Bypassing dynamical systems: a simple way to get the box-counting dimension of the graph of the Weierstrass function

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Abstract. In the following, bypassing dynamical systems tools, we propose a simple means of computing the box dimension of the graph of the classical Weierstrass function defined, for any real number x, by $\mathscr{W}(x) = \displaystyle\sum_{n=0}^{+\infty} \lambda^n \cos\left(2\,\pi\,N_b^n\,x\right)$, where λ and N_b are two real numbers such that $0 < \lambda < 1$, $N_b \in \mathbb{N}$ and $\lambda\,N_b > 1$, using a sequence a graphs that approximate the studied one.

INTRODUCTION

The determination of the box and Hausdorff dimension of the graph of the Weierstrass function has, since long been, a topic of interest. In the following, we show that the box-counting dimension (or Minskowski dimension) can be obtained directly, without using dynamical systems tools.

Let us recall that, given $\lambda \in\,]0, 1[$, and b such that $\lambda\,b > 1 + \dfrac{3\,\pi}{2}$, the Weierstrass function

$$x \in \mathbb{R} \mapsto \mathscr{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \cos\left(\pi\,b^n\,x\right)$$

is continuous everywhere, while nowhere differentiable. The original proof, by K. Weierstrass [18], can also be found in [17]. It has been completed by the one, now a classical one, in the case where $\lambda\,b > 1$, by G. Hardy [5].

After the works of A. S. Besicovitch and H. D. Ursell [2], it is Benoît Mandelbrot [12] who particularly highlighted the fractal properties of the graph of the Weierstrass function. He also conjectured that the Hausdorff dimension of the graph is $D_{\mathscr{W}} = 2 + \dfrac{\ln \lambda}{\ln b}$. Interesting discussions in relation to this question have been given in the book of K. Falconer [4]. A series of results for the box dimension can be found in the works of J.-L. Kaplan et al. [8] (where the authors show that it is equal to the Lyapunov dimension of the equivalent attracting torus), in the one of F. Przytycki and M. Urbański [14], and in those by T-Y. Hu and K-S. Lau [6]. As for the Hausdorff dimension, a proof was given by B. Hunt [7] in 1998 in the case where arbitrary phases are included in each cosinusoidal term of the summation. Recently, K. Barańsky, B. Bárány and J. Romanowska [1] proved that, for any value of the real number b, there exists a threshold value λ_b belonging to the interval $\left] \dfrac{1}{b}, 1 \right[$ such that the aforementioned dimension is equal to $D_{\mathscr{W}}$ for every b in $]\lambda_b, 1[$. Results by W. Shen [16] go further than the ones of [1]. In [9], G. Keller proposes what appears as a much simpler and very original proof.

May one wish to understand the proofs mentioned above, it requires theoretical background in dynamic systems theory. For instance, in the work of J.-L. Kaplan et al. [8], the authors call for results that cannot be understood without knowledge on the Lyapunov dimension. One may also note that their proof, which enables one to obtain the box-counting dimension of the aforementioned graph, involves sequences revolving around the Gamma Function, Fourier coefficients, integration in the complex plane, definition of a specific measure, the solving of several equations, thus, a lot of technical manipulations (on eleven pages), to yield the result.

Following those results, F. Przytycki and M. Urbański [14] give a general method leading to the value of this box-counting dimension. It was initially devoted to the calculation of the Hausdorff dimension of the graph. It appears simpler than the one by Kaplan et al., calling for Frostman's lemma [15], [13]. The authors deal with continuous functions f satisfying conditions of the form:

$\forall \, (x_1, x_2) \, \in \, [0, 1]^2 \; :$

$\quad \sup \left\{ |f(a_1) - f(a_2)| \; ; \; x_1 \leq a_1 \leq a_2 \leq x_2 \right\} \geq C \, |x_1 - x_2|^{\alpha} \qquad (\star)$

where C and $\alpha < 1$ denote strictly positive real constants.

In order to apply the results by F. Przytycki and M. Urbański, one thus requires the estimate ($\star$), which is not that easy to prove. The same kind of estimate is required to obtain the Hausdorff dimension of the graph. As evoked above, existing works in the literature all call for the theory of dynamical systems.

Until now, the simplest calculation is the one by G. Keller [9], where the author bypasses the Ledrappier-Young theory on hyperbolic measures [11], [10], embedding the graph into an attractor of a dynamical system. The proof requires b-baker maps, acting on the unit square. It also requires results on stable and unstable manifolds, as well as results on related fibers.

In our work [3], where we build a Laplacian on the graph of the Weierstrass function $\mathscr{W}$, we came across a simpler means of computing the box dimension of the graph, using a sequence a graphs that approximate the studied one, bypassing all the aforementioned tools. The main computation, which, for any small interval $[x_1, x_2] \subset [0, 1]$, leads to an estimate of the form:

$$C_{inf} \, |x_1 - x_2|^{2 - D_{\mathscr{W}}} \leq |\mathscr{W}(x_1) - \mathscr{W}(x_2)| \leq C_{sup} \, |x_1 - x_2|^{2 - D_{\mathscr{W}}}$$

where $D_{\mathscr{W}} = 2 + \dfrac{\ln \lambda}{\ln b}$, and where C_{inf} and C_{sup} denote strictly positive constants, is done in barely two pages, and does not require specific knowledge, putting the result at the disposal of a wider audience. The key results are exposed in the sequel.

1. FRAMEWORK OF THE STUDY

In this section, we recall results that are developed in [3]. We consider the case when the real number b is an integer, that we thus choose to denote by N_b.

Notation

We will denote by $\mathbb{N}$ the set of natural integers.

