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An accurate modelling of piezoelectric plates. Single-layered plate

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Abstract - The paper presents an efficient two-dimensional approach to piezoelectric plates in the framework of linear theory of piezoelectricity. The approximation of the through-the-thickness variations then considered accounts for the shear effects and a refinement of the electric potential. On using a variational formalism the fully electromechanically coupled plate equations are obtained for the generalized stress resultants as well as for the generalized electric inductions. The latter are, in fact, deduced from the conservative electric charge equation which plays a crucial role in the present model. The emphasis is also placed on the boundary conditions on the plate faces. The model is then used to examine some problems for the cylindrical bending of a single plate simply supported. In this configuration a number of situations are examined : a piezoelectric plate subject to (i) an applied electric potential, (ii) a surface density of force and (iii) a surface density of electric charge. The through-thickness distributions of electromechanical quantities (displacements, stresses, electric potential and displacement) are obtained and compared to the results provided by finite element simulations and by a simplified plate model without shear effects as well. A good agreement is observed between the results coming from the present plate model and finite element computations, which ascertains the effectiveness of the proposed approach to piezoelectric plates.

1. Introduction

The important economic and technical developments of piezoelectricity have attracted the attention of lot of interest in the research of theoretical and computational models of *piezoelectric composites*. Among interesting materials capable of being viable candidates for actuators or sensors, piezoelectric materials have received the most attention [1]. One of the key factors for this choice is that piezoelectric materials act either as *actuators* or *sensors* and relate electric signals directly to material strains or stresses and vice versa. Piezoelectric materials and especially piezoelectric composites are usefully utilized for multi-purpose devices or smart materials and numerous technological applications have been proposed, running from aerospace structures (shape control of large space antennas, active control of vibrations, etc.) [2,3] to miniature medical apparatus (micro-robots, pumps, micro-positioning devices, etc.) [4]. Nevertheless, it turns out that producing practically meaningful actuation or sensing capabilities significant piezoelectric materials should be included into the structures in the form of *laminated plates* such as bimorphs, piezoelectric layers joined to a non piezoelectric slab or sandwich structures or even embedded piezoelectric patches in elastic materials. A great deal of attention should be given to the piezoelectric plate as a basic component of multi-layered structures. In the present work, a special attention is then devoted to the *single piezoelectric plate* formulation.

Although quite a number of recent studies have been shown considerable progress toward establishing correct and efficient plate models along with their corresponding equations, a better modelling of electromechanical field distribution through the thickness coordinate becomes now a necessity for engineering. The objective of the present work is twofold (i) to construct an accurate model based on an *approximation of the elastic displacements and electric potential* as function of the thickness coordinate of the plate and (ii) to assess the *capability of the model* to describe the global plate response, local variations of both mechanical and electric variables, stresses as well as the limitations of the model. At this end, we propose an efficient plate model which accounts for *shearing effects* of the "sine" type. Such a refinement satisfies the vanishing condition of shear stress at the top and bottom faces of the plate [5]. Moreover, our piezoelectric plate approach is valid for any kind of electromechanical loads (surface density of force, electric potential or charges). The present version of plate model is quite complete in comparison to the most piezoelectric plate models considering only applied electric potential as load. A significant number of works have been devoted to piezoelectric plates. These works attempt to incorporate various representations of approximation through-the-thickness of laminated piezoelectric beams, plates and shells [6]. One of the first piezoelectric plate approach was the work of H.F. Tiersten [7] follows by R.D. Mindlin [8] and C.K. Lee [9]. The simplest model is based on the kinematic assumptions of the Love's first approximation including the electric degree of freedom. R.D. Mindlin has considered an expansion of the elastic displacements and electric potential as polynomial functions, the level of truncation of the expansion leading to the order of the plate theory. X.D. Zhang and C.T. Sun [10] have derived governing plate equations using the kinematic of the first order shear deformation theory assumption developed by E. Reissner and R.D. Mindlin for purely elastic plates. Some formulations of piezoelectric plates assume, *a priori*, that the normal component of the electric displacement is constant through the thickness [11]. As a consequence, the electric charge conservative law or the Gauss equation is dropped out. Nevertheless, it turns out that such an assumption is no longer satisfied in most situations and we must consider the approximation of the *electric charge equation*. Due to the limitation of the standard plate theory, it seems to be necessary to investigate more refined and efficient approach to piezoelectric plates.

In the present study we attempt of developing a consistent approach to piezoelectric plates based on approximate equations deduced from a generalized variational formulation. The latter involves the prescribed electromechanical loads on the plate boundary in a natural way. Moreover, from the variational formulation, equations for the generalized stress and electric charge or induction resultants are then obtained. The final set of two-dimensional effective equations governs the extensional (or membrane deformation), flexural and shear deformations coupled to the applied and induced electric potential. Various benchmark tests are then proposed in order to valid the present approach. Especially, differents situations with particular electromechanical loads are retained (i) *density of applied forces*, (ii) *applied electric potential* on the top and bottom faces of the plate and (iii) *applied surface density of electric charges* on both faces of the plate. In order to draw the attention on the capabilities of the model some comparisons to *finite element computations* for the identical situations performed directly on the 3D problem are proposed. In addition, the results are compared to those coming from a classical plate model based on the Love's first approximation (no shear effects), which underlines the performance of the present approach. The practical situation is the *cylindrical bending of simply supported piezoelectric plate* for which solutions to the plate equations are looked for as Fourier series.

The prerequisite of the piezoelectric formulation (variational principle, equations of conservation and constitutive laws) is briefly stated in the next section. The through-the-thickness approximation of the elastic displacements and electric potential is presented in Section 3 and some comments are also given. The equations for the two-dimensional approach to piezoelectric plate as well as the associated mechanical and electric boundary conditions are presented in Section 4. The problem of a single piezoelectric plate under cylindrical bending is given in Section 5 in the cases of applied electric charge and applied electric potential. In Section 6, the numerical results for different kinds of electromechanical loads and slenderness ratios are discussed, the results are also compared to those provided by the finite element simulations and by an elementary plate model. Finally, Section 7 is devoted to the discussion on the results and extensions of the model to piezoelectric laminated plates.

2. Formulation of piezoelectricity

In this section, we briefly recall the necessary ingredients of linear piezoelectricity and the associated variational formulation based on the *Hamilton's principle.* Assuming that the deformations are infinitesimal and electric field is small enough and there are no body forces, the Hamilton's principle can be stated as follows

$$\delta \int_{t_1}^{t_2} \int_{\Omega} \mathcal{L} dv dt + \int_{t_1}^{t_2} \delta W dt = 0 \quad . \tag{1}$$

In Eq.(1), the Lagrangian functional is given by

$$\mathcal{L} = \frac{1}{2}\rho \dot{u}_i \dot{u}_i - H(S_{ij}, E_i) \quad , \tag{2}$$

where u_i is the displacement component, ρ is the mass density, $H(S_{ij}, E_i)$ is called the electric enthalpy density function with $S_{ij} = u_{(i,j)} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the linear part of the strain tensor component and E_i is the electric field vector. For the linear piezoelectricity the enthalpy density function takes on the form [12]

$$H(S_{ij}, E_i) = \frac{1}{2}\sigma_{ij} S_{ij} - \frac{1}{2} D_i E_i \quad . \tag{3}$$

In Eq.(3), σ_{ij} and D_i represent the components of the stress tensor and electric displacement vector, respectively. Furthermore, in Eq.(1), the virtual work of the prescribed mechanical and electric quantities on the domain boundary is given by

$$\delta W = \int_{\partial\Omega} T_i \delta u_i dS + \int_{\partial\Omega} Q \delta\phi dS \quad . \tag{4}$$

The above virtual work involves the surface traction vector $\mathbf{T}$ and applied surface density of electric charge Q on the boundary $\partial\Omega$. The quantity ϕ is the electric potential. A further step in the simplification can be made by assuming a *quasi electrostatic approximation*, which allows for the electric field to be derived from the electric potential as follows

$$E_i = -\phi_{,i} \quad . \tag{5}$$

Moreover, it is also supposed that the piezoelectric material is a perfect isolator (no volumic electric charges) and that the magnetic field and magnetization have no influence. On using the classical integration by part and assuming the variations δu_i and $\delta\phi$ are arbitrary throughout the domain Ω, field equations in Ω are

$$\begin{aligned} \sigma_{ij,j} &= \rho\ddot{u}_i \quad , \\ D_{i,i} &= 0 \quad . \end{aligned} \tag{6}$$

