{E}}_{e_j} = m^{\mathcal{S}}(K) - \frac{1}{2}\nabla^{\mathcal{E}}_{K^X}. \tag{5.71}$$

Since the G-action preserves the splitting (1.4), $\langle [K^X, f^H_p], e_j\rangle = 0$. Thus from (1.9), (1.19) and (1.22),

$$\begin{aligned}[\nabla^{\mathbb{E},u}, c(K^X)] = f^p \wedge c\left(\nabla^{TX}_{f^H_p} K^X\right) &= -\langle \nabla^{TW,L}_{f^H_p} K^X, e_j\rangle c(e_j) f^p \wedge \\ &= \langle \nabla^{TW,L}_{K^X} e_j, f^H_p\rangle c(e_j) f^p \wedge = \langle S(K^X)e_j, f^H_p\rangle c(e_j) f^p \wedge .\end{aligned} \tag{5.72}$$

From (1.10)-(1.12), we get

$$S(e_j)e_k = S(e_k)e_j, \quad \langle S(e_j) f^H_p, f^H_q\rangle = \frac{1}{2}\langle T(f^H_p, f^H_q), e_j\rangle. \tag{5.73}$$

Thus from (1.22),

$$\begin{aligned}[c(T^H), c(K^X)] &= \langle S(e_j) f^H_p, f^H_q\rangle f^p \wedge f^q \wedge [c(e_j), c(K^X)] \\ &= -2\langle S(K^X) f^H_p, f^H_q\rangle f^p \wedge f^q \wedge .\end{aligned} \tag{5.74}$$

Thus from (5.1), (5.2) and (5.70)-(5.74), we get (5.69) without dt term. By comparing directly the coefficient of dt on both sides of (5.69) as in [18, Theorem 2.10], we get (5.69).

The proof of Proposition 5.18 is completed. □

5.5. **A local coordinate system near X^g.** Take $x \in W^g$. Then the fiberwise exponential map $Z \in T_xX \to \exp_x^X(Z) \in X$ identifies $B^{T_xX}(0, 16\varepsilon_0)$ with $B^X(x, 16\varepsilon_0)$. With this identification, there exists a smooth function $k'_x(Z)$, $Z \in B^{T_xX}(0, a_X/2)$ such that

$$dv_X(Z) = k'_x(Z) dv_{TX}(Z), \quad \text{with } k'_x(0) = 1. \tag{5.75}$$

We may and we will assume that ε_0 is small enough so that if $Z \in T_xX, |Z| \leq 4\varepsilon_0$,

$$\frac{1}{2} g_x^{TX} \leq g_Z^{TX} \leq \frac{3}{2} g_x^{TX}. \tag{5.76}$$

Assume from now, $K \in \mathfrak{z}(g)$. Recall that ϑ_K is the one form dual to K^X defined in (3.1).

Definition 5.19. Let ${}^1\nabla^{\mathcal{E},t}$ be the connection on $\Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}$ along the fibers,

$$\begin{aligned} {}^1\nabla^{\mathcal{E},t}_{\cdot} := \nabla^{\mathcal{E}}_{\cdot} + \frac{1}{2\sqrt{t}} \langle S(\cdot)e_j, f_p^H\rangle c(e_j) f^p \wedge + \frac{1}{4t} \langle S(\cdot) f_p^H, f_q^H\rangle f^p \wedge f^q \wedge \\ - \frac{z\vartheta_K(\cdot)}{4t} - dt \wedge \frac{c(\cdot)}{4\sqrt{t}}. \end{aligned} \tag{5.77}$$

In the sequel, we will trivialize $\Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}$ by parallel transport along $u \in [0,1] \to uZ$ with respect to the connection ${}^1\nabla^{\mathcal{E},t}$. Observe that the above connection is g-invariant.

From (1.10) and (1.13), we have $S(e_i)e_j = S(e_j)e_i$. Let L be a trivial line bundle over W. We equip a connection on L by

$$\nabla^L = d - \frac{z\vartheta_K}{4}. \tag{5.78}$$

Thus

$$R^L = (\nabla^L)^2 = -\frac{zd\vartheta_K}{4}. \tag{5.79}$$

From (1.30), (3.3), (3.5), (5.73), (5.78) and (5.79), we could calculate that

$$\begin{aligned} \left({}^1\nabla^{\mathcal{E},1}\right)^2(e_i, e_j) = \frac{1}{4}\langle R^{TX}(e_k, e_l)e_i, e_j\rangle c(e_k)c(e_l) + \frac{1}{2}\langle R^{TX}(e_k, f_p^H)e_i, e_j\rangle c(e_k) f^p \wedge \\ + \frac{1}{4}\langle R^{TX}(f_p^H, f_q^H)e_i, e_j\rangle f^p \wedge f^q \wedge + R^E(e_i, e_j) - \frac{z}{2}\langle m^{TX}(K)e_i, e_j\rangle. \end{aligned} \tag{5.80}$$

Note that when $K = 0$, (5.80) is [8, Theorem 4.14], [3, Theorem 10.11] or [11, Theorem 11.8]. Note that from (5.77), $\left({}^1\nabla^{\mathcal{E},t}\right)^2$ could be obtained from $\left({}^1\nabla^{\mathcal{E},1}\right)^2$ by replacing $f^p\wedge$, $f^q\wedge$ and K by $\frac{f^p}{\sqrt{t}}\wedge$, $\frac{f^q}{\sqrt{t}}\wedge$ and $\frac{K}{t}$.

Let A, A' be smooth sections of TX. By (5.77),

$${}^1\nabla^{\mathcal{E},1}_A c(A') = c(\nabla^{TX}_A A') + \langle S(A)A', f_p^H\rangle f^p \wedge + \frac{1}{2}\langle A, A'\rangle dt. \tag{5.81}$$

Let $c^1(TX) \simeq TX$ be the set of elements of length 1 in $c(TX)$. It follows from (5.81) that parallel transport along the fiber X with respect to ${}^1\nabla^{\mathcal{E},1}$ maps $c^1(TX)$ into $c^1(TX) \oplus T^*B \oplus T^*\mathbb{R}$, while leaving $\Lambda(T^*B)\widehat{\otimes}\Lambda(T^*\mathbb{R})$ invariant.

5.6. **Replacing X by T_xX.** Let $\gamma(u)$ be a smooth even function from $\mathbb{R}$ into $[0,1]$ such that

$$\gamma(u) = \begin{cases} 1 & \text{if } |u| \leq 1/2; \\ 0 & \text{if } |u| \geq 1. \end{cases} \tag{5.82}$$

If $Z \in T_xX$, put

$$\rho(Z) = \gamma\left(\frac{|Z|}{4\varepsilon_0}\right). \tag{5.83}$$

Then

$$\rho(Z) = \begin{cases} 1 & \text{if } |Z| \leq 2\varepsilon_0; \\ 0 & \text{if } |Z| \geq 4\varepsilon_0. \end{cases} \tag{5.84}$$

For $x \in W^g$, let $\mathbf{H}_x$ be the vector space of smooth sections of $\Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*(\Lambda(T^*B))\widehat{\otimes}\mathcal{E}_x$ over T_xX. Let Δ^{TX} be the (negative) standard Laplacian on the fiber of TX.

Let $L^{1,t}_{x,zK}$ be the differential operator acting on $\mathbf{H}_x$,

$$L^{1,t}_{x,zK} = (1-\rho^2(Z))(-t\Delta^{TX}) + \rho^2(Z)\mathcal{A}_{zK,t}. \tag{5.85}$$

Let $\widetilde{F}_t(L^{1,t}_{x,zK})(Z,Z')$ be the smooth kernel of $\widetilde{F}_t(L^{1,t}_{x,zK})$ with respect to $dv_{TX}(Z')$. Using the finite propagation speed for solutions of hyperbolic equations [48, Appendix D.2] and (5.75), we find that if $Z \in N_{X^g/X,x}$, $|Z| \leq \varepsilon_0$, then

$$\widetilde{F}_t(\mathcal{A}_{zK,t})(g^{-1}Z,Z)k'_x(Z) = \widetilde{F}_t(L^{1,t}_{x,zK})(g^{-1}Z,Z). \tag{5.86}$$

Thus in our proof of Theorem 5.17, we can then replace $\mathcal{A}_{zK,t}$ by $L^{1,t}_{x,zK}$.

5.7. **The Getzler rescaling.** Let Op_x be the set of scalar differential operators on T_xX acting on $\mathbf{H}_x$. Then by (5.62),

$$L^{1,t}_{x,zK} \in (\Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*(\Lambda(T^*B))\widehat{\otimes}c(TX)\otimes \mathrm{End}(E))_x \otimes \mathrm{Op}_x. \tag{5.87}$$

For $t>0$, let $H_t : \mathbf{H}_x \to \mathbf{H}_x$ be the linear map

$$H_th(Z) = h(Z/\sqrt{t}). \tag{5.88}$$

Let $L^{2,t}_{x,zK}$ be the differential operator acting on $\mathbf{H}_x$ defined by

$$L^{2,t}_{x,zK} = H_t^{-1}L^{1,t}_{x,zK}H_t. \tag{5.89}$$

By (5.87),

$$L^{2,t}_{x,zK} \in (\Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*(\Lambda(T^*B))\widehat{\otimes}c(TX)\otimes \mathrm{End}(E))_x \otimes \mathrm{Op}_x. \tag{5.90}$$

Recall that $\dim X^g = \ell$ and $\dim N_{X^g/X} = n-\ell$. Let $(e_1,\cdots,e_\ell)$ be an orthonormal oriented basis of T_xX^g, let $(e_{\ell+1},\cdots,e_n)$ be an orthonormal oriented basis of $N_{X^g/X}$, so that $(e_1,\cdots,e_n)$ is an orthonormal oriented basis of T_xX. We denote with an superscript the corresponding dual basis.

For $1 \leq j \leq \ell$, the operators $e^j\wedge$ and i_{e_j} act as odd operators on $\Lambda(T^*X^g)$.

Definition 5.20. For $t>0$, put

$$c_t(e_j) = \frac{1}{\sqrt{t}}e^j \wedge -\sqrt{t}i_{e_j}, \quad 1 \leq j \leq \ell. \tag{5.91}$$

Let $L^{3,t}_{x,zK}$ be the differential operator acting on $\mathbf{H}_x$ obtained from $L^{2,t}_{x,zK}$ by replacing $c(e_j)$ by $c_t(e_j)$ for $1 \leq j \leq \ell$.

For $A \in (\Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*(\Lambda(T^*B))\widehat{\otimes}c(TX)\otimes \mathrm{End}(E))_x \otimes \mathrm{Op}_x$, we denote by $[A]^{(3)}_t$ the differential operator obtained from A by using the Getzler rescaling of the Clifford variables which is given in Definition 5.20.

Let $\tau e_j(Z)$ be the parallel transport of e_j along the curve $t \in [0,1] \to tZ$ with respect to the connection ∇^{TX}. Let $\mathcal{O}_1(|Z|^2)$ be any object in $\Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*(\Lambda(T^*B))\widehat{\otimes}c(TX)$ which is of length at most 1 and is also $\mathcal{O}(|Z|^2)$. By (5.81), in the trivialization associated with ${}^1\nabla^{\mathcal{E},t}$,

$$c(\tau e_j(Z)) = c(e_j) + \frac{1}{\sqrt{t}}\langle S(Z)e_j, f^H_p\rangle f^p \wedge + \frac{1}{2\sqrt{t}}\langle Z, e_j\rangle dt \wedge + \mathcal{O}_1(t^{-1/2}|Z|^2). \tag{5.92}$$

From (5.92), for $1 \leq j \leq \ell$,

$$\Big[\sqrt{t}c(\tau e_j(\sqrt{t}Z))\Big]^{(3)}_t = e^j \wedge + \mathcal{O}(\sqrt{t}|Z|); \tag{5.93}$$

for $\ell + 1 \leq j \leq n$,

$$\Big[c(\tau e_j(\sqrt{t}Z))\Big]^{(3)}_t = c(e_j) + \langle S(Z)e_j, f^H_p\rangle f^p \wedge + \frac{1}{2}\langle Z, e_j\rangle dt \wedge + \mathcal{O}(\sqrt{t}|Z|^2). \tag{5.94}$$