Notation

In the following, λ and N_b are two real numbers such that:

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N} \quad \text{and} \quad \lambda\,N_b > 1.$$

We will consider the Weierstrass function $\mathscr{W}$, defined, for any real number x, by:

$$\mathscr{W}(x) = \sum_{n=0}^{+\infty} \lambda^n \, \cos\left(2\,\pi\,N_b^n\,x\right).$$

Property 2.1. Periodic properties of the Weierstrass function
For any real number x:

$$\mathscr{W}(x+1) = \sum_{n=0}^{+\infty} \lambda^n \, \cos\left(2\,\pi\,N_b^n\,x + 2\,\pi\,N_b^n\right) = \sum_{n=0}^{+\infty} \lambda^n \, \cos\left(2\,\pi\,N_b^n\,x\right) = \mathscr{W}(x).$$

The study of the Weierstrass function can be restricted to the interval $[0,1[$.

In the sequel, we place ourselves in the Euclidean plane of dimension 2, referred to a direct orthonormal frame. The usual Cartesian coordinates are (x,y).

The restriction $\Gamma_{\mathscr{W}}$ to $[0,1[\times\mathbb{R}$, of the graph of the Weierstrass function, is approximated by means of a sequence of graphs, built through an iterative process. For this purpose, we introduce the iterated function system, i.e. the family of C^∞ contractions from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$\{T_0, \ldots, T_{N_b-1}\}$$

where, for any integer i belonging to $\{0, \ldots, N_b - 1\}$, and any (x,y) in $\mathbb{R}^2$:

$$T_i(x,y) = \left(\frac{x+i}{N_b},\, \lambda\,y + \cos\left(2\,\pi\,\left(\frac{x+i}{N_b}\right)\right)\right).$$

Notation

We will denote by:

$$D_{\mathscr{W}} = 2 + \frac{\ln \lambda}{\ln N_b}$$

the Hausdorff dimension of $\Gamma_{\mathscr{W}}$ (see [1], [9]).

Proposition 1.1.

$$\Gamma_{\mathscr{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathscr{W}}).$$

Definition 1.2. For any integer i belonging to $\{0, \ldots, N_b - 1\}$, let us denote by:

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b - 1}, \frac{1}{1 - \lambda} \cos \left(\frac{2 \pi i}{N_b - 1} \right) \right)$$

the fixed point of the contraction T_i.

We will denote by V_0 the ordered set (according to increasing abscissae), of the points:

$$\{P_0, \ldots, P_{N_b-1}\}.$$

The set of points V_0, where, for any i of $\{0, \ldots, N_b - 2\}$, the point P_i is linked to the point P_{i+1}, constitutes an oriented graph (according to increasing abscissa)), that we will denote by $\Gamma_{\mathscr{W}_0}$. V_0 is called the set of vertices of the graph $\Gamma_{\mathscr{W}_0}$.

For any natural integer m, we set:

$$V_m = \bigcup_{i=0}^{N_b-1} T_i \left(V_{m-1} \right).$$

The set of points V_m, where two consecutive points are linked, is an oriented graph (according to increasing abscissa), which we will denote by $\Gamma_{\mathscr{W}_m}$. V_m is called the set of vertices of the graph $\Gamma_{\mathscr{W}_m}$. We will denote, in the sequel, by

$$\mathscr{N}_m^{\mathscr{S}} = 2 N_b^m + N_b - 2$$

the number of vertices of the graph $\Gamma_{\mathscr{W}_m}$, and we will write:

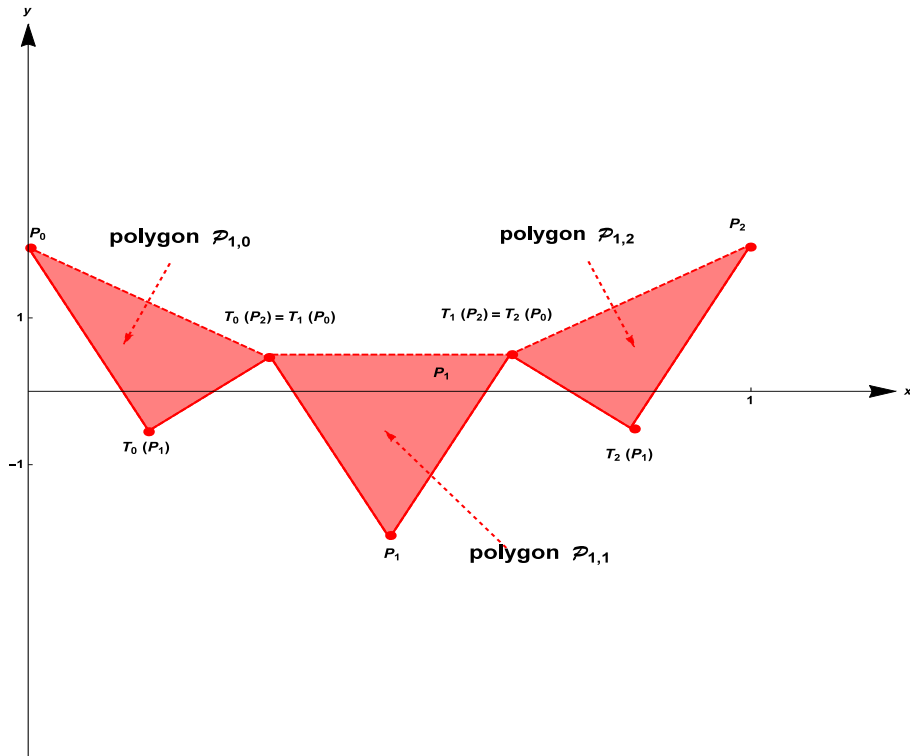$$V_m = \left\{ P_0^m, P_1^m, \ldots, P_{\mathscr{N}_m^{\mathscr{S}}-1}^m \right\}.$$

