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STRATIFYING SYSTEMS AND g-VECTORS

OCTAVIO MENDOZA, CORINA SÁENZ AND HIPOLITO TREFFINGER

Abstract. In this paper we study the Cartan matrix associated to the Ext-projective stratifying system induced by a basic and τ-rigid object M in $\operatorname{mod}(A)$ by means of the g-vectors of the indecomposable direct summands of M. In particular we show that the Cartan group of a stratifying system associated to a τ-rigid module can be calculated directly using these vectors. Moreover we characterise the stratifying systems coming from τ-rigid modules that have a diagonal Cartan matrix.

1. Introduction

The notion of an Ext-injective stratifying system was introduced in [11] as an axiomatisation of the standard objects of the standardly stratified algebras defined in [9]. It was quickly realised that Ext-injective stratifying systems are equivalent to the notions of Ext-projective stratifying systems [17] and what is now known simply as stratifying systems [16]. Since then, stratifying systems were intensively studied in [10, 23, 15, 20, 21, 24, 13]. Throughout this paper, A will denote to a basic finite dimensional k-algebra over an algebraically closed filed k. We will work on the abelian category $\operatorname{mod}(A)$ of all the finitely generated left A-modules. Given a positive integer t, we denote by $[1,t]$ the set of all the natural numbers between 1 and t, equipped with the natural order $\leq$ on the set $[1,t]$.

The notion of Ext-projective stratifying system in $\operatorname{mod}(A)$ is defined in [17]. We recall this notion as follows.

Definition 1.1. [17, Definition 2.1] An Ext-projective stratifying system in $\operatorname{mod}(A)$, of size t, is a triple $(\Theta, \underline{Q}, \leq)$, where $\Theta := \{\Theta(i)\}_{i=1}^t$ is a family of non-zero A-modules, $\underline{Q} := \{Q(i)\}_{i=1}^t$ is a family of indecomposable A-modules and $\leq$ is the natural order on $[1,t]$ satisfying the following conditions:

(a) $\operatorname{Hom}_A(\Theta(j), \Theta(i)) = 0$ for $j > i$;
(b) for each $i \in [1,t]$, there is an exact sequence $0 \to U(i) \xrightarrow{\alpha_i} Q(i) \xrightarrow{\beta_i} \Theta(i) \to 0$ in $\operatorname{mod}(A)$ such that $U(i)$ is filtered by the set $\{\Theta(j) : j > i\}$;
(c) $\operatorname{Ext}_A^1(Q, X) = 0$ where $Q := \bigoplus_{i=1}^t Q(i)$ and X is an A-module filtered by $\{\Theta(i)\}_{i=1}^t$.

Although it has been shown that stratifying systems are objects having very nice properties, until very recently there was no systematic method to construct stratifying systems in the module category of a given algebra A. This problem has been addressed in [22] and [25] using tools from τ-tilting theory and higher homological algebra, respectively.

It was shown in [18] that it can be associated to every Ext-projective stratifying system $(\Theta, \underline{Q}, \leq)$ two invariants C_Θ and G_Θ, known as the *Cartan matrix* and the *Cartan group* of Θ, respectively (see Section 3). In this paper, we study the Cartan matrix C_{Δ_M} and the Cartan group G_{Δ_M} for a stratifying system Δ_M associated to a (basic) τ-rigid module $M \in \operatorname{mod}(A)$. In

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particular, we show that the matrix C_{Δ_M} can be obtained using the g-matrix G^M (formed by the g-vectors of the indecomposable direct summands of M), the M-standard matrix S^M (formed by the dimension vectors of the M-standard modules Δ_M) and the M-residual matrix R^M (see Section 5). This result can be summarized as follows, for the complete version, see Theorem 5.3 and Proposition 5.4.

Theorem 1.2. *Let $M \in \operatorname{mod}(A)$ be basic and τ-rigid with a TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$, and let $(\{\Delta_M\}_{i=1}^{t}, \underline{Q}, \leq)$ be its Ext-projective stratifying system. Then,*

$$C_{\Delta_M} = (G^M)^{tr}\, S^M + R^M \quad \text{and} \quad |G_{\Delta_M}| = \prod_{i=1}^{t} [(G^M)^{tr}\, S^M]_{i,i}$$

Moreover, if M is filtered by the family Δ_M, then $R^M = 0$ and $Q(i) \simeq M_i \ \forall\, i$.

Then we apply this theorem to characterise algebras A and stratifying systems in $\operatorname{mod}(A)$ coming from τ-tilting modules having a diagonal Cartan matrix, generalising [18, Theorem 5.9]. In what follows, we summarize this result. For more details, see Theorem 5.7.

Theorem 1.3. *For a basic τ-tilting module $M \in \operatorname{mod}(A)$ with a TF-admissible decomposition $M = \bigoplus_{i=1}^{n} M_i$ such that $M \in \mathcal{F}(\Delta_M)$, the following statements are equivalent.*

(a) *The matrix C_{Δ_M} is diagonal.*
(b) $\operatorname{Hom}_A(M_i, M_j) = 0$ *for $i < j$.*
(c) $\operatorname{Hom}_A(M_i, \Delta_M(j)) = 0$ *for $i < j$.*

Moreover, if one of the above conditions holds true, then $M \simeq {}_AA$ and A is a weakly triangular algebra.

The paper is organised as follows. In Section 2 we establish the setting and notation in which we work and we recall some basic representation theory. Then, in Section 3 we recall some theory on stratifying systems and we prove some preparatory results. Later, in Section 4 we give a brief overview of the necessary results on τ-tilting theory that we need in Section 5 where we state and prove the main results of this paper. Finally, in Section 6 we give some examples illustrating Theorem 5.3 and we comment about the relationship between stratifying systems and stability conditions.

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2. Setting and background

For $M \in \operatorname{mod}(A)$, we denote by $rk(M)$ the number of pairwise non-isomorphic indecomposable direct summands of M. Finally, we say that $M \in \operatorname{mod}(A)$ is *basic* if there is a decomposition $M = \bigoplus_{i=1}^{t} M_i$, with M_i indecomposable for every i and $M_i \not\simeq M_j$ if $i \neq j$.

Given an algebra A, we denote by $M \twoheadrightarrow N$ an epimorphism from M to N in $\operatorname{mod}(A)$; and for each $M \in \operatorname{mod}(A)$, we consider the class

$$\operatorname{Fac}(M) := \{X \in \operatorname{mod}(A) \ : \ \exists\, M^n \twoheadrightarrow X \text{ for some } n \in \mathbb{N}\}.$$

Dually, the arrow $\hookrightarrow$ stands for a monomorphism in $\operatorname{mod}(A)$, and we have the class

$$\operatorname{Sub}(M) := \{X \in \operatorname{mod}(A) \ : \ \exists\, X \hookrightarrow M^n \text{ for some } n \in \mathbb{N}\}.$$

For any subclass $\mathcal{X} \subseteq \operatorname{mod}(A)$, we have the right Hom-perpendicular complement of $\mathcal{X}$

$$\mathcal{X}^{\perp} := \{M \in \operatorname{mod}(A) \ : \ \operatorname{Hom}_A(-, M)|_{\mathcal{X}}) = 0\}.$$

Dually, we have the left Hom-perpendicular complement ${}^{\perp}\mathcal{X}$ of $\mathcal{X}$. For a single element class $\mathcal{X} = \{M\}$, we just write $M^{\perp}$ and ${}^{\perp}M$. It is said that $N \in \operatorname{mod}(A)$ admits an $\mathcal{X}$-*filtration* if there is a chain of submodules $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_{n-1} \subseteq M_n = M$ satisfying that each quotient $M_i/M_{i-1} \simeq X_i \in \mathcal{X}$. The class of all the $N \in \operatorname{mod}(A)$ which admits an $\mathcal{X}$-filtration will be denoted by $\mathcal{F}(\mathcal{X})$.

