A slow review of the AGT correspondence

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January 2022

Abstract

Starting with a gentle approach to the Alday–Gaiotto–Tachikawa (AGT) correspondence from its 6d origin, these notes provide a wide (albeit shallow) survey of the literature on numerous extensions of the correspondence up to early 2020. This is an extended writeup of the lectures given at the Winter School "YRISW 2020" to appear in a special issue of JPhysA.

Class S is a wide class of 4d $\mathcal{N} = 2$ supersymmetric gauge theories (ranging from super-QCD to non-Lagrangian theories) obtained by twisted compactification of 6d $\mathcal{N} = (2, 0)$ superconformal theories on a Riemann surface C. This 6d construction yields the Coulomb branch and Seiberg–Witten geometry of class S theories, geometrizes S-duality, and leads to the AGT correspondence, which states that many observables of class S theories are equal to 2d conformal field theory (CFT) correlators. For instance, the four-sphere partition function of a 4d $\mathcal{N} = 2$ SU(2) superconformal quiver theory is equal to a Liouville CFT correlator of primary operators.

Extensions of the AGT correspondence abound: asymptotically-free gauge theories and Argyres–Douglas theories correspond to irregular CFT operators, quivers with higher-rank gauge groups and non-Lagrangian tinkertoys such as T_N correspond to Toda CFT correlators, and nonlocal operators (Wilson–'t Hooft loops, surface operators, domain walls) correspond to Verlinde networks, degenerate primary operators, braiding and fusion kernels, and Riemann surfaces with boundaries.

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1 Introduction and outline

quantum field theories (QFTs) arise from many different constructions, be it Lagrangian descriptions, dimensional reduction or geometric engineering. The resulting building blocks can then be further deformed (e.g. partially Higgsed), coupled (e.g. by gauging symmetries), or reduced by decoupling a subsector. Theories living in different dimensions can also be fruitfully coupled together.

We explore these constructions, and some computation techniques, in the world of 4d $\mathcal{N}=2$ supersymmetric theories, specifically class S theories [1] which are dimensional reduction of a 6d theory ("S" stands for "Six"). Class S includes the most commonly studied 4d $\mathcal{N}=2$ Lagrangian gauge theories (super-Yang–Mills (SYM), super-QCD (SQCD), quiver gauge theories, $\mathcal{N}=4$ SYM and its mass deformation) and non-Lagrangian ones such as Argyres–Douglas (AD) theories [2], but also a plethora of previously unknown ones that have considerably broadened the set of known 4d $\mathcal{N}=2$ theories.

To construct a class S theory we start from a 6d $\mathcal{N}=(2,0)$ superconformal field theory (SCFT) denoted by $\mathcal{X}(\mathfrak{g})$, which is characterized by a simply-laced[1] Lie algebra $\mathfrak{g}$, for instance $\mathfrak{su}(N)$. We then reduce $\mathcal{X}(\mathfrak{g})$ on a Riemann surface C called the ultraviolet (UV) curve[2], while preserving 4d $\mathcal{N}=2$ supersymmetry thanks to a procedure called partial topological twist. The Riemann surface can have punctures (removed points, so that $C=\overline{C}\setminus\{z_1,\dots,z_n\}$ with $\overline{C}$ being compact) at which boundary conditions must be prescribed. Each choice of punctured Riemann surface, and data D_i describing the boundary condition at z_i, leads to one 4d $\mathcal{N}=2$ class S theory $\mathrm{T}(\mathfrak{g},C,D)$.

Due to their 6d origin, nonperturbative dynamics of class S theories are encoded in the geometry of C. For example the Seiberg–Witten (SW) curve [3, 4] of a theory, which determines the low-energy effective action in a given Coulomb branch vacuum, is a branched cover of C. Strikingly, this idea extends to many observables of the class S theory. The AGT correspondence [5] concerns the four-sphere (and ellipsoid) partition function:

$$Z_{S^4_b}(\mathrm{T}(\mathfrak{g},C,D)) = \Big\langle \widehat{V}_{D_1}(z_1)\dots\widehat{V}_{D_n}(z_n)\Big\rangle_{\overline{C}}^{\mathrm{Toda}(\mathfrak{g})} \tag{1.1}$$

where the right-hand side is a correlator of vertex operators in the Liouville CFT (for $\mathfrak{g}=\mathfrak{su}(2)$) or its generalization, Toda CFT. The vertex operators are inserted at each puncture z_i and depend on the data D_i characterizing punctures.

The rest of the introduction summarizes this review quickly: the reader should feel free to skip to the main text. Sections 2, 3, and 4 (summarized in subsection 1.1) describe the theories $\mathrm{T}(\mathfrak{g},C,D)$ and the puncture data D_i. Sections 5 and 6 (summarized in subsection 1.2) explain how to define and compute both sides of (1.1), namely $Z_{S^4_b}$ and Liouville CFT correlators. Finally, sections 7, 8, 9, and 10 describe numerous extensions of the correspondence, with pointers to the literature. In subsection 1.3 we present the preexisting reviews on topics related to AGT.[3]

[1]A simple Lie algebra $\mathfrak{g}$ is simply-laced if all its roots have the same length. Such algebras have an ADE classification: concretely, $\mathfrak{g}$ is one of $\mathfrak{su}(N)$, $\mathfrak{so}(2N)$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, or $\mathfrak{e}_8$.

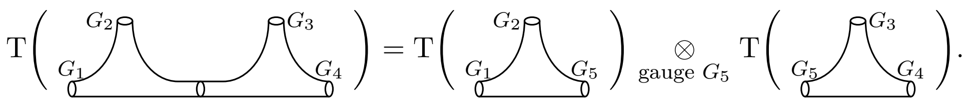
[2]The two-dimensional Riemann surface C is a complex curve: it has complex dimension 1.

[3]References (out before January 31, 2020) and comments on more recent developments welcome.

1.1 Class S theories

In the main text we study the 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{g})$ (section 2), its twisted dimensional reductions to class S theories (section 3), and Lagrangian descriptions of some of these 4d $\mathcal{N}=2$ theories (section 4). Here we only give some outcomes of these discussions. We often reduce to $\mathfrak{g}=\mathfrak{su}(N)$ for simplicity, but extensions to $\mathfrak{g}=\mathfrak{so}(2N)$ are also well-understood [6, 7].

Building blocks for $\mathrm{T}(\mathfrak{g},C,D)$**.** A Riemann surface C of genus g with n punctures can be cut into $2g-2+n$ three-punctured spheres, also called trinions or pairs of pants[4] glued together by tubes that connect pairs of punctures. Such a description is often called a pants decomposition of C. Correspondingly, the general class S theory $\mathrm{T}(\mathfrak{g},C,D)$ can be decomposed into class S theories called *tinkertoys* that correspond to each three-punctured sphere (tinkertoys range in complexity from free hypermultiplets to previously unknown non-Lagrangian isolated SCFTs). Each puncture is associated to a flavour symmetry, and connecting two punctures by a tube amounts to identifying the two associated flavour symmetries and gauging them using the same 4d $\mathcal{N}=2$ vector multiplet. For instance a four-punctured sphere can be split into two three-punctured spheres (for suitable groups $G_1,\dots,G_5$):



(1.2)

In simple cases where tinkertoys are collections of hypermultiplets, this results in gauge theories with an explicit Lagrangian made of hypermultiplets and vector multiplets. Thanks to the partial topological twist, the 4d theory does not depend on the metric of C [8, 9] but only on the complex structure of C, which can be described by the "length" and "angle" of each tube. These two parameters control the complexified gauge coupling ($q=e^{2\pi i\tau}$ with $\tau=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2}$) that combines the Yang–Mills coupling g with the theta angle θ of the 4d vector multiplet corresponding to the tube. Weak coupling $g\to 0$ corresponds to a very long tube.

Of course, C can be decomposed in many ways into three-punctured spheres: correspondingly, $\mathrm{T}(\mathfrak{g},C,D)$ has many equivalent dual descriptions involving completely different sets of fields and gauge groups. The weak gauge coupling regime of these descriptions correspond to regimes where the complex structure on C is well-described by one pants decomposition where three-punctured spheres are joined by very long tubes. These regimes, which are different cusps of the space of complex structures on C, are continuously connected by varying the gauge couplings. In this way, gauge theories at strong coupling in one description may admit a different weakly-coupled description. This phenomenon [1] generalizes S-duality of the SU(2) $N_f=4$ theory and of $\mathcal{N}=4$ SYM. The 6d construction thus makes these S-dualities manifest through C.

[4] That number is zero or negative for the sphere with 0, 1 or 2 punctures and the torus with no punctures: these Riemann surfaces cannot be cut into three-punctured spheres, and the class S construction does not give a 4d theory, see [8].

In the 6d construction, the punctures at $z_i \in \overline{C}$ are codimension 2 defects that wrap the 4d spacetime on which the class S theory is defined. To preserve supersymmetry of the 4d theory the defects should be half-BPS, namely preserve half of the original supersymmetry. One must classify such defects (typically by moving along the Coulomb branch), and then the tinkertoys corresponding to three-punctured spheres. Incidentally, the 6d theory also admits interesting half-BPS codimension 4 operators supported on 2d subspaces, which enrich the correspondence.

Coulomb branch and Seiberg–Witten curve. One way to get a handle on the theory $\mathrm{T}(\mathfrak{g}, C, D)$ is to describe its supersymmetric vacua, especially its Coulomb branch, and give the low-energy behaviour of the theory near each vacuum. This branch is spanned by Coulomb branch operators, namely local operators annihilated by all 4d antichiral supercharges.

Semiclassically, vacua of the 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{g})$ are parametrized modulo gauge transformations by some (commuting, diagonalizable) adjoint-valued scalars Φ_I, where $I = 6, \ldots, 10$ is an index for the $\mathfrak{so}(5)$ R-symmetry[5]. Alternatively they are parametrized by (consistent) values for gauge-invariant polynomials (Casimirs) such as $\mathrm{Tr}(\Phi_I \Phi_J)$. Coulomb branch vacua of the 4d theory are then configurations of the Φ_I (or rather of the invariant polynomials) allowed to vary along the curve C. More precisely, tracking down how 4d $\mathcal{N} = 2$ antichiral supercharges embed into 6d $\mathcal{N} = (2,0)$, we find two restrictions: the Casimirs depend holomorphically on the coordinate $z \in C$, and among all the Φ_I only $\Phi_z := \Phi_6 + i\Phi_7$ is non-zero. Because the partial topological twist mixes a subalgebra $\mathfrak{so}(2)$ of R-symmetry (under which Φ_z is charged) into the rotation group on C, $\Phi_z \mathrm{d}z$ is tensorial (specifically a one-form) on C. Roughly speaking, then, the 4d Coulomb branch is parametrized by the adjoint-valued holomorphic one-form $\Phi_z \mathrm{d}z$ on C modulo gauge transformations, and more invariantly by vacuum expectation values (VEVs) $\langle P_k(\Phi_z) \rangle \mathrm{d}z^k$ of Casimir polynomials.

In the $\mathfrak{g} = \mathfrak{su}(N)$ case we repackage them as $\phi_k(z) = u_k(z)\mathrm{d}z^k$, $k = 2, \ldots, N$, defined in local coordinates $z \in C$ by expanding

$$\langle \det(x - \Phi_z) \rangle = x^N + \sum_{k=2}^{N} u_k(z) x^{N-k}. \tag{1.3}$$

It is then useful to consider the zeros of this determinant

$$\Sigma = \left\{ (z, x) \;\middle|\; x^N + \sum_{k=2}^{N} u_k(z) x^{N-k} = 0 \right\} \subset T^*C, \tag{1.4}$$

where z is a coordinate on C and x parametrizes the fiber of the cotangent bundle T^*C, the bundle of one-forms[6] on C. The complex curve Σ depends on the choice of vacuum (specified by $\phi_2, \ldots, \phi_N$) and turns out to be the SW curve of $\mathrm{T}(\mathfrak{g}, C, D)$, presented as an N-fold (ramified) cover of C. It is equipped with a natural one-form $\lambda = x\mathrm{d}z$, the SW

[5] In the M-theory construction of the 6d theory, $\mathfrak{so}(5)$ rotates coordinates $x^6, \ldots, x^{10}$.

[6] This just means $x\mathrm{d}z$ transforms as a tensor under changing the coordinate z on C.

differential.[7] From the SW curve and differential (Σ, λ) of $\mathrm{T}(\mathfrak{g}, C, D)$ in a given Coulomb branch vacuum one can derive the infrared effective action (the prepotential). Masses of BPS particles can also be extracted as integrals of λ along closed contours.

Tame punctures and tinkertoys. A puncture at $z_i \in \overline{C}$ is described in this language as a singularity of the gauge-invariants ϕ_k. An important example is the *full tame puncture* which imposes a first order pole $\Phi_z \sim m_i(z-z_i)^{-1}\mathrm{d}z + O(1)$ at z_i, up to conjugation, where the residue $m_i \in \mathfrak{g}_{\mathbb{C}}$ is a suitably generic element of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. This mass[8] parameter m_i can be understood as a constant value for the background vector multiplet scalar that couples to the flavour symmetry $\mathfrak{g}$ corresponding to the puncture at z_i. In gauge-invariant terms this first order pole translates to

$$\langle P_k(\Phi_z)\rangle = \frac{P_k(m_i)}{(z-z_i)^k} + \dots, \tag{1.5}$$

or equivalently $\phi_k \propto \mathrm{d}z^k/(z-z_i)^k + \dots$ with a leading-order coefficient determined from m_i, using (1.3) in the $\mathfrak{su}(N)$ case.

In fact, when C gets pinched and split into two in the limit where a tube becomes infinitely thin, this type of singularity generically occurs. The main building block of class S theories is thus *the tinkertoy* $T_{\mathfrak{g}}$, namely the theory associated to a sphere with three full tame punctures. A frequent notation is $T_N \coloneqq T_{\mathfrak{su}(N)}$. By matching SW curves and SW differentials of $\mathrm{T}(\mathfrak{su}(2), C, D)$ theories to previously known theories such as SU(2) $N_f = 4$ SQCD, one checks that T_2 is simply a collection of 4 free hypermultiplets [1]. In general, however, the theory $T_{\mathfrak{g}}$ is a non-Lagrangian theory, with (at least) one flavour symmetry $\mathfrak{g}$ for each puncture. For instance, $T_{\mathfrak{su}(3)}$ is the Minahan–Nemeschansky SCFT with flavour symmetry $\mathfrak{e}_6 \supset \mathfrak{su}(3)^3$.

There are more general *tame punctures*, defined as points where one imposes a first order pole of Φ_z with a residue m that may be non-generic. The resulting tinkertoys amount to a *partial Higgsing*: moving onto the Higgs branch of $T_{\mathfrak{g}}$ by turning on a nilpotent VEV for (the moment map of) the symmetry carried by the puncture, thus reducing the symmetry. For $\mathfrak{su}(N)$ they are characterized by the pattern of equal eigenvalues of m, encoded as a partition of N, and they lead to lower-order poles for the ϕ_k. The partition for a full tame puncture is $N = 1+1+\dots+1$, also denoted by $[1^N]$; it carries $\mathfrak{su}(N)$ flavour symmetry (broken explicitly by the mass m). At the other extreme, the puncture corresponding to the partition $[N]$ (all eigenvalues equal, hence vanishing) is a trivial absence of puncture since it is a pole with zero residue. The next "smallest" puncture, called a *simple tame puncture* corresponds to the partition $[N-1,1]$ so $m = \mathrm{diag}(m_1, \dots, m_1, -(N-1)m_1)$; it carries $\mathfrak{u}(1)$ flavour symmetry, enhanced to $\mathfrak{su}(2)$ for $N=2$ since in that case the simple and full punctures are identical. Both the full and the simple tame punctures appear in the class S description of $\mathrm{SU}(N)$ $N_f = 2N$ SQCD, as depicted in Figure 1.

[7] Equation (1.4) is often reformulated as $\lambda^N + \sum_{k=2}^N \phi_k(z)\lambda^{N-k} = 0$.

[8] For $\mathfrak{su}(N)$, the N eigenvalues of m_i give residues of λ at each of the N points of Σ projecting to z_i. Integrating λ to compute masses of BPS particles picks up such residues, which are thus mass parameters.

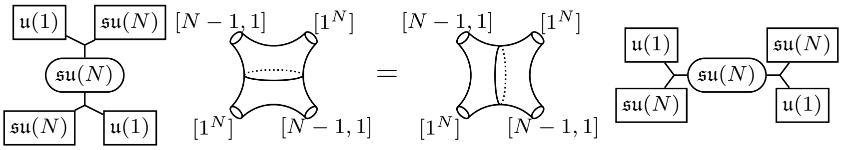
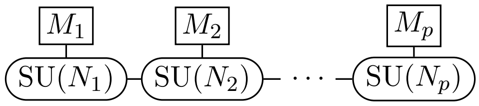


Figure 1: The $\mathfrak{su}(N)$ class S theory corresponding to a sphere with two full tame punctures (labelled $[1^N]$, flavour symmetry $\mathfrak{su}(N)$) and two simple tame punctures (labelled $[N-1,1]$, symmetry $\mathfrak{u}(1)$). We depict two pants decompositions constructed from spheres with one simple and two full punctures, whose corresponding tinkertoy is a collection of hypermultiplets. The decompositions lead to two S-dual Lagrangian descriptions of the theory as $\mathrm{SU}(N)$ SQCD with $N_f = 2N$. The third pants decomposition (not depicted here) involves non-Lagrangian tinkertoys (for $N > 2$).

While the gauge algebra carried by each tube is $\mathfrak{g}$ when all punctures are full tame punctures, more general tame punctures may lead to smaller gauge algebras. For example, $\mathfrak{su}(N)$ class S includes *linear quiver gauge theories* with gauge group $\prod_i \mathrm{SU}(N_i)$ (with $N_i \leq N$), one hypermultiplet in each bifundamental representation $N_i \otimes \overline{N_{i+1}}$, and $M_i \leq 2N_i - N_{i-1} - N_{i+1}$ hypermultiplets[9] in fundamental representations N_i of each $\mathrm{SU}(N_i)$. This is summarized in the quiver diagram



(1.6)

1.2 Basic AGT correspondence

We summarize here two sections that build up to the full AGT correspondence. First, section 5 describes how the (squashed) sphere partition function $Z_{S^4_b}$ of quiver gauge theories is computed using supersymmetric localization, and especially the issue of instanton counting. Then, section 6 explains basic aspects of Liouville CFT and gives the precise statement of the AGT correspondence for $\mathfrak{g} = \mathfrak{su}(2)$ generalized quivers.

Supersymmetric localization. In section 5 we explain how to place class S theories on the (squashed) four-sphere $S^4_b \coloneqq \{y_5^2 + b^2(y_1^2 + y_2^2) + b^{-2}(y_3^2 + y_4^2) = r^2\} \subset \mathbb{R}^5$ supersymmetrically, which incidentally requires masses to be purely imaginary. We also explain how to evaluate the partition function on this ellipsoid using supersymmetric localization [10, 11]. This path integral technique applies to each 4d $\mathcal{N} = 2$ Lagrangian description of $\mathrm{T}(\mathfrak{g}, C, D)$—if such a description exists.[10] Supersymmetric localization can

[9]When this bound is saturated the gauge coupling of that group does not run. When it is obeyed but not saturated (so $M_i < 2N_i - N_{i-1} - N_{i+1}$) we get an asymptotically free gauge theory, which can be realized in class S using wild punctures. When the bound is violated instead, the theory is only an effective theory and does not have a class S construction.

