h-Laplacians on Singular Sets

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June 19, 2022

Abstract

Until now, the correspondence between the Alexander-Kolmogorov Complex, and the De Rham one, by means of a small scale parameter, has not gone that far as passing to the limit of the resolvent of the associated Laplacian, when the small parameter tends towards zero. In this line, a result proving a complete Hodge decomposition was missing. We bridge this gap by means of our own rescaled h-cohomology, h being a very small parameter. Passing to the limit of the resolvent enables us to consider the extension to singular spaces, in particular, our h-differential operators also enable us to also make the connection with those of analysis on fractals, as introduced by Jun Kigami, and taken up by Robert S. Strichartz.

MSC Classification: 28A80-35R02.

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1 Introduction

How could one define differentiation and integration on general topological sets? This is the problem that James Waddell Alexander and Andreï Nikolaïevitch Kolmogorov tried to solve with chains and cochains [Buc07].

The underlying idea was given by Kolmogorov himself [Buc07], [Kol37]:

"The author's goal is to construct a particular difference calculus which, on the one hand, leads to differential operators acting on antisymmetric tensors (multivectors) by a limit process, and on the other hand is closely related to the concepts of combinatorial topology.

In particular, it is possible to define new invariants of complexes and closed sets using this difference calculus, and to prove some generalizations of the known duality theorems."

One may see the underlying perspectives, especially, defining and handling differential on non-smooth objects, by means of simplices and the associated cohomology groups.

This is only the first step. What happens when those objects are very small, either since their measure tends towards zero, or when they belong to an everywhere singular set, of fractal type ?

Let us recall, first, the context and the existing works on connected subjects. It is often taken for granted that de Rham differential forms are limits of suitably rescaled Alexander-Spanier cochains. One has to be be much more precise as soon as one ventures to this terrain. In the work by Alain Connes and Henri Moscovici [CM90] (mainly devoted to a proof of the Novikov conjecture for hyperbolic groups), the authors review the Alexander-Spanier realization of the cohomology of a smooth manifold M, as it can be found in the original work by Edwin H. Spanier [Spa95] (Chapter 6, *General Cohomology Theory and Duality*). The main topic is the definition of an homomorphism of complexes between the (quotient) Alexander-Spanier complex associated to the cohomology, $\bar{C}^\star(M)$, and the De Rham one, $\Lambda^\star(M)$ (recall that given a cochain complex $C^\star(M) = \{C^p(M), \delta\}$, and the subcomplex $C_0^\star(M) = \{C_0^p(M), \delta\} \subset C^\star(M)$, where $C_0^p(M)$ denotes the set of functions from M^{p+1} to $\mathbb{R}$ which vanish on a neighborhood of the p^{th} diagonal of M, $\bar{C}^\star(M)$ is simply the complex quotient of $\bar{C}^\star(M)$ by $C_0^\star(M)$ – a very natural way of doing, the fact that a function vanishes in a given region necessary implying the same feature for the differential). To the aforementioned purpose, the manifold is endowed with a Riemannian metric, while considering an open covering $\mathcal{B}$ which satisfies specific properties. The rescaling is obtained by means of this covering. However, the isomorphism is not explicited. The second work one might think of is the one by Laurent Bartholdi, Thomas Schick, Stephen Smale, and Nathan Smale, on abstract and classical Hodge-de Rham theory in [BSSS12], followed up by the results from the last two authors in [SS12], where, given a compact Riemannian manifold M, the authors build cochain maps between the de Rham complex of M, $\Lambda^\star(M)$, and the Alexander-Spanier one $\bar{C}^\star(M)$ at a scale $\alpha > 0$, for sufficiently small values of the parameter α. The authors go as far as comparing the Hodge Laplacian on differential forms, and a suitably rescaled one on the space of cochains at scale α. It is shown, in the case of functions, and when α tends towards zero, that the rescaled Laplacian on cochains converges towards the Hodge operator.

Note that the techniques used in the aforementioned Connes and Moscovici paper are different than the ones of our work. First, A. Connes and H. Moscovici integrate differential forms on simplices. Then, they involve the exponential map, which result in numerous and heavy computations. As for the work by Smale et al., it is, also, based on the use of integral operators. Given a metric d, and the scale parameter α, they consider the α-neighborhood of the diagonal in the product manifold X^p, denoted by U_α, and the associated spaces $L^2_{alt}\left(U^p_\alpha\right)$ and $C^\infty\left(U^p_\alpha\right)$ of alternating functions of respectively L^2 and C^∞_{alt} class on U^p_α. They prove, for any integer p, the isomorphism between the α-

scale subspace $\mathrm{Harm}_\alpha{}^p(M) \subset L^2_{alt}\left(U^p_\alpha\right)$ of harmonic p-forms on the manifold, and the cohomology in degree p of the respective complexes

$$0 \longrightarrow L^2\left(M\right) \xrightarrow{\delta^1} L^2_{alt}\left(U^2_\alpha\right) \xrightarrow{\delta^2} \ldots \xrightarrow{\delta^{p-1}} L^2_{alt}\left(U^p_\alpha\right) \xrightarrow{\delta^p} L^{p+1}_{alt}\left(U^{p+1}_\alpha\right) \xrightarrow{\delta^{p+1}} \ldots$$

and

$$0 \longrightarrow C^\infty\left(M\right) \xrightarrow{\delta^1} C^\infty_{alt}\left(U^2_\alpha\right) \xrightarrow{\delta^2} \ldots \xrightarrow{\delta^{p-1}} C^\infty_{alt}\left(U^p_\alpha\right) \xrightarrow{\delta^p} C^\infty_{alt}\left(U^{p+1}_\alpha\right) \xrightarrow{\delta^{p+1}} \ldots$$

where δ denotes the classical Alexander-Spanier coboundary operator. They also show the isomorphism with the De Rham cohomology. As is not the case in the Connes and Moscovici paper, the isomorphism is given explicitely. However, the result only concerns the cohomology, i.e., the quotiented kernels of the coboundary operator. Moreover, in so far as they solely deal with harmonic forms, they do not have the associated Hodge theory, which can only be obtained by means of a suitable renormalization. Things are easier to handle, when only dealing with harmonic forms. In this line, there is thus no result regarding the limit of the resolvent when the scale parameter tends towards zero.

As for a general theory of differential operators on fractals, the problem was tackled by Fabio Cipriani and Jean-Luc Sauvageot in [CS09]. The authors place themselves in the line of noncommutative geometry *à la Connes*, where, given a compact topological space K, a continuous function f on K is represented by means of a bounded operator $\pi(f)$, which acts on a Hilbert space $\mathcal{H}$. If F denotes a self-adjoint operator of square 1, acting on $\mathcal{H}$, the (commutator) operator $df = i\,[F, \pi(f)]$, where $i^2 = -1$, stands for a "substitute" of the differential of f. In the case of post critically finite (p.c.f) fractals [1], the authors build Fredholm modules, in relation with the self-similar Dirichlet form $\mathcal{E}$ associated to the self-similar fractal [2]. A key result of the Cipriani and Sauvageot paper is their Proposition 3.1., where they exhibit the existence of a "essentially unique derivation", denoted by ∂, defined on the Dirichlet algebra $C(K) \cap \mathcal{F}$, taking its values in a real Hilbert module $\mathcal{H}$, and which is a differential square root of the Dirichlet form $\mathcal{E}$. In other words, this means that the algebra of continuous functions on K acts in a continuous way, and that the classical Leibniz rule for the derivative of a product is true. Then, by using the Fredholm modules, the authors are able to associate, to each harmonic structure, a topological invariant of the considered compact topological space K, the "K-homology class of the Fredholm module".

In [IRT12], Marius Ionescu, Luke G. Rogers and Alexander Teplyaev go further, and give an explicit description of the elements of the aforementioned Hilbert module $\mathcal{H}$. A very interesting feature of this work is the authors are able to give "a direct sum decomposition of this module to piecewise harmonic components that correspond to the cellular structure of the fractal". They go as far as giving an analog of the Hodge decomposition for $\mathcal{H}$.

A completely different approach has been developped by Michel L. Lapidus and Machiel van Frankenhuisjen in [LvF16], [LvF13] and [LvF00], where the authors suggest that there should exist a fractal cohomology having direct links with the theory of Complex Dimensions, introduced by Michel L. Lapidus and his collaborators in [Lap91], [Lap92], [Lap93], [LP93], [LM95], [LvF00], [LP06], [Lap08], [LPW11], [ELMR15], [LvF13], [LRŽ17], [LRŽ18], [Lap19], [HL21] and [Lap22]. Further results have been obtained by Michel L. Lapidus and Tim Cobler in [CL17], where they study the properties of the derivative operator $D = \dfrac{d}{dz}$ on a particular family of weighted Bergman space constituted of entire functions on $\mathbb{C}$.

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[1] For the reader who might not be familiar with those notions, we refer to the book of Jun Kigami [Kig01], Chapter 1, Section 1.3., Definition 1.3.13, page 23.

[2] See the seminal works of Arne Beurling and Jacques Deny in [BD85b], along with the aforementioned book [Kig01], Chapter 2.

We hereafter place ourselves in the same kind of perspective. To begin with, we generalize the algebraic notion of chains (instead of cochains), to what we call fermions. Then, we redefine the concept of h-differentiation, where h denotes a very small real parameter. We go so far as connecting the associated h-cohomology to the classical De Rham one, much more simpler than what can be found in the existing work. If we rely on analogous tools that happen to be the same as the ones that can be found in the Smale et al. work (for instance, the diagonal of the product manifold, and the same explicit isomorphism), our approach takes a completely different turn: in fact, we are the only ones to pass to the limit of the resolvent, when the scale parameter tends towards zero.

This very powerful result enables us to consider the special cases of singular spaces. In fact, the h-differentiation, connected, as one could foresee, to the notion of boundary, leads to a local operator, equivalent to the classical Riemanniann Laplacian, which may act on singular objects. When the parameter h tends towards zero, one recovers the usual Laplacian.

A natural question that may be asked is wether this Laplacian the same as the one of fractal analysis introduced by Jun Kigami [Kig01] and Robert S. Strichartz [Str06]? This question is all the more interesting, as laplacians on fractals are defined by means of local differences – the starting point being graph Laplacians. More precisely, one uses Dirichlet forms, built by induction on a sequence of prefractals, i.e., a sequence of finite graphs which converge towards the considered fractal set. For a continuous function on this set, and subject to existence, its Laplacian is obtained as the renormalized limit of the sequence of graph Laplacians. At first sight, one cannot be sure that this operator is the same as the usual Riemannian one – one understands that it is an operator of the same nature, but further? Another concern comes from the fact that changing the measure also changes the Laplacian! The problem is even less obvious as such an operator is not of order two: existing works on the Sierpiński Gasket [Str03], or on the Weierstrass Curve [DL20], show that the order is greater than two.

Our differential is completely different from the one of Cipriani and Sauvageot, hence, also from the differential of Ionescu, Rogers and Teplyaev. It relyes the use of paths across the consecutive prefractal graphs. The detailed study of these differentials is the object of our following work [DL22].

The main results obtained in this paper can be found in the following places:

i. In Definition 5.7, where, for the small parameter $h > 0$, we define the h-Laplacian, Δ_h, acting on the space of p-forms, for $p \in \mathbb{N}$.

ii. In Theorem 5.8, where we pass to the limit of the resolvent of the h-Laplacian, $(z - |\Delta_h|)^{-1}$, when the scale parameter h tends towards zero. This results requires the introduction of a modified scalar product on the space of p-forms (see Proposition 5.5), a compulsory step in order to otain the full rescaled Hodge decomposition, and not only the part associated to harmonic forms as in [SS12].

iii. In Definition 6.1, where we extend the definition of the h-Laplacian to smooth functions.

iv. In Property 6.7, where we explicit the connection between the h-Laplacian, and the now classical Laplacian of fractal analysis [Kig01].

Henceforth, in the light of h-cohomology, the link is obvious: the h-Laplacian can be either obtained through De Rham differentiation, but also through local differences. So, modulo a multiplicative constant, which value will also be discussed and questioned, this is the same operator as the Laplacian on fractals.

In doing so, one falls back on the results exposed by R. S. Strichartz et al. in [ACSY14], where the authors build k-forms and de Rham differential operators d and δ on prefractals, k-forms being considered as k-ones on graphs, a natural approach in the light that "a k-form is an object that can be integrated over k-dimensional subjects". Passing to the limit – which calls for ad hoc renormalization – shows that their Laplacian on 0-forms – functions – is the same as the one of J. Kigami.

The circle is thus complete, one is on a closed path. Strichartz et al. made the connection with Hodge-De Rham Theory, the missing one with the Alexander-Kolmogorov Complex reinforces the legitimacy of differential operators on fractals. And last but not least, one also falls on random walks, which occur through the normalization process required to obtain the limit of the h-Laplacian.

2 Geometric Context

Notation. In the sequel, we will denote by $\mathcal{A}$ a ring of characteristics different from 2, and by X a general space.

Definition 2.1 (p**-Fermion**).

By analogy with particle physics, given a positive integer p, we will call p-fermion on X, with values in A, any antisymmetric map f from X^{p+1} to $\mathcal{A}$, i.e., such that, for any transposition τ, and any $(x_0, \ldots, x_p)$ in X^{p+1},

$$f(x_0, \ldots, x_p) = -f\left(x_{\tau(0)}, \ldots, x_{\tau(p)}\right) \cdot$$

A 0-fermion on X is simply a map f from X to A.

Remark 2.1. p-fermions are simply the generalization of p-chains.

Definition 2.2 ($\mathcal{A}$**-Module of** p**-Fermions on** X).

Given a positive integer p, we will denote by $\mathcal{F}^p(X, \mathcal{A})$ the $\mathcal{A}$-module of p-fermions on X with values in $\mathcal{A}$, which makes it an abelian group with respect to the addition, with an external law from $A \times \mathcal{F}^p(X, \mathcal{A})$ to $\mathcal{F}^p(X, \mathcal{A})$ where:

$$\forall (a, b) \in \mathcal{A}^2, \forall (f, g) \in (\mathcal{F}^p(X, \mathcal{A}))^2 \quad : \quad \begin{cases} a(f+g) &= a\,f + a\,g \\ (a+b)\,f &= a\,f + b\,f \end{cases}$$

Notation (**Constant**).

In the sequel, given a positive integer p, we wil denote by $c_p \in \mathcal{A}$ a constant, the value of which will be defined when necessary.

Definition 2.3 (*p*-**Differential**).

