Collective motion driven by nutrient consumption

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Abstract

A classical problem describing the collective motion of cells, is the movement driven by consumption/depletion of a nutrient. Here we analyze one of the simplest such model written as a coupled Partial Differential Equation/Ordinary Differential Equation system which we scale so as to get a limit describing the usually observed pattern. In this limit the cell density is concentrated as a moving Dirac mass and the nutrient undergoes a discontinuity.

We first carry out the analysis without diffusion, getting a complete description of the unique limit. When diffusion is included, we prove several specific a priori estimates and interpret the system as a heterogeneous monostable equation. This allow us to obtain a limiting problem which shows the concentration effect of the limiting dynamics.

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1 Introduction

A classical problem describing the collective motion of cells, is the movement driven by consumption/depletion of a nutrient [11, 15, 16]. The simplest description uses a number density of cells u_ε and a nutrient concentration v_ε. It is written

$$\begin{cases} \partial_t u_\varepsilon - \varepsilon \partial^2_{xx} u_\varepsilon = \frac{1}{\varepsilon} u_\varepsilon (v_\varepsilon - \mu), & t \geq 0, \ x \in \mathbb{R}, \\ \partial_t v_\varepsilon = -u_\varepsilon v_\varepsilon, \end{cases} \tag{1}$$

completed with initial data u^0_ε, v^0_ε, such that

$$\varepsilon u^0_\varepsilon \in L^1_+(\mathbb{R}), \qquad 0 < v_m \leq v^0_\varepsilon \leq v_M < \infty, \qquad v_m < \mu < v_M.$$

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We have introduced a parameter ε which measures the time scale of the cell random motion compared to nutrient consumption. Our interest here is when this parameter is small because it is a case when a pattern is produced under the form of a high concentration of cells despite the parabolic character of Equation (1). In fact this phenomena is closely related to concentration effects in non-local semi-linear parabolic equations as studied intensively recently, see [9, 17, 6, 14] and the references therein. This analogy leads us to postulate that u_ε concentrate as Dirac masses at points where v_ε undergoes a discontinuity.
The scale proposed here, which is chosen to produce a distinguished limit, is usual for semi-linear diffusion equations and has been studied for local problems in classical works, [10, 1]. The most efficient method is to use the Hopf-Cole transform and viscosity solutions of Hamilton-Jacobi equations [8, 7]. We restrict our analysis to one dimension to explain the solution structure as depicted in Figure 1 but significant parts of our analysis can be extended to several dimensions.

Several related studies can be mentioned. Coupling Ordinary and Partial Differential Equation is rather classical in different areas: for pattern formation, see [12] (study of existence and stability of stationary solutions), modeling of two species dynamics with an unmotile specie [5, 18] for instance. Traveling waves with a non-motile phase have also been studied, see [19] and the references therein. However we are not aware of any analytical study related to the scaling proposed here.

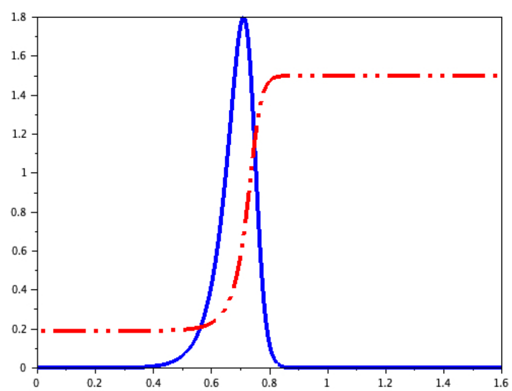

Figure 1: Traveling wave solution of Equation (1). In blue/solid line the component u_ε. In red/dashed line, the nutrient v_ε.

Our approach combines a reformulation of the problem which leads to a heterogeneous monostable equation for v_ε and the standard Hopf-Cole transform as mentioned above. This a convenient tool to represent the Dirac concentration of u_ε under the form $\exp(-\varphi(t,x)/\varepsilon)$ where φ behaves like a quadratic function.

In Section 2, we begin with a simpler case where we omit the diffusion on u_ε, arriving to a system of ordinary differential equations which can be solved nearly explicitly. This allows us to introduce the tools which are used in Section 3 where we state and prove the concentration effect. Several related questions are detailed in appendices: the particular case of traveling waves, and some Sobolev regularity results.

2 The problem without diffusion

We begin with the simpler case where diffusion is ignored and where we can give a complete description while introducing the main tools for the general problem. We are reduced to a system of two differential equations with a parameter x, which solutions however behave as a front propagation in space, namely

$$\begin{cases} \partial_t u_\varepsilon(t,x) = \frac{1}{\varepsilon} u_\varepsilon(v_\varepsilon - \mu), \qquad t \geq 0,\ x \in \mathbb{R}, \\ \partial_t v_\varepsilon = -u_\varepsilon v_\varepsilon. \end{cases} \tag{2}$$

We define

$$Q(v) = v - \mu \ln v.$$

We assume there are constants Q_m and Q_M such that

$$\begin{cases} \text{For } x < 0, \quad v_\varepsilon^0(x) < \mu, \qquad \text{and for } x > 0 \quad v_\varepsilon^0(x) > \mu, \\ Q(v_\varepsilon^0(x)) + \varepsilon u_\varepsilon^0 \leq Q_M, \qquad Q(v_\varepsilon^0(x)) > Q_m > Q(\mu), \\ v_\varepsilon^0 \to v^0 \quad \text{pointwise for } x \neq 0. \end{cases} \tag{3}$$