FIGURE 1.1. The polygons $\mathscr{P}_{1,0}$, $\mathscr{P}_{1,1}$, $\mathscr{P}_{1,2}$, in the case where $\lambda = \dfrac{1}{2}$, and $N_b = 3$.
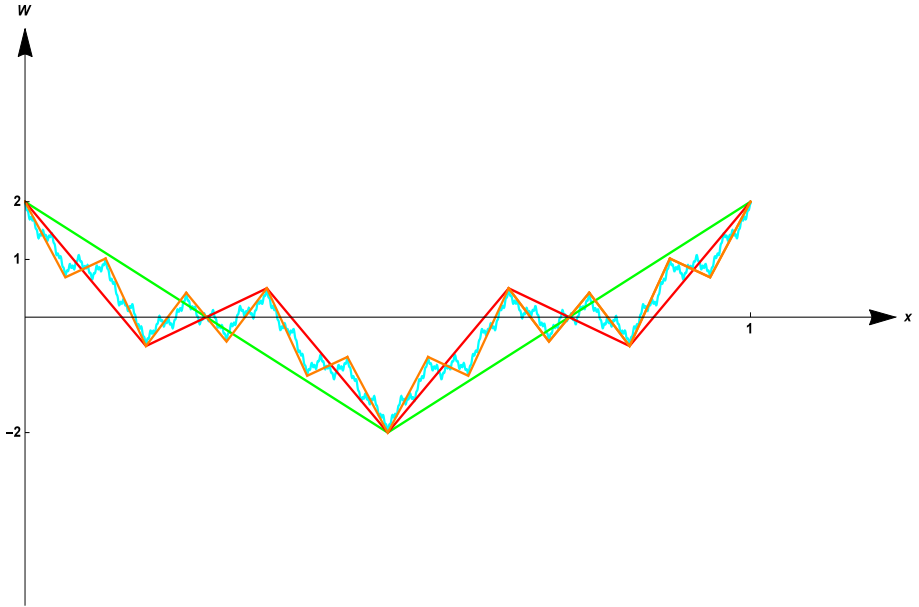
FIGURE 1.2. The graphs $\Gamma_{\mathscr{W}_0}$, $\Gamma_{\mathscr{W}_1}$ (in red), $\Gamma_{\mathscr{W}_2}$, $\Gamma_{\mathscr{W}}$, in the case where $\lambda = \dfrac{1}{2}$, and $N_b = 3$.

Definition 1.3. Consecutive vertices on the graph $\Gamma_{\mathscr{W}}$

Two points X and Y of $\Gamma_{\mathscr{W}}$ will be called **_consecutive vertices_** of the graph $\Gamma_{\mathscr{W}}$ if there exists a natural integer m, and an integer j of $\{0, ..., N_b - 2\}$, such that:

$$\begin{cases} X & = & (T_{i_1} \circ \ldots \circ T_{i_m})\,(P_j) \\ Y & = & (T_{i_1} \circ \ldots \circ T_{i_m})\,(P_{j+1}) \end{cases} \qquad (i_1, \ldots, i_m) \in \{0, \ldots, N_b - 1\}^m$$

or:

$$\begin{cases} X & = & (T_{i_1} \circ T_{i_2} \circ \ldots \circ T_{i_m})\,(P_{N_b - 1}) \\ Y & = & (T_{i_1+1} \circ T_{i_2} \ldots \circ T_{i_m})\,(P_0) \end{cases} \; .$$

Definition 1.4. For any natural integer m, the $\mathscr{N}_m^{\mathscr{S}}$ consecutive vertices of the graph $\Gamma_{\mathscr{W}_m}$ are, also, the vertices of N_b^m simple polygons $\mathscr{P}_{m,j}$, $0 \leq j \leq N_b^m - 1$, with N_b sides. For any integer j such that $0 \leq j \leq N_b^m - 1$,

one obtains each polygon by linking the point number j to the point number $j + 1$ if $j = i \bmod N_b$, $0 \leq i \leq N_b - 2$, and the point number j to the point number $j - N_b + 1$ if $j = -1 \bmod N_b$. These polygons generate a Borel set of $\mathbb{R}^2$.

Definition 1.5. Word, on the graph $\Gamma_{\mathscr{W}}$

Let m be a strictly positive integer. We will call **number-letter** any integer $\mathscr{M}_i$ of $\{0, \dots, N_b - 1\}$, and **word of length** $|\mathscr{M}| = m$, on the graph $\Gamma_{\mathscr{W}}$, any set of number-letters of the form:

$$\mathscr{M} = (\mathscr{M}_1, \dots, \mathscr{M}_m).$$

We will write:

$$T_{\mathscr{M}} = T_{\mathscr{M}_1} \circ \dots \circ T_{\mathscr{M}_m}.$$

Definition 1.6. Edge relation, on the graph $\Gamma_{\mathscr{W}}$

Given a natural integer m, two points X and Y of $\Gamma_{\mathscr{W}_m}$ will be called **adjacent** if and only if X and Y are two consecutive vertices of $\Gamma_{\mathscr{W}_m}$. We will write:

$$X \underset{m}{\sim} Y.$$

This edge relation ensures the existence of a word $\mathscr{M} = (\mathscr{M}_1, \dots, \mathscr{M}_m)$ of length m, such that X and Y both belong to the iterate:

$$T_{\mathscr{M}} V_0 = (T_{\mathscr{M}_1} \circ \dots \circ T_{\mathscr{M}_m}) V_0.$$

Given two points X and Y of the graph $\Gamma_{\mathscr{W}}$, we will say that X and Y are **adjacent** if and only if there exists a natural integer m such that:

$$X \underset{m}{\sim} Y.$$

Proposition 1.7. *Adresses, on the graph of the Weierstrass function*

Given a strictly positive integer m, and a word $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_m)$ of length $m \in \mathbb{N}^\star$, on the graph $\Gamma_{\mathcal{W}_m}$, for any integer j of $\{1, \ldots, N_b - 2\}$, any $X = T_{\mathcal{M}}(P_j)$ of $V_m \setminus V_0$, i.e. distinct from one of the N_b fixed point P_i, $0 \leq i \leq N_b - 1$, has exactly two adjacent vertices, given by:

$$T_{\mathcal{M}}(P_{j+1}) \quad and \quad T_{\mathcal{M}}(P_{j-1})$$

where:

$$T_{\mathcal{M}} = T_{\mathcal{M}_1} \circ \ldots \circ T_{\mathcal{M}_m}.$$

By convention, the adjacent vertices of $T_{\mathcal{M}}(P_0)$ are $T_{\mathcal{M}}(P_1)$ and $T_{\mathcal{M}}(P_{N_b-1})$, those of $T_{\mathcal{M}}(P_{N_b-1})$, $T_{\mathcal{M}}(P_{N_b-2})$ and $T_{\mathcal{M}}(P_0)$.