Given a positive integer p, we define the p-differential δ^p from $\mathcal{F}^p\,(X,\mathcal{A})$ to $\mathcal{F}^{p+1}\,(X,\mathcal{A})$, for any f in $\mathcal{F}^p\,(X,\mathcal{A})$ through:

$$\forall\,(x_0,\ldots,x_{p+1})\,\in\,X^{p+2}:\quad \delta^p(f)\,(x_0,\ldots,x_{p+1})=c_p\,\left(\sum_{q=0}^{p}(-1)^q\,f\,(\ldots,x_{q-1},x_{q+1},\ldots)\right)\cdot$$

As for the 0-differential δ^0, from $\mathcal{F}^0\,(X,\mathcal{A})$ to $\mathcal{F}^1\,(X,\mathcal{A})$, it is defined, for any f in $\mathcal{F}^0\,(X,\mathcal{A})$ through:

$$\forall\,(x_0,x_1)\,\in\,X^2:\quad \delta^0(f)\,(x_0,x_1)=c_0\,\left(f\,(x_1)-f(x_0)\right)$$

Remark 2.2. The kernel of the 0-differential δ^0 is the subset $\mathcal{F}^0_{\text{constant}}\,(X,\mathcal{A})\subset\mathcal{F}^0\,(X,\mathcal{A})$ of constant 0-fermions on X. For the sake of simplicity, we will from now on identify it to $\mathcal{A}$:

$$\ker\delta^0\equiv\mathcal{A}\cdot$$

Property 2.1.

$$\forall\,p\,\in\,\mathbb{N}:\quad \delta^{p+1}\circ\delta^p=0\cdot$$

Proof. i. For $p=0$, given f in $\mathcal{F}^0\,(X,\mathcal{A})$, and $(x_0,x_1)\,\in\,X^2$, we have that

$$\delta^0(f)\,(x_0,x_1)=c_0\,\left(f\,(x_0)\right)-f\,(x_1))$$

which yields, for any $(x_0,x_1,x_2)\,\in\,X^3$,

$$\begin{aligned}\delta^1\left(\delta^0(f)\right)(x_0,x_1,x_2)&=c_1\left\{\delta^0(f)\,(x_0,x_1)-\delta^0(f)\,(x_0,x_2)+\delta^0(f)\,(x_1,x_2)\right\}\\&=c_0\,c_1\left\{f\,(x_0)-f\,(x_1)-f\,(x_0)+f\,(x_2)+f\,(x_1)-f\,(x_2)\right\}\\&=0\end{aligned}$$

ii. For $p\geqslant 0$, given f in $\mathcal{F}^p\,(X,\mathcal{A})$, and $(x_0,\ldots,x_{p+2})\,\in\,X^{p+3}$:

$$\begin{aligned}\delta^{p+1}\left(\delta^p(f)\right)(x_0,\ldots,x_{p+2})&=c_{p+1}\left\{\sum_{q=0}^{p+1}(-1)^q\,\delta^p(f)\,(\ldots,x_{q-1},x_{q+1},\ldots)\right\}\\&=c_p\,c_{p+1}\sum_{q=0}^{p+1}(-1)^q\left\{\sum_{q'=0}^{p}(-1)^{q'}\,f\left(\ldots,x_{q-1},x_{q+1},\ldots,x_{q'-1},x_{q'+1},\ldots\right)\right\}\end{aligned}$$

To understand how things are going in this double sum: this amounts, in the $(p+3)$-uple $(x_0, \dots, x_{p+2})$, to suppress two terms x_q, and $x_{q'}$. So, the following configurations occur:

- $\rightsquigarrow$ Either $q < q'$, in which case, one first takes out $x_{q'}$, which occupies the place number q'. One then takes out x_q, which still occupies its original place number q.
  The resulting term is thus:

$$(-1)^q\,(-1)^{q'}\,f\left(\dots, x_{q-1}, x_{q+1}, \dots, x_{q'-1}, x_{q'+1}, \dots\right)$$

which can be illustrated as:

$\rightsquigarrow$ **Step 0**

$$\begin{array}{ccccccccc}\hline \multicolumn{1}{|c|}{\dots} & \multicolumn{1}{c|}{x_{q-1}} & \multicolumn{1}{c|}{x_q} & \multicolumn{1}{c|}{x_{q-1}} & \multicolumn{1}{c|}{\dots} & \multicolumn{1}{c|}{x_{q'-1}} & \multicolumn{1}{c|}{x'_q} & \multicolumn{1}{c|}{x_{q'-1}} & \multicolumn{1}{c|}{\dots} \\ \hline & & \uparrow & & & & \uparrow & & \\ & & \text{place number } q & & & & \text{place number } q' & & \end{array}$$

$\rightsquigarrow$ **Step 1**

$$\begin{array}{ccccccccc} & & & & & \text{place number } q'-1 & & & \\ & & & & & \downarrow & & & \\ \hline \multicolumn{1}{|c|}{\dots} & \multicolumn{1}{c|}{x_{q-1}} & \multicolumn{1}{c|}{x_q} & \multicolumn{1}{c|}{x_{q-1}} & \multicolumn{1}{c|}{\dots} & \multicolumn{1}{c|}{x_{q'-1}} & \multicolumn{1}{c|}{x_{q'+1}} & \multicolumn{1}{c|}{\dots} & \\ \cline{1-8} & & \uparrow & & & & \uparrow & & \\ & & \text{place number } q & & & & \text{place number } q' & & \end{array}$$

- $\rightsquigarrow$ Either $q > q'$, in which case, one first takes out $x_{q'}$, which occupies the place number q'. One then takes out x_q, which this time occupies the place number $q-1$, due to the shift induced by suppressing $x_{q'}$.

  The resulting term is thus exactly the opposite of the previous one:

$$(-1)^{q-1}\,(-1)^{q'}\,f\left(\dots, x_{q-1}, x_{q+1}, \dots, x_{q'-1}, x_{q'+1}, \dots\right)$$

which can be illustrated as:

$\rightsquigarrow$ **Step 0**

$$\begin{array}{ccccccccc}\hline \multicolumn{1}{|c|}{\dots} & \multicolumn{1}{c|}{x_{q'-1}} & \multicolumn{1}{c|}{x'_q} & \multicolumn{1}{c|}{x_{q'+1}} & \multicolumn{1}{c|}{\dots} & \multicolumn{1}{c|}{x_{q-1}} & \multicolumn{1}{c|}{x_q} & \multicolumn{1}{c|}{x_{q-1}} & \multicolumn{1}{c|}{\dots} \\ \hline & & \uparrow & & & & \uparrow & & \\ & & \text{place number } q' & & & & \text{place number } q & & \end{array}$$

$\rightsquigarrow$ **Step 1**

place number $q-2$

$\downarrow$

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\dots & x_{q'-1} & x_{q'+1} & \dots & x_{q-1} & x_q & x_{q+1} & \dots \\
\hline
\end{array}
$$

$\uparrow$ $\uparrow$

place number $q'-1$ place number $q-1$

All quantities of the above double sum are thus simplified two by two.

$\square$

Remark 2.3. The above definition can be understood in the following sense: p-fermions act on a collection $(x_0, \dots, x_p)$ of points in X^{p+1}, which are the vertices of n-simplices. Those simplices are themselves n-faces of $(n+1)$-simplices, the vertices $(x_0, \dots, x_{p+1})$ of which are then in X^{p+2}. Thus, the p-differential stands out as a map acting on the co-boundary of the elements of X^{p+1}.

It could seem strange that the p-differential takes values to $\mathcal{F}^{p+1}(X, \mathcal{A})$: in classical analysis, one loses informations in the differentiation process. In our case, it is just a generalization, in order to enable one to handle all (oriented) paths between given extremities, bearing in mind that differentiation is deeply linked to the increasing rate. The definition makes all the more sense may one introduce a metric, and consider very close points, as we will do it further.

Definition 2.4 (**p-Cycle, Closed p-Fermion**).

A p-fermion f will be called p-cycle, or closed $p-$fermion, if

$$\delta^p f = 0 \cdot$$

Definition 2.5 (**Exact p-Fermion**).

A p-fermion f will be called exact if there exists a $(p-1)$-fermion g such that

$$f = \delta^p g \cdot$$

Definition 2.6 (**p-Homology Group**).

Given a positive integer p, the quotient group $\ker \delta^p / Im\, \delta^{p+1}$ will be called p-homology group of X over $\mathcal{A}$. It thus corresponds to the equivalence classes of closed p-fermions, modulo exact p-fermions.

Definition 2.7 (**p-Cohomology Group**).

Given a positive integer p, the quotient group $\ker \delta^p / Im\, \delta^{p-1}$ will be called p-cohomology group of X over $\mathcal{A}$.

Since $Im\,\delta^{-1} = \{0\}$, the zero cohomology quotient group $\ker \delta^0 / Im\,\delta^{-1}$ is simply $\ker \delta^0$.

Definition 2.8 (**Complex of Fermions**).

The Complex $\left(\mathcal{F}^{\bullet}(X, \mathcal{A}), \delta^{\bullet}\right)$ is

$$\mathcal{F}^0 \xrightarrow{\delta^0} \ldots \mathcal{F}^p \xrightarrow{\delta^p} \mathcal{F}^{p+1} \xrightarrow{\delta^{p+1}} \ldots$$

where, for any natural integer p,

$$\delta^{p+1} \circ \delta^p = 0 \cdot$$

Notation. We set

$$\mathcal{F}^{\bullet}(X, \mathcal{A}) = \bigoplus_{p=0}^{\infty} \mathcal{F}^p(X, \mathcal{A}) \cdot$$

The associated cohomology, i.e., the set constituted of $\ker \delta^0$ and of the p-cohomology groups $\ker \delta^{p+1} / Im\,\delta^p$, $p \in \mathbb{N}$, will be denoted by

$$H^{\bullet}\left(\mathcal{F}^{\bullet}(X, \mathcal{A}), \delta^{\bullet}\right) \cdot$$

Property 2.2 (**Acyclic Complex of Fermions**).

The Complex $\mathcal{F}^{\bullet}(X, \mathcal{A})$ is acyclic: its cohomology is constant, i.e.,

$$\forall\, p \in \mathbb{N} : \quad \ker\, \delta^{p+1} / Im\,\delta^p = \{0\}$$

which amounts to

$$\forall\, p \in \mathbb{N} : \quad \ker\, \delta^{p+1} = Im\,\delta^p \cdot$$

We set

$$H^0\left(\mathcal{F}^{\bullet}(X, \mathcal{A}), \delta^{\bullet}\right) = \ker \delta^0 \,,$$

i.e.,

$$H^0\left(\mathcal{F}^{\bullet}(X, \mathcal{A}), \delta^{\bullet}\right) = \mathcal{A} \cdot$$

This implies, for the associated general cohomology, that

$$H^{\bullet}\left(\mathcal{F}^{\bullet}(X, \mathcal{A}), \delta^{\bullet}\right) = H^0\left(\mathcal{F}^{\bullet}(X, \mathcal{A}), \delta^{\bullet}\right) = \mathcal{A} \cdot$$

Remark 2.4. Since, for any $p \in \mathbb{N}^{\star}$

$$\delta^p \circ \delta^{p-1} = 0$$

and

$$Im\,\delta^{p-1} \subset \ker\,\delta^p$$

this simply amounts to

$$\ker\,\delta^p = Im\,\delta^{p-1}\cdot$$

When $p \geqslant 1$, the p-cohomology group of X over $\mathcal{A}$ reduces to the trivial quotient group:

$$\ker\delta^p/Im\,\delta^{p-1} = \{0\}$$

Thus, for $p \geqslant 1$, the p-cohomology groups $\ker\delta^p/Im\,\delta^{p-1}$ do not play any part in the Complex $\mathcal{F}^\bullet(X,\mathcal{A})$. Hence:

$$H^\bullet\left(\mathcal{F}^\bullet(X,\mathcal{A}),\delta^\bullet\right) = H^0\left(\mathcal{F}^\bullet(X,\mathcal{A}),\delta^\bullet\right) = \mathcal{A}\cdot$$

Proof. *i.* For $p=0$:

$Im\,\delta^0$ is the set of 1-fermions f^1 such that there exists a 0-fermion f^0 such that:

$$\forall\,(x,y)\,\in\,X^2:\quad f^1(x,y) = c_1\left\{f^0(x)-f^0(y)\right\}$$

Recalling now that the 1-differential δ, from $\mathcal{F}^1(X,\mathcal{A})$ to $\mathcal{F}^2(X,\mathcal{A})$, is defined, for any f^1 in $\mathcal{F}^1(X,\mathcal{A})$ through

$$\forall\,(x,y,z)\,\in\,X^3:\qquad \delta(f^1)(x,y,z) \quad=\quad c_2\left\{f^1(y,z)-f^1(x,z)+f^1(x,y)\right\}\,,$$

its kernel is thus the set of 1-fermions f^1 such that

$$\forall\,(x,y,z)\,\in\,X^3:\qquad f^1(x,y) \quad=\quad f^1(x,z)-f^1(y,z)$$

which can also be written as

$$\forall\,(x,y)\,\in\,X^2:\qquad f^1(x,y) \quad=\quad f^1(x,z)-f^1(y,z) \quad \forall\,z\,\in\,X\cdot$$

One can see that, given a pair (x,y) in X^2, $f^1(x,y)$ does not depend on the third variable. Given z in X, let us set

$$\tilde f(x) = f^1(x,z)\quad,\quad \tilde f(y) = f^1(y,z)$$

Then, $\tilde f$ is a 1-fermion, and

$$f^1(x,y) = \tilde f(x) - \tilde f(y)\cdot$$

Thus, we have that

$$\ker\,\delta \subset Im\,\delta^0\,,$$

which yields

$$\ker\,\delta = Im\,\delta^0\cdot$$

ii. <u>For a given integer $p \geqslant 1$</u>:

Let us prove that

$$\ker\, \delta^{p+1} = Im\, \delta^p \cdot$$

Hence, $Im\, \delta^p$ is the set of $(p+1)$-fermions f^{p+1} such that there exists a p-fermion f^p such that

$$\forall\, (x_0, \ldots, x_{p+1}) \,\in\, X^{p+2} : \quad f^{p+1}\,(x_0, \ldots, x_{p+1}) = c_{p+1} \left(\sum_{q=0}^{p} (-1)^q\, f^p\,(\ldots, x_{q-1}, x_{q+1}, \ldots) \right) \cdot$$

Recalling now that the $(p+1)$-differential δ^{p+1}, from $\mathcal{F}^{p+1}\,(X, \mathcal{A})$ to $\mathcal{F}^{p+2}\,(X, \mathcal{A})$, is defined, for any f in $\mathcal{F}^{p+3}\,(X, \mathcal{A})$ through

$$\forall\, (x_0, \ldots, x_{p+2}) \,\in\, X^{p+3} : \quad \delta^{p+1}(f)\,(x_0, \ldots, x_{p+2}) = c_p \left(\sum_{q=0}^{p+1} (-1)^q\, f\,(\ldots, x_{q-1}, x_{q+1}, \ldots) \right) ,$$

its kernel is thus the set of $(p+2)$-fermions f^{p+2} such that

$$\forall\, (x_0, \ldots, x_{p+2}) \,\in\, X^{p+3} : \qquad \sum_{q=0}^{p+1} (-1)^q\, f^{p+2}\,(\ldots, x_{q-1}, x_{q+1}, \ldots) \quad = \quad 0$$

which can also be written as

$$\forall\, (x_0, \ldots, x_{p+1}) \,\in\, X^{p+2} : \qquad (-1)^{p+2}\, f^{p+2}\,(x_0, \ldots, x_{p+1}) \quad = \quad - \sum_{q=0}^{p} (-1)^q\, f^{p+2}\,(\ldots, x_{q-1}, x_{q+1}, \ldots) \cdot$$

One can see that, given a $(p+2)$-uple $(x_0, \ldots, x_{p+1})$ in X^{p+2}, $f^{p+2}\,(x_0, \ldots, x_{p+1})$ does not depend on the variable x_{p+2}. It can thus be written in the following form

$$\begin{aligned} f^{p+2}\,(x_0, \ldots, x_{p+1}) \quad &= \quad (-1)^{p+3} \sum_{q=0}^{p} (-1)^q\, f^{p+2}\,(\ldots, x_{q-1}, x_{q+1}, \ldots) \\ &= \quad \tilde{c}_{p+1} \sum_{q=0}^{p} (-1)^q\, \tilde{f}^{p+1}\,(\ldots, x_{q-1}, x_{q+1}, \ldots) \end{aligned}$$

where $\tilde{f}^{p+1}$ denotes a $(p+1)$-fermion.

Thus, we thave that

$$\ker\, \delta^{p+1} \subset Im\, \delta^p ,$$

which yields

$$\ker\, \delta^{p+1} = Im\, \delta^p \cdot$$

$\square$

3 De Rham Cohomology

For the benefit of the reader who may not be familiar with mathematical notions devoted to the De Rham Cohomology, we shall first recall several definitions.