From [3, Proposition 1.18], (5.69), (5.80), (5.85), (5.89) and (5.91), we calculate that

$$\begin{aligned} L^{3,t}_{x,zK} = (1-\rho^2(\sqrt{t}Z))(-\Delta^{TX}) + \rho^2(\sqrt{t}Z)\cdot \Big\{ -g^{ij}(\sqrt{t}Z)\Big(\nabla'_{e_i}\nabla'_{e_j} - \Gamma^k_{ij}(\sqrt{t}Z)\sqrt{t}\nabla'_{e_k}\Big) \\ + \frac{t}{4}H_{\sqrt{t}Z} + \frac{t}{2}R^E_{\sqrt{t}Z}(\tau e_i, \tau e_j)\Big[c(\tau e_i(\sqrt{t}Z))c(\tau e_j(\sqrt{t}Z))\Big]^{(3)}_t \\ + \sqrt{t}R^E_{\sqrt{t}Z}(\tau e_j, f^H_p)\Big[c(\tau e_j(\sqrt{t}Z))\Big]^{(3)}_t f^p \wedge + \frac{1}{2}R^E_{\sqrt{t}Z}(f^H_p, f^H_q) f^p \wedge f^q \wedge - m^E_{\sqrt{t}Z}(zK)\Big\}, \end{aligned} \tag{5.95}$$

where $(g^{ij}(Z))$ is the inverse matrix of $(g_{ij}(Z) = \langle e_i, e_j\rangle_Z)$, $\big(\nabla^{TX}_{e_i} e_j\big)_Z = \Gamma^k_{ij}(Z)e_k$ and

$$\begin{aligned} \nabla'_{e_i} = \nabla_{\tau e_i(\sqrt{t}Z)} + \frac{t}{8}\Big(\langle R^{TX}_x(e_k, e_l)Z, e_i\rangle + \mathcal{O}(\sqrt{t}|Z|^2)\Big)\Big[c(\tau e_i(\sqrt{t}Z))c(\tau e_j(\sqrt{t}Z))\Big]^{(3)}_t \\ + \frac{\sqrt{t}}{4}\Big(\langle R^{TX}_x(e_j, f^H_p)Z, e_i\rangle + \mathcal{O}(\sqrt{t}|Z|^2)\Big)\Big[c(\tau e_j(\sqrt{t}Z))\Big]^{(3)}_t f^p \wedge \\ + \frac{1}{8}\Big(\langle R^{TX}_x(f^H_p, f^H_q)Z, e_i\rangle + \mathcal{O}(\sqrt{t}|Z|^2)\Big) f^p \wedge f^q \wedge + \frac{t}{2}\Big(R^E_x(Z, e_i) + \mathcal{O}(\sqrt{t}|Z|^2)\Big) \\ - \frac{1}{4}\langle m^{TX}_x(zK)Z, e_i\rangle + \frac{1}{\sqrt{t}}h_i(zK, \sqrt{t}Z). \end{aligned} \tag{5.96}$$

Here ∇_U is the ordinary differentiation operator on TX in the direction U, $h_i(zK, Z)$ is a function depending linearly on zK and $h_i(zK, Z) = \mathcal{O}(|Z|^2)$ for $|zK| < \beta$.

Let $\widetilde{F}_t(L^{3,t}_{x,zK})(Z, Z')$ be the smooth kernel associated with $\widetilde{F}_t(L^{3,t}_{x,zK})$ with respect to $dv_{TX}(Z')$.

From the finite propagation speed argument explained before (5.62), we could also assume that TX^g and $N_{X^g/X}$ are spin. Let $\mathcal{S}_{X^g}$ and $\mathcal{S}_N$ be the spinors of TX^g and $N_{X^g/X}$ respectively. Then $\mathcal{S}_X = \mathcal{S}_{X^g}\widehat{\otimes}\mathcal{S}_N$. Recall that g acts on $(\mathcal{S}_N \otimes E)_x$.

We may write $\widetilde{F}_t(L^{3,t}_{x,zK})(Z, Z')$ in the form

$$\begin{aligned} \widetilde{F}_t(L^{3,t}_{x,zK})(Z, Z') = \sum \widetilde{F}^{j_1\cdots j_q}_{t, i_1\cdots i_p}(Z, Z') e^{i_1} \wedge \cdots \wedge e^{i_p} i_{e_{j_1}} \cdots i_{e_{j_q}}, \\ 1 \leq i_1 < \cdots < i_p \leq \ell,\ 1 \leq j_1 < \cdots < j_q \leq \ell, \end{aligned} \tag{5.97}$$

and $\widetilde{F}^{j_1\cdots j_q}_{t,i_1\cdots i_p}(Z,Z') \in \Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*\Lambda(T^*B)\otimes\big(c(N_{X^g/X})\otimes \mathrm{End}(E)\big)_x$. As explained in Section 1.3, $\ell = \dim X^g$ has the same parity as $n = \dim X$. As in (5.64), put

$$[\widetilde{F}_t(L^{3,t}_{x,zK})(Z,Z')]^{\max} = \widetilde{F}_{t,1\cdots\ell}(Z,Z'). \tag{5.98}$$

In other words, $\widetilde{F}_{t,1\cdots\ell}(Z,Z')$ is the coefficient of $e^1\wedge\cdots\wedge e^\ell$ in (5.97).

Proposition 5.21. *If $Z\in T_xX$, $|Z|\le \varepsilon_0/\sqrt{t}$,*

$$\begin{aligned} t^{(n-\ell)/2}\operatorname{Tr}_s[g\widetilde{F}_t(\mathcal{A}_{zK,t})(g^{-1}(\sqrt{t}Z),\sqrt{t}Z)]k'_x(\sqrt{t}Z)& \\ = (-i)^{\ell/2}2^{\ell/2}\operatorname{Tr}_s^{\mathcal{S}_N\otimes E}[g\widetilde{F}_t(L^{3,t}_{x,zK})(g^{-1}Z,Z)]^{\max}&. \end{aligned} \tag{5.99}$$

Proof. As $K\in\mathfrak{z}(g)$, ${}^1\nabla^{\mathcal{E},t}$ is g-equivariant. Thus the trivialization $\Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}$ is g-equivariant and the action of g on $\big(\Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}\big)_{g^{-1}Z}$ is the action of g on $\big(\Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}\big)_x$, which is an element in $\big(c(N_{X^g/X})\otimes\mathrm{End}(E)\big)_x$. Now we get Proposition 5.21 by the same proof of [19, Proposition 7.25]. □

Remark 5.22. As in [17, (1.6) and (1.7)], if $n=\dim X$ is even,

$$\begin{aligned} &\operatorname{Tr}_s^{\mathcal{S}_X}[c(e_{i_1})\cdots c(e_{i_p})] = 0, \text{ for } p<n, 1\le i_1<\cdots<i_p\le n,\\ &\operatorname{Tr}_s^{\mathcal{S}_X}[c(e_1)\cdots c(e_n)] = (-2i)^{n/2}; \end{aligned} \tag{5.100}$$

if $n=\dim X$ is odd,

$$\operatorname{Tr}^{\mathcal{S}_X}[1] = 2^{(n-1)/2},\quad \operatorname{Tr}^{\mathcal{S}_X}[c(e_1)\cdots c(e_n)] = (-i)^{(n+1)/2}2^{(n-1)/2}, \tag{5.101}$$

and the trace of the other monomials is zero.

If $n=\dim X$ is odd, since (5.101) holds and the total degree of $\widetilde{F}_t(\mathcal{A}_{zK,t})$ is even, we only take the trace for the odd degree Clifford part. In this case, (5.65) is replaced by

$$\begin{aligned} \Bigg| t^{(n-\ell)/2}\int_{Z\in N,|Z|\le\frac{\varepsilon_0}{\sqrt{t}}}\psi_{\mathbb{R}\times B}\operatorname{Tr}^{\mathrm{odd}}[g\widetilde{F}_t(\mathcal{A}_{zK,t})(g^{-1}(x,\sqrt{t}Z),(x,\sqrt{t}Z))]& \\ \times k(x,\sqrt{t}Z)dv_{N_{X^g/X}}(Z) - \Big\{\widehat{\mathrm{A}}_{g,zK}(TX,\nabla^{TX})\operatorname{ch}_{g,zK}(E,\nabla^E)\Big\}^{\max}\Bigg| &\le Ct^\delta. \end{aligned} \tag{5.102}$$

In particular, since $n-\ell$ is even,

$$\begin{aligned} &\operatorname{Tr}^{\mathcal{S}_X}[c(e_1)\cdots c(e_n)] \\ &\qquad = (-i)^{(\ell+1)/2}2^{(\ell-1)/2}t^{\ell/2}\left\{\operatorname{Tr}_s^{\mathcal{S}_N}[c_t(e_1)\cdots c_t(e_\ell)c(e_{\ell+1})\cdots c(e_n)]\right\}^{\max}, \end{aligned} \tag{5.103}$$

the analogue of (5.99) is

$$\begin{aligned} t^{(n-\ell)/2}\operatorname{Tr}^{\mathrm{odd}}[g\widetilde{F}_t(\mathcal{A}_{zK,t})(g^{-1}(\sqrt{t}Z),\sqrt{t}Z)]k'_x(\sqrt{t}Z)& \\ = (-i)^{(\ell+1)/2}2^{(\ell-1)/2}\{\operatorname{Tr}_s^{\mathcal{S}_N\otimes E}[g\widetilde{F}_t(L^{3,t}_{x,zK})(g^{-1}Z,Z)]\}^{\max}&. \end{aligned} \tag{5.104}$$

Let $\jmath: W^g\to W$ be the obvious embedding.

Definition 5.23. Let $L^{3,0}_{x,zK}$ be the operator in

$$(\pi^*(\Lambda(T^*B))\widehat{\otimes}\Lambda(T^*X^g)\widehat{\otimes}c(N_{X^g/X})\otimes\mathrm{End}(E))_x\otimes\mathrm{Op}_x, \tag{5.105}$$

under the notation (5.68), given by

$$L^{3,0}_{x,zK} = -\left(\nabla_{e_i} + \frac{1}{4}\left\langle (\jmath^* R^{TX}_x - m^{TX}(zK)_x)Z, e_i\right\rangle\right)^2 + \jmath^* R^E_x - m^E(zK)_x. \tag{5.106}$$

In the sequel, we will write that a sequence of differential operators on T_xX converges if its coefficients converge together with their derivatives uniformly on the compact subsets in T_xX.

Comparing with [19, Proposition 7.27], from (5.93)-(5.96), we have

Proposition 5.24. *As $t \to 0$,*

$$L^{3,t}_{x,zK} \to L^{3,0}_{x,zK}. \tag{5.107}$$

5.8. **A family of norms.** For $x \in W^g$, let $\mathbf{I}_x$ be the vector space of smooth sections of $(\Lambda(T^*\mathbb{R})\widehat{\otimes}\pi^*\Lambda(T^*B)\widehat{\otimes}\Lambda(T^*X^g)\widehat{\otimes}\mathcal{S}_N \otimes E)_x$ on T_xX, let $\mathbf{I}_{(r,q),x}$ be the vector space of smooth sections of

$$\Big(\big(T^*\mathbb{R}\widehat{\otimes}\pi^*\Lambda^{r-1}(T^*B) \oplus \pi^*\Lambda^r(T^*B)\big)\widehat{\otimes}\Lambda^q(T^*X^g)\widehat{\otimes}\mathcal{S}_N \otimes E\Big)_x$$

on T_xX. We denote by $\mathbf{I}^0_x = \bigoplus_{r,q} \mathbf{I}^0_{(r,q),x}$ the corresponding vector space of square-integrable sections. Put $k = \dim B$.

Definition 5.25. If $s \in \mathbf{I}_{(r,q),x}$ has compact support, put

$$|s|^2_{t,x,0} = \int_{T_xX} |s(Z)|^2 \left(1 + |Z|\,\rho\left(\frac{\sqrt{t}Z}{2}\right)\right)^{2(k+\ell+1-q-r)} dv_{TX}(Z). \tag{5.108}$$

Recall that by (5.84), if $\rho(\sqrt{t}Z) > 0$, then $|\sqrt{t}Z| \leq 4\varepsilon_0$. If $\sqrt{t}|Z| \leq 4\varepsilon_0$, then $\rho(\sqrt{t}Z/2) = 1$. By the same arguments as in [21, Proposition 11.24], for $t \in (0,1]$, the following family of operators acting on $(\mathbf{I}^0_x, |\cdot|_{t,x,0})$ are uniformly bounded:

$$\begin{aligned} &1_{|\sqrt{t}Z|\leq 4\varepsilon_0}\sqrt{t}c_t(e_j), \quad 1_{|\sqrt{t}Z|\leq 4\varepsilon_0}|Z|\sqrt{t}c_t(e_j), \quad \text{for } 1 \leq j \leq \ell, \\ &1_{|\sqrt{t}Z|\leq 4\varepsilon_0}|Z|f^p\wedge, \quad 1_{|\sqrt{t}Z|\leq 4\varepsilon_0}|Z|dt\wedge. \end{aligned} \tag{5.109}$$

Definition 5.26. If $s \in \mathbf{I}_x$ has compact support, put

$$|s|^2_{t,x,1} = |s|^2_{t,x,0} + \sum_{i=1}^n |\nabla_{e_i}s|^2_{t,x,0}, \tag{5.110}$$

and

$$|s|_{t,x,-1} = \sup_{0\neq s'\in\mathbf{I}_x} \frac{|\langle s, s'\rangle_{t,x,0}|}{|s'|_{t,x,1}}. \tag{5.111}$$

Let $(\mathbf{I}^1_x, |\cdot|_{t,x,1})$ be the Hilbert closure of the above vector space with respect to $|\cdot|_{t,x,1}$. Let $(\mathbf{I}^{-1}_x, |\cdot|_{t,x,-1})$ be the antidual of $(\mathbf{I}^1_x, |\cdot|_{t,x,1})$. Then $(\mathbf{I}^1_x, |\cdot|_{t,x,1})$ and $(\mathbf{I}^0_x, |\cdot|_{t,x,0})$ are densely embedded in $(\mathbf{I}^0_x, |\cdot|_{t,x,0})$ and $(\mathbf{I}^{-1}_x, |\cdot|_{t,x,-1})$ with norms smaller than 1 respectively.

Comparing with [19, Proposition 7.31], by (5.95) and (5.109), we get the following estimates.

