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Abstract

A classical problem describing the collective motion of cells, is the movement driven by consumption/depletion of a nutrient. Here we analyze one of the simplest such model written as a coupled Partial Differential Equation/Ordinary Differential Equation system which we scale so as to get a limit describing the usually observed pattern. In this limit the cell density is concentrated as a moving Dirac mass and the nutrient undergoes a discontinuity.

We first carry out the analysis without diffusion, getting a complete description of the unique limit. When diffusion is included, we prove several specific a priori estimates and interpret the system as a heterogeneous monostable equation. This allow us to obtain a limiting problem which shows the concentration effect of the limiting dynamics.

1 Introduction

A classical problem describing the collective motion of cells, is the movement driven by consumption/depletion of a nutrient [11, 15, 16]. The simplest description uses a number density of cells u_ε and a nutrient concentration v_ε. It is written

$$\begin{cases} \partial_t u_\varepsilon - \varepsilon \partial_{xx}^2 u_\varepsilon = \frac{1}{\varepsilon} u_\varepsilon (v_\varepsilon - \mu), & t \geq 0,\ x \in \mathbb{R}, \\ \partial_t v_\varepsilon = -u_\varepsilon v_\varepsilon, \end{cases} \tag{1}$$

completed with initial data u_ε^0, v_ε^0, such that

$$\varepsilon u_\varepsilon^0 \in L_+^1(\mathbb{R}), \qquad 0 < v_m \leq v_\varepsilon^0 \leq v_M < \infty, \qquad v_m < \mu < v_M.$$

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We have introduced a parameter ε which measures the time scale of the cell random motion compared to nutrient consumption. Our interest here is when this parameter is small because it is a case when a pattern is produced under the form of a high concentration of cells despite the parabolic character of Equation (1). In fact this phenomena is closely related to concentration effects in non-local semi-linear parabolic equations as studied intensively recently, see [9, 17, 6, 14] and the references therein. This analogy leads us to postulate that u_ε concentrate as Dirac masses at points where v_ε undergoes a discontinuity.

The scale proposed here, which is chosen to produce a distinguished limit, is usual for semi-linear diffusion equations and has been studied for local problems in classical works, [10, 1]. The most efficient method is to use the Hopf-Cole transform and viscosity solutions of Hamilton-Jacobi equations [8, 7]. We restrict our analysis to one dimension to explain the solution structure as depicted in Figure 1 but significant parts of our analysis can be extended to several dimensions.

Several related studies can be mentioned. Coupling Ordinary and Partial Differential Equation is rather classical in different areas: for pattern formation, see [12] (study of existence and stability of stationary solutions), modeling of two species dynamics with an unmotile specie [5, 18] for instance. Traveling waves with a non-motile phase have also been studied, see [19] and the references therein. However we are not aware of any analytical study related to the scaling proposed here.
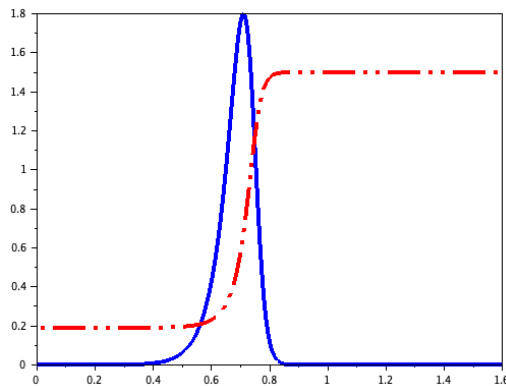
Figure 1: Traveling wave solution of Equation (1). In blue/solid line the component u_ε. In red/dashed line, the nutrient v_ε.

Our approach combines a reformulation of the problem which leads to a heterogeneous monostable equation for v_ε and the standard Hopf-Cole transform as mentioned above. This a convenient tool to represent the Dirac concentration of u_ε under the form $\exp(-\varphi(t,x)/\varepsilon)$ where φ behaves like a quadratic function.

In Section 2, we begin with a simpler case where we omit the diffusion on u_ε, arriving to a system of ordinary differential equations which can be solved nearly explicitly. This allows us to introduce the tools which are used in Section 3 where we state and prove the concentration effect. Several related questions are detailed in appendices: the particular case of traveling waves, and some Sobolev regularity results.

2 The problem without diffusion

We begin with the simpler case where diffusion is ignored and where we can give a complete description while introducing the main tools for the general problem. We are reduced to a system of two differential equations with a parameter x, which solutions however behave as a front propagation in space, namely

$$\begin{cases} \partial_t u_\varepsilon(t,x) = \frac{1}{\varepsilon} u_\varepsilon(v_\varepsilon - \mu), & t \geq 0, \ x \in \mathbb{R}, \\ \partial_t v_\varepsilon = -u_\varepsilon v_\varepsilon. \end{cases} \tag{2}$$

We define

$$Q(v) = v - \mu \ln v.$$

We assume there are constants Q_m and Q_M such that

$$\begin{cases} \text{For } x < 0, \quad v_\varepsilon^0(x) < \mu, \qquad \text{and for } x > 0 \quad v_\varepsilon^0(x) > \mu, \\ Q(v_\varepsilon^0(x)) + \varepsilon u_\varepsilon^0 \leq Q_M, \qquad Q(v_\varepsilon^0(x)) > Q_m > Q(\mu), \\ v_\varepsilon^0 \to v^0 \quad \text{pointwise for } x \neq 0. \end{cases} \tag{3}$$