In other words v_ε^0 has an initial increasing discontinuity at $x = 0$. For the cell density, we assume that in the weak sense of measures, as $\varepsilon \to 0$,

$$\begin{cases} u_\varepsilon^0 = \exp(\frac{\varphi_\varepsilon^0}{\varepsilon}) \rightharpoonup \rho^0 \delta(x) \quad \text{in the weak sense of measures}, \\ \varphi_\varepsilon^0 \to \varphi^0 \quad \text{in } C(\mathbb{R}), \qquad \varphi^0 < 0 \text{ for } x \neq 0. \end{cases} \tag{4}$$

In this framework, we can describe the behavior of solutions as follows

Theorem 1 *Assume (3)– (4). The solution of (2) has limits $u_\varepsilon \rightharpoonup u$ (weak measures) and $v_\varepsilon \to v$ (strongly in L^p_{loc}) and, for $x > 0$ there is a time $\tau(x)$ and $v_-^0(x) < \mu < v^0(x)$ such that*
(i) $v(t,x) = v_-^0(x)$ for $t < \tau(x)$, $\qquad v(t,x) = v^0(x)$ for $t > \tau(x)$,
(ii) $u(t,x) = [\ln v^0(x) - \ln v_-^0(x)]\delta(t - \tau(x))$,
(iii) $\tau(x) = -\frac{\varphi^0(x)}{v^0(x) - \mu}$ with φ defined by (8).

For $x < 0$, we have $v(t,x) = v^0(x)$.

Proof. A remarkable property of the system (2) is the identity

$$\partial_t[\varepsilon u_\varepsilon + v_\varepsilon - \mu \ln v_\varepsilon] = 0,$$

which implies

$$\varepsilon u_\varepsilon + Q(v_\varepsilon) = Q(v_\varepsilon^0(x)) + \varepsilon u_\varepsilon^0 \leq Q_M, \quad \forall x \in \mathbb{R}. \tag{5}$$

Using this inequality and the fact that v_ε decreases, we first conclude for $x < 0$, $v_\varepsilon - \mu < 0$ and thus $u_\varepsilon(t,x) \to 0$ as $\varepsilon \to 0$, therefore $v_\varepsilon(t,x) \to v^0(x)$

For $x > 0$, we conclude from (5) and assumption (3), that there are constants such that

$$v_m \leq v_\varepsilon(t,x) \leq v_M. \tag{6}$$

Also, integrating the second equation of (2), we have for all $T > 0$,

$$\int_0^T u_\varepsilon(t,x)dt = \ln v_\varepsilon^0(x) - \ln v_\varepsilon(T,x) \le C(T),$$

and since we expect that u_ε is a concentrated measure, we define $\varphi_\varepsilon = \varepsilon \ln u_\varepsilon$. It satisfies

$$\partial_t \varphi_\varepsilon = v_\varepsilon - \mu \quad \text{ is bounded in } t \text{ and } x. \tag{7}$$

We can argue x by x and define as $\varepsilon \to 0$ (after extraction of subsequences, but the uniqueness of the limit shows that it is the full family) the strong limits

$$v_\varepsilon(t,x) \to v(t,x), \qquad \varphi_\varepsilon \to \varphi(t,x).$$

We can also define the weak limit

$$u_\varepsilon(t,x) \rightharpoonup m(t,x) \ge 0 \in \mathcal{M}^1(0,T).$$

These limits satisfy the equations obtained passing to the limit in (2) and (7)

$$\begin{cases} \partial_t \varphi(t,x) = v - \mu, \qquad \varphi(t,x) \le 0 \\ \partial_t \ln(v) = -m(t,x), \qquad \text{supp}(m) \subset \{\varphi(t) = 0\}. \\ Q(v(t,x)) = Q(v^0(x)). \end{cases} \tag{8}$$

Because of this last equality, $v(t,x)$ belongs to one of the two branches $v_-^0(x) < v_+^0(x) := v^0(x)$ of roots of $Q(v(t,x)) = Q(v^0(x))$. Therefore $v - \mu$ is away from 0 and there is a first time $\tau(x)$ such that $\varphi\big(\tau(x),x\big) = 0$ and $\varphi(t,x) < 0$ for $t \ne \tau(x)$. This time $\tau(x)$ is also the jump time from one branch to the other for v as stated in (i).

We can also compute, from the above equation on $\ln v$, and this gives (ii), namely

$$u_\varepsilon = -\partial_t \ln v_\varepsilon \rightharpoonup u = [\ln v_+(x) - \ln v_-(x)]\delta(t - \tau(x)).$$

The time $\tau(x)$ is fully identified from the limiting system: By integrating the equation on $\varphi(t,x)$, we find that

$$\varphi(t,x) = \begin{cases} \varphi(0,x) + t\,(v^0(x) - \mu), \quad t < \tau(x), \\ \varphi(0,x) + \tau(x)(v^0(x) - \mu) + (t - \tau(x))\,(v_-^0(x) - \mu), \quad t > \tau(x). \end{cases}$$

For $x > 0$, as $v_-^0(x) < \mu < v^0(x)$, $\varphi(t,x)$ is strictly increasing in time for $t < \tau(x)$, and strictly decreasing for $t > \tau(x)$.