Notation

For any integer j belonging to $\{0, \ldots, N_b - 1\}$, any natural integer m, and any word $\mathcal{M}$ of length m, we set:

$$T_{\mathcal{M}}(P_j) = (x(T_{\mathcal{M}}(P_j)), y(T_{\mathcal{M}}(P_j)))$$

$$T_{\mathcal{M}}(P_{j+1}) = (x(T_{\mathcal{M}}(P_{j+1})), y(T_{\mathcal{M}}(P_{j+1})))$$

$$L_m = x(T_{\mathcal{M}}(P_{j+1})) - x(T_{\mathcal{M}}(P_j)) = \frac{1}{(N_b - 1) N_b^m}$$

$$h_{j,m} = y(T_{\mathcal{M}}(P_{j+1})) - y(T_{\mathcal{M}}(P_j)).$$

2. Main results

Theorem 2.1. *An upper bound and a lower bound, for the box-dimension of the graph* $\Gamma_{\mathscr{W}}$

For any integer j belonging to $\{0, 1, \ldots, N_b - 2\}$, each natural integer m, and each word $\mathscr{M}$ of length m, let us consider the rectangle, whose sides are parallel to the horizontal and vertical axes, of width:

$$L_m = x\left(T_{\mathscr{M}}\left(P_{j+1}\right)\right) - x\left(T_{\mathscr{M}}\left(P_j\right)\right) = \frac{1}{(N_b - 1)\, N_b^m}$$

and height $|h_{j,m}|$, such that the points $T_{\mathscr{M}}\left(P_j\right)$ and $T_{\mathscr{M}}\left(P_{j+1}\right)$ are two vertices of this rectangle.
Then:

i. When the integer N_b is odd:

$$L_m^{2-D_{\mathscr{W}}}\,(N_b - 1)^{2-D_{\mathscr{W}}}\left\{\frac{2}{1-\lambda}\,\sin\left(\tfrac{\pi}{N_b - 1}\right)\min_{0 \leq j \leq N_b - 1}\left|\sin\left(\tfrac{\pi\,(2\,j+1)}{N_b - 1}\right)\right| - \frac{2\,\pi}{N_b\,(N_b - 1)}\,\frac{1}{\lambda\,N_b - 1}\right\} \leq |h_{j,m}|$$

ii. When the integer N_b is even:

$$L_m^{2-D_{\mathscr{W}}}\,(N_b - 1)^{2-D_{\mathscr{W}}}\max\left\{\frac{2}{1-\lambda}\,\sin\left(\tfrac{\pi}{N_b - 1}\right)\min_{0 \leq j \leq N_b - 1}\left|\sin\left(\tfrac{\pi\,(2\,j+1)}{N_b - 1}\right)\right| - \frac{2\,\pi}{N_b\,(N_b - 1)}\,\frac{1}{\lambda\,N_b - 1},\,\frac{4}{N_b^2}\,\frac{1-N_b^{-2}}{N_b^2 - 1}\right\} \leq |h_{j,m}|$$

Also:

$$|h_{j,m}| \leq \eta_{\mathscr{W}}\,L_m^{2-D_{\mathscr{W}}}\,(N_b - 1)^{2-D_{\mathscr{W}}}$$

where the real constant $\eta_{\mathscr{W}}$ is given by :

$$\eta_{\mathscr{W}} = 2\,\pi^2\left\{\frac{(2\,N_b - 1)\,\lambda\,(N_b^2 - 1)}{(N_b - 1)^2\,(1-\lambda)\,(\lambda\,N_b^2 - 1)} + \frac{2\,N_b}{(\lambda\,N_b^2 - 1)\,(\lambda\,N_b^3 - 1)}\right\}.$$

Corollary 2.2. *The box-dimension of the graph $\Gamma_{\mathscr{W}}$ is exactly $D_{\mathscr{W}}$.*

Proof. By definition of the box-counting dimension $D_{\mathscr{W}}$ (we refer, for instance, to [4]), the smallest number of squares, the side length of which is

at most equal to L_m, that can cover the graph $\Gamma_{\mathscr{W}}$ on $[0, 1[$, obeys, approximately, a power law of the form:

$$c\, L_m^{-D_{\mathscr{W}}} \quad , \quad c > 0.$$

Let us set:

$$C = \max\left\{\left\{\frac{2}{1-\lambda}\,\sin\left(\frac{\pi}{N_b-1}\right)\min_{0\le j\le N_b-1}\left|\sin\left(\frac{\pi\,(2\,j+1)}{N_b-1}\right)\right| - \frac{2\,\pi}{N_b\,(N_b-1)}\,\frac{1}{\lambda\,N_b-1}\right\}, \frac{4}{N_b^2}\,\frac{1-N_b^{-2}}{N_b^2-1}, \eta_{\mathscr{W}}\right\}$$

and consider the subdivision of the interval $[0, 1[$ into:

$$N_m = \frac{1}{L_m} = (N_b - 1)\, N_b^m$$

sub-intervals of length L_m. One has to determine a natural integer $\tilde{N}_m$ such that the graph of $\Gamma_{\mathscr{W}}$ on $[0, 1[$ can be covered by $N_m \times \tilde{N}_m$ squares of side L_m. By considering, the vertical amplitude of the graph, one gets:

$$\tilde{N}_m = \left\lfloor \frac{C\, L_m^{2-D_{\mathscr{W}}}}{L_m} \right\rfloor + 1 \quad \text{i.e.} \quad \tilde{N}_m = \left\lfloor C\, L_m^{1-D_{\mathscr{W}}} \right\rfloor + 1.$$