In other words v_ε^0 has an initial increasing discontinuity at $x = 0$. For the cell density, we assume that in the weak sense of measures, as $\varepsilon \to 0$,

$$\begin{cases} u_\varepsilon^0 = \exp(\frac{\varphi_\varepsilon^0}{\varepsilon}) \rightharpoonup \rho^0 \delta(x) \quad \text{in the weak sense of measures,} \\ \varphi_\varepsilon^0 \to \varphi^0 \quad \text{in } C(\mathbb{R}), \qquad \varphi^0 < 0 \text{ for } x \neq 0. \end{cases} \tag{4}$$

In this framework, we can describe the behavior of solutions as follows

Theorem 1 *Assume* (3)*–* (4)*. The solution of* (2) *has limits* $u_\varepsilon \rightharpoonup u$ *(weak measures) and* $v_\varepsilon \to v$ *(strongly in L_{loc}^p) and, for $x > 0$ there is a time $\tau(x)$ and $v_-^0(x) < \mu < v^0(x)$ such that*
(i) $v(t,x) = v_-^0(x)$ *for* $t < \tau(x)$, $\qquad v(t,x) = v^0(x)$ *for* $t > \tau(x)$,
(ii) $u(t,x) = [\ln v^0(x) - \ln v_-^0(x)]\delta(t - \tau(x))$,
(iii) $\tau(x) = -\frac{\varphi^0(x)}{v^0(x) - \mu}$ *with φ defined by* (8)*.*

For $x < 0$, we have $v(t,x) = v^0(x)$.

Proof. A remarkable property of the system (2) is the identity

$$\partial_t[\varepsilon u_\varepsilon + v_\varepsilon - \mu \ln v_\varepsilon] = 0,$$

which implies

$$\varepsilon u_\varepsilon + Q(v_\varepsilon) = Q(v_\varepsilon^0(x)) + \varepsilon u_\varepsilon^0 \leq Q_M, \quad \forall x \in \mathbb{R}. \tag{5}$$

Using this inequality and the fact that v_ε decreases, we first conclude for $x < 0$, $v_\varepsilon - \mu < 0$ and thus $u_\varepsilon(t,x) \to 0$ as $\varepsilon \to 0$, therefore $v_\varepsilon(t,x) \to v^0(x)$

For $x > 0$, we conclude from (5) and assumption (3), that there are constants such that

$$v_m \leq v_\varepsilon(t,x) \leq v_M. \tag{6}$$

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Also, integrating the second equation of (2), we have for all $T > 0$,

$$\int_0^T u_\varepsilon(t, x)dt = \ln v_\varepsilon^0(x) - \ln v_\varepsilon(T, x) \le C(T),$$

and since we expect that u_ε is a concentrated measure, we define $\varphi_\varepsilon = \varepsilon \ln u_\varepsilon$. It satisfies

$$\partial_t \varphi_\varepsilon = v_\varepsilon - \mu \quad \text{is bounded in } t \text{ and } x. \tag{7}$$

We can argue x by x and define as $\varepsilon \to 0$ (after extraction of subsequences, but the uniqueness of the limit shows that it is the full family) the strong limits

$$v_\varepsilon(t, x) \to v(t, x), \qquad \varphi_\varepsilon \to \varphi(t, x).$$

We can also define the weak limit

$$u_\varepsilon(t, x) \rightharpoonup m(t, x) \ge 0 \in \mathcal{M}^1(0, T).$$

These limits satisfy the equations obtained passing to the limit in (2) and (7)

$$\begin{cases} \partial_t \varphi(t, x) = v - \mu, & \varphi(t, x) \le 0 \\ \partial_t \ln(v) = -m(t, x), & \text{supp}(m) \subset \{\varphi(t) = 0\}. \\ Q(v(t, x)) = Q(v^0(x)). \end{cases} \tag{8}$$

Because of this last equality, $v(t, x)$ belongs to one of the two branches $v_-^0(x) < v_+^0(x) := v^0(x)$ of roots of $Q(v(t, x)) = Q(v^0(x))$. Therefore $v - \mu$ is away from 0 and there is a first time $\tau(x)$ such that $\varphi(\tau(x), x) = 0$ and $\varphi(t, x) < 0$ for $t \neq \tau(x)$. This time $\tau(x)$ is also the jump time from one branch to the other for v as stated in (i).

We can also compute, from the above equation on $\ln v$, and this gives (ii), namely

$$u_\varepsilon = -\partial_t \ln v_\varepsilon \rightharpoonup u = [\ln v_+(x) - \ln v_-(x)]\delta(t - \tau(x)).$$

The time $\tau(x)$ is fully identified from the limiting system: By integrating the equation on $\varphi(t, x)$, we find that

$$\varphi(t, x) = \begin{cases} \varphi(0, x) + t\,(v^0(x) - \mu), & t < \tau(x), \\ \varphi(0, x) + \tau(x)(v^0(x) - \mu) + (t - \tau(x))\,(v_-^0(x) - \mu), & t > \tau(x). \end{cases}$$

For $x > 0$, as $v_-^0(x) < \mu < v^0(x)$, $\varphi(t, x)$ is strictly increasing in time for $t < \tau(x)$, and strictly decreasing for $t > \tau(x)$.

Furthermore by the second equation, we have that $\varphi(\tau(x), x) = 0$. Hence this characterizes $\tau(x)$ as

$$\tau(x) = -\frac{\varphi^0(x)}{v^0(x) - \mu}.$$

This gives (iii) and identifies completely the limiting solution x by x.