The associated boundary conditions read as

$$\begin{aligned} \sigma_{ij}n_j &= T_i \quad \text{or} \quad u_i = \bar{u}_i \quad \text{on} \quad \partial\Omega \quad , \\ D_i n_i &= Q \quad \text{or} \quad \phi = \bar{\phi} \quad \text{on} \quad \partial\Omega \quad . \end{aligned} \tag{7}$$

The equations of field are completed by the constitutive equations. The latter for the linear piezoelectricity can be deduced from the following form for the enthalpy density function [12]

$$H(S_{ij}, E_i) = \frac{1}{2}C^E_{ijpq}S_{ij}S_{pq} - e_{ipq}E_iS_{pq} - \frac{1}{2}\epsilon^S_{ij}E_iE_j \quad , \tag{8}$$

where C^E_{ijpq}, e_{ipq} and ϵ^S_{ij} are called the elastic, piezoelectric and dielectric permittivity constants, respectively. Accordingly, the constitutive laws for linear piezoelectricity are

$$\begin{aligned} \sigma_{ij} &= \frac{\partial H}{\partial S_{ij}} = C^E_{ijpq}S_{pq} - e_{lij}E_l \quad , \\ D_i &= -\frac{\partial H}{\partial E_i} = e_{ipq}S_{pq} + \epsilon^S_{ij}E_j \quad . \end{aligned} \tag{9}$$

The set of Eqs(5), (6), (7) and (9) are the essentially basic equations of linear piezoelectricity which are going to be used in the following.

3. The plate approximation for displacement field and electric potential

It is commonly considered, in plate theory, an expansion of the displacement in power series of the thickness coordinate. Different refined models can be introduced according to the form of the expansion approximation. Other approaches to plates are based on an asymptotic theory of the full 3D problem [13]. In the present work, the displacement field and electric potential are assumed to be of the form

$$\begin{aligned} u_\alpha(x,y,z,t) &= U_\alpha(x,y,t) - zw_{,\alpha}(x,y,t) + f(z)\gamma_\alpha(x,y,t) \quad , \quad \alpha \in \{1,2\} \quad , \\ u_3(x,y,z,t) &= w(x,y,t) \quad , \\ \phi(x,y,z,t) &= \phi_0(x,y,t) + z\phi_1(x,y,t) + P(z)\phi_2(x,y,t) + g(z)\phi_3(x,y,t) \quad . \end{aligned} \tag{10}$$

It is important to discuss the above expansion in detail.
(i) In the case of purely elastic media, if $f(z) = 0$ we recover the classical theory of Love-Kirchhoff of elastic thin plates [14]. Particular forms of the function $f(z)$ give rise to different models which have been investigated by E. Reissner [15], S.A. Ambartsumian [16] or J.N. Reddy [17] to quote just only the most known approaches. In the first equation

of Eq.(10) U_α represents the *middle plane displacement components*, w the *deflection* and γ_α is associated with the *shearing effects*. The subscript α takes the value 1 or 2. All the functions are defined at the middle plane coordinate $(x, y, 0)$. In the present model, the through-thickness distribution of the shearing effect is approximated by a *trigonometric function*.
(ii) Insofar as the electric potential is concerned, the first two terms in Eq.(10) (linear part) hold for the influence of the *applied electric potential* on the plate faces. The third term can be referred as to the *induced electric potential* by the elastic deformation mediated by the *piezoelectric coupling*. The last term is due to the shearing effect through the piezoelectric coupling. As a consequence, we adopt the following functions

$$f(z) = \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right), \qquad g(z) = \frac{h}{\pi}\cos\left(\frac{\pi z}{h}\right), \qquad P(z) = z^2 - \left(\frac{h}{2}\right)^2 \quad , \tag{11}$$

where h is the plate thickness which is supposed to be uniform. The case of the purely elastic plates has been examined by M. Touratier in detail for the single and multi-layered plates [5]. Extension to elastic shells has been also considered [18]. In the classical theory of piezoelectric plates, the shearing effect is removed and the second order term in the approximation of the electric potential is often neglected. Nevertheless, most applications of the piezoelectric adaptive plates are based mainly on the first-order shear deformation assumption [10]. We are going to see the implications of the present approximation in the plate equations.

Boundary conditions - Three kinds of electromechanical conditions are considered on the plate boundaries.
(a) We assume that the plate undergoes a *force density per unit area* on the top face of the plate and perpendicular to this face.
(b) The second kind of boundary conditions that we are interested in is an *applied electric potential* on the top and bottom faces of the plate such as

$$\phi(x, y, z = \pm h/2, t) = V^{\pm}(x, y, t) \quad . \tag{12}$$

From Eq.(11), we note that $P(\pm h/2) = 0$ and $g(\pm h/2) = 0$. Then we deduce that

$$\phi_0 = \frac{1}{2}\left(V^+ + V^-\right) \quad \text{and} \quad \phi_1 = \frac{1}{h}\left(V^+ - V^-\right) \quad . \tag{13}$$

Accordingly, the unknown functions ϕ_0 and ϕ_1 are no longer arbitrary and they depend on the applied electric potential. For practical cases, it is more convenient to take $V^+ = +V$ and $V^- = -V$ so that we have $\phi_0 = 0$ and $\phi_1 = 2V/h$.
(c) An other possible electric boundary condition is *electric charges* imposed on the top and bottom faces of the plates. In this situation the electric boundary condition on the electric displacement is given by $[\![\mathbf{D}]\!] \cdot \mathbf{n} = q$, where $\mathbf{n}$ is the unit outward normal vector to the boundary and q is the surface density of electric charge. In the case of a plate geometry, the condition reads as

$$D_3\left(x, y, z = \pm h/2, t\right) = q\left(x, y, t\right) \quad . \tag{14}$$

Comments
(i) We should underline the boundary conditions associated with the electric field. This boundary condition deduced from the formulation of the Maxwell equations reads as

$[\![\mathbf{E}]\!] \times \mathbf{n} = \mathbf{0}$, which means that the tangential components of the electric field must be continuous through the interface [19]. In order to apply an electric potential to the plate faces, the latter are coated with thin metallic electrodes of negligeable thickness and playing no role mechanically. Moreover, it is assumed that the stresses and displacements are perfectly transmitted through the electrodes. Since in a conductor the electric field is zero (or the electric potential is constant), the boundary condition on the electric field can be written as $E_1 = E_2 = 0$ on the top and bottom faces of the piezoelectric plate.
(ii) It is worthwhile specifying that the surface density of electric charge $-q$ is applied on the bottom face while we have $+q$ on the top. Then, the boundary condition on the electric displacement is $-D_3 = -q$ at $z = -h/2$ and $D_3 = +q$ at $z = +h/2$, whence Eq.(14).