3.1 A Few Recalls

Notation. In the sequel, X denotes a smooth manifold, of dimension $n \in \mathbb{N}^\star$. We will hereafter use the classical $\wedge$ notation for exterior derivatives.

Definition 3.1. Given a natural integer p, we will denote by $\Omega^p(X)$ the space of p-forms on X.

Notation (Partial Derivative).

Given a strictly positive integer p, a smooth p-form f on X, and k in $\{0, \ldots, p\}$, the partial derivative $\partial_k f$ is defined, for any

$$(x_0, \ldots, x_p) = \left((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, (x_{p,i_p})_{1 \leqslant i_p \leqslant n} \right) \in X^{p+1}$$

through

$$\partial_k f(x_0, \ldots, x_p) = \sum_{i_k=1}^{n} \frac{\partial f}{\partial x_{k,i_k}} \left((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, (x_{p,i_p})_{1 \leqslant i_p \leqslant n} \right) dx^{k,i_k} .$$

Definition 3.2 (De Rham Differential).

Given a p-form $\omega \in \Omega^p(X)$, such that, for any $x = (x_1, \ldots, x_n) \in X$,

$$\omega(x) = \sum_{1 \leqslant i_1 < \ldots < i_p \leqslant n} f_{i_1,\ldots,i_p}(x)\, dx^{i_1} \wedge \ldots dx^{i_p} ,$$

and where, for any $(i_1, \ldots, i_p) \in \{1, \ldots, n\}^p$, the $f_{i_1,\ldots,i_p}$ denote smooth functions on X, the De Rham differential $d\omega$ is defined through

$$d\omega(x) = \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \ldots < i_p \leqslant n} \frac{\partial f_{i_1,\ldots,i_p}}{\partial x_k}(x)\, dx^k \wedge dx^{i_1} \wedge \ldots dx^{i_p} \cdot$$

Definition 3.3 (Diagonal).

Given a natural integer p, the diagonal of X^{p+1} is defined as the following set,

$$\Delta_X = \{\underline{x} = (x, \ldots, x)\} \subset X^{p+1} \cdot$$

Definition 3.4 (**De Rham Complex on** X).

The De Rham Complex on X is the cochain complex of differential forms

$$0 \xrightarrow{d} \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \dots$$

that we will denote by $\Omega^{\bullet,d}$.

Property 3.1.

$$d^2 = 0 \cdot$$

3.2 Natural Correspondence Between Fermions and Differential Forms

Definition 3.5 (p-**Linear Forms on the Tangent Space** $T\,X$).

Given a strictly positive integer p, a smooth p-fermion f on X, and x in X, we define a p-linear form $r_p(f)(x)$ on $T_x\,X$ through

$$\forall\;(u_1,\dots,u_p)\;\in\;(T_x\,X)^p\;:\quad r_p(f)(x)\,(u_1,\dots,u_p) = \partial_1\dots\partial_p f(x,\dots,x)\;(u_1,\dots,u_p)$$

In the case where $p=0$, we simply set

$$r_0(f)(x) = f(x,\dots,x)\cdot$$

Proposition 3.2. *Given a strictly positive integer p, a smooth p-fermion f on X, and x in X, we have that*

$$r_p(f)\;\in\;\Omega^p(X)$$

and

$$(p+1)\,\partial_0\,\dots\partial_p f|_{\Delta_X} = d\,r_p(f)\,,$$

i.e.,

$$\forall\;(u_0,u_1,\dots,u_p)\;\in\;(T_x\,X)^{p+1}\;:\quad (p+1)\,\partial_0\,\dots\partial_p f\,(x,\dots,x)\;(u_0,u_1,\dots,u_p) = d\,(r_p(f))\,(x)\;(u_0,u_1,\dots,u_p)\cdot$$

Proof. As introduced in Notation 3.1,for any

$$(x_0,\dots,x_p) = \Big(\left(x_{0,i_0}\right)_{1\leqslant i_0\leqslant n},\dots,\left(x_{p,i_p}\right)_{1\leqslant i_p\leqslant n}\Big)\;\in\;X^{p+1}$$

we have that

$$\partial_0\dots\partial_p\,f\,(x_0,\dots,x_p)\quad=\quad\sum_{i_0=1}^{n}\dots\sum_{i_p=1}^{n}\frac{\partial^{p+1}f}{\partial x_{0,i_0}\dots\partial x_{p,i_p}}\Big(\left(x_{0,i_0}\right)_{1\leqslant i_0\leqslant n},\dots,\left(x_{p,i_p}\right)_{1\leqslant i_p\leqslant n}\Big)\;dx^{0,i_0}\wedge\dots\wedge dx^{p,i_p}$$

which yields, on the diagonal,

$$\partial_0 \ldots \partial_p f(x, \ldots, x) = \sum_{i_0=1}^{n} \ldots \sum_{i_p=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_0} \ldots \partial x_{i_p}} \left((x_i)_{1 \leqslant i \leqslant n}, \ldots, (x_i)_{1 \leqslant i \leqslant n} \right) dx^{i_0} \wedge \ldots \wedge dx^{i_p} \cdot$$

As previously, for any

$$(x_0, \ldots, x_p) = \left((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, \left(x_{p,i_p}\right)_{1 \leqslant i_p \leqslant n} \right) \in X^{p+1}$$

we have that

$$\partial_1 \ldots \partial_p f(x_0, \ldots, x_p) = \sum_{i_1=1}^{n} \ldots \sum_{i_p=1}^{n} \frac{\partial^{p} f}{\partial x_{1,i_1} \ldots \partial x_{p,i_p}} \left((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, \left(x_{p,i_p}\right)_{1 \leqslant i_p \leqslant n} \right) dx^{1,i_1} \wedge \ldots \wedge dx^{p,i_p} \cdot$$

Thus,

$$d\left[\partial_1 \ldots \partial_p f(x_0, \ldots, x_p)\right] = \sum_{k=0}^{p} \sum_{i_k=1}^{n} \sum_{i_1=1}^{n} \ldots \sum_{i_p=1}^{n} \frac{\partial^{p+1} f}{\partial x_{k,i_k} \partial x_{1,i_1} \ldots \partial x_{p,i_p}} \left((x_{0,i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, \left(x_{p,i_p}\right)_{1 \leqslant i_p \leqslant n} \right) dx^{k,i_k} \wedge dx^{1,i_1} \wedge \ldots \wedge dx^{p,i_p} ,$$

which yields, on the diagonal

$$d\left[\partial_1 \ldots \partial_p f(x, \ldots, x)\right] = \sum_{k=0}^{p} \sum_{i_k=1}^{n} \sum_{i_1=1}^{n} \ldots \sum_{i_p=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_k} \partial x_{i_1} \ldots \partial x_{i_p}} \left((x_{i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, \left(x_{i_p}\right)_{1 \leqslant i_p \leqslant n} \right) dx^{i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \cdot$$

One may note that, given an integer k in $\{0, \ldots, p\}$, the exterior product

$$dx^{i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

vanishes for $i_k = i_1, \ldots, i_p$. Thus, the nonzero terms depend on the values of $i_1, \ldots, i_p$, and not on k, which enables us to write

$$\begin{aligned}
d\left[\partial_1 \ldots \partial_p f(x, \ldots, x)\right] &= \sum_{k=0}^{p} \sum_{i_0=1}^{n} \sum_{i_1=1}^{n} \ldots \sum_{i_p=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_0} \partial x_{i_1} \ldots \partial x_{i_p}} \left((x_{i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, \left(x_{i_p}\right)_{1 \leqslant i_p \leqslant n} \right) dx^{i_0} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \\
&= (p+1) \sum_{i_0=1}^{n} \sum_{i_1=1}^{n} \ldots \sum_{i_p=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_0} \partial x_{i_1} \ldots \partial x_{i_p}} \left((x_{i_0})_{1 \leqslant i_0 \leqslant n}, \ldots, \left(x_{i_p}\right)_{1 \leqslant i_p \leqslant n} \right) dx^{i_0} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p} \\
&= (p+1)\, \partial_0 \ldots \partial_p f(x, \ldots, x)
\end{aligned}$$

□

Corollary 3.3 (**Correspondence Between Fermions and Differential Forms**).

By choosing $c_p = p + 1$*, we thus obtain that*

$$d\, r_p = r_{p+1}\, \delta^p \cdot$$

Proof.

$$\begin{tikzcd}
\mathcal{F}^p \arrow[r, "\delta^p"] \arrow[d, "r_p"] & \mathcal{F}^{p+1} \arrow[d, "r_{p+1}"] \\
\Omega^p \arrow[r, "d"] & \Omega^{p+1}
\end{tikzcd}$$

Let us consider a p fermion f, which operates on a $(x_0, \ldots, x_p)$ of X^{p+1}. Indifferently, one may handle the variables as $(x_0, \ldots, x_p)$, or as $(x_1, \ldots, x_{p+1})$, thus, writing $(\partial_0, \ldots, \partial_p)$ or $(\partial_1, \ldots, \partial_{p+1})$ is equivalent.

Then, we trivially hve that

$$\partial_1 \ldots \partial_p \, \partial_{p+1} \, f = \partial_0 \, \partial_1 \ldots \partial_p \, f \cdot$$

Due to our previous result, we also have that

$$(d \, r_p)(f) \, (x, \ldots, x) \quad = \quad (p+1) \, \partial_0 \ldots \partial_p \, f \, (x \ldots, x) \; \cdot$$

At the same time, for any $(x_0, \ldots, x_{p+1}) \in X^{p+2}$, we have that

$$\begin{aligned}
(r_{p+1} \, \delta^p)(f) \, (x_0, \ldots, x_{p+1}) &= \partial_1 \ldots \partial_p \, \partial_{p+1} \, \delta^p(f) \, (x_0, \ldots, x_{p+1}) \\
&= \partial_1 \ldots \partial_p \, \partial_{p+1} \, c_p \left\{ \sum_{q=0}^{p} (-1)^q \, f \, (\ldots, x_{q-1}, x_{q+1}, \ldots) \right\} \\
&= c_p \sum_{i_1=1}^{n} \ldots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1}}{\partial x_{1,i_1} \ldots \partial x_{p+1,i_{p+1}}} \left[\sum_{q=0}^{p} (-1)^q \, f \, (\ldots, x_{q-1}, x_{q+1}, \ldots) \right] dx^{1,i_1} \wedge \ldots \wedge dx^{p+1,i_{p+1}} \\
&= c_p \sum_{q=0}^{p} (-1)^q \sum_{i_1=1}^{n} \ldots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1}}{\partial x_{1,i_1} \ldots \partial x_{p+1,i_{p+1}}} \left[f \, (\ldots, x_{q-1}, x_{q+1}, \ldots) \right] dx^{1,i_1} \wedge \ldots \wedge dx^{p+1,i_{p+1}} \, .
\end{aligned}$$

One may note that, given an integer q in $\{1, \ldots, p\}$, the derivative

$$\sum_{i_1=1}^{n} \ldots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1}}{\partial x_{1,i_1} \ldots \partial x_{p+1,i_{p+1}}} \left[f \, (\ldots, x_{q-1}, x_{q+1}, \ldots) \right]$$

takes the value zero, since there is no x_q!

Thus,

$$(r_{p+1} \, \delta^p)(f) \, (x_0, \ldots, x_{p+1}) \quad = \quad c_p \, (-1)^0 \sum_{i_1=1}^{n} \ldots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1}}{\partial x_{1,i_1} \ldots \partial x_{p+1,i_{p+1}}} \left[f \, (x_1, \ldots, x_{p+1}) \right] dx^{1,i_1} \wedge \ldots \wedge dx^{p+1,i_{p+1}} \, ,$$

which yields, on the diagonal,

$$(r_{p+1} \, \delta^p)(f) \, (x, \ldots, x) \quad = \quad c_p \sum_{i_1=1}^{n} \ldots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_1} \ldots \partial x_{i_{p+1}}} \, (x, \ldots, x) \; dx^{i_1} \wedge \ldots \wedge dx^{i_{p+1}} \, ,$$

which, by means of a change of indices, can also be written as

$$(r_{p+1} \, \delta^p)(f) \, (x, \ldots, x) \quad = \quad c_p \sum_{i_0=1}^{n} \ldots \sum_{i_{p+1}=1}^{n} \frac{\partial^{p+1} f}{\partial x_{i_0} \ldots \partial x_{i_p}} \, (x, \ldots, x) \; dx^{i_0} \wedge \ldots \wedge dx^{i_p} \, ,$$

i.e.,

$$(r_{p+1}\,\delta^p)(f)\,(x,\ldots,x) \quad = \quad c_p\,\partial_0\,\ldots\partial_p\,f$$

□

Remark 3.1. This result is all the more important, since it enables us to make a connection between the Alexander-Kolmogorov cohomology, based upon differences, and the De Rham one, which is the usual one.

Notation (Complex of Smooth Fermions on X).

We will denote by

$$\left(\mathcal{F}^\bullet\,(X)\,,\delta^\bullet\right) = \bigoplus_{p=0}^{\infty} \mathcal{F}^p\,(X)$$

the acyclic complex of smooth fermions on X, and by $\delta^\bullet$ the associated differential (which means that, in practice, one deals with a δ^p, for a given value of the integer p).

4 (h,p)-Fermions

Notation. In the sequel, we will denote by $(X, dist)$ a metric space.

Definition 4.1 ((h,p)-Fermions on X).

Given a strictly positive number h, and a natural integer p, we will denote by $\mathcal{F}_h^p\,(X,\mathcal{A})$ the set of p-fermions on X, with values in $\mathcal{A}$, defined on

$$X_h^{p+1} = \left\{(x_0,\ldots,x_p)\,\in\,X^{p+1}\,,\forall\,(i,j)\,\in\,\{0,\ldots,p\}^2\,:\,dist\,(x_i,x_j) < h\right\}$$

and by

$$\left(\mathcal{F}_h^\bullet\,(X)\,,\delta^\bullet\right) = \bigoplus_{p=0}^{\infty} \mathcal{F}_h^p\,(X)$$

the associated complex.

Definition 4.2 (h-Cohomology).

We will call $H_h^\bullet\,(X, dist, \mathcal{A})$ the h-cohomology of $(X, dist)$ at scale h, with values in $\mathcal{A}$.

Definition 4.3 (Radius of Injectivity).

Given a Riemannian manifold (M, g), the injectivity radius on M is defined as

$$\rho(M, g) = \mathrm{inj}\,(M, g) = \inf_{x \in M} \mathrm{inj}_{x \in M}\,(M, g)$$

where, for any $x \in M$,

$$\mathrm{inj}_{x \in M}\,(M, g) = \sup\,\{r \geqslant 0\,,\, \exp_x \text{ is a diffeomorphism on the ball } \mathcal{B}\,(x, r) \subset T_x M\} \cdot$$

5 h-Hodge Theory

5.1 Geometric Context

Notation. In the sequel, we denote by:

i. $(X, dist, \mu)$ a metric space, of dimension d_X, with a measure μ on Borel sets, such that:

$$\forall\, x \in X, \forall\, \varepsilon > 0 : \quad \mu\,(\mathcal{B}(x, \varepsilon)) > 0$$

ii. $h > 0$ a real parameter.

iii. μ^{p+1} the product measure on X^{p+1}.

Definition 5.1 (Measure on X_h^{p+1}).

We define a measure μ_h^{p+1} on X_h^{p+1} through:

$$\mu_h^{p+1} = C_p\,(\cdot, h)\ \mu^{p+1}$$

where the normalization factor $= C_p\,(\cdot, h)$ stands as a parameter.

Definition 5.2.

We set

$$L^2 \mathcal{F}_h^p = L^2\left(X_h^{p+1}, \mu^{p+1}\right) \cdot$$

Theorem 5.1. *Given a compact analytic Riemannian manifold (X, g), there exists a finite number of real numbers*

$$0 = h_0 < h_1 < \ldots < h_{max} = diam\, X$$

such that the fibration

$$H_h^\bullet\left(L^2, X, g\right) \mapsto h \cdot$$

is constant on each interval $]h_i, h_{i+1}[$.