[10]Factorization properties of $Z_{S^4_b}$ that we find upon cutting the Riemann surface also hold for non-Lagrangian class S theories. They are obtained by applying supersymmetric localization to the vector

reduce the infinite-dimensional path integral down to a finite-dimensional integral over supersymmetric configurations of the hypermultiplets and vector multiplets. One finds configurations labeled by the (purely imaginary) constant value a of a vector multiplet scalar, which can be gauge-fixed to lie in the Cartan subalgebra of the gauge algebras. These configurations are additionally dressed by point-like instantons at one pole ($y_5 = r$) and anti-instantons at the other pole ($y_5 = -r$) of S^4_b. The partition function then reads

$$Z_{S^4_b}(q,\overline{q}) = \int \mathrm{d}a\, Z_{\mathrm{cl}}(a,q,\overline{q})\, Z_{\text{one-loop}}(a) Z_{\mathrm{inst}}(a,q) Z_{\mathrm{inst}}(a,\overline{q}), \tag{1.7}$$

where we omit the dependence on $\mathfrak{g}$ and data D at the punctures but write explicitly the dependence on complex structure parameters q of the curve C. Here, Z_{cl} comes from the classical action of supersymmetric configurations; it depends non-holomorphically on the complex gauge couplings q, but factorizes as $Z_{\mathrm{cl}}(a,q,\overline{q}) = Z_{\mathrm{cl}'}(a,q) Z_{\mathrm{cl}'}(a,\overline{q})$. Quadratic fluctuations around these configurations yield $Z_{\text{one-loop}}(a)$, a straightforward product of special functions that is completely independent of the shape (complex structure) of C. Finally, (anti-)instantons at each pole bring a factor of Z_{inst} that depends (anti-)holomorphically on gauge couplings q.

The factor $Z_{\mathrm{inst}}(a,q) = \sum_{k\geq 0} q^k Z_{\mathrm{inst},k}(a) = 1 + O(q^1)$ is Nekrasov's instanton partition function [12, 13] with parameters $\epsilon_1 = b/r$, $\epsilon_2 = 1/(rb)$, computable in favourable cases. The main difficulty is to compute each k-instanton contribution $Z_{\mathrm{inst},k}(a)$, which is an integral over the k-instanton moduli space. This space is finite-dimensional but very singular, and its singularities are understood best for unitary gauge groups. For linear quivers of unitary groups, which are obtained from (1.6) by replacing all $\mathrm{SU}(N_i)$ gauge groups by $\mathrm{U}(N_i)$, the Nekrasov partition function can be determined by equivariant localization or through IIA brane constructions. The instanton partition function of the SU theories (1.6) that we care about can then be derived by an appropriate decoupling of the U(1) factors, which divides $Z_{\mathrm{inst}}(a,q)$ by simple factors such as powers of $(1-q)$ [5]. Various other methods have been devised, but there is as of yet no complete first principles derivation of Z_{inst} for general class S theories, and even when restricting to $\mathfrak{g} = \mathfrak{su}(2)$ with tame punctures.[11]

S-dual Lagrangian descriptions of the same theory, obtained by different pants decompositions of C, should have the same partition function if S-duality is to hold. The equality of explicit integral expressions (1.7) is extremely challenging to prove, even for the SU(2) $N_f = 4$ theory. In fact the easiest way I know is to derive the AGT correspondence in that case (e.g. [14]) and then rely on modularity properties on the 2d CFT side shown in [15–17].

Liouville CFT correlators and basic AGT correspondence. In section 6 we move on to the other side of the correspondence for $\mathfrak{g} = \mathfrak{su}(2)$, namely Liouville CFT correlators. Liouville CFT depends on a "background charge" $Q = b + 1/b \geq 2$ (the central charge is $c = 1 + 6Q^2 \geq 25$), which translates on the 4d side to a deformation parameter of S^4 into

multiplets only, and not to the tinkertoys.

[11] I thank Jaewon Song for clarifications on this point.

the ellipsoid S^4_b. As in any 2d CFT, local operators organize into conformal families constructed by acting with the Virasoro algebra on primary operators. In the Liouville CFT these primaries are the vertex operators $\widehat{V}_\alpha$, labeled by a continuous parameter $\alpha = Q/2 + iP$ with $P \in \mathbb{R}/\mathbb{Z}_2$ (called momentum), and they have equal holomorphic and antiholomorphic dimension $h(\alpha) = \alpha(Q - \alpha) = Q^2/4 + P^2$. In the $\mathfrak{su}(2)$ case the data D_j for each tame puncture reduces to specifying a mass $m_j \in i\mathbb{R}/\mathbb{Z}_2$ (imaginary), naturally identified with a Liouville momentum (up to the sphere's radius r): the AGT correspondence then states

$$Z_{S^4_b}\big(\mathrm{T}(\mathfrak{su}(2), C, m)\big) = \langle \widehat{V}_{Q/2+rm_1}(z_1) \dots \widehat{V}_{Q/2+rm_n}(z_n)\rangle^{\text{Liouville}}_{\overline{C}}. \tag{1.8}$$

As in any 2d CFT, n-point functions of Virasoro primary operators on the Riemann surface $\overline{C}$ have a useful expression for each pants decomposition of the punctured Riemann surface C. The idea is to insert a complete set of states along each cut in the decomposition, then use Virasoro symmetry to rewrite all resulting three-point functions in terms of those of primaries. Schematically this gives

$$\Big\langle \widehat{V}_{\mu_1}(z_1, \bar{z}_1) \dots \widehat{V}_{\mu_n}(z_n, \bar{z}_n)\Big\rangle^{\text{Liouville}}_{\overline{C}} = \int \mathrm{d}\alpha\, C(\mu, \alpha)\mathcal{F}(\mu, \alpha, q)\mathcal{F}(\mu, \alpha, \overline{q}). \tag{1.9}$$

Here we integrate over all internal momenta α labelling the conformal family in each inserted complete set of states. The factor $C(\mu, \alpha)$ is a combination of structure constants of Liouville CFT. The other two factors are conformal blocks, which are purely about representation theory of the Virasoro algebra, and which depend (anti-)holomorphically on the complex structure parameters q of C, including (cross-ratios of) z_i.

Both sides of the AGT correspondence admit the same kind of expressions (1.7) and (1.9) for each pants decomposition of C, with one integration variable a or α for each tube, and a factorization of the dependence on q into holomorphic and antiholomorphic. In fact these expressions match factor by factor: $Z_{\text{one-loop}}(m, a) = C(\mu, \alpha)$ and $Z_{\text{cl}'}(a, q)Z_{\text{inst}}(m, a, q) = \mathcal{F}(\mu, \alpha, q)$. An additional entry in the dictionary is that ϕ_2 on the 4d side corresponds to the holomorphic stress-tensor $T(z)$ on the Liouville side in the classical limit $r \to \infty$: the leading term in an operator product expansion (OPE) with $T(z)$ matches $r^2\phi_2(z)$,

$$T(z)\widehat{V}_\mu(0) = \frac{h(\mu)\widehat{V}_\mu(0)}{z^2} + \cdots \underset{r\to\infty}{\simeq} -\frac{r^2m^2}{z^2}\widehat{V}_\mu(0) + \cdots \simeq -r^2\phi_2(z)\widehat{V}_\mu(0) + \dots. \tag{1.10}$$

We end section 6 by outlining the technical derivation of how Liouville CFT appears upon reducing the 6d theory on S^4 [18].

Extensions of the AGT correspondence. The AGT correspondence is generalized in two ways in section 7. First, asymptotically free theories and AD theories are described by replacing tame punctures by wild punctures, which replaces primary vertex operators by irregular ones on the CFT side. Second, $\mathfrak{su}(2)$ is replaced by $\mathfrak{g} = \mathfrak{su}(N)$: hypermultiplets are then replaced by non-Lagrangian building blocks T_N and Liouville CFT by Toda CFT.

In section 8 we investigate how to include in the AGT correspondence various gauge theory operators (local operators, Wilson–'t Hooft loops, …). The CFT side features Verlinde loops, degenerate vertex operators, fusion and braiding kernels, and Riemann surfaces with boundaries. The dictionary and references are summarized in Table 1.

Table 1: AGT correspondence for extended operators, sorted by codimension of the 6d operator or orbifold that yields them, and sorted by dimension on the 4d side. Most entries are hyperlinked to the main text.

		Operator in class S theory	Liouville/Toda CFT	References
Codimension 4	0d	Coulomb branch operator	Integrated current	[19]
Codimension 4	0d	Orbifold $\mathbb{C}^2/\mathbb{Z}_M$	Change CFT to coset	[20–37]
Codimension 4	1d	Dyonic loop: Wilson loop/'t Hooft loop	Degenerate Verlinde loop: around a tube/transverse	$\mathfrak{su}(2)$ [38–44], $\mathfrak{g}$ [45–54]
Codimension 4	2d	Vortex string operator	Degenerate vertex operator	[39, 55–71]
Codimension 2	2d	Gukov–Witten surface defect or orbifold $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$	Change CFT by Drinfeld–Sokolov reduction	[72–85]
Codimension 2	3d	Symmetry-breaking wall	Verlinde loop	[43]
Codimension 2	3d	S-duality domain wall	Modular kernel	[86–90]
Codimension 2	3d	Boundary	Boundary CFT	[91, 92]
Codimension 2	4d	Coupling to a tinkertoy	Vertex operator	[5, 93–109]

We discuss some offshoots of the AGT correspondence in section 9. Placing the 6d theory onto other product spaces $M \times C$ (with some twist) leads to interesting relations between theories on M and on C: the index/qYang–Mills (YM) correspondence [110], the 3d/3d correspondence [111], the 2d/4d correspondence [112]. In another direction, some class S theories (especially linear quiver gauge theories) can be realized as reductions of 5d $\mathcal{N} = 1$ theories. Instanton partition functions have direct analogues in 5d as certain q-deformations of the 4d results. This leads to a q-deformed AGT correspondence [113] equating these 5d instanton partition functions to chiral correlators ("conformal blocks") of q-deformed Virasoro or W_N algebras. The S^5 partition function involves *three* instanton partition functions, and its proper translation to non-chiral correlators of a complete q-Toda theory is still under investigation [114]. We end in section 10 with a quick outline of many topics omitted in this review, such as matrix models, topological strings, quantum integrable systems, etc.

1.3 Earlier reviews

There have been many good reviews related to the AGT correspondence, including in several PhD theses. I particularly recommend Tachikawa's very clear collection of reviews [115–118].

- **6d $(2,0)$ SCFTs.** These theories, and more generally 6d $(1,0)$ SCFT, are reviewed in [119] from an F-theory perspective. For codimension 2 defects, which are central in the AGT correspondence, see [120].

- **4d $\mathcal{N}=2$ and Seiberg–Witten.** While there are nice introductions from the late 1990's [121, 122] to the SW solution of 4d $\mathcal{N}=2$ theories, I recommend more modern explanations such as Martone's notes in this school [123], and the well-known review "for pedestrians" [115] which covers a lot of ground, including how AD theories arise from limits of SQCD. The book [124] discusses many modern relations between 4d $\mathcal{N}=2$ theories and other topics. The review [117] is focussed on the very important non-Lagrangian 4d $\mathcal{N}=2$ theory T_N.

- **Localization and instanton counting.** Supersymmetric localization is reviewed in the book [125], and in particular the squashed four-sphere partition function in [126]. Its expression involves Nekrasov's instanton partition function, for which a good starting point is [116], followed by [127] which discusses all gauge groups, subtleties regarding the U(1) factor, and the choice of renormalization scheme.

- **Toda CFT and W-algebras.**[12] Liouville CFT is reviewed in [128, 129] among many others, and it is worth reading [130] for some subtleties. There are no recent reviews on Toda CFT or on W-algebras. For W-algebras see the old [131, 132] (and possibly [133]) or the truncations of $W_{1+\infty}$ in [134, 135]. For Toda CFT perhaps the early article [136] or my thesis [137][13].

- **AGT for physicists.** See [118] (or perhaps [138], in Japanese) for a brief review, and the longer [139] ranging from SW basics to AD theories arising from degenerations of SQCD. The matrix model approach to AGT is reviewed in [140, 141].

- **AGT for mathematicians.** Possible starting points for mathematicians include the introductory seminar notes [142], a "pseudo-mathematical pseudo-review" [143, 144], incomplete (nevertheless 200 pages long) lecture notes [145], a categorical version of the correspondence [146], a review that focuses on moduli spaces of flat connections [147] and one discussing instanton counting on asymptotically locally Euclidean (ALE) spaces [148]. There are also notes on mathematical applications of the 6d $(2,0)$ SCFT to geometric representation theory, symplectic duality, knot homology, and Hitchin systems [149].

- **Generalizations of AGT.** These include the 3d/3d correspondence reviewed in [150] and the AGT relation between 5d $\mathcal{N}=1$ gauge theories and q-Toda correlators in [151].

Given these numerous reviews, writing yet another set of notes is perhaps futile, but hopefully the rather different approach taken here, starting from the 6d theory,

[12] I thank Ioana Coman for pointers.

[13] Better reference very welcome: only the A_{N-1} case is considered there, and only full, simple, and degenerate punctures rather than general tame punctured labeled by partitions of N.

is the right one for some readers. I apologize for omitting many directions from this review, listed in the conclusion section 10, especially a broader discussion of the BPS/CFT correspondence and of the deep links to quantization of integrable models underlying SW geometry, matrix models, and topological strings.

Part I
Class S theories

2 6d $(2,0)$ SCFT of ADE type

Superconformal algebras exist in dimensions up to 6, and there is by now ample evidence for the existence of 6d $\mathcal{N}=(2,0)$ (maximally supersymmetric) SCFTs $\mathcal{X}(\mathfrak{g})$, labelled by a Lie algebra $\mathfrak{g}$ that is simply-laced[14,15]. Nobody knows how to actually define $\mathcal{X}(\mathfrak{g})$ directly in a QFT language, for instance through a Lagrangian formulation. It is instead obtained as a decoupling limit of certain string theory or M-theory brane setups. Despite its stringy construction, the theory is expected to be a bona-fide local QFT, for instance having a local conserved stress-tensor.[16] These constructions entail three important properties detailed to the right.

Properties of $\mathcal{X}(\mathfrak{g})$

- $\mathcal{X}(\mathfrak{g})$ has vacua whose infra-red (IR) description is an abelian 6d $(2,0)$ theory of self-dual two-form gauge fields valued in the Cartan algebra of $\mathfrak{g}$ modulo the Weyl group.
- $\mathcal{X}(\mathfrak{g})$ is a UV-completion of 5d $\mathcal{N}=2$ SYM in the sense that SYM with gauge algebra $\mathfrak{g}$ and gauge coupling $g_{5\mathrm{d}}$ gives an IR description of $\mathcal{X}(\mathfrak{g})$ compactified on a circle of radius $g_{5\mathrm{d}}^2$.
- $\mathcal{X}(\mathfrak{g})$ admits codimension 2 half-BPS defects labeled by nilpotent orbits in $\mathfrak{g}$, and codimension 4 half-BPS defects labeled by representations of $\mathfrak{g}$.

The first two properties are compatible because both 5d $\mathcal{N}=2$ SYM on its Coulomb branch, and the abelian 6d theory on a circle, are described by 5d abelian vector multiplets in the Cartan of $\mathfrak{g}$. The last property is compatible as well, as the defects have rather explicit descriptions when one moves along the Coulomb branch or when one places the theory on a circle. The existence of $\mathcal{X}(\mathfrak{g})$ with these properties is confirmed by many consistency checks involving better-understood theories. A major set of consistency checks is the AGT correspondence obtained by placing these theories on the product $M_4\times C_2$ of a 4d and a 2d manifolds.

[14]As a reminder, simply-laced Lie algebras are $\mathfrak{a}_{N-1}=\mathfrak{su}(N)$, $\mathfrak{so}(2N)=\mathfrak{d}_N$, and the three exceptional algebras $\mathfrak{e}_6,\mathfrak{e}_7,\mathfrak{e}_8$ (in each case the subscript is the rank). This ADE classification has several beautiful avatars in theoretical physics but we will not get to explore them in this review.

[15]While different constructions of $\mathcal{X}(\mathfrak{g})$ give the same condition that $\mathfrak{g}$ is simply-laced, including some field theoretic arguments [152], it has not been proven that $\mathcal{X}(\mathfrak{g})$ exhaust all 6d $\mathcal{N}=(2,0)$ SCFTs. The situation is the same in 4d $\mathcal{N}=4$ SCFT: there might possibly be such theories other than $\mathcal{N}=4$ SYM theories.

[16]To be precise, it is a relative quantum field theory [153].

Table 2: Nahm classification of superconformal algebras in Lorentzian signature. Here we list the superconformal algebras in each dimension, the two bosonic factors (conformal algebra and R-symmetry algebra), and the representations (of these bosonic factors) in which Poincaré and conformal supercharges Q and S transform.

	Superalgebra	Conformal	R-symmetry	$Q\&S$
3d $\mathcal{N} \leq 8$	$\mathfrak{osp}(\mathcal{N}\|4)$	$\mathfrak{sp}(4,\mathbb{R})$	$\mathfrak{so}(\mathcal{N})$	$(4,\mathcal{N})$
4d $\mathcal{N} \leq 3$	$\mathfrak{su}(2,2\|\mathcal{N})$	$\mathfrak{su}(2,2)$	$\mathfrak{su}(\mathcal{N}) \oplus \mathfrak{u}(1)$	$(4,\overline{\mathcal{N}}) \oplus (\overline{4},\mathcal{N})$
4d $\mathcal{N} = 4$	$\mathfrak{psu}(2,2\|4)$	$\mathfrak{su}(2,2)$	$\mathfrak{su}(4)$	$(4,\overline{4}) \oplus (\overline{4},4)$
5d $\mathcal{N} = 1$	$\mathfrak{f}^2(4)$	$\mathfrak{so}(2,5)$	$\mathfrak{su}(2)$	$(8,2)$
6d $\mathcal{N} \leq 2$	$\mathfrak{osp}(8^*\|2\mathcal{N})$	$\mathfrak{so}^*(8)$	$\mathfrak{usp}(2\mathcal{N})$	$(8,2\mathcal{N})$

In this section we describe the symmetry algebra $\mathfrak{osp}(8^*|4)$ (subsection 2.1), properties of self-dual two-form gauge fields (subsection 2.2), string/M-theory constructions (subsection 2.3) and extended operators (subsection 2.4) of $\mathcal{X}(\mathfrak{g})$.

2.1 Superconformal algebras

Superconformal algebras in dimensions $d > 2$ have been classified by Nahm [154] under certain conditions. Their even (bosonic) part consists of the conformal algebra $\mathfrak{so}(2,d)$ (in Lorentzian signature) and an R-symmetry algebra, and their odd (fermionic) part consists of supercharges that must transform in the spinor representation of $\mathfrak{so}(2,d)$, and such that translations are realized as anticommutators of supercharges.

The classification is in Table 2. In dimensions $d = 3,4,6$ the conformal algebra coincides with the expected $\mathfrak{so}(2,d)$ thanks to accidental isomorphisms[17] $\mathfrak{so}(2,3) = \mathfrak{sp}(4,\mathbb{R})$ and $\mathfrak{so}(2,4) = \mathfrak{su}(2,2)$ and $\mathfrak{so}(2,6) = \mathfrak{so}^*(8)$. In each case, the spinor representation of $\mathfrak{so}(2,d)$ is the fundamental (vector) representation of the other group. It is known that SCFTs with more than 16 Poincaré supercharges do not exist for $d \geq 4$ (and are free for $d = 3$) [155], and this leads to the bounds on $\mathcal{N}$ given in the table.

For the 6d case of interest to us, minimal spinor representation of the Lorentz algebra $\mathfrak{so}(2,6)$ are chiral, and the superconformal algebras contain $\mathcal{N} = 1$ or 2 such chiral spinors (technically, symplectic Majorana–Weyl spinors) with the same chirality. These algebras are thus called 6d $\mathcal{N} = (1,0)$ and 6d $\mathcal{N} = (2,0)$ superconformal algebras. There is no 6d $\mathcal{N} = (1,1)$ superconformal algebra. We are interested in the largest superconformal algebra of all: the 6d $(2,0)$ algebra $\mathfrak{osp}(8^*|4)$.

Supercharges of this algebra transform in the $(\mathbf{8}_s,\mathbf{4})$ representation[18] of the conformal

[17] There are no such accidental isomorphisms for $d > 6$, which more or less explains the lack of higher-dimensional superconformal algebras.