4. Plate equations

The plate equations are deduced by using the variational formulation presented in Section 2. By taking the approximation of the displacement field and electric potential as defined by Eq.(10), the dependency of the field (u_1, u_2, u_3, ϕ) upon the thickness coordinate z can be cancelled out by integrating over the plate thickness. The procedure leads, in a natural way, to the definition of the generalized stresses and electric charges or inductions. More precisely, the equations of motion and the associated boundary conditions are obtained by, first, substituting the approximation (10) into the variational principle (1)-(4) and, next, assuming independant variations of the unknown functions $((U_\alpha, w, \gamma_\alpha, \phi_A)\,;\ \alpha \in \{1,2\}$ and $A \in \{0,1,2,3\})$. After a straightforward algebras, the Hamiltonian's principle can be put in the sum of integrals

$$\int_{t_1}^{t_2} (\delta K - \delta U + \delta W_1 + \delta W_2)\, dt = 0 \quad . \tag{15}$$

The first part holds for the kinetic energy that we do not write down here since we deal only with static processes in the following. The second term in Eq.(15) is the variation of the internal force work defined on the middle plane surface Σ of the plate

$$\begin{aligned} \delta U = \int_\Sigma \{ & N_{\alpha\beta} \left(\delta U_\alpha\right)_{,\beta} - M_{\alpha\beta} \left(\delta w\right)_{,\alpha\beta} + \hat{M}_{\alpha\beta} \left(\delta\gamma_\alpha\right)_{,\beta} + \hat{Q}_\alpha \delta\gamma_\alpha \\ & + D_\alpha^{(0)} \left(\delta\phi_0\right)_{,\alpha} + D_\alpha^{(1)} \left(\delta\phi_1\right)_{,\alpha} + D_\alpha^{(2)} \left(\delta\phi_2\right)_{,\alpha} + D_\alpha^{(3)} \left(\delta\phi_3\right)_{,\alpha} \\ & + D_3^{(1)} \delta\phi_1 + D_3^{(2)} \delta\phi_2 + D_3^{(3)} \delta\phi_3 \} dS \quad . \end{aligned} \tag{16}$$

Where the generalized stresses and electric inductions are computed using the three dimensional stresses σ_{ij} and electric displacement D_i

$$\left(N_{\alpha\beta}, M_{\alpha\beta}, \hat{M}_{\alpha\beta}\right) = \int_{-h/2}^{+h/2} \left(1, z, f(z)\right) \sigma_{\alpha\beta} dz \quad , \tag{17}$$

$$\hat{Q}_\alpha = \int_{-h/2}^{+h/2} f'(z) \sigma_{\alpha 3} dz \quad , \tag{18}$$

$$\left(D_\alpha^{(0)}, D_\alpha^{(1)}, D_\alpha^{(2)}, D_\alpha^{(3)}\right) = \int_{-h/2}^{+h/2} \left(1, z, P(z), g(z)\right) D_\alpha dz \quad , \tag{19}$$

$$\left(D_3^{(1)}, D_3^{(2)}, D_3^{(3)}\right) = \int_{-h/2}^{+h/2} (1, P'(z), g'(z))\, D_3 dz \quad , \tag{20}$$

with $\alpha, \beta \in \{1, 2\}$ and $f'(z) = \frac{df(z)}{dz}$, $P'(z) = \frac{dP(z)}{dz}$, $g'(z) = \frac{dg(z)}{dz}$.
The last two terms in Eq.(15) denotes the variational works of the applied force densities and electric charges applied on the upper and lower faces of the plate as well as those applied to the lateral boundary of the same plate. These variational works take on the form

$$\delta W_1 = \int_\Sigma (f_\alpha \delta U_\alpha - p\delta w + \hat{m}_\alpha \delta\gamma_\alpha + q_1 \delta\phi_1)\, dS \quad , \tag{21}$$

$$\delta W_2 = \int_{\mathcal{C}} \left(F_\alpha \delta U_\alpha + T\delta w + C_\alpha \delta\gamma_\alpha - M_f\, (\delta w)_{,n}\right) d\ell - \sum_p Z_p \delta w_p \quad . \tag{22}$$

In Eq.(21), f_α and p are the surface force densities, $\hat{m}_\alpha$ is a surface moment density and q_1 is the surface electric charge density. In Eq.(22) F_α and T are lineic force densities, M_f and C_α are lineic torque densities defined along the plate contour. Z_p are transverse forces at the angular points of the edge boundary $\mathcal{C}$ of the plate, $\mathbf{n}$ is the unit normal to $\mathcal{C}$. Moreover, it has been assumed that there is no electric charge on the lateral plate boundary, especially because the dielectric constant of the piezoelectric material is much larger than the dielectric constant of the outside air for electric fields of the same order.

Remarks - In Eq.(21), concerning the applied electromechanical loads on the plate faces, the electric charge density q_1 is only considered, because the other generalized electric charges associated with the electric potential variations $\delta\phi_0$, $\delta\phi_2$ and $\delta\phi_3$ disappear. Indeed, these generalized electric charges can be connected with the electric boundary conditions on the top and bottom faces through the integration over the plate thickness as follows $(q_0, q_1, q_2, q_3) = [(1, z, P(z), g(z))\, D_3]_{-h/2}^{+h/2}$. Owing to $P(\pm h/2) = g(\pm h/2) = 0$ and the boundary conditions (14), therefore q_1 is only the non vanishing electric charge.

On using the variational calculus arguments, Eq.(15) along with the variations (16), (21) and (22) must be satisfied for any arbitrary variations $(\delta U_\alpha, \delta w, \delta\gamma_\alpha, \delta\phi_A)$; $\alpha \in \{1, 2\}$, $A \in \{0, 1, 2, 3\}$. After some cumbersome but straighforward computations we arrive at (static case)

$$\begin{cases} N_{\alpha\beta,\beta} + f_\alpha = 0 \quad , \\ M_{\alpha\beta,\alpha\beta} - p = 0 \quad , \\ \hat{M}_{\alpha\beta,\beta} - \hat{Q}_\alpha + \hat{m}_\alpha = 0 \quad , \end{cases} \tag{23}$$

and

$$\begin{cases} D_{\alpha,\alpha}^{(0)} = 0 \quad , \\ D_{\alpha,\alpha}^{(1)} - D_3^{(1)} + q_1 = 0 \quad , \\ D_{\alpha,\alpha}^{(2)} - D_3^{(2)} = 0 \quad , \\ D_{\alpha,\alpha}^{(3)} - D_3^{(3)} = 0 \quad . \end{cases} \tag{24}$$

The associated boundary conditions on the lateral plate contour $\mathcal{C}$ are

$$\left\{\begin{array}{lllllll} F_\alpha & = & N_{\alpha\beta}n_\beta & \text{or} & U_\alpha & \text{given} & , \\ T & = & \left(\tau_\alpha M_{\alpha\beta}n_\beta\right)_{,s} + n_\alpha M_{\alpha\beta,\beta} & \text{or} & w & \text{given} & , \\ M_f & = & n_\alpha M_{\alpha\beta}n_\beta & \text{or} & w_{,n} & \text{given} & , \\ C_\alpha & = & \hat{M}_{\alpha\beta}n_\beta & \text{or} & \gamma_\alpha & \text{given} & , \\ D_\alpha^{(A)}n_\alpha & = & 0 \quad (A \in \{0,1,2,3\}) & \text{or} & \phi_A & \text{given} & , \end{array}\right. \tag{25}$$

The vector $\boldsymbol{\tau}$ is the unit tangent vector to $\mathcal{C}$ and s is the curvilinear coordinate along the contour $\mathcal{C}$. In addition to Eq.(25), we have $[\![\tau_\alpha M_{\alpha\beta}n_\beta - Z_p]\!]_{A_p} = 0$ the condition at the angular points of the contour. The last equation of Eq.(25) are the boundary conditions along $\mathcal{C}$ associated with the electric quantities. In addition, the right hand side of $(25)_5$ is zero according to the above remarks, there is no electric charge and no electrode on the lateral surface of the plate.

Comments

(i) The first two Eq.(23) are similar to those of the Love-Kirchhoff first-order theory of thin plates, third equation represents the equation of the shearing effects. The set of equations (24) is, in fact, deduced from the conservation of the electric charge or the Gauss equation, these equations govern the generalized electric displacements or electric charges associated with the electric potential functions ϕ_A $(A \in \{0,1,2,3\})$ introduced in the third equation of the expansion (10). We note, in the second equation (24), the generalized electric charge due to the surface density of electric charge applied to the top and bottom faces of the plate.

(ii) In the case of an electric potential applied to the top and bottom faces of the plate, the functions ϕ_0 and ϕ_1 are no longer unknown, but they are related to the applied electric potential through (13). Therefore, the first two equations of Eq.(24) do not appear and the number of unknown functions as well as the equations are then reduced in this situation.