Thus:

$$N_m \times \tilde{N}_m = \frac{\tilde{N}_m}{L_m} = \frac{1}{L_m}\left\lfloor C\, L_m^{1-D_{\mathscr{W}}} \right\rfloor + \frac{1}{L_m}.$$

The integer $N_m \times \tilde{N}_m$ then obeys a power law of the form

$$N_m \times \tilde{N}_m \approx c\, L_m^{-D_{\mathscr{W}}}$$

where c denotes a strictly positive constant. $\qquad\square$

Proof of Theorem 2.1.

i. Preliminary computations.

For any pair of integers (i_m, j) of $\{0, \ldots, N_b - 2\}^2$:

$$T_{i_m}(P_j) = \left(\frac{x_j + i_m}{N_b}, \lambda\, y_j + \cos\left(2\,\pi\left(\frac{x_j + i_m}{N_b} \right) \right) \right).$$

For any triple of integers (i_m, i_{m-1}, j) of $\{0, \ldots, N_b - 2\}^3$:

$$T_{i_{m-1}}(T_{i_m}(P_j)) =$$

$$= \left(\frac{\frac{x_j + i_m}{N_b} + i_{m-1}}{N_b}, \lambda^2\, y_j + \lambda\,\cos\left(2\,\pi\left(\frac{x_j + i_m}{N_b} \right) \right) + \cos\left(2\,\pi\left(\frac{\frac{x_j + i_m}{N_b} + i_{m-1}}{N_b} \right) \right) \right)$$

$$= \left(\frac{x_j + i_m}{N_b^2} + \frac{i_{m-1}}{N_b}, \lambda^2\, y_j + \lambda\cos\left(2\,\pi\left(\frac{x_j + i_m}{N_b} \right) \right) + \cos\left(2\,\pi\left(\frac{x_j + i_m}{N_b^2} + \frac{i_{m-1}}{N_b} \right) \right) \right).$$

For any quadruple of integers $(i_m, i_{m-1}, i_{m-2}, j)$ of $\{0, \ldots, N_b - 2\}^4$:

$$T_{i_{m-2}}\left(T_{i_{m-1}}(T_{i_m}(P_j)) \right) =$$

$$= \left(\frac{x_j + i_m}{N_b^3} + \frac{i_{m-1}}{N_b^2} + \frac{i_{m-2}}{N_b}, \right.$$
$$\left. \lambda^3\, y_j + \lambda^2\,\cos\left(2\,\pi\left(\frac{x_j + i_m}{N_b} \right) \right) + \lambda\cos\left(2\,\pi\left(\frac{x_j + i_m}{N_b^2} + \frac{i_{m-1}}{N_b} \right) \right) + \cos\left(2\,\pi\left(\frac{x_j + i_m}{N_b^3} + \frac{i_{m-1}}{N_b^2} + \frac{i_{m-2}}{N_b} \right) \right) \right).$$

Given a strictly positive integer m, and two points X and Y of V_m such that:

$$X \underset{m}{\sim} Y$$

there exists a word $\mathscr{M}$ of length $|\mathscr{M}| = m$, on the graph $\Gamma_{\mathscr{W}}$, and an integer j of $\{0, \ldots, N_b - 2\}^2$, such that:

$$X = T_{\mathscr{M}}(P_j) \quad, \quad Y = T_{\mathscr{M}}(P_{j+1}).$$

Let us write $T_{\mathscr{M}}$ under the form:

$$T_{\mathscr{M}} = T_{i_m} \circ T_{i_{m-1}} \circ \ldots \circ T_{i_1}$$

where $(i_1, \ldots, i_m) \in \{0, \ldots, N_b - 1\}^m$.

One has then:

$$x\left(T_{\mathscr{M}}\left(P_j\right)\right) = \frac{x_j}{N_b^m} + \sum_{k=1}^{m} \frac{i_k}{N_b^k} \quad , \quad x\left(T_{\mathscr{M}}\left(P_{j+1}\right)\right) = \frac{x_{j+1}}{N_b^m} + \sum_{k=1}^{m} \frac{i_k}{N_b^k}$$

and:

$$\begin{cases} y\left(T_{\mathscr{M}}\left(P_j\right)\right) & = & \lambda^m\, y_j + \sum_{k=1}^{m} \lambda^{m-k}\,\cos\left(2\,\pi\,\left(\frac{x_j}{N_b^k} + \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right) \\[2em] y\left(T_{\mathscr{M}}\left(P_{j+1}\right)\right) & = & \lambda^m\, y_{j+1} + \sum_{k=1}^{m} \lambda^{m-k}\,\cos\left(2\,\pi\,\left(\frac{x_{j+1}}{N_b^k} + \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right) \end{cases} .$$

ii. Determination of a lower bound.