[18] We denote irreps (irreducible representations) of a simple Lie algebra by their dimension in bold face. When ambiguities arise there are standard decorations to distinguish them, such as overlines for conjugating the representation, or primes when there are several irreps of the same dimension and they are not related by conjugation. A peculiar example is $\mathfrak{so}(8)$ and other real forms thereof like $\mathfrak{so}(p,8-p)$ as they have three dimension 8 irreps: the defining representation of $\mathfrak{so}(8)$ called 8_v, and two conjugate

and R-symmetry algebras $\mathfrak{so}(6,2) \times \mathfrak{so}(5)_{\mathrm{R}}$, with a reality condition. Decomposing this into representations of the Lorentz algebra $\mathfrak{so}(1,5)$ gives $(\mathbf{4},\mathbf{4}) \oplus (\mathbf{4},\mathbf{4})$, with a symplectic reality condition. One set $(\mathbf{4},\mathbf{4})$ consists of Poincaré supercharges and the other of superconformal transformations.

2.2 Self-dual forms

The 6d $\mathcal{N}=(2,0)$ SCFT $\mathcal{X}(\mathfrak{g})$ is roughly speaking a theory of self-dual two-forms gauge fields for a gauge Lie algebra $\mathfrak{g}$ among $\mathfrak{a}_{N-1}, \mathfrak{d}_N, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, as we explain next.

Abelian self-dual forms. In even dimension d there exists an interesting notion of (anti)[19] self-dual k-form for $k = d/2 - 1$: a k-form B with components $B_{\alpha_1\dots\alpha_k}$ (antisymmetric in $\alpha_1,\dots,\alpha_k$) such that the field strength $H = dB$ is mapped to a multiple of itself by the Hodge star, that is,

$$H_{\alpha_0\alpha_1\dots\alpha_k} \coloneqq (k+1)!\, \partial_{[\alpha_0} B_{\alpha_1\dots\alpha_k]} = \pm i^{d/2+s} \epsilon_{\alpha_0\dots\alpha_k\beta_0\dots\beta_k} \partial^{[\beta_0} B^{\beta_1\dots\beta_k]}. \tag{2.1}$$

Here indices within square brackets are antisymmetrized and the power of $i = \sqrt{-1}$ involves $s=0$ for Euclidean and $s=1$ for Lorentzian signature. The self-duality condition regards the field strength hence is invariant under gauge transformations $B \to B + d\Lambda$ for any k-form Λ: explicitly this adds $k!\, \partial_{[\alpha_1} \Lambda_{\alpha_2\dots\alpha_k]}$ to the component $B_{\alpha_1\dots\alpha_k}$ of the k-form gauge field B.

From (2.1) we see that real self-dual k-forms exist only if $d/2+s$ is even. In 2d this happens in Lorentzian signature, and it corresponds to a real scalar field propagating only in one lightlike direction. (In the Euclidean case it is a complex chiral boson depending on one holomorphic coordinate.) In 4d with Euclidean signature, (2.1) defines self-dual gauge field configurations, also called instantons, which play a crucial role on the 4d side of the AGT correspondence. (In the Lorentzian case they are complex saddle-points.) In 6d with Lorentzian signature we get a real self-dual two-form gauge field $B_{\alpha\beta}$.

We care about 6d $(2,0)$ supersymmetry, in which case the multiplet containing $B_{\alpha\beta}$ consists of B, spinors λ, and scalars Φ that transform respectively as the singlet, the 4-dimensional, and the 5-dimensional representations of R-symmetry $\mathfrak{usp}(4) = \mathfrak{so}(5)$.

Compactifying on a circle. Let us place this 6d $(2,0)$ abelian theory of (B,λ,Φ) on a circle and decompose into Kaluza–Klein (KK) modes. As determined in the following exercise, the five scalars Φ_I remain scalars, the spinors λ as well, and the self-dual two-form gauge field B becomes a usual gauge field A in 5d. Altogether this gives abelian 5d $\mathcal{N}=2$ SYM.

We review dimensional reduction in Exercise 2.1 below. An important aspect for the reduction from $\mathcal{X}(\mathfrak{g})$ to 5d is that 5d SYM has instanton particles, namely gauge field configurations with non-trivial topological number $\int \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \mathrm{d}^4 x$ on each spatial slice.

spinor representations 8_s and 8_c, related by the triality automorphism of $\mathfrak{so}(8)$.

[19] Self-dual and anti-self-dual cases differ by a sign, and we shall just write "self-dual" for simplicity.

These excitations of the gauge field A play the role of the tower of KK modes: their mass (proportional to) $1/g_{5d}^2$ is correctly identified with the mass $1/R$ of KK modes.

Exercise 2.1. *1. Consider a D-dimensional scalar field φ, with Lagrangian $\mathcal{L}(\varphi) = \partial_\alpha\varphi\partial^\alpha\varphi - V(\varphi)$ (you can take $V = 0$ for simplicity). Consider it on a d-dimensional Minkowski space times a $(D-d)$-dimensional torus of radius R (you can take $D - d = 1$ for simplicity). Write a Fourier decomposition of φ along the circle direction and rewrite the action of φ as an action for these components. In the limit $R \to 0$ notice that all Fourier modes become infinitely massive except the zero mode.*

2. Repeat the exercise for an abelian vector field A_α ($\alpha = 0, \ldots, D-1$) with Lagrangian $F_{\alpha\beta}F^{\alpha\beta}$, where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. Check that the dimensionally-reduced theory has both a vector field A_μ ($\mu = 0, \ldots, d-1$) and $D-d$ scalar fields. These can be gauge-invariantly understood for finite R as Wilson loops of A_α around coordinate circles of the torus. How do D-dimensional gauge transformations act? Deduce that the scalar fields are circle-valued.

3. Repeat the exercise for a two-form $B_{\alpha\beta}$ reduced from 6d to 5d. This results in a two-form $B_{\mu\nu}$ and a one-form A_μ. By imposing the self-duality condition on $B_{\alpha\beta}$ find that $B_{\mu\nu}$ can be reconstructed (up to gauge transformations) from A_μ.

Nonabelian theory. Recall the Bianchi identity $\partial_{[\mu}F_{\nu\rho]} = 0$ in 4d. It generalizes to $dH = ddB = 0$. For a self-dual form this implies the free equations of motion $d\star dB = 0$, namely $\partial^\mu H_{\mu\nu\ldots} = 0$. How can we add interactions? In 4d, the equation $F_{\mu\nu} = \mp\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ defining instantons makes sense even for the field strength of non-abelian gauge fields, $F = \mathrm{d}A + A\wedge A$, explicitly $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. The non-abelian version of the Bianchi identity is $\epsilon^{\lambda\mu\nu\rho}D_\mu F_{\nu\rho} = 0$. When the gauge field configuration is self-dual this implies the standard Yang–Mills equations of motion $D_\mu F^{\mu\lambda} = 0$. In contrast, for other self-dual k-forms, $k \neq 1$, there is no obvious non-abelian generalization of the relation $H = dB$, hence no obvious way to introduce interactions. Instead, we use two stringy constructions.

2.3 Brane construction of 6d theories

The 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{g})$ naturally arises as the zero string tension limit of a 6d $(2,0)$ little string theory, whose excitations are self-dual strings. These self-dual strings were uncovered as supergravity solutions [156], then in IIB string theory at ADE orbifold singularities [157], then as D2 branes ending on NS5 branes (or M2 ending on M5) [158]. (There also exist analogous 6d $(1,1)$ little string theories, but they have no conformal limit so we do not discuss them further.) The 6d $(2,0)$ little string theories are labeled by a simply-laced[20] Lie algebra $\mathfrak{g}$, just like $\mathcal{X}(\mathfrak{g})$, and we discuss two string theory constructions reviewed in [160].

- In the zero string coupling limit $g_s \to 0$ of a stack of N coincident NS5 branes in IIA string theory (or of M5 branes in M-theory), bulk degrees of freedom decouple, and one gets the 6d $(2,0)$ little string theory with $\mathfrak{g} = \mathfrak{su}(N)$.

[20] The ADE classification comes here from anomaly cancellation on the string worldsheet [159].

- General 6d $(2,0)$ little string theories arise in the zero string coupling limit $g_s \to 0$ of IIB string theory on a $\mathbb{C}^2/\Gamma$ singularity for the discrete subgroup $\Gamma \subset \mathrm{SU}(2)$ corresponding to $\mathfrak{g}$.

These approaches teach us that $\mathcal{X}(\mathfrak{g})$ on a circle is equivalent to 5d $\mathcal{N}=2$ SYM, and how to describe $\mathcal{X}(\mathfrak{g})$ upon moving on the tensor branch[21].

Fivebranes in IIA or M-theory. We begin with the M-theory (or equivalently IIA) construction, applicable to $\mathfrak{g}=\mathfrak{su}(N)$: $\mathcal{X}(\mathfrak{su}(N))$ is the world-volume theory of a stack of N coincident M5 branes in M-theory, with the decoupled center of mass degrees of freedom removed.

M-theory is an 11 dimensional theory (Lorentzian signature) with 32 supersymmetries (one Majorana spinor). It is related by various dualities to better-understood string theories and supergravity, for instance its low-energy limit is described by 11-dimensional supergravity. For our purposes, the most interesting aspect is that M-theory on a circle times a 10-dimensional spacetime is equivalent to IIA strings on that spacetime. The aim of this review is not to discuss the intricate web of dualities relating M-theory to IIA and other string theories, so we are quite schematic.

A standard comment on terminology: p branes are $(p+1)$ dimensional objects, with p space and 1 time directions, so for instance the M5 brane is 6-dimensional and has Lorentzian signature, as we wanted. M-theory has two such half-BPS objects: the M5 brane and the M2 brane. Stacks of flat[22] parallel branes of the same type preserve the same half of supersymmetry (see Exercise 2.2), and there is no energy cost to moving the branes while keeping them flat and parallel. While the world-volume theory of a stack of N Dp branes has been known for a long time to be maximally supersymmetric SYM in $p+1$ dimensions (see the review [161]), the world-volume theory of stacks of branes in M-theory has proven more difficult to pin down.

- The world-volume theory of a stack of coincident M2 branes is now known[23] to be the Aharony–Bergman–Jafferis–Maldacena (ABJM) Chern–Simons matter theory, an SCFT with an explicit 3d $\mathcal{N}=2$ Lagrangian description, whose supersymmetry enhances to the expected 3d $\mathcal{N}=8$ superconformal algebra $\mathfrak{osp}(8|4)$ preserved by the branes (see the review [162]). The R-symmetry $\mathfrak{so}(8)$ rotates the 11-dimensional

[21] The 6d $\mathcal{N}=(2,0)$ tensor multiplet splits into a 6d $\mathcal{N}=(1,0)$ tensor multiplet and a hypermultiplet. The tensor branch and Higgs branch are vacua where scalar fields in tensor or hyper multiplets acquire a VEV (with $(2,0)$ supersymmetry the two branches combine). The tensor branch is sometimes called Coulomb branch because it reduces to Coulomb branches in 5d and 4d. In 6d $\mathcal{N}=(1,0)$ theories one also has vector multiplets but they contain no scalars so there is no corresponding branch.

[22] Here we work as if spacetime were flat; the backreaction of branes on the geometry does not invalidate the conclusions.

[23] Depending on one's point of view, most words "known" in this review should be replaced by "conjectured". Ultimately, since the path integral has not been properly defined in most cases of interest to physicists, almost all non-perturbative QFT results are conjectural. One can think about how much "evidence" there is for one result or another. Results that are consistent with many others should then serve as a guide to determine if a given mathematical definition of the theories is acceptable.

space around the M2 branes. Its holographic dual is $\mathrm{AdS}_4 \times S^7$. That is all we will say in this review.

- The world-volume theory of a stack of N coincident M5 branes is what we call $\mathcal{X}(\mathfrak{su}(N))$, a 6d $(2,0)$ SCFT with no Lagrangian description.[24] More precisely, this would give $\mathfrak{u}(N)$, but the $\mathfrak{u}(1)$ center of mass of the branes decouples. The R-symmetry $\mathfrak{so}(5)$ rotates space around the M5 branes. The holographic dual is $\mathrm{AdS}_7 \times S^4$, which has the expected symmetry algebra $\mathfrak{osp}(8^*|4)$, differing only from the 3d case by some signs in the 7d and 4d parts.

Consequences of the M-theory construction. Consider now $\mathcal{X}(\mathfrak{su}(N))$ on a circle (times five-dimensional Minkowski space). M-theory on a circle is equivalent to IIA string theory, and M5 branes wrapping the circle become D4 branes. Thus, $\mathcal{X}(\mathfrak{su}(N))$ on a circle is equivalent to the world-volume theory of N D4 branes, which is 5d $\mathcal{N}=2$ SYM, as announced at the start of this section 2. Dimensional analysis shows that the 5d gauge coupling scales as $g_{5\mathrm{d}} \sim L_5^{1/2}$ in terms of the compactification circle length L_5.

We move on to describing the vacua of $\mathcal{X}(\mathfrak{g})$ from its M-theory construction. Supersymmetric vacua are parametrized by the positions of the N M5 branes in the 5 transverse directions, modulo relabelling of the branes since they are indistinguishable. The vacua are thus $(\mathbb{R}^N)^5/S_N$. At any generic vacuum, all degrees of freedom are massive (with mass proportional to the separation between the branes), except fluctuations around each individual brane, which are known to be described by one 6d abelian theory of (B,λ,Φ) for each brane. The scalar fields Φ_I, $I=6,\dots,10$, describe fluctuations of each of the N M5 branes in the transverse directions (except the $\mathfrak{u}(1)$ trace part).

IIB strings. The M-theory construction gives a lot of insight on $\mathcal{X}(\mathfrak{g})$ for $\mathfrak{g}=\mathfrak{a}_{N-1}$, and can be extended to $\mathfrak{d}_N$ by orbifolding, but it cannot realize the exceptional cases $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$. For this a dual IIB description is needed.

As understood in [163], T-duality transverse to a stack of N NS5 branes in IIA theory produces IIB strings on an A_{N-1} singularity. This second construction of the 6d $(2,0)$ little string theory, which we will not use much, generalizes readily to all ADE cases. Place IIB string theory on Minkowski space $\mathbb{R}^{1,5}$ times a quotient $\mathbb{C}^2/\Gamma$ by a finite subgroup $\Gamma \subset \mathrm{SU}(2)$. Such subgroups are classified by ADE Lie algebras $\mathfrak{g}$. For instance, the A_{N-1} case is $\Gamma = \mathbb{Z}_N$ acting as $(z,w) \mapsto (e^{2\pi i/N}z, e^{-2\pi i/N}w)$ on coordinates of $\mathbb{C}^2$. The zero-coupling limit $g_s \to 0$ of this set-up yields little string theory, and the further zero-tension limit gives the 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{g})$.

Moving along the vacuum moduli space of the 6d theory corresponds to blowing up $\mathbb{C}^2/\Gamma$ into ALE space, namely resolving the singularity at the origin of $\mathbb{C}^2/\Gamma$ into a collection of $r = \operatorname{rank}\mathfrak{g}$ finite-size two-cycles. The sizes of these two-cycles become r scalar fields Φ of the 6d theory on $\mathbb{R}^{1,5}$, in the Cartan subalgebra of $\mathfrak{g}$. In fact their VEV parametrizes the vacua of $\mathcal{X}(\mathfrak{g})$. In any vacuum, the IR degrees of freedom are: these

[24] Instead of $\mathfrak{g}=\mathfrak{a}_{N-1}=\mathfrak{su}(N)$ one can realize $\mathfrak{g}=\mathfrak{d}_N=\mathfrak{so}(2N)$ by including an O5 orbifold plane on top of the M5 branes.

scalar fields Φ, the two-form B obtained by integrating the chiral four-form of IIB string theory around each of the two-cycles, and some spinors. We end up as wanted with the 6d $(2,0)$ theory of an abelian self-dual two-form gauge field multiplet (B, λ, Φ) in the Cartan subalgebra of $\mathfrak{g}$.

In this description only SO(4) R-symmetry is manifest, and the reduction to 5d $\mathcal{N}=2$ SYM is also nontrivial to see.

2.4 Codimension 2 and 4 defects

We return to the M-theory construction of $\mathcal{X}(\mathfrak{su}(N))$ and consider intersecting brane configurations with branes placed along the following directions inside $\mathbb{R}^{1,10}$.

M5	0	1	2	3	4	5	.	.	.	.	.	→ 6d (2,0) theory
M5'	0	1	2	3	.	.	6	7	.	.	.	→ codimension 2 defect
M2	0	1	.	.	.	.	.	.	.	.	10	→ codimension 4 defect

Each column is a direction in $\mathbb{R}^{1,10}$; a dot indicate that a stack of branes is localized at a given value of that coordinate, and a number indicates that the brane extends along the corresponding direction. For instance the M5 branes are at given values of $x^6, x^7, x^8, x^9, x^{10}$ and extend in all other coordinates. The prime on M5' branes just helps us distinguish them from the M5 branes on which $\mathcal{X}(\mathfrak{g})$ lives. Each additional stack of branes in this table breaks half of supersymmetry (see Exercise 2.2). There can be several stacks of the same kind of branes parallel to each other, in which case they don't break further supersymmetry.

The M2 branes extend only in one direction transverse to the M5 branes. In this direction x^{10} they can either be infinite, or semi-infinite ending on one M5 brane, or finite stretching between two M5 branes. Either way, from the point of view of $\mathcal{X}(\mathfrak{g})$, stacks of M2 branes insert a half-BPS codimension 4 operator, namely an operator supported on a two-dimensional slice of the 6d theory [164].

The way it is written here, it would seem the M5 and M5' branes intersect in codimension 2. In truth they turn out to merge into a smooth complex manifold that asymptotes at large distances to the configuration we wrote. For this to happen, the x^6, x^7 positions of the M5 branes should grow to infinity as x^4, x^5 get closer to the positions of M5' branes, as depicted in Figure 2. We return to this in section 4 for concrete cases. From the point of view of $\mathcal{X}(\mathfrak{g})$, at large distance, the intersection with M5' branes has an effective description as a four-dimensional (codimension 2) half-BPS operator [165].

As we explore the AGT correspondence in this review we learn various properties of these defects, and especially the data that describes them. We find that:

- Codimension 2 operators are labeled by nilpotent orbits of $\mathfrak{g}$ [7, 100]. In the $\mathfrak{su}(N)$ case, these amount to partitions of N specifying the way in which the N M5 branes cluster into different groups as they go to infinity in the x^6, x^7 directions. Additional continuous data describes the length scales in these directions.

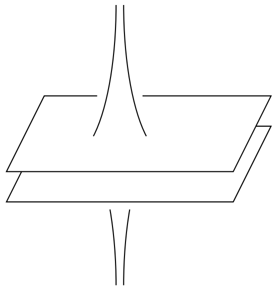


Figure 2: Configuration of a pair of M5 branes spanning the x^4, x^5 directions (depicted horizontally) in the presence of an M5' brane at a point in the x^4, x^5 plane. The M5 and M5' branes merge into a complex manifold. The x^6, x^7 positions (depicted vertically) diverge at one point in the x^4, x^5 plane. We depicted the situation after decoupling the center of mass modes, which is why the branes diverge symmetrically.

- Codimension 4 operators are labeled by representations of $\mathfrak{g}$. For the $\mathfrak{su}(N)$, recall that to each representation is associated a Young diagram, such that $\square$ is the fundamental N-dimensional representation, $\square\!\square$ is the symmetric representation, etc. The total number of boxes is the number of M2 branes necessary to describe the operator in M-theory. Roughly speaking, the number of boxes in each row of the Young diagram indicates how many M2 branes can end on the same M5 brane.