Furthermore by the second equation, we have that $\varphi(\tau(x),x) = 0$. Hence this characterizes $\tau(x)$ as

$$\tau(x) = -\frac{\varphi^0(x)}{v^0(x) - \mu}.$$

This gives (iii) and identifies completely the limiting solution x by x.

□

3 The full problem

When diffusion is included, the previous analysis, which uses fundamentally x by x convergence, does not apply and the assumptions have to take into account the diffusion term. We also make more general assumptions. For the initial data, we assume that

$$u^0_\varepsilon > 0, \qquad \varepsilon u^0_\varepsilon \leq C, \qquad \int_{\mathbb{R}} u^0_\varepsilon dx \leq C, \tag{9}$$

and, $\varphi^0_\varepsilon = \varepsilon \ln u^0_\varepsilon$ satisfies

$$\varepsilon \partial^2_{xx} \varphi^0_\varepsilon \leq C, \qquad |\partial_x \varphi^0_\varepsilon| \leq C, \qquad \varphi^0_\varepsilon(x) \geq -C(1+|x|), \tag{10}$$

$$\varepsilon\, |\partial^2_{xx} \ln v^0_\varepsilon| \leq C, \qquad |\partial_x \ln v^0_\varepsilon| + |\ln v^0_\varepsilon| \leq C. \tag{11}$$

Then, recalling the definition $Q(v) = v - \mu \ln v$, we also use

$$w = \ln v, \qquad \widetilde{Q}(w) = Q(v) = e^w - \mu w, \tag{12}$$

and we can define uniquely two smooth branches of initial data $w^0_{\pm,\varepsilon}$ by

$$\widetilde{Q}(w^0_\varepsilon) = \widetilde{Q}(w^0_{-,\varepsilon}) = \widetilde{Q}(w^0_{+,\varepsilon}), \qquad w^0_{-,\varepsilon} \leq \ln \mu \leq w^0_{+,\varepsilon}. \tag{13}$$

Note that assumption (11) provides corresponding bounds on w^0_ε and $w^0_{\pm,\varepsilon}$, which are

$$|w^0_\varepsilon| + |w^0_{\pm,\varepsilon}| \leq C, \quad |\partial_x w^0_\varepsilon| + |\partial_x w^0_{\pm,\varepsilon}| \leq C, \quad \varepsilon\, |\partial^2_{xx} w^0_\varepsilon| + \varepsilon |\partial^2_{xx} w^0_{\pm,\varepsilon}| \leq C. \tag{14}$$

And we also use, for an unessential result, the stronger condition

$$\varepsilon \partial^2_{xx}(w^0_\varepsilon - w^0_{-,\varepsilon}) + u^0_\varepsilon \leq C. \tag{15}$$

Finally, we use the notation $|W|_- = \max(0, -W)$.

Our main theorem now reads

Theorem 2 *Assume that $u^0_\varepsilon \in C^2 \cap L^1(\mathbb{R})$ and that assumptions (9)–(11) hold. The solution of (1) has limits $u_\varepsilon \rightharpoonup u$ (weak measures) and $v_\varepsilon \to v$ (strongly in L^p_{loc}) and, there is a time $\tau(x)$ and $v^0_-(x) < \mu < v^0(x)$ such that*
(i) $v(t,x) = v^0_-(x)$ for $t < \tau(x)$, $\qquad v(t,x) = v^0(x)$ for $t > \tau(x)$,
(ii) $u(t,x) = [\ln v^0(x) - \ln v^0_-(x)]\delta(t - \tau(x))$.

Compared to the case without diffusion in Theorem 1, there is no simple explicit formula for the jump time $\tau(x)$. However, here it is also characterized by $\varphi(\tau(x), x) = 0$ for the solution of an Eikonal equation.

The end of this section is devoted to the proof of Theorem 1.

Our analysis is based on the observation that $\partial_t \ln v_\varepsilon = -u_\varepsilon$, thus, from (1), we get the identity

$$\partial_t[\varepsilon u_\varepsilon + v_\varepsilon - \mu \ln v_\varepsilon + \varepsilon^2 \partial^2_{xx} \ln v_\varepsilon] = 0,$$

and consequently

$$\varepsilon\partial_t \ln v_\varepsilon - \varepsilon^2\partial^2_{xx}\ln v_\varepsilon - Q(v_\varepsilon) = -Q(v^0_\varepsilon) - \varepsilon u^0_\varepsilon - \varepsilon^2\partial^2_{xx}\ln v^0_\varepsilon,$$

which we can write in terms of w_ε as

$$\varepsilon\partial_t w_\varepsilon - \varepsilon^2\partial^2_{xx} w_\varepsilon - \widetilde{Q}(w_\varepsilon) = -\widetilde{Q}(w^0_\varepsilon) - \varepsilon u^0_\varepsilon - \varepsilon^2\partial^2_{xx} w^0_\varepsilon. \tag{16}$$

Notice that this is a monostable equation of Fisher/KPP type, where the steady states depend on x.

A priori estimates on v_ε.

Lemma 3 *The inequalities hold*

$$0 < C \le w_\varepsilon(t,x) \le w^0_{+,\varepsilon}, \tag{17}$$

and for any R, there is a constant C_R such that,

$$\int_{|x|\le R} |\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)|\mathbb{1}_{\{w_\varepsilon - w^0_{-,\varepsilon}\le 0\}}\,dx \le C_R\,\varepsilon, \qquad \forall t\ge 0. \tag{18}$$

Finally, with the additional assumption (15), we have $-C\sqrt{\varepsilon} + w^0_{-,\varepsilon} \le w_\varepsilon(t,x)$.