Moreover,

i. For $h > diam\, X$*:*

$$H_h^\bullet\left(L^2, X, g\right) = \mathbb{C}\cdot$$

ii. For $h < h_1$*:*

$$H_h^\bullet\left(L^2, X, g\right) \simeq H^\bullet\cdot$$

Property 5.2. *The p-differential* δ^p *is a bounded operator from* $L^2\mathcal{F}_h^p$ *to* $L^2\mathcal{F}_h^{p+1}$*, which norms obviously depends on* h*.*

Proof. This immediately comes from the fact that the space X is compact, while δ^p is a difference operator acting on continuous functions on X.

□

Notation (Normalized Differential).

From now on, given a strictly positive real number h, we will denote by δ_h the normalized differential

$$\delta_h = h^{-1}\,\delta\cdot$$

Remark 5.1. As explained in the above, the differential δ is bounded independantly of h. The interesting point is that in the normalized one δ_h, the h^{-1} coefficient allows h to tend towards zero, which enables one to recover the usual De Rham differential and infinitesimal calculus.

Definition 5.3 (Hodge Star Operator).

Let E be a finite-dimensional oriented euclidean space, endowed with a nondegenerate symmetric bilinear form $\wedge$. We set

$$\dim E = n \,\in\, \mathbb{N}^\star\cdot$$

Given a natural integer $p \leqslant n$, $\wedge^p E$ and $\wedge^{n-p} E$ respectively denote the subspaces of p and $n-p$ vectors. One trivially has:

$$\dim \wedge^p E = \dim \wedge^{n-p} E = \binom{n}{p}$$

(the choice of a basis amounts to choose p vectors among the n of any basis of E)

The Hodge star operator $\star$ is simply the natural isomorphism between $\wedge^p E$ and $\wedge^{n-p} E$. For any orthonormal basis $\{e_1, \ldots, e_n\}$, we have that

$$\star\left(e_1 \wedge \ldots \wedge e_p\right) = e_{p+1} \wedge \ldots \wedge e_n\cdot$$

Property 5.3. *Given a natural integer $p \leqslant n$, and a p-vector $\eta \in \wedge^p E$:*

$$\star \star \eta = (-1)^{k\,(n-p)}\, \eta \cdot$$

Remark 5.2. We thus have that

$$\begin{tikzcd}
\Lambda^p \arrow[rr, "\star"] & & \Lambda^{n-p} \arrow[d, "d"] \\
\Lambda^{p-1} & & \Lambda^{n-p+1} \arrow[ll, "\star"']
\end{tikzcd}$$

Definition 5.4 (**Hodge Star Operator on the De Rham Complex**).

The above definition of the Hodge star operator naturally extends to the De Rham Complex $\Omega^{\bullet,d}$ on the smooth manifold X, as the natural isomorphism between Ω^p and Ω^{n-p} through

$$\star\,(\partial_1 \ldots \partial_p) = \partial_{p+1} \ldots \partial_n \cdot$$

Definition 5.5 ($d^\star$ **Operator on the De Rham Complex**).

Given a strictly positive integer $p \leqslant n$, we define the codifferential $d^\star$ by

$$d^\star\,: \Omega^p \longrightarrow \Omega^{p-1}$$

through

$$d^\star = (-1)^{n\,(p-1)+1} \star d \star \cdot$$

$$\begin{tikzcd}
\Omega^p \arrow[rr, "\star"] \arrow[d, "d^\star"'] & & \Omega^{n-p} \arrow[d, "d"] \\
\Omega^{p-1} & & \Omega^{n-p+1} \arrow[ll, "\star"']
\end{tikzcd}$$

Definition 5.6 (**Hodge Laplacian**).

The Hodge Laplacian on $\Omega^\bullet(X)$ is given by

$$\Box = (d + d^\star)^2 = d\,d^\star + d^\star\,d \cdot$$

Notation (**Space of Harmonic Forms**).

For any positive integer p, we will denote by $\mathcal{H}_{|\Omega^p}$ the space of harmonic forms on Ω^p, i.e., the forms f such that

$$\Box\, f = 0 \cdot$$

Theorem 5.4 (Hodge Decomposition).

Given a compact analytic Riemannian manifold X, then, for any strictly positive integer p, we have that

$$\Omega^{p-1} \xrightarrow{d} \Omega^p \xrightarrow{d} \Omega^{p+1}$$

and

$$\Omega^{p+1} \xrightarrow{d^\star} \Omega^p \xrightarrow{d^\star} \Omega^{p-1}$$

To facilitate understanding, the following diagram might be helpful:

$$\begin{tikzcd}
\Omega^{p-1} \arrow[rr, "d"] & & \Omega^p \arrow[d, "d"] \arrow[dll, "d\,d^\star"'] \\
\Omega^p \arrow[u, "d^\star"] & & \Omega^{p+1} \arrow[ll, "d^\star"']
\end{tikzcd}$$

Also, we have the following orthogonal, direct sum decompositions,

$$\left\{ \begin{array}{lcl} \ker d_{|\Omega^p} & = & Im\, d_{|\Omega^{p-1}} \oplus \mathcal{H}_{|\Omega^p} \\ \\ \ker d^\star_{|\Omega^p} & = & Im\, d^\star_{|\Omega^{p+1}} \oplus \mathcal{H}_{|\Omega^p} \end{array} \right. ,$$

and

$$\left\{ \begin{array}{lcl} \Omega^p(X) & = & Im\, d_{|\Omega^{p-1}} \oplus \mathcal{H}_{|\Omega^p} \oplus \left(\ker d_{|\Omega^p}\right)^\perp \\ \\ \Omega^p(X) & = & Im\, d^\star_{|\Omega^{p+1}} \oplus \mathcal{H}_{|\Omega^p} \oplus \left(\ker d^\star_{|\Omega^p}\right)^\perp \end{array} \right.$$

which naturally yields

$$\square = \left(\begin{array}{ccc} d\,d^\star & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d^\star\,d \end{array} \right) .$$

Moreover, $d_{|\Omega^p}$ induces an isomorphism j_p from $\left(\ker d_{|\Omega^p}\right)^\perp$ onto $Im\, d_{|\Omega^p}$:

$$\left(d_{|\Omega^p}\right)\Big|_{\left(\ker d_{|\Omega^p}\right)^\perp} = j_p$$

At the same time, $d^\star_{|\Omega^p}$ induces an isomorphism $j^\star_p$ from $Im\, d_{|\Omega^p} \subset \Omega^{p+1}$ onto $Im\, d^\star_{|\Omega^{p+1}}$:

$$\left(d^\star_{|\Omega^p}\right)\Big|_{Im\, d_{|\Omega^p}} = j^\star_p$$

In the same way, $d^\star_{|\Omega^p}$ induces an isomorphism $j^\star_{p-1}$ from $Im\, d_{|\Omega^{p-1}}$ onto $Im\, d^\star_{|\Omega^p}$:

$$\left(d^\star_{|\Omega^p}\right)\Big|_{Im\, d_{|\Omega^p}} = j^\star_{p-1}$$

while $d_{|\Omega^p}$ induces an isomorphism j_{p-1} from $\left(\ker d_{|\Omega^{p-1}}\right)^\perp = Im\, d^\star_{|\Omega^p} \subset \Omega^{p-1}$ onto $Im\, d_{|\Omega^{p-1}}$:

$$\left(d_{|\Omega^p}\right)\Big|_{Im\, d^\star_{|\Omega^p}} = j_{p-1}$$

This yields the Hodge decomposition

$$\square = \left(\begin{array}{ccc} j_{p-1}\, j^\star_{p-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & j^\star_p\, j_p \end{array} \right) .$$

5.2 Main Result: Limit of the Resolvent of the h-Laplacian

Proposition 5.5 (**Modified Scalar Product on** $\Omega^p(X)$, $p \in \mathbb{N}$).

In the sequel, given a natural integer p, we modify the usual scalar product $(\cdot,\cdot)$ *on* $\Omega^p(X)$ *by means of a multiplicative strictly positive constant* α_p*, setting, for any pair* (u,v) *of smooth p-fermions on X:*

$$\widetilde{(u,v)}_p = \alpha_p \, (u,v)_p \cdot$$

Since

$$d^\star : \Omega^{p+1} \longrightarrow \Omega^p$$

we can naturally introduce the operator

$$\tilde{d}^\star = \frac{\alpha_{p+1}}{\alpha_p} \, d^\star$$

(the multiplicative constant α_{p+1} *comes from the modified scalar product on* $\Omega^{p+1}(X)$*, while the division by* α_p *stands as a normalization one.)*

We then set

$$\Delta_0 = \left(d + \tilde{d}^\star\right)^2 = \underset{p \in \mathbb{N}}{\oplus} \begin{pmatrix} \dfrac{\alpha_p}{\alpha_{p-1}} \, j_p \, j_p^\star & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{\alpha_{p+1}}{\alpha_p} \, j_p^\star \, j_p \end{pmatrix} \cdot$$

Definition 5.7 (**h-Laplacian**).

Let us recall that, in the above, given a strictly positive real number h, we have introduced the normalized differential

$$\delta_h = h^{-1} \, \delta \cdot$$

The natural correspondence of Corollary 3.3, by means of p and $(p+1)$-linear forms:

$$\forall \, p \in \mathbb{N} : \quad d \, r_p = r_{p+1} \, \delta$$

naturally induces the existence of the operator $\delta^\star$,

$$\forall \, p \in \mathbb{N} : \quad d^\star \, r_p = r_{p+1} \, \delta^\star \, ,$$

and its normalized versionn

$$\delta_h^\star = h^{-1} \, \delta^\star \cdot$$

We now define the h-Laplacian by

$$\Delta_h = (\delta_h + \delta_h^\star)^2 \cdot$$

Notation (Spectrum of a Laplacian Operator).

Given a Laplacian operator among the ones previously encountered, $\Box$, Δ_0, Δ_h, we denote by Spec $(\cdot)$ its spectrum:

$$\mathrm{Spec}\,(\Box) \quad \text{or} \quad \mathrm{Spec}\,(\Delta_0) \quad \text{or} \quad \mathrm{Spec}\,(\Delta_h) \cdot$$

Notation (Canonical projectors).

In the sequel, we will denote by:

i. $\Pi_{\mathcal{F}_h^\bullet,\Omega^\bullet}$ the canonical projector from $L^2\mathcal{F}_h^\bullet$ on $L^2\left(X,\Omega^\bullet\right)$.

ii. $\Pi_{\Omega^\bullet,\mathcal{F}_h^\bullet}$ the canonical injection from $L^2\left(X,\Omega^\bullet\right)$ on $L^2\mathcal{F}_h^\bullet$ such that, on smooth functions,

$$\Pi_{\mathcal{F}_h^\bullet,\Omega^\bullet} \circ \Pi_{\Omega^\bullet,\mathcal{F}_h^\bullet} = Id_{L^2(X,\Omega^\bullet)} \,,$$

and such that $\Pi_{\Omega^\bullet,\mathcal{F}_h^\bullet} \circ \Pi_{\mathcal{F}_h^\bullet,\Omega^\bullet}$ is an orthogonal projector, for the Hilbert structure of $L^2\mathcal{F}_h^\bullet$ – which simply comes from the fact that

$$\Pi_{\Omega^\bullet,\mathcal{F}_h^\bullet} \circ \underbrace{\Pi_{\mathcal{F}_h^\bullet,\Omega^\bullet} \circ \Pi_{\Omega^\bullet,\mathcal{F}_h^\bullet}}_{Id_{L^2(X,\Omega^\bullet)}} \circ \Pi_{\mathcal{F}_h^\bullet,\Omega^\bullet} = \Pi_{\Omega^\bullet,\mathcal{F}_h^\bullet} \circ \Pi_{\mathcal{F}_h^\bullet,\Omega^\bullet} \cdot$$

Proposition 5.6. *Let us denote by $r_{p,h}$ the restriction to $\mathcal{F}_h$ of the p-linear form r_p introduced in Definition 3.5.*
Since $\mathcal{F}_h^p \subset \mathcal{F}^p$, we can then use the diagram given in the proof of Corollary 3.3, which yields:

$$\mathcal{F}_h^p \xrightarrow{\quad r_{p,h} \quad} \Omega^p$$

This provides a further understanding of the aforementioned canonical operators:

i. the first one simply arises as

$$\Pi_{\mathcal{F}_h^\bullet,\Omega^\bullet} = r_{p,h} \cdot$$

ii. As for the second one, it is uniquely determined by the following condition:

$$\Pi_{\mathcal{F}_h^\bullet,\Omega^\bullet} \circ \Pi_{\Omega^\bullet,\mathcal{F}_h^\bullet} = Id_{L^2(X,\Omega^\bullet)} \,,$$

along with the fact that $\Pi_{\Omega^\bullet,\mathcal{F}_h^\bullet} \circ \Pi_{\mathcal{F}_h^\bullet,\Omega^\bullet}$ is self-adjoint.

It also happens that the restriction $r_{p,h}$ to $\mathcal{F}_h^p \subset \mathcal{F}^p$ of course does not depend on h.

Property 5.7. *Given a strictly positive real number h, $|\Delta_h|$ is bounded, self-adjoint, and non-negative on $L^2\mathcal{F}_h^\bullet$.*

Proof. This directly comes from the definition of Δ_h:

$$\Delta_h = (\delta_h + \delta_h^\star)^2$$

where the differential δ_h is bounded, as shown in Property 5.2.

□

Theorem 5.8 (Limit of the h-Laplacian).

Given a compact subset $K \subset \mathbb{C} \setminus Spec\,(\Delta_0)$, there exists a strictly positive constant h_K such that, for any h in $]0, h_K[$, the resolvent $(z - |\Delta_h|)^{-1}$ exists, and

$$\lim_{h \to 0} (z - |\Delta_h|)^{-1} = \lim_{h \to 0} \Pi_{\Omega^\bullet, \mathcal{F}_h^\bullet} \, (z - |\Delta_0|)^{-1} \, \Pi_{\mathcal{F}_h^\bullet, \Omega^\bullet} \cdot$$

Proof. In the case of smooth functions, a direct computation, by means of a Taylor expansion on the diagonal of the involved matrices, show that the result is true.

Now, given a strictly positive real number h, we obviously have that $(z - |\Delta_h|)^{-1}$ is defined for $z \in \mathbb{C} \setminus \mathrm{Spec}\,(\Delta_h)$.

We thus have to determine the spectrum of the h-Laplacian Δ_h, which can be done by using the definition, i.e.,

$$\Delta_h = (\delta_h + \delta_h^\star)^2 \cdot$$

We can naturally write that

$$\Pi_{\mathcal{F}_h^\bullet, \Omega^\bullet} \, (\delta_h + \delta_h^\star)^2 = \bigoplus_{p \in \mathbb{N}} \begin{pmatrix} \dfrac{\alpha_p}{\alpha_{p-1}} \, j_p \, j_p^\star & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{\alpha_{p+1}}{\alpha_p} \, j_p^\star \, j_p \end{pmatrix},$$

which resolvent exists. One then goes back to the one of the h-Laplacian by applying the projector $\Pi_{\mathcal{F}_h^\bullet, \Omega^\bullet}$.