Let us note that:

$$h_{j,m} - \lambda^m\,\left(y_{j+1} - y_j\right) =$$

$$= \sum_{k=1}^{m} \lambda^{m-k}\,\left\{\cos\left(2\,\pi\,\left(\frac{x_{j+1}}{N_b^k} + \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right) - \cos\left(2\,\pi\,\left(\frac{x_j}{N_b^k} - \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right)\right\}$$

$$= -2 \sum_{k=1}^{m} \lambda^{m-k}\,\sin\left(\pi\,\left(\frac{x_{j+1}-x_j}{N_b^k}\right)\right)\,\sin\left(2\,\pi\,\left(\frac{x_{j+1}+x_j}{2\,N_b^k} + \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right).$$

Taking into account:

$$\begin{aligned} \lambda^m\,\left(y_{j+1} - y_j\right) & = & \frac{\lambda^m}{1 - \lambda}\,\left(\cos\left(\frac{2\,\pi\,(j+1)}{N_b-1}\right) - \cos\left(\frac{2\,\pi\,j}{N_b-1}\right)\right) \\ & = & -2\,\frac{\lambda^m}{1 - \lambda}\,\sin\left(\frac{2\,\pi\,(j+1-j)}{2\,(N_b-1)}\right)\,\sin\left(\frac{2\,\pi\,(j+1+j)}{2\,(N_b-1)}\right) \\ & = & -2\,\frac{\lambda^m}{1 - \lambda}\,\sin\left(\frac{\pi}{N_b-1}\right)\,\sin\left(\frac{\pi\,(2\,j+1)}{N_b-1}\right) \end{aligned}$$

the triangular inequality leads then to:

$$|y\left(T_{\mathscr{M}}\left(P_{j+1}\right)\right) - y\left(T_{\mathscr{M}}\left(P_j\right)\right)| =$$

$$= \left| \lambda^m \left(y_{j+1} - y_j\right) - 2\sum_{k=1}^{m} \lambda^{m-k} \sin\left(\frac{\pi}{N_b^{k+1}\left(N_b-1\right)}\right) \sin\left(\frac{\pi\left(2\,j+1\right)}{N_b^{k+1}\left(N_b-1\right)} + 2\,\pi\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right) \right|$$

$$\geq \left| \lambda^m \left|\left(y_{j+1} - y_j\right)\right| - 2\sum_{k=1}^{m} \lambda^{m-k} \left|\sin\left(\frac{\pi}{N_b^{k+1}\left(N_b-1\right)}\right) \sin\left(\frac{\pi\left(2\,j+1\right)}{N_b^{k+1}\left(N_b-1\right)} + 2\,\pi\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right| \right|$$

$$= \lambda^m \left| \frac{2}{1-\lambda} \sin\left(\frac{\pi}{N_b-1}\right) \left|\sin\left(\frac{\pi\left(2\,j+1\right)}{N_b-1}\right)\right| \right.$$

$$\left. -2\sum_{k=1}^{m} \lambda^{-k} \sin\left(\frac{\pi}{N_b^{k+1}\left(N_b-1\right)}\right) \left|\sin\left(\frac{\pi\left(2\,j+1\right)}{N_b^{k+1}\left(N_b-1\right)} + 2\,\pi\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right| \right|.$$

One also has:

$$2\sum_{k=1}^{m} \lambda^{-k} \sin\left(\frac{\pi}{N_b^{k+1}\left(N_b-1\right)}\right) \left|\sin\left(\frac{\pi\left(2\,j+1\right)}{N_b^{k+1}\left(N_b-1\right)} + 2\,\pi\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right| \leq$$

$$\leq 2\sum_{k=1}^{m} \lambda^{-k} \sin\left(\frac{\pi}{N_b^{k+1}\left(N_b-1\right)}\right)$$

$$\leq 2\sum_{k=1}^{m} \lambda^{-k} \frac{\pi}{N_b^{k+1}\left(N_b-1\right)}$$

$$= \frac{2\,\pi}{N_b\left(N_b-1\right)} \sum_{k=1}^{m} \frac{1}{\lambda^k\,N_b^k}$$

$$= \frac{2\,\pi}{N_b\left(N_b-1\right)} \lambda^{-1}\,N_b^{-1} \frac{1-\lambda^{-m}\,N_b^{-m}}{1-\lambda^{-1}\,N_b^{-1}}$$

$$\leq \frac{2\,\pi}{N_b\left(N_b-1\right)} \lambda^{-1}\,N_b^{-1} \frac{1}{1-\lambda^{-1}\,N_b^{-1}}$$

$$= \frac{2\,\pi}{N_b\left(N_b-1\right)} \frac{1}{\lambda\,N_b-1}$$

which yields:

$$-2\sum_{k=1}^{m} \lambda^{-k} \sin\left(\frac{\pi}{N_b^{k+1}\left(N_b-1\right)}\right) \left|\sin\left(\frac{\pi\left(2\,j+1\right)}{N_b^{k+1}\left(N_b-1\right)} + 2\,\pi\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right| \geq -\frac{2\,\pi}{N_b\left(N_b-1\right)} \frac{1}{\lambda\,N_b-1}.$$

Thus:

$$\frac{2}{1-\lambda} \sin\left(\frac{\pi}{N_b - 1}\right) \left|\sin\left(\frac{\pi\,(2\,j+1)}{N_b - 1}\right)\right|$$
$$-2 \sum_{k=1}^{m} \lambda^{-k} \sin\left(\frac{\pi}{N_b^{k+1}\,(N_b-1)}\right) \left|\sin\left(\frac{\pi\,(2\,j+1)}{N_b^{k+1}\,(N_b-1)} + 2\,\pi \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right| \geq$$
$$\geq \frac{2}{1-\lambda} \sin\left(\frac{\pi}{N_b-1}\right) \left|\sin\left(\frac{\pi\,(2\,j+1)}{N_b-1}\right)\right| - \frac{2\,\pi}{N_b\,(N_b-1)}\, \frac{1}{\lambda\,N_b - 1}.$$

Lemma 2.3. *The following results hold:*

For $0 \leq j \leq N_b - 1$:

$$\sin\left(\frac{\pi\,(2\,j+1)}{N_b - 1}\right) = 0 \quad \text{if and only if } N_b \text{ is even and } j = \frac{N_b}{2} - 1.$$

Proof of Lemma 2.3.