Lemma 5.27. *There exist constants $C_i > 0$, $i = 1,2,3,4$, such that if $t \in (0,1]$, $z \in \mathbb{C}$, $|zK| \leq 1$, if $n \in \mathbb{N}$, $x \in X^g$, if the support of $s, s' \in \mathbf{I}_x$ is included in $\{Z \in T_xX : |Z| \leq n\}$,* *then*

$$\begin{aligned}
&\mathrm{Re}\langle L^{3,t}_{x,zK}s,s\rangle_{t,x,0}\geq C_1|s|^2_{t,x,1}-C_2(1+|nzK|^2)|s|^2_{t,x,0},\\
&|\mathrm{Im}\langle L^{3,t}_{x,zK}s,s\rangle_{t,x,0}|\leq C_3\Big((1+|nzK|)|s|_{t,x,1}|s|_{t,x,0}+|nzK|^2|s|^2_{t,x,0}\Big),\\
&|\langle L^{3,t}_{x,zK}s,s'\rangle_{t,x,0}|\leq C_4(1+|nzK|^2)|s|_{t,x,1}|s'|_{t,x,1}.
\end{aligned}\tag{5.112}$$

Proof. We only need to observe that the terms containing $|nzK|^2$ come from terms

$$\left|\left\langle\left(\rho(\sqrt{t}Z)\left(-\frac{1}{4}\langle m^{TX}(zK)Z,e_i\rangle+\frac{1}{\sqrt{t}}h_i(zK,\sqrt{t}Z)\right)\right)^2 s,s\right\rangle_{t,x,0}\right|,\tag{5.113}$$

which can be dominated by $C(1+|nzK|^2)|s|^2_{t,x,0}$.

The proof of Lemma 5.27 is completed. □

5.9. **The kernel $\widetilde{F}_t(L^{3,t}_{x,K})$ as an infinite sum.** Let h be a smooth even function from $\mathbb{R}$ into $[0,1]$ such that

$$h(u)=\begin{cases}1 & \text{if } |u|\leq\frac{1}{2};\\ 0 & \text{if } |u|\geq 1.\end{cases}\tag{5.114}$$

For $n\in\mathbb{N}$, put

$$h_n(u)=h\left(u+\frac{n}{2}\right)+h\left(u-\frac{n}{2}\right).\tag{5.115}$$

Then h_n is a smooth even function whose support is included in $\left[-\frac{n}{2}-1,-\frac{n}{2}+1\right]\cup\left[\frac{n}{2}-1,\frac{n}{2}+1\right]$.

Set

$$\mathcal{H}(u)=\sum_{n\in\mathbb{N}}h_n(u).\tag{5.116}$$

The above sum is locally finite, and $\mathcal{H}(u)$ is a bounded smooth even function which takes positive values and has a positive lower bound on $\mathbb{R}$.

Put

$$k_n(u)=\frac{h_n}{\mathcal{H}}(u).\tag{5.117}$$

Then the k_n are bounded even smooth functions with bounded derivatives, and moreover

$$\sum_{n\in\mathbb{N}}k_n=1.\tag{5.118}$$

Note that here we use n as an index for the natural numbers, not the $\dim X$ in the previous sections.

Definition 5.28. For $t\in[0,1]$, $n\in\mathbb{N}$, $a\in\mathbb{C}$, put

$$F_{t,n}(a)=\int_{-\infty}^{+\infty}\exp(\sqrt{2}iua)\exp\left(-\frac{u^2}{2}\right)f(\sqrt{t}u)k_n(u)\frac{du}{\sqrt{2\pi}}.\tag{5.119}$$

By (5.54), (5.118) and (5.119),

$$F_t(a)=\sum_{n\in\mathbb{N}}F_{t,n}(a).\tag{5.120}$$

Also, given $m, m' \in \mathbb{N}$, there exist $C > 0$, $C' > 0$, $C'' > 0$ such that for any $t \in [0,1]$, $n \in \mathbb{N}$, $c > 0$,

$$\sup_{a\in\mathbb{C},|\mathrm{Im}(a)|\leq c} |a|^m \left| F_{t,n}^{(m')}(a) \right| \leq C\exp(-C'n^2 + C''c^2). \tag{5.121}$$

Let $\widetilde{F}_{t,n}(a)$ be the unique holomorphic function such that

$$F_{t,n}(a) = \widetilde{F}_{t,n}(a^2). \tag{5.122}$$

Recall that V_c was defined in (5.35). By (5.121), given $m, m' \in \mathbb{N}$, there exist $C > 0$, $C' > 0$, $C'' > 0$ such that for any $t \in [0,1]$, $n \in \mathbb{N}$, $c > 0$, $\lambda \in V_c$,

$$|\lambda|^m \left| \widetilde{F}_{t,n}^{(m')}(\lambda) \right| \leq C\exp(-C'n^2 + C''c^2). \tag{5.123}$$

By (5.120),

$$\widetilde{F}_t(a) = \sum_{n\in\mathbb{N}} \widetilde{F}_{t,n}(a). \tag{5.124}$$

Using (5.124), we get

$$\widetilde{F}_t(L_{x,zK}^{3,t}) = \sum_{n\in\mathbb{N}} \widetilde{F}_{t,n}(L_{x,zK}^{3,t}). \tag{5.125}$$

More precisely, by (5.123) and using standard elliptic estimates, given $t \in (0,1]$, we have the identity

$$\widetilde{F}_t(L_{x,zK}^{3,t})(Z,Z') = \sum_{n\in\mathbb{N}} \widetilde{F}_{t,n}(L_{x,zK}^{3,t})(Z,Z') \tag{5.126}$$

and the series in the right-hand side of (5.126) converges uniformly together with its derivatives on the compact sets in $T_xX \times T_xX$.

Definition 5.29. For γ in (5.82), put

$$L_{x,zK,n}^{3,t} = -\left(1 - \gamma^2\left(\frac{|Z|}{2(n+2)}\right)\right)\Delta^{TX} + \gamma^2\left(\frac{|Z|}{2(n+2)}\right)L_{x,zK}^{3,t}. \tag{5.127}$$

Observe that if $k_n(u) \neq 0$, then $|u| \leq \frac{n}{2}+1$. Using finite propagation speed and (5.76), we find that if $Z \in T_xX$, the support of $\widetilde{F}_{t,n}(L_{x,zK}^{3,t})(Z,\cdot)$ is included in $\{Z' \in T_xX : |Z'-Z| \leq n+2\}$. Therefore, given $p \in \mathbb{N}$, if $Z \in T_xX$, $|Z| \leq p$, the support of $\widetilde{F}_{t,n}(L_{x,zK}^{3,t})(Z,\cdot)$ is included in $\{Z' \in T_xX : |Z'| \leq n+p+2\}$.

If $|Z| \leq n+p+2$, then $\gamma(|Z|/2(n+p+2)) = 1$. Using finite propagation speed again, we see that by (5.127), for $Z \in T_xX$, $|Z| \leq p$,

$$\widetilde{F}_{t,n}(L_{x,zK}^{3,t})(Z,Z') = \widetilde{F}_{t,n}(L_{x,zK,n+p}^{3,t})(Z,Z'). \tag{5.128}$$

From Lemma 5.27, we have

$$\begin{aligned}
&\mathrm{Re}\langle L_{x,zK,n}^{3,t}s, s\rangle_{t,x,0} \geq C_1|s|_{t,x,1}^2 - C_2(1+|nzK|^2)|s|_{t,x,0}^2,\\
&|\mathrm{Im}\langle L_{x,zK,n}^{3,t}s, s\rangle_{t,x,0}| \leq C_3\Big((1+|nzK|)|s|_{t,x,1}|s|_{t,x,0} + |nzK|^2|s|_{t,x,0}^2\Big),\\
&|\langle L_{x,zK,n}^{3,t}s, s'\rangle_{t,x,0}| \leq C_4(1+|nzK|^2)|s|_{t,x,1}|s'|_{t,x,1}.
\end{aligned} \tag{5.129}$$

Put

$$L^{3,0}_{x,zK,n} = -\left(1-\gamma^2\left(\frac{|Z|}{2(n+2)}\right)\right)\Delta^{TX} + \gamma^2\left(\frac{|Z|}{2(n+2)}\right)L^{3,0}_{x,zK}. \tag{5.130}$$

By Proposition 5.24, as $t\to 0$,

$$L^{3,t}_{x,zK,n} \to L^{3,0}_{x,zK,n}. \tag{5.131}$$

By (5.129), the functional analysis arguments in [19, §7.10-7.12] work perfectly here. We have the following uniform estimates, which is formally the same as [19, Theorem 7.38]. In particular, since the estimates in (5.112) and (5.129) are the analogue of [19, (7.131) and (7.148)], the proof of the following theorem is exactly the same as that of [19, Theorem 7.38].

Theorem 5.30. *There exist $C'>0$, $C''>0$, $C'''>0$ such that for $\eta>0$ small enough, there is $c_\eta\in(0,1]$ such that for any $m\in\mathbb{N}$, there are $C>0$, $r\in\mathbb{N}$ such that for $t\in(0,1]$, $|zK|\le c_\eta$, $n\in\mathbb{N}$, $x\in X^g$, $Z,Z'\in T_xX$,*

$$\begin{aligned}\sup_{|\alpha|,|\alpha'|\le m}&\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\widetilde{F}_{t,n}(L^{3,t}_{x,zK})(Z,Z')\right| \\ &\le C(1+|Z|+|Z'|)^r\exp\Big(-C'n^2/4+2C''\eta^2\sup(|Z|^2,|Z'|^2)-C'''|Z-Z'|^2\Big).\end{aligned} \tag{5.132}$$

5.10. **A proof of Theorem 5.17.** Remark that as explained in the introduction of [19], $L^{3,t}_{x,zK}$ does not have a fixed lower bound. So it is not possible to define a priori a honest heat kernel for $\exp(-L^{3,t}_{x,zK})$. So we cannot prove Theorem 5.17 following the arguments in [21, §11].

Since $L^{3,0}_{x,zK,n+p}$ coincides with $-\Delta^{TX}$ near infinity, the operator $\widetilde{F}_{0,n}(L^{3,0}_{x,zK,n+p})$ is well-defined. Also, by proceeding as in (5.128), if $|Z|,|Z'|\le p$, using finite propagation speed, we find that the kernel $\widetilde{F}_{0,n}(L^{3,0}_{x,zK,n+p})(Z,Z')$ does not depend on p. Finally this kernel verifies estimates similar to (5.132) for $\eta>0$ small enough and $|zK|\le c_\eta$. Therefore we may define the kernel $\exp(-L^{3,0}_{x,zK})(Z,Z')$ by

$$\exp(-L^{3,0}_{x,zK})(Z,Z') = \sum_{n\in\mathbb{N}}\widetilde{F}_{0,n}(L^{3,0}_{x,zK,n+p})(Z,Z'), \quad \text{for } |Z|,|Z'|\le p. \tag{5.133}$$

Note that the estimate in (5.132) also works for $t=0$. Thus the series in (5.133) converges uniformly on compact subsets of $T_xX\times T_xX$ together with its derivatives.

From (5.95), (5.106), (5.127) and (5.130), there exists $C>0$ such that for $t\in(0,1]$, $z\in\mathbb{C}$, $|zK|\le 1$, $n\in\mathbb{N}$, $x\in X^g$, if $s\in\mathbf{I}_x$ has compact support, then

$$\left|(L^{3,t}_{x,zK,n}-L^{3,0}_{x,zK,n})s\right|_{t,x,-1}\le C\sqrt{t}(1+n^4)|s|_{0,x,1}. \tag{5.134}$$

From Theorem 5.30, (5.133) and (5.134), the proof of the following theorem is exactly the same as that of [19, Theorem 7.43].