Let $\mathcal{X} \subseteq \operatorname{mod}(A)$ and $M \in \operatorname{mod}(A)$. A morphism $f : X \to M$ in $\operatorname{mod}(A)$ is called $\mathcal{X}$-precover (or right $\mathcal{X}$-approximation) of M if $X \in \mathcal{X}$ and $\operatorname{Hom}_A(X', f) : \operatorname{Hom}_A(X', X) \to \operatorname{Hom}_A(X', M)$ is surjective, for any $X' \in \mathcal{X}$. Moreover, if the equality $fh = f$ holds true only for an automorphism $h : X \to X$, it is said that the $\mathcal{X}$-precover is an $\mathcal{X}$-cover. The class $\mathcal{X}$ is precovering (or contravariantly finite) if any $N \in \operatorname{mod}(A)$ admits an $\mathcal{X}$-precover. Dually, we have the notions of $\mathcal{X}$-preenvelope (or left $\mathcal{X}$-approximation), $\mathcal{X}$-envelope and preenveloping (or covariantly finite) class. If $\mathcal{X}$ is precovering and preenveloping, it is said that $\mathcal{X}$ is functorially finite.

We recall that a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories in $\operatorname{mod}(A)$ is a *torsion pair* if $\mathcal{X} = {}^{\perp}\mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^{\perp}$. Given a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\operatorname{mod}(A)$, it is said that $\mathcal{X}$ is a torsion class and $\mathcal{Y}$ is a torsion free class. Moreover, in this case, it can be shown that $\mathcal{X}$ is closed under quotients and extensions, while $\mathcal{Y}$ is closed under submodules and extensions. It is also well known that, for every subclass $\mathcal{X} \subseteq \operatorname{mod}(A)$ which is closed under quotients and extensions, there exists a subclass $\mathcal{Y} \subseteq \operatorname{mod}(A)$ such that $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\operatorname{mod}(A)$. In particular, if $\mathcal{X} = \operatorname{Fac}(M)$ for some $M \in \operatorname{mod}(A)$, then $\mathcal{Y} = M^{\perp}$.

It is also well known that each torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\operatorname{mod}(A)$ has associated two additive functors $\mathrm{t}, \mathrm{f} : \operatorname{mod}(A) \to \operatorname{mod}(A)$. The *torsion functor* t is a subfunctor of the identity functor $1_{\operatorname{mod}(A)}$, and the *torsion free functor* f is equal to the quotient functor $1/\mathrm{t}$. Moreover, for every $M \in \operatorname{mod}(A)$, there exists the so-called canonical short exact sequence

$$0 \to \mathrm{t}(M) \to M \to \mathrm{f}(M) \to 0,$$

where $\mathrm{t}(M) \in \mathcal{X}$ and $\mathrm{f}(M) \in \mathcal{Y}$ and such exact sequence is unique up to isomorphisms.

Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod}(A)$. It is said that $X \in \mathcal{X}$ is *Ext-projective* in $\mathcal{X}$ if $\operatorname{Ext}^1_A(X, -)|_{\mathcal{X}} = 0$. Moreover, if $\mathcal{X}$ is functorially finite and a torsion class, it is well known that there are only finitely many (up to isomorphisms) indecomposable Ext-projective modules in $\mathcal{X}$ and we denote by $\mathbb{P}(\mathcal{X})$ their direct sum.

3. Stratifying systems

The concept of stratifying system for module categories was first introduced in [11], and further studied in [16, 17]. The original definition is stated in Definition 3.2, but it was shown in [16] that this definition is equivalent to the following one.

Definition 3.1. [16, Characterisation 1.6] A stratifying system of size t, in $\operatorname{mod}(A)$, consists of a pair $(\Theta, \leq)$, where $\Theta := \{\Theta(i)\}_{i=1}^{t}$ is a family of indecomposable objects in $\operatorname{mod}(A)$ and $\leq$ is the natural order on the set $[1,t] := \{1, 2, \ldots, t\}$ satisfying the following conditions:

(a) $\operatorname{Hom}_A(\Theta(j), \Theta(i)) = 0$ if $j > i$;
(b) $\operatorname{Ext}^1_A(\Theta(i), \Theta(j)) = 0$ if $j \geq i$.

Let $(\Theta, \leq)$ be a stratifying system in $\operatorname{mod}(A)$. The theory of stratifiying systems, developed in [11, 16, 17], shows that $\mathcal{F}(\Theta)$ is equivalent to the full subcategory $\mathcal{F}(\Delta)$ of $\operatorname{mod}(B)$, where B is a standardly stratified algebra and $\mathcal{F}(\Delta)$ is the category of the Δ-filtered objects, of key importance in the theory of quasi-hereditary and standardly stratified algebras. In order to determine the standardly stratified algebra B, it is necessary to consider the so-called *Ext-projective* and *Ext-injective* stratifying systems. The notion of Ext-projective stratifying system was given in Definition 1.1.

Definition 3.2. [11, Definition 1.1] An Ext-injective stratifying system in $\operatorname{mod}(A)$, of size t, is a triple $(\Theta, \underline{Y}, \leq)$, where $\Theta := \{\Theta(i)\}_{i=1}^{t}$ is a family of non-zero A-modules, $\underline{Y} := \{Y(i)\}_{i=1}^{t}$ is a family of indecomposable A-modules and $\leq$ is the natural order on $[1,t]$ satisfying the following conditions:

(a) $\operatorname{Hom}_A(\Theta(j), \Theta(i)) = 0$ for $j > i$;
(b) for each $i \in [1,t]$, there is an exact sequence $0 \to \Theta(i) \xrightarrow{\lambda_i} Y(i) \xrightarrow{\gamma_i} Z(i) \to 0$ in $\operatorname{mod}(A)$ such that $Z(i) \in \mathcal{F}(\Theta(j) : j < i)$;
(c) $\operatorname{Ext}^1_A(-, Y)|_{\mathcal{F}(\Theta)} = 0$, for $Y := \bigoplus_{i=1}^{t} Y(i)$.

Given a stratifying system $(\Theta, \leq)$, it is well known [11, 16, 17] that there exist a unique (up to isomorphism) Ext-projective stratifying system $(\Theta, \underline{Q}, \leq)$ and an Ext-injective stratifying system $(\Theta, \underline{Y}, \leq)$. Furthermore, for a given Ext-projective stratifying system $(\Theta, \underline{Q}, \leq)$, we have that the pair $(\Theta, \leq)$ is a stratifying system. Moreover the same statement holds true for Ext-injective stratifying systems.

Let $(\Theta, \{Q(i)\}_{i=1}^{t}, \leq)$ be an Ext-projective stratifying system in $\operatorname{mod}(A)$. By following [18, Section 4], we consider the Θ-Cartan matrix $C_\Theta \in \operatorname{Mat}_{t\times t}(\mathbb{Z})$, naturally associated to this system, which is defined as follows: the (i,j)-coordinate of C_Θ is $[C_\Theta]_{i,j} := \dim_k \operatorname{Hom}_A(Q(i), \Theta(j))$.

The main idea in [18] behind the introduction of the Cartan matrix C_Θ, for an Ext-projective stratifying system $(\Theta, \{Q(i)\}_{i=1}^{t}, \leq)$, is that we can use it to construct a finite abelian group G_Θ, known as the *Cartan Group* of Θ which is defined as $G_\Theta := \operatorname{CoKer}(C_\Theta)$, where the matrix C_Θ is identified with the $\mathbb{Z}$-linear map $\mathbb{Z}^t \to \mathbb{Z}^t,\ X \mapsto C_\Theta\, X$. It is shown in [18], that $|G_\Theta| = 1$ if and only if the standardly stratified algebra $\operatorname{End}_A(Q)^{op}$ is quasi-hereditary. Then the size $|G_\Theta|$ of G_Θ give us a measure of how far is $\operatorname{End}_A(Q)^{op}$ from being quasi-hereditary.