Since $0 \leq j \leq N_b - 1$, one has:

$$1 \leq 2\,j + 1 \leq 2\,N_b - 1 \quad \text{and thus} \quad 0 < \frac{2\,j+1}{N_b - 1} \leq 2 + \frac{1}{N_b - 1}.$$

Then, $\sin\left(\frac{\pi\,(2\,j+1)}{N_b-1}\right) = 0$ if and only if:

$$\frac{2\,j+1}{N_b - 1} = 1 \quad \text{or} \quad \frac{2\,j+1}{N_b - 1} = 2.$$

The second case has to be rejected, since it would lead to:

$$j = N_b - \frac{3}{2} \notin \mathbb{N}$$

The only possibility is thus when N_b is an even number:

$$j = \frac{N_b}{2} - 1.$$

The converse is obvious.

$\square$

<u>First case:</u> $\sin\left(\frac{\pi(2j+1)}{N_b-1}\right) \neq 0$. One has then:

$$\left|\sin\left(\frac{\pi(2j+1)}{N_b-1}\right)\right| \geq \min_{0\leq j\leq N_b-1} \left|\sin\left(\frac{\pi(2j+1)}{N_b-1}\right)\right| = \left|\sin\left(\frac{\pi}{N_b-1}\right)\right| \geq \frac{2}{N_b-1}.$$

This leads to:

$$\frac{2}{1-\lambda}\sin\left(\frac{\pi}{N_b-1}\right)\left|\sin\left(\frac{\pi(2j+1)}{N_b-1}\right)\right|$$
$$-2\sum_{k=1}^{m}\lambda^{-k}\sin\left(\frac{\pi}{N_b^{k+1}(N_b-1)}\right)\left|\sin\left(\frac{\pi(2j+1)}{N_b^{k+1}(N_b-1)}+2\pi\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right| \geq$$

$$\geq \frac{2}{1-\lambda}\sin\left(\frac{\pi}{N_b-1}\right)\sin\left(\frac{\pi}{N_b-1}\right) - \frac{2\pi}{N_b(N_b-1)}\frac{1}{\lambda N_b-1}$$

$$\geq \frac{2}{1-\lambda}\frac{4}{(N_b-1)^2} - \frac{2\pi}{N_b(N_b-1)}\frac{1}{\lambda N_b-1}$$

$$= \frac{2}{N_b-1}\left\{\frac{4}{1-\lambda}\frac{1}{N_b-1} - \frac{\pi}{N_b}\frac{1}{\lambda N_b-1}\right\}$$

$$= \frac{2}{N_b(N_b-1)(1-\lambda)(\lambda N_b-1)}\left\{4N_b(\lambda N_b-1) - \pi(1-\lambda)(N_b-1)\right\}.$$

The function

$$\lambda \mapsto 4N_b(\lambda N_b-1) - \pi(1-\lambda)(N_b-1)$$

is affine and strictly increasing in λ, and quadratic and strictly increasing in N_b, for strictly positive values of N_b. This ensures the positivity of:

$$\frac{2}{1-\lambda}\sin\left(\frac{\pi}{N_b-1}\right)\left|\sin\left(\frac{\pi(2j+1)}{N_b-1}\right)\right|$$
$$-2\sum_{k=1}^{m}\lambda^{-k}\sin\left(\frac{\pi}{N_b^{k+1}(N_b-1)}\right)\left|\sin\left(\frac{\pi(2j+1)}{N_b^{k+1}(N_b-1)}+2\pi\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right|.$$

<u>Second case:</u> $\sin\left(\frac{\pi(2j+1)}{N_b-1}\right) = 0$.

One has then:

$$|y(T_{\mathcal{M}}(P_{j+1})) - y(T_{\mathcal{M}}(P_j))| \geq 2\lambda^m\left|\sum_{k=1}^{m}\lambda^{-k}\sin\left(\frac{\pi}{N_b^{k+1}}\right)\left|\sin\left(\frac{\pi}{N_b^{k+1}}+2\pi\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right)\right|\right|.$$

Thanks to the periodic properties of the sine function, one may only consider the case when:

$$0 \le \frac{\pi}{N_b^{k+1}} + 2\pi \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}} \le \frac{\pi}{2}.$$

Thus:

$$
\begin{aligned}
|y\left(T_{\mathscr{M}}\left(P_{j+1}\right)\right) - y\left(T_{\mathscr{M}}\left(P_j\right)\right)| \quad \ge \quad & \sum_{k=1}^{m} \lambda^{-k} \frac{2}{N_b^{k+1}} \left\{ \frac{2}{N_b^{k+1}} + 2 \sum_{\ell=0}^{k} \frac{i_{m-\ell}}{N_b^{k-\ell}} \right\} \\
\ge \quad & \sum_{k=1}^{m} \lambda^{-k} \frac{2}{N_b^{k+1}} \left\{ \frac{2}{N_b^{k+1}} \right\} \\
= \quad & \frac{4\lambda^{-1}}{N_b^4} \frac{1 - \lambda^{-m} N_b^{-2m}}{1 - \lambda^{-1} N_b^{-2}} \\
= \quad & \frac{4}{N_b^2 \left(N_b - 1\right)^2} \frac{1 - \lambda^{-m} N_b^{-m}}{\lambda N_b - 1} \\
= \quad & \frac{4}{N_b^2} \frac{1 - N_b^{-2}}{N_b^2 - 1}.
\end{aligned}
$$