Exercise 2.2. *A flat M5 brane along directions $x^0, x^1, \dots, x^5$ preserves supersymmetries with $\Gamma^{012345}\epsilon = \epsilon$ while a flat M2 brane along directions x^0, x^1, x^{10} preserves supersymmetries with $\Gamma^{23456789}\epsilon = \epsilon$. Check that the brane configurations above are such that each additional stack of branes breaks half of supersymmetry. (Hint: check that Γ^{01}, Γ^{23}, Γ^{45} etc. commute with each other.) What other relative orientations of the stacks of branes preserve half of the supersymmetry?*

3 Class S theories from 6d

Our next task is to dimensionally reduce the 6d theory $\mathcal{X}(\mathfrak{g})$ on a Riemann surface C_2. We explain in subsection 3.1 a partial topological twist such that the reduced theory has 4d $\mathcal{N} = 2$ supersymmetry [1]. The Coulomb branch and SW curves giving the IR physics are worked out in subsection 3.2 and subsection 3.3. We then explain in subsection 3.4 how the 4d theory decomposes into building blocks called tinkertoys [95].

6d viewpoint on 4d $\mathcal{N} = 2$ class S theories

- Reducing (with a twist) the 6d theory $\mathcal{X}(\mathfrak{g})$ on a punctured Riemann surface C gives a 4d $\mathcal{N} = 2$ theory.
- Its Coulomb branch is parametrized by differentials $\phi_k = u_k \mathrm{d}z^k$ of the same degrees k as the Casimir invariants of $\mathfrak{g}$.
- Its SW curve $\Sigma \subset T^*C$ is a multiple cover of C.
- Gluing punctures amounts to gauging symmetries.
- Cutting C amounts to decoupling gauge fields.

3.1 Partial topological twist

Our aim is to place the 6d $(2,0)$ theory on $\mathbb{R}^4 \times C_2$, where C_2 is an arbitrary punctured Riemann surface. Doing this too naively would not preserve any symmetry beyond the Poincaré symmetry of $\mathbb{R}^4$. We explain a procedure, the *partial topological twist*, that allows 4d $\mathcal{N}=2$ supersymmetry to be preserved regardless of C_2.

Generalities on topological twist. First we comment on the topological twist of supersymmetric theories [166] in general terms from several point of views.

When placing a field theory on a curved background, the metric $g_{\mu\nu}$ acts as a source for the stress tensor $T^{\mu\nu}$. For a supersymmetric field theory, $T^{\mu\nu}$ typically belongs to a multiplet together with supersymmetry currents S^μ_α and R-symmetry currents J^μ. These can also be coupled to sources ψ^α_μ and A_μ. The partial topological twist consists of setting $S^\mu_\alpha = 0$ and choosing A_μ equal to the spin connection derived from $g_{\mu\nu}$. Schematically, at linearized order around some background values of g, ψ, A, when these fields are changed the Lagrangian varies by

$$\delta\mathcal{L} = T\delta g + J\delta A = T\delta g - J\partial(\delta g) \simeq (T + \partial J)\delta g. \tag{3.1}$$

In the second step we used our choice that A is related to derivatives of the metric, and in the last step we integrate by parts.

In this way the topological twist amounts to redefining the stress-tensor from T to $T_{\text{twist}} = T + \partial J$ before placing the theory on a non-trivial background metric. The twist mixes the stress-tensor $T^{\mu\nu}$ with the R-symmetry current J^μ, but it is good to remember that it does not affect any observables of the theory in flat space, only what we call the stress-tensor. Through the change of stress-tensor it changes how the theory is put on curved spaces.

One job of the stress-tensor is to keep track of Poincaré symmetries: $T^{\mu\nu}$ is the conserved current of translation symmetries, while $x^{[\mu}T^{\nu]\rho}$ is the conserved current of rotations. Since the twist shifts T by a total derivatives it is simply an improvement transformation of the translation symmetry current, and it does not change the corresponding conserved charge, the momentum operator. In contrast, it has a non-trivial effect on what we call rotations: twisted rotation acts by a rotation plus an R-symmetry transformation, because schematically $xT \mapsto xT + x\partial J = xT - J + \partial(xJ)$ where $\partial(xJ)$ is an improvement transformation. Commutators between translations and the twisted rotations nevertheless coincide with those in the standard Poincaré algebra.

What happens to supercharges? They typically transform as spinors under the original rotations and under R-symmetry transformations. Under the new rotations embedded diagonally, the supercharges typically split into a scalar supercharge Q and a vector. By virtue of Q being a scalar, the stress tensor is Q-closed, so that placing the theory on a curved manifold using the twisted stress-tensor preserves the supersymmetry Q. The next step is typically to restrict to operators in the Q-cohomology. In many cases, the twisted stress-tensor is Q-exact, namely $T^{\mu\nu}_{\text{twist}} = \{Q, G^{\mu\nu}\}$ for some supersymmetry generator $G^{\mu\nu}$, so that it vanishes in Q-cohomology and the correlators are described by a topological quantum field theory (TQFT). For the twist we consider, this will not happen and there will remain non-trivial local dynamics instead.

Partial topological twist of 6d theories. The *partial* topological twist we use consists of only mixing *some* of the R-symmetries into *some* of the rotation symmetries. To define the specific twist we use, consider rotations $\mathfrak{so}(1,3) \times \mathfrak{so}(2)_{\text{old}}$ preserving separately the two factors of a product $\mathbb{R}^{1,3} \times \mathbb{R}^2$, and consider the block-diagonal subalgebra $\mathfrak{so}(2)_{\text{R}} \times \mathfrak{so}(3)_{\text{R}} \subset \mathfrak{so}(5)_{\text{R}}$ of R-symmetry. We define twisted rotations to be embedded diagonally into $\mathfrak{so}(2)_{\text{old}} \times \mathfrak{so}(2)_{\text{R}}$, namely we treat the following symmetries as our (twisted) Lorentz and R-symmetries:

$$\mathfrak{so}(1,3) \times \mathfrak{so}(2)_{\text{twist}} \times \mathfrak{so}(3)_{\text{R}}. \tag{3.2}$$

This is done by changing the stress-tensor to

$$T^{\mu\nu}_{\text{twist}} = T^{\mu\nu}_{\text{old}} + \frac{1}{4}(\epsilon^{\mu\rho}\partial_\rho J^\nu_{12} + \epsilon^{\nu\rho}\partial_\rho J^\mu_{12}), \tag{3.3}$$

where J_{12} is the R-symmetry rotation generator of $\mathfrak{so}(2)_{\text{R}}$ and $\epsilon^{\mu\nu} = \delta^\mu_4\delta^\nu_5 - \delta^\nu_5\delta^\mu_4$ is the Levi-Civita tensor on the $\mathbb{R}^2$ factor.

Exercise 3.1. *Check that* (3.3) *shifts the* x^4, x^5 *rotation current* $x^{[4}T^{5]\mu}$ *by* J_{12} *up to total derivatives (an improvement term), so that the twisted rotation is a combination of rotation and R-symmetry.*

Let us track supersymmetries as we twist and then compactify. Under the $\mathfrak{so}(1,5)$ rotations of $\mathbb{R}^{1,5}$ and $\mathfrak{so}(5)_{\text{R}}$ R-symmetry, the Poincaré supersymmetries transform in the spinor representation of each, denoted $(\mathbf{4},\mathbf{4})$, with a symplectic reality condition that we hide for simplicity. Each 6d Weyl spinor, namely each representation $\mathbf{4}$ of $\mathfrak{so}(1,5)$ decomposes into a pair of 4d Weyl spinors of opposite chirality $(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$ under $\mathfrak{so}(1,3)$, and these spinors have opposite charges $1/2$ and $-1/2$ under $\mathfrak{so}(2)_{\text{old}}$. Each spinor $\mathbf{4}$ of $\mathfrak{so}(5)$ decomposes into two $\mathbf{2}$ of $\mathfrak{so}(3)_{\text{R}}$ with $\mathfrak{so}(2)_{\text{R}}$ charges $\pm 1/2$. Altogether we denote this as follows, with subscripts denoting charges under the two $\mathfrak{so}(2)$ algebras:

$$\begin{aligned}(\mathbf{4},\mathbf{4}) &= \Big((\mathbf{2},\mathbf{1})_{\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2})_{-\frac{1}{2}}\Big) \otimes \Big(\mathbf{2}_{\frac{1}{2}} \oplus \mathbf{2}_{-\frac{1}{2}}\Big) \\ &= (\mathbf{2},\mathbf{1},\mathbf{2})_{\frac{1}{2},\frac{1}{2}} \oplus (\mathbf{2},\mathbf{1},\mathbf{2})_{\frac{1}{2},-\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-\frac{1}{2},\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-\frac{1}{2},-\frac{1}{2}}.\end{aligned} \tag{3.4}$$

By construction the charge under $\mathfrak{so}(2)_{\text{twist}}$ is the sum of those under $\mathfrak{so}(2)_{\text{old}}$ and $\mathfrak{so}(2)_{\text{R}}$. Thus, under the $\mathfrak{so}(1,3) \times \mathfrak{so}(3)_{\text{R}} \times \mathfrak{so}(2)_{\text{twist}}$ symmetry of $\mathbb{R}^{1,5}$ that we are concentrating on, Poincaré supercharges transform as

$$(\mathbf{2},\mathbf{1};\mathbf{2})_1 \oplus (\mathbf{2},\mathbf{1};\mathbf{2})_0 \oplus (\mathbf{1},\mathbf{2};\mathbf{2})_0 \oplus (\mathbf{1},\mathbf{2};\mathbf{2})_{-1}. \tag{3.5}$$

We denote them respectively as

$$Q^{\alpha A}_z, Q^{\alpha A}, \bar{Q}^{\dot{\alpha} A}, \bar{Q}^{\dot{\alpha} A}_{\bar{z}}, \tag{3.6}$$

where $\alpha, \dot{\alpha}, A$, ranging from 1 to 2, are indices for spinors of $\mathfrak{so}(1,3)$ of the two chiralities and spinors of $\mathfrak{so}(3)_{\mathrm{R}}$, respectively, while z is a complex coordinate on the $\mathbb{R}^2$ factor that keeps track of $\mathfrak{so}(2)_{\mathrm{twist}}$ charges ± 1 of the first and last supercharges $Q_z, \bar{Q}_{\bar{z}}$.

The middle two supercharges $Q, \bar{Q}$ are scalars under $\mathfrak{so}(2)_{\mathrm{twist}}$ rotations, so that deforming the metric on $\mathbb{R}^2$ to any curved metric preserves these supercharges. Altogether, upon compactifying on $\mathbb{R}^{1,3} \times C$ with the partial topological twist we obtain a system that preserves $\mathfrak{iso}(1,3)$ Poincaré symmetry, supercharges $Q^{\alpha A}$ and $\bar{Q}^{\dot{\alpha} A}$, and the $\mathfrak{so}(3)_{\mathrm{R}} = \mathfrak{su}(2)$ R-symmetry. Together these form the 4d $\mathcal{N} = 2$ Poincaré supersymmetry algebra.

In the limit where C has zero size, we thus obtain a 4d $\mathcal{N} = 2$ theory, generically.[25] Twisting (3.3) does not preserve the tracelessness of T, so even though the original 6d rotation symmetry extends to conformal symmetry, this is not the case of the twisted rotation symmetry. In the zero area limit, 4d conformal symmetry can be restored and we get an SCFT unless data at punctures of C carry an intrinsic mass scale.

3.2 Coulomb branch

The Coulomb branch of a 4d $\mathcal{N} = 2$ theory is described by giving a VEV to Coulomb branch operators, namely (gauge-invariant) operators of the 4d theory that are annihilated by all antichiral Poincaré supercharges $\bar{Q}^{\dot{\alpha} A}$. Let us identify these operators starting from the 6d theory $\mathcal{X}(\mathfrak{g})$, following roughly [167, section 3].

Importantly, the resulting Coulomb branch $\mathcal{B}$ obtained in (3.13) *only depends on the complex structure of* C, not on its metric. This lets us deform the Riemann surface in various ways to understand the resulting 4d theory, and it underlies the appearance of 2d CFT objects on C in the AGT correspondence.

General supersymmetry considerations allow the vacuum moduli space of 4d $\mathcal{N} = 2$ theories [168] to be a union of mixed branches $\mathcal{C}_\alpha \times \mathcal{H}_\alpha$, which include the pure Coulomb and pure Higgs branches as special cases. The special Kähler manifolds $\mathcal{C}_\alpha$ are parametrized by Coulomb branch operators and the hyper-Kähler manifolds $\mathcal{H}_\alpha$ are parametrized by Higgs branch operators. Higgs branch chiral ring relations for class S theories were explored in [169–172]. Determining all branches as done in [173, 174] for class S theories is in general difficult, so we will concentrate solely on the Coulomb branch (for which $\mathcal{H}_\alpha$ is a point).

Coulomb branch operators. The vacua of $\mathcal{X}(\mathfrak{g})$ are parametrized by the VEV of scalar fields Φ_I, $I = 6, \dots, 10$, in the Cartan subalgebra of $\mathfrak{g}$ (modulo the Weyl group). The low-energy theory in a given vacuum is described by fluctuations of these fields as well as spinors λ and a self-dual two-form B. Under the $(\mathfrak{so}(1,3) \times \mathfrak{so}(3)_{\mathrm{R}}) \times \mathfrak{so}(2)_{\mathrm{old}} \times \mathfrak{so}(2)_{\mathrm{R}}$

[25]The system at finite area of C has a certain moduli space of vacua, and in the scaling limit where the area is sent to zero one must specify around which vacuum to expand. If C has "enough" handles or punctures, then its Higgs branch has a maximally symmetric point around which it is natural to expand, and the 4d $\mathcal{N} = 2$ limit is well-defined. If C is a sphere with "too few punctures" or is a torus without punctures, there is no maximally symmetric point and the situation is more subtle, as explained in [8].

symmetry algebra of interest to us just before the twist, these fields transform as

$$\begin{aligned}
&B \in (\mathbf{3},\mathbf{1},\mathbf{1})_{0,0} \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_{0,0} \oplus (\mathbf{2},\mathbf{2},\mathbf{1})_{\pm 1,0} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{0,0}, \\
&\lambda \in (\mathbf{2},\mathbf{1},\mathbf{2})_{\frac{1}{2},\pm\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_{-\frac{1}{2},\pm\frac{1}{2}}, \qquad \Phi_z := \Phi_6 + i\Phi_7 \in (\mathbf{1},\mathbf{1},\mathbf{1})_{0,1}, \\
&\Phi_8, \Phi_9, \Phi_{10} \in (\mathbf{1},\mathbf{1},\mathbf{3})_{0,0}, \qquad \Phi_{\bar z} := \Phi_6 - i\Phi_7 \in (\mathbf{1},\mathbf{1},\mathbf{1})_{0,-1}.
\end{aligned} \tag{3.7}$$

On the other hand the supercharges $\bar{Q}^{\dot\alpha A}$ transform as $(\mathbf{1},\mathbf{2},\mathbf{2})_{-1/2,1/2}$. We deduce that

$$\bar{Q}^{\dot\alpha A}\Phi_z = 0 \tag{3.8}$$

because no component of λ has the appropriate $\mathfrak{so}(2)_{\mathrm{R}}$ charge $3/2$.

Really, we should be working with the corresponding gauge-invariant operators, such as traces $\mathrm{Tr}(\Phi_z^j)$ in classical cases $\mathfrak{su}(N)$ and $\mathfrak{so}(2N)$. These are the Casimirs of $\mathfrak{g}$, polynomials $P_k(\Phi_z)$ of various degrees d_k for $k = 1, \dots, \operatorname{rank}\mathfrak{g}$. Concretely, for classical gauge groups these gauge-invariant operators annihilated by $\bar{Q}^{\dot\alpha A}$ are

$$\begin{aligned}
&\mathrm{Tr}(\Phi_z^j), \quad j = 2,3,4,\dots,N && \text{for } \mathfrak{su}(N), \\
&\mathrm{Tr}(\Phi_z^j), \quad j = 2,4,6,\dots,2N-2, \text{ and } \quad \mathrm{Pfaff}(\Phi_z) && \text{for } \mathfrak{so}(2N).
\end{aligned} \tag{3.9}$$

(We recall that the Pfaffian is a square root of the determinant.) For reference, the degrees of Casimirs of $\mathfrak{su}(N)$ are $2,3,\dots,N$; of $\mathfrak{so}(2N)$ are $2,4,6,\dots,2N-2$ and N; of $\mathfrak{e}_6$ are $2,5,6,8,9,12$; of $\mathfrak{e}_7$ are $2,6,8,10,12,14,18$; of $\mathfrak{e}_8$ are $2,8,12,14,18,20,24,30$.

It is often convenient to replace Φ_z by $\Phi_z \mathrm{d}z$ to soak up the z index and obtain a tensor. Then we work with the order d_k differentials $P_k(\Phi_z)\mathrm{d}z^{d_k}$ on the holomorphic curve (aka Riemann surface) C. A somewhat different basis is more practical: for instance for $\mathfrak{su}(N)$ one expands

$$\det(X - \Phi_z \mathrm{d}z) = X^N - \sum_{j=2}^{N} \mathcal{O}_j \, X^{N-j}. \tag{3.10}$$

Exercise 3.2. *Check that* $\mathcal{O}_2 = \mathrm{Tr}(\Phi_z^2/2)\mathrm{d}z^2$, $\mathcal{O}_3 = \mathrm{Tr}(\Phi_z^3/3)\mathrm{d}z^3$, *and perhaps check that* $\mathcal{O}_4 = \mathrm{Tr}(\Phi_z^4/4)\mathrm{d}z^4 - \mathcal{O}_2^2/2$. *Why is there no* $\mathcal{O}_1$*?*

Coulomb branch. What VEV can we give $\mathcal{O}_j$ to define a vacuum? Denote it by[26]

$$\phi_j := \langle \mathcal{O}_j \rangle. \tag{3.11}$$

It should be constant along $\mathbb{R}^{1,3}$ to avoid breaking Poincaré symmetry. Next we use the anticommutator $\{\bar{Q}^{\dot\alpha A}, \bar{Q}_{\bar z}^{\dot\beta B}\} \sim \epsilon^{\dot\alpha\dot\beta}\epsilon^{AB}\partial_{\bar z}$ to deduce that ϕ_j must depend holomorphically on z:

$$\epsilon^{\dot\alpha\dot\beta}\epsilon^{AB}\partial_{\bar z}\phi_j \sim \left\langle \bar{Q}^{\dot\alpha A}\big(\bar{Q}_{\bar z}^{\dot\beta B}\mathcal{O}_j\big)\right\rangle + \left\langle \bar{Q}_{\bar z}^{\dot\beta B}\big(\bar{Q}^{\dot\alpha A}\mathcal{O}_j\big)\right\rangle = 0. \tag{3.12}$$

[26]The notation is slightly ill-defined in the case of $\mathfrak{so}(4K)$ because there are then two Casimirs of the same degree $2K$, leading to two order $2K$ differentials: ϕ_{2K} defined from traces of powers of Φ_z, and $\tilde\phi_{2K} = \langle \mathrm{Pfaff}(\Phi_z)\rangle \, \mathrm{d}z^{2K}$.

The first term vanishes because the twisted compactification preserves the supercharge $\bar{Q}$. The second term vanishes by construction of $\mathcal{O}_j$.

The Coulomb branch of the 4d $\mathcal{N} = 2$ theory is thus parametrized by degree d_k differentials ϕ_{d_k} on C for $k = 1, \dots, \operatorname{rank} \mathfrak{g}$. When there are no punctures,

$$\mathcal{B} = \bigoplus_{k=1}^{r} H^0(C, K^{\otimes d_k}), \qquad \phi_j \in H^0(C, K^{\otimes j}), \tag{3.13}$$

where K is the canonical bundle on the curve C and $H^0(C, \mathcal{L})$ is the vector space of sections of the line bundle $\mathcal{L}$ on C. When C has punctures, the sections ϕ_{d_k} have prescribed behaviours at each puncture. Starting in section 4 we explain, for concrete choices of C giving usual 4d $\mathcal{N} = 2$ gauge theories, how to relate the parametrization (3.13) to the usual description in terms of scalars in 4d $\mathcal{N} = 2$ vector multiplets.