Lemma 3.3. [17, 18, 19] *Let $(\Theta, \{Q(i)\}_{i=1}^{t}, \leq)$ be an Ext-projective stratifying system in* $\operatorname{mod}(A)$. *Then, C_Θ is an upper triangular matrix and* $\operatorname{Hom}_A(Q(i), \Theta(i)) \simeq \operatorname{End}_A(\Theta(i))\ \forall\, i \in [1,t]$. *Moreover* $|G_\Theta| = \prod_{i=1}^{n} [C_\Theta]_{i,i}$.

Proof. From [19, Lemma 2.5 (b)], we have that the matrix C_Θ is upper triangular. On the other hand, for each $i \in [1,t]$, the isomorphism $\operatorname{Hom}_A(Q(i), \Theta(i)) \simeq \operatorname{End}_A(\Theta(i))$ is given by [17, Lemma 2.6 (b)]. Finally, the equality $|G_\Theta| = \prod_{i=1}^{n} [C_\Theta]_{i,i}$ is obtained from [18, Theorem 4.2 a.] □

Let $(\Theta, \{Q(i)\}_{i=1}^{t}, \leq)$ be an Ext-projective stratifying system in $\operatorname{mod}(A)$. In what follows, for each $i \in [1,t]$, we consider the exact sequence $\eta_i :\ 0 \to U(i) \xrightarrow{\alpha_i} Q(i) \xrightarrow{\beta_i} \Theta(i) \to 0$ associated to each $\Theta(i)$, see Definition 1.1 (b), and the map

$$\partial_{i,j} : \operatorname{Hom}_A(U(i), \Theta(j)) \to \operatorname{Ext}^1_A(\Theta(i), \Theta(j)),\ f \mapsto f \cdot \eta_i,$$

where $f \cdot \eta_i$ is the push-out of f and η_i.

Proposition 3.4. *For an Ext-projective stratifying system* $(\Theta, \{Q(i)\}_{i=1}^t, \leq)$ *in* $\operatorname{mod}(A)$, *the following statements are equivalent.*

(a) *The matrix* C_Θ *is diagonal.*
(b) $\operatorname{Hom}_A(Q(i), Q(j)) = 0$ *for* $i < j$.
(c) $\operatorname{Hom}_A(Q(i), \Theta(j)) = 0$ *for* $i < j$.
(d) *For* $i < j$, *we have that* $\operatorname{Hom}_A(\Theta(i), \Theta(j)) = 0$ *and*

$$\partial_{i,j} : \operatorname{Hom}_A(U(i), \Theta(j)) \to \operatorname{Ext}_A^1(\Theta(i), \Theta(j)),\ f \mapsto f \cdot \eta_i$$

is an isomorphism.

Moreover, if one of the above conditions holds true, then $\operatorname{Hom}_A(Q(i), U(j)) = 0$ *for* $i < j$.

Proof. (a) $\Leftrightarrow$ (c): It follows from Lemma 3.3.

(b) $\Rightarrow$ (c): Let $\operatorname{Hom}_A(Q(i), Q(j)) = 0$ for $i < j$. We prove now that $\operatorname{Hom}_A(Q(i), \Theta(j)) = 0$. If $Q(j) = \Theta(j)$ there is nothing to prove. Otherwise, take $f : Q(i) \to \Theta(j)$ and consider the exact sequence

$$\eta_j : \quad 0 \to U(j) \xrightarrow{\alpha_j} Q(j) \xrightarrow{\beta_j} \Theta(j) \to 0$$

associated to $\Theta(j)$, where $U(j)$ is nonzero and $U(j) \in \mathcal{F}(\Theta(k) : k > j)$. Since $Q(i)$ is Ext-projective in $\mathcal{F}(\Theta)$, there exist $f' : Q(i) \to Q(j)$ such that $f'\beta_j = f$. By hypothesis, $f' = 0$. Hence $f = 0$.

(c) $\Rightarrow$ (b): Let $\operatorname{Hom}_A(Q(i), \Theta(j)) = 0$ for $i < j$. We carry on the proof by reverse induction on j. If $j = n$ then $\Theta(n) = Q(n)$ and thus (b) follows. Therefore, we can assume that $j < n$. Let us prove that $\operatorname{Hom}_A(Q(i), Q(j)) = 0$. In order to do that, take a map $g : Q(i) \to Q(j)$ and consider the exact sequence η_j. Since $g\beta_j = 0$, there exist $h : Q(i) \to U(j)$ such that $\alpha_j h = g$. Moreover, since $U(j) \in \mathcal{F}(\{\Theta(k) : k > j\})$, [17, Proposition 2.10] implies the existence of an exact sequence

$$0 \to N \to Q_0(U(j)) \xrightarrow{\epsilon_{U(j)}} U(j) \to 0$$

in $\mathcal{F}(\Theta)$, where $Q_0(U(j)) \in \operatorname{add}\left(\bigoplus_{k>j} Q(k)\right)$. Once again, by using that $Q(i)$ is Ext-projective in $\mathcal{F}(\Theta)$, there exist $h' : Q(i) \to Q_0(U(j))$ such that $\epsilon_{U(j)} h' = h$. By induction, $h' = 0$. Therefore $g = \alpha_j \epsilon_{U(j)} h' = 0$.

(c) $\Leftrightarrow$ (d): Let $i < j$. By applying the functor $\operatorname{Hom}_A(-, \Theta(j))$ to the exact sequence η_i, we get the following exact sequence

$$0 \to \operatorname{Hom}_A(\Theta(i), \Theta(j)) \to \operatorname{Hom}_A(Q(i), \Theta(j)) \to \operatorname{Hom}_A(U(i), \Theta(j)) \to \operatorname{Ext}_A^1(\Theta(i), \Theta(j)) \to 0$$

Thus, the equivalence between (c) and (d) follows from the above exact sequence.

Finally, assume that (b) holds true. Then, by applying the functor $\operatorname{Hom}_A(Q(i), -)$ to η_j we get that $\operatorname{Hom}_A(Q(i), U(j)) = 0$ for $i < j$.

□

Corollary 3.5. *For an Ext-projective stratifying system* $(\Theta, \{Q(i)\}_{i=1}^t, \leq)$ *in* $\operatorname{mod}(A)$, *such that the matrix* C_Θ *is diagonal, the following statements are equivalent.*

(a) $[U(i) : \Theta(j)] = 0$ *for* $i < j$.
(b) $\operatorname{Ext}_A^1(\Theta(i), \Theta(j)) = 0\ \forall\, i, j$.
(c) $Q(i) \simeq \Theta(i)$ *as* A*-modules* $\forall\, i$.
(d) $U(i) = 0\ \forall\, i$.

Moreover, if one of the above conditions holds true, then $\mathcal{F}(\Theta) = \operatorname{add}(Q)$ *and* $\operatorname{End}_A(Q) \simeq \times_{i=1}^t \operatorname{End}_A(Q(i))$ *as* k*-algebras.*

Proof. By Proposition 3.4, we get that $\mathrm{Hom}_A(\Theta(i),\Theta(j)) = 0\ \forall\, i \neq j$, and moreover $\partial_{i,j}$: $\mathrm{Hom}_A(U(i),\Theta(j)) \xrightarrow{\simeq} \mathrm{Ext}^1_A(\Theta(i),\Theta(j))$ for $i<j$.