A delicate point is to ensure the existence of the limit

$$\lim_{h \to 0} \Pi_{\Omega^\bullet, \mathcal{F}_h^\bullet} \, (z - |\Delta_0|)^{-1} \, \Pi_{\mathcal{F}_h^\bullet, \Omega^\bullet} \cdot$$

This directly comes from Proposition 5.6, since the canonical projectors involved do not depend on h. Thus, the operator

$$\Pi_{\Omega^\bullet, \mathcal{F}_h^\bullet} \, (z - |\Delta_0|)^{-1} \, \Pi_{\mathcal{F}_h^\bullet, \Omega^\bullet} \quad (\star)$$

does not depend on h. In fact, since, on smooth functions,

$$\Pi_{\mathcal{F}_h^\bullet, \Omega^\bullet} \circ \Pi_{\Omega^\bullet, \mathcal{F}_h^\bullet} = Id_{L^2(X, \Omega^\bullet)} \,,$$

we can observe that the operator defined in $(\star)$ above, is (uniformly in h), continuous on $L^2\,(\mathcal{F}_h^\bullet)$. Henceforth, we obtain the sought for result on $L^2\,(\mathcal{F}_h^\bullet)$.

□

Property 5.9 (*h*-**Normalization Constant**).

Given a natural integer p, the normalization constant C_p introduced in Definition 5.1 enables us to connect the measure μ_h^{p+1} on X_h^{p+1} to the one on X^{p+1}. It is simply given by

$$C_p = h^{-p\, d_X} \,.$$

Proof. This just comes from the definition of X_h^{p+1}. In fact, we have that

$$X_h^{p+1} = \Big\{ (x_0, \ldots, x_p) \,\in\, X^{p+1} \,, \forall\, (i,j) \,\in\, \{0, \ldots, p\}^2 \,:\, dist\,(x_i, x_j) < h \Big\} \,,$$

which can be interpretated as a change of variables.

□

6 h-Laplacian, Random walks, Singular Sets

6.1 h-Laplacian, and Random Walks

Notation.

In the sequel, we denote by (X, d, μ) a metric measured space.

Definition 6.1 (h-**Laplacian**).

Given a strictly positive real number h, we define the h-Laplacian as the operator

$$|\Delta_h| = \delta_h^\star \, \delta_h$$

which acts on smooth functions f on X through:

$$\forall\, x \,\in\, X \,: \quad |\Delta_h|\,(f)(x) = \frac{2\, c_0^2}{h^2} \int_{\mathcal{B}(x,h)} \{f(y) - f(x)\}\; C_0\,(x, y, h)\; d\mu_X(y)$$

where C_0 denotes a function defined on $X^2 \times]0, +\infty[$, and where c_0^2 denotes a strictly positive constant.

Remark 6.1.

i. The $\dfrac{1}{h^2}$ term comes from the definition of $\delta_h = \dfrac{1}{h}\, \delta$.

ii. In Proposition 2.2, we showed that the Complex $\left(\mathcal{F}^\bullet\,(X, \mathbb{C})\,, \delta^\bullet\right)$ is acyclic, and that

$$H^0\left(\mathcal{F}^\bullet\,(X, \mathbb{C})\,, \delta^\bullet\right) = \mathbb{C}\,\cdot$$

Thus, in the Laplacian decomposition given in 5.5, the sole term that plays a part now is the one that corresponds to $p = 0$, from which one gets the term C_0.

Remark 6.2. Following a comment in our introduction, we may note that the h-Laplacian depends on the choice of the measure μ_X. Somewhere this is not surprising - at the very beginning, which means, the cohomology, there were sums. Now, as we will see it in the sequel, the part played by the normalization factor C_0 will, in a certain sense, counterbalance this choice. So, finally, we perfectly fall on our feet, with an operator that is just defined up to multiplicative constants.

Notation (Set of Smooth Functions on X).

We will from now on denote by $C(X, \mathbb{C})$ the set of smooth functions on X, which take values in $\mathbb{C}$.

The subset of smooth functions on X, which take values in $\mathbb{R}^+ \times \mathbb{R}^+$, will be denoted by $C(X, \mathbb{R}^+ \times \mathbb{R}^+)$.

Definition 6.2 (**h-Markov Operator**).

Given a strictly positive real number h, we introduce the operator M_h, given by

$$\frac{Id - M_h}{h^2} = \frac{1}{2\,c_0^2}\,|\Delta_h| \cdot$$

Property 6.1. *Given a strictly positive real number h, we trivially have, for the constant function on X which takes the value 1, that*

$$M_h(1) = 1 \cdot$$

Given $x \in X$, let us denote by $M_h(x, \cdot)\, d\mu_X(\cdot)$ the measure such that, for any continuous function f on X:

$$M_h(f)(x) = \int_X f(y)\, M_h(x, y)\, d\mu_X(y) \cdot$$

For any $(y, z) \in X^2$, we have that

$$M_h(x, y)\, d\mu_X(y) = \left\{ \mathbb{1}_X - \mathbb{1}_{\mathcal{B}(x,h)}\, C_0(x, y, h)\, d\mu_X(y) \right\} \delta_x + \mathbb{1}_{\mathcal{B}(x,h)}\, C_0(x, y, h)\, d\mu_X(y) \cdot$$

Since the M_h operator is Markov if and only if, for any $x \in X$:

$$0 \leqslant \int_X M_h(x, y)\, d\mu(y) = 1$$

a necessary condition is thus that, for any $x \in X$:

$$\int_X \mathbb{1}_{\mathcal{B}(x,h)}\, C_0(x, z, h)\, d\mu(z) = \int_{\mathcal{B}(x,h)} C_0(x, z, h)\, d\mu(z) \leqslant 1 \cdot$$

Property 6.2 (Metropolis-Hastings Algorithm [MRR+53], [Has70]).

We recall that the Metropolis-Hastings algorithm is a Markov chain Monte Carlo method (MCMC), which enables one to generate a collection of sample states from a probability distribution $P(x)$, by means of a Markov process, which enables one to asymptotically reach a unique stationary distribution $\mathbb{P}_M(x) = \mathbb{P}(x)$.

The transition probabilities, from a given state x, to another y one, which are involved in the Markov process, have to satisfy the following necessary conditions:

i. *Existence of a stationary distribution $P_M(x)$, which requires the so-called detailed balance condition, in terms of conditional probabilities:*

$$\mathbb{P}\left[y|x\right]\,\mathbb{P}\left[x\right] = \mathbb{P}\left[x|y\right]\,\mathbb{P}\left[y\right]$$

which means that the process at stake is a reversible one.

ii. *Uniqueness of stationary distribution, which directly comes from the ergodicity (aperiodic and positive recurrent in time) of the Markov process. One easily sees that the aperiodicity garantees that the system does not return to the same state at fixed intervals, while the positive recurrence ensures that the expected number of steps for returning to the same state is finite.*

Remark 6.3. We can note that since

$$\frac{\mathbb{P}\left[y|x\right]}{\mathbb{P}\left[y\right]} = \frac{\mathbb{P}\left[x|y\right]}{\mathbb{P}\left[x\right]}$$

the transition is thus separated to, first, the proposal of a transition state, second, its acceptance/or rejection. The proposal distribution $\mathbb{P}rop\left[y|x\right]$ is thus the conditional probability of proposing a state y given the original one x. It is naturally connected to the probability of acceptance of the new state y with regard to x by

$$\mathbb{P}\left[y|x\right] = \mathbb{P}rop\left[y|x\right]\,\mathbb{A}\left[y|x\right]\,\cdot$$

At the same time, since one deals with a reversible process, we can write that

$$\mathbb{P}\left[x|y\right] = \mathbb{P}rop\left[x|y\right]\,\mathbb{A}\left[x|y\right]\,\cdot$$

Those two relations yield that

$$\frac{\mathbb{A}\left[y|x\right]}{\mathbb{A}\left[x|y\right]} = \frac{\mathbb{P}\left[y|x\right]}{\mathbb{P}\left[x|y\right]}\,\frac{\mathbb{P}rop\left[x|y\right]}{\mathbb{P}rop\left[y|x\right]}\,.$$

One of the probabilities of acceptance has to take the value 1 (either one stays in x, either one moves to y). States x and y playing symmetric parts, we can concentrate on the probabilty of acceptance of y:

$$\mathbb{A}\left[y|x\right] = \min\left\{1, \frac{\mathbb{P}\left[y|x\right]}{\mathbb{P}\left[x|y\right]}\,\frac{\mathbb{P}rop\left[x|y\right]}{\mathbb{P}rop\left[y|x\right]}\right\}\,.$$

One clearly sees a very useful advantage of such a method: bypassing the determination of normalization constants.

The algorithm itself is implemented according to the following steps:

i. *At time $t = 0$, one chooses an initial state x_0.*

ii. *At time $t > 0$, one generates a random candidat state y, and compute the acceptance probability:*

$$\min \left\{ 1, \frac{\mathbb{P}\,[y|x]}{\mathbb{P}\,[x|y]}\,\frac{\mathbb{P}rop\,[x|y]}{\mathbb{P}rop\,[y|x]} \right\}$$

and accept, or reject.

Remark 6.4. Why MCM methods? As recalled in the generalization paper by W. K. Hastings [Has70], such methods appear as more efficient than conventional ones once one deals with problems in "a large number of dimensions". Such a choice thus seems interesting for an upcoming potential application to fractal based structures, especially, when they are approximated by means of prefractal graphs, where iterations quickly yield very large number of points.

Property 6.3 (**h-Metropolis Operator**).

Given a strictly positive number h, a natural choice for the normalization factor C_0 involved in Definition 6.1 of the h-Laplacian is such that, for any $(x, y) \in X^2$,

$$C_0(x, y, h) = \min \left\{ \frac{1}{\mu\left(\mathcal{B}\left(x, h\right)\right)}, \frac{1}{\mu\left(\mathcal{B}\left(y, h\right)\right)} \right\} .$$

One thus recovers the Metropolis operator associated to the Markov kernel $(Id - M_h)$.

The associated random walk is the following: if the walk is in x, one chooses y in $\mathcal{B}\,(x, h)$ for the probability

$$\mathbb{1}_{\mathcal{B}(x,h)}\,\frac{1}{\mu\left(\mathcal{B}\left(x, h\right)\right)}\,d\mu_X(y)\,\cdot$$

Then, depending wether $\mu\left(\mathcal{B}\left(y, h\right)\right) \geqslant \mu\left(\mathcal{B}\left(x, h\right)\right)$ or not, one moves to y, or stay in x.

6.2 Singular Sets

We hereafter place ourselves in the euclidian plane of dimension 2, referred to a direct orthonormal frame.

6.2.1 Frame of the Study - Prefractal Graph Approximation

Notation. In the sequel, we will denote by $\mathcal{S}$ a singular set, of fractal type. Examples of such sets are the classical Sierpiński Gasket, the Koch Curve, the Weierstrass Curve.

By following the method developed by J. Kigami [Kig03], we approximate $\mathcal{S}$ by a sequence $(\mathcal{S}_m)_{m\in N}$ of finite graphs, the so-called *prefractals*. In classical cases, those graphs can be built through an iterative process, by means of an iterated function system (i.f.s) $\mathcal{T} = \{\mathcal{T}_0, \ldots, \mathcal{T}_{N-1}\}$ of N maps, $N \in \mathbb{N}$, such that

$$\mathcal{S} = \bigcup_{i=0}^{N-1} \mathcal{T}_i(\mathcal{S}) \cdot$$

When the maps of the i.f.s. are contractive, this latter property is the so-called *Collage Theorem* [BD85a]. When the maps are not contractive, one can, under specific conditions, have an equivalent result (see [Dav19]).

The process is more or less complicated, depending on wether the maps of the i.f.s. are affine (Sierpiński Gasket, the Koch Curve), or not (Weierstrass Curve).

Example 6.1.

i. In the case of Sierpiński Gasket, the iterated function system is constituted of three affine contractive maps (similarities), all with the same contraction ratio $\frac{1}{2}$, and fixed points P_0, P_1, P_2 located at the vertices of the initial equilateral triangle (see [Str06], and Figure 1 in the sequel):

$$\forall j \in \{0,1,2\}, \forall x \in \mathbb{R}^2 : \quad T_j(x) = \frac{1}{2}(x - P_j) + P_j \cdot$$

ii. In the case of a non-affine fractal curve, as the Weierstrass one, the iterated function system is constituted of $N_b \geqslant 3$ nonlinear maps, which, if they cannot be said contractive in the classical sense, bear an equivalent property (see [Dav19]). The fixed points are located at the vertices of the initial graph.

Definition 6.3 (Prefractal Graph Approximation).

Let us consider a sequence of finite discrete graphs $(\mathcal{S}_m)_{m\in\mathbb{N}}$. For any natural integer m, we denote by V_m the set of vertices of $\mathcal{S}_m$. In agreement with definition **??**, the initial set of points V_0 stands as the boundary of any $\partial\mathcal{S}_m$, $m \in \mathbb{N}$.

We suppose that:

i. The sequence $(V_m)_{m\in N}$ is increasing, i.e.,

$$\forall m \in \mathbb{N} : \quad V_m \subset V_{m+1} \cdot$$

ii. For any natural integer m, the graph $\mathcal{S}_m$ is equipped with an edge relation $\underset{m}{\sim}$: two vertices x and y of $\mathcal{S}_m$, i.e., two points belonging to V_m, will be said adjacent (or neighboring points) (see Figure 2) if and only if the line segment $[x,y]$ is an edge of $\mathcal{S}_m$. Note that this edge relation depends on m, which means that points connected in V_m might not stay connected in V_{m+1}.

iii. The euclidean distance between adjacent points tends towards zero when m goes to infinity, and the union $\bigcup_{m\in\mathbb{N}} V_m$ is dense in $\mathcal{S}$.

The sequence $(\mathcal{S}_m)_{m\in N}$ will then be called prefractal graph approximation to $\mathcal{S}$ (see Figure 1 for an example, in the case of the Sierpiński Gasket).

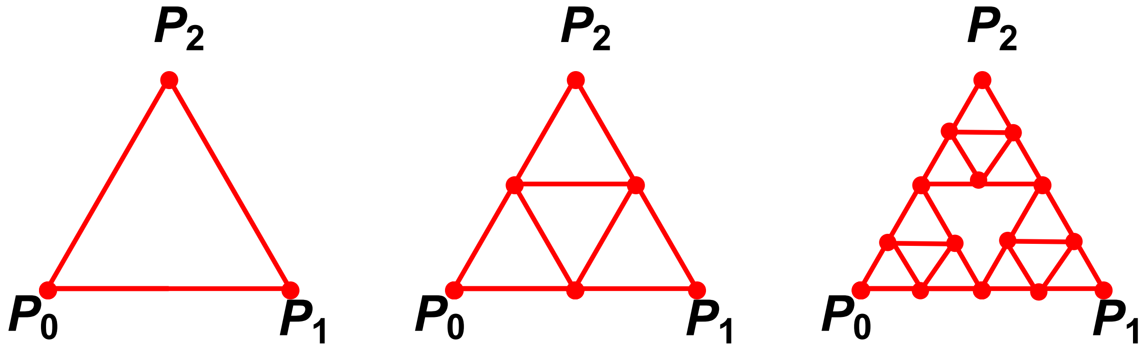

Figure 1: In the case of the Sierpiński Gasket, the graphs $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$, with $\partial\mathcal{S}_0 = V_0 = \{P_0, P_1, P_2\}$.

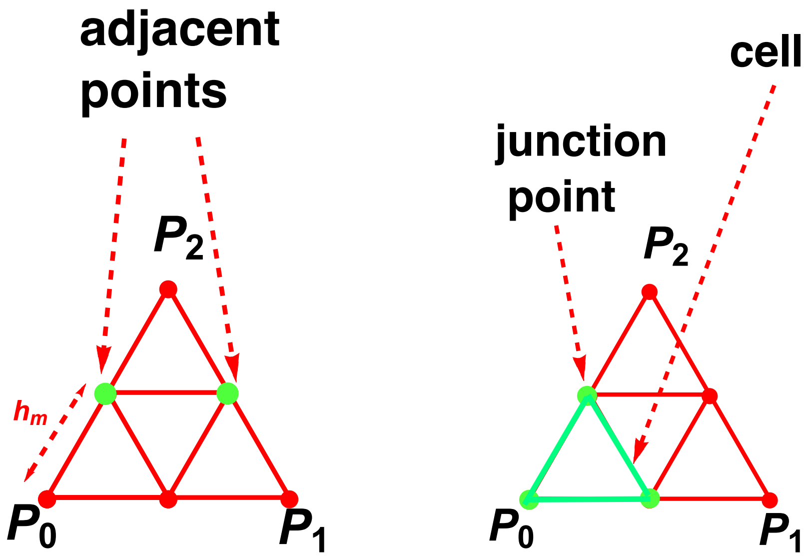

Figure 2: In the case of the Sierpiński Gasket, adjacent points, junction points, cells.