5. The plate constitutive laws

A - Applied electric charges - We restrict the study to constitutive laws for linear piezoelectricity (see Eqs.(9)) of materials with an orthotropic symmetry [12, 20]. At this end, we compute the generalized stress and electric inductions resultants defined by Eqs(17)-(20) by using the constitutive laws (9). The results can be put in the matrix form

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ D_3^{(1)} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & a_{31} \\ Q_{12} & Q_{22} & 0 & a_{32} \\ 0 & 0 & Q_{66} & 0 \\ a_{31} & a_{32} & 0 & f_{33} \end{bmatrix} \begin{bmatrix} S_1^{(0)} \\ S_2^{(0)} \\ S_6^{(0)} \\ \phi_1 \end{bmatrix} \quad , \tag{26}$$

The other constitutive laws take on the matrix form

$$
\begin{bmatrix} M_1 \\ M_2 \\ M_6 \\ \hat{M}_1 \\ \hat{M}_2 \\ \hat{M}_6 \\ D_3^{(2)} \\ D_3^{(3)} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 & d_{11} & d_{12} & 0 & R_{31} & r_{31} \\ & D_{22} & 0 & d_{12} & d_{22} & 0 & R_{32} & r_{32} \\ & & D_{66} & 0 & 0 & d_{66} & 0 & 0 \\ & & & \hat{D}_{11} & \hat{D}_{12} & 0 & \hat{R}_{31} & \hat{r}_{31} \\ & & & & \hat{D}_{22} & 0 & \hat{R}_{32} & \hat{r}_{32} \\ & & \text{(sym.)} & & & \hat{D}_{66} & 0 & 0 \\ & & & & & & P_{33} & \overline{P}_{33} \\ & & & & & & & \overline{\overline{P}}_{33} \end{bmatrix} \begin{bmatrix} S_1^{(1)} \\ S_2^{(1)} \\ S_6^{(1)} \\ S_1^{(2)} \\ S_2^{(2)} \\ S_6^{(2)} \\ \phi_2 \\ \phi_3 \end{bmatrix} , \tag{27}
$$

The shear and electric induction resultants are

$$
\begin{bmatrix} \hat{Q}_1 \\ \hat{Q}_2 \\ D_1^{(0)} \\ D_2^{(0)} \\ D_1^{(2)} \\ D_2^{(2)} \\ D_1^{(3)} \\ D_2^{(3)} \end{bmatrix} = \begin{bmatrix} \hat{A}_{55} & 0 & l_{15} & 0 & L_{15} & 0 & \overline{L}_{15} & 0 \\ & \hat{A}_{44} & 0 & l_{24} & 0 & L_{24} & 0 & \overline{L}_{24} \\ & & f_{11} & 0 & F_{11} & 0 & \overline{F}_{11} & 0 \\ & & & f_{22} & 0 & F_{22} & 0 & \overline{F}_{22} \\ & & & & B_{11} & 0 & \overline{B}_{11} & 0 \\ & \text{(sym.)} & & & & B_{22} & 0 & \overline{B}_{22} \\ & & & & & & \overline{\overline{B}}_{11} & 0 \\ & & & & & & & \overline{\overline{B}}_{22} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \phi_{0,1} \\ \phi_{0,2} \\ \phi_{2,1} \\ \phi_{2,2} \\ \phi_{3,1} \\ \phi_{3,2} \end{bmatrix} . \tag{28}
$$

At last, we have as well

$$
\begin{bmatrix} D_1^{(1)} \\ D_2^{(1)} \end{bmatrix} = \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix} \begin{bmatrix} \phi_{1,1} \\ \phi_{1,2} \end{bmatrix} . \tag{29}
$$

All the coefficients introduced in the matrices are defined in Appendix A. These coefficients depend on the material constants of the piezoelectric plate and its thickness.
In addition, the strain tensors which has been introduced in the above constitutive laws (26)-(27) are defined from Eq.(10) by

$$
S_{\alpha\beta}^{(0)} = U_{(\alpha,\beta)} \quad , \qquad S_{\alpha\beta}^{(1)} = -w_{,\alpha\beta} \quad , \qquad S_{\alpha\beta}^{(2)} = \gamma_{(\alpha,\beta)} \quad . \tag{30}
$$

It is worthwhile observing that we have electromechanical couplings between some generalized stress and electric induction resultants.

B - Applied electric potential - In the case of an applied electric potential the functions ϕ_0 and ϕ_1, in the approximation (10) are no longer arbitrary and they are given by $\phi_0 = 0$ and $\phi_1 = 2V/h$. Consequently, the constitutive law defined by Eq.(26) must be replaced by

$$
\begin{bmatrix} N_1 \\ N_2 \\ N_6 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} S_1^{(0)} \\ S_2^{(0)} \\ S_6^{(0)} \end{bmatrix} + \begin{bmatrix} e_{31}^* \\ e_{32}^* \\ 0 \end{bmatrix} 2V \quad . \tag{31}
$$

The effective piezoelectric constants e_{31}^* and e_{32}^* are given in Appendix A. The constitutive law (27) is unchanged. However, in Eq.(28) the lines and columns corresponding to the function ϕ_0 disappear and the matrix in Eq.(28) is of the 6×6 order, in addition, Eq.(29) is not considered. We note that, according to Eq.(31), if an electric potential is applied on the top and bottom faces of the plate, an elongation or compression of the plate will be produced only.

6. Piezoelectric plate in cylindrical bending

Now, we intend to solve the problem of piezoelectric plate undergoing applied surface force density or surface electric charge or electric potential in the *cylindrical bending* configuration. In addition, the shear traction is zero on the plate faces ($f_\alpha = 0$) and there is no surface moment density ($\hat{m}_\alpha = 0$). The *simple support conditions* for a rectangular plate of length L are simulated by (see Fig.1)

$$\begin{array}{lllll} \sigma_{11}(0,z) & = & \sigma_{11}(L,z) & = & 0 \quad , \\ \sigma_{13}(0,z) & = & \sigma_{13}(L,z) & = & 0 \quad , \\ u_3(0,z) & = & u_3(L,z) & = & 0 \quad . \end{array} \tag{32}$$

It is noticed that the boundary conditions along the contour $\mathcal{C}$ are obviously satisfied in the cylindrical bending configuration.

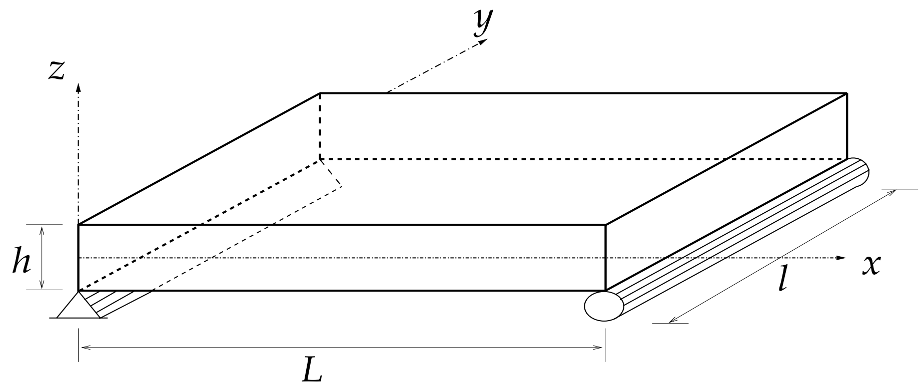

Fig.1 : *Piezoelectric plate on simple supports*

All the stresses, strains, displacements, electric field and potential do not depend on the y variable and the displacement u_2 plays no role in the problem, it can be cancelled out and we set $u_2 = 0$, $\gamma_2 = 0$. With a view toward satisfying the boundary conditions (32) the load functions are expanded in a Fourier series over the segment $[0, L]$. Consequently, the surface density of force, surface density of electric charge and electric potential applied to the plate faces can be expressed in the form

$$(p(x), Q(x), V(x)) = \sum_{n=1}^{\infty} (S_n, Q_n, V_n) \sin \lambda_n x \quad , \tag{33}$$

with

$$\lambda_n = n\pi/L, \qquad (S_n, Q_n, V_n) = \frac{4}{n\pi}(S_0, Q_0, V_0) \quad . \tag{34}$$

Such loads defined by the above equations represent uniform applied surface density of force S_0, density of electric charge Q_0 and electric potential V_0, respectively. On account of the load functions (33) and the boundary conditions of the cylindrical bending of a plate simply supported (32), it is natural to search for a solution to the plate problem given by Eqs(23)-(24) along with the constitutive laws (26)-(29) as also Fourier series as follows

$$\begin{aligned} (U_1(x), \gamma_1(x)) &= \sum_{n=1}^{\infty} (U_n, \Gamma_n) \cos \lambda_n x \quad , \\ (w(x), \phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x)) &= \sum_{n=1}^{\infty} (W_n, \Phi_{0,n}, \Phi_{1,n}, \Phi_{2,n}, \Phi_{3,n}) \sin \lambda_n x \quad . \end{aligned} \tag{35}$$

We should recall that, in the case of an applied electric potential, the number of unknown functions is reduced, especially, ϕ_0 and ϕ_1 are not accounted for. In this situation, the applied electric charge $Q(x)$ is not considered as a load.