(a) $\Rightarrow$ (b): Let $i,j \in [1,t]$. By [17, Lemma 2.6 (b)], we know that $\mathrm{Ext}^1_A(\Theta(i),\Theta(j)) = 0$ for $i \geq j$. Hence, we can assume that $i<j$. Since $U(i) \in \mathcal{F}(\Theta(l) : l > i)$, by (a) we have that $\mathrm{Hom}_A(U(i),\Theta(j)) = 0$. Therefore $\mathrm{Ext}^1_A(\Theta(i),\Theta(j)) = 0$ for $i<j$.

(b) $\Rightarrow$ (c): Let $i \in [1,t]$. Then, by (b), we get that the exact sequence $\eta_i :\quad 0 \to U(i) \xrightarrow{\alpha_i} Q(i) \xrightarrow{\beta_i} \Theta(i) \to 0$ splits. Thus $Q(i) \simeq \Theta(i)$ as A-modules since $Q(i)$ and $\Theta(i)$ are indecomposable (see [17, Corollary 2.13]).

(c) $\Rightarrow$ (d): From (c) and the exact sequence η_i, we have that $\dim_k U(i) = 0$ and hence $U(i) = 0$.

(d) $\Rightarrow$ (a): It is trivial.

Finally, If one of the above conditions holds true, then we get that $Q(i) \simeq \Theta(i)$ as A-modules $\forall\, i$. Hence $\mathrm{Hom}_A(Q(i),Q(j)) = 0$ for $i \neq j$ and thus $\mathrm{End}_A(Q) \simeq \times_{i=1}^t \mathrm{End}_A(Q(i))$ as k-algebras. □

4. τ-TILTING THEORY

Most of the results of this paper are proven using the tools provided by τ-tilting theory [1]. We now give a brief summary of the definitions and results on τ-tilting theory that will be needed later. For a broader survey on this topic, the reader is encouraged to see [26].

Definition 4.1. [1, Definition 0.1] Let $M \in \mathrm{mod}\,(A)$. If $\mathrm{Hom}_A(M,\tau M) = 0$, it is said that M is *τ-rigid*. A τ-rigid module M is *τ-tilting* if $rk(M) = rk({}_AA)$. A pair (M,P) is called τ-rigid if $P \in \mathrm{mod}\,(A)$ is projective, M is τ-rigid and $\mathrm{Hom}_A(P,M) = 0$. Finally, the τ-rigid pair (M,P) is τ-tilting if $rk(M) + rk(P) = rk({}_AA)$.

One of the main features of τ-tilting theory is that all functorially finite torsion classes in $\mathrm{mod}\,(A)$ can be described by using τ-tilting pairs, as stated in the following result that has been taken from [1, Theorem 2.7] and [3, Theorem 5.10].

Theorem 4.2. *For any algebra A, there is a well defined function*

$$\Phi : \tau\text{-rig}(A) \to \text{f-tors}(A),\ M \mapsto \mathrm{Fac}(M),$$

from τ-rigid basic pairs to functorially finite torsion classes in $\mathrm{mod}\,(A)$. *Moreover, Φ is a bijection if we restrict it to the class* $\mathrm{s}\tau\text{-tilt}(A)$ *of τ-tilting pairs, and in this case* $\Phi^{-1}(\mathcal{X}) = \mathbb{P}(\mathcal{X})$.

In [22], it was shown how to produce a stratifying system from a basic τ-rigid module $M \in \mathrm{mod}\,(A)$. We start by recalling the notion of torsion free admissible decomposition.

Definition 4.3. [22, Definition 3.1] Let $M \in \mathrm{mod}\,(A)$ be basic and τ-rigid. A decomposition $M = \bigoplus_{i=1}^t M_i$ as the direct sum of indecomposables A-modules is called *torsion free admissible* (TF-admissible, for short) if $M_i \notin \mathrm{Fac}\left(\bigoplus_{j>i} M_j\right)$, for every $i \in [1,t]$.

The existence of such TF-admissible decompositions is supported by the following proposition proved in [22].

Proposition 4.4. [22, Proposition 3.2] *Every basic and τ-rigid A-module $M \in \mathrm{mod}\,(A)$ admits a TF-admissible decomposition.*

The following result, proved in [22], tells us how to build a stratifying system $(\Delta_M, \leq)$ from a TF-admissible decomposition of $M \in \mathrm{mod}\,(A)$. The stratifying system $(\Delta_M, \leq)$ is known as the *M-standard stratifying system* associated to this TF-admissible decomposition.

Theorem 4.5. [22, Theorem 3.4] *Let $M \in \operatorname{mod}(A)$ be a basic and τ-rigid with a TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$, and let f_k be the torsion free functor associated to the torsion pair $\left(\operatorname{Fac}(\bigoplus_{j\geq k} M_j), (\bigoplus_{j\geq k} M_j)^{\perp}\right)$. Then, the family $\Delta_M := \{\Delta_M(i) := \mathrm{f}_{i+1}(M_i)\}_{i=1}^{t}$ and the natural order on $[1,t]$ form a stratifying system in $\operatorname{mod}(A)$ of size t.*

In this paper we study the relation between the homological properties of a basic and τ-rigid module $M \in \operatorname{mod}(A)$ and its induced M-stratifying system $(\Delta_M, \leq)$. In order to do that, we need to introduce a vector with integer coordinates, known as the g-vector of M, associated to each module M in mod (A). For defining such g-vector, we denote by $P(1), \ldots, P(n)$ the iso-classes of indecomposable projective A-modules.

Definition 4.6. [1, Section 5] Let $M \in \operatorname{mod}(A)$ and $P_1 \to P_0 \to M \to 0$ be the minimal projective presentation of M, where $P_0 = \bigoplus_{i=1}^{n} P(i)^{a_i}$ and $P_1 = \bigoplus_{i=1}^{n} P(i)^{a'_i}$. The g-vector of M is set to be $g^M := (a_1 - a'_1, a_2 - a'_2, \ldots, a_n - a'_n) \in \mathbb{Z}^n$.

Another useful vector associated with $M \in \operatorname{mod}(A)$ is the *dimension vector* $\underline{\dim}(M) \in \mathbb{Z}^n$ whose i-th coordinate is the composition factor of the simple $S(i)$ in M, where $S(i)$ is the simple top of $P(i)$. There is important homological information that arises when g-vectors and dimension vectors interplay, as shown in [2]. For $u, v \in \mathbb{Z}^n$, we denote by $\langle u, v\rangle$ the standard inner product $\sum_{i=1}^{n} u_i v_i$ of these vectors.

Theorem 4.7. [2, Theorem 1.4.(a)] *Let M and N be in $\operatorname{mod}(A)$. Then*

$$\langle g^M, \underline{\dim}(N)\rangle = \dim_k(\operatorname{Hom}_A(M,N)) - \dim_k(\operatorname{Hom}_A(N, \tau_A M)).$$

5. Main results

In this section we state and prove our main results. From now on, let $n := rk({}_AA)$ and fix some basic and τ-rigid module M in $\operatorname{mod}(A)$ with some TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$. As we mentioned in the previous section, this determines a stratifying system $(\Delta_M, \leq)$ which, in turn, induces an Ext-projective stratifying system $(\Delta_M, \underline{Q}, \leq)$. Moreover, for each $i \in [1,t]$, we consider the following exact sequences:

(1) the canonical exact sequence

$$\varepsilon_i :\ 0 \to K(i) \xrightarrow{\iota_i} M_i \xrightarrow{\pi_i} \Delta_M(i) \to 0$$

of M_i with respect to the torsion pair $\left(\operatorname{Fac}(\bigoplus_{l>i} M_l), (\bigoplus_{l>i} M_l)^{\perp}\right)$;