$\square$

3 The full problem

When diffusion is included, the previous analysis, which uses fundamentally x by x convergence, does not apply and the assumptions have to take into account the diffusion term. We also make more general assumptions. For the initial data, we assume that

$$u_\varepsilon^0 > 0, \qquad \varepsilon u_\varepsilon^0 \le C, \qquad \int_{\mathbb{R}} u_\varepsilon^0 dx \le C, \tag{9}$$

and, $\varphi_\varepsilon^0 = \varepsilon \ln u_\varepsilon^0$ satisfies

$$\varepsilon \partial_{xx}^2 \varphi_\varepsilon^0 \le C, \qquad |\partial_x \varphi_\varepsilon^0| \le C, \qquad \varphi_\varepsilon^0(x) \ge -C(1+|x|), \tag{10}$$

$$\varepsilon |\partial_{xx}^2 \ln v_\varepsilon^0| \le C, \qquad |\partial_x \ln v_\varepsilon^0| + |\ln v_\varepsilon^0| \le C. \tag{11}$$

Then, recalling the definition $Q(v) = v - \mu \ln v$, we also use

$$w = \ln v, \qquad \widetilde{Q}(w) = Q(v) = e^w - \mu w, \tag{12}$$

and we can define uniquely two smooth branches of initial data $w_{\pm,\varepsilon}^0$ by

$$\widetilde{Q}(w_\varepsilon^0) = \widetilde{Q}(w_{-,\varepsilon}^0) = \widetilde{Q}(w_{+,\varepsilon}^0), \qquad w_{-,\varepsilon}^0 \le \ln \mu \le w_{+,\varepsilon}^0. \tag{13}$$

Note that assumption (11) provides corresponding bounds on w_ε^0 and $w_{\pm,\varepsilon}^0$, which are

$$|w_\varepsilon^0| + |w_{\pm,\varepsilon}^0| \le C, \quad |\partial_x w_\varepsilon^0| + |\partial_x w_{\pm,\varepsilon}^0| \le C, \quad \varepsilon |\partial_{xx}^2 w_\varepsilon^0| + \varepsilon |\partial_{xx}^2 w_{\pm,\varepsilon}^0| \le C. \tag{14}$$

And we also use, for an unessential result, the stronger condition

$$\varepsilon \partial_{xx}^2 (w_\varepsilon^0 - w_{-,\varepsilon}^0) + u_\varepsilon^0 \le C. \tag{15}$$

Finally, we use the notation $|W|_- = \max(0, -W)$.

Our main theorem now reads

Theorem 2 *Assume that $u_\varepsilon^0 \in C^2 \cap L^1(\mathbb{R})$ and that assumptions (9)–(11) hold. The solution of (1) has limits $u_\varepsilon \rightharpoonup u$ (weak measures) and $v_\varepsilon \to v$ (strongly in L^p_{loc}) and, there is a time $\tau(x)$ and $v_-^0(x) < \mu < v^0(x)$ such that*
(i) $v(t,x) = v_-^0(x)$ for $t < \tau(x)$, $\qquad v(t,x) = v^0(x)$ for $t > \tau(x)$,
(ii) $u(t,x) = [\ln v^0(x) - \ln v_-^0(x)]\delta(t - \tau(x))$.

Compared to the case without diffusion in Theorem 1, there is no simple explicit formula for the jump time $\tau(x)$. However, here it is also characterized by $\varphi(\tau(x), x) = 0$ for the solution of an Eikonal equation.

The end of this section is devoted to the proof of Theorem 1.

Our analysis is based on the observation that $\partial_t \ln v_\varepsilon = -u_\varepsilon$, thus, from (1), we get the identity

$$\partial_t[\varepsilon u_\varepsilon + v_\varepsilon - \mu \ln v_\varepsilon + \varepsilon^2 \partial_{xx}^2 \ln v_\varepsilon] = 0,$$

and consequently

$$\varepsilon \partial_t \ln v_\varepsilon - \varepsilon^2 \partial_{xx}^2 \ln v_\varepsilon - Q(v_\varepsilon) = -Q(v_\varepsilon^0) - \varepsilon u_\varepsilon^0 - \varepsilon^2 \partial_{xx}^2 \ln v_\varepsilon^0,$$

which we can write in terms of w_ε as

$$\varepsilon \partial_t w_\varepsilon - \varepsilon^2 \partial_{xx}^2 w_\varepsilon - \widetilde{Q}(w_\varepsilon) = -\widetilde{Q}(w_\varepsilon^0) - \varepsilon u_\varepsilon^0 - \varepsilon^2 \partial_{xx}^2 w_\varepsilon^0. \tag{16}$$

Notice that this is a monostable equation of Fisher/KPP type, where the steady states depend on x.

A priori estimates on v_ε.

Lemma 3 *The inequalities hold*

$$0 < C \leq w_\varepsilon(t,x) \leq w_{+,\varepsilon}^0, \tag{17}$$

and for any R, there is a constant C_R such that,

$$\int_{|x| \leq R} |\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w_\varepsilon^0)| \mathbb{1}_{\{w_\varepsilon - w_{-,\varepsilon}^0 \leq 0\}} \, dx \leq C_R \, \varepsilon, \qquad \forall t \geq 0. \tag{18}$$

Finally, with the additional assumption (15), *we have* $-C\sqrt{\varepsilon} + w_{-,\varepsilon}^0 \leq w_\varepsilon(t,x)$.