Notation (Adjacent Consecutive Vertices of the m^{th} Level Prefractal Approximation, $m \in \mathbb{N}$).

For the sake of clarity, given a natural integer m, two adjacent, consecutive vertices of the m^{th} level prefractal approximation $\mathcal{S}_m$ will be denoted in the following form

$$x_{m,k} \quad \text{and} \quad x_{m,k+1} \quad , \quad 0 \leqslant k \leqslant N-1 \,,$$

where N is the number of maps of the iterated function system.

The qualifier "consecutive" is to be understood in the sense that such points are obtained by means of consecutive maps of the iterated function system. We refer to [Str06] or [Dav18] for further details and examples.

Definition 6.4 (m**-Radius (**m**-Heigth)).**

Given a natural integer m, we will call m-radius (or m-heigth) of $\mathcal{S}_m$ the maximal euclidean distance between two connected vertices of $\mathcal{S}_m$, which we will denote by

$$h_m = \max_{(x,y)\,\in\, V_m^2,\, x \underset{m}{\sim} y} d_{eucl}(x,y)\cdot$$

Property 6.4 (Polygonal Domain [DL20]**).**

For any natural integer m, the $\#_m$ consecutive vertices of the graph $\mathcal{S}_m$ are, also, the vertices of N^m simple polygons $\mathcal{P}_{m,j}$, for $0 \leqslant j \leqslant N^m-1$, with N sides (see Figure 3). For any integer j such that $0 \leqslant j \leqslant N^m-1$, one obtains each polygon $\mathcal{P}_{m,j}$ by connecting the point number j to the point number $j+1$ if $j = i \bmod N$, for $0 \leqslant i \leqslant N-2$, and the point number j to the point number $j-N+1$ if $j = -1 \bmod N$.

To go further, and as required in the specific case of a fractal Curve (in order to have a complete polygonal neighborhood of the Curve), the $\#_m - 1$ consecutive vertices of the graph $\mathcal{S}_m$, distinct of P_0 and P_{N-1}, are the vertices of N^m-1 simple polygons $\mathcal{Q}_{m,j}$, $1 \leqslant j \leqslant N^m-2$, with maximum N sides. For any integer j such that $1 \leqslant j \leqslant N^m-2$, one obtains each polygon $\mathcal{Q}_{m,j}$ by linking the point number j to the point number $j+1$ if $j = i \bmod N$, for $1 \leqslant i \leqslant N-1$, and the point number j to the point number $j-N+1$ if $j = 0 \bmod N$.

Of course, those latter polygons are not to be taken to account when the considered singular set is not a fractal curve. If such is the case, we have that

$$\left\{\mathcal{Q}_m^j\,,\ 1 \leqslant j \leqslant N^m-2\right\} = \emptyset\cdot$$

The above polygons generate a Borel set of $\mathbb{R}^2$.

Example 6.2.

i. In the case of Sierpiński Gasket, the polygonal domain is constituted of equilateral triangles, as it can be seen in Figure 1.

ii. In the case of the Weierstrass Curve, the polygonal domain is constituted of N-gons, as it can be seen in Figure 3.

Definition 6.5 (m**-Cell).**

Given a natural integer m, we call m-cell, any simple polygon $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N^m-1$, or, when necessary, $\mathcal{Q}_{m,j}$, $1 \leqslant j \leqslant N^m-2$.

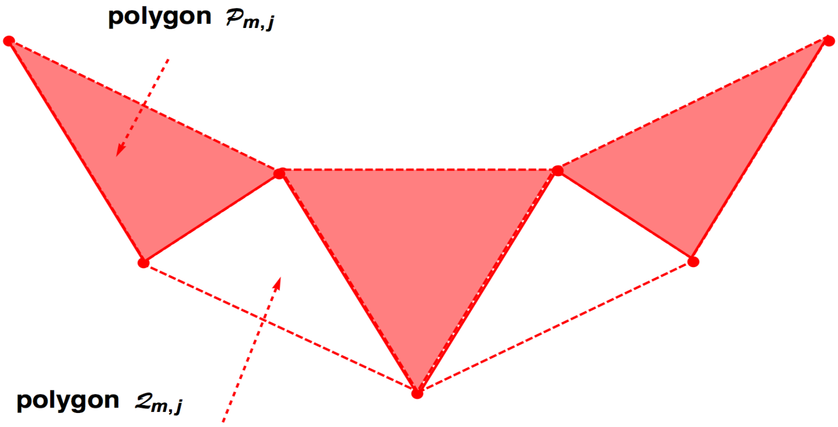

Figure 3: $\mathcal{P}_{m,j}$ and $\tilde{\mathcal{Q}}_{m,j}$ polygons/cells, in the case of the Weierstrass Curve.

Notation. For the sake of simplicity, given a natural integer m, the set of cells of $\mathcal{S}_m$ will be denoted by $\mathcal{C}_m$.

Remark 6.5.

i. Except for intersection points (i.e., junction ones), m-cells are disjoint.

ii. Despite the sequence $(V_m)_{m\in N}$ is increasing, the set of cells of $\mathcal{S}_{m+1}$, $m \in \mathbb{N}$ is not necessarily contained in the one of $\mathcal{S}_m$. For instance, one clearly see it is the case for the Sierpiński Gasket, since a $(m+1)$-cell is obtained by dividing a m-one to three. In a different configuration, let us say, the Weierstrass Curve (see Figure 4 - we refer to [Dav18], [DL20] for further details), this is not the case.

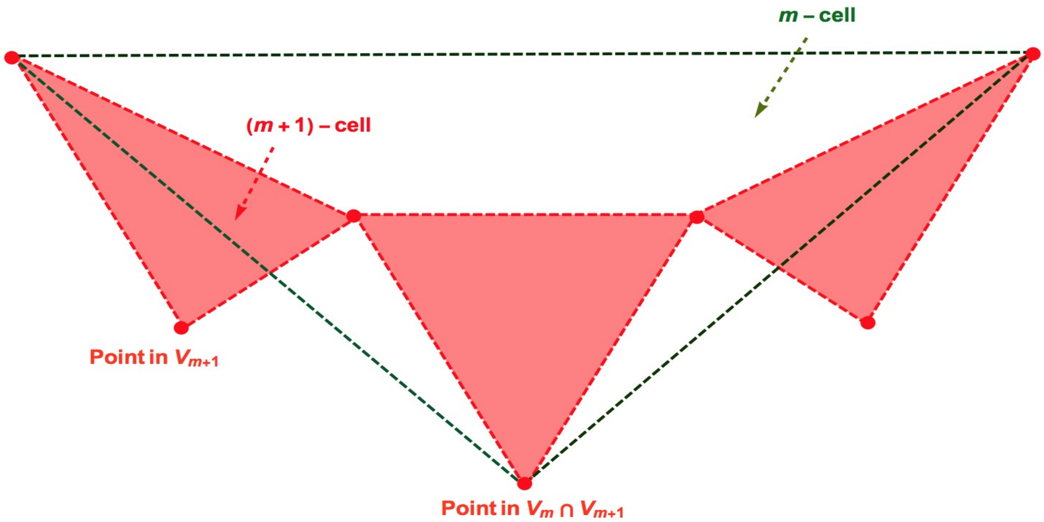

Figure 4: $(m+1)$-cells and m-cells, in the case of the Weierstrass Curve, for $N = 3$.

Definition 6.6. Power of a Vertex of the Prefractal Graph $\mathcal{S}_m$, $m \in \mathbb{N}^\star$ with Regard to the Polygonal Family $\left\{\mathcal{C}_m^j \, , \, 0 \leqslant j \leqslant \#\mathcal{C}_m^j - 1\right\}$

Given a strictly positive integer m, a vertex x of the prefractal graph $\mathcal{S}_m$ will be said:

i. of power one with regard to the polygonal family $\left\{\mathcal{C}_m^j \, , \, 0 \leqslant j \leqslant \#\mathcal{C}_m^j\right\}$ if x belongs to one and only one m-cell $\mathcal{C}_{m,j}$, $0 \leqslant j \leqslant \#\mathcal{C}_m^j - 1$;

ii. of power $\dfrac{1}{k}$, $k \in \mathbb{N}^\star$, with regard to the polygonal family $\left\{\mathcal{C}_m^j \, , \, 0 \leqslant j \leqslant \#\mathcal{C}_m^j - 1\right\}$ if x is a common vertex to k cells $\mathcal{C}_{m,j}$, $0 \leqslant j \leqslant \#\mathcal{C}_m^j - 1$;

Remark 6.6.

i. The above power is required when defining a measure (see [DL20], in the case of the Weierstrass Curve, or [Str06], in the case of the Sierpiński Gasket. It acts as a kind of pound.

ii. In the case of the Sierpiński Gasket, except for boundary points (the fixed points of the affine maps of the associated i.f.s., P_0, P_1, P_2,), each vertex point at a given level $m \in \mathbb{N}^\star$ belongs to exactly two m-cells, and thus has power $\dfrac{1}{2}$. As explained in [Str06], one can get rid of the part played by the boundary points when computing a measure, since the sum at stake goes to zero when the integer m tends towards infinity.

iii. In the case of the Weierstrass Curve, except again for boundary points, each vertex point at a given level $m \in \mathbb{N}^\star$ belongs to at most two m-cells, in which case it has power $\dfrac{1}{2}$ also.

iv. The associated power coefficient $\dfrac{1}{2}$ thus plays the part of a multiplicative constant. For the sake of simplicity, we will consider it as contained in the one at stake in the definition of our Laplacians (r^{-m}, 6.6, or c_m, 6.4).

Definition 6.7 (**m-Path**).

Given two vertices in $\bigcup\limits_{m\in\mathbb{N}} V_m$, i.e., two vertices $x_{m,k}$ and $x_{m,k+p}$, for $m \in \mathbb{N}$, $0 \leqslant k \leqslant \#V_m$ and $0 \leqslant p \leqslant \#V_m - k$, we call m-path between $x_{m,k}$ and $x_{m,k+p}$ the ordered set of vertices given by

$$\mathcal{P}_m\left(x_{m,k}, x_{m,k+p}\right) = \left\{x_{m,k+j} \, , \, 0 \leqslant k \leqslant p\right\} \cdot$$

An example is given Figure 5.

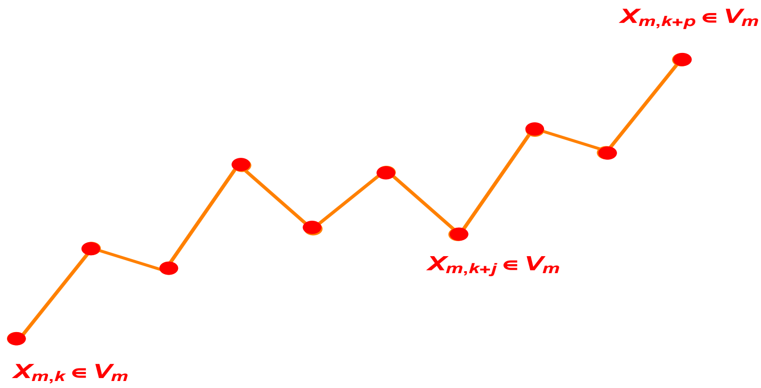

Figure 5: A m-path.

Definition 6.8 ((m,n)**-Path**).

i. Given a natural integer m, and two adjacent vertices $x_{m,k}$ and $x_{m,k+1} \underset{m}{\sim} x_{m,k}$ of V_m, for $0 \leqslant k \leqslant \# V_m - 1$, we call (m,m)-path between $x_{m,k}$ and $x_{m,k+1}$ the ordered set of vertices

$$\mathcal{P}_{m,n}\left(x_{m,k}, x_{m,k+1}\right) = \left\{x_{m+n,k+j}\,,\, 0 \leqslant j \leqslant N^{n-m}\right\}$$

where

$$x_{m+n,k} = x_{m,k} \quad \text{and} \quad x_{m+n,k+n} = x_{m,k+1}\,.$$

(We recall that N denotes the number of maps of the iterated function system introduced at the beginning of subsection 6.2.1. N^{n-m} simply means that N^{n-m} new points have been introduced between $x_{m,k}$ and $x_{m,k+1}$)

An example is given Figure 6.

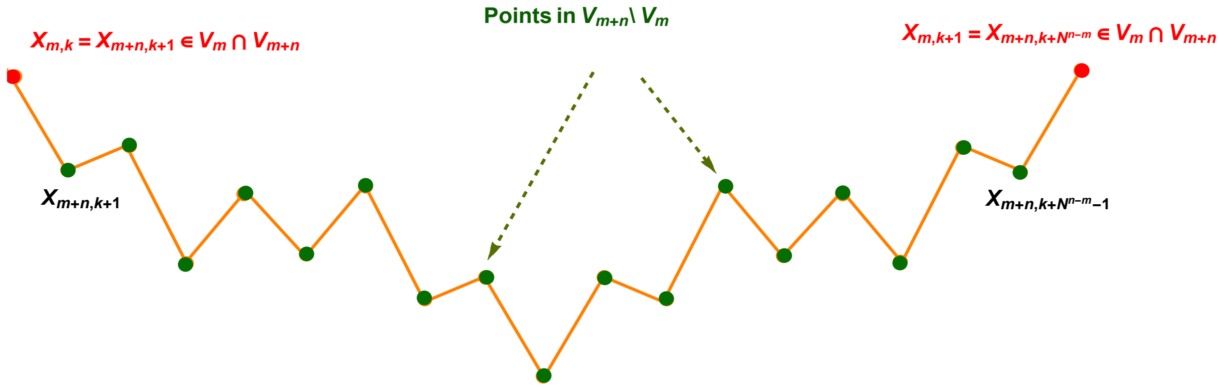

Figure 6: A (m,n)-path.

ii. Given a natural integer m, and two vertices $x_{m,k}$ and $x_{m,k+p}$ of V_m, for $0 \leqslant p \leqslant \# V_m$ and $0 \leqslant k \leqslant \# V_m - p$, we call (m,m)-path between $x_{m,k}$ and
$x_{m,k+p}$ the ordered set of vertices given by

$$\mathcal{P}_{m,n}\left(x_{m,k}, x_{m,k+p}\right) = \bigcup_{j=0}^{p-1} \mathcal{P}_{m,n}\left(x_{m,k+j}, x_{m,k+j+1}\right)\,.$$

Remark 6.7. Given two vertices x and y in $\bigcup\limits_{m\,\in\,\mathbb{N}} V_m$, i.e., two vertices x and y in V_m, for a given value of the integer m, there exists an infinity of (m,n)-paths between x and y. It is clear that the minimal one - the simplest one, is the m-one.

Definition 6.9 (m**-Edge Distance**).