(2) the exact sequence

$$\eta_i :\ 0 \to U(i) \xrightarrow{\alpha_i} Q(i) \xrightarrow{\beta_i} \Delta_M(i) \to 0$$

associated to each $\Delta_M(i)$, see Definition 1.1 (b). We also consider the following integer matrices:

(3) the **G-matrix** $G^M \in \operatorname{Mat}_{n\times t}(\mathbb{Z})$ of M whose i-th column is $[G^M]^i := (g^{M_i})^{tr}$;

(4) the **M-standard matrix** $S^M \in \operatorname{Mat}_{n\times t}(\mathbb{Z})$ whose i-th column is $[S^M]^i := \underline{\dim}(\Delta_M(i))^{tr}$;

(5) the **M-residual matrix** $R^M \in \operatorname{Mat}_{t\times t}(\mathbb{Z})$ whose (i,j)-coordinate is

$$[R^M]_{i,j} := \dim_k \operatorname{Hom}_A(U(i), \Delta_M(j)) - \dim_k \operatorname{Hom}_A(K(i), \Delta_M(j))$$

Remark 5.1. We note that in (3) we are incurring in an abuse of notation, since the G-matrices are only defined for τ-tilting pairs, see [8, 12, 27].

An important lemma in the proof of our main theorem is the following.

Lemma 5.2. *For a basic and τ-rigid module $M \in \operatorname{mod}(A)$ with a TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$, and the Ext-projective stratifying system $(\Delta_M, \underline{Q}, \leq)$ induced by $(\Delta_M, \leq)$, the following statements hold true.*

(a) $\mathrm{Hom}_A(U(i), \Delta_M(j)) = 0$ *and* $\mathrm{Hom}_A(K(i), \Delta_M(j)) = 0$ *if* $i \geq j$.

(b) *We have the isomorphisms of k-vector spaces*

$$\mathrm{Hom}_A(M_i, \Delta_M(i)) \simeq \mathrm{Hom}_A(Q(i), \Delta_M(i)) \simeq \mathrm{End}_A(\Delta_M(i)) \quad \forall\, i \in [1, t].$$

(c) *For all $i, j \in [1, t]$, we have*

$$[C_{\Delta_M}]_{i,j} = \dim_k \mathrm{Hom}_A(M_i, \Delta_M(j)) + [R^M]_{i,j}$$

(d) $\mathrm{Hom}_A(\Delta_M(j), \tau M_i) = 0$ *and* $\mathrm{Ext}^1_A(M_i, \Delta_M(j)) = 0\ \forall\, i, j$.

Proof. Fix some $i \in [1, t]$. Then, we have the exact sequence

$$\eta_i :\ 0 \to U(i) \xrightarrow{\alpha_i} Q(i) \xrightarrow{\beta_i} \Delta_M(i) \to 0,$$

where $U(i) \in \mathcal{F}(\Delta_M(j) : j > i)$, and the canonical exact sequence

$$\varepsilon_i :\ 0 \to K(i) \xrightarrow{\iota_i} M_i \xrightarrow{\pi_i} \Delta_M(i) \to 0$$

of M_i with respect to the torsion pair $(\mathcal{X}_i, \mathcal{Y}_i) := \left(\mathrm{Fac}(\bigoplus_{l>i} M_l), (\bigoplus_{l>i} M_l)^{\perp}\right)$. Moreover, by [22, Corallary 3.6(c)], we have that $\mathcal{F}(\Delta_M)$ is a subcategory of $\mathrm{Fac}(M)$.

We know that M_i is Ext-projective in $\mathrm{Fac}(M)$. Since $\mathcal{F}(\Delta_M) \subseteq \mathrm{Fac}(M)$, we have in particular that M_i is Ext-projective in $\mathcal{F}(\Delta_M)$. Hence, we can complete the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K(i) \arrow[r, "\iota_i"] \arrow[d, dotted, "q"] & M_i \arrow[r, "\pi_i"] \arrow[d, dotted, "p"] & \Delta_M(i) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U(i) \arrow[r, "\alpha_i"] & Q(i) \arrow[r, "\beta_i"] & \Delta_M(i) \arrow[r] & 0.
\end{tikzcd}$$

Note that the first square of the above diagram is the push-out of q and α_i. Hence, there exist an exact sequence $0 \to K(i) \to U(i) \bigoplus M_i \to Q(i) \to 0$ to which we apply the functor $\mathrm{Hom}_A(-, \Delta_M(j))$ to obtain the following one

$$(*) :\ 0 \to \mathrm{Hom}_A(Q(i), \Delta_M(j)) \to \mathrm{Hom}_A(U(i) \oplus M_i, \Delta_M(j)) \to \mathrm{Hom}_A(K(i), \Delta_M(j)) \to 0.$$

(a) Let $i \geq j$. Since $U(i) \in \mathcal{F}(\Delta_M(r) : r > i)$ and $\mathrm{Hom}_A(\Delta_M(r), \Delta_M(j)) = 0\ \forall\, r > i$, we can conclude that $\mathrm{Hom}_A(U(i), \Delta_M(j)) = 0$. On the other hand, by using that

$$K(i) \in \mathrm{Fac}\left(\bigoplus_{l>i} M_l\right) \subseteq \mathrm{Fac}\left(\bigoplus_{l>j} M_l\right),$$

it follows that $\mathrm{Hom}_A(K(i), \Delta_M(j)) = 0$.

(b) Take $i = j$ in $(*)$. Then, from (a), $\mathrm{Hom}_A(K(i), \Delta_M(i)) = 0$ and $\mathrm{Hom}_A(U(i), \Delta_M(i)) = 0$. Hence, by $(*)$ we get that $\mathrm{Hom}_A(Q(i), \Delta_M(i)) \simeq \mathrm{Hom}_A(M_i, \Delta_M(i))$. Finally, the isomorphism $\mathrm{Hom}_A(Q(i), \Delta_M(i)) \simeq \mathrm{End}_A(\Delta_M(i))$ follows from Lemma 3.3.

(c) It follows by applying $\dim_k$ on the exact sequence $(*)$.

(d) Since $\tau M \simeq \bigoplus_{r=1}^{t} \tau M_r$, it is enough to show that $\mathrm{Hom}_A(\Delta_M(j), \tau(M)) = 0\ \forall\, j \in [1, t]$. But, the aforementioned statement holds true since $\Delta_M(j) \in \mathrm{Fac}(M_j) \subseteq \mathrm{Fac}(M)$ and M is τ-rigid. Finally, we have $\mathrm{Ext}^1_A(M_i, \Delta_M(j)) \simeq D\overline{\mathrm{Hom}}_A(\Delta_M(j), \tau M_i) = 0$. □

Now, we are ready to announce and prove our first main result.

Theorem 5.3. *For a basic and τ-rigid module $M \in \mathrm{mod}\,(A)$ with a TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$, and the Ext-projective stratifying system $(\Delta_M, \underline{Q}, \leq)$ induced by $(\Delta_M, \leq)$, the following statements hold true.*

(a) *The matrix $(G^M)^{tr}\, S^M$ is upper triangular and $[(G^M)^{tr}\, S^M]_{i,j} = \dim_k \mathrm{Hom}_A(M_i, \Delta_M(j))$ $\forall\, i, j$.*

(b) *R^M is an upper triangular matrix with zeros in its diagonal. Moreover, we have that*
$$R^M = 0 \Leftrightarrow \mathrm{Hom}_A(M_i, \Delta_M(j)) \simeq \mathrm{Hom}_A(Q(i), \Delta_M(j)) \ \forall\, i < j.$$
(c) $C_{\Delta_M} = (G^M)^{tr}\, S^M + R^M$.
(d) $R^M = 0$ *if* $M \in \mathcal{F}(\Delta_M)$.
(e) $|G_{\Delta_M}| = \prod_{i=1}^{t} [(G^M)^{tr}\, S^M]_{i,i}$.