Proof. For the first statement, on the one hand, we note that we necessarily have that for any x, $w_\varepsilon^0(x) = w_{-,\varepsilon}^0(x)$ or $w_\varepsilon^0(x) = w_{+,\varepsilon}^0(x)$. In both cases, that implies that $w_\varepsilon^0(x) \leq w_{+,\varepsilon}^0(x)$. Since v_ε is non-increasing in time, so is w_ε and

$$w_\varepsilon(t,x) \leq w_\varepsilon^0(x) \leq w_{+,\varepsilon}^0(x).$$

On the other hand, as $\partial_t w_\varepsilon \leq 0$, we can deduce from (16), that

$$-\varepsilon^2 \partial_{xx}^2(w_\varepsilon - w_{-,\varepsilon}^0) + [\widetilde{Q}(w_{-,\varepsilon}^0) - \widetilde{Q}(w_\varepsilon)] \geq -\varepsilon^2 \partial_{xx}^2(w_\varepsilon^0 - w_{-,\varepsilon}^0) - \varepsilon u_\varepsilon^0. \tag{19}$$

Using the maximum principle and assumptions (9), (11), we conclude that at a minimum value of $w_\varepsilon - w_{-,\varepsilon}^0$, the quantity $\widetilde{Q}(w_{-,\varepsilon}^0) - \widetilde{Q}(w_\varepsilon)$ is controlled from below and the lower bound on w_ε follows.

Furthermore, from the usual convex inequalities, we also observe that

$$-\varepsilon^2 \partial_{xx}^2(w_\varepsilon - w_{-,\varepsilon}^0)_- + [\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w_{-,\varepsilon}^0)] \mathbb{1}_{\{w_\varepsilon - w_{-,\varepsilon}^0 < 0\}} \leq \varepsilon \, u_\varepsilon^0 + \varepsilon^2 |\partial_{xx}^2(w_\varepsilon^0 - w_{-,\varepsilon}^0)|. \tag{20}$$

Because the set $\{W \, s.t. \, W - w_{-,\varepsilon}^0 < 0\}$ lies in the decreasing branch of $\widetilde{Q}$, the quantity $[\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w_{-,\varepsilon}^0)]\mathbb{1}_{\{w_\varepsilon - w_{-,\varepsilon}^0 < 0\}}$ is positive. We integrate against some smooth non-negative ψ with $\psi = 1$ on $B(0,R)$ and ψ compactly supported in $B(0,2R)$. Since we have already proved that $w_\varepsilon - w_{-,\varepsilon}^0$ is bounded, we obtain the estimate

$$\int_{|x| \leq R} |\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w_\varepsilon^0)| \mathbb{1}_{\{w_\varepsilon - w_{-,\varepsilon}^0 \leq 0\}} \, dx \leq C_R \, \varepsilon^2 + \varepsilon \int_{|x| \leq R} u_\varepsilon^0 \, dx + \varepsilon^2 \int_{|x| \leq R} |\partial_{xx}^2(w_\varepsilon^0 - w_{-,\varepsilon}^0)| \, dx.$$

Using assumption (9) and assumption (11) concludes the second point of the lemma.

We may also use the specific assumption (15) in (20), we obtain that

$$-\varepsilon^2 \partial_{xx}^2(w_\varepsilon - w_{-,\varepsilon}^0)_- + [\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w_{-,\varepsilon}^0)] \mathbb{1}_{\{w_\varepsilon - w_{-,\varepsilon}^0 < 0\}} \leq C\varepsilon.$$

Recalling that $[\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w_{-,\varepsilon}^0)]\mathbb{1}_{\{w_\varepsilon - w_{-,\varepsilon}^0 < 0\}}$ is positive, we conclude that

$$Q(w_\varepsilon) - \widetilde{Q}(w_{-,\varepsilon}^0) \mathbb{1}_{\{w_\varepsilon - w_{-,\varepsilon}^0 < 0\}} \leq C\varepsilon,$$

and thus the third statement of Lemma 3 holds, namely $w_\varepsilon - w_{-,\varepsilon} \geq -C\sqrt{\varepsilon}$.

$\square$

Concentration dynamics of u_ε. We turn to the study of u_ε and begin with some simple estimates.

• Since $u_\varepsilon = -\partial_t w_\varepsilon$, and using the bound (17), we find

$$\sup_x \int_0^T u_\varepsilon(t,x)dt = \sup_x [w_\varepsilon^0(x) - w_\varepsilon(x,T)] \le C(T), \tag{21}$$

and thus, integrating the equation on u_ε, we also get

$$\varepsilon \int_{\mathbb{R}} u_\varepsilon(t,x)\,dx = \varepsilon \int_{\mathbb{R}} u_\varepsilon^0(x)\,dx + \int_{\mathbb{R}} \int_0^t u_\varepsilon(s,x)(v_\varepsilon(s,x) - \mu)ds\,dx \le C(t). \tag{22}$$

• Next, we use the Hopf-Cole transform

$$\varphi_\varepsilon(t,x) = \varepsilon \ln u_\varepsilon(t,x).$$

As usual, we compute that φ_ε satisfies the Eikonal equation

$$\partial_t \varphi_\varepsilon = \varepsilon \partial_{xx}^2 \varphi_\varepsilon + |\partial_x \varphi_\varepsilon|^2 + v_\varepsilon - \mu. \tag{23}$$

We are going to show some uniform bounds on φ_ε.