Proof. For the first statement, on the one hand, we note that we necessarily have that for any x, $w^0_\varepsilon(x) = w^0_{-,\varepsilon}(x)$ or $w^0_\varepsilon(x) = w^0_{+,\varepsilon}(x)$. In both cases, that implies that $w^0_\varepsilon(x) \le w^0_{+,\varepsilon}(x)$. Since v_ε is non-increasing in time, so is w_ε and

$$w_\varepsilon(t,x) \le w^0_\varepsilon(x) \le w^0_{+,\varepsilon}(x).$$

On the other hand, as $\partial_t w_\varepsilon \le 0$, we can deduce from (16), that

$$-\varepsilon^2\partial^2_{xx}(w_\varepsilon - w^0_{-,\varepsilon}) + [\widetilde{Q}(w^0_{-,\varepsilon}) - \widetilde{Q}(w_\varepsilon)] \ge -\varepsilon^2\partial^2_{xx}(w^0_\varepsilon - w^0_{-,\varepsilon}) - \varepsilon u^0_\varepsilon. \tag{19}$$

Using the maximum principle and assumptions (9), (11), we conclude that at a minimum value of $w_\varepsilon - w^0_{-,\varepsilon}$, the quantity $\widetilde{Q}(w^0_{-,\varepsilon}) - \widetilde{Q}(w_\varepsilon)$ is controlled from below and the lower bound on w_ε follows.

Furthermore, from the usual convex inequalities, we also observe that

$$-\varepsilon^2\partial^2_{xx}(w_\varepsilon - w^0_{-,\varepsilon})_- + [\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_{-,\varepsilon})]\mathbb{1}_{\{w_\varepsilon - w^0_{-,\varepsilon}<0\}} \le \varepsilon\, u^0_\varepsilon + \varepsilon^2|\partial^2_{xx}(w^0_\varepsilon - w^0_{-,\varepsilon})|. \tag{20}$$

Because the set $\{W\, s.t.\, W - w^0_{-,\varepsilon} < 0\}$ lies in the decreasing branch of $\widetilde{Q}$, the quantity $[\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_{-,\varepsilon})]\mathbb{1}_{\{w_\varepsilon - w^0_{-,\varepsilon}<0\}}$ is positive. We integrate against some smooth non-negative ψ with $\psi = 1$ on $B(0,R)$ and ψ compactly supported in $B(0,2R)$. Since we have already proved that $w_\varepsilon - w^0_{-,\varepsilon}$ is bounded, we obtain the estimate

$$\int_{|x|\le R} |\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)|\mathbb{1}_{\{w_\varepsilon - w^0_{-,\varepsilon}\le 0\}}\,dx \le C_R\,\varepsilon^2 + \varepsilon\int_{|x|\le R} u^0_\varepsilon\,dx + \varepsilon^2\int_{|x|\le R}|\partial^2_{xx}(w^0_\varepsilon - w^0_{-,\varepsilon})|\,dx.$$

Using assumption (9) and assumption (11) concludes the second point of the lemma.

We may also use the specific assumption (15) in (20), we obtain that

$$-\varepsilon^2\partial^2_{xx}(w_\varepsilon - w^0_{-,\varepsilon})_- + [\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_{-,\varepsilon})]\mathbb{1}_{\{w_\varepsilon - w^0_{-,\varepsilon}<0\}} \le C\varepsilon.$$

Recalling that $[\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_{-,\varepsilon})]\mathbb{1}_{\{w_\varepsilon - w^0_{-,\varepsilon}<0\}}$ is positive, we conclude that

$$Q(w_\varepsilon) - \widetilde{Q}(w^0_{-,\varepsilon})\mathbb{1}_{\{w_\varepsilon - w^0_{-,\varepsilon}<0\}} \le C\varepsilon,$$

and thus the third statement of Lemma 3 holds, namely $w_\varepsilon - w_{-,\varepsilon} \ge -C\sqrt{\varepsilon}$.

□

Concentration dynamics of u_ε. We turn to the study of u_ε and begin with some simple estimates.
• Since $u_\varepsilon = -\partial_t w_\varepsilon$, and using the bound (17), we find

$$\sup_x \int_0^T u_\varepsilon(t,x)dt = \sup_x [w^0_\varepsilon(x) - w_\varepsilon(x,T)] \le C(T), \tag{21}$$

and thus, integrating the equation on u_ε, we also get

$$\varepsilon \int_{\mathbb{R}} u_\varepsilon(t,x)\,dx = \varepsilon \int_{\mathbb{R}} u^0_\varepsilon(x)\,dx + \int_{\mathbb{R}} \int_0^t u_\varepsilon(s,x)(v_\varepsilon(s,x)-\mu)ds\,dx \le C(t). \tag{22}$$

• Next, we use the Hopf-Cole transform

$$\varphi_\varepsilon(t,x) = \varepsilon \ln u_\varepsilon(t,x).$$

As usual, we compute that φ_ε satisfies the Eikonal equation

$$\partial_t \varphi_\varepsilon = \varepsilon \partial^2_{xx}\varphi_\varepsilon + |\partial_x \varphi_\varepsilon|^2 + v_\varepsilon - \mu. \tag{23}$$

We are going to show some uniform bounds on φ_ε.