Hitchin system. We would like to say intuitively that the 4d Coulomb branch is parametrized by the "VEV" of the adjoint-valued holomorphic one-form $\Phi_z \mathrm{d}z$, a putative element of $H^0(C, K \otimes \mathfrak{g})$, modulo gauge transformations. Of course, VEVs of non-gauge-invariant operators don't make sense (or are automatically zero, depending on your point of view) so talking about them is an abuse of language. Nevertheless in our case there is a construction of the so-called Hitchin field (or Higgs field), a holomorphic one-form $\varphi = \varphi_z \mathrm{d}z$ with component $\varphi_z \in \mathfrak{g}$, whose Casimirs give $\mathrm{Tr}(\Phi_z^j)\mathrm{d}z^j$ in the $\mathfrak{su}(N)$ case and likewise in other cases. For convenience we occasionally use φ rather than the gauge-invariants ϕ_j in some explanations.

The story is a bit longer: one compactifies the 4d theory further on S^1. Coulomb branch vacua of the 3d theory are given by solutions (A, φ) of the Hitchin system on C,

$$F + [\varphi, \bar{\varphi}] = 0, \quad \bar{\partial}_A \varphi = 0, \quad \partial_A \bar{\varphi} = 0, \tag{3.14}$$

modulo G gauge transformations. The resulting Coulomb branch $\mathcal{M}$ (the Hitchin moduli space) admits a projection onto the Coulomb branch $\mathcal{B}$ of the 4d theory by mapping (A, φ) to Casimirs of φ. The Hitchin equations (3.14) are equivalent to flatness of the $G_{\mathbb{C}}$ connection $A + \varphi_z \mathrm{d}z + \bar{\varphi}_{\bar{z}} d\bar{z}$ together with a gauge-fixing condition, so that $\mathcal{M}$ can also be described as the moduli space of complex $G_{\mathbb{C}}$ flat connections on C modulo $G_{\mathbb{C}}$ gauge transformaions.

Comment on the IR behaviour. The low-energy limit of the 6d theory in a generic vacuum is given by an abelian 6d $(2,0)$ theory valued in the vacuum moduli space. Likewise, in a Coulomb branch vacuum described by a given choice of differentials ϕ_j in (3.13), the effective description of the 4d $\mathcal{N} = 2$ theory includes massless scalar fields describing fluctuations of the ϕ_j. Together with similar dimensional reductions of $B_{\mu\nu}$ and λ, these scalar fields form 4d $\mathcal{N} = 2$ abelian vector multiplets.

How many? The scalar fields have the Coulomb branch $\mathcal{B}$ as their target, so we should expect an infrared description as a 4d $\mathcal{N} = 2$ gauge theory with gauge group $\mathrm{U}(1)^{\dim_{\mathbb{C}} \mathcal{B}}$. At particular points on the Coulomb branch there are additional massless particles charged under this gauge group. The Coulomb branch typically features points that are even more singular, where the low-energy dynamics are not abelian.

3.3 Seiberg–Witten curve

Seiberg–Witten curve. In the $\mathfrak{su}(N) = \mathfrak{a}_{N-1}$ case we can repackage the data of ϕ_k in a geometric way in terms of the SW curve Σ and SW differential λ defined next.

Consider the canonical line bundle $T^*C \twoheadrightarrow C$, whose fiber at a point in C consists of one-forms at that point. In a local coordinate z on C the total space T^*C admits coordinates (z,x) where $x \in \mathbb{C}$ describes a one-form $x\mathrm{d}z$. There is a natural injection $C \hookrightarrow T^*C$, the "zero section", that maps $z \in C$ to $(z, x=0)$. We define the (complex) curve $\Sigma \subset T^*C$ as the locus (z,x) such that

$$\langle \det(x - \Phi_z)\rangle = \underbrace{\det(x - \varphi_z)}_{\text{see (3.14)}} = x^N - \sum_{j=2}^{N} u_j(z) x^{N-j} = 0 \tag{3.15}$$

where we used the construction (3.10) of $\mathcal{O}_j$ and wrote $\phi_j = \langle \mathcal{O}_j \rangle = u_j(z)\mathrm{d}z^j$ for each exponent j of $\mathfrak{g}$. Note that (3.15) is consistent with transformation properties of x and of the u_j since each term is (the sole component of) a holomorphic N-form. At generic points $z \in C$ this equation (3.15) has N solutions, which locally gives an N sheeted cover of C. Generically, at certain isolated points on C two sheets intersect with a branch point of order 2. We have constructed in this way an N-sheeted ramified cover Σ of C.

As we will see in concrete examples, Σ turns out to be the SW curve of the 4d $\mathcal{N}=2$ theory in the given Coulomb branch vacuum, and the SW differential is the holomorphic one-form λ defined as $\lambda = x\mathrm{d}z$ in coordinates (z,x) of T^*C. The fact that our (Σ, λ) matches the usual one is easier to see for concrete theories later on, but we can give some intuition. Besides indirectly giving the prepotential for the low-energy $\mathrm{U}(1)^{\dim_{\mathbb{C}} \mathcal{B}}$ vector multiplets, one of the jobs of the SW curves is to calculate the central charge of particles (which puts a BPS lower bound on their mass) in terms of their electric, magnetic, and flavour charges: it should be obtained by integrating λ along closed contours in Σ. Let us confirm this from the M-theory perspective in the A-type case.

M-theory perspective on SW curve. We recall that $\mathcal{X}(\mathfrak{su}(N))$ is the world-volume theory of N M5 branes (with the decoupled center of mass modes removed). The R-symmetry is then realized geometrically as transverse rotations. The topological twist corresponds to combining the 2d rotations with 2d transverse rotations, and one finds that the full geometrical set-up corresponding to $\mathcal{X}(\mathfrak{su}(N))$ partially twisted on $\mathbb{R}^{1,3} \times C$ is to consider M-theory on $\mathbb{R}^{1,3} \times T^*C \times \mathbb{R}^3$ and to place M5 branes along $\mathbb{R}^{1,3} \times C$, the zero section.[27]

Moving onto the Coulomb branch corresponds to shifting the M5 branes away from each other along the fibers of T^*C. Since the branes are indistinguishable they generically

[27]More generally, T^*C can be replaced by a four-dimensional hyper-Kähler manifold and C by a holomorphic cycle inside Q.

reconnect into an N-sheeted ramified cover $\Sigma \subset T^*C$ of C. Supersymmetry requires it to be holomorphic and we thus reproduce the above classification of Coulomb branch vacua. We emphasize that the UV curve C characterizes the theory, while the IR curve (or SW curve) Σ depends on (and characterizes) the given Coulomb branch vacuum.

Excitations of the brane system include massless fluctuations along the Coulomb branch of course, but also very interesting massive excitations coming from M2 branes ending on the M5 branes. Consider a two-dimensional surface $D \subset T^*C$ whose boundary lies in the SW curve, $\partial D \subset \Sigma$, and let us place an M2 brane along $D \times \mathbb{R}$ where $\mathbb{R}$ is the time direction in 4d Minkowski space. From the 4d point of view this describes a particle sitting still as time passes. Its mass m is simply the area of D,

$$m = \int_D |\mathrm{d}z\mathrm{d}x| \geq \Bigl|\int_D \mathrm{d}z\mathrm{d}x\Bigr| = \Bigl|\int_D \mathrm{d}(x\mathrm{d}z)\Bigr| = \Bigl|\int_{\partial D} \lambda\Bigr|. \tag{3.16}$$

This reproduces the BPS lower bound expected from the SW curve and differential (Σ, λ). In fact, realizing the SW curve Σ as a fibration over C gives slightly finer control of the BPS spectrum than just knowing Σ (and λ). Indeed, some closed curves on Σ are not the boundary of any two-dimensional $D \subset T^*C$, so that the M-theory setup "knows" that no BPS state with these charges exist, while the data of (Σ, λ) only would not know it.

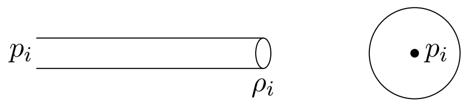
These M-theory considerations suggest that we found the right notion of SW curve and differential for class S theories. But we have yet to explain any concrete description of the 4d theories, rather than only their IR behaviour on the Coulomb branch. We turn to this next.

3.4 Tubes and tinkertoys

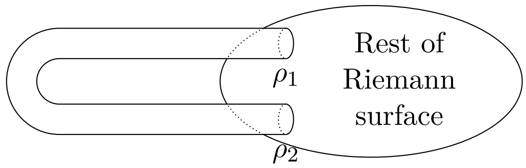
So far we only discussed the low-energy effective description of $\mathrm{T}(\mathfrak{g}, C, D)$ on its Coulomb branch. We now study how the class S theory can be described without moving along its Coulomb branch. Our guide to find such a description is that it should reproduce the aforementioned IR physics (it also reproduces some protected observables), and that different descriptions we find should be (exactly) dual to each other. Recall that the partial twist ensures that 4d physics we are interested in only depends on the complex structure of the Riemann surface C on which we compactify. We can thus pick any metric compatible with this complex structure.

Gluing. Consider two punctures $p_1, p_2 \in \overline{C}$ of the same (or of different) punctured Riemann surface $C = \overline{C} \setminus \{p_i\}$ and consider disks around p_1 and p_2. As far as the complex structure is concerned, these punctured disks are the same as semi-infinite cylinders thanks to the exponential map (expressed here in coordinates centered at p_i)

$$\exp\colon (-\infty, \rho_i] \times (\mathbb{R}/(2\pi\mathbb{Z})) \xrightarrow{\sim} \{z \mid |z| \leq e^{\rho_i}\} \setminus \{0\}. \tag{3.17}$$



We can glue two such semi-infinite cylinders by cutting their infinite end off at some finite distance and identifying the cutoff points on the left side of the following diagram:



(3.18)

where the "rest" can remain connected or be disconnected upon removing the tube. In terms of complex coordinates w and z around p_1 and p_2 respectively (with p_1 at $w = 0$ and p_2 at $z = 0$), the identification is

$$zw = q \tag{3.19}$$

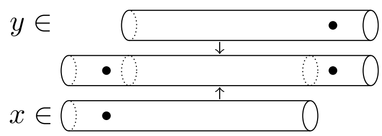
for some parameter q. The modulus $|q|$ gives the aspect ratio (length over circumference) $(-\log|q|)/2\pi$ of the tube, while the phase of q indicates how the cylinders are rotated before gluing. The coordinates w, z are only locally defined so $|q|$ cannot be too big: the tube can be arbitrarily long/thin but not too short/thick, as the description otherwise breaks down.

Exercise 3.3 (On punctured spheres)**.** *Choose a coordinate w on the complex projective plane $\mathbb{CP}^1$ (the two-sphere), where $w \in \mathbb{C} \cup \{\infty\}$.*

1. For $n \geq 3$ arbitrary distinct points $w_j \in \mathbb{C} \cup \{\infty\}$, $j = 1, \ldots, n$, define a new coordinate $z(w) := \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$. Check that $w \mapsto z$ is bijective on $\mathbb{CP}^1$ so that the definition gives a good coordinate on $\mathbb{CP}^1$. Check that w_1, w_2, w_3 are mapped to $0, 1, \infty$. The coordinate $z(w_j)$ for $j > 3$ is called cross-ratio of w_1, w_2, w_3, w_j.

2. In the four-punctured sphere, how does the cross-ratio q change when w_1, w_2, w_3, w_4 are permuted?

3. Construct the four-punctured sphere $\mathbb{CP}^1 \setminus \{0, q, 1, \infty\}$ by gluing two three-punctured spheres $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$. (Hint: let x, y be coordinates on the two three-punctured spheres; identify $qx = y$ for some region $1 < |x| < 1/|q|$.) Generalize to the n-punctured sphere.



Vector multiplets. Despite how it is drawn in (3.18), the cylinder connecting the two punctures is flat and of constant circumference $2\pi L_5$ (for some metric). Let us denote the directions along and around the cylinder as x^4, x^5. We know that the 6d theory $\mathcal{X}(\mathfrak{g})$ reduced on a circle gives 5d $\mathcal{N} = 2$ SYM with gauge algebra $\mathfrak{g}$ and coupling $g^2_{5\mathrm{d}} \simeq L_5$. We should thus expect that part of the system obtained by reducing $\mathcal{X}(\mathfrak{g})$ on the glued surface (3.18) is 5d $\mathcal{N} = 2$ SYM on an interval of length $L_4 \sim (-\log|q|)L_5$. In the limit where C shrinks to a point, the 5d term $\mathrm{Tr}(F^2)$ does not depend much on the x^4 direction, thus the 4d Lagrangian ought to have a term

$$\frac{1}{g_{5\mathrm{d}}^2}\int_I \mathrm{Tr}(F^2) = \frac{1}{g_{4\mathrm{d}}^2}\mathrm{Tr}(F^2), \qquad \frac{1}{g_{4\mathrm{d}}^2} = \frac{L_4}{g_{5\mathrm{d}}^2} \simeq \frac{L_4}{L_5} = -\log|q|. \tag{3.20}$$

What about the phase of q, which implements a translation along the circle direction, namely a rotation around the cylinder? The instanton current of a 5d gauge field is defined as

$$J_\mu^{\mathrm{inst}} = \epsilon_{\mu\nu\rho\sigma\tau}\,\mathrm{Tr}(F^{\nu\rho}F^{\sigma\tau}). \tag{3.21}$$

As we mentioned earlier, in the reduction from $\mathcal{X}(\mathfrak{g})$ to 5d $\mathcal{N}=2$ SYM the KK (Kaluza–Klein) modes correspond to instanton particles of the 5d theory, namely the KK momentum in x^5 is equal to the charge under J^{inst}. Thinking of x^4 as Euclidean time, the translation operator P_5 is given as an integral of the "time" component J_4^{inst} over the 4d "spatial" directions $x^0,\dots,x^3$. Twisting the cylinder by an angle $\theta = \mathrm{Im}\log|q|$ thus contributes a term $\theta\,\mathrm{Tr}(F\wedge F)$ to the 4d Lagrangian when we eventually reduce C to a point.

Altogether we expect that a long cylinder as in (3.18) should yield a 4d $\mathcal{N}=2$ vector multiplet with complexified gauge coupling τ given by $\log q$:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}, \qquad q \sim e^{2\pi i\tau}. \tag{3.22}$$

The relation is made more precise later in concrete geometries.

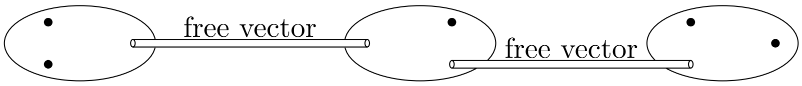
Pants decomposition and S-duality. Vector multiplets must gauge flavour symmetries of some matter sector, and our next task is to understand where that matter comes from. For this, the key is to send gauge couplings to zero, because in this limit the vector multiplet decouples and leaves behind the matter sector with its flavour symmetries.

Exercise 3.4. *As a toy model, consider a scalar field ϕ transforming in some representation of a group G, and gauge the symmetry G using a gauge field A. We denote by $D = \mathrm{d} + A$ the covariant derivative and $F = \mathrm{d}A + A\wedge A$, and ignore numerical factors. By introducing a field $\tilde{A} = g^{-1}A$ with canonically normalized kinetic term, show how*

$$\mathcal{L} = \frac{1}{g^2}\mathrm{Tr}(F^2) + |D\phi|^2 \xrightarrow{g\to 0} \mathrm{Tr}(\mathrm{d}\tilde{A})^2 + |\mathrm{d}\phi|^2. \tag{3.23}$$

Note that in the limit the flavour symmetry G of ϕ is not gauged any longer. The original gauge theory can be then restored (up to the free gauge field $\tilde{A}$) by gauging this flavour symmetry with a new gauge field. Check the same decoupling happens for spinors ($\bar{\psi}\gamma^\mu D_\mu\psi$).

Any punctured Riemann surface C with genus g and n punctures, except for (g,n) among $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, can be decomposed into three-punctured spheres (pairs of pants) glued as described above. Such a decomposition is called a pants decomposition. For each pants decomposition of C there is a corresponding cusp in the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces with genus g and n punctures. At this cusp, C is described by three-punctured spheres joined by infinitely thin tubes. Each such tube yields an infinitely weakly coupled vector multiplet in the 4d theory, so that in this limit we can expect 6d fields "localized" on each pair of pants to decouple from each other since the 4d vector multiplets joining them become free:



(3.24)

As in the toy model, the symmetries gauged by the vector multiplet are restored as flavour symmetries in the zero coupling limit.

The picture that emerges is as follows. The building blocks of $\mathrm{T}(\mathfrak{g}, C, D)$ are class S theories called tinkertoys associated to three-punctured spheres. These (4d $\mathcal{N} = 2$) tinkertoys have flavour symmetries associated to each puncture, which we study carefully later. For each tube, consider the flavour symmetry groups F_1 and F_2 associated to the two punctures that it connects, and gauge a suitable diagonal subgroup $F \subset F_1 \times F_2$ using a 4d $\mathcal{N} = 2$ vector multiplet. This yields $\mathrm{T}(\mathfrak{g}, C, D)$. This description of $\mathrm{T}(\mathfrak{g}, C, D)$ for each pants decomposition of C can be written schematically as

$$\mathrm{T}(\mathfrak{g}, C, D) = \Big(\prod_{\text{pants}} \mathrm{T}(\mathfrak{g}, \text{sphere} \setminus 3\text{pt})\Big) \Big/ \Big(\prod_{\text{tubes}} \text{gauge group}\Big). \tag{3.25}$$

A large part of the work in understanding the AGT correspondence is to classify tinkertoys obtained from three punctured spheres with different types of punctures. Their flavour symmetry can be rather intricate, which is why we cannot make the gauge groups more explicit in (3.25) in such generality.

When all punctures are so-called full tame punctures (explained later), all building blocks are the same tinkertoy $T_{\mathfrak{g}}$. This theory is an isolated[28] SCFT with (at least) $\mathfrak{g}^3$ flavour symmetry associated to its three punctures. For $\mathfrak{g} = \mathfrak{su}(2)$ it consists of four free hypermultiplets, while for other $\mathfrak{g}$ it has no 4d $\mathcal{N} = 2$ Lagrangian description.

Of course, a given Riemann surface has many inequivalent decompositions into pairs of pants. Each one leads to a description of $\mathrm{T}(\mathfrak{g}, C, D)$ as a weakly coupled gauge theory in one corner of parameter space. At strong coupling (short tubes) another description may be weakly coupled hence more useful. In concrete cases this reproduces known 4d $\mathcal{N} = 2$ S-dualities. Here is an exercise to get an intuition about pants decompositions.

Exercise 3.5 (Combinatorics of pants decompositions)**.** *1. Given a surface $C_{g,n}$ with genus g and n punctures, check that all pants decompositions use the same number of three-punctured spheres.*

2. Draw the three topologically different[29] pants decompositions of a four-punctured sphere (assuming punctures are distinguishable). How many pants decompositions does an n-punctured sphere have? Does a once-punctured torus have a finite number of pants decompositions?

[28] An SCFT with a certain amount of supersymmetry is *isolated* if it does not have any exactly marginal deformation with the same supersymmetry (such as gauge couplings in 4d).

[29] Two pants decompositions are the same in this sense if the closed curves cutting the surface into pieces with three boundaries can be deformed into each other without (self-)intersection or crossing punctures.

3. Return to point 3 of Exercise 3.3 and construct the sphere with $n = 4$ (or 5) punctures by gluing three-punctured spheres in all possible ways.

4. We don't need to degenerate the Riemann surface completely down to pairs of pants: as soon as C involves one long tube the theory $\mathrm{T}(\mathfrak{g}, C, D)$ can be written in terms of a weakly coupled vector multiplet gauging flavour symmetries of a "smaller" class S theory. What Riemann surface (genus, punctures) underlies the latter theory? There are two cases depending on whether the surface disconnects.

4 Lagrangians for class S theories

After discussing the tame punctures that arise when pinching tubes, we argue in subsection 4.1 that $\mathcal{X}(\mathfrak{su}(2))$ on a sphere with three tame punctures yields 4 free hypermultiplets, with a flavour symmetry $\mathrm{SU}(2)^3$ made manifest. In subsection 4.2 we glue two such building blocks to learn how $\mathcal{X}(\mathfrak{su}(2))$ on a sphere with four tame punctures reproduces known aspects of 4d $\mathcal{N} = 2$ SQCD with gauge group SU(2) and $N_f = 4$ flavours. This is the conventional starting point of AGT reviews: one usually studies S-duality of SU(2) SQCD [4] and of quivers gauge theories [175], before explaining the unifying 6d point of view [1]. We extend the discussion in subsection 4.3 to generalized SU(2) quivers arising from $\mathcal{X}(\mathfrak{su}(2))$ on arbitrary punctured Riemann surfaces.