Lemma 4 *We have*

$$\partial_t \varphi_\varepsilon(t,x) \le C, \qquad \forall x \in \mathbb{R}, \ t \ge 0 \tag{24}$$

$$\|\partial_x \varphi_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \le C, \qquad \|\partial_t \varphi_\varepsilon\|_{L_{t,x}^p} \le C(p), \quad \forall p \in [1,\infty), \tag{25}$$

$$-C(t)(1+|x|) \le \varphi_\varepsilon(t,x) \le 2\,\varepsilon \ln \frac{1}{\varepsilon} + C(t)\,\varepsilon, \qquad \forall x \in \mathbb{R}, \ t \ge 0. \tag{26}$$

Proof. For the time derivative, differentiating (23) and using the equation on v_ε, we find

$$\partial_t(\partial_t \varphi_\varepsilon) = \varepsilon \partial_{xx}^2(\partial_t \varphi_\varepsilon) + 2\partial_x \varphi_\varepsilon \partial_x(\partial_t \varphi_\varepsilon) - u_\varepsilon v_\varepsilon \le \varepsilon \partial_{xx}^2(\partial_t \varphi_\varepsilon) + 2\partial_x \varphi_\varepsilon \partial_x(\partial_t \varphi_\varepsilon),$$

so that the maximum principle gives $\partial_t \varphi_\varepsilon(t,x) \le \max_x \partial_t \varphi_\varepsilon^0(x)$ which gives (24) thanks to the assumption (10).

Next, we prove the Lipschitz bound. Consider any point x_ε that is a maximum in x of $\partial_x \varphi_\varepsilon$ at any time t (standard arguments apply if the maximum is not achieved, see [7, 2]). Then $\partial_{xx}^2 \varphi_\varepsilon(x_\varepsilon) = 0$ and we conclude, still using (24), that

$$|\partial_x \varphi_\varepsilon(t,x_\varepsilon)|^2 = \partial_t \varphi_\varepsilon(t,x_\varepsilon) + v_\varepsilon - \mu \le C. \tag{27}$$

Once $\partial_x \varphi_\varepsilon \in L_{t,x}^\infty$ uniformly, standard parabolic estimates provide a uniform bound on $\partial_t \varphi_\varepsilon$ in $L_{t,x}^p$ for any $1 \le p < \infty$.

Finally, since φ_ε is uniformly Lipschitz in x, let x_ε be a maximum of φ_ε, then

$$\varphi_\varepsilon(t,x) \ge \max \varphi_\varepsilon(t,.) - C\,|x - x_\varepsilon|,$$

so that, using (22),

$$\frac{C}{\varepsilon} \ge \int_{\mathbb{R}} u_\varepsilon(t,x)\,dx \ge \int_{\mathbb{R}} e^{\max \varphi_\varepsilon(t,.)/\varepsilon}\, e^{-C\,|x-x_\varepsilon|/\varepsilon}\,dx \ge \frac{\varepsilon}{C}\, e^{\max \varphi_\varepsilon(t,.)/\varepsilon},$$

which proves the upper bound in (26). The lower bound relies, as it is standard [2, 7, 3], on the construction of a sub-solution. Here one can immediately check that $-C(t+1) - \frac{Cx^2}{\sqrt{1+|x|^2}}$ will work.

□

Compactness of v_ε. We introduce the quantity $\Phi(x, w)$, defined up to a constant, by

$$\Phi_w(x, w) = |\widetilde{Q}(w_\varepsilon^0(x)) - \widetilde{Q}(w)| \geq 0.$$

Lemma 5 *With assumptions* (11)–(9), *we have*

$$\sup_{x \in \mathbb{R},\, 0 \leq t \leq T} |\partial_x w_\varepsilon| \leq \frac{C_T}{\varepsilon}, \qquad \int_0^T \int_{|x| \leq R} \left| \partial_x \Phi(x, w_\varepsilon(t, x)) \right| \leq C_{T,R}.$$

Consequently, by monotonicity of Φ in w, v_ε is locally compact in $L^p((0, \infty) \times \mathbb{R})$ for any $1 \leq p < \infty$.

Remark 6 *It is also possible to conclude from this lemma that w_ε is uniformly bounded in $L^1([0,\,T],\, W^{\theta,1}(\mathbb{R}))$ for some $\theta > 0$; see Appendix B.*

Proof. First of all, calculate

$$\partial_t \partial_x w_\varepsilon = -\partial_x u_\varepsilon = -\frac{\partial_x \varphi_\varepsilon}{\varepsilon} u_\varepsilon,$$

which yields, from the Lipschitz bound on φ_ε in (27) and the estimate (21),

$$|\partial_x w_\varepsilon(t, x)| \leq |\partial_x w_\varepsilon^0(x)| + \frac{C}{\varepsilon} \int_0^t u_\varepsilon(s, x)\, ds \leq \frac{C(t)}{\varepsilon}.$$

Next, we write $\partial_x[\Phi(x, w_\varepsilon(t, x))] = \partial_x \Phi(x, w_\varepsilon) + [\widetilde{Q}(w_\varepsilon^0) - \widetilde{Q}(w_\varepsilon)]\partial_x w_\varepsilon$. Since $\partial_x \Phi(x, w_\varepsilon)$ is bounded in L^∞, and thanks to the second bound in Lemma 3, we conclude that

$$\int_0^T \int_{|x| \leq R} |\partial_x \Phi(x, w_\varepsilon)| \leq C_{T,R} + \varepsilon \sup_{|x| \leq R,\, 0 \leq t \leq T} |\partial_x w_\varepsilon| \int_0^T \int_{|x| \leq R} \left| \frac{Q(v_\varepsilon^0) - Q(v_\varepsilon)}{\varepsilon} \right| \leq C_{T,R}. \qquad (28)$$