Lemma 4 *We have*

$$\partial_t \varphi_\varepsilon(t,x) \le C, \qquad \forall x \in \mathbb{R},\ t \ge 0 \tag{24}$$

$$\|\partial_x \varphi_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \le C, \qquad \|\partial_t \varphi_\varepsilon\|_{L^p_{t,x}} \le C(p), \quad \forall p \in [1,\infty), \tag{25}$$

$$-C(t)(1+|x|) \le \varphi_\varepsilon(t,x) \le 2\,\varepsilon\,\ln\frac{1}{\varepsilon} + C(t)\,\varepsilon, \qquad \forall x \in \mathbb{R},\ t \ge 0. \tag{26}$$

Proof. For the time derivative, differentiating (23) and using the equation on v_ε, we find

$$\partial_t(\partial_t\varphi_\varepsilon) = \varepsilon\partial^2_{xx}(\partial_t\varphi_\varepsilon) + 2\partial_x\varphi_\varepsilon\partial_x(\partial_t\varphi_\varepsilon) - u_\varepsilon v_\varepsilon \le \varepsilon\partial^2_{xx}(\partial_t\varphi_\varepsilon) + 2\partial_x\varphi_\varepsilon\partial_x(\partial_t\varphi_\varepsilon),$$

so that the maximum principle gives $\partial_t\varphi_\varepsilon(t,x) \le \max_x \partial_t\varphi^0_\varepsilon(x)$ which gives (24) thanks to the assumption (10).

Next, we prove the Lipschitz bound. Consider any point x_ε that is a maximum in x of $\partial_x\varphi_\varepsilon$ at any time t (standard arguments apply if the maximum is not achieved, see [7, 2]). Then $\partial^2_{xx}\varphi_\varepsilon(x_\varepsilon) = 0$ and we conclude, still using (24), that

$$|\partial_x\varphi_\varepsilon(t,x_\varepsilon)|^2 = \partial_t\varphi_\varepsilon(t,x_\varepsilon) + v_\varepsilon - \mu \le C. \tag{27}$$

Once $\partial_x\varphi_\varepsilon \in L^\infty_{t,x}$ uniformly, standard parabolic estimates provide a uniform bound on $\partial_t\varphi_\varepsilon$ in $L^p_{t,x}$ for any $1 \le p < \infty$.

Finally, since φ_ε is uniformly Lipschitz in x, let x_ε be a maximum of φ_ε, then

$$\varphi_\varepsilon(t,x) \ge \max \varphi_\varepsilon(t,.) - C\,|x - x_\varepsilon|,$$

so that, using (22),

$$\frac{C}{\varepsilon} \ge \int_{\mathbb{R}} u_\varepsilon(t,x)\,dx \ge \int_{\mathbb{R}} e^{\max\varphi_\varepsilon(t,.)/\varepsilon}\,e^{-C\,|x-x_\varepsilon|/\varepsilon}\,dx \ge \frac{\varepsilon}{C}\,e^{\max\varphi_\varepsilon(t,.)/\varepsilon},$$

which proves the upper bound in (26). The lower bound relies, as it is standard [2, 7, 3], on the construction of a sub-solution. Here one can immediately check that $-C(t+1) - \frac{Cx^2}{\sqrt{1+|x|^2}}$ will work.
□

Compactness of v_ε. We introduce the quantity $\Phi(x,w)$, defined up to a constant, by

$$\Phi_w(x,w) = |\widetilde{Q}(w_\varepsilon^0(x)) - \widetilde{Q}(w)| \geq 0.$$

Lemma 5 *With assumptions* (11)–(9)*, we have*

$$\sup_{x\in\mathbb{R},\, 0\leq t\leq T} |\partial_x w_\varepsilon| \leq \frac{C_T}{\varepsilon}, \qquad \int_0^T \int_{|x|\leq R} \big|\partial_x \Phi(x, w_\varepsilon(t,x))\big| \leq C_{T,R}.$$

Consequently, by monotonicity of Φ in w, v_ε is locally compact in $L^p((0,\infty)\times\mathbb{R})$ for any $1 \leq p < \infty$.

Remark 6 *It is also possible to conclude from this lemma that w_ε is uniformly bounded in $L^1([0,\ T],\ W^{\theta,1}(\mathbb{R}))$ for some $\theta > 0$; see Appendix B.*

Proof. First of all, calculate

$$\partial_t \partial_x w_\varepsilon = -\partial_x u_\varepsilon = -\frac{\partial_x \varphi_\varepsilon}{\varepsilon}\, u_\varepsilon,$$

which yields, from the Lipschitz bound on φ_ε in (27) and the estimate (21),

$$|\partial_x w_\varepsilon(t,x)| \leq |\partial_x w_\varepsilon^0(x)| + \frac{C}{\varepsilon} \int_0^t u_\varepsilon(s,x)\, ds \leq \frac{C(t)}{\varepsilon}.$$

Next, we write $\partial_x[\Phi(x, w_\varepsilon(t,x))] = \partial_x \Phi(x, w_\varepsilon) + [\widetilde{Q}(w_\varepsilon^0) - \widetilde{Q}(w_\varepsilon)]\partial_x w_\varepsilon$. Since $\partial_x \Phi(x, w_\varepsilon)$ is bounded in L^∞, and thanks to the second bound in Lemma 3, we conclude that

$$\int_0^T \int_{|x|\leq R} |\partial_x \Phi(x, w_\varepsilon)| \leq C_{T,R} + \varepsilon \sup_{|x|\leq R,\, 0\leq t\leq T} |\partial_x w_\varepsilon| \int_0^T \int_{|x|\leq R} \Big|\frac{Q(v_\varepsilon^0) - Q(v_\varepsilon)}{\varepsilon}\Big| \leq C_{T,R}. \tag{28}$$