Proof. Let $i, j \in [1, t]$. Then, by Theorem 4.7 we get
$$[(G^M)^{tr}\, S^M]_{i,j} = \langle g^{M_i}, \underline{\dim}(\Delta_M(j))\rangle = \dim_k(\mathrm{Hom}_A(M_i, \Delta_M(j))) - \dim_k(\mathrm{Hom}_A(\Delta_M(j), \tau_A M_i)).$$
On the other hand, from Lemma 5.2 (d), we know that $\mathrm{Hom}_A(\Delta_M(j), \tau_A M_i) = 0$, proving that
$$[(G^M)^{tr}\, S^M]_{i,j} = \dim_k(\mathrm{Hom}_A(M_i, \Delta_M(j))).$$
Thus, from Lemma 5.2 (c), we also get the equality $C_{\Delta_M} = (G^M)^{tr}\, S^M + R^M$, proving (c). Moreover, by Lemma 5.2 (a) and the exact sequence $(*)$ in the proof of Lemma 5.2, we can obtain (b). Also, since C_{Δ_M} is upper triangular (see Lemma 3.3), then by (b) and (c) it follows that the matrix $(G^M)^{tr}\, S^M$ is upper triangular. Finally, (e) follows directly from (b), (c) and Lemma 3.3.

Assume now that $M \in \mathcal{F}(\Delta_M)$. Then, by the proof of [22, Corollary 3.8], we get that the exact sequences η_i and ε_i are isomorphic, for each $i \in [1, t]$. Therefore we obtain (d). □

As we have seen in Theorem 5.3 (d), the condition $M \in \mathcal{F}(\Delta_M)$ is enough to get that $R^M = 0$. In the following, we characterise when $M \in \mathcal{F}(\Delta_M)$.

Proposition 5.4. *Let $M \in \mathrm{mod}\,(A)$ be a basic and τ-rigid module with a TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$, and let $(\Delta_M, \underline{Q}, \leq)$ be the Ext-projective stratifying system induced by $(\Delta_M, \leq)$. Then, for any $i \in [1, t]$, the following statements are equivalent.*
(a) $M_i \in \mathcal{F}(\Delta_M)$.
(b) *The exact sequences η_i and ε_i are isomorphic.*
(c) $Q(i) \simeq M_i$.
(d) $\mathrm{Ext}^1_A(Q(i), K(i)) = 0$.
(e) $K(i) \simeq U(i)$.

Proof. (a) $\Rightarrow$ (b) It follows by the proof of [22, Corollary 3.8].
(b) $\Rightarrow$ (c) It is trivial.
(c) $\Rightarrow$ (d) It follows from $\mathrm{Ext}^1_A(M, \mathrm{Fac}(M)) = 0$ since $K(i) \in \mathrm{Fac}(M)$.
(d) $\Rightarrow$ (e) Since $\mathrm{Ext}^1_A(Q(i), K(i)) = 0$, there is some $q : Q(i) \to M_i$ with $\pi_i q = \beta_i$. On the other hand by the proof of Lemma 5.2, there is some $p : M_i \to Q(i)$ with $\beta_i p = \pi_i$. Since $\beta_i(pq) = \beta_i$ and $\beta_i : Q(i) \to \Delta_M(i)$ is right minimal [17, Lemma 2.3], we get that pq is an isomorphism, and thus $q : Q(i) \xrightarrow{\sim} M_i$ since $Q(i)$ and M_i are indecomposable. Thus, we have that the exact sequences η_i and ε_i are isomorphic. In particular, $K(i) \simeq U(i)$.
(e) $\Rightarrow$ (a) By the exact sequence ε_i and the fact that $\mathcal{F}(\Delta_M)$ is closed under extensions, we conclude that $M_i \in \mathcal{F}(\Delta_M)$. □

Let Λ be a finite dimensional k-algebra. We recall from [18] that the Cartan group G_Λ of Λ is the Cokernel of the Cartan map $C_\Lambda : K_0(\mathrm{proj}(\Lambda)) \to K_0(\Lambda)$, $[P_i] \mapsto \underline{\dim}(P_i)$, where $\{P_i\}_{i=1}^{n}$ is a representative set of pairwise non-isomorphic indecomposable projective Λ-modules.

Corollary 5.5. *Let $M \in \mathrm{mod}\,(A)$ be basic and τ-tilting and consider a TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$ such that $M \in \mathcal{F}(\Delta_M)$. Then, for the algebra $B := \mathrm{End}_A(M)^{op}$, we have that $G_B \simeq \mathrm{CoKer}(S^M)$.*

Proof. Let $(\Delta_M, \underline{Q}, \leq)$ be the Ext-projective stratifying system induced by $(\Delta_M, \leq)$. By Theorem 5.3 and Proposition 5.4, we get that $C_{\Delta_M} = (G^M)^{tr}\, S^M$ and $M_i \simeq Q(i)\ \forall\, i \in [1,t]$. On the other hand, by [1, Theorem 5.1], we conclude that the matrix $G^M \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$ induces an isomorphism $\mathbb{Z}^n \to \mathbb{Z}^n,\ X \mapsto G^M\, X$, of abelian groups. Therefore, $\mathrm{CoKer}(C_{\Delta_M}) \simeq \mathrm{CoKer}(S^M)$ and then the result follows from [18, Theorem 4.2]. □

As we have already seen, the Cartan matrix C_Θ associated to a stratifying system Θ is an upper-triangular matrix. We now will characterise in Theorem 5.7 the stratifying systems coming from τ-tilting modules M whose associated Cartan matrix C_Θ is diagonal. Before proving our second main theorem, we need the following intermediate result.

Proposition 5.6. *For a basic and τ-rigid module $M \in \mathrm{mod}\,(A)$, with a TF-admissible decomposition $M = \bigoplus_{i=1}^t M_i$, the following statements (a), (b) and (c) are equivalent.*

(a) *The matrix $(G^M)^{tr}\, S^M$ is diagonal and $\mathrm{Hom}_A(M_i, K(j)) = 0$ for $i < j$.*
(b) $\mathrm{Hom}_A(M_i, M_j) = 0$ *for $i < j$.*
(c) *For $i < j$, we have that $\mathrm{Hom}_A(\Delta_M(i), \Delta_M(j)) = 0$, $\mathrm{Hom}_A(M_i, K(j)) = 0$ and the map*

$$\mathrm{Hom}_A(K(i), \Delta_M(j)) \to \mathrm{Ext}^1_A(\Delta_M(i), \Delta_M(j)),\ f \mapsto f \cdot \varepsilon_i$$

is an isomorphism.

Moreover, for $B := \mathrm{End}_A(M)^{op}$ and $d_i := \dim_k \mathrm{End}_A(\Delta_M(i))$, we have that $G_B \simeq \bigoplus_{i=1}^t \mathbb{Z}/d_i\mathbb{Z}$ if $M \in \mathcal{F}(\Delta_M)$ and $(G^M)^{tr}\, S^M$ is diagonal.

Proof. Let $i, j \in [1,t]$. Consider the canonical exact sequence

$$\varepsilon_j :\ 0 \to K(j) \to M_j \to \Delta_M(j) \to 0$$

of M_j with respect to the torsion pair $\left(\mathrm{Fac}(\bigoplus_{l>j} M_l), (\bigoplus_{l>j} M_l)^\perp\right)$. Note that ε_j lies in $\mathrm{Fac}(M)$; and since $\mathrm{Ext}^1_A(M, \mathrm{Fac}(M)) = 0$, from ε_j we get the exact sequence

$$(*) :\ 0 \to \mathrm{Hom}_A(M_i, K(j)) \to \mathrm{Hom}_A(M_i, M_j) \to \mathrm{Hom}_A(M_i, \Delta_M(j)) \to 0.$$

(a) ⇔ (b): It follows from (*) and Theorem 5.3 (a).