Now, we have all the ingredients in view of solving the cylindrical bending of a piezoelectric plate simply supported. The Fourier coefficients in the series (35) are determined by first substituting the solution (35) into the constitutive laws (26)-(29) and the results into the plate equations (23)-(24). The Fourier coefficients are then solution to a set of linear algebraic equations for each n, which can be written in matrix form.

A - Applied surface density of force and/or electric charge. The set of linear equations for the Fourier coefficients takes on the form

$$A_n^Q X_n^Q = B_n^Q \quad , \tag{36}$$

with the matrix and vectors

$$A_n^Q =$$

$$\begin{bmatrix} -\lambda_n^2 C_{11}^* & 0 & 0 & 0 & \lambda_n e_{31}^* & 0 & 0 \\ & -\frac{\lambda_n^4}{12} C_{11}^* & \frac{\lambda_n^3}{\pi^3} C_{11}^* & 0 & 0 & -\frac{\lambda_n^2}{6} e_{31}^* & \frac{2\lambda_n^2}{\pi^2} e_{31}^* \\ & & -\frac{1}{2}\left(C_{55}^* + \frac{\lambda_n^2}{\pi^2} C_{11}^*\right) & \frac{2\lambda_n}{\pi} e_{15}^* & 0 & \frac{4\lambda_n}{\pi^3}\left(e_{31}^* + e_{15}^*\right) & -\frac{\lambda_n}{2\pi}\left(e_{31}^* + e_{15}^*\right) \\ & & & \lambda_n^2 \epsilon_{11}^* & 0 & -\frac{\lambda_n^2}{6} \epsilon_{11}^* & \frac{2\lambda_n^2}{\pi^2} \epsilon_{11}^* \\ & & & & \epsilon_{33}^* + \frac{\lambda_n^2}{12} \epsilon_{11}^* & 0 & 0 \\ & & \text{(sym.)} & & & \frac{1}{3}\left(\epsilon_{33}^* + \frac{\lambda_n^2}{10} \epsilon_{11}^*\right) & -\frac{4}{\pi^2}\left(\epsilon_{33}^* + \frac{\lambda_n^2}{\pi^2} \epsilon_{11}^*\right) \\ & & & & & & \frac{1}{2}\left(\epsilon_{33}^* + \frac{\lambda_n^2}{\pi^2} \epsilon_{11}^*\right) \end{bmatrix}$$

$$[X_n^Q]^T = (U_n, W_n, \Gamma_n, \Phi_{0,n}, \Phi_{1,n}, \Phi_{2,n}, \Phi_{3,n}) \quad ,$$

$$[B_n^Q]^T = (0, S_n, 0, 0, -Q_n, 0, 0) \quad .$$

B - Applied surface density of force and/or electric potential. In this case the set of linear algebraic equations is simpler since the number of equations is thus reduced

$$A_n^V X_n^V = B_n^V \quad , \tag{37}$$

with the matrix and vectors

$$A_n^V = \begin{bmatrix} -\lambda_n^2 C_{11}^* & 0 & 0 & 0 & 0 \\ & -\frac{\lambda_n^4}{12} C_{11}^* & \frac{\lambda_n^3}{\pi^3} C_{11}^* & -\frac{\lambda_n^2}{6} e_{31}^* & \frac{2\lambda_n^2}{\pi^2} e_{31}^* \\ & & -\frac{1}{2}\left(C_{55}^* + \frac{\lambda_n^2}{\pi^2} C_{11}^*\right) & \frac{4\lambda_n}{\pi^3}\left(e_{31}^* + e_{15}^*\right) & -\frac{\lambda_n}{2\pi}\left(e_{31}^* + e_{15}^*\right) \\ & \text{(sym.)} & & \frac{1}{3}\left(\epsilon_{33}^* + \frac{\lambda_n^2}{10}\epsilon_{11}^*\right) & -\frac{4}{\pi^2}\left(\epsilon_{33}^* + \frac{\lambda_n^2}{\pi^2}\epsilon_{11}^*\right) \\ & & & & \frac{1}{2}\left(\epsilon_{33}^* + \frac{\lambda_n^2}{\pi^2}\epsilon_{11}^*\right) \end{bmatrix}$$

$$\begin{aligned} [X_n^V]^T &= (U_n, W_n, \Gamma_n, \Phi_{2,n}, \Phi_{3,n}) \quad , \\ [B_n^V]^T &= (-2\lambda_n e_{31}^* V_n, S_n, 0, 0, 0) \quad . \end{aligned}$$

In both cases the matrices possess real elements and are symmetric. Now the way of solving the plate problem is straightforward, first solve Eq.(36) or (37) to find the Fourier coefficients, then substitute the results into the Fourier series (35), go back to the displacement field and electric potential by means of Eq.(10). Afterward, the stresses and electric displacement are computed through the constitutive laws of linear piezoelectricity specialized for orthotropic materials.

7. Numerical investigations and comparisons

Numerical simulations of the present plate model are considered for a single plate made of PZT-4 ceramics whose nonzero material constants are listed in Table 1 [20]. The geometry of the plate is $h = 0.001$m and two slenderness ratios are considered $L/h = 10$ and $L/h = 50$. The numerical results for the mechanical and electric quantities are given with the following dimensionless units.
(i) for a surface density of normal force $S_0 \neq 0$ ($S_0 = 1000\,N/m^2$), we set

$$(U, W, \Phi) = \frac{C_{11}^E}{hS_0}\left(u_1, u_3, \phi/E_0\right), \quad (T_{ij}, \mathcal{D}_l) = \frac{1}{S_0}\left(\sigma_{ij}, E_0 D_l\right) \quad ,$$

(ii) for a surface density of electric charge $Q_0 \neq 0$ ($Q_0 = 10\,C/m^2$), we set

$$(U, W, \Phi) = \frac{C_{11}^E}{hE_0 Q_0}\left(u_1, u_3, \phi/E_0\right), \quad (T_{ij}, \mathcal{D}_l) = \frac{1}{E_0 Q_0}\left(\sigma_{ij}, E_0 D_l\right) \quad ,$$

(iii) for an applied electric potential $V_0 \neq 0$ ($V_0 = 50$ volts), we have

$$(U, W, \Phi) = \frac{E_0}{V_0}\left(u_1, u_3, \phi/E_0\right), \quad (T_{ij}, \mathcal{D}_l) = \frac{hE_0}{C_{11}^E V_0}\left(\sigma_{ij}, E_0 D_l\right) \quad .$$

For the numerical simulations we take $E_0 = 10^{10}$ volts/m. Moreover, the number of terms retained in series (35) are adjusted according to the slenderness ratios and electromechanical loads then considered in order to ensure the convergence. The finite element computations for comparison are performed with ABAQUS code by using plane strain elements of 8-node biquadratic type and 800 elements are considered for both $L/h = 10$ and $L/h = 50$.