Theorem 5.31. *There exist $C''>0$, $C'''>0$ such that for $\eta>0$ small enough, there exist $c_\eta\in(0,1]$, $r\in\mathbb{N}$, $C>0$, such that for $t\in(0,1]$, $z\in\mathbb{C}$, $|zK|\le c_\eta$, $x\in X^g$, $Z,Z'\in T_xX$,*

$$\begin{aligned}\left|\left(\widetilde{F}_t(L^{3,t}_{x,zK})-\exp(-L^{3,0}_{x,zK})\right)(Z,Z')\right| &\le Ct^{\frac{1}{4(\dim X+1)}}(1+|Z|+|Z'|)^r \\ &\cdot\exp\left(2C''\eta^2\sup(|Z|^2,|Z'|^2)-C'''|Z-Z'|^2/2\right).\end{aligned} \tag{5.135}$$

Now there is $C>0$ such that if $Z\in N_{X^g/X}$, then

$$|g^{-1}Z-Z|\geq C|Z|. \tag{5.136}$$

By (5.135) and (5.136), we find that there exists $C'''' > 0$ such that if $Z\in N_{X^g/X}$,

$$\begin{aligned}\Big|(\widetilde{F}_t(L^{3,t}_{x,zK})-\exp(-L^{3,0}_{x,zK}))(g^{-1}Z,Z)\Big| \\ \leq Ct^{\frac{1}{4(\dim X+1)}}(1+|Z|)^r\cdot\exp\left(2C''\eta^2|Z|^2-C''''|Z|^2\right).\end{aligned} \tag{5.137}$$

For $\eta>0$ small enough,

$$2C''\eta^2-C''''\leq -C''''/2. \tag{5.138}$$

So by (5.137), if $Z\in N_{X^g/X}$,

$$\Big|\left(\widetilde{F}_t(L^{3,t}_{x,zK})-\exp(-L^{3,0}_{x,zK})\right)(g^{-1}Z,Z)\Big|\leq Ct^{\frac{1}{4(\dim X+1)}}\exp\left(-C''''|Z|^2/4\right). \tag{5.139}$$

For $K\in\mathfrak{z}(g)$, put

$$H^{TX}=\jmath^*R^{TX}-m^{TX}(zK). \tag{5.140}$$

Clearly H^{TX} splits under $TX=TX^g\oplus N_{X^g/X}$ as

$$H^{TX}=H^{TX^g}+H^N. \tag{5.141}$$

Using the Mehler's formula (cf. e.g., [43, (1.34)]), by (5.106), for $|zK|$ small enough,

$$\begin{aligned}\exp(-L^{3,0}_{x,zK})(g^{-1}Z,Z)=(4\pi)^{-\dim X/2}{\det}^{\frac{1}{2}}\left(\frac{H^{TX}/2}{\sinh(H^{TX}/2)}\right)\\ \cdot\exp\left(-\frac{1}{2}\left\langle\frac{H^N/2}{\sinh(H^N/2)}\big(\cosh(H^N/2)-\exp(H^N/2)g^{-1}\big)Z,Z\right\rangle\right)\\ \cdot\exp(-\jmath^*R^E+m^E(zK)).\end{aligned} \tag{5.142}$$

Observe that for $z\in\mathbb{C}$, $|zK|$ small enough, the right-hand side of (5.142) is well-defined. Using (5.142), comparing with [43, (1.37)], if $|zK|$ is small enough,

$$\begin{aligned}\int_{N_{X^g/X}}\exp(-L^{3,0}_{x,zK})(g^{-1}Z,Z)dv_N(Z)=(4\pi)^{-\ell/2}{\det}^{\frac{1}{2}}\left(\frac{H^{TX^g}/2}{\sinh(H^{TX^g}/2)}\right)\\ \cdot\left({\det}^{1/2}(1-g^{-1}|_N){\det}^{1/2}(1-g\exp(-H^N))\right)^{-1}\cdot\exp(-\jmath^*R^E+m^E(zK)).\end{aligned} \tag{5.143}$$

Also compare with [43, (1.38)],

$$\begin{aligned}\mathrm{Tr}_s^{\mathcal{S}_N\otimes E}[g\exp(-\jmath^*R^E+m^E(zK))]\\ =(-i)^{(\dim X-\ell)/2}{\det}^{1/2}(1-g^{-1}|_N)\,\mathrm{Tr}^E[g\exp(-\jmath^*R^E+m^E(zK))].\end{aligned} \tag{5.144}$$

Using (2.15), (2.16), (5.143) and (5.144), we get

$$\begin{aligned}\psi_{\mathbb{R}\times B}\int_{N_{X^g/X}}(-i)^{\ell/2}2^{\ell/2}\left\{\mathrm{Tr}_s^{\mathcal{S}_N\otimes E}[g\exp(-L^{3,0}_{x,zK})(g^{-1}Z,Z)]\right\}^{\max}dv_N(Z)\\ =\left\{\widehat{\mathrm{A}}_{g,zK}(TX,\nabla^{TX})\,\mathrm{ch}_{g,zK}(E,\nabla^E)\right\}^{\max}.\end{aligned} \tag{5.145}$$

From (5.99), (5.139) and (5.145), we obtain Theorem 5.17 for $\dim X$ even.

If $\dim X$ is odd, following the explanation in Remark 5.22, the proof is the same.

The proof of Theorem 5.17 is completed.

5.11. **A proof of Theorem 4.2.** Since $v \geq t > 0$, we have

$$0 \leq t^{-1} - v^{-1} < t^{-1}. \tag{5.146}$$

Set

$$\mathcal{A}'_{K,t,v} = \left(\mathbb{B}_t + \frac{\sqrt{t}c(K^X)}{4}\left(\frac{1}{t} - \frac{1}{v}\right) + t \cdot dt \wedge \frac{\partial}{\partial t}\right)^2 + \mathcal{L}_K. \tag{5.147}$$

Let $\mathcal{A}'^{(0)}_{K,t,v}$ be the piece of $\mathcal{A}'_{K,t,v}$ which has degree 0 in $\Lambda(T^*(\mathbb{R} \times B))$. Then from (5.146), $\mathcal{A}'^{(0)}_{K,t,v}$ satisfies the same estimate in Lemma 5.2 and the estimate (5.15) of $\mathcal{A}_{K,t} - \mathcal{A}^{(0)}_{K,t}$ also holds for $\mathcal{A}'_{K,t,v} - \mathcal{A}'^{(0)}_{K,t,v}$ uniformly on $v \geq t \geq 1$. Since $v \geq t$, as $t \to +\infty$, we have

$$\left| \frac{\partial}{\partial t}\left(\frac{\sqrt{t}c(K^X)}{4}\left(\frac{1}{t} - \frac{1}{v}\right) - \frac{c(T^H)}{4\sqrt{t}}\right)\right| = \mathcal{O}(t^{-3/2}). \tag{5.148}$$

Then the analogue of Propositions 5.5 and 5.8 holds for $\mathcal{A}'_{K,t,v}$ uniformly for $v \geq t \geq 1$. Thus replacing $\mathcal{A}_{zK,t}$ by $\mathcal{A}'_{K,t,v}$ in the proof of Theorem 5.1, we obtain Theorem 4.2.

6. A proof of Theorem 4.3

In this section, we prove Theorem 4.3. This section is organized as follows. In Section 6.1, we establish a Lichnerowicz formula for $\mathcal{B}_{K,t,v}$ in (4.11). In Section 6.2, we prove Theorem 4.3 a). In Sections 6.3 - 6.8, we prove Theorem 4.3 b). In Section 6.9, we prove Theorem 4.3 c). In Section 6.10, we prove Theorem 4.3 d). In this section, we use the assumptions and the notations in Section 4.

6.1. **A Lichnerowicz formula.** Let L be a trivial line bundle over W. We equip a connection on L by

$$\nabla^L_v = d - \frac{\vartheta_K}{4v}. \tag{6.1}$$

Thus

$$R^L_v = (\nabla^L_v)^2 = -\frac{d\vartheta_K}{4v}. \tag{6.2}$$

Let $\nabla^{\mathcal{E}\otimes L}_v$ be the connection on $\mathcal{E} \otimes L$ induced by $\nabla^{\mathcal{E}}$ and ∇^L_v. The corresponding Dirac operator is

$$D_v = \sum_{i=1}^n c(e_i)\nabla^{\mathcal{E}\otimes L}_{v,e_i} = D - \frac{c(K^X)}{4v}. \tag{6.3}$$

Since

$$\nabla^{\mathcal{E}\otimes L}_{v,f^H_p} = \nabla^{\mathcal{E}}_{f^H_p}, \tag{6.4}$$

from (6.3), the new Bismut superconnection associated with $\mathcal{E} \otimes L$ is

$$\mathbb{B}^v_t = \mathbb{B}_t - \frac{\sqrt{t}c(K^X)}{4v}. \tag{6.5}$$

Theorem 6.1. *The following identity holds,*

$$\begin{aligned}(6.6)\quad \mathcal{B}_{K,t,v} = -t\Bigg(\nabla^{\mathcal{E}}_{e_i} + \frac{1}{2\sqrt{t}}\langle S(e_i)e_j, f^H_p\rangle c(e_j) f^p\wedge \\ +\frac{1}{4t}\langle S(e_i)f^H_p, f^H_q\rangle f^p\wedge f^q\wedge - \frac{\langle K^X, e_i\rangle}{4}\left(\frac{1}{t}+\frac{1}{v}\right)\Bigg)^2 \\ +\frac{t}{4}H + \frac{t}{2}\left(R^{\mathcal{E}/\mathcal{S}}(e_i,e_j) - \frac{1}{2v}\langle \nabla^{TX}_{e_i}K^X, e_j\rangle\right)c(e_i)c(e_j) \\ +\sqrt{t}\left(R^{\mathcal{E}/\mathcal{S}}(e_i, f^H_p) - \frac{1}{2v}\langle T(e_i, f^H_p), K^X\rangle\right)c(e_i)f^p\wedge \\ +\frac{1}{2}\left(R^{\mathcal{E}/\mathcal{S}}(f^H_p, f^H_q) - \frac{1}{8v}\langle T(f^H_p, f^H_q), K^X\rangle\right) f^p\wedge f^q\wedge - m^{\mathcal{E}/\mathcal{S}}(K^X) + \frac{1}{4v}|K^X|^2.\end{aligned}$$

Proof. From (4.11), (5.1), (5.2), (5.69) and (6.5), we have

$$\begin{aligned}(6.7)\quad \mathcal{B}_{K,t,v} = \left(\mathbb{B}^v_t + \frac{c(K^X)}{4\sqrt{t}}\right)^2 + \mathcal{L}_K = -t\Bigg(\nabla^{\mathcal{E}\otimes L}_{v,e_i} + \frac{1}{2\sqrt{t}}\langle S(e_i)e_j, f^H_p\rangle c(e_j)f^p\wedge \\ +\frac{1}{4t}\langle S(e_i)f^H_p, f^H_q\rangle f^p\wedge f^q\wedge - \frac{\langle K^X, e_i\rangle}{4t}\Bigg)^2 + \frac{t}{4}H + \frac{t}{2}R^{\mathcal{E}\otimes L/\mathcal{S}}(e_i,e_j)c(e_i)c(e_j) \\ +\sqrt{t}R^{\mathcal{E}\otimes L/\mathcal{S}}(e_i, f^H_p)c(e_i)f^p\wedge + \frac{1}{2}R^{\mathcal{E}\otimes L/\mathcal{S}}(f^H_p, f^H_q)f^p\wedge f^q\wedge - m^{\mathcal{E}\otimes L/\mathcal{S}}(K^X).\end{aligned}$$

Since G acts trivially on L, the corresponding $m^L(K)$ in the sense of (2.4) is given by

$$(6.8)\qquad m^L(K^X) = -K^X + \nabla^L_{v,K^X} = -\frac{|K^X|^2}{4v}.$$

Then (6.6) follows from (3.3)-(3.5), (6.2), (6.7) and (6.8).

The proof of Theorem 6.1 is completed. □

6.2. **A proof of Theorem 4.3 a).** Comparing with (5.77), we set

$$(6.9)\quad {}^2\nabla^{\mathcal{E},t}_{\cdot} := \nabla^{\mathcal{E}}_{\cdot} + \frac{1}{2\sqrt{t}}\langle S(\cdot)e_j, f^H_p\rangle c(e_j)f^p\wedge + \frac{1}{4t}\langle S(\cdot)f^H_p, f^H_q\rangle f^p\wedge f^q\wedge - \frac{\vartheta_K(\cdot)}{4t}\left(1+\frac{t}{v}\right).$$

We trivialize $\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}$ by parallel transport along $u\in[0,1]\to uZ$ with respect to the connection ${}^2\nabla^{\mathcal{E},t}$. Observe that the above connection is g-equivariant as $K\in\mathfrak{z}(g)$. Let A, A' be smooth sections of TX. As in (5.81), from (6.9),

$$(6.10)\qquad {}^2\nabla^{\mathcal{E},1}_A c(A') = c(\nabla^{TX}_A A') + \langle S(A)A', f^H_p\rangle f^p\wedge.$$

For $x\in W^g$, in this section, we denote by $\mathbf{H}_x$ the vector space of smooth sections of $\pi^*(\Lambda(T^*B))\widehat{\otimes}\mathcal{E}_x$. Let $L^{1,(t,v)}_{x,K}$ be the differential operator acting on $\mathbf{H}_x$,

$$(6.11)\qquad L^{1,(t,v)}_{x,K} = (1-\rho^2(Z))(-t\Delta^{TX}) + \rho^2(Z)\mathcal{B}_{K,t,v}.$$

We define $L^{2,(t,v)}_{x,K} := H_t^{-1}L^{1,(t,v)}_{x,K}H_t$ and $L^{3,(t,v)}_{x,K} := \left[L^{2,(t,v)}_{x,K}\right]^{(3)}_t$ as in Section 5.7. By Proposition 5.24 for $(\mathcal{E}\otimes L, \nabla^{\mathcal{E}\otimes L}_v)$, we have

$$(6.12)\qquad L^{3,(0,v)}_{x,K} = -\left(\nabla_{e_i} + \frac{1}{4}\left\langle (\jmath^*R^{TX}_x - m^{TX}(K)_x)Z, e_i\right\rangle\right)^2 + \jmath^*R^{E\otimes L}_x - m^{E\otimes L}(K)_x,$$

and as $t \to 0$,

$$L^{3,(t,v)}_{x,K} \to L^{3,(0,v)}_{x,K}. \tag{6.13}$$

By (6.10), as in (5.92) and (5.94),

$$\left[\sqrt{t}c(K^X)(\sqrt{t}Z)\right]^{(3)}_t = \jmath^*\vartheta_K + \mathcal{O}(\sqrt{t}Z + \sqrt{t}). \tag{6.14}$$

By (2.9), (2.17), (3.2), (6.2) and (6.8), we get

$$\jmath^* R^L_{v,K} = \jmath^* R^L_v - 2i\pi m^L(K) = -\frac{1}{4v}(d^{W^g}\vartheta_K - 2i\pi|K^X|^2) = -\frac{d^{W^g}_K\vartheta_K}{4v}. \tag{6.15}$$

Then by (2.18),

$$\mathrm{ch}_{g,K}(L, \nabla^L_v) = \exp\left(\frac{d^{W^g}_K\vartheta_K}{8\pi i v}\right). \tag{6.16}$$

From (2.15), (2.16), (3.1), (3.2) and (3.6), set

$$\gamma_{K,v} = -\frac{\vartheta_K}{8vi\pi}\exp\left(\frac{d_K\vartheta_K}{8vi\pi}\right)\widehat{\mathrm{A}}_{g,K}(TX, \nabla^{TX})\,\mathrm{ch}_{g,K}(\mathcal{E}/\mathcal{S}, \nabla^{\mathcal{E}/\mathcal{S}}) \in \Omega(W^g, \det(N_{X^g/X})). \tag{6.17}$$

By (4.9), (6.12) and (6.17), if $\dim X$ is even, as in (5.145), we get

$$\begin{aligned}\phi\int_{N_{X^g/X}}(-i)^{\ell/2}2^{\ell/2}\left\{\mathrm{Tr}_s^{\mathcal{S}_N\otimes E\otimes L}\left[g\frac{\jmath^*\vartheta_K}{4v}\exp(-L^{3,(0,v)}_{x,K})(g^{-1}Z, Z)\right]\right\}^{\max} dv_N(Z)&\\ = -\left\{\gamma_{K,v}\right\}^{\max}_x&.\end{aligned} \tag{6.18}$$

By (3.6), (6.13)-(6.18), from the same argument of Section 5.7 and (5.139) for $(\mathcal{E}\otimes L, \nabla^{\mathcal{E}\otimes L}_v)$, we obtain Theorem 4.3 a) for $\dim X$ even.