Since $\partial_t w_\varepsilon \leq 0$ and w_ε bounded provide us with compactness in time, we conclude that $\Phi(x, w_\varepsilon)$ is compact and thus converges a.e. By monotonicity of Φ in w_ε, we also conclude that w_ε converges a.e. $\square$

Convergence as $\varepsilon \to 0$. We are now ready to study the limit as ε vanishes.
• The bounds in Lemma 4 show that φ_ε is locally compact in $C(\mathbb{R}_+ \times \mathbb{R})$ and hence, after extraction of a subsequence that we still denote by ε, there exists φ which is Lipschitz in space and with time derivatives in L^p such that

$$\|\varphi_\varepsilon - \varphi\|_{L^\infty((0,T) \times (-R,R))} \to 0, \quad \text{as } \varepsilon \to 0, \qquad \forall T > 0,\ R > 0.$$

We note that

$$-C(1 + t) - C|x| \leq \varphi(t, x) \leq 0. \qquad (29)$$

• From Lemma 5, we also conclude that v_ε converges locally; for any $1 \leq p < \infty$,

$$v_\varepsilon \to v, \quad w_\varepsilon \to w = \ln v \quad \text{in} \quad L^p((0, T) \times (-R, R)), \qquad \forall T > 0,\ R > 0.$$

• From the bound (21), we can also extract a subsequence such that, in the weak sense of measures,

$$u_\varepsilon \rightharpoonup u, \quad \text{in} \quad \mathcal{M}((0, T) \times (-R, R)), \qquad \forall T > 0,\ R > 0.$$

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We may pass to the limit, in distributional sense, in Equations ([1]) and get, as $\varepsilon \to 0$,

$$u_\varepsilon v_\varepsilon \rightharpoonup u\mu, \qquad \partial_t w = -u \qquad \partial_t v = \text{w-lim} \ (-u_\varepsilon v_\varepsilon) = -u\mu,$$

which expresses that the concentration of the measure u_ε is exactly at the point where $v_\varepsilon = \mu$. And from Lemma [3], we know that

$$\widetilde{Q}(w) = \widetilde{Q}(w^0), \quad i.e., \quad w(t,x) = w_-^0(x) \quad or \quad w_+^0(x).$$

Since w is non-decreasing in time, we conclude that, for all $x \in \mathcal{S}$, a subset of $\mathbb{R}$ where $w^0 = w_-^0$, there is a unique time $\tau(x)$ such that w jumps from $w_-^0(x)$ to $w_+^0(x)$ (with $\tau(x) = \infty$ for $x < 0$), and

$$u(t,x) = [w_+^0(x) - w_-^0(x)]\delta(t - \tau(x))\mathbb{1}_{\{x \in \mathcal{S}\}}, \qquad \varphi(\tau(x), x) = 0.$$

Open questions. Uniqueness for the limit problem, which we proved when diffusion is ignored (Section [2], is an open question in full generality. In particular it seems hard to determine more properties about the set $\mathcal{S}$, which depends on the initial data. In the monotone case when $w^0(x) = w_-^0(x)$ for $x < 0$ and $w^0(x) = w_+^0(x)$ for $x > 0$ with $u^0 = [[w^0]]\delta(x)$, one can expect $\tau(x)$ be invertible and to obtain a front located at some value $x = X(t)$.

A Traveling wave

Traveling waves are an intuitive way to understand, in a very particular case, the general behavior of system ([1]). considering solutions of the form $u_\varepsilon(x - \sigma t)$, $v_\varepsilon(x - \sigma t)$, $w_\varepsilon = \ln v_\varepsilon$. Recalling the notation $\widetilde{Q}(w) = e^w - \mu w$, we arrive at an equation on the single quantity $w_\varepsilon(y)$,

$$-\sigma\varepsilon w_\varepsilon' - \varepsilon^2 w_\varepsilon'' = \widetilde{Q}(w_\varepsilon) - A,$$

with the conditions at infinity

$$w_\varepsilon(-\infty) = w_- < \ln\mu, \qquad w_\varepsilon(+\infty) = w_+ > \ln\mu, \qquad A = \widetilde{Q}(w_-) = \widetilde{Q}(w_+).$$

This is just a Fisher/KPP monostable equation with w_+ the unstable state and we know from the general theory [13] that there is a traveling wave with minimal speed σ_* which is characterized by the property of a double root for the polynomial

$$-\sigma\varepsilon\lambda - \varepsilon^2\lambda^2 = \widetilde{Q}'(w_+),$$

that is $\sigma_* = 2\sqrt{\widetilde{Q}'(w_+)}$. In our analysis, this value σ_* also appears in the limit of Equation (23), that is the Eikonal equation

$$\partial_t \varphi = |\partial_x \varphi|^2 + v - \mu,$$

which for the traveling wave problem generates a solution $\varphi(x - \sigma t)$ with

$$\sigma \varphi'(y) = |\varphi'(y)|^2 + v_\pm(y) - \mu,$$

where $v_\pm(y) = v_-$ for $y < 0$ and $v_\pm(y) = v_+$ for $y > 0$. The limiting minimal speed traveling wave solution is

$$\varphi(y) = \begin{cases} p_- y < 0 & \text{for} \quad y < 0, \\ p_+ y < 0 & \text{for} \quad y > 0, \end{cases}$$

and p_+ is the double root of the polynomial $-\sigma_* \lambda - 2\lambda^2 = \widetilde{Q}'(w_+) = v_+ - \mu$. This approach based on the concentration as a Dirac measure of u_ε differs (but is restricted to one dimension) from the general front propagation theory in [1, 4] based on the quantity v_ε.