	C_{11}^E (GPa)	C_{12}^E	C_{33}^E	C_{13}^E	C_{44}^E	e_{31} (C/m^2)	e_{33}	e_{15}	ϵ_{11} (nF/m)	ϵ_{33}
PZT-4	139.	77.8	115.	74.3	25.6	-5.2	15.1	12.7	13.06	11.51

Table 1: *Independent elastic, piezoelectric and dielectric constants of a piezoelectric material (transversely isotropic symmetry).*

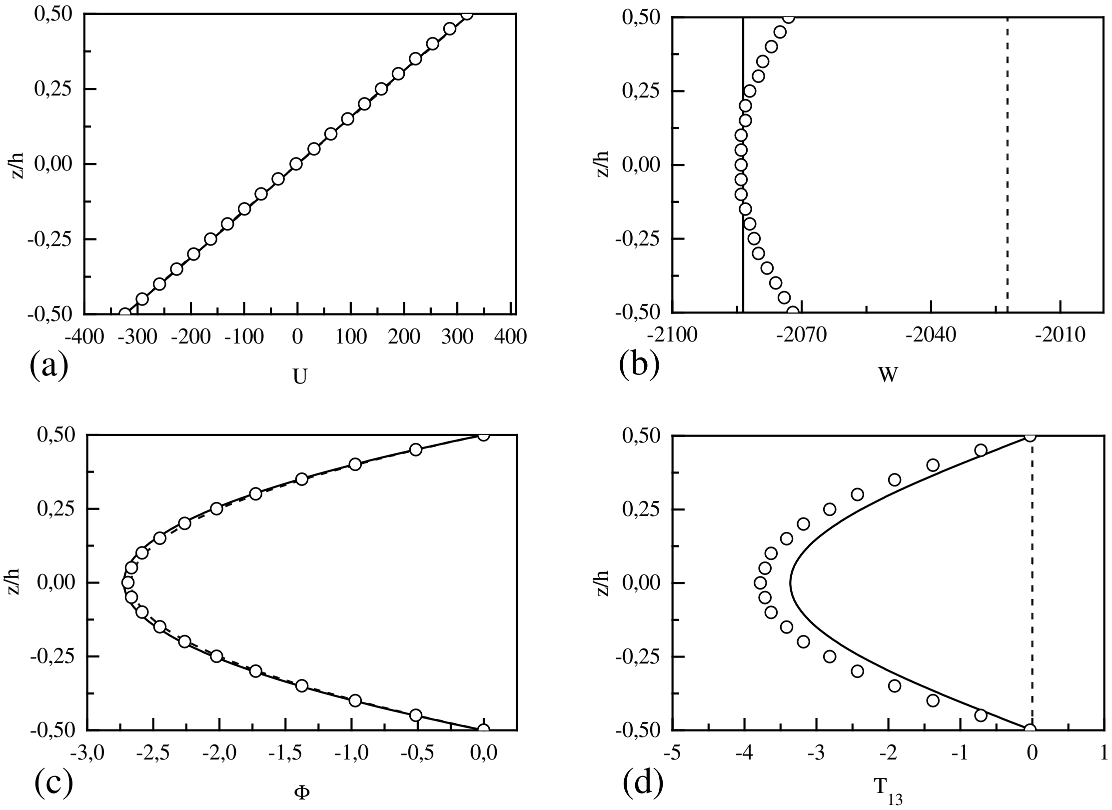

Fig.2 : *Force density applied on the top face of a piezoelectric single plate in closed circuit for $L/h = 10$. Plate model (full line), finite element (small circles) and simplified plate model (dashed-line)*

Case 1.a - Surface density of normal force applied to the top face of the plate, closed circuit. It means that $V_0 = 0$. In the present situation, the set of linear algebraic equations (37) is only considered. The through-thickness distribution for U, W, Φ and T_{11} are collected together in Fig.2 for the ratio $L/h = 10$. The displacement U at $x = 0$ is plotted in Fig.2.a and it is almost linear through the plate thickness. The flexural displacement W at $x = L/2$ is given in Fig.2.b, the straight line corresponds to the present plate approach while the small circles are the finite element computations and the straight dashed-line is the result provided by the classical thin plate theory based on the Love's assumption

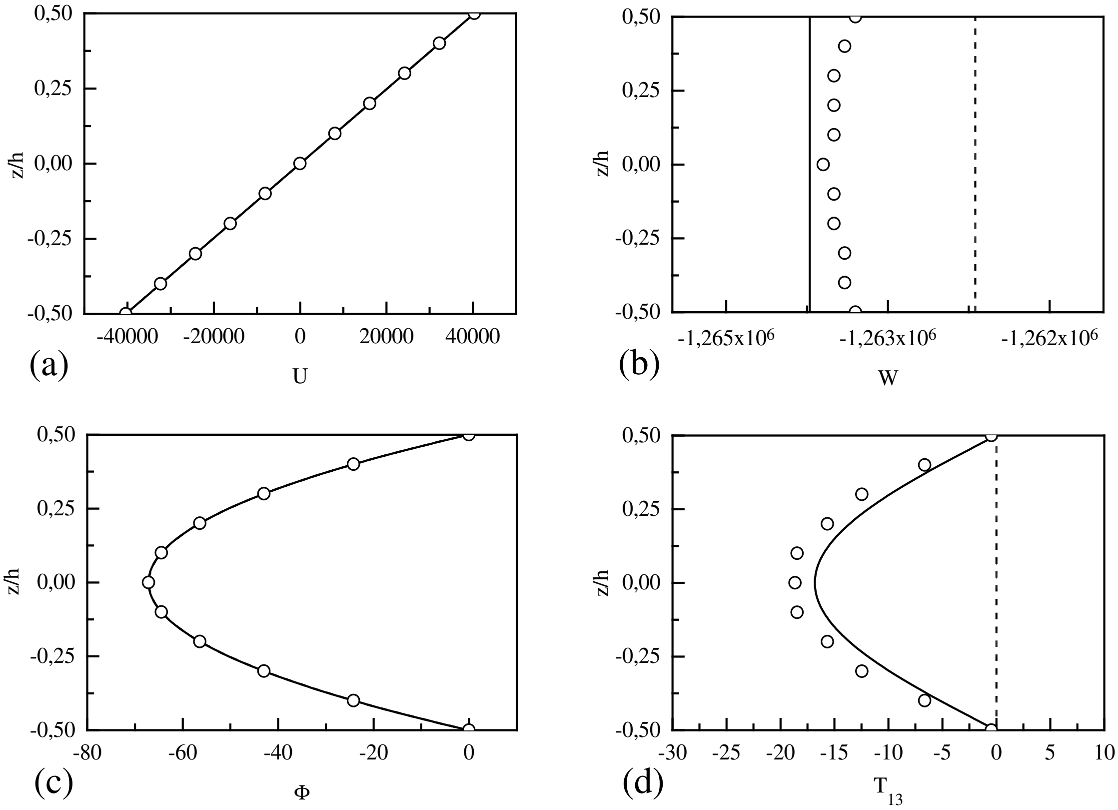

Fig.3 : *Force density applied on the top face of a piezoelectric single plate in closed circuit for $L/h = 50$. Plate model (full line), finite element (small circles) and simplified plate model (dashed-line)*

(no shear correction, that is, $f(z) = 0$ and $g(z) = 0$ in Eq.(10)). We observe that the discrepancy between the maximum values of the deflection (at $z = 0$) for the 3D computation and that of the plate model is less than 0.02 %, however the difference becomes bigger, about 3 %, for the classical thin plate model. Nevertheless, the most interesting result is the electric potential at $x = L/2$ plotted in Fig.2.c. The electric potential is, in fact, induced through the piezoelectric coupling by the elastic deformation. We first note a very good accuracy with the finite element method and next the result ascertains the existence of the ϕ_2 term in the electric potential approximation (10). However, if the ϕ_2 term is absent from the expansion (10) there is no induced electric potential through the plate thickness. At last, the shear stress component T_{13} at $x = L/4$ is drawn in Fig.2.d. The identical simulations are performed with the slenderness ratio $L/h = 50$, the results are presented in Fig.3 for the same electromechanical quantities as in the previous figure. The difference between the results coming from the plate model and those of the finite element simulations are very small since we are closer to thin plate assumption. Especially, the discrepancy between the maxima of the deflection displacement at the plate center for the present plate model and finite element results is now less than 0.01 %, whereas this difference is about 0.1 % for the classical plate theory. Going back to the physical units, we have an estimate of $9\,\mu m$ for the deflection at the plate center and the maximum of the induced electric potential is about 4.8 volts for the slenderness ratio $L/h = 50$.