If n is odd, following the explanation in Remark 5.22, the proof is the same.

The proof of Theorem 4.3 a) is completed.

6.3. **Localization of the problem.** The proof of Theorem 4.3 b) is devoted to Sections 6.3-6.8.

Let $\mathcal{B}^0$ be the piece of $\mathcal{B}_{K,t,v}$ which has degree 0 in $\Lambda(T^*B)$. Then for $t \in (0,1]$, $v \in [t,1]$, by (5.146), $t\mathcal{B}^0$ satisfies the same estimates as Lemma 5.11 uniformly for $v \in [t,1]$.

Thus following the same arguments in the proof of Lemma 5.16, we have

Theorem 6.2. *There exist $\beta > 0$, $C > 0$, $C' > 0$ such that if $K \in \mathfrak{g}$, $|K| \le \beta$, $t \in (0,1]$, $v \in [t,1]$,*

$$\|\widetilde{I}_t(t\mathcal{B}_{K,t,v})\|_{(1)} \le C\exp(-C'/t). \tag{6.19}$$

So our proof of inequality (4.15) in Theorem 4.3 can be localized near X^g. As in Section 5.3, we may and we will assume that $W = B \times X$, TX is spin and $\mathcal{E} = \mathcal{S}_X \otimes E$.

6.4. **A rescaling of the normal coordinate to $X^{g,K}$ in X^g.** In the sequel, we fix $g \in G$, $0 \neq K_0 \in \mathfrak{z}(g)$ and

$$K = zK_0, \quad z \in \mathbb{R}^*. \tag{6.20}$$

Recall that X^g and $X^{g,K}$ are totally geodesic in X. Given $\varepsilon > 0$, let $\mathcal{U}''_\varepsilon$ be the ε-neighbourhood of $X^{g,K}$ in $N_{X^{g,K}/X^g}$ (cf. the notation in the proof of Lemma 3.1). By zooming out $\varepsilon_0 \in (0, a_X/32]$ in Section 5.3, we can assume that the map $(y_0, Z_0) \in N_{X^{g,K}/X^g} \to \exp^{X^g}_{y_0}(Z_0) \in X^g$ is a diffeomorphism from $\mathcal{U}''_\varepsilon$ into the tubular neighbourhood $\mathcal{V}''_\varepsilon$ of $X^{g,K}$ in X^g for any $0 < \varepsilon \leq 16\varepsilon_0$.

Since X^g is totally geodesic in X, the connection ∇^{TX} induces the connection $\nabla^{N_{X^g/X}}$ on $N_{X^g/X}$ (cf. (1.31) and (3.8)).

For $(y_0, Z_0) \in \mathcal{U}''_\varepsilon$, we identify $N_{X^g/X,(y_0,Z_0)}$ with $N_{X^g/X,y_0}$ by parallel transport along the geodesic $u \in [0,1] \to uZ_0$ with respect to ∇^{TX}. If $y_0 \in X^{g,K}$, $Z_0 \in N_{X^{g,K}/X^g,y_0}$, $Z \in N_{X^g/X,y_0}$, $|Z_0|, |Z| \leq 4\varepsilon_0$, we identify (y_0, Z_0, Z) with $\exp^X_{\exp^{X^g}_{y_0}(Z_0)}(Z) \in X$. Therefore, (y_0, Z_0, Z) defines a coordinate system on X near $X^{g,K}$.

From (2.15), (2.16) and (6.17), for $|z|$ small enough, $\gamma_{K,v}$ is a smooth form on W^g. Recall that the function k is defined in (5.63) and $\ell' = \dim X^{g,K}$.

Theorem 6.3. *There exist $\beta \in (0,1]$, $\delta \in (0,1]$ such that for $p \in \mathbb{N}$, there is $C > 0$ such that if $z \in \mathbb{R}^*$, $|z| \leq \beta$, $t \in (0,1]$, $v \in [t,1]$, $y_0 \in X^{g,K}$, $Z_0 \in N_{X^{g,K}/X^g,y_0}$, $|Z_0| \leq \varepsilon_0/\sqrt{v}$, then for $K = zK_0$,*

$$\begin{aligned}\Bigg| v^{\frac{1}{2}\dim N_{X^{g,K}/X^g}} \Bigg(\phi \int_{Z \in N_{X^g/X,y_0}, |Z| \leq \varepsilon_0} \widetilde{\mathrm{Tr}}' \Bigg[g \frac{\sqrt{t} c(K^X)}{4v} \widetilde{F}_t\left(-\mathcal{B}_{K,t,v}\right) \\ (g^{-1}(y_0, \sqrt{v}Z_0, Z), (y_0, \sqrt{v}Z_0, Z)) \Bigg] \cdot k(y_0, \sqrt{v}Z_0, Z) dv_{N_{X^g/X}}(Z) \\ + \{\gamma_{K,v}\}^{\max}(y_0, \sqrt{v}Z_0) \Bigg) \Bigg| \leq C \frac{(1+|Z_0|)^{\ell'+1}}{(1+|zZ_0|)^p} \left(\frac{t}{v}\right)^\delta .\end{aligned} \tag{6.21}$$

Proof. Sections 6.5-6.7 will be devoted to the proof of Theorem 6.3. □

6.5. **A new trivialization and Getzler rescaling near $X^{g,K}$.** Since g preserves geodesics and the parallel transport, in the coordinate system in above subsection,

$$g(Z_0, Z) = (Z_0, gZ). \tag{6.22}$$

By an abuse of notation, we will often write $Z_0 + Z$ instead of $\exp^X_{\exp^{X^g}_{y_0}(Z_0)}(Z)$.

Firstly, we fix $Z_0 \in N_{X^{g,K}/X^g,y_0}$, $|Z_0| \leq \varepsilon_0$, and we take $Z \in T_{y_0}X, |Z| \leq 4\varepsilon_0$. The curve $u \in [0,1] \to Z_0 + uZ$ lies in $B^X_{y_0}(0, 5\varepsilon_0)$. Moreover we identify TX_{Z_0+Z}, $\pi^*\Lambda(T^*B) \otimes \mathcal{E}_{Z_0+Z}$ with TX_{Z_0}, $\pi^*\Lambda(T^*B) \otimes \mathcal{E}_{Z_0}$ by parallel transport with respect to the connections ∇^{TX}, ${}^2\nabla^{\mathcal{E},t}$ along the curve.

When $Z_0 \in N_{X^{g,K}/X^g,y_0}$ is allowed to vary, we identify TX_{Z_0}, $\pi^*\Lambda(T^*B) \otimes \mathcal{E}_{Z_0}$ with TX_{y_0}, $\pi^*\Lambda(T^*B) \otimes \mathcal{E}_{y_0}$ by parallel transport with respect to the connections ∇^{TX}, $\nabla^{\mathcal{E}}$ along the curve $u \in [0,1] \to uZ_0$. Then $\mathbf{H}_{Z_0}$ is identified with $\mathbf{H}_{y_0}$ associated with this trivialization. Furthermore the fiber of $\pi^*\Lambda(T^*B) \otimes \mathcal{E}$ at $Z_0 + Z$ and y_0 are identified by parallel transport along the broken curve $u \in [0,1] \to 2uZ_0$, for $0 \leq u \leq \frac{1}{2}$; $Z_0 + (2u-1)Z$ for $\frac{1}{2} \leq u \leq 1$.

Note that here we use the trick in [11, Section 11.4] (cf. also [19, Section 9.5]) and the trivialization here is different from that in the proof of Theorem 4.3 a) in Section 6.2. Under this new trivialization, the identification between $\mathbf{H}_{y_0}$ and $\mathbf{H}_{Z_0}$ is an isometry with respect to (1.17).

For $Z_0 \in N_{X^{g,K}/X^g,y_0}$, $|Z_0| \leq \varepsilon_0$, the considered trivializations depend explicitly on Z_0. We denote by $(\mathcal{B}_{K,t,v})_{Z_0}$ the action of $\mathcal{B}_{K,t,v}$ centered at Z_0. Thus the operator $(\mathcal{B}_{K,t,v})_{Z_0}$ acts on $\mathbf{H}_{Z_0}$. As $\mathbf{H}_{Z_0}$ is identified with $\mathbf{H}_{y_0}$, so that ultimately, $(\mathcal{B}_{K,t,v})_{Z_0}$ acts on $\mathbf{H}_{y_0}$.

We may and we will assume that ε_0 is small enough so that if $|Z_0| \leq \varepsilon_0$, $|Z| \leq 4\varepsilon_0$, then

$$\frac{1}{2} g^{TX}_{y_0} \leq g^{TX}_{Z_0+Z} \leq \frac{3}{2} g^{TX}_{y_0}. \tag{6.23}$$

We define $k'_{(y_0,Z_0)}(Z)$ as in (5.75). Recall that $\rho(Z)$ is defined in (5.84).

Definition 6.4. Let $L'^{1,(t,v)}_{Z_0,K}$ be the differential operator acting on $\mathbf{H}_{y_0}$,

$$L'^{1,(t,v)}_{Z_0,K} = -(1-\rho^2(Z))(-t\Delta^{TX}) + \rho^2(Z)(\mathcal{B}_{K,t,v})_{Z_0}. \tag{6.24}$$

By proceeding as in (5.86), and using Theorem 6.2 and (6.22), we find that if $Z_0 \in N_{X^{g,K}/X^g,y_0}$, $Z \in N_{X^g/X,y_0}$, $|Z|, |Z_0| \leq \varepsilon_0$,

$$\widetilde{F}_t(\mathcal{B}_{K,t,v})(g^{-1}(Z_0,Z),(Z_0,Z))k'_{(y_0,Z_0)}(Z) = \widetilde{F}_t(L^{1,(t,v)}_{Z_0,K})(g^{-1}Z, Z). \tag{6.25}$$

We still define H_t as in (5.88). Let

$$L'^{2,(t,v)}_{Z_0,K} = H_t^{-1} L'^{1,(t,v)}_{Z_0,K} H_t. \tag{6.26}$$

Let $(e_1, \cdots, e_{\ell'})$, $(e_{\ell'+1}, \cdots, e_\ell)$, $(e_{\ell+1}, \cdots, e_n)$ be orthonormal basis of $T_{y_0}X^{g,K}$, $N_{X^{g,K}/X^g,y_0}$, $N_{X^g/X,y_0}$ respectively.

Definition 6.5. Let $L'^{3,(t,v)}_{Z_0,K}$ be the differential operator acting on $\mathbf{H}_{y_0}$ obtained from $L'^{2,(t,v)}_{Z_0,K}$ by replacing $c(e_j)$ by $c_t(e_j)$ (cf. (5.91)) for $1 \leq j \leq \ell'$, by $c_{t/v}(e_j)$ for $\ell'+1 \leq j \leq \ell$, while leaving unchanged the $c(e_j)$'s for $\ell+1 \leq j \leq n$.

For $A \in (\pi^*(\Lambda(T^*B))\widehat{\otimes} c(TX)\otimes \mathrm{End}(E))_x \otimes \mathrm{Op}_x$, we denote by $[A]^{(3)}_{(t,v)}$ the differential operator obtained from A by using the Getzler rescaling of the Clifford variables which is given in Definition 6.5.

If $Z_0 \in N_{X^{g,K}/X^g,y_0}$, $|Z_0| \leq \varepsilon_0$, $Z \in T_{y_0}X$, $|Z| \leq 4\varepsilon_0$, if $U \in T_{y_0}X$, let $\tau^{Z_0}U(Z) \in TX_{Z_0+Z}$ be the parallel transport of U along the curve $u \to 2uZ_0$, $0 \leq u \leq \frac{1}{2}$, $u \to \exp^X_{Z_0}((2u-1)Z)$, $\frac{1}{2} \leq u \leq 1$, with respect to ∇^{TX}.