Since $\partial_t w_\varepsilon \leq 0$ and w_ε bounded provide us with compactness in time, we conclude that $\Phi(x, w_\varepsilon)$ is compact and thus converges a.e. By monotonicity of Φ in w_ε, we also conclude that w_ε converges a.e.
□

Convergence as $\varepsilon \to 0$. We are now ready to study the limit as ε vanishes.
• The bounds in Lemma 4 show that φ_ε is locally compact in $C(\mathbb{R}_+ \times \mathbb{R})$ and hence, after extraction of a subsequence that we still denote by ε, there exists φ which is Lipschitz in space and with time derivatives in L^p such that

$$\|\varphi_\varepsilon - \varphi\|_{L^\infty((0,T)\times(-R,R))} \to 0, \quad \text{as } \varepsilon \to 0, \qquad \forall T > 0,\ R > 0.$$

We note that

$$-C(1+t) - C|x| \leq \varphi(t,x) \leq 0. \tag{29}$$

• From Lemma 5, we also conclude that v_ε converges locally; for any $1 \leq p < \infty$,

$$v_\varepsilon \to v, \quad w_\varepsilon \to w = \ln v \quad \text{in} \quad L^p((0,T)\times(-R,R)), \qquad \forall T > 0,\ R > 0.$$

• From the bound (21), we can also extract a subsequence such that, in the weak sense of measures,

$$u_\varepsilon \rightharpoonup u, \quad \text{in} \quad \mathcal{M}((0,T)\times(-R,R)), \qquad \forall T > 0,\ R > 0.$$

We may pass to the limit, in distributional sense, in Equations (1) and get, as $\varepsilon \to 0$,

$$u_\varepsilon v_\varepsilon \rightharpoonup u\mu, \qquad \partial_t w = -u \qquad \partial_t v = \text{w-lim } (-u_\varepsilon v_\varepsilon) = -u\mu,$$

which expresses that the concentration of the measure u_ε is exactly at the point where $v_\varepsilon = \mu$. And from Lemma 3, we know that

$$\widetilde{Q}(w) = \widetilde{Q}(w^0), \quad i.e., \quad w(t,x) = w_-^0(x) \quad or \quad w_+^0(x).$$

Since w is non-decreasing in time, we conclude that, for all $x \in \mathcal{S}$, a subset of $\mathbb{R}$ where $w^0 = w_-^0$, there is a unique time $\tau(x)$ such that w jumps from $w_-^0(x)$ to $w_+^0(x)$ (with $\tau(x) = \infty$ for $x < 0$), and

$$u(t,x) = [w_+^0(x) - w_-^0(x)]\delta(t - \tau(x))\mathbb{1}_{\{x \in \mathcal{S}\}}, \qquad \varphi(\tau(x), x) = 0.$$

Open questions. Uniqueness for the limit problem, which we proved when diffusion is ignored (Section 2, is an open question in full generality. In particular it seems hard to determine more properties about the set $\mathcal{S}$, which depends on the initial data. In the monotone case when $w^0(x) = w_-^0(x)$ for $x < 0$ and $w^0(x) = w_+^0(x)$ for $x > 0$ with $u^0 = [[w^0]]\delta(x)$, one can expect $\tau(x)$ be invertible and to obtain a front located at some value $x = X(t)$.

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A Traveling wave

Traveling waves are an intuitive way to understand, in a very particular case, the general behavior of system (1). considering solutions of the form $u_\varepsilon(x - \sigma t)$, $v_\varepsilon(x - \sigma t)$, $w_\varepsilon = \ln v_\varepsilon$. Recalling the notation $\widetilde{Q}(w) = e^w - \mu w$, we arrive at an equation on the single quantity $w_\varepsilon(y)$,

$$-\sigma\varepsilon w_\varepsilon' - \varepsilon^2 w_\varepsilon'' = \widetilde{Q}(w_\varepsilon) - A,$$

with the conditions at infinity

$$w_\varepsilon(-\infty) = w_- < \ln \mu, \qquad w_\varepsilon(+\infty) = w_+ > \ln \mu, \qquad A = \widetilde{Q}(w_-) = \widetilde{Q}(w_+).$$

This is just a Fisher/KPP monostable equation with w_+ the unstable state and we know from the general theory [13] that there is a traveling wave with minimal speed σ_* which is characterized by the property of a double root for the polynomial

$$-\sigma\varepsilon\lambda - \varepsilon^2\lambda^2 = \widetilde{Q}'(w_+),$$

that is $\sigma_* = 2\sqrt{\widetilde{Q}'(w_+)}$. In our analysis, this value σ_* also appears in the limit of Equation (23), that is the Eikonal equation

$$\partial_t \varphi = |\partial_x \varphi|^2 + v - \mu,$$

which for the traveling wave problem generates a solution $\varphi(x - \sigma t)$ with

$$\sigma \varphi'(y) = |\varphi'(y)|^2 + v_\pm(y) - \mu,$$

where $v_\pm(y) = v_-$ for $y < 0$ and $v_\pm(y) = v_+$ for $y > 0$. The limiting minimal speed traveling wave solution is

$$\varphi(y) = \begin{cases} p_- y < 0 & \text{for} \quad y < 0, \\ p_+ y < 0 & \text{for} \quad y > 0, \end{cases}$$

and p_+ is the double root of the polynomial $-\sigma_* \lambda - 2\lambda^2 = \widetilde{Q}'(w_+) = v_+ - \mu$. This approach based on the concentration as a Dirac measure of u_ε differs (but is restricted to one dimension) from the general front propagation theory in [1, 4] based on the quantity v_ε.