In subsection 4.4 we realize as class S theories some $\mathrm{SU}(N)$ linear quiver gauge theories including $\mathrm{SU}(N)$ SQCD with $N_f = 2N$ flavours. This teaches us that there are several types of tame punctures hence several types of codimension 4 operators in the 6d theory. All theories we consider in this section are such that gauge couplings have vanishing one-loop beta function, and this implies that the couplings have vanishing beta function at all orders [176].

Description of punctures and some tinkertoys

- The $\mathfrak{su}(2)$ tinkertoy consist of a trifundamental half-hypermultiplet of $\mathrm{SU}(2)^3$.
- The $\mathfrak{su}(N)$ tinkertoy with two full and one simple puncture is an $\mathrm{SU}(N)^2$ bifundamental hypermultiplet.
- For $\mathfrak{su}(N)$, at a full tame puncture z_0 the differentials obey $\phi_k = (\frac{\sigma_k}{(z-z_0)^k} + O(\frac{1}{(z-z_0)^{k-1}}))\mathrm{d}z^k$ for $2 \leq k \leq N$ where σ_k has mass dimension k. Other tame punctures arise by restricting the pole coefficients down to orders $< k - 1$.
- Many linear quivers have both class S and IIA descriptions; Coulomb branches and SW curves work out.

4.1 Trifundamental tinkertoy

We discuss tame punctures; for $\mathfrak{su}(2)$ there is only one type. We then consider $\mathcal{X}(\mathfrak{su}(2))$ on a sphere $\mathbb{CP}^1$ with three tame punctures at $0, 1, \infty$ and we argue that the resulting tinkertoy $T_2 = T_{\mathfrak{su}(2)}$, which is the main building block of $\mathfrak{su}(2)$ class S theories, is a collection of four free hypermultiplets. There is no first principle derivation of that fact, but we will see many checks of it, especially correct postdictions of Coulomb branch and SW curves of many gauge theories, as well as consistency with string theory dualities.

Tame punctures. We describe punctures in terms of their effect on the Hitchin field $\varphi(z)$ parametrizing the Coulomb branch, or gauge-invariantly in terms of the higher-order differentials ϕ_{d_k}, $k = 1, \dots, \operatorname{rank}\mathfrak{g}$.

Punctures can arise from pinching a thin tube. In a complex coordinate $w \in \mathbb{R} \times S^1$ describing this tube, the ϕ_{d_k} often tend to constants (times dw^{d_k}) inside the thin tube. Cutting the cylinder (the opposite of what we did in (3.18)) and applying the exponential map (3.17) $z = e^w$, we generically expect

$$\phi_{d_k} \simeq \frac{\mathrm{d}z^{d_k}}{z^{d_k}} + \dots \tag{4.1}$$

with some coefficients, in a local coordinate z in which the puncture is at $z = 0$.

This motivates us to work with *tame punctures*, namely points where $\varphi(z)\mathrm{d}z$ has a first order pole with a prescribed residue, of course up to gauge conjugation: the prescribed residue translates generically to a prescribed leading coefficient in (4.1) —a *full* tame puncture. We study other punctures later in section 7: tame punctures in which ϕ_{d_k} have lower-order poles instead of (4.1), and wild punctures defined as having higher-order poles.

Massive and massless tame punctures for $\mathfrak{su}(2)$. For now we focus on $\mathfrak{su}(2)$: there is then a single type of tame puncture.

This case has a single Casimir, the quadratic differential $\phi_2 = \frac{1}{2}\operatorname{Tr}(\varphi^2)\mathrm{d}z^2$. We impose the residue of the Hitchin field φ up to conjugation (which we denote $\sim$): for non-zero $m \in \mathbb{C}$,

$$\varphi(z) \sim \left(\frac{\operatorname{diag}(m, -m)}{z - z_i} + O(1)\right)\mathrm{d}z \implies \phi_2(z) = \left(\frac{m^2}{(z - z_i)^2} + O\left(\frac{1}{z - z_i}\right)\right)\mathrm{d}z^2. \tag{4.2}$$

We call $m \neq 0$ the *mass parameter* of the puncture for the following reason. The sheets of Σ defined in (3.15) behave as $x_\pm(z) = \pm m/(z - z_i) + O(1)$, and integrating the SW differential λ around z_i on one of the two sheets picks up the residue $\pm m$. This means m appears as a contribution to the central charge hence to masses of BPS particles.

Naively, taking the $m \to 0$ limit in the $\varphi(z)$ asymptotics changes z_i into a regular point. In the ϕ_2 equation however, the puncture remains as a first order pole. This is explained from the $\varphi(z)$ point of view by noting that it is only defined up to conjugation. Conjugating the diagonal matrix $\operatorname{diag}(m, -m)$ before taking the $m \to 0$ limit can yield a non-zero value,

$$\begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} \sim \begin{pmatrix} m & 1 \\ 0 & -m \end{pmatrix} \xrightarrow{m \to 0} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.3}$$

Indeed, we find a consistent *massless* tame puncture

$$\varphi(z) \sim \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{z - z_i} + O(1)\right)\mathrm{d}z \implies \phi_2(z) = O\left(\frac{1}{z - z_i}\right)\mathrm{d}z^2 \tag{4.4}$$

where the pole has a free coefficient. Interestingly, the sheets of Σ defined by $x^2\mathrm{d}z^2 = \phi_2$ admit a branch point at such a massless puncture.

Exercise 4.1. *1. For any* $\alpha \in \mathbb{C}$ *find an invertible matrix* $g \in \mathrm{SL}(2,\mathbb{C})$ *such that* $g^{-1}\operatorname{diag}(m,-m)g = \left(\begin{smallmatrix} m & \alpha \\ 0 & -m \end{smallmatrix}\right)$.

2. Check that all $\left(\begin{smallmatrix} 0 & \alpha \\ 0 & 0 \end{smallmatrix}\right)$, $\alpha \neq 0$, *are conjugate to each other.*

3. How does the coefficient of $1/(z-z_i)$ *arise in* (4.2) *and* (4.4) *from components of the* $(z-z_i)^0$ *term in the expansion of* φ*?*

We interpret (4.2) as follows: the massless puncture (4.4) carries SU(2) flavour symmetry, and turning on a constant scalar $\phi_{\text{background}} = m$ in a background vector multiplet coupled to that symmetry changes the puncture to the massive one (4.2).

Three-punctured sphere and symmetries. Now consider T_2, the result of placing $\mathcal{X}(\mathfrak{su}(2))$ on a sphere $\mathbb{CP}^1$ with three tame punctures. What do we know for sure about T_2?

First, it should have at least $\mathrm{SU}(2)^3$ flavour symmetry, one SU(2) per puncture. We learn in the next exercise that 4 free hypermultiplets indeed have an USp(8) flavour symmetry, which contains $\mathrm{SU}(2) \times \mathrm{Spin}(4) = \mathrm{SU}(2)^3$. The USp(8) flavour symmetry is in this context an emergent symmetry that is only present in the limit where C shrinks to a point; it is not a symmetry of the 6d setup.

Exercise 4.2. *1. Check that* k *free scalar fields carry* $\mathrm{O}(k)$ *flavour symmetry. Check that* p *free hypermultiplets contain* $4p$ *free scalars hence have* $\mathrm{O}(4p)$ *symmetries (and more from spinors). Out of these, check a* $\mathrm{USp}(2p)$ *subgroup commutes with* $\mathrm{SU}(2)_{\mathrm{R}}$ *R-symmetry. (Hint: as an intermediate step, the* $\mathrm{U}(2p)$ *subgroup commutes with* $J_3 \in \mathrm{SU}(2)_{\mathrm{R}}$*.)*

2. Now gauge an $\mathrm{SU}(2) = \mathrm{USp}(2)$ *flavour symmetry embedded diagonally into* $\mathrm{USp}(2)^p \subset \mathrm{USp}(2p)$*. The gauged* $\mathfrak{su}(2)$ *times* $\mathfrak{su}(2)_{\mathrm{R}}$ *combine into* $\mathfrak{so}(4)$*. Check that the* $4p$ *scalars organize as* p *copies of the fundamental representation of* $\mathfrak{so}(4)$*. Deduce that the remaining flavour symmetry is* $SO(p)$*. It is in fact* $\mathrm{Spin}(p)$ *because of the action on spinors in the hypermultiplets.*

The trifundamental half-hypermultiplet. All three SU(2) symmetries of the four hypermultiplets can be made manifest, at the cost of hiding $\mathcal{N}=2$ supersymmetry. Split each hypermultiplet into a pair of $\mathcal{N}=1$ chiral multiplets $(q,\tilde{q})$. The four hypermultiplets split into eight $\mathcal{N}=1$ chiral multiplets q_{aiu} where a,i,u (ranging from 1 to 2) are indices for the three independent SU(2). To reconstruct the hypermultiplets as $(q,\tilde{q})$ simply introduce the notation

$$\tilde{q}^{aiu} = \epsilon^{ab}\epsilon^{ij}\epsilon^{uv} q_{bjv}. \tag{4.5}$$

The hypermultiplets are thus in a trifundamental representation of $\mathrm{SU}(2)^3$ with a reality property (4.5) that halves the number of components. This set of matter fields is called a half-hypermultiplet.

If background vector multiplet scalars (i.e., masses m_1, m_2, m_3) are turned on for the three $\mathrm{SU}(2)^3$, then the underlying 8 chiral multiplets have complex masses $\pm m_1 \pm m_2 \pm m_3$ for all choices of signs. In particular one of the 4 hypermultiplets becomes massless when $m_2 = \pm m_1 \pm m_3$. This is important later.

Seiberg–Witten curve of T_2. We denote by m_1, m_2, m_3 the mass parameters of punctures at $0, 1, \infty$ in the sense of (4.2) or (4.4).[30] The Coulomb branch (if any) of the 4d theory is parametrized by holomorphic quadratic differential $\phi_2(z)$ that have second order poles (4.2) or first-order in the massless case (4.4) at each of the punctures, and no other pole. The puncture at infinity translates to a condition as $z \to \infty$:

$$\phi_2(z) = \left(\frac{m_3^2}{z^2} + O\left(\frac{1}{z^3}\right)\right)\mathrm{d}z^2. \tag{4.6}$$

We recall Liouville's theorem regarding entire functions (holomorphic functions on $\mathbb{C}$ with no pole): if an entire function f is bounded as $|f(z)| < Kz^p$ for some constant K and exponent p then f is a polynomial of degree at most p.

Exercise 4.3. *Find a quadratic differential $\phi_2(z) = u_2(z)\mathrm{d}z^2$ that has the prescribed second order poles at $0, 1, \infty$ and no other singularity and show it is unique. (Hint: write it as $u_2(z) = f(z)/(z^2(z-1)^2)$, change variables to $w = 1/z$ to polynomially bound f at infinity and use Liouville's theorem to bound the degree of f, then compare with the prescribed asymptotics to fix coefficients.)*

The Coulomb branch is thus a single point, which is consistent with the lack of vector multiplet in our description of T_2 as free hypermultiplets. Explicitly,

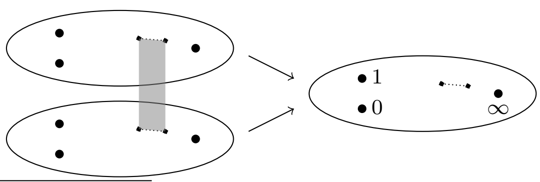
$$\phi_2(z) = u_2(z)\mathrm{d}z^2, \qquad u_2(z) = \frac{-m_1^2}{z^2(z-1)} + \frac{m_2^2}{z(z-1)^2} + \frac{m_3^2}{z(z-1)}. \tag{4.7}$$

Let us find the IR description of T_2 at this unique Coulomb branch vacuum. As we commented on page 24, the low-energy theory is generically a 4d $\mathcal{N} = 2$ abelian gauge theory with the vector multiplet scalars living in the Coulomb branch $\mathcal{B}$. Here there is no Coulomb branch hence no gauge fields in the IR. There may be hypermultiplets: for this we have to study the SW curve Σ defined by $x^2 = u_2$ and the SW differential $\lambda = x\mathrm{d}z$. The integral of λ over closed cycles tells us about masses of BPS states.

The curve Σ is a ramified double cover of $\mathbb{CP}^1$. How many branch points does it have? Branch points are where the two sheets $x = \pm\sqrt{u_2}$ rejoin, namely where $u_2 = 0$. This happens at the (generically) two roots of the quadratic polynomial

$$z^2(z-1)^2 u_2(z) = -(z-1)m_1^2 + zm_2^2 + z(z-1)m_3^2. \tag{4.8}$$

Altogether, Σ wraps the sphere twice, with a single branch cut. It is thus topologically a sphere. The three punctures at $0, 1, \infty \in \mathbb{CP}^1$ become six point on Σ where the SW differential λ blows up:

Σ:


$: C$ (4.9)

[30] We don't know at this stage that they are the same parameters as in the last paragraph about the trifundamental half-hypermultiplet.

BPS spectrum. Contour integrals of λ give integer[31] linear combinations of residues of $\lambda = x\mathrm{d}z = \pm\sqrt{u_2}\,\mathrm{d}z$ at the poles $z = 0, 1, \infty$. By construction these residues are $\pm m_1, \pm m_2, \pm m_3$, so we find that masses (or rather central charges) of BPS states take the form $Z = f_1 m_1 + f_2 m_2 + f_3 m_3$ for $f_1, f_2, f_3 \in \mathbb{Z}$. On the other hand, the trifundamental half-hypermultiplet only has BPS states with integer linear combinations of $\pm m_1 \pm m_2 \pm m_3$: this imposes the further restriction that $f_1 = f_2 = f_3 \mod 2$. Does the tinkertoy T_2 also have that restriction?

In subsection 3.3 we learned that M-theory instructs us to only integrate λ over contours γ in $\Sigma \subset T^*C$ that can be written as the boundary $\gamma = \partial D$ of some two-dimensional surface $D \subset T^*C$.

Exercise 4.4. *1. First choose D to be a small circle around one of the punctures, times the interval connecting the two sheets of Σ. Its boundary is a pair of circles picking up twice the same residue m_i (from different sheets). Deduce that $2m_1, 2m_2, 2m_3$ and all their integer linear combinations are in the spectrum.*

2. Next, choose D such that ∂D is a contour from one branch point to the other (on one sheet) and back via the other sheet. Note that the contour ∂D can be deformed to a contour staying on one sheet and surrounding the cut. Deduce that the integral of λ is one of the combinations $\pm m_1 \pm m_2 \pm m_3$ (three poles are on each side of the contour) and conclude that the BPS spectrum of T_2 contains all $Z = f_1 m_1 + f_2 m_2 + f_3 m_3$ with $f_1 = f_2 = f_3 \mod 2$.

3. (Mathematical.) For any $D \subset T^*C$ with boundary $\partial D \subset \Sigma$, consider the projection $\pi\colon T^*C \to C$ and deduce that $\pi(\partial D) = \partial(\pi(D))$ cannot surround a pole. Deduce that the BPS spectrum of T_2 is exactly that of the trifundamental half-hypermultiplet.

Generically, all of these BPS particles are massive so that the low-energy theory is empty. An interesting case is the limit $m_2 \to \pm(m_1 \pm m_3)$ where one of the four hypermultiplets in the trifundamental half-hypermultiplet becomes massless. Then the SW curve degenerates because the two branch points collide: indeed, u_2 has a double root (4.8)

$$z^2(z-1)^2 u_2(z) = (m_1 \pm z m_3)^2. \tag{4.10}$$

The contour we considered in point 2 of the above exercise shrinks to zero size while λ itself remains finite, so the integral is indeed zero, consistent with the vanishing mass. We will run this kind of easy consistency checks for the more complicated theories.

4.2 4d $\mathcal{N} = 2$ SU(2) $N_f = 4$ SQCD

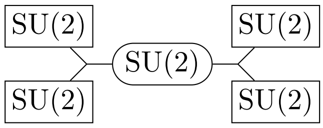
After so many generalities let us study the Coulomb branch, SW curve and S-dualities of our first interesting concrete theory: $\mathcal{X}(\mathfrak{su}(2))$ on a sphere with four tame punctures.

Identifying the 4d theory (spoilers in the title above). We place the four punctures at $z_1 = 0$, $z_2 = q$, $z_3 = 1$, $z_4 = \infty$ on the two-sphere $\mathbb{CP}^1$. The three degeneration limits $q \to 0, 1, \infty$ of the four-punctured sphere correspond to all ways of

[31] We ignore the factor of $2\pi i$ in the residue theorem.

clustering the punctures pairwise. Since the three limits are identical up to permuting the punctures we concentrate on $q \to 0$. In this limit, we expect on general grounds that the 4d theory consists of two tinkertoys $T_{\mathfrak{su}(2)}$ and one SU(2) vector multiplet gauging an SU(2) flavour symmetry of each tinkertoy. After this gauging each tinkertoy should still carry at least $\mathfrak{su}(2) \times \mathfrak{su}(2)$ flavour symmetries associated to its two remaining punctures. We can depict this as a (generalized) quiver making all symmetries explicit:

$$\mathrm{T}(\mathfrak{su}(2), \mathbb{CP}^1 \setminus \{0, q, 1, \infty\}) = \tag{4.11}$$



Here the round node denotes a gauge group while square nodes denote flavour symmetries. Each junction $\succ$ represents our favorite tinkertoy T_2, the trifundamental half-hypermultiplet, i.e., four hypermultiplets transforming as two doublet representations of the SU(2) gauge group.

We thus get two flavours from the left junction and two flavours from the right junction, hence the theory is SU(2) SQCD with $N_f = 4$ flavours. While each tinkertoy in (4.11) has Spin(4) flavour symmetry after gauging SU(2), the full theory has $N_f = 4$ doublets of SU(2) on an equal footing hence has flavour symmetry Spin(8), larger than $\mathrm{Spin}(4)^2$. This symmetry of the 4d theory only emerges in the limit where C shrinks to a point.

Coulomb branch. We denote by m_1, m_2, m_3, m_4 the mass parameters at each of these punctures in the sense of (4.2) or (4.4). Coulomb branch vacua of the 4d theory are parametrized by holomorphic quadratic differential $\phi_2(z)$ that have second order poles (4.2) or first-order in the massless case (4.4) at each of the punctures, and no other pole. The puncture at infinity translates to a condition as $z \to \infty$:

$$\phi_2(z) = \left(\frac{m_4^2}{z^2} + O\left(\frac{1}{z^3}\right)\right) \mathrm{d}z^2. \tag{4.12}$$

We parametrize the possible $\phi_2(z)$ in the next exercise, starting with the massless case $m_1 = m_2 = m_3 = m_4 = 0$ for which ϕ_2 has first-order poles.

Exercise 4.5. *1. Find all quadratic differentials $\phi_2(z) = u_2(z)\mathrm{d}z^2$ that have first order poles at $0, q, 1, \infty$ and no other. (Hint: after writing $u_2(z) = f(z)/(z(z-q)(z-1))$, change variables to $w = 1/z$ to deduce a polynomial bound on $f(z)$, then use Liouville's theorem mentioned above.)*

2. Find one quadratic differential ϕ_2 that has leading behaviour $m_i^2/(z-z_i)^2$ for $i = 1, 2, 3$ and m_4^2/z^2 at infinity as per (4.12). Combining with the massless case deduce all such quadratic differentials.