(a) ⇔ (c): Let $i < j$. Consider the exact sequence $\varepsilon_i :\ 0 \to K(i) \to M_i \to \Delta_M(i) \to 0$. Then, we get the exact sequence

$$0 \to \mathrm{Hom}_A(\Delta_M(i), \Delta_M(j)) \to \mathrm{Hom}_A(M_i, \Delta_M(j)) \to \mathrm{Hom}_A(K(i), \Delta_M(j)) \to \mathrm{Ext}^1_A(\Delta_M(i), \Delta_M(j)) \to 0$$

Since $\mathrm{Ext}^1_A(M_i, \Delta_M(j)) = 0$ (see Lemma 5.2 (d)). Thus, the equivalence between (a) and (c) follows from the above exact sequence and Theorem 5.3 (a).

Let $M \in \mathcal{F}(\Delta_M)$ and $(G^M)^{tr}\, S^M$ be a diagonal matrix. By Theorem 5.3 and Proposition 5.4, we get that $C_{\Delta_M} = (G^M)^{tr}\, S^M$ and $M_i \simeq Q(i)\ \forall\, i$. Then, by [18, Theorem 4.2] and Lemma 5.2 (b), we conclude that $G_B \simeq \bigoplus_{i=1}^t \mathbb{Z}/d_i\mathbb{Z}$. □

Theorem 5.7. *For a basic τ-tilting module $M \in \mathrm{mod}\,(A)$, with a TF-admissible decomposition $M = \bigoplus_{i=1}^n M_i$ such that $M \in \mathcal{F}(\Delta_M)$, the following statements are equivalent.*

(a) *The matrix C_{Δ_M} is diagonal.*
(b) $\mathrm{Hom}_A(M_i, M_j) = 0$ *for $i < j$.*
(c) $\mathrm{Hom}_A(M_i, \Delta_M(j)) = 0$ *for $i < j$.*
(d) *For $i < j$, we have that $\mathrm{Hom}_A(\Delta_M(i), \Delta_M(j)) = 0$ and the map*

$$\mathrm{Hom}_A(K(i), \Delta_M(j)) \to \mathrm{Ext}^1_A(\Delta_M(i), \Delta_M(j)),\ f \mapsto f \cdot \eta_i$$

is an isomorphism.

Moreover, if one of the above conditions holds true, then $M \simeq {}_AA$, A is a weakly triangular algebra and $\mathrm{Hom}_A(M_i, K(j)) = 0$ for $i < j$.

Proof. Let $(\Delta_M, \underline{Q}, \leq)$ be the Ext-projective stratifying system induced by $(\Delta_M, \leq)$. By Theorem 5.3 and Proposition 5.4, we get that $C_{\Delta_M} = (G^M)^{tr}\, S^M$, $U(i) \simeq K(i)$ and $M_i \simeq Q(i)\ \forall\, i$. Thus, the fact that $\mathrm{Hom}_A(M_i, K(j)) = 0$ for $i < j$ (by assuming one of the above conditions) and the equivalences between (a), (b), (c) and (d) follow from Proposition 3.4.

Now, assume that one of the above equivalent conditions holds true. Let us proof, firstly, that $M \simeq {}_AA$. Indeed, since the decomposition $M = \bigoplus_{i=1}^n M_i$ is TF-admissible, we get from (b) that $M_i \notin \mathrm{Fac}(\bigoplus_{j\neq i} M_j)\ \forall\, i \in [1, n]$. On the other hand, it is known that $\bigoplus_{j\neq i} M_j$ admits two torsion classes, namely, $\mathrm{Fac}(\bigoplus_{j\neq i} M_j)$ and ${}^{\perp}(\bigoplus_{j\neq i} \tau M_j)$. Moreover, since $\mathrm{Fac}(\bigoplus_{j\neq i} M_j) \subsetneq {}^{\perp}(\bigoplus_{j\neq i} \tau M_j)$ and $M_i \notin \mathrm{Fac}(\bigoplus_{j\neq i} M_j)$, we have that $M_i \in {}^{\perp}(\bigoplus_{j\neq i} \tau M_j)$ and thus the mutation of $M = \bigoplus_{i=1}^t M_i$ at M_i is descendent, for all $i \in [1, n]$. Let us show that $\mathrm{Fac}(M) = \mathrm{mod}(A)$. Suppose that $\mathrm{Fac}(M) \neq \mathrm{mod}(A)$. Since $(\mathrm{Fac}(M), M^{\perp})$ is a torsion pair in $\mathrm{mod}(A)$, it follows from [8, Theorem 3.1], that there exists a mutation M' of M such that $\mathrm{Fac}(M) \subsetneq \mathrm{Fac}(M')$ contradicting that any mutation of M is descendent. Therefore $\mathrm{Fac}(M) = \mathrm{mod}(A)$ and then $M \simeq {}_AA$. In particular, A is a standardly stratified algebra since $M_i \simeq Q(i)\ \forall\, i$; and by [18, Remark 4.5], we conclude that A is a weakly triangular algebra. □

6. Examples and relation with the wall-and-chamber structure

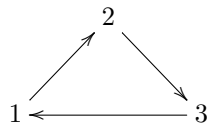
In this last section we calculate the Cartan matrix of stratifying systems induced by τ-rigid modules and we explore some connections between stratifying systems and stability conditions.

6.1. **Examples.** Before we go to the examples, we recall the notation of Lemma 5.2 of the two short exact sequences naturally associated to $\Delta_M(i)$ for $i \in \{1, \ldots, t\}$.

$$\varepsilon_i :\ 0 \to K(i) \xrightarrow{\iota_i} M_i \xrightarrow{\pi_i} \Delta_M(i) \to 0,$$

$$\eta_i :\ 0 \to U(i) \xrightarrow{\alpha_i} Q(i) \xrightarrow{\beta_i} \Delta_M(i) \to 0.$$

Example 6.1. Let A be the quotient path k-algebra given by the quiver



and the third power of the ideal generated by all the arrows. The Auslander-Reiten quiver of A can be seen in Figure 1. Note that every module is represented by its Loewy series and both

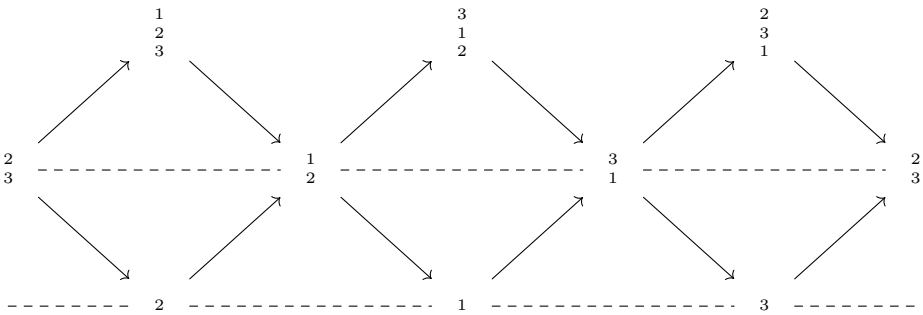


Figure 1. The Auslander-Reiten quiver of A

copies of $\begin{smallmatrix}2\\3\end{smallmatrix}$ should be identified. Thus, the Auslander-Reiten quiver of A has the shape of a cylinder.