By (6.10), under the identification of $\pi^*\Lambda(T^*B)\otimes\mathcal{E}_{Z_0+Z}$ and $\pi^*\Lambda(T^*B)\otimes\mathcal{E}_{y_0}$ at the beginning of this subsection, in the trivialization

$$c\left(\tau^{Z_0}e_j(Z)\right) = c(e_j) + \frac{1}{\sqrt{t}}\big(\langle S(Z)e_j, f^H_p\rangle_{Z_0} + \mathcal{O}(|Z|^2)\big) f^p \wedge . \tag{6.27}$$

Then comparing with (5.95) and (5.96), from (6.6), we have

$$
\begin{aligned}
(6.28)\quad L^{\prime 3,(t,v)}_{Z_0,K} = &-(1-\rho^2(\sqrt{t}Z))\Delta^{TX} + \rho^2(\sqrt{t}Z)\cdot\Big\{-g^{ij}(\sqrt{t}Z)\left(\nabla''_{e_i}\nabla''_{e_j} - \Gamma^k_{ij}(\sqrt{t}Z)\sqrt{t}\nabla''_{e_k}\right)\\
&+\frac{t}{2}\left(R^{\mathcal{E}/\mathcal{S}}_{(Z_0,\sqrt{t}Z)}(e_i,e_j) - \frac{1}{2v}\langle\nabla^{TX}_{e_i}K^X, e_j\rangle_{(Z_0,\sqrt{t}Z)}\right)\left[c\left(\tau^{Z_0}e_i(\sqrt{t}Z)\right)c\left(\tau^{Z_0}e_j(\sqrt{t}Z)\right)\right]^3_{(t,v)}\\
&+\sqrt{t}\left(R^{\mathcal{E}/\mathcal{S}}_{(Z_0,\sqrt{t}Z)}(e_i,f^H_p) - \frac{1}{2v}\langle T(e_i,f^H_p),K^X\rangle_{(Z_0,\sqrt{t}Z)}\right)\left[c\left(\tau^{Z_0}e_i(\sqrt{t}Z)\right)\right]^3_{(t,v)} f^p\wedge\\
&+\frac{1}{2}\left(R^{\mathcal{E}/\mathcal{S}}_{(Z_0,\sqrt{t}Z)}(f^H_p,f^H_q) - \frac{1}{8v}\langle T(f^H_p,f^H_q),K^X\rangle_{(Z_0,\sqrt{t}Z)}\right) f^p\wedge f^q\wedge\\
&+\frac{t}{4}H_{(Z_0,\sqrt{t}Z)} - m^{\mathcal{E}/\mathcal{S}}_{(Z_0,\sqrt{t}Z)}(K^X) + \frac{1}{4v}|K^X(Z_0,\sqrt{t}Z)|^2\Big\},
\end{aligned}
$$

where

(6.29)

$$
\begin{aligned}
\nabla''_{e_i} = &\nabla_{\tau^{Z_0}e_i(\sqrt{t}Z)} + \frac{t}{8}\Big(\langle R^{TX}_{Z_0}(e_k,e_l)Z,e_i\rangle + \mathcal{O}(\sqrt{t}|Z|^2)\Big)\left[c\left(\tau^{Z_0}e_k(\sqrt{t}Z)\right)c\left(\tau^{Z_0}e_l(\sqrt{t}Z)\right)\right]^3_{(t,v)}\\
&+\frac{\sqrt{t}}{4}\Big(\langle R^{TX}_{Z_0}(e_k,f^H_p)Z,e_i\rangle + \mathcal{O}(\sqrt{t}|Z|^2)\Big)\left[c\left(\tau^{Z_0}e_k(\sqrt{t}Z)\right)\right]^3_{(t,v)} f^p\wedge\\
&+\frac{1}{8}\Big(\langle R^{TX}_{Z_0}(f^H_p,f^H_q)Z,e_i\rangle + \mathcal{O}(\sqrt{t}|Z|^2)\Big)f^p\wedge f^q\wedge + \frac{t}{2}\left(R^E_{Z_0}(Z,e_i) + \mathcal{O}(\sqrt{t}|Z|^2)\right)\\
&-\frac{1}{4}\left(1+\frac{t}{v}\right)\langle m^{TX}_{Z_0}(K)Z,e_i\rangle + \sqrt{t}h_i(K,\sqrt{t}Z)\left(\frac{1}{t}+\frac{1}{v}\right).
\end{aligned}
$$

Here $h_i(K,Z)$ is a function depending linearly on K and $h_i(K,Z)=\mathcal{O}(|Z|^2)$ for $|K|$ bounded.

Let $\psi_v \in \operatorname{End}(\Lambda(T^*X^g))$ be the morphism of exterior algebras such that

$$
(6.30)\qquad \begin{aligned}
&\psi_v(e^j) = e^j, \quad 1\le j\le \ell',\\
&\psi_v(e^j) = \sqrt{v}e^j, \quad \ell'+1\le j\le \ell.
\end{aligned}
$$

Recall that for $x=(y_0,Z_0)\in X^g$, $\Lambda(T^*X^g)_{(y_0,Z_0)}$ has been identified with $\Lambda(T^*X^g)_{y_0}$.

Definition 6.6. Let $L^{\prime 3,(0,v)}_{x,K}$ be the operator

$$
(6.31)\qquad L^{\prime 3,(0,v)}_{x,K} = \psi_v L^{3,(0,v)}_{x,K}\psi_v^{-1}.
$$

By Definitions 6.5 and 6.6, (6.13) and (6.30), as $t\to 0$,

$$
(6.32)\qquad L^{\prime 3,(t,v)}_{Z_0,K} \to L^{\prime 3,(0,v)}_{(y_0,Z_0),K}.
$$

6.6. **A family of norms.** For $0\le p\le \ell'$, $0\le q\le \ell-\ell'$, put

$$
(6.33)\qquad \Lambda^{(p,q)}(T^*X^g)_{y_0} = \Lambda^p(T^*X^{g,K})_{y_0}\widehat{\otimes}\Lambda^q(N^*_{X^{g,K}/X^g})_{y_0}.
$$

The various $\Lambda^{(p,q)}(T^*X^g)_{y_0}$ are mutually orthogonal in $\Lambda(T^*X^g)_{y_0}$. Let $\mathbf{I}_{y_0}$ be the vector space of smooth sections of $(\pi^*\Lambda(T^*B)\otimes\Lambda(T^*X^g)\widehat{\otimes}\mathcal{S}_N\otimes E)_{y_0}$ on $T_{y_0}X$, let $\mathbf{I}_{(r,p,q),y_0}$ be the vector space of smooth sections of $(\pi^*\Lambda^r(T^*B)\otimes\Lambda^{(p,q)}(T^*X^g)\widehat{\otimes}\mathcal{S}_N\otimes E)_x$ on $T_{y_0}X$. Let $\mathbf{I}^0_{y_0}, \mathbf{I}^0_{(r,p,q),y_0}$ be the corresponding vector spaces of square-integrable sections.

Now we imitate constructions in [21, §11]. Recall that $\dim B = k$.

Definition 6.7. For $t\in[0,1]$, $v\in\mathbb{R}_+^*$, $y_0\in X^{g,K}$, $Z_0\in N_{X^{g,K}/X^g,y_0}$, $|Z_0|\le \varepsilon_0/\sqrt{v}$, $s\in\mathbf{I}_{(r,p,q),y_0}$, set

$$|s|^2_{t,v,Z_0,0}=\int_{T_{y_0}X}|s(Z)|^2\left(1+\big(|Z_0|+|Z|\big)\rho\left(\frac{\sqrt{t}Z}{2}\right)\right)^{2(k+\ell'-p-r)}\cdot\left(1+\sqrt{v}|Z|\rho\left(\frac{\sqrt{t}Z}{2}\right)\right)^{2(\ell-\ell'-q)}dv_{TX}(Z). \tag{6.34}$$

Then (6.34) induces a Hermitian product $\langle\cdot,\cdot\rangle_{t,v,Z_0,0}$ on $\mathbf{I}^0_{(r,p,q),y_0}$. We equip $\mathbf{I}^0_{y_0}=\bigoplus\mathbf{I}^0_{(r,p,q),y_0}$ with the direct sum of these Hermitian metrics.

Recall that by (5.84), if $\rho(\sqrt{t}Z)>0$, then $|\sqrt{t}Z|\le 4\varepsilon_0$. The proof of the following proposition is almost the same as that of [19, Proposition 8.16] (cf. also [21, Proposition 11.24]).

Proposition 6.8. *For $t\in(0,1]$, $v\in[t,1]$, $y_0\in X^{g,K}$, $Z_0\in N_{X^{g,K}/X^g,y_0}$, $|Z_0|\le\varepsilon_0/\sqrt{v}$, the following family of operators acting on $(\mathbf{I}^0_{y_0},|\cdot|_{t,v,Z_0,0})$ are uniformly bounded:*

$$\begin{aligned}&1_{|\sqrt{t}Z|\le 4\varepsilon_0}\sqrt{t}c_t(e_j),\ 1_{|\sqrt{t}Z|\le 4\varepsilon_0}|Z|\sqrt{t}c_t(e_j),\ 1_{|\sqrt{t}Z|\le 4\varepsilon_0}|Z_0|\sqrt{t}c_t(e_j),\ \textit{for}\ 1\le j\le\ell',\\ &1_{|\sqrt{t}Z|\le 4\varepsilon_0}|Z_0|f^p\wedge,\quad 1_{|\sqrt{t}Z|\le 4\varepsilon_0}|Z|f^p\wedge,\\ &1_{|\sqrt{t}Z|\le 4\varepsilon_0}\sqrt{\frac{t}{v}}c_{\frac{t}{v}}(e_j),\quad 1_{|\sqrt{t}Z|\le 4\varepsilon_0}|Z|\sqrt{t}c_{\frac{t}{v}}(e_j),\ \ \textit{for}\ \ell'+1\le j\le\ell.\end{aligned} \tag{6.35}$$

Definition 6.9. For $t\in[0,1]$, $v\in\mathbb{R}_+^*$, $y_0\in X^{g,K}$, $Z_0\in N_{X^{g,K}/X^g,y_0}$, $|Z_0|\le\varepsilon_0/\sqrt{v}$, if $s\in\mathbf{I}_{y_0}$ has compact support, set

$$|s|^2_{t,v,Z_0,1}=|s|^2_{t,v,Z_0,0}+\frac{1}{v}\left|\rho(\sqrt{t}Z)|K^X|(\sqrt{v}Z_0+\sqrt{t}Z)s\right|^2_{t,v,Z_0,0}+\sum_{i=1}^n|\nabla_{e_i}s|^2_{t,v,Z_0,0}. \tag{6.36}$$

Note that $|s|_{t,v,Z_0,1}$ depends explicitly on $K=zK_0$. In fact, $|s|_{t,v,Z_0,1}$ depends on $z\in\mathbb{R}^*$.

Theorem 6.10. *There exist constants $C_i>0$, $i=1,2,3,4$, such that if $t\in(0,1]$, $v\in[t,1]$, $n\in\mathbb{N}$, $y_0\in X^{g,K}$, $Z_0\in N_{X^{g,K}/X^g,y_0}$, $|Z_0|\le\varepsilon_0/\sqrt{v}$, $z\in\mathbb{R}$, $|z|\le 1$, and if the support of $s,s'\in\mathbf{I}_{y_0}$ is included in $\{Z\in T_{y_0}X:|Z|\le n\}$, then*

$$\begin{aligned}&\mathrm{Re}\langle L'^{3,(t,v)}_{\sqrt{v}Z_0,zK_0}s,s\rangle_{t,v,Z_0,0}\ge C_1|s|^2_{t,v,Z_0,1}-C_2(1+|nz|^2)|s|^2_{t,v,Z_0,0},\\ &|\mathrm{Im}\langle L'^{3,(t,v)}_{\sqrt{v}Z_0,zK_0}s,s\rangle_{t,v,Z_0,0}|\le C_3((1+|nz|)|s|_{t,v,Z_0,1}|s|_{t,v,Z_0,0}+|nz|^2|s|^2_{t,v,Z_0,0}),\\ &|\langle L'^{3,(t,v)}_{\sqrt{v}Z_0,zK_0}s,s'\rangle_{t,v,Z_0,0}|\le C_4(1+|nz|^2)|s|_{t,v,Z_0,1}|s'|_{t,v,Z_0,1}.\end{aligned} \tag{6.37}$$

Proof. Comparing with $L^{3,t}_{x,K}$ in (5.95) and (5.112), there are four additional terms in (6.28) which should be estimated:

$$\frac{1}{4v}\left|\rho(\sqrt{t}Z)|zK_0^X|(\sqrt{v}Z_0+\sqrt{t}Z)s\right|^2_{t,v,Z_0,0}, \tag{6.38}$$

$$-\rho^2(\sqrt{t}Z)\frac{t}{4v}\Big\langle\langle\nabla^{TX}_{\tau^{\sqrt{v}Z_0}e_i(\sqrt{t}Z)}zK_0^X(\sqrt{v}Z_0+\sqrt{t}Z),\tau^{\sqrt{v}Z_0}e_j(\sqrt{t}Z)\rangle\cdot\left[c\left(\tau^{\sqrt{v}Z_0}e_i(\sqrt{t}Z)\right)c\left(\tau^{\sqrt{v}Z_0}e_j(\sqrt{t}Z)\right)\right]^3_{(t,v)}s,s\Big\rangle_{t,v,Z_0,0}, \tag{6.39}$$

(6.40)
$$-\rho^2(\sqrt{t}Z)\frac{\sqrt{t}}{2v}\left\langle \langle T(e_i,f_p^H),zK_0^X\rangle(\sqrt{v}Z_0+\sqrt{t}Z)\left[c\left(\tau^{\sqrt{v}Z_0}e_i(\sqrt{t}Z)\right)\right]^3_{(t,v)}f^p\wedge s,s\right\rangle_{t,v,Z_0,0},$$

and

$$-\rho^2(\sqrt{t}Z)\frac{1}{16v}\left\langle \langle T(f_p^H,f_q^H),zK_0^X\rangle(\sqrt{v}Z_0+\sqrt{t}Z)f^p\wedge f^q\wedge s,s\right\rangle_{t,v,Z_0,0}. \tag{6.41}$$

The first term is controlled by (6.36) and the second term was estimated in the proof of [19, Theorem 8.18]. We only need to estimate (6.40) and (6.41), which are new terms in the family case.