Case 1.b - Surface density of normal force applied to the top face of the plate, open circuit. In this situation, the algebraic equations are given by Eq.(36) with $S_0 \neq 0$ and $Q_0 = 0$. The electromechanical response of the piezoelectric plate is shown in Fig.4 for the profiles of U, W, Φ and $\mathcal{D}_3$ as functions of the thickness coordinate for the thickness aspect ratio $L/h = 10$. Figure 4.a presents the displacement U at $x = 0$. The deflection W at $x = L/2$ is given in Fig.4.b. The difference between maxima of the deflection at the plate center for the present plate approach (straight line) and finite element simulation (small circles) is evaluated at 0.1 %, whereas the same comparison to the simplified plate theory (dashed-line curve) yields an error of 2.8 %. The induced electric potential at $x = L/2$ is plotted in Fig.4.c and the potential variation possesses a parabolic profile. Finally, Fig.4.d shows the normal component of the electric displacement or induction $\mathcal{D}_3$ at $x = L/2$. The comparison of the latter electrical quantity particularly speaks for itself, indeed, we have an excellent agreement with the finite element results, whereas the classical thin plate theory (dashed-line curve) does not give the correct through-the-thickness profile. Similar results are presented in Fig.5 for the same quantities with the slenderness ratio $L/h = 50$. The present plate model gives accurate predictions for the different electromechanical variables, displacements, stresses, electric potential and displacement. Especially, the estimate error between the deflection at the plate center for the finite element method and our improved plate model is about 0.01 % and 0.1 % for the simplified thin plate approach. The comparison between the results provided by the finite element method

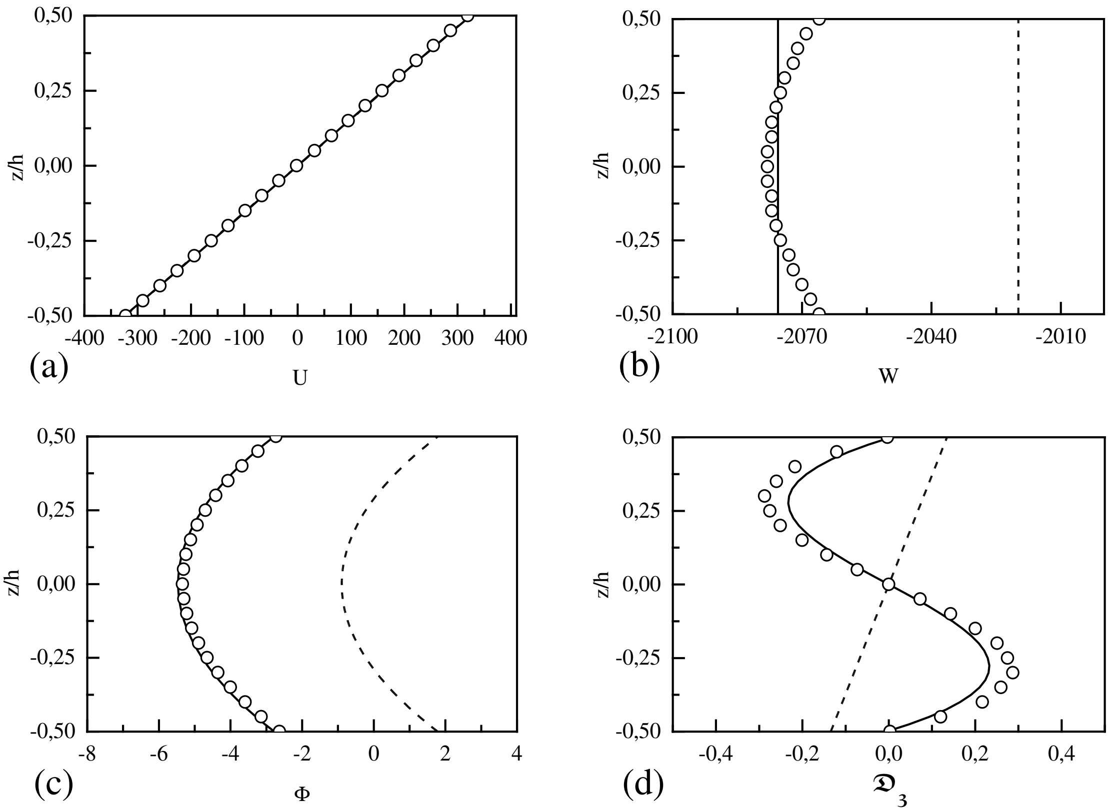

Fig.4 : *Force density applied on the top face of a piezoelectric single plate in open circuit for $L/h = 10$. Plate model (full line), finite element (small circles) and simplified plate model (dashed-line)*

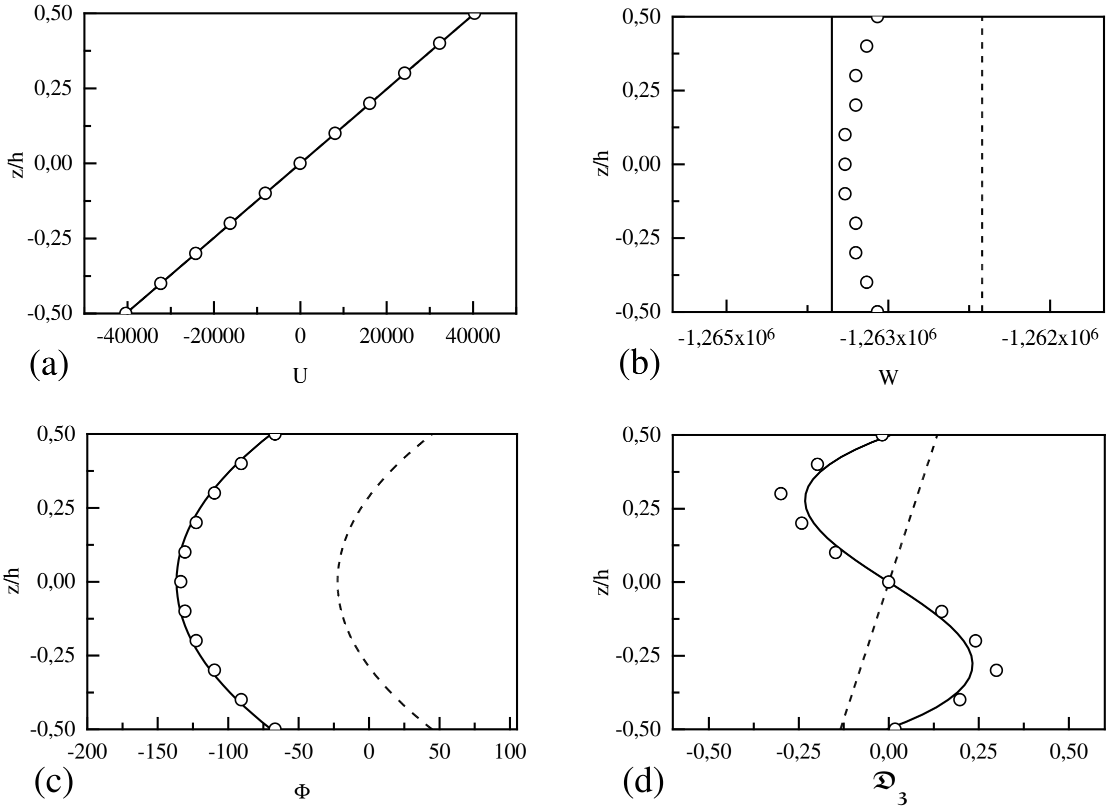

Fig.5 : *Force density applied on the top face of a piezoelectric single plate in open circuit for $L/h = 50$. Plate model (full line), finite element (small circles) and simplified plate model (dashed-line)*

and those coming from the present model illustrates the performance of the plate model and gives very good predictions with minor differences. The comparison made with the classical thin plate theory (no shear effect) ascertains the efficiency of our plate modelling.