Now take the τ-rigid modules $A = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix}$ and $M = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1$. It can be shown that, in both cases, the τ-tilting modules are written in a TF-admissible fashion. Hence, their corresponding induced stratifying system are $\Delta_A = \left\{1, 2, \begin{smallmatrix}3\\1\\2\end{smallmatrix}\right\}$ and $\Delta_M = \left\{\begin{smallmatrix}1\\2\\3\end{smallmatrix}, 1, 2\right\}$.

i	$\Delta_A(i)$	M_i	$K(i)$	$Q(i)$	$U(i)$
1	1	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$	$\begin{smallmatrix}2\\3\end{smallmatrix}$	$\begin{smallmatrix}1\\2\end{smallmatrix}$	2
2	2	$\begin{smallmatrix}2\\3\\1\end{smallmatrix}$	$\begin{smallmatrix}3\\1\end{smallmatrix}$	2	0
3	$\begin{smallmatrix}3\\1\\2\end{smallmatrix}$	$\begin{smallmatrix}3\\1\\2\end{smallmatrix}$	0	$\begin{smallmatrix}3\\1\\2\end{smallmatrix}$	0

TABLE 1. Modules associated to the stratifying system Δ_A

i	$\Delta_M(i)$	M_i	$K(i)$	$Q(i)$	$U(i)$
1	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$	0	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}$	0
2	1	$\begin{smallmatrix}1\\2\end{smallmatrix}$	2	$\begin{smallmatrix}1\\2\end{smallmatrix}$	1
3	2	2	0	2	0

TABLE 2. Modules associated to the stratifying system Δ_M

Using the modules in Table 1 and Table 2 and the Auslander-Reiten quiver of A, one can calculate easily the following matrices.

$$C_{\Delta_A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad G^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad S^A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad R^A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$C_{\Delta_M} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad G^M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \qquad S^M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad R^M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular, it is straightforward to verify that $C_{\Delta_A} = (G^A)^{tr} S^A + R^A$ and $C_{\Delta_M} = (G^M)^{tr} S_M + R^M$, as shown in Theorem 5.3.

Example 6.2. Let A be the quotient path k-algebra given by the quiver

$$\begin{tikzcd} 3 \arrow[r, shift left, "\beta_1"] & 1 \arrow[l, shift left, "\alpha_1"] \arrow[r, shift left, "\beta_2"] & 2 \arrow[l, shift left, "\alpha_2"] \end{tikzcd}$$

and the ideal generated by the set of relations $\{\beta_1\alpha_1\beta_1,\ \beta_1\alpha_1 - \alpha_2\beta_2,\ \alpha_2\beta_2\alpha_2\}$. Now take the τ-rigid module $A = \begin{smallmatrix}1\\3\\1\end{smallmatrix}\begin{smallmatrix}2\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\3\end{smallmatrix}$. Hence, its corresponding induced stratifying system are $\Delta_A = \left\{1, \begin{smallmatrix}2\\1\\2\end{smallmatrix}, \begin{smallmatrix}3\\1\\3\end{smallmatrix}\right\}$. A quick calculation shows that the modules associated to the stratifying system Δ_A are the following.

i	$\Delta_A(i)$	M_i	$K(i)$	$Q(i)$	$U(i)$
1	1	1 3 2 1	3 2 1	1	0
2	2 1 2	2 1 2	0	2 1 2	0
3	3 1 3	3 1 3	0	3 1 3	0

TABLE 3. Modules associated to the stratifying system Δ_A

From Table 3 we can calculate the matrices associated to Δ_A, which are below. In particular, it is straightforward to verify that $C_{\Delta_A} = (G^A)^{tr} S^A + R^A$, as shown in Theorem 5.3.

$$C_{\Delta_A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad G^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad S^A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad R^A = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

6.2. **Stratifying systems and wall-and-chamber structures.** Another important instance of the use of g-vectors was made to relate τ-tilting theory and stability conditions. Recall from [14] that given $M \in \mathrm{mod}(A)$ and a vector $v \in \mathbb{R}^n$, we say that M is v-semistable if $\langle v, \underline{\dim} M\rangle = 0$ and $\langle v, \underline{\dim} L\rangle \leq 0$ for all submodule L of M. Using this notion of stability, one can construct (for every finite dimesional algebra A) a geometric invariant known as the *wall-and-chamber structure* of A. We briefly recall this construction.

The union of the *walls* in the wall-and-chamber structure of A is the set of all the vectors $v \in \mathbb{R}^n$ such that there exists at least one nonzero module M which is v-semistable. A *chamber* in the wall-and-chamber structure of A is a open connected component of the complement of the union of the set of walls.

Now, let $M = \bigoplus_{i=1}^t M_i$ be a τ-rigid module, where M_i is indecomposable for all $1 \leq i \leq t$ and M_i is not isomorphic to M_j if $i \neq j$. Then we denote by $\mathcal{C}_M^\circ$ the cone spanned by the g-vectors of M_i for all $1 \leq i \leq t$, that is,

$$\mathcal{C}_M^\circ := \left\{ \sum_{i=1}^t \alpha_i g^{M_i} \mid \alpha_i \in \mathbb{R}_{>0} \right\}.$$

The explicit relationship between τ-tilting theory and stability conditions is given in the following proposition.

Proposition 6.3. [4, Proposition 3.13] *Let $M = \bigoplus_{i=1}^t M_i$ be a τ-rigid module and let $v \in \mathcal{C}_M^\circ$. Then a module X is v-semistable if and only if $X \in M^\perp \cap {}^\perp\tau M$.*

Remark 6.4. It follows from the previous proposition and the results of [1] that $\mathcal{C}_M^\circ$ is a chamber if and only if M is a τ-tilting module.

Based on Proposition 6.3 we can show the following property of stratifying systems induced by τ-rigid modules.

Proposition 6.5. *Let M be a τ-rigid module with TF-admissible decomposition $M = \bigoplus_{i=1}^t M_i$ and let $\Delta_M := \{\Delta_M(i) \mid 1 \leq i \leq t\}$ be the stratifying system induced by M. Then $\Delta_M(i)$ is v-semistable for all v in the cone $\mathcal{C}^\circ_{\bigoplus_{j>i} M_j}$ of $\bigoplus_{j>i} M_j$.*

Proof. By Proposition 6.3 we only need to show that $\Delta_M(i) \in \bigcap_{j>i} \left(M_j^\perp \cap {}^\perp\tau M_j\right)$. On the one hand, we have that $\Delta(i) \in \mathrm{Fac}M \subset {}^\perp\tau M \subset {}^\perp\tau M_j$ for all $j > i$. On the other hand, Theorem 4.5 states that $\Delta_M(i) \in \left(\bigoplus_{j>i} M_j\right)^\perp$ and the result follows. □

Note that in the statement of Proposition 6.5 if we fix $i = t$ we are saying that $M_t = \Delta_M(t)$ is v-semistable for all vector v in a cone spanned by the empty set. By convention, the cone spanned by the empty set is the cone $\{0\}$ consisting only of the origin 0 in $\mathbb{R}^n$. In this case, Proposition 6.5 is trivial since every module is 0-semistable.

If we assume that M is a τ-tilting module, then the Proposition 6.5 indicates that the stratifying system Δ_M induced by M can be interpreted as a distinguished way to construct the chamber $\mathcal{C}_M^\circ$ associated to M. Moreover, note that $\bigcap_{j>i}\left(M_j^\perp \cap {}^\perp\tau M_j\right)$ is a wide subcategory of $\operatorname{mod}(A)$ for all $1 \leq i \leq n$. This indicates that a construction of the *category of wide subcategories* introduced in [6] or even the *τ-cluster morphism category* [5, 7] can be constructed using the wall-and-chamber structure of the algebra.

Given the close relationship between g-vectors, stability conditions and Cartan matrices, it would be interesting to see if the latter has a geometric realisation in the wall-and-chamber structure of A.

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