We find a one-dimensional Coulomb branch $\mathcal{B} = \mathbb{C}$ with vacua[32]

$$\phi_2 = u_2 \mathrm{d}z^2, \qquad u_2(z) = \frac{\frac{q}{z} m_1^2 + \frac{q(q-1)}{z-q} m_2^2 + \frac{z-q}{z-1} m_3^2 + z m_4^2 - u}{z(z-q)(z-1)} \tag{4.13}$$

[32]The variable u parametrizing the Coulomb branch can be freely redefined, hence you may have gotten a slightly different expression in Exercise 4.5.

labeled by $u \in \mathcal{B} = \mathbb{C}$. A zero-th order check that we did not go astray is that we got the correct dimension (namely 1) for the Coulomb branch of SU(2) SQCD with $N_f = 4$ flavours.

Degeneration limit. As $q \to 0$, the surface degenerates, and we should obtain in a suitable sense two disconnected three-punctured spheres. For $|q|, |z| \ll 1$ at fixed masses, (4.13) behaves as

$$\phi_2(z) \simeq \frac{\frac{-q}{z} m_1^2 + \frac{q}{z-q} m_2^2 + u}{z(z-q)} \mathrm{d}z^2, \tag{4.14}$$

which is precisely the quadratic differential on a sphere with three tame punctures of masses squared m_1^2, m_2^2, and u. Likewise, for $|q| \ll |z|, 1$ (4.13) behaves as the quadratic differential on a three-punctured sphere with masses squared u, m_3^2 and m_4^2. This is consistent with how we introduced tame punctures in subsection 4.1.

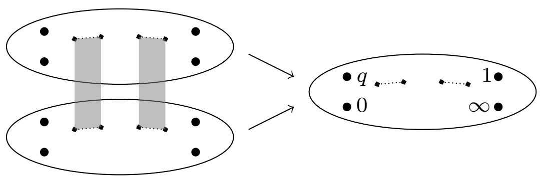
Since masses are background values of vector multiplet scalars, we learn from (4.14) the identification

$$u = \frac{1}{2} \left\langle \operatorname{Tr} \phi^2 \right\rangle \tag{4.15}$$

in the weakly-coupled limit $|q| \ll 1$, where ϕ is the (dynamical) vector multiplet scalar corresponding to the SU(2) gauge group. In other words u is the usual parametrization of the Coulomb branch of SQCD.

Seiberg–Witten curve. We now return to general q. The SW curve and differential are defined by $\Sigma = \{(x, z) \in T^*\mathbb{CP}^1 \mid x^2 = u_2(z)\}$ and $\lambda = x\mathrm{d}z$.

The curve Σ is a ramified double cover of $\mathbb{CP}^1$. How many branch points does it have? Branch points are where the two sheets $x = \pm\sqrt{u_2}$ rejoin, namely where $u_2 = 0$. This happens at the (generically) four roots of the polynomial $z^2(z-q)^2(z-1)^2 u_2(z)$, which is quartic. Altogether, Σ wraps the sphere twice, with four branch points joined by two branch cuts. It is thus topologically a torus. In addition to these branch cuts we have four punctures at $0, q, 1, \infty \in \mathbb{CP}^1$, hence eight point on Σ where the SW differential λ blows up:

Σ:  : C (4.16)

Exercise 4.6. *1. By changing coordinates as $x = \tilde{x}/z + m_2/(z-q) + m_3/(z-1)$, rewrite the curve $x^2 = u_2$ in a form that only has simple poles at $z = 0, q, 1$. Show that $\tilde{\lambda} := \tilde{x}\mathrm{d}z/z$ differs from the SW differential $\lambda = x\mathrm{d}z$ by a u-independent term whose contour integrals (residues) are linear combinations of masses. Recall the BPS mass formula $\oint \lambda = na + ma_D + f_i m_i$ and check what changing λ to $\tilde{\lambda}$ amounts to a redefinition of flavour charges. Up to simple changes of coordinates perhaps*[33] *match with more*

[33] I have not checked yet what form Martone uses in his notes.

conventional expressions of the SW *curve and differential of* SU(2) $N_f = 4$ *SQCD given in [123]. The match confirms that we correctly identified the tinkertoy* $T_{\mathfrak{su}(2)}$.

Singularities on the Coulomb branch. As we described on page 24, the low-energy theory is generically a 4d $\mathcal{N} = 2$ abelian gauge theory with the vector multiplet scalars living in the Coulomb branch $\mathcal{B}$. For generic values of u and of masses, we thus get a U(1) vector multiplet, but at special values of the parameters some branch points may collide, which leads to interesting low-energy behaviours. We already saw that near (4.10) in our study of the tinkertoy: we found particular values of the masses where a pair of branch points of the SW curve collide. This collision made a certain contour shrink to zero size, hence lead to a massless BPS particle which remains present in the IR theory. For SQCD such collisions of branch points enrich the IR theory by adding one or more massless hypermultiplets charged under the low-energy U(1).

Exercise 4.7 (On the discriminant)**.** *The discriminant of a degree d polynomial $P(z) = p_d \prod_{a=1}^{d}(z - z_a)$ is $\Delta_P = p_d^{2d-2} \prod_{a<b}(z_a - z_b)^2$. It vanishes by construction exactly when $P(z)$ has double roots. It is known that Δ_P can be expressed as a polynomial of degree $2d - 2$ in the coefficients p_j of $P(z) = \sum_{j=0}^{d} p_j z^j$. Check this for quadratic polynomials.*

Our question is to find when $P(z) = z^2(z - q)^2(z - 1)^2 u_2(z)$, which is a quartic polynomial given explicitly in (4.13), has double roots (hence when two branch points collide). The discriminant Δ_P is then of degree 6 in the coefficients of P. Since P depends linearly on u we find that Δ_P is of degree 6 in u (the leading coefficient turns out to be nonzero). We should thus expect 6 singularities on the Coulomb branch.

Four of these can be seen concretely in the weak coupling limit (with fixed masses). Then ϕ_2 is roughly given by the quadratic differential on each pair of pants, connected by a long tube where ϕ_2 is suitably constant, see (4.14). Each three-punctured sphere has two zeros of ϕ_2, hence one branch cut of the SW curve. Consider the pair of pants with masses squared m_1^2, m_2^2, u, for definiteness. Its branch cut shrinks to zero size whenever any combination $\pm m_1 \pm m_2 \pm \sqrt{u}$ of the mass parameters vanishes. We thus find four of the six singular points of the Coulomb branch:

$$u = (m_1 \pm m_2)^2 + O(q) \quad \text{and} \quad u = (m_3 \pm m_4)^2 + O(q). \tag{4.17}$$

The remaining two points are not so easy to determine from the explicit quadratic differential (4.13) of the class S theory, partly because they correspond to the collision of branch points that sit in different pair of pants in our decomposition above. A tedious series expansion shows that at[34]

$$u = \pm 2\big(q(m_2^2 - m_1^2)(m_3^2 - m_4^2)\big)^{1/2} + O(q) \tag{4.18}$$

two branch points collide at $z = \mp 2\big(q(m_2^2 - m_1^2)/(m_3^2 - m_4^2)\big)^{1/2} + O(q)$, with the sign being correlated to that of u.

From the point of view of SQCD with $N_f = 4$ flavours, what happens is as follows. The four doublet hypermultiplets have mass parameters $m_1 + m_2, m_1 - m_2, m_3 + m_4, m_3 - m_4$,

[34] It would be nice to understand the formulae better from our 6d construction.

so when the "VEV" of the vector multiplet ϕ matches one of these we get a massless hypermultiplet; its charge is $+1$ or -1 under the low-energy U(1) because that is how the diagonal $U(1) \subset SU(2)$ acts on a doublet. At low energies $|u| \ll |m_i|, |m_i \pm m_j|$, all hypermultiplets are massive and can be integrated out, leaving behind pure SU(2) SYM, whose Coulomb branch is known to have two singular points at $u = \pm 2\Lambda$, the monopole and dyon points. Incidentally we learn that the dynamically generated scale is at 1/2 times the value (4.18). The main takeaway for our purposes is that the 6d perspective reproduces all the expected physics of SQCD.

By tuning more than just u we can get more than two branch points to collide, hence more than one set of fields to become massless. Such limits can lead in the IR to non-trivial SCFT including the AD theory, which we return to in section 7. The limits are also interesting on the 2d side.

Coupling constants. It may be puzzling how $q = e^{2\pi i \tau}$, which ranges over $\mathbb{C} \setminus \{0, 1\}$, reproduces the complexified gauge coupling $\tau_{\text{Lag}} = \theta/(2\pi) + 4\pi i/g^2$ that appears in a Lagrangian description of SU(2) SQCD with $N_f = 4$, especially the fact that $\operatorname{Im} \tau_{\text{Lag}} > 0$. To relate them, we consider the massless limit where all $m_i \to 0$, such that the theory is an SCFT and the couplings do not run. An equivalent limit is the large $|u|$ region of the Coulomb branch, specifically $|u| \gg |m_i|^2$. From this point of view the running is stopped very quickly at the large energy scale $|u|$. Thus, the IR gauge coupling τ_{IR} captured by the SW curve obeys $\tau_{\text{IR}} = 2\tau_{\text{Lag}}$, with a factor due to how U(1) embeds into SU(2).

The SW curve Σ is a torus double-covering the sphere, with four branch points found at zeros of u_2. At most one of the first four terms in u_2 given in (4.13) can be large at once, so to compensate for the large value of u one must have z close to one of the punctures $0, q, 1, \infty$: the four branch points are thus at

$$z = O\left(\frac{m_1^2}{u}\right), \quad z = q + O\left(\frac{m_2^2}{u}\right), \quad z = 1 + O\left(\frac{m_3^2}{u}\right), \quad z = O\left(\frac{u}{m_4^2}\right). \tag{4.19}$$

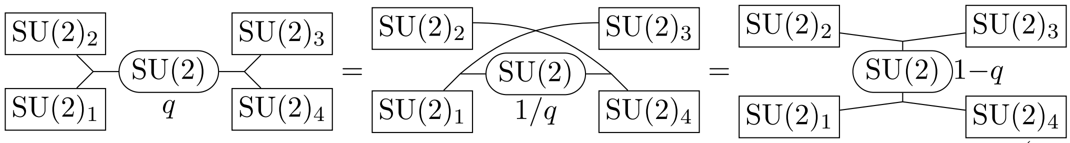
The modular parameter τ_{IR} of the torus Σ gives the complexified gauge coupling, which automatically obeys $\operatorname{Im} \tau_{\text{IR}} > 0$. In terms of the modular lambda function λ, one has [177]

$$\begin{aligned} q &= \lambda(\tau_{\text{IR}}) = 16 e^{\pi i \tau_{\text{IR}}} - 128 e^{2\pi i \tau_{\text{IR}}} + \dots, \\ \tau &= \frac{1}{2}\tau_{\text{IR}} + \frac{\log 16}{2\pi i} - \frac{8}{2\pi i} e^{\pi i \tau_{\text{IR}}} + \dots, \end{aligned} \tag{4.20}$$

where the expansion holds for small gauge coupling g. Both τ and $\tau_{\text{Lag}} = \frac{1}{2}\tau_{\text{IR}}$ are perfectly good definitions of gauge coupling, which amount to two different renormalization scheme, differing by a constant shift and by instanton corrections. The freedom to redefine couplings appears again in (5.38).

S-duality. The four-punctured sphere $\mathbb{CP}^1 \setminus \{0, q, 1, \infty\}$ has three pants decompositions hence three Lagrangian descriptions. The descriptions are identical except for permutations of masses m_1, m_2, m_3, m_4 and changing $q \to 1/q$ or $q \to 1 - q$. This is S-duality of

SQCD [4]. In the notation of (4.11),



(4.21)

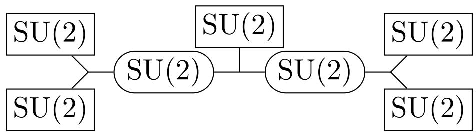
While these equalities are manifest in the 6d perspective they hide deep non-perturbative physics, as they are equalities between QFTs involving completely different elementary gauge fields and matter fields (the gauge field A_μ in some description is unrelated to A_μ in another).

4.3 Generalized SU(2) quivers

We have given all the ingredients to determine $\mathfrak{su}(2)$ class S theories arising from $\mathcal{X}(\mathfrak{su}(2))$ on an arbitrary punctured Riemann surface C with tame punctures.[35] This subsection will thus consist essentially of exercises.

Five-punctured sphere. We consider here $C = \mathbb{CP}^1 \setminus \{0, z_1, z_2, 1, \infty\}$; note that we shifted indices of punctures z_i a bit compared to our earlier conventions. For any decomposition into three-punctured spheres the Lagrangian has the form

$$\mathrm{T}(\mathfrak{su}(2), \mathbb{CP}^1 \setminus \{0, z_1, z_2, 1, \infty\}) =$$



(4.22)

Contrarily to spheres with fewer punctures, the $\mathrm{SU}(2)^5$ flavour symmetry manifest from 6d does not enhance in the 4d theory (as far as I know). S-dualities of this theory were studied in [175] before class S theories and their S-dualities were uncovered in [1].

Exercise 4.8. *Write C as the gluing of three pairs of pants with gluing parameters z_1/z_2 and z_2 following Exercise 3.3.*

Exercise 4.9. *For each pants decomposition of C check that the Lagrangian description is (4.22), with gauge group $\mathrm{SU}(2)^2$ and twelve hypermultiplets. In what representations of the $\mathrm{SU}(2)^2$ gauge group do they transform? What flavour symmetries do these representations carry?*

Exercise 4.10. *1. Each SU(2) gauge group is coupled to four doublet hypermultiplets. When the other gauge group is weakly coupled the theory is thus SQCD coupled to further matter by a weakly coupled gauge field. "Apply" S-duality to this SQCD theory and check that the resulting description is the description one would have written for some pair of pants of the five-punctured sphere.*

[35] We exclude the sphere with no puncture, one puncture (a plane), or two punctures (a cylinder), and the torus without punctures, as they are pathological.

2. Check that elementary S-dualities (4.21) *applied to different gauge nodes do not commute so that the S-duality group(oid) of the* $\mathrm{SU}(2)^2$ *gauge theory is not the product of S-duality groups of two* SQCD *theories.*

Punctured sphere. Next we consider $\mathbb{CP}^1 \setminus \{z_0, \ldots, z_{n-1}\}$ with n punctures, with $z_0 = 0$, $z_{n-2} = 1$, $z_{n-1} = \infty$. Denote by $m_0, m_1, \ldots, m_{n-1}$ the mass parameters of the punctures.

Exercise 4.11. *1. By using Liouville's theorem as in Exercise 4.3 and Exercise 4.5, find all quadratic differentials* $\phi_2(z)$ *that have the prescribed second order poles at punctures. Deduce that the Coulomb branch is* $\mathcal{B} = \mathbb{C}^{n-3}$.

2. Write a $\mathrm{SU}(2)^{n-3}$ *linear quiver description of the theory that is weakly coupled in the regime* $|z_1| \ll |z_2| \ll \cdots \ll |z_{n-3}| \ll 1$.

2. Expand $\phi_2(z)$ *in this regime for* z *in an annulus* $|z_{i-1}| \ll |z| \ll |z_{i+1}|$ *(*$i = 1, \ldots, n-2$*). Check that* ϕ_2 *reduces to the differential of* T_2 *on each of these pair of pants building blocks. Check that* $\mathcal{B} = \mathbb{C}^{n-3}$ *can be parametrized by the parameters* u_i*,* $i = 1, \ldots, n-3$ *of punctures in these pants. Identify* $u_i = \frac{1}{2} \operatorname{Tr} \phi_i^2$ *where* ϕ_i *is the vector multiplet scalar of the* i*-th vector multiplet.*

3. Check that starting at $n = 6$ *pants decompositions can be topologically distinct beyond just the permutation of punctures.*

Punctured torus. We repeat a similar exercise for genus $g = 1$. One could also study theories associated to higher-genus curves, but the relevant mathematics are out of scope of this review.

Exercise 4.12. *1. The once-punctured torus is obtained by gluing two punctures of the same pair of pants. Write the theory as an* $\mathrm{SU}(2)$ *gauge theory and note that there is a decoupled gauge singlet in addition to the adjoint hypermultiplet.*[36]

2. Write the theory associated to an n*-punctured torus as a circular* $\mathrm{SU}(2)^n$ *quiver with a bifundamental hypermultiplet for each pair of neighboring groups. The weak gauge coupling regime corresponds to a long torus with well-separated punctures.*

3. If you know enough about elliptic functions determine all quadratic differentials with prescribed second order poles at the punctures. Expand them in the weak gauge coupling limit as in Exercise 4.11.

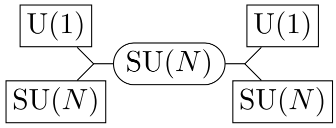
4.4 Linear quiver $\mathfrak{su}(N)$ theories

We move on briefly to $\mathfrak{su}(N)$ class S theories, specifically a particular subclass that is ad-hoc from the 6d perspective but leads to linear quiver gauge theories in 4d, as can be understood using brane constructions. These will be useful for our discussion of instantons.

[36] It is not immediately clear to me how such gauge singlets work out when considering different Lagrangian descriptions of n-punctured tori or of higher genus surfaces. Indeed, some channels include adjoint hypermultiplets, hence gauge singlets, while for others the singlets are not manifest.

Conformal SQCD. Let us try and realize SU(N) SQCD with $N_f = 2N$ flavours (the number of flavours needed for a vanishing beta function) as a class S theory. Its flavour symmetry is $\mathfrak{u}(N_f) = \mathfrak{u}(2N)$ (enhanced to $\mathfrak{so}(8)$ when $N = 2$). In analogy to the $N = 2$ case we expect the gauge group to correspond to a tube joining two three-punctured spheres, so we split the $2N$ flavours as two groups of N, where each group should come from one of the two three-punctured sphere. The flavour symmetry of each group is $\mathfrak{u}(N) = \mathfrak{u}(1) \times \mathfrak{su}(N)$, so that this split makes $\mathfrak{su}(N)^2 \times \mathfrak{u}(1)^2$ flavour symmetry manifest. In analogy to the $N = 2$ case we associate each of the four factors to one puncture and write an analogue of (4.11):[37]

$$\mathrm{T}(\mathfrak{su}(N), \mathbb{CP}^1 \setminus 4\mathrm{pt}, \text{suitable data}) = \tag{4.23}$$



In the $N = 2$ case the $\mathfrak{u}(N) = \mathfrak{u}(2)$ symmetry enhances to $\mathfrak{so}(4)$, namely the $\mathfrak{u}(1)$ factor enhances to $\mathfrak{su}(2)$. For $N > 2$ in contrast we have to deal with the presence of different kinds of punctures. We delay the full story to subsection 7.2. For now we shall be content with using two types of tame punctures: full punctures that carry $\mathfrak{su}(N)$ flavour symmetry and simple punctures that carry $\mathfrak{u}(1)$.

From the 6d point of view, the $\mathfrak{u}(2N)$ flavour symmetry of conformal $\mathfrak{su}(N)$ SQCD is an accidental IR symmetry, as it is not a symmetry of the 6d $\mathcal{N} = (2, 0)$ setup.

Free hypermultiplets. The left and right sides of the quiver (4.23) consist of N^2 hypermultiplets that each have $\mathfrak{u}(1) \times \mathfrak{su}(N)^2$ flavour symmetry (actually more before gauging), of which one $\mathfrak{su}(N)$ factor is gauged. This collection of N^2 free hypermultiplets is the tinkertoy associated to a sphere with two full and one simple puncture.

Punctures. Following the general ideas from the $\mathfrak{su}(2)$ case the full punctures are described as a boundary condition like (4.2):

$$\varphi(z) \sim \Big(\frac{m_i}{z - z_i} + O(1)\Big)\mathrm{d}z \implies \phi_k(z) = \Big(\frac{(-1)^{k+1}\sigma_k(m_i)}{(z - z_i)^k} + O\Big(\frac{1}{(z - z_i)^{k-1}}\Big)\Big)\mathrm{d}z^k, \tag{4.24}$$

for $m_i \in \mathfrak{su}(N)$, where the symmetric polynomials $\sigma_k(m_i)$ are defined by expanding $\det(X - m_i) = X^N + \sum_{k\geq 2} X^{N-k}(-1)^k\sigma_k(m_i)$. The condition on $\varphi(z)$ should be understood modulo conjugation, hence only the conjugacy class of m_i is important.