Given a natural integer m, and two vertices $x_{m,k}$ and $x_{m,k+p}$ in $\bigcup\limits_{m\,\in\,\mathbb{N}} V_m$, for $0 \leqslant k \leqslant \# V_m$ and $0 \leqslant p \leqslant \# V_{m-k}$, the m-edge distance between $x_{m,k}$ and $x_{m,k+p}$ is defined as the length of the minimal path connecting $x_{m,k}$ and $x_{m,k+p}$ in V_m, i.e.,

$$d_{m,edge}\left(x_{m,k}, x_{m,k+p}\right) = \sum_{k=0}^{p-1} d_{eucl}\left(x_{m,k+j}, x_{m,k+j+1}\right) \cdot$$

In the case of adjacent vertices $x_{m,k}$ and $x_{m,k+1}$, we simply have that

$$d_{m,edge}\left(x_{m,k}, x_{m,k+1}\right) = d_{eucl}\left(x_{m,k+j}, x_{m,k+1}\right) \cdot$$

Remark 6.8.

i. This edge distance between two vertices corresponds, in a sense, tothe distance at a given level m of the prefractal graph approximation. Adjacent points at the same level are close, but become very distant as far as the level increases.

ii. Defining (m,n)-paths enables one to switch, when necessary, from a level m to higher $n > m$. Such a situation happens when handling our forthcoming m-balls.

The next problem that arises now is: how can one define balls in our context?
Of course, euclidean ones could do the job – namely, the important point is that given a radius $r > 0$, and a point x, we still have that

$$\forall \varepsilon \in]0,1[: \quad \mathcal{B}(x, \varepsilon\, r) \subset \mathcal{B}(x, r) \ ,$$

i.e., bigger balls contain smaller ones.

An important thing is that we deal with discrete balls. This specific point has to be taken to account when defining balls – in so far as we will further consider random walks moving on a given state m of the sequence of prefractal graphs, which ends by switching from V_m to V_{m+1}, in a lack of memory process. The change of state – the m^{th} to the $(m+1)^{th}$ state – comes from the fact that $V_m \subset V_{m+1}$ and that $\#\left(V_{m+1} \setminus V_m\right) > \# V_m$ – in a sense, the probability of reaching the new state $m+1$ is higher.

What we would like, thus, is that the definition of balls could account for this specificity. Bearing in mind that when m increases, the edge distance between adjacent vertices become more and more small, the solution is that balls could have more points near their origin, i.e., with a distribution of points proportional to their position.

Definition 6.10 (**m-Ball**).

Given a natural integer m, a strictly positive number r, and a vertex x of V_m, the m-ball of center x and radius r is defined by

$$\mathcal{B}_m(x,r) = \{y \in V_m \,,\, d_{m,edge}(x,y) < r\} \cdot$$

The associated closed ball will be denoted $\bar{\mathcal{B}}_m(x,r)$.

Remark 6.9. The above definition 6.10 enable us to deal with the best suited ball, depending on the considered structure:

i. In the case of Sierpiński Gasket, we will handle m-balls of radius $\dfrac{1}{2^m}$, which coincide with m-cells.

ii. In the case of the Weierstrass Curve, we will handle m-balls of radius $j \times h_m$, for $1 \leqslant j \leqslant N-1$ (see Definition 6.4). For a center of the ball located in a junction point x_m (between m-cells), m-balls of radius $(N-1) \times h_m$ enable us to encompass the m-cells with the same vertex x_m. One can also simply want to take to account the immediate (adjacent) neighbors of a vertex, in which case m-balls of radius h_m are enough.

Remark 6.10. Another interesting point that may be noted is that our definition of m-balls yields, for any vertex x of V_m, inclusion relations of the form

$$\mathcal{B}_{m+1}(x,h_{m+1}) \subset \mathcal{B}_{m+1}(x,h_m) \cdot$$

Property 6.5. *Since the sequence $(V_m)_{m\in N}$ is increasing, we of course have, for any strictly positive number r, any natural integer m, and any vertex x of V_m, that*

$$\mathcal{B}_m(x,r) \subset \mathcal{B}_{m+1}(x,r) \cdot$$

This can be refined, for $r' < r$, in:

$$\mathcal{B}_m\left(x,r'\right) \subset \mathcal{B}_{m+1}(x,r) \cdot$$

Definition 6.11 (**Regular Probability Measure on $\mathcal{S}$ [Str06]**).

A regular probability measure on $\mathcal{S}$ is a measure μ that assigns weights $\mu(\mathcal{C}_m^j)$ to any m-cell of $\mathcal{S}_m$, $m \in \mathbb{N}$, for $\left\{\mathcal{C}_m^j \,,\, 0 \leqslant j \leqslant \#\mathcal{C}_m - 1\right\}$, in an additive way:

i. $\forall\, m \in \mathbb{N}\,, \forall\, j \in \{0,\ldots,\#\mathcal{C}_m-1\}:\quad \mu\left(\mathcal{C}_m^j\right) > 0.$

ii. Given two m-cells $\mathcal{C}_m^j$ and $\mathcal{C}_m^{j+1}$, $\left\{\mathcal{C}_m^j \,,\, 0 \leqslant j \leqslant \#\mathcal{C}_m - 2\right\}$ which intersect only at junction points:

$$\mu\left(\mathcal{C}_m^j \cup \mathcal{C}_m^{j+1}\right) = \mu\left(\mathcal{C}_m^j\right) + \mu\left(\mathcal{C}_m^{j+1}\right) \cdot$$

iii. $\lim\limits_{m\to+\infty}\left(\mu\left(\mathcal{C}_m^j\right)\right)_{0\leqslant j\leqslant \#\mathcal{C}_m-1}=0.$

iii. $\mu\left(\mathcal{S}\right)=\lim\limits_{m\to+\infty}\sum\limits_{j=0}^{\#\mathcal{C}_m-1}\mu\left(\mathcal{C}_m^j\right)=1.$

Given a continuous function f on $\mathcal{S}$, we set, from now,

$$\int_{\mathcal{S}} f\,d\mu=\lim_{m\to+\infty}\sum_{j=0}^{\#\mathcal{C}_m-1}\sum_{x \text{ vertex of } \mathcal{C}_m^j}\frac{\mu\left(\mathcal{C}_m^j\right)}{\#\text{vertices of } \mathcal{C}_m^j}\,f\left(x\right)\cdot$$

Notation. From now on, we will denote by μ a measure on $\mathcal{S}$.

6.2.2 h_m-Laplacian

Definition 6.12 (h_m**-Laplacian,** $m\in\mathbb{N}$**).**

Following Definition 5.7, given a natural integer m, we define the h_m-Laplacian as the operator

$$|\Delta_{h_m}|=\delta^\star_{h_m}\,\delta_{h_m}\,,$$

which acts on smooth functions f on V_m through:

$$\forall\,x\,\in\,V_m\,:\quad |\Delta_{h_m}|\,(f)(x)=\frac{2\,c^2_{0,m}}{h_m^2}\int_{\bar{\mathcal{B}}_m(x,h_m)}\left\{f(y)-f(x)\right\}\,C_0\left(x,y,m\right)\,d\mu(y)\,,$$

where

$$C_0\left(x,y,m\right)=\min\left\{\frac{1}{\mu\left(\bar{\mathcal{B}}_m\left(x,h_m\right)\right)},\frac{1}{\mu\left(\bar{\mathcal{B}}_m\left(y,h_m\right)\right)}\right\}\,,$$

and where $c^2_{0,m}$ denotes a strictly positive constant.

Remark 6.11. It is clear that, when $m\to\infty$,

$$\frac{1}{\mu\left(\bar{\mathcal{B}}_m\left(x,h_m\right)\right)}\gg 1\quad\text{and}\quad\frac{1}{\mu\left(\bar{\mathcal{B}}_m\left(y,h_m\right)\right)}\gg 1\cdot$$

Definition 6.13. Topological Laplacian of Order $m\in\mathbb{N}^\star$

For any strictly positive integer m, and any real-valued function f, defined on the set V_m of the vertices of the graph $\mathcal{S}_m$, we introduce the topological Laplacian of order m, $\Delta^\tau_m(f)$, by

$$\Delta^\tau_m f(x)=\sum_{y\in V_m,\,y\underset{m}{\sim}x}\left(f(y)-f(x)\right)\quad\forall\,x\,\in\,V_m\setminus\partial V_m\cdot$$

Property 6.6 (Pointwise Formula - Kigami-Strichartz Laplacian [Str06]).

Given a strictly positive integer m, and a vertex $x \in V_m \setminus V_0$, we introduce the piecewise harmonic (with respect to the topological Laplacian Δ_m^τ) spline function $\psi_X^m \in \mathcal{S}(\mathcal{H}_0, V_k)$ such that

$$\psi_x^m(y) = \left\{ \begin{array}{ccc} \delta_{xy} & \forall & y \in V_m \\ 0 & \forall & y \notin V_m \end{array} \right. \quad , \quad where \quad \delta_{xy} = \left\{ \begin{array}{ccc} 1 & if & x = y \\ 0 & else & \end{array} \right. .$$

Provided the fractal $\mathcal{S}$ is self-similar, we the obtain a Laplacian, defined, for any continuous function f on $\mathcal{S}$, which belongs to its domain $dom\,\Delta$, through

$$\forall\, x \notin V_0 : \quad \Delta f(x) = \lim_{m\to\infty} \frac{r^{-m}}{\displaystyle\int_{\mathcal{S}} \psi_x^m \, d\mu} \, \Delta_m^\tau f(X) \,,$$

where, for any strictly positive integer m, r^{-m} is a normalization constant.

Property 6.7 (Back to the h_m-Laplacian).

The definition of the measure on $\mathcal{S}$ yields, for any vertex $x \in V_m \setminus V_0$:

$$\begin{aligned} |\Delta_{h_m}|\,(f)(x) &= \frac{2\,c_{0,m}^2}{h_m^2} \int_{\bar{\mathcal{B}}_m(x,h_m)} \{f(y) - f(x)\} \,\min\left\{ \frac{1}{\mu\left(\bar{\mathcal{B}}_m\,(x,h_m)\right)}, \frac{1}{\mu\left(\bar{\mathcal{B}}_m\,(y,h_m)\right)} \right\} d\mu(y) \\ &= \frac{2\,c_{0,m}^2}{h_m^2} \sum_{y\in\mathcal{C}_m^j,\, y \underset{m}{\sim} x} \frac{\mu\left(\mathcal{C}_m^j\right)\,\{f(y)-f(x)\}}{\#vertices\ of\ \mathcal{C}_m^j} \,\min\left\{ \frac{1}{\mu\left(\bar{\mathcal{B}}_m\,(x,h_m)\right)}, \frac{1}{\mu\left(\bar{\mathcal{B}}_m\,(y,h_m)\right)} \right\} . \end{aligned}$$

Up to a multiplicative constant that depends on the geographic position of x, which impacts the number of its neighbors and, thus, the measures of the (closed) balls $\bar{\mathcal{B}}_m\,(x,h_m)$ and $\bar{\mathcal{B}}_m\,(y,h_m)$, we have that

$$\mu\left(\bar{\mathcal{B}}_m\,(x,h_m)\right) = \mu\left(\bar{\mathcal{B}}_m\,(y,h_m)\right) = \mu\left(\mathcal{C}_m^j\right) \,,$$

which yields

$$\begin{aligned} |\Delta_{h_m}|\,(f)(x) &= \frac{2\,c_{0,m}^2}{h_m^2\ \#vertices\ of\ \mathcal{C}_m^j} \sum_{y\in\mathcal{C}_m^j,\, y \underset{m}{\sim} x} \{f(y)-f(x)\} \\ &= \frac{2\,c_{0,m}^2}{h_m^2\ \#vertices\ of\ \mathcal{C}_m^j}\, \Delta_m^\tau(f)(x) \cdot \end{aligned}$$

Since $\lim_{m\to\infty} h_m = 0$, we have that

$$\begin{aligned} \lim_{m\to\infty} |\Delta_{h_m}|\,(f)(x) &= \lim_{h\to 0} |\Delta_h|\,(f)(x) \\ &= |\Delta_0|\,(f)(x) \cdot \end{aligned}$$

Henceforth, under the condition

$$\frac{r^{-m}}{\displaystyle\int_{\mathcal{S}} \psi_x^m \, d\mu} = \frac{2\,c_{0,m}^2}{h_m^2\ \#vertices\ of\ \mathcal{C}_m^j} \,,$$

it can also be written as

$$\frac{r^{-m}}{\frac{\mu(\mathcal{C}_m^j)}{\#vertices\ of\ \mathcal{C}_m^j}} = \frac{2\,c_{0,m}^2}{h_m^2\,\#vertices\ of\ \mathcal{C}_m^j}$$

i.e.,

$$c_{0,m}^2 = \frac{r^{-m}\,h_m^2\,\left(\#vertices\ of\ \mathcal{C}_m^j\right)^2}{2\,\mu\left(\mathcal{C}_m^j\right)}$$

in order to recover the same Laplacian, i.e., the one of classical analysis.

Remark 6.12. The above condition makes sense, in so far as

$$\mu\left(\mathcal{C}_m^j\right) \lesssim \frac{1}{h_m^2}\cdot$$

Then, one just has, up to a multiplicative constant, the equality of the normalization constants.

Remark 6.13. Henceforth, Laplacians on singular sets can be equivalently obtained, either through the now classical analysis tools on fractals introduced by J. Kigami, either using our h-Laplacians. There is here an interesting point to note, due to the fact that the sequence $(V_m)_{m\in N}$ is increasing. It thus happens that the h_{m+1}-Laplacian can be obtained may one consider the modified MCMC method where, given a state $x \in V_m \subset V_{m+1}$, the transition probability towards a new state $y \in V_m \subset V_{m+1}$ depends on wether $y \underset{m+1}{\sim} x$, or not (i.e., an edge relation between x and y can only exist at level m):

$$\mathbb{P}\,[y|x] = \mathbb{P}\left[y \underset{m+1}{\sim} x|x\right] + \mathbb{P}\left[y \underset{m+1}{\not\sim} x|x\right]\cdot$$

The acceptance probability is then given by

$$\min\left\{1, \frac{\#\left(V_{m+1}\setminus V_m\right)}{\#V_{m+1}}\,\frac{\mu\left(\mathcal{B}_m\left(y,h_m\right)\right)}{\mu\left(\mathcal{B}_m\left(x,h_m\right)\right)}, \frac{\#V_m}{\#V_{m+1}}\,\frac{\mu\left(\mathcal{B}_{m+1}\left(y,h_m\right)\right)}{\mu\left(\mathcal{B}_{m+1}\left(x,h_m\right)\right)}\right\}\cdot$$

Since

$$\#\left(V_{m+1}\setminus V_m\right) > \#V_m\,,$$

and, more precisely, when $m \to \infty$,

$$\#\left(V_{m+1}\setminus V_m\right) \gg \#V_m\,,$$

when, at the same time, $h_m \to 0$, which means that the random walk will naturally end in switching to the $(m+1)^{th}$ level of the prefractal graph approximation.

As seen previously (see Property 6.5), we cannot write a comparison-inclusion relation between the balls $\mathcal{B}_m\left(x,h_m\right)$ and $\mathcal{B}_{m+1}\left(x,h_{m+1}\right)$, as the one that exists for the euclidean ones, i.e.,

$$\mathcal{B}_{eucl}\left(x,h_{m+1}\right) \subset \mathcal{B}_{eucl}\left(x,h_m\right)\cdot$$

Yet, the switching is natural, since

$$\mathcal{B}_m\left(x,h_m\right) \subset \mathcal{B}_{m+1}\left(x,h_m\right) \quad \text{and} \quad \mathcal{B}_{m+1}\left(x,h_{m+1}\right) \subset \mathcal{B}_{m+1}\left(x,h_m\right)\cdot$$

In fact, the random walk is initially in $\mathcal{B}_m(x, h_m)$, but already in $\mathcal{B}_{m+1}(x, h_m)$. It then naturally switches to $\mathcal{B}_{m+1}(x, h_{m+1})$.