B   A Sobolev estimate

We may use the bound (28) to obtain Sobolev regularity on v_ε by controlling

$$\sup_{|h| \leq 1} \int_0^T \int_{|x| \leq R} \frac{|v_\varepsilon(x + h) - v_\varepsilon(x)|}{|h|^\theta} \, dx \, dt, \tag{30}$$

for some appropriate value of θ.

This requires to be a bit more precise on the set where the initial data v_ε crosses μ. Specifically, we assume that there exists some constants $C > 0$ and $\kappa > 0$ such that for any $\delta > 0$

$$|\{x, \ |v_\varepsilon^0(x) - \mu| \leq \delta\}| \leq C \, \delta^\kappa. \tag{31}$$

Observe that when $|h| \leq \varepsilon^{1/(1-\theta)}$ then by the Lipschitz bound on v_ε (which follows immediately from that on w_ε), then

$$\int_0^T \int_{|x| \leq R} \frac{|v_\varepsilon(x + h) - v_\varepsilon(x)|}{|h|^\theta} \, dx \, dt \leq C_{T,R} \|\partial_x v_\varepsilon\|_{L^\infty} |h|^{1-\theta} \ \leq C_R,$$

so that we can limit ourselves to $h \geq \varepsilon^{1/(1-\theta)}$.

For some $\alpha > 0$ which we later relate to θ and some constant C, denote

$$\Omega_- = \{(t, x) \in [0, \ T] \times B(0, R), \ w_{-,\varepsilon}^0 - |h|^\alpha \leq w_\varepsilon(t, x) \leq \ln \mu - |h|^{\alpha/2}\},$$

$$\Omega_+ = \{(t, x) \in [0, \ T] \times B(0, R), \ w_\varepsilon(t, x) \geq \ln \mu + |h|^{\alpha/2}\},$$

$$\Omega_0 = \{(t, x) \in [0, \ T] \times B(0, R), \ \ln \mu - |h|^{\alpha/2} \leq w_\varepsilon(t, x) \leq \ln \mu + |h|^{\alpha/2} \text{ or } w_\varepsilon(t, x) \leq w_{-,\varepsilon}^0 - |h|^\alpha\}.$$

Observe that when $(t, x) \in \Omega_0$ then

$$\sup_{\varepsilon^{1/(1-\theta)} \leq |h| \leq 1} \int_{(t,x) \in \Omega_0 \text{ or } (t,x+h) \in \Omega_0} \frac{|v_\varepsilon(t, x + h) - v_\varepsilon(t, x)|}{|h|^\theta} \, dx \, dt$$

$$\leq 2 \sup_{\varepsilon^{1/(1-\theta)} \leq |h| \leq 1} \frac{\|v_\varepsilon\|}{|h|^\theta} \int_{(t,x) \in \Omega_0 \text{ or } (t,x+h) \in \Omega_0} dx dt$$

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We note that if $w_\varepsilon(t, x) \leq w^0_{-,\varepsilon}(x) - |h|^\alpha$ then

$$|\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)| \geq |h|^\alpha/C.$$

Similarly if $\ln \mu - |h|^{\alpha/2} \leq w_\varepsilon(t, x) \leq \ln \mu + |h|^{\alpha/2}$ but $w^0_\varepsilon(x) < \ln \mu - 2\,|h|^{\alpha/2}$ or $w^0_\varepsilon(x) > \ln \mu + 2|h|^{\alpha/2}$, then we have again

$$|\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)| \geq |h|^\alpha/C,$$

as $\widetilde{Q}(w)$ has a minimum at $w = \ln \mu$ but is strictly convex.

Hence by (31)

$$|\Omega_0| \leq \frac{C}{|h|^\alpha} \int_0^T \int_{B(0,R)} |\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)| \, dx \, dt + \{x, \, |w^0_\varepsilon(x) - \ln \mu| \leq 2\,|h|^{\alpha/2}\}$$

$$\leq \frac{C}{|h|^\alpha} \int_0^T \int_{B(0,R)} |\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)| \, dx \, dt + C\,|h|^{\kappa\,\alpha/2}.$$

We therefore obtain that

$$\sup_{\varepsilon^{1/(1-\theta)} \leq |h| \leq 1} \int_{(t,x) \in \Omega_0 \text{ or } (t,x+h) \in \Omega_0} \frac{|v_\varepsilon(t, x+h) - v_\varepsilon(t, x)|}{|h|^\theta} \, dx \, dt$$

$$\leq \sup_{\varepsilon^{1/(1-\theta)} \leq |h| \leq 1} \frac{C}{|h|^\theta} \frac{1}{|h|^\alpha} \int_0^T \int_{B(0,R)} |\widetilde{Q}(w_\varepsilon) - \widetilde{Q}(w^0_\varepsilon)| \, dx \, dt + C \frac{|h|^{\kappa\,\alpha/2}}{|h|^\theta} \leq C_{T,R},$$

by Lemma 3 and provided that $\kappa\,\alpha/2 \geq \theta$ leading us to take $\alpha = 2\,\theta/\kappa$ and $(\alpha + \theta)/(1 - \theta) = (1 + 2/\kappa)\,\theta/(1-\theta) \leq 1$. It is always possible to satisfy these inequalities provided that $\theta/(1-\theta) \leq \frac{1}{1+2/\kappa}$.