By (3.4), $\widetilde{T}$ is G-invariant, thus $[K^X,\widetilde{T}]=0$. Since $m^{TX}(K)$ is skew-adjoint, by (2.5),

$$Z\langle\widetilde{T},K^X\rangle=\langle\nabla^{TX}_Z\widetilde{T},K^X\rangle+\langle\widetilde{T},\nabla^{TX}_ZK^X\rangle=\langle\nabla^{TX}_Z\widetilde{T},K^X\rangle-\langle\nabla^{TX}_{K^X}\widetilde{T},Z\rangle. \tag{6.42}$$

As $y_0\in X^{g,K}\subset X^K$, we know $K^X_{y_0}=0$. Thus from (6.42), we have

$$\frac{\partial}{\partial s}\langle\widetilde{T},K^X\rangle_{(y_0,sZ)}|_{s=0}=0. \tag{6.43}$$

From (6.43), we have

$$\langle\widetilde{T},K^X\rangle_{(y_0,Z)}=\mathcal{O}(|Z|^2). \tag{6.44}$$

Thus we have

$$\begin{aligned}\rho^2(\sqrt{t}Z)\frac{\sqrt{t}}{2v}\langle T(e_i,f_p^H),zK_0^X\rangle(\sqrt{v}Z_0+\sqrt{t}Z)\left[c\left(\tau^{\sqrt{v}Z_0}e_i(\sqrt{t}Z)\right)\right]^3_{(t,v)}f^p\wedge\\ =\rho^2(\sqrt{t}Z)v^{-1}\sqrt{t}|z|\left[c\left(\tau^{\sqrt{v}Z_0}e_i(\sqrt{t}Z)\right)\right]^3_{(t,v)}f^p\wedge\cdot\mathcal{O}((\sqrt{v}|Z_0|+\sqrt{t}|Z|)^2),\end{aligned} \tag{6.45}$$

and

$$\begin{aligned}\rho^2(\sqrt{t}Z)\frac{1}{16v}\langle T(f_p^H,f_q^H),zK_0^X\rangle(\sqrt{v}Z_0+\sqrt{t}Z)f^p\wedge f^q\wedge\\ =\rho^2(\sqrt{t}Z)|z|f^p\wedge f^q\wedge\cdot\mathcal{O}((|Z_0|+\sqrt{t/v}|Z|)^2).\end{aligned} \tag{6.46}$$

Using the fact that $v\leq 1$ and $t/v\leq 1$ and also Proposition 6.8, from (6.27), we find that the operators in (6.45) and (6.46) remain uniformly bounded with respect to $|\cdot|_{t,v,Z_0,0}$.

The proof of Theorem 6.10 is completed. □

Definition 6.11. Put

$$L^{3,(t,v)}_{Z_0,K,n}=-\left(1-\gamma^2\left(\frac{|Z|}{2(n+2)}\right)\right)\Delta^{TX}+\gamma^2\left(\frac{|Z|}{2(n+2)}\right)L^{3,(t,v)}_{Z_0,K}. \tag{6.47}$$

Let $\widetilde{F}_t(L^{3,(t,v)}_{Z_0,K})(Z,Z')$ and $\widetilde{F}_t(L^{3,(t,v)}_{Z_0,K,n})(Z,Z')$ be the smooth kernels associated with $\widetilde{F}_t(L^{3,(t,v)}_{Z_0,K})$ and $\widetilde{F}_t(L^{3,(t,v)}_{Z_0,K,n})$ with respect to $dv_{TX}(Z')$. Using (6.23) and proceeding as in (5.128), i.e., using finite propagation speed, we see that if $Z\in T_{y_0}X$, $|Z|\leq p$,

$$\widetilde{F}_{t,n}(L^{3,(t,v)}_{Z_0,K})(Z,Z')=\widetilde{F}_{t,n}(L^{3,(t,v)}_{Z_0,K,n+p})(Z,Z'). \tag{6.48}$$

Clearly, when replacing $L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0}$ in (6.37) by $L^{3,(t,v)}_{\sqrt{v}Z_0,zK_0,n}$, the estimates (6.37) still hold.

6.7. **A Proof of Theorem 6.3.** Since W is a compact manifold, there exists a finite family of smooth functions $f_1, \cdots, f_r : W \to [-1, 1]$ which have the following properties:

- $W^K = \cap_{j=1}^r \{x \in W : f_j(x) = 0\}$;
- On W^K, $df_1 \cdots, df_r$ span $N_{X^{g,K}/X}$.

Definition 6.12. Let $\mathcal{Q}_{t,v,Z_0}$ be the family of operators

$$\mathcal{Q}_{t,v,Z_0} = \left\{ \nabla_{e_i}, 1 \le i \le \dim X; \frac{z}{\sqrt{v}} \rho(\sqrt{t}Z) f_j(\sqrt{v}Z_0 + \sqrt{t}Z), 1 \le j \le r \right\}. \tag{6.49}$$

For $j \in \mathbb{N}$, let $\mathcal{Q}^j_{t,v,Z_0}$ be the set of operators $Q_1 \cdots Q_j$, with $Q_i \in \mathcal{Q}_{t,v,Z_0}$, $1 \le i \le j$.

Following the arguments in [19, §8.8-8.10], we have the following uniform estimate, which is formally the same as [19, (8.76)]. We only need to take care that in the proof of the analogue of [19, Proposition 8.22 and Theorems 8.23, 8.24], there are two new terms like (6.40) and (6.41) appear in our family case. However, they are easy to be controlled as in (6.45) and (6.46).

Theorem 6.13. *There exist $C > 0$, $C' > 0$ such that given $m > 0$, there exists $\beta_1 > 0$ such that if $t \in (0, 1]$, $v \in [t, 1]$, $z \in \mathbb{R}$, $|z| \le \beta_1$, $y_0 \in X^{g,K}$, $Z_0 \in N_{X^{g,K}/X^g, y_0}$, $|Z_0| \le \varepsilon_0/\sqrt{v}$, $Z \in N_{X^g/X, y_0}$, $|Z| \le \varepsilon_0/\sqrt{t}$,*

$$\begin{aligned} \left| \left(\widetilde{F}_t \left(L^{3,(t,v)}_{\sqrt{v}Z_0, zK_0} \right) - \exp\left(-L^{3,(0,v)}_{\sqrt{v}Z_0, zK_0} \right) \right) (g^{-1}Z, Z) \right| \\ \le C \left(\frac{t}{v} \right)^{\frac{1}{4(\dim X+1)}} \cdot \frac{(1 + |Z_0|)^{\ell'+1}}{(1 + |zZ_0|)^m} \exp\left(-C'|Z|^2/4 \right). \end{aligned} \tag{6.50}$$

The kernel $\exp(-L^{3,(0,v)}_{\sqrt{v}Z_0, zK_0}))(g^{-1}Z, Z)$ here is defined in the same way as in (5.133).

From (6.27), we get

$$\sqrt{\frac{t}{v}} \left[c\left(\tau^{\sqrt{v}Z_0} e_j(\sqrt{t}Z) \right) \right]^3_{(t,v)} = \begin{cases} \frac{1}{\sqrt{v}} e^j \wedge + \sqrt{\frac{t}{v}} \Big(\mathcal{O}(\sqrt{t}) + \mathcal{O}(|Z|) \Big), & \text{if } 1 \le j \le \ell'; \\ e^j \wedge + \sqrt{\frac{t}{v}} \mathcal{O}(1 + |Z|), & \text{if } \ell' + 1 \le j \le \ell; \\ \sqrt{\frac{t}{v}} \big(c(e_j) + \mathcal{O}(|Z|) \big), & \text{if } \ell + 1 \le j \le n. \end{cases} \tag{6.51}$$

Moreover as K^X vanishes on $W^{g,K}$, we have

$$\begin{aligned} \langle K_0^X(\sqrt{v}Z_0 + \sqrt{t}Z), \tau^{\sqrt{v}Z_0} e_j(\sqrt{t}Z) \rangle &= \langle K_0^X(\sqrt{v}Z_0), \tau^{\sqrt{v}Z_0} e_j \rangle_{(y_0, \sqrt{v}Z_0)} + \mathcal{O}(\sqrt{t}|Z|), \\ \langle K_0^X(\sqrt{v}Z_0), \tau^{\sqrt{v}Z_0} e_j \rangle_{(y_0, \sqrt{v}Z_0)} &= \mathcal{O}(\sqrt{v}|Z_0|). \end{aligned} \tag{6.52}$$

By (3.1), (6.30), (6.51) and (6.52), we get

$$\frac{\sqrt{t}}{4v} \left[c\left(K^X_{\sqrt{v}|Z_0| + \sqrt{t}|Z|} \right) \right]^3_{(t,v)} = \left(\psi_v \frac{j^*\vartheta_K}{4v} \psi_v^{-1} \right)_{\sqrt{v}Z_0} + \frac{\sqrt{t}}{v} z \mathcal{O}(\sqrt{v}|Z_0| + \sqrt{t}|Z|) \mathcal{O}(1 + |Z|). \tag{6.53}$$

Note that we have $(\delta_v^* \alpha)_{Z_0} = (\psi_v \alpha \psi_v^{-1})_{\sqrt{v}Z_0}$ for any $\alpha \in \Omega(W^g)$ with δ_v defined above (3.12). Therefore, from (3.18), (6.50) and (6.53), we get Theorem 6.3.

6.8. **A Proof of Theorem 4.3 b).** Theorem 4.3 b) follows directly from the following theorem.

Theorem 6.14. *There exist $\beta_1 > 0$, $r \in \mathbb{N}$, $C > 0$, $\delta \in (0,1]$, such that if $t \in (0,1]$, $v \in [t,1]$, if $z \in \mathbb{R}\backslash\{0\}$, $|z| \leq \beta_1$, then*

$$|z|^r \left| \phi \widetilde{\mathrm{Tr}}' \left[g \frac{\sqrt{t} c(K^X)}{4v} \exp\left(-\mathcal{B}_{K,t,v}\right) \right] + \widetilde{e}_v \right| \leq C \left(\frac{t}{v} \right)^{\delta}. \tag{6.54}$$

Proof. Recall that $\mathcal{U}_\varepsilon$, $\mathcal{U}'_\varepsilon$, $\mathcal{U}''_\varepsilon$ are ε-neighborhoods of X^g, $X^{g,K}$, $X^{g,K}$ in $N_{X^g/X}$, $N_{X^{g,K}/X}$, $N_{X^{g,K}/X^g}$ respectively. Let $\bar{k}(y_0, Z_0)$ be the function defined on $X^g \cap \mathcal{U}''_\varepsilon$ by the relation

$$dv_{X^g}(y_0, Z_0) = \bar{k}(y_0, Z_0) dv_{X^{g,K}}(y_0) dv_{N_{X^{g,K}/X^g}}(Z_0). \tag{6.55}$$

Then

$$\bar{k}|_{X^{g,K}} = 1. \tag{6.56}$$

Recall that $\widetilde{F}_t(\mathcal{B}_{K,t,v})(g^{-1}x, x)$ vanishes on $X\backslash\mathcal{U}_{\varepsilon_0}$. Using (5.75), (6.55), we get

$$\begin{aligned}
\phi \int_{\mathcal{U}'_{\varepsilon_0}} \widetilde{\mathrm{Tr}}' \left[g \frac{\sqrt{t} c(K^X)}{4v} \exp\left(-\mathcal{B}_{K,t,v}\right)(g^{-1}x, x) \right] dv_X(x) + \int_{X^g \cap \mathcal{U}'_{\varepsilon_0}} \{\gamma_{K,v}\}^{\max} dv_{X^g} \\
= \int_{y_0 \in X^{g,K}} v^{(\ell-\ell')/2} \int_{|Z_0| \leq \varepsilon_0/\sqrt{v}} \Bigg[\phi \int_{|Z| \leq \varepsilon_0} \widetilde{\mathrm{Tr}}' \Bigg[g \frac{\sqrt{t} c(K^X)}{4v} \exp\left(-\mathcal{B}_{K,t,v}\right) \\
(g^{-1}(y_0, \sqrt{v} Z_0, Z), (y_0, \sqrt{v} Z_0, Z)) \Bigg] \cdot k(y_0, \sqrt{v} Z_0, Z) dv_{N_{X^g/X}}(Z) \\
+ \{\gamma_{K,v}\}^{\max}(y_0, \sqrt{v} Z_0) \Bigg] \bar{k}(y_0, \sqrt{v} Z_0) dv_{N_{X^{g,K}/X^g}}(Z_0) dv_{X^{g,K}}(y_0).
\end{aligned} \tag{6.57}$$