Henceforth, the h_{m+1}-Laplacian can be seen as an extension of the h_m-one to V_{m+1}. We can then draw a parallel with the decimation process of Fukushima and Shima [FS92], [Shi96], where, given an eigenfunction u_m on $V_m \setminus \partial V_m$, for the eigenvalue Λ_m, one extends u_m on $V_{m+1} \setminus \partial V_{m+1}$ in a function u_{m+1}, which will itself be an eigenfunction of the $(m+1)^{th}$ graph Laplacian Δ_{m+1}, for the eigenvalue Λ_m.

In other words, this can be seen as a sort of "continuity" of the sequence of discrete Laplacians.

Definition 6.14 (**Modified h_m-Laplacian,** $m \in \mathbb{N}$).

Following Definition 6.12, given a natural integer m, we define the modified h_m-Laplacian as the operator $\left|\tilde{\Delta}_{h_m}\right|$, which acts on smooth functions f on V_m, through:

$$\forall\, x \in V_m \,:\quad \left|\tilde{\Delta}_{h_m}\right|(f)(x) = \frac{2\,\tilde{c}_{0,m}^2}{h_m^2} \int_{\bar{\mathcal{B}}_m(x,h_m)} \{f(y) - f(x)\}\, \tilde{C}_0(x, y, m)\, d\mu(y)\,,$$

where

$$\tilde{C}_0(x,y,m) = \min\left\{ \frac{\#(V_{m+1}\setminus V_m)}{\#V_{m+1}} \frac{1}{\mu(\mathcal{B}_m(x,h_m))}, \frac{\#(V_{m+1}\setminus V_m)}{\#V_{m+1}} \frac{1}{\mu(\mathcal{B}_m(y,h_m))}, \frac{\#V_m}{\#V_{m+1}} \frac{1}{\mu(\mathcal{B}_{m+1}(x,h_m))}, \frac{\#V_m}{\#V_{m+1}} \frac{1}{\mu(\mathcal{B}_{m+1}(y,h_m))} \right\}$$

and where $\tilde{c}_{0,m}^2$ denotes a strictly positive constant.

As for the correspondence of Property 6.7, it is obtained thanks to the following property:

Property 6.8 (**Recovering the Modified h_m-Laplacian,** $m \in \mathbb{N}$).

The definition of the measure on $\mathcal{S}$ yields, for any vertex $x \in V_m \setminus V_0$, for $m \in \mathbb{N}$, that

$$\begin{aligned} |\Delta_{h_m}|(f)(x) &= \frac{2\,\tilde{c}_{0,m}^2}{h_m^2} \int_{\bar{\mathcal{B}}_m(x,h_m)} \{f(y) - f(x)\}\, \tilde{C}_0(x,y,m)\, d\mu(y) \\ &= \frac{2\,\tilde{c}_{0,m}^2}{h_m^2} \sum_{y \in \mathcal{C}_m^j,\, y \underset{m}{\sim} x} \frac{\mu\left(\mathcal{C}_m^j\right) \{f(y) - f(x)\}}{\#vertices\ of\ \mathcal{C}_m^j}\, \tilde{C}_0(x,y,m)\,. \end{aligned}$$

Under the condition

$$r^{-m} = \frac{2\,\tilde{c}_{0,m}^2}{h_m^2} \sum_{\substack{y \in \mathcal{C}_m^j,\ y \underset{m}{\sim} x \\ or\ \ y \in \mathcal{C}_{m+1}^{j'},\ y \underset{m+1}{\sim} x}} \left(\frac{\mu\left(\mathcal{C}_m^j\right)}{\#vertices\ of\ \mathcal{C}_m^j} \right)^2 \tilde{C}_0(x,y,m)\,,$$

we then obtain the new correspondence between the modified h_m-Laplacian, and the Kigami-Strichartz Laplacian (see [Kig01], [Str06]).

6.2.3 Prefractal Cohomology

At the beginning of our study (see Definition 2.3), given a natural integer p, we have introduced the concept of p-differential δ^p, from the set of p-fermions $\mathcal{F}^p\left(X,\mathcal{A}\right)$ to the set of $(p+1)$-fermions $\mathcal{F}^{p+1}\left(X,\mathcal{A}\right)$, by means of differences.

In the case of prefractals, if differential operators – local ones, are also defined by means of differences, we have to be more subtle, in so far as it depends on edge relations. For instance, given $m\in\mathbb{N}^\star$, and a real-valued function f, defined on the set of vertices V_m, the topological Laplacian of order m is defined through:

$$\Delta^{\tau}_m f(x)=\sum_{y\in V_m,\, y\underset{m}{\sim}x}\left(f(y)-f(x)\right)\quad\forall\, x\,\in\,V_m\setminus\partial V_m\,\cdot$$

Thus, local differences between adjacent points–vertices are concerned.

Now, since the sequence $(V_m)_{m\in N}$ is increasing, a local difference of the form

$$f(x)-f(y)\quad\text{for}\quad y\underset{m}{\sim}x\,,$$

can be more explicitely written as

$$f\left(x_{m,k}\right)-f\left(x_{m,k+1}\right)\,,$$

or, thanks to an equivalent of a Chasles relation along the path $\mathcal{P}_{m,N}\left(x,y\right)$, as

$$f(x)-f(y)=\sum_{(z,t)\,\in\,\left(\mathcal{P}_{m,N}(x,y)\right)^2}\left(f(z)-f(t)\right)\,\cdot$$

It can then be explicited, in the case of the example displayed Figure 7:

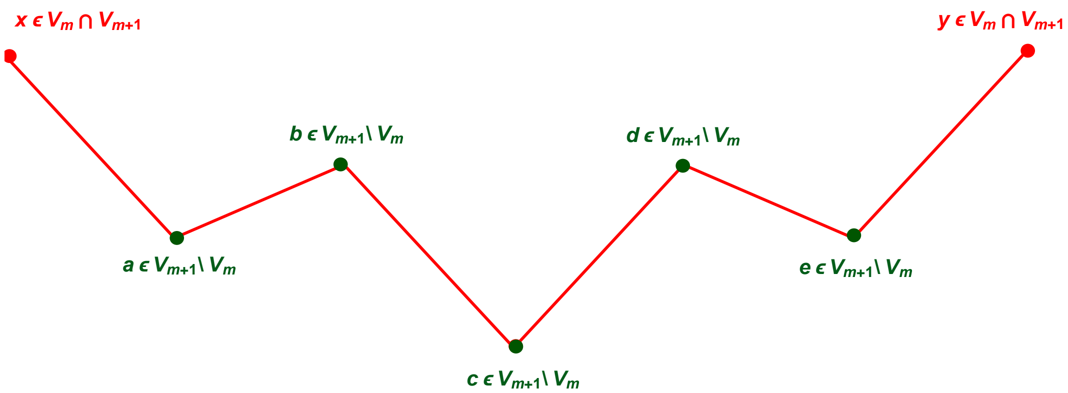

Figure 7

$$\begin{aligned} f(x)-f(y) \quad &= \quad \{f(x)-f(a)\}+\{f(a)-f(b)\}+\{f(b)-f(c)\}\\ &\quad\; +\{f(c)-f(d)\}+\{f(d)-f(e)\}+\{f(e)-f(y)\}\end{aligned}$$

Thus, p-differentials map the set of m-fermions to the set of $N\times p$ fermions:

$$\mathcal{F}^p\xrightarrow{\delta^p}\mathcal{F}^{N\,p}$$

It is then legitimate to question the real meaning of the associated cohomology.

As in [ACSY14], we consider that p-fermions act on p-dimensional structures. Given a natural integer m, the set of vertices V_m has $\#V_m$ points, and can be considered as $\#V_m$-dimensional. In fact, the kernel $\ker \delta^{\#V_m}$ corresponds to the fermions that stay on the m^{th}-level approximation to the prefractal sequence $(\mathcal{S}_m)_{m\in N}$. The image $Im\,\delta^{\#V_{m-1}}$ consists in the fermions coming from the $(m-1)^{th}$-level approximation to the prefractal sequence. The cohomology is thus constituted of the quotient groups

$$\ker \delta^{\#V_m}/Im\,\delta^{\#V_{m-1}} \quad , \quad m \in \mathbb{N}^\star \cdot$$

In a sense, this amounts to a kind of "hierarchy" in the structure.

To this point, we would like to concentrate upon the fact that, in [ACSY14], the authors mainly deal with low dimensional forms (0-,1-,2-). We find it interesting to handle $\#V_m$-fermions, acting on the whole set of vertices of V_m, which appears as rather natural, in so far as the points belong to the same m^{th}-order prefractal graph.

This is of course a math paper. Yet, the following quote seems very appropriate to close this point:

"It is by no means obvious how to realize these intuitions in a precise theory, and there are perhaps more than one way to do this." [ACSY14]

Example 6.3 (**The Specific Case of the Sierpiński Gasket**).

In the case of the Sierpiński Gasket, given a natural integer m, we have that

$$h_m = \frac{1}{2^m} \cdot$$

For the natural probability measure μ, which assigns the value 1 to the Gasket, the measure of a m-cell of the prefractal graph $\mathcal{S}_m$ is given, for any integer j in $\{0,\ldots,N^m-1\}$, by

$$\mu\left(\mathcal{C}_m^j\right) = \mathcal{A}_m = \frac{1}{3^m} \cdot$$

For any vertex $x \in V_m$, the number of points in the closed ball $\bar{\mathcal{B}}_m(x,h_m)$ depends on the geographic location of x:

i. If x belongs to V_0: x has exactly two neigbors, at distance h_m. The ball $\mathcal{B}_m(x,h_m)$ contains exactly three points, x and its two neighbors.

The measure of the ball is then exactly the measure of a m-cell, i.e.,

$$\mu\left(\mathcal{B}_m(x,h_m)\right) = \mathcal{A}_m \cdot$$

ii. If x does not belong to V_0: x has exactly four neigbors, at distance h_m. The ball $\mathcal{B}_m(x,h_m)$ contains exactly five points, x and its four neighbors, and, thus, three m-cells.

The measure of the ball is then

$$\mu\left(\mathcal{B}_m(x,h_m)\right) = 3\,\mathcal{A}_m \cdot$$

Meanwhile, for a m^{th}-order triangular cell of the Gasket, with respective vertices x, y, z, we have that

$$\int_{\mathcal{S}} \left\{ \psi_x^m + \psi_y^m + \psi_z^m \right\} \, d\mu = \mathcal{A}_m \cdot$$

Thus,

$$\int_{\mathcal{S}} \psi_x^m \, d\mu = \frac{1}{3} \, \mathcal{A}_m \cdot$$

Since (we refer to [Str06]),

$$r^{-m} = \left(\frac{5}{3} \right)^m ,$$

we then obtain that

$$\frac{r^{-m}}{\mathcal{A}_m} = \frac{2 \, c_{0,m}^2}{9 \, h_m^2}$$

i.e.,

$$c_{0,m}^2 = \frac{9 \times 3^m \, r^{-m}}{2 \times 4^m} = \frac{9}{2} \, \frac{5^m}{4^m} \cdot$$

As for the detailed Hodge-De Rham calculus, one may find it, in an explicit way, in [ACSY14]. Now, as for the modified h_m-Laplacian, we have that

$$\# V_m = \frac{3^{m+1} + 3}{2} \quad , \quad \# V_{m+1} = \frac{3^{m+2} + 3}{2} \cdot$$

At the same time, for any integer j' in $\left\{ 0, \ldots, N^{m+1} - 1 \right\}$, we also have that

$$\mu \left(\mathcal{C}_m^{j'} \right) = \mathcal{A}_{m+1} = \frac{1}{3^{m+1}} \cdot$$

This yields

$$\tilde{c}_{0,m} = \frac{3^{m+2} + 3}{3^{m+1} + 3} \, \frac{3^{m+1}}{2 \times 4^m} \, r^{-m} > c_{0,m}^2 \cdot$$

Example 6.4 (**The Specific Case of the Weierstrass Curve**).

This case is slightly different from the one of the preceeding Gasket, in so far as we deal with a curve. The existing results [Dav18], [DL20] enable us to handle a specific two-dimensional measure, in so far as the Curve is approached by means of a polygonal neighborhood.

We hereafter denote by $N = N_b \geqslant 3$ the number of maps of the involved iterated function system (see 6.2.1), and by $D_{\mathcal{W}}$ the box-dimension of the Curve.

Given a natural integer m, we have that

$$h_m = \frac{N_b^{(D_{\mathcal{W}} - 2) \, m}}{(N_b - 1)^{2 - D_{\mathcal{W}}}} \cdot$$

A m-cell has N_b vertices, while its measure is given by (we refer to [Dav19], [DL20])

$$\mu_m \lesssim N_b^{(D_{\mathcal{W}}-3)\,m} \,.$$

For a continuous function f on the Curve, belonging to the domain of the Laplacian, its Laplacian is obtained, for any $x \notin V_0$, through

$$\Delta\, f(x) = \lim_{m\to\infty} \Delta_m f(x) = \lim_{m\to\infty} \frac{c_m}{h_m^2}\, \Delta_m^\tau f(X)\,,$$

where

$$c_m = h_m^{-2\left(\frac{D_{\mathcal{W}}-1}{2-D_{\mathcal{W}}}\right)} \,.$$

So, in a sense, the definition of the Laplacian already resembles the one of the h_m-Laplacian, which is thus obtained when

$$c_m = \frac{2\, c_{0,m}^2}{N_b}\,,$$

i.e.,

$$c_{0,m}^2 = \frac{N_b}{2}\, h_m^{-2\left(\frac{D_{\mathcal{W}}-1}{2-D_{\mathcal{W}}}\right)} \,.$$

Now, as for the modified h_m-Laplacian, we have that

$$\# V_m = N_b^{m+1} + 1 - N_b^m \quad , \quad \# V_{m+1} = N_b^{m+2} + 1 - N_b^{m+1} \,.$$

At the same time, for any integer j' in $\left\{0, \ldots, N^{m+1}-1\right\}$,

$$\mu\left(\mathcal{C}_m^{j'}\right) = \mathcal{A}_{m+1} = \frac{1}{3^{m+1}} \,.$$

This yields

$$c_m = 2\,\tilde{c}_{0,m} \sum_{\substack{y\in\mathcal{C}_m^j,\; y \underset{m}{\sim} x \\ \text{or } y\in\mathcal{C}_{m+1}^{j'},\; y \underset{m+1}{\sim} x}} \frac{\mu\left(\mathcal{C}_m^j\right)}{\#\text{vertices of } \mathcal{C}_m^j} \min\left\{ \frac{\#\left(V_{m+1}\setminus V_m\right)}{\# V_{m+1}}\, \frac{1}{\mu\left(\mathcal{C}_m^j\right)}, \frac{\# V_m}{\# V_{m+1}}\, \frac{1}{\mu\left(\mathcal{C}_{m+1}^{j'}\right)} \right\} .$$

If we cannot presently have the exact value, we can nonetheless write

$$c_m \sim \frac{2\,\tilde{c}_{0,m}\, N_b^{(D_{\mathcal{W}}-3)\,m}}{N_b} \min\left\{ \frac{N_b^{m+2}+1-N_b^{m+1}-N_b^{m+1}-1+N_b^m}{N_b^{m+2}+1-N_b^{m+1}}\, \frac{1}{N_b^{(D_{\mathcal{W}}-3)\,m}}, \frac{N_b^{m+1}+1-N_b^m}{N_b^{m+2}+1-N_b^{m+1}}\, \frac{1}{N_b^{(D_{\mathcal{W}}-3)\,(m+1)}} \right\},$$

which yields

$$\tilde{c}_{0,m}^2 \sim \frac{N_b^{m+2}+1-N_b^{m+1}}{2\left(N_b^{m+1}-2\,N_b^m+N_b^{m-1}\right)}\, c_m\,,$$

and

$$\tilde{c}_{0,m} > c_{0,m} \,.$$

Thanks

The authors would like to thank the anonymous referee to have brought to their attention articles that have made it possible to better value their own original contributions.

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