We can consequently exclude the case where $(t, x) \in \Omega_0$ or $(t, x + h) \in \Omega_0$ when bounding (30).

The BV bound (28) also directly controls the case where $(t, x) \in \Omega_-$ and $(t, x + h) \in \Omega_+$ (or vice-versa). Indeed in that case, we necessarily have that $w^0_{+,\varepsilon} \geq w^0_{-,\varepsilon} + 2\,|h|^{\alpha/2}$ and $\partial_w \Phi(x, w_\varepsilon) = \widetilde{Q}(w^0_\varepsilon) - \widetilde{Q}(w_\varepsilon)$ has a sign between $w_{-,\varepsilon}$ and $w_{+,\varepsilon}$ so that there exists a constant C s.t. (again at least locally)

$$|\Phi(x, w^0_{-,\varepsilon}) - \Phi(x, w^0_{+,\varepsilon})| \geq \frac{|h|^{3\,\alpha/2}}{C}.$$

By our definitions of Ω_- and Ω_+ and taking C large enough, this implies that

$$|\Phi(x, w_\varepsilon(t, x+h)) - \Phi(x, w_\varepsilon(t, x))| \geq \frac{|h|^{3\,\alpha/2}}{C} \quad \text{if } (t, x) \in \Omega_- \text{ and } (t, x + h) \in \Omega_+.$$

Therefore

$$\int_{(t,x) \in \Omega_- \text{ and } (t,x+h) \in \Omega_+} \frac{|v_\varepsilon(t, x+h) - v_\varepsilon(t, x)|}{|h|^\theta} \, dx \, dt \leq 2 \frac{\|v_\varepsilon\|}{|h|^\theta} \int_{(t,x) \in \Omega_- \text{ and } (t,x+h) \in \Omega_+} dx$$

$$\leq \frac{C}{|h|^{\theta + 3\alpha/2}} \int_0^T \int_{B(0,R)} |\Phi(x, w_\varepsilon(t, x+h)) - \Phi(x, w_\varepsilon(t, x))| \, dx \, dt \leq C_{T,R} \, |h|^{1 - \theta - 3\alpha/2},$$

by (28). This is bounded as long as $\theta + 3\alpha/2 = \theta\,(1 + 3/\kappa) \leq 1$.

We are finally able to focus on the case where for example both $(t, x) \in \Omega_-$ and $(t, x + h) \in \Omega_-$. Note again that $\partial_w \Phi(x, w_\varepsilon) = \widetilde{Q}(w^0_\varepsilon) - \widetilde{Q}(w_\varepsilon)$ vanishes once with a change of sign at $w_\varepsilon = w_{-,\varepsilon}$ for $(t, x) \in \Omega_-$. Therefore $w \to \Phi(x, w)$ is injective on $w \in [w^0_{-,\varepsilon}, \ln \mu - |h|^{\alpha/2}]$ for some constant C and

$$|w_1 - w_2|^3 \leq C\,|\Phi(x, w_1) - \Phi(x, w_2)|, \quad w_1, w_2 \in [w^0_{-,\varepsilon}, \ln \mu - |h|^{\alpha/2}].$$

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Since $w_\varepsilon \geq w^0_{-,\varepsilon} - |h|^\alpha$ on Ω_-, this implies that for both $(t,x) \in \Omega_-$ and $(t,x+h) \in \Omega_-$, we have

$$|w_\varepsilon(t,x) - w_\varepsilon(t,x+h)|^3 \leq C\,|\Phi(x,w_\varepsilon(t,x)) - \Phi(x,w_\varepsilon(t,x+h))| + C\,\varepsilon^\alpha.$$

By a straightforward Hölder inequality, we get

$$\int_{(t,x)\in\Omega_- \text{ and } (t,x+h)\in\Omega_+} \frac{|v_\varepsilon(t,x+h) - v_\varepsilon(t,x)|}{|h|^\theta}\,dx\,dt$$

$$\leq C_{T,R} \left(\int_{(t,x)\in\Omega_- \text{ and } (t,x+h)\in\Omega_-} \frac{|w_\varepsilon(t,x+h) - w_\varepsilon(t,x)|^3}{|h|^{3\theta}}\,dx\,dt \right)^{1/2}$$

$$\leq C_{T,R} \left(\int_{(t,x)\in\Omega_- \text{ and } (t,x+h)\in\Omega_-} \frac{|\Phi(x,w_\varepsilon(t,x)) - \Phi(x,w_\varepsilon(t,x+h))| + |h|^\alpha}{|h|^{3\theta}}\,dx\,dt \right)^{1/2} \leq C_{T,R},$$

by (28) again, provided that $3\,\theta \leq 1$ and $\alpha \geq 3\,\theta$. Since we chose $\alpha = 2\,\theta/\kappa$, this last inequality holds provided that $\kappa \leq 2/3$, which we can always impose.

To summarize, we have obtained the desired bound (30) provided that $\theta \leq 1/3$ and $\theta \leq 1/(1+3/\kappa)$ together with $\theta/(1-\theta) \leq 1/(1+2/\kappa)$.

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