General case: the above results enable us to obtain the predominant term of the lower bound of $|y\left(T_{\mathscr{M}}\left(P_{j+1}\right)\right) - y\left(T_{\mathscr{M}}\left(P_j\right)\right)|$, which is thus:

$$\lambda^m = e^{m\left(D_{\mathscr{W}}-2\right)\ln N_b} = N_b^{m\left(D_{\mathscr{W}}-2\right)} = L_m^{2-D_{\mathscr{W}}}\left(N_b - 1\right)^{2-D_{\mathscr{W}}}.$$

ii. <u>Determination of an upper bound.</u>

One has:

$$
\begin{aligned}
|h_{j,m}| \quad \leq \quad & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2} + 2\sum_{k=1}^{m}\lambda^{m-k}\,\pi\left\{\frac{2\,j+1}{(N_b-1)\,N_b^k} + 2\sum_{\ell=0}^{k}\frac{i_{m-\ell}}{N_b^{k-\ell}}\right\}\frac{\pi}{(N_b-1)\,N_b^k} \\[2ex]
= \quad & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2} + \frac{2\,\pi^2\,\lambda^m}{N_b-1}\sum_{k=1}^{m}\left\{\frac{(2\,j+1)\,\lambda^{-k}}{(N_b-1)\,N_b^{2k}} + 2\sum_{\ell=0}^{k}\frac{i_{m-\ell}\,\lambda^{-k}}{N_b^{2k-\ell}}\right\} \\[2ex]
= \quad & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,j+1)}{(N_b-1)^2} \\
& +\frac{2\,\pi^2\,\lambda^m}{N_b-1}\left\{\frac{\lambda^{-1}\,N_b^{-2}\,(2\,j+1)}{(N_b-1)}\,\frac{(1-\lambda^{-m}\,N_b^{-2m})}{1-\lambda^{-1}\,N_b^{-2}} + 2\sum_{k=1}^{m}\frac{(N_b-1)\,\lambda^{-k}}{N_b^{2k}}\,\frac{1-N_b^{-k-1}}{1-N_b^{-1}}\right\} \\[2ex]
\leq \quad & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2} + \frac{2\,\pi^2\,\lambda^m}{N_b-1}\,\frac{(2\,N_b-1)}{(N_b-1)}\,\frac{(1-\lambda^{-m}\,N_b^{-2m})}{\lambda\,N_b^2-1} \\[2ex]
& +\frac{2\,\pi^2\,\lambda^m}{N_b-1}\,2\,\frac{\lambda^{-1}\,N_b^{-2}\,(N_b-1)\,(1-\lambda^{-m}\,N_b^{-2m})}{(1-N_b^{-1})\,(1-\lambda^{-1}\,N_b^{-2})} \\[2ex]
& -\frac{2\,\pi^2\,\lambda^m}{N_b-1}\,2\,\frac{\lambda^{-1}\,N_b^{-3}\,(N_b-1)\,(1-\lambda^{-m}\,N_b^{-3m})}{(1-N_b^{-1})\,(1-\lambda^{-1}\,N_b^{-3})} \\[2ex]
\leq \quad & \frac{2\,\lambda^m}{1-\lambda}\,\frac{\pi^2\,(2\,N_b-1)}{(N_b-1)^2} + \frac{2\,\pi^2\,\lambda^m}{N_b-1}\,\frac{(2\,N_b-1)}{(N_b-1)}\,\frac{1}{\lambda\,N_b^2-1} \\[2ex]
& +\frac{4\,\pi^2\,N_b\,\lambda^m}{N_b-1}\left\{\frac{1}{\lambda\,N_b^2-1} - \frac{1}{\lambda\,N_b^3-1}\right\} \\[2ex]
= \quad & 2\,\pi^2\,\lambda^m\left\{\frac{(2\,N_b-1)\,\lambda\,(N_b^2-1)}{(N_b-1)^2\,(1-\lambda)\,(\lambda\,N_b^2-1)} + \frac{2\,N_b}{(\lambda\,N_b^2-1)\,(\lambda\,N_b^3-1)}\right\}.
\end{aligned}
$$

Since:

$$
x\left(T_{\mathcal{M}}\left(P_{j+1}\right)\right) - x\left(T_{\mathcal{M}}\left(P_{j}\right)\right) = \frac{1}{(N_b-1)\,N_b^m}
$$

and:

$$
D_{\mathscr{W}} = 2 + \frac{\ln\lambda}{\ln N_b} \quad , \quad \lambda = e^{(D_{\mathscr{W}}-2)\,\ln N_b} = N_b^{(D_{\mathscr{W}}-2)}
$$

one has thus:

$$|h_{j,m}| \leq 2\pi^2 L_m^{2-D_{\mathscr{W}}} (N_b - 1)^{2-D_{\mathscr{W}}} \left\{ \frac{(2N_b - 1)\lambda(N_b^2 - 1)}{(N_b - 1)^2(1-\lambda)(\lambda N_b^2 - 1)} + \frac{2N_b}{(\lambda N_b^2 - 1)(\lambda N_b^3 - 1)} \right\}.$$

$\square$

Remark 2.4. In [7], B. Hunt uses the fact that the Hausdorff dimension of a fractal set $\mathscr{F}$ can be obtained by means of what is called the t-energy, $t \in \mathbb{R}$, of a Borel measure supported on $\mathscr{F}$ (one may refer to [4], for instance):

$$I_t(\mu) = \iint \frac{d\mu(x)\, d\mu(x')}{|x - x'|^t}$$

which enables one to obtain:

$$\dim \mathscr{F} = \sup \{ t \in \mathbb{R},\ \mu \text{ supported on } \mathscr{F} \,;\, I_t(\mu) < +\infty \}$$

A lower bound t_0 of the Hausdorff dimension can thus be obtained by building a measure μ supported on $\mathscr{F}$ such that:

$$I_{t_0}(\mu) < +\infty.$$

B. Hunt proceeds as follows: he introduces the measure $\mu_{\mathscr{W}}$ supported on $\Gamma_{\mathscr{W}}$, induced by the Lebesgue measure μ on $[0,1]$. Thus:

$$I_t(\mu_{\mathscr{W}}) = \iint \frac{d\mu_{\mathscr{W}}(x)\, d\mu_{\mathscr{W}}(x')}{\{|x - x'|^2 + |\mathscr{W}(x) - \mathscr{W}(x')|^2\}^{\frac{t}{2}}}.$$

We could also have used a similar argument since, in our case:

$$|x - x'|^{2-D_{\mathscr{W}}} \lesssim |\mathscr{W}(x) - \mathscr{W}(x')| \lesssim |x - x'|^{2-D_{\mathscr{W}}}.$$

Thanks

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