B A Sobolev estimate

We may use the bound (28) to obtain Sobolev regularity on v_ε by controlling

$$\sup_{|h| \le 1} \int_0^T \int_{|x| \le R} \frac{|v_\varepsilon(x+h) - v_\varepsilon(x)|}{|h|^\theta} \, dx \, dt, \tag{30}$$

for some appropriate value of θ.

This requires to be a bit more precise on the set where the initial data v_ε crosses μ. Specifically, we assume that there exists some constants $C > 0$ and $\kappa > 0$ such that for any $\delta > 0$

$$|\{x, \ |v^0_\varepsilon(x) - \mu| \le \delta\}| \le C\, \delta^\kappa. \tag{31}$$

Observe that when $|h| \le \varepsilon^{1/(1-\theta)}$ then by the Lipschitz bound on v_ε (which follows immediately from that on w_ε), then

$$\int_0^T \int_{|x| \le R} \frac{|v_\varepsilon(x+h) - v_\varepsilon(x)|}{|h|^\theta} \, dx \, dt \le C_{T,R} \, \|\partial_x v_\varepsilon\|_{L^\infty} \, |h|^{1-\theta} \ \le C_R,$$

so that we can limit ourselves to $h \ge \varepsilon^{1/(1-\theta)}$.

For some $\alpha > 0$ which we later relate to θ and some constant C, denote

$$\Omega_- = \{(t,x) \in [0, \ T] \times B(0,R), \ w^0_{-,\varepsilon} - |h|^\alpha \le w_\varepsilon(t,x) \le \ln \mu - |h|^{\alpha/2}\},$$
$$\Omega_+ = \{(t,x) \in [0, \ T] \times B(0,R), \ w_\varepsilon(t,x) \ge \ln \mu + |h|^{\alpha/2}\},$$
$$\Omega_0 = \{(t,x) \in [0, \ T] \times B(0,R), \ \ln \mu - |h|^{\alpha/2} \le w_\varepsilon(t,x) \le \ln \mu + |h|^{\alpha/2} \text{ or } w_\varepsilon(t,x) \le w^0_{-,\varepsilon} - |h|^\alpha\}.$$

Observe that when $(t,x) \in \Omega_0$ then

$$\sup_{\varepsilon^{1/(1-\theta)} \le |h| \le 1} \int_{(t,x) \in \Omega_0 \ \text{or} \ (t,x+h) \in \Omega_0} \frac{|v_\varepsilon(t,x+h) - v_\varepsilon(t,x)|}{|h|^\theta} \, dx \, dt$$
$$\le 2 \sup_{\varepsilon^{1/(1-\theta)} \le |h| \le 1} \frac{\|v_\varepsilon\|}{|h|^\theta} \int_{(t,x) \in \Omega_0 \ \text{or} \ (t,x+h) \in \Omega_0} dx dt$$

We note that if $w_\varepsilon(t,x) \le w^0_{-,\varepsilon}(x) - |h|^\alpha$ then

$$|\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)| \ge |h|^\alpha / C.$$

Similarly if $\ln\mu - |h|^{\alpha/2} \le w_\varepsilon(t,x) \le \ln\mu + |h|^{\alpha/2}$ but $w^0_\varepsilon(x) < \ln\mu - 2\,|h|^{\alpha/2}$ or $w^0_\varepsilon(x) > \ln\mu + 2|h|^{\alpha/2}$, then we have again

$$|\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)| \ge |h|^\alpha / C,$$

as $\widetilde{Q}(w)$ has a minimum at $w = \ln\mu$ but is strictly convex.

Hence by (31)

$$\begin{aligned}|\Omega_0| &\le \frac{C}{|h|^\alpha} \int_0^T \int_{B(0,R)} |\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)|\,dx\,dt + \{x,\ |w^0_\varepsilon(x) - \ln\mu| \le 2\,|h|^{\alpha/2}\} \\ &\le \frac{C}{|h|^\alpha} \int_0^T \int_{B(0,R)} |\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)|\,dx\,dt + C\,|h|^{\kappa\,\alpha/2}.\end{aligned}$$

We therefore obtain that

$$\begin{aligned}&\sup_{\varepsilon^{1/(1-\theta)} \le |h| \le 1} \int_{(t,x)\in\Omega_0 \text{ or } (t,x+h)\in\Omega_0} \frac{|v_\varepsilon(t,x+h) - v_\varepsilon(t,x)|}{|h|^\theta}\,dx\,dt \\ &\quad\le \sup_{\varepsilon^{1/(1-\theta)} \le |h| \le 1} \frac{C}{|h|^\theta}\,\frac{1}{|h|^\alpha} \int_0^T \int_{B(0,R)} |\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)|\,dx\,dt + C\,\frac{|h|^{\kappa\,\alpha/2}}{|h|^\theta} \le C_{T,R},\end{aligned}$$

by Lemma 3 and provided that $\kappa\,\alpha/2 \ge \theta$ leading us to take $\alpha = 2\,\theta/\kappa$ and $(\alpha+\theta)/(1-\theta) = (1+2/\kappa)\,\theta/(1-\theta) \le 1$. It is always possible to satisfy these inequalities provided that $\theta/(1-\theta) \le \frac{1}{1+2/\kappa}$.