Case 2 - Applied electric potential. In this situation the set of linear algebraic equations (37) is solved with $V_0 \neq 0$ and $S_0 = 0$. The results are collected together in Table 2 in dimensionless unit for the displacement U at $x = L$, stress T_{22} at the plate center and the normal component of the electric displacement for two slenderness ratios $L/h = 10$ and $L/h = 50$. Only an elongational deformation along the $x-$axis is obviously produced and, in this case, the shear effect does not play any role. A comparison is done with the finite element computations and the plate model providing rather quite good estimates. We note that the thickness aspect ratio has obviously no influence on the stress T_{22} and the electric displacement component $\mathcal{D}_3$. In addition, the electric potential is linear through the plate thickness going from $-V$ at $z = -h/2$ to $+V$ at $z = +h/2$. The elongation of the plate produced by the applied electric potential is about $0.4\,\mu m$ for the ratio $L/h = 50$.

Case 3 - Applied electric charges. For this case, we solve the set of linear algebraic equations (36) with $S_0 = 0$ and $Q_0 \neq 0$. As in the problem of an applied electric potential, elongational deformation is only obtained. Table 3 gives the essential numerical results in dimensionless unit for U, T_{22} and the electric potential Φ for two characteristic thickness

	L/h=10			L/h=50		
	Plate Model	Finite Elements	Error (%)	Plate Model	Finite Elements	Error (%)
U at $x = L$	16.41	16.42	0.05	82.06	82.11	0.06
T_{22} at $x = L/2$	−1.447	−1.446	0.09	−1.447	−1.446	0.09
$\mathcal{D}_3$ at $x = L/2$	−22.96	−22.94	0.08	−22.96	−22.94	0.08

Table 2: *Single piezoelectric plate, applied electric potential.*

aspect ratios $L/h = 10$ and $L/h = 50$. The comparison with the finite element method performed on the 3D problem shows a rather good accuracy of the present plate model except for the elongational displacement. As in the case of an applied electric potential, the shear effect is obviouly zero and there is no difference with the simplified model (Love-Kirchhoff approach).

	L/h=10			L/h=50		
	Plate Model	Finite Elements	Error (%)	Plate Model	Finite Elements	Error (%)
U at $x = L$	0.6785	0.716	5.2	3.54	3.58	1.0
T_{22} at $x = L/2$	0.063043	0.063044	0.001	0.063043	0.063044	0.001
Φ at $x = L/2$	0.043597	0.0436	0.007	0.043597	0.0436	0.007

Table 3: *Single piezoelectric plate, applied electric charges.*

In the case of an applied electric potential or charges on the plate faces, a thickness deformation is obviously produced for the real 3D plate. Such a thickness variation is not accounted for in the present approach since the deflection displacement w is constant through the plate thickness, which explains the rather small discrepancy observed in the cases 2 and 3. In spite of this limitation, the thickness variation represents however less than 1 % of the elongation or compression in the direction of the plate length.
Additional comparisons can be done, but not shown here, to exact solutions for laminated piezoelectric plates in cylindrical bending which are merely an extension of the Pagano's works for elastic laminates [21] to piezoelectric plates [22].

8. Closing remarks and future directions

In the present work we attempt to promote an efficient and interesting approach to piezoelectric plates. The field approximation accounts for the shear effects modelled by a "sine" function and a refined electric potential distribution through the plate thickness. Some comparative tests between, first, our improved plate model and the finite element computations and, next, the classical thin plate theory allow us to ascertain the validity and the capability of the piezoelectric plate model then considered. Especially, the different benchmark tests have been carried out for different kinds of electromechanical loads (i) an applied normal force at the top surface of the plate, (ii) an applied electric potential at the top and bottom faces of the plate and (iii) applied electric charges on both faces of the plate. In the case of an applied normal force density for an open and closed circuits, the through-thickness distribution of the most pertinent electromechanical variables have been computed for the present model, for the finite element computations for the 3D plate and for the simplified plate model (no shear correction) and compared each other. The comparisons yield an excellent agreement of the present model with the finite element method. Furthermore, the efficient approach to piezolectric plates provides very accurate predictions (error less than 0.1 % for the deflection displacement), whereas the classical thin plate theory gives less accurate results. It should be underlined (i) *the influence of the shear correction* described by a "sine" function on the computation of the *shear stress through the plate thickness*, (ii) the *accurate approximation of the electric potential* (see the third expansion in Eq.(10)) giving rise to the induced electric potential (often absent in most oversimplified model) and (iii) the usefulness of the *approximate charge equation* for the generalized electric charges or inductions (see Eq.(24)), which escapes from assuming constant electric displacement through the plate thickness.

An important extension of the present work concerns laminated piezoelectric plates, that is, plates made of piezoelectric and purely elastic layers. This extension will be presented in the second part of the present work. Here, a particular attention will be paid to the continuity conditions of the electromechanical quantities at the layer interfaces [23, 24]. Moreover, in view of the results for the single-layered plate, we are encouraged to study the vibrations of laminated piezoelectric plates [25], which is particularly usefull for active or passive control of vibrations. Some other challenging problems that must still be considered include the geometric and material nonlinearities. The nonlinear strain becomes a necessity if large deflections of the plate are produced and nonlinear electric and piezoelectric behaviors can appear if rather large electric fields are applied to the piezoelectric plate due to hysteresis of ferroelectric materials [26]. At last, the edge effects, such as electric field concentration, can be interesting to investigate, especially for plates partly coated with piezoelectric slabs.

Appendix A

All the different coefficients introduced in the matrices (26)-(29) are defined by

$$\begin{aligned}
&\big(Q_{ab}, D_{ab}, d_{ab}, \hat{D}_{ab}\big) = \big(1, \tfrac{h^2}{12}, \tfrac{2h^2}{\pi^3}, \tfrac{h^3}{2\pi^2}\big) h C^*_{ab} \quad , \\
&\hat{A}_{MN} = \tfrac{h}{2} C^*_{MN} \quad , \\
&\big(a_{2\alpha}, R_{3\alpha}, r_{3\alpha}, \hat{R}_{3\alpha}, \hat{r}_{3\alpha}\big) = \big(1, \tfrac{h^2}{6}, -\tfrac{2h}{\pi^2}, \tfrac{4h^3}{\pi^3}, -\tfrac{h}{2\pi}\big) h e^*_{3\alpha} \quad , \\
&\big(l_{\alpha N}, L_{\alpha N}, \overline{L}_{\alpha N}\big) = \big(\tfrac{2}{\pi}, -\tfrac{4h^2}{\pi^3}, \tfrac{h}{2\pi}\big) h e^*_{\alpha N} \quad , \\
&\big(b_{\alpha\alpha}, B_{\alpha\alpha}, \overline{B}_{\alpha\alpha}, \overline{\overline{B}}_{\alpha\alpha}, f_{\alpha\alpha}, F_{\alpha\alpha}, \overline{F}_{\alpha\alpha}\big) = \big(-\tfrac{h^2}{12}, -\tfrac{h^4}{30}, \tfrac{4h^3}{\pi^4}, -\tfrac{h^2}{2\pi^2}, -1, \tfrac{h^2}{6}, -\tfrac{2h}{\pi^2}\big) h \epsilon^*_{\alpha\alpha} \quad , \\
&\big(f_{33}, P_{33}, \overline{P}_{33}, \overline{\overline{P}}_{33}\big) = \big(-1, -\tfrac{h^2}{3}, \tfrac{4h}{\pi^2}, -\tfrac{1}{2}\big) h \epsilon^*_{33} \quad ,
\end{aligned}$$

with the definitions $(ab) \in \{(11),(22),(12),(66)\}$, $\alpha \in \{1,2\}$, $(MN) \in \{(44),(55)\}$ and $(\alpha N) \in \{(24),(15)\}$ (the Voigt notation is used for convenience). The modulus of elasticity due to the normal shear stress hypothesis (σ_{33} negligeable in comparison to the other stress components) are then given by $C^*_{ab} = C^E_{ab} - C^E_{a3}C^E_{3b}/C^E_{33}$. On using the same argument, we have the effective piezoelectric and dielectric coefficients $e^*_{ja} = e_{ja} - e_{j3}C^E_{a3}/C^E_{33}$ and $\epsilon^*_{ij} = \epsilon_{ij} + e_{i3}e_{j3}/C^E_{33}$ (with $a \in \{1, ..., 6\}, \quad j \in \{1, 2, 3\}$).

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