Using Theorem 6.3 and (6.57), we find that there exist $C > 0$ and $\beta_1 > 0$ such that for $z \in \mathbb{R}^*$, $|z| \leq \beta_1$,

$$\begin{aligned}
|z|^{\ell+1} \left| \phi \int_{\mathcal{U}'_{\varepsilon_0}} \widetilde{\mathrm{Tr}}' \left[g \frac{\sqrt{t} c(K^X)}{4v} \exp\left(-\mathcal{B}_{K,t,v}\right)(g^{-1}x, x) \right] dv_X(x) + \int_{X^g \cap \mathcal{U}'_{\varepsilon_0}} \gamma_{K,v} \right| \\
\leq C |z|^{\ell-\ell'} \int_{y_0 \in X^{g,K}} \int_{Z_0 \in N_{X^{g,K}/X^g}, |Z_0| < \varepsilon_0} (1 + |z Z_0|)^{-(\ell-\ell')-1} dZ_0 \cdot \left(\frac{t}{v} \right)^{\delta} \leq C \left(\frac{t}{v} \right)^{\delta}.
\end{aligned} \tag{6.58}$$

Similar estimates can be obtained for

$$\left| \phi \int_{X\backslash\mathcal{U}'_{\varepsilon_0}} \widetilde{\mathrm{Tr}}' \left[g \frac{\sqrt{t} c(K^X)}{4v} \exp\left(-\mathcal{B}_{K,t,v}\right)(g^{-1}x, x) \right] dv_X(x) + \int_{X^g \backslash \mathcal{U}'_{\varepsilon_0}} \gamma_{K,v} \right|. \tag{6.59}$$

In fact, on $X\backslash\mathcal{U}'_{\varepsilon_0}$, we observe that $|K^X|^2/2v$ has a positive lower bound. Then we adopt the above techniques to the case where $X^{g,K} = \emptyset$. The potentially annoying term $\frac{\sqrt{t}c(K^X)}{4v}$ can be controlled by the term $|K^X|^2/2v$.

The proof of Theorem 6.14 is completed. ∎

6.9. **A Proof of Theorem 4.3 c).** When $v \in [1,+\infty)$, $\frac{1}{v}$ remains bounded. By using the methods of the last section and of the present section, one sees easily that for $K_0 \in \mathfrak{z}(g)$, $K = zK_0$, there exist $C > 0$, $\beta > 0$ such that for $t \in (0,1]$, $v \in [1,+\infty)$, $|zK_0| < \beta$, we have

$$\left|\widetilde{\mathrm{Tr}}'\left[g\sqrt{t}c(K^X)\exp\left(-\mathcal{B}_{K,t,v}\right)\right]\right| \leq C, \tag{6.60}$$

which is equivalent to Theorem 4.3 c).

The proof of Theorem 4.3 c) is completed.

6.10. **A Proof of Theorem 4.3 d).** In this subsection, we will prove Theorem 4.3 d) by using the method in [19, §9]. Since the singular term there does not appear here, our proof is in fact much easier.

We fix $g \in G$, $0 \neq K_0 \in \mathfrak{z}(g)$, and take $K = zK_0$ with $z \in \mathbb{R}^*$.

From Theorem 6.1, we have

$$\begin{aligned}\mathcal{B}_{K,t,tv} = -t\Bigg(\nabla^{\mathcal{E}}_{e_i} &+ \frac{1}{2\sqrt{t}}\langle S(e_i)e_j, f^H_p\rangle c(e_j)f^p\wedge \\ &+\frac{1}{4t}\langle S(e_i)f^H_p, f^H_q\rangle f^p\wedge f^q\wedge - \frac{\langle K^X, e_i\rangle}{4t}\left(1+\frac{1}{v}\right)\Bigg)^2 \\ &+\frac{t}{4}H + \frac{t}{2}\left(R^{\mathcal{E}/\mathcal{S}}(e_i,e_j) - \frac{1}{2vt}\langle\nabla^{TX}_{e_i}K^X, e_j\rangle\right)c(e_i)c(e_j) \\ &+\sqrt{t}\left(R^{\mathcal{E}/\mathcal{S}}(e_i,f^H_p) - \frac{1}{2vt}\langle T(e_i,f^H_p),K^X\rangle\right)c(e_i)f^p\wedge \\ +\frac{1}{2}\left(R^{\mathcal{E}/\mathcal{S}}(f^H_p,f^H_q) - \frac{1}{8vt}\langle T(f^H_p,f^H_q),K^X\rangle\right)&f^p\wedge f^q\wedge - m^{\mathcal{E}/\mathcal{S}}(K^X) + \frac{1}{4vt}|K^X|^2.\end{aligned} \tag{6.61}$$

As in sections 5.3 and 6.3, the proof of Theorem 4.3 d) can be localized near X^g. In the following, we will concentrate on the estimates near $X^{g,K}$. As in (6.59), the proof of the estimates near X^g and far from $X^{g,K}$ is much easier.

We may assume that for ε_0 taken in Section 6.4, if $\varepsilon \in (0, 8\varepsilon_0]$, the map $(y_0, Z) \in N_{X^{g,K}/X} \to \exp^X_{y_0}(Z) \in X$ induces a diffeomorphism from the ε-neighborhood $\mathcal{U}'_\varepsilon$ of $X^{g,K}$ in $N_{X^{g,K}/X}$ on the tubular neighborhood $\mathcal{V}'_\varepsilon$ of $X^{g,K}$ in X as in the proof of Theorem 6.14.

As in (5.77) and (6.9), we put

$$\begin{aligned}{}^3\nabla^{\mathcal{E},t}_{\cdot} := \nabla^{\mathcal{E}}_{\cdot} &+ \frac{1}{2\sqrt{t}}\langle S(\cdot)e_j, f^H_p\rangle c(e_j)f^p\wedge \\ &+\frac{1}{4t}\langle S(\cdot)f^H_p, f^H_q\rangle f^p\wedge f^q\wedge - \frac{\vartheta_K(\cdot)}{4t}\left(1+\frac{1}{v}\right).\end{aligned} \tag{6.62}$$

Take $y_0 \in W^{g,K}$ in (2.12). If $Z \in N_{X^{g,K}/X,y_0}$, $|Z| \leq 4\varepsilon_0$, we identify $\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}_Z$ with $\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}_{y_0}$ by parallel transport with respect to the connection ${}^3\nabla^{\mathcal{E},t}$ along the curve $u \in [0,1] \to uZ$.

Recall that ρ is the cut-off function in (5.84). Let

$$L^{1,(t,v)}_{y_0,K} = (1-\rho^2(Z))(-t\Delta^{TX}) + \rho^2(Z)(\mathcal{B}_{K,t,tv}). \tag{6.63}$$

We still define H_t as in (5.88) and define $L^{2,(t,v)}_{y_0,K}$ as in (5.89) from $L^{1,(t,v)}_{y_0,K}$. Let $L^{3,(t,v)}_{y_0,K}$ be the operator obtained from $L^{2,(t,v)}_{y_0,K}$ by replacing $c(e_j)$ by $c_t(e_j)$ as in (5.91) for $1 \leq j \leq \ell'$ (cf. (2.14)), while leaving the $c(e_j)$'s unchanged for $\ell' + 1 \leq j \leq n$.

As in (3.21), we have

$$|K^X(y_0, Z)|^2 = |m^{TX}_{y_0}(K)Z|^2 + \mathcal{O}(|Z|^3). \tag{6.64}$$

Let $\jmath' : W^{g,K} \to W$ be the obvious embedding. Put

$$\begin{aligned} L^{3,(0,v)}_{y_0,K} = &- \left(\nabla_{e_i} + \frac{1}{4}\left\langle \left(\jmath'^* R^{TX} - \left(1 + \frac{1}{v}\right) m^{TX}(K)\right) Z, e_i \right\rangle\right)^2 \\ &+ \jmath'^* R^E_{y_0} - m^E(K)_{y_0} - \frac{1}{4v} \sum_{j,k \geq \ell'+1} \langle m^{TX}(K) e_j, e_k\rangle_{y_0} c(e_j) c(e_k) \\ &+ \frac{1}{4v} \langle \jmath'^* R^{TX}(m^{TX}(K)Z), Z\rangle_{y_0} + \frac{1}{4v} |m^{TX}_{y_0}(K)Z|^2. \end{aligned} \tag{6.65}$$

From (3.5), (3.19), (3.20), (6.44), (6.61) and (6.65), as Proposition 5.24, we have

$$L^{3,(t,v)}_{y_0,K} \to L^{3,(0,v)}_{y_0,K}. \tag{6.66}$$

Now we take a new trivialization as in Section 6.5. Take $Z_0 \in N_{X^{g,K}/X^g, y_0}, |Z_0| \leq \varepsilon_0$. If $Z \in T_{y_0}X$, $|Z| \leq 4\varepsilon_0$, we identify $\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}_{Z+Z_0}$ with $\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}_{Z_0}$ by parallel transport along the curve $u \in [0,1] \to \exp^X_{Z_0}(uZ)$ with respect to the connection ${}^3\nabla^{\mathcal{E},t}$. Also we identify $\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}_{Z_0}$ with $\pi^*\Lambda(T^*B)\widehat{\otimes}\mathcal{E}_{y_0}$ by parallel transport along the curve $u \in [0,1] \to uZ_0$ with respect to the connection $\nabla^{\mathcal{E}}$. Using this trivialization, the analogues of [19, Theorems 9.19 and 9.22] hold here following the same arguments except for replacing the norm in [19, (9.43)] by

$$|s|^2_{t,Z_0,0} = \int_{T_{y_0}X} |s(Z)|^2 \left(1 + (|Z| + |Z_0|)\,\rho\left(\frac{\sqrt{t}Z}{2}\right)\right)^{2(k+\ell'-p-r)} dv_{TX}(Z). \tag{6.67}$$

Here s is a square integrable section of $\left(\pi^*\Lambda^r(T^*B)\widehat{\otimes}\Lambda^p(T^*X^{g,K})\widehat{\otimes}\mathcal{S}_{N_{X^{g,K}/X}} \otimes E\right)_{y_0}$ over $T_{y_0}X$, and $\dim B = k$.

As in [19, (9.52)-(9.57)], combining with (3.18), if n is even, there exists $\beta > 0$, if $z \in \mathbb{R}^*$, $|z| \leq \beta$, for $t \to 0$,

$$\begin{aligned} &\int_{X^{g,K}} \int_{\substack{(Z_0,Z)\in N_{X^{g,K}/X^g}\times N_{X^g/X}, \\ |Z_0|,|Z|\leq\varepsilon_0}} \mathrm{Tr}_s\left[g \frac{c(K^X)}{4\sqrt{tv}} \widetilde{F}_t(\mathcal{B}_{zK_0,t,tv})(g^{-1}(y_0,Z_0,Z),\right. \\ &\qquad\qquad \left.(y_0,Z_0,Z))\right] dv_X(y_0,Z_0,Z) \\ &\to \int_{X^{g,K}} \int_{N_{X^{g,K}/X}} (-i)^{\ell'/2} 2^{\ell'/2}\, \mathrm{Tr}_s\left[g \frac{c\left(m^{TX}_{y_0}(K)Z\right)}{4v} \exp\left(-L^{3,(0,v)}_{y_0,zK_0}\right)(g^{-1}Z, Z)\right] dv_{N_{X^{g,K}/X}}(Z). \end{aligned} \tag{6.68}$$

The heat kernel $\exp\left(-L^{3,(0,v)}_{y_0,zK_0}\right)(g^{-1}Z,Z)$ could be calculated as in (5.142) by [19, Theorem 4.13], which is an even function on Z and can be controlled by $C\exp(-C'|Z|^2)$. So the right-hand side of (6.68) is an integral of an odd function on Z over $N_{X^{g,K}/X}$, which is zero.

If n is odd, by Remark 5.22, from the same argument above, as $t \to 0$,

$$\int_{\mathcal{U}'_{\varepsilon_0}} \mathrm{Tr}^{\mathrm{even}} \left[g \frac{c(K^X)}{4\sqrt{t}v} \exp\left(-\mathcal{B}_{K,t,tv}\right) \right] \to 0. \tag{6.69}$$

After adopting the above technique to the case where $X^{g,K} = \emptyset$, for $z \in \mathbb{R}^*$, $|z|$ small enough, as $t \to 0$, we have

$$\int_{X \setminus \mathcal{U}'_{\varepsilon_0}} \widetilde{\mathrm{Tr}}' \left[g \frac{c(K^X)}{4\sqrt{t}v} \exp\left(-\mathcal{B}_{K,t,tv}\right) \right] \to 0. \tag{6.70}$$

The proof of Theorem 4.3 d) is completed.

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