We can consequently exclude the case where $(t,x) \in \Omega_0$ or $(t,x+h) \in \Omega_0$ when bounding (30).

The BV bound (28) also directly controls the case where $(t,x) \in \Omega_-$ and $(t,x+h) \in \Omega_+$ (or vice-versa). Indeed in that case, we necessarily have that $w^0_{+,\varepsilon} \ge w^0_{-,\varepsilon} + 2\,|h|^{\alpha/2}$ and $\partial_w \Phi(x,w_\varepsilon) = \widetilde{Q}(w^0_\varepsilon) - \widetilde{Q}(w_\varepsilon)$ has a sign between $w_{-,\varepsilon}$ and $w_{+,\varepsilon}$ so that there exists a constant C s.t. (again at least locally)

$$|\Phi(x, w^0_{-,\varepsilon}) - \Phi(x, w^0_{+,\varepsilon})| \ge \frac{|h|^{3\,\alpha/2}}{C}.$$

By our definitions of Ω_- and Ω_+ and taking C large enough, this implies that

$$|\Phi(x, w_\varepsilon(t,x+h)) - \Phi(x, w_\varepsilon(t,x))| \ge \frac{|h|^{3\,\alpha/2}}{C} \quad \text{if } (t,x)\in\Omega_- \text{ and } (t,x+h)\in\Omega_+.$$

Therefore

$$\begin{aligned}&\int_{(t,x)\in\Omega_- \text{ and } (t,x+h)\in\Omega_+} \frac{|v_\varepsilon(t,x+h) - v_\varepsilon(t,x)|}{|h|^\theta}\,dx\,dt \le 2\,\frac{\|v_\varepsilon\|}{|h|^\theta} \int_{(t,x)\in\Omega_- \text{ and } (t,x+h)\in\Omega_+} dx \\ &\quad\le \frac{C}{|h|^{\theta+3\alpha/2}} \int_0^T \int_{B(0,R)} |\Phi(x, w_\varepsilon(t,x+h)) - \Phi(x, w_\varepsilon(t,x))|\,dx\,dt \le C_{T,R}\,|h|^{1-\theta-3\alpha/2},\end{aligned}$$

by (28). This is bounded as long as $\theta + 3\alpha/2 = \theta\,(1+3/\kappa) \le 1$.

We are finally able to focus on the case where for example both $(t,x)\in\Omega_-$ and $(t,x+h)\in\Omega_-$. Note again that $\partial_w \Phi(x,w_\varepsilon) = \widetilde{Q}(w^0_\varepsilon) - \widetilde{Q}(w_\varepsilon)$ vanishes once with a change of sign at $w_\varepsilon = w_{-,\varepsilon}$ for $(t,x)\in\Omega_-$. Therefore $w \to \Phi(x,w)$ is injective on $w \in [w^0_{-,\varepsilon},\ \ln\mu - |h|^{\alpha/2}]$ for some constant C and

$$|w_1 - w_2|^3 \le C\,|\Phi(x,w_1) - \Phi(x,w_2)|, \quad w_1,\, w_2 \in [w^0_{-,\varepsilon},\ \ln\mu - |h|^{\alpha/2}].$$

Since $w_\varepsilon \geq w^0_{-,\varepsilon} - |h|^\alpha$ on Ω_-, this implies that for both $(t,x) \in \Omega_-$ and $(t,x+h) \in \Omega_-$, we have

$$|w_\varepsilon(t,x) - w_\varepsilon(t,x+h)|^3 \leq C\,|\Phi(x,w_\varepsilon(t,x)) - \Phi(x,w_\varepsilon(t,x+h))| + C\,\varepsilon^\alpha.$$

By a straightforward Hölder inequality, we get

$$\begin{aligned}
&\int_{(t,x)\in\Omega_- \text{ and } (t,x+h)\in\Omega_+} \frac{|v_\varepsilon(t,x+h) - v_\varepsilon(t,x)|}{|h|^\theta}\,dx\,dt \\
&\leq C_{T,R}\left(\int_{(t,x)\in\Omega_- \text{ and } (t,x+h)\in\Omega_-} \frac{|w_\varepsilon(t,x+h) - w_\varepsilon(t,x)|^3}{|h|^{3\theta}}\,dx\,dt\right)^{1/2} \\
&\leq C_{T,R}\left(\int_{(t,x)\in\Omega_- \text{ and } (t,x+h)\in\Omega_-} \frac{|\Phi(x,w_\varepsilon(t,x)) - \Phi(x,w_\varepsilon(t,x+h))| + |h|^\alpha}{|h|^{3\theta}}\,dx\,dt\right)^{1/2} \leq C_{T,R},
\end{aligned}$$

by (28) again, provided that $3\,\theta \leq 1$ and $\alpha \geq 3\,\theta$. Since we chose $\alpha = 2\,\theta/\kappa$, this last inequality holds provided that $\kappa \leq 2/3$, which we can always impose.

To summarize, we have obtained the desired bound (30) provided that $\theta \leq 1/3$ and $\theta \leq 1/(1+3/\kappa)$ together with $\theta/(1-\theta) \leq 1/(1+2/\kappa)$.

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