We return in subsection 7.2 to a description of conjugacy classes in $\mathfrak{su}(N)_{\mathbb{C}} = \mathfrak{sl}(N, \mathbb{C})$. For now, it suffices to mention simple punctures, whose mass parameters take the form $m_i = \mathrm{diag}((N - 1)\mu, -\mu, \dots, -\mu)$. As a result, the massless limit where m_i becomes

[37] As in various other places in this review there are inaccuracies about the global structure of groups. Corrections welcome.

nilpotent (see (4.4) for the $N=2$ case) is different for full and simple punctures:

$$\begin{aligned}\phi_k(z) &= O\Big(\frac{1}{(z-z_i)^{k-1}}\Big)\mathrm{d}z^k \qquad \text{(massless full puncture)},\\ \phi_k(z) &= O\Big(\frac{1}{(z-z_i)}\Big)\mathrm{d}z^k \qquad \text{(massless simple puncture)}.\end{aligned} \tag{4.25}$$

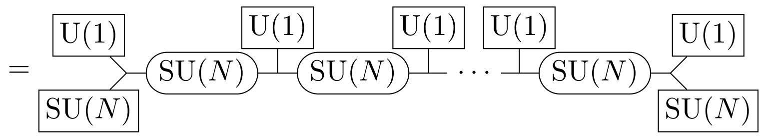
Exercise 4.13. *1. Massless case. Find the most general degree $k\geq 2$ differential on the three-punctured sphere with a simple pole at $z=1$ and poles of order $k-1$ at $z=0,\infty$. Deduce the SW curve of the class S theory corresponding to a sphere with one simple and two full punctures and deduce the theory has no Coulomb branch, consistent with free hypermultiplets.*

2. Massive case. For $N=3$, evaluate $\det(x-\varphi_z)$ near a simple puncture $\varphi(z)\sim((z-1)^{-1}m+p)\mathrm{d}z$ with $m=\mathrm{diag}(2\mu,-\mu,-\mu)$. Observe that the $(z-1)^{-k}$ terms in $u_d(z)$, for $k,d=2,3$, are expressed in terms of the $(z-1)^{-1}$ terms and of μ. Deduce that there is again no Coulomb branch.

3. Massless case. Write the most general degree $k\geq 2$ differential on the n-punctured sphere with order $k-1$ poles (full punctures) at $0,\infty$ and simple poles (simple punctures) at $q_1,\dots,q_{n-3},1$. Write the SW curve and check that the Coulomb branch has dimension $(n-3)(N-1)$, consistent with the quiver (4.26) below.

Linear quiver gauge theory. Starting with collections of N^2 free hypermultiplets, identifying pairs of $\mathfrak{su}(N)$ symmetries, and gauging them using vector multiplets, we find

$$\mathrm{T}(\mathfrak{su}(N),\mathbb{CP}^1\setminus\{0,\underline{z_1},\dots,\underline{z_{n-2}},\infty\})$$



(4.26)

where we have underlined the simple punctures (so that only 0 and ∞ are full punctures). This linear quiver gauge theory description corresponds to a specific degeneration limit of the Riemann surface. We emphasize that other limits would involve more elaborate tinkertoys, which do not typically have any Lagrangian description unless every pair of pants involves a simple puncture. As found in Exercise 4.13, the SW curve in the massless case reads

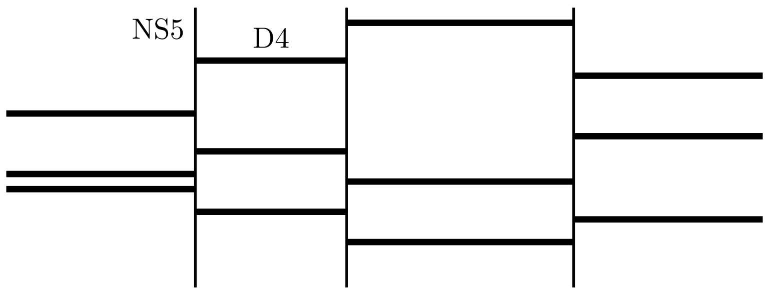
$$x^N=\sum_{k=2}^{N}\frac{P^{(k)}_{n-4}(z)}{(z-z_1)\cdots(z-z_{n-2})z^{k-1}}x^{N-k} \tag{4.27}$$

where $P^{(k)}_{n-4}$ denote polynomials of degree $n-4$.

Exercise 4.14. *Consider the degeneration limit where each z_{i+1}/z_i is kept fixed (and $|z_{i+1}/z_i|>1$) except $z_{j+1}/z_j\to+\infty$ for some j. Check that the part of the curve with finite x/z_j takes the form (4.27) with n replaced by $j+2$ and additional mass terms. In particular the new puncture is a full puncture.*

Brane construction. We know that the 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{su}(N))$ is the world-volume theory of a stack of N coincident M5 branes (minus the center of mass). The Riemann surface in (4.26) can be taken to be a cylinder, with some punctures. Then the brane setup can be described by N M5 branes wrapping the cylinder, with the insertion of transverse M5' branes at $n-2$ points on the cylinder.

Now, M-theory on a circle is IIA string theory, M5 branes wrapping the circle become D4 branes, while M5' branes at points on the circle become NS5 branes. This gives a well-known brane set-up [178] with N D4 brane segments stretching between each pair of neighboring NS5 branes:



(4.28)

The world-volume description of this brane diagram is known to be the linear quiver gauge theory (4.26). Mass parameters of the two $\mathrm{SU}(N)$ flavour symmetries are positions (vertically in the figure) of the semi-infinite D4 branes on either end. Mass parameters of all $\mathrm{U}(1)$ flavour symmetries are distances between centers of masses of each collection of N D4 branes. The remaining vertical positions are dynamical and appear on the gauge theory side as Coulomb branch parameters. The SW curve and differential of the linear quiver can be extracted from this construction [178] and coincides with what is found from the 6d perspective.

It is very easy in the brane diagram to accomodate gauge group ranks that are not all the same. We outline in subsection 7.2 how to realize such theories in class S.

Part II
AGT correspondence

5 Localization for 4d quivers

Up to this point we have been working with 4d $\mathcal{N}=2$ class S theories in Minkowski space. We now turn[38] to Euclidean signature. Our aim in this section and the next is to explain both sides of the AGT relation (1.1) for the case $\mathfrak{g}=\mathfrak{su}(2)$ with tame punctures:

$$Z_{S^4_b}(\mathrm{T}(\mathfrak{su}(2),C,m)) = \left\langle \widehat{V}_{\alpha_1}(z_1)\ldots\widehat{V}_{\alpha_n}(z_n)\right\rangle^{\text{Liouville}}_{\overline{C}}. \tag{5.1}$$

[38]We shall ignore possible difficulties with Wick rotation.

We postpone to section 6 the description of the right-hand side, a 2d Liouville CFT correlator on the Riemann surface C. For now we concentrate on the sphere (and squashed sphere) partition function of $\mathrm{T}(\mathfrak{su}(2), C, m)$. We explain how 4d $\mathcal{N}=2$ theories are placed on this curved background geometry in subsection 5.1, by coupling with supergravity [179–181] (see also [182, 183] for other early explorations). In subsection 5.2 we reduce the infinite-dimensional path integral to a finite-dimensional one (a matrix model) in the Lagrangian case using supersymmetric localization. The resulting expression is built from Nekrasov instanton partition functions, which we explore in subsection 5.3. Supersymmetric localization implies that some factorization properties remain true even for non-Lagrangian theories, see subsection 5.4.

Localization and factorization

- 4d $\mathcal{N}=2$ theories can be put supersymmetrically on $S_b^4 = \{y_5^2 + b^2(y_1^2+y_2^2) + b^{-2}(y_3^2+y_4^2) = r^2\} \subset \mathbb{R}^5$.
- For any gauge theory, supersymmetric localization reduces the S_b^4 partition function exactly to an integral $Z_{S_b^4} = \int \mathrm{d}a\, Z_{\mathrm{cl}}(a, q\overline{q})\, Z_{\text{one-loop}}(a)\, Z_{\mathrm{inst}}(a,q)\, Z_{\mathrm{inst}}(a,\overline{q})$ over the gauge group's Cartan algebra.
- The classical contribution is $Z_{\mathrm{cl}} = |q|^{r^2|a|^2}$.
- The one-loop contribution $Z_{\text{one-loop}}$ only depends on matter and is known for any Lagrangian theory.
- The instanton contribution Z_{inst} is holomorphic in q and known for many theories such as linear quivers.

Localization on the round sphere was done in [10] and extended to the squashed sphere in [11] based on some analogous developments on 3d squashed spheres [184, 185]. The supergravity background of [11] was generalized to complex b in [186]. The partition function admits alternate "Higgs branch localization" expressions [187–190] which we will not need. See [191] for a review of curved-space localization, and [126] more specifically for 4d $\mathcal{N}=2$ theories.

5.1 Theories on an ellipsoid

An easy case: the round sphere. Placing an SCFT on a round sphere S^4 is in principle[39] straightforward: just apply a conformal transformation to the flat space theory since the sphere is conformally flat. The 4d $\mathcal{N}=2$ superconformal algebra on S^4 is then the same as on $\mathbb{R}^4$, namely $\mathfrak{su}^*(4|2)$, whose bosonic part is the conformal algebra $\mathfrak{su}^*(4) = \mathfrak{so}(5,1)$ times the R-symmetry algebras $\mathfrak{u}(1)$ and $\mathfrak{su}^*(2) = \mathfrak{su}(2)$.

The class S theories we study (for tame punctures) are mass deformations of SCFTs. They can thus be placed on S^4 by conformally mapping the SCFT to the sphere, then turning on masses as background values for vector multiplet scalars coupled to the various flavour symmetries. Mass terms turn out to break half of supersymmetry, break the conformal algebra to the rotation algebra $\mathfrak{so}(5) = \mathfrak{usp}(4)$, and the R-symmetry to $\mathfrak{so}(2) = \mathfrak{so}^*(2)$. Altogether one can work out that massive theories preserve the supersymmetry subalgebra

$$\mathfrak{osp}^*(2|4) \subset \mathfrak{su}^*(2|4). \tag{5.2}$$

[39] Identifying operators in flat space and on the sphere is subtle, as there can be some mixing involving curvature tensors. This was understood for 4d $\mathcal{N}=2$ theories in [192].

Note that this differs quite a bit from the Poincaré algebra preserved by massive theories on $\mathbb{R}^4$: for instance spatial isometries of $\mathbb{R}^4$ are $\mathfrak{iso}(4) = \mathbb{R}^4 \rtimes \mathfrak{so}(4)$ while here we have the $\mathfrak{usp}(4) = \mathfrak{so}(5)$ rotation algebra.

The AGT correspondence involves an ellipsoid (often called squashed sphere) S^4_b and not only S^4. The squashed sphere is not conformally flat, and defining theories on this background requires technology that we now explain.

Conformal Killing vectors. We are interested in QFTs on rigid curved spaces (no dynamical gravity). Placing a Poincaré-invariant QFT on a curved space is done by coupling the theory to gravity and freezing the value of the metric[40]. Invariance under changes of coordinates leads to the existence of a conserved stress tensor ($\nabla_\nu T^{\mu\nu} = 0$). As the next exercise shows, the resulting curved-space theory preserves some space-time (Poincaré) symmetries provided the metric admits *Killing vectors* Y_μ, defined by the Killing vector equation $\nabla_\mu Y_\nu + \nabla_\nu Y_\mu = 0$. More generally if the flat-space QFT is conformal, isometry and conformal symmetries of the CFT are given by *conformal Killing vectors*

$$\nabla_\mu Y_\nu + \nabla_\nu Y_\mu = \frac{2}{d} g_{\mu\nu} \nabla_\rho Y^\rho. \tag{5.3}$$

Exercise 5.1. *1. A Poincaré-invariant local QFT has a conserved stress-tensor $T^{\mu\nu}$ that is symmetric. Check that the current $Y_\mu T^{\mu\nu}$ is conserved if Y is a Killing vector.*

2. If the flat-space QFT is conformally invariant, $T^{\mu\nu}$ is traceless as well. Check that $Y_\mu T^{\mu\nu}$ is then conserved provided Y is a conformal Killing vector. Hint: explain the factor $2/d$ in (5.3) by taking the trace of the equation.

Conformal Killing spinors. Consider now a supersymmetric theory. This means that there are conserved supersymmetry currents G^μ_α and $\tilde{G}^{\dot\alpha\mu}$, where μ is a vector index of the SO(4) rotation group, and $\alpha = 1, 2$ and $\dot\alpha = 1, 2$ are spinor indices with both chiralities. Leaving the spinor index of G^μ implicit, the conservation equation reads

$$D_\mu G^\mu := \nabla_\mu G^\mu + \frac{1}{4} \omega_\mu{}^{ab} \gamma_a \gamma_b G^\mu = 0 \tag{5.4}$$

where ∇ is the Levi–Civita connection of the given metric, ω its spin connection, a, b are vielbein indices, and γ are Dirac matrices.

To get a usual conserved translation current from the conserved stress-tensor in flat space one contracts $T^{\mu\nu}$ with a constant translation vector a_μ. Likewise here we have usual currents $\xi G^\mu = \xi^\alpha G^\mu_\alpha$ and $\tilde\xi \tilde G^\mu = \tilde\xi_{\dot\alpha} G^{\dot\alpha\mu}$ for constant[41] spinors ξ. In curved space we can check that ξG^μ is conserved provided ξ is a Killing spinor:

$$D_\mu \xi := \left(\partial_\mu + \frac{1}{4} \omega^{ab}_\mu \gamma_a \gamma_b \right) \xi = 0. \tag{5.5}$$

[40] To be precise, if the theory has spinors one must additionally give a spin structure rather than only the metric (for instance giving a vielbein is enough).

[41] Here we use a common abuse of language: talking about constant spinors requires a choice of vielbein, for which we choose the standard Cartesian one on flat space.

(Note that we put ∂ instead of ∇ because ξ has no vector index.)

Just as a theory is conformally invariant if $x_\mu T^{\mu\nu}$ is conserved, a theory is superconformally invariant if $x^\nu \gamma_\nu G^\mu$ is conserved in the same sense as (5.4). When put on curved space, the theory now has super(conformal) symmetries if the spacetime admits a conformal Killing spinor

$$D_\mu \xi = \frac{1}{d} \gamma_\mu \gamma^\nu D_\nu \xi. \tag{5.6}$$

Exercise 5.2. *Check that* (5.6) *indeed leads to a conserved current ξG if the theory is superconformal.*

Generalized Killing spinors. We defined the (partial) topological twist in subsection 3.1 as a mixing of some rotations and R-symmetries, which allowed us to compactify the 6d $(2,0)$ theory on a Riemann surface C. This idea is refined and generalized as follows whenever the flat-space theory has an R-symmetry current J^μ. When placing the QFT on curved space we can turn on a background gauge field V_μ coupled to J^μ in addition to the metric $g_{\mu\nu}$ that is coupled to $T^{\mu\nu}$ (we typically don't turn on fermionic backgrounds coupled to supersymmetry currents).

In such a background, the Killing spinor equation (5.5) generalizes by including the R-symmetry gauge field in the covariant derivative:

$$D_\mu \xi \coloneqq \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_a \gamma_b + V_\mu\right) \xi = 0. \tag{5.7}$$

The conformal Killing spinor equation generalizes in the same way to (5.6) with the new D_μ. Here $V_\mu \xi$ should be suitably weighted by the R-charge under the given R-symmetry, as is standard for covariant derivatives in the presence of a gauge field. The (partial) topological twist consists of choosing V_μ to cancel some component of ω_μ^{ab} so that the corresponding component of ξ can simply be chosen to be constant (along C). More general choices for the background gauge field V can make it possible to preserve some supersymmetries even if the curved manifold of interest does not have any (conformal) Killing spinors.

Squashed sphere. The supergravity background found in [11] to place 4d $\mathcal{N} = 2$ theories on S_b^4 is rather complicated and gives non-zero values to most bosonic fields in the supergravity multiplet. We point to the review [126] for actual expressions. For our purposes we only need two aspects. The metric is the one induced from that of Euclidean $\mathbb{R}^5$ in the embedding

$$S_b^4 \coloneqq \{y_5^2 + b^2(y_1^2 + y_2^2) + b^{-2}(y_3^2 + y_4^2) = r^2\} \subset \mathbb{R}^5. \tag{5.8}$$

Parts of 4d $\mathcal{N} = 2$ supersymmetry remain: $\mathrm{U}(1)^2$ rotations M_{12} and M_{34} in the y_1, y_2 and y_3, y_4 planes, and a supercharge Q (and its conjugate) such that

$$Q^2 = \frac{b}{r}(M_{12} - \tfrac{1}{2} J_3^{\mathrm{R}}) + \frac{1}{br}(M_{34} - \tfrac{1}{2} J_3^{\mathrm{R}}) \tag{5.9}$$

where J_3^{R} is the Cartan generator of $\mathfrak{su}(2)_{\mathrm{R}}$.

5.2 Supersymmetric localization

The idea of supersymmetric localization is several decades old when applied to scalar supercharges of topologically twisted field theories. It received a new life since Pestun's calculation in 2007 [10] of the sphere partition function of 4d $\mathcal{N} = 2$ theories, and of Wilson loop expectation values. In the following decade the technique was sucessfully applied to many dimensions (from 1d to 7d and even continuous dimensions) and geometries (such as spheres S^d, products $S^{d-1} \times S^1$, hemispheres and other spaces with boundaries), as summarized in the 2016 review volume [125]. Besides the applications to understanding supersymmetric theories and black hole state counting, the resulting expressions are often complicated special functions with an interest of their own. We introduce here the technique and present in subsection 5.4 a variant that explains various factorization properties.

Set-up for supersymmetric localization. Our goal is to compute a path integral

$$\langle \mathcal{O} \rangle = \int [\mathrm{D}\phi] e^{-S} \mathcal{O} \tag{5.10}$$

that is invariant under some supercharge Q (we denote collectively all the fields as ϕ). This means that the action and path integration measure are Q-invariant ($QS = 0$ and $\int [\mathrm{D}\phi] Q(\text{anything}) = 0$) and that the observable also is ($Q\mathcal{O} = 0$). Roughly speaking, the integrand is invariant along orbits of Q in the space of field configurations, so the integral on each non-trivial orbit is the Grassmann integral of a constant hence vanishes. Intuitively, this ought to reduce the integral to Q-invariant field configurations.

Recipe for supersymmetric localization

- Choose a supergravity background on M with at least one generalized conformal Killing spinor ξ, so that the theory on M has at least one supersymmetry Q. Realize it off-shell.
- Choose a fermionic functional V that is Q^2-invariant and has $(QV)_{\text{bosonic}} \geq 0$ on the path integral contour.
- Find zeros of $(QV)_{\text{bosonic}}$, which will be the resulting integration locus, often finite-dimensional.
- Expand $(QV)_{\text{bosonic}}$ to quadratic order around these zeros and compute the Gaussian integral (one-loop determinants).

Supersymmetric localization is based on the path integral so we need the supersymmetry Q to be realized *off-shell*, not only on-shell. For highly supersymmetric theories (such as 4d $\mathcal{N} = 2$), realizing all supersymmetries on-shell requires infinitely many auxiliary fields, but only a finite number are typically needed to realize a single supercharge off-shell.

The key idea of supersymmetric localization is to deform the action S in a way that does not affect $\langle \mathcal{O} \rangle$ yet suppresses contributions from most configurations, thus reducing the path integral down to a smaller space of configurations. Typically one arranges to make this smaller space finite-dimensional so as to get a well-defined finite-dimensional integral, but it can also be interesting to get a lower-dimensional field theory.

Concretely, one needs some functional of the fields with three properties: it is Q-exact (namely of the form QV), Q-closed (namely V is invariant under the bosonic symmetry Q^2), and has nonnegative bosonic part on the path integration contour we wish to consider. A typical choice is roughly speaking a sum over all fermions of the theory (collectively denoted as ψ) of the form

$$V = \sum_{\text{all spinors } \psi} \psi \overline{Q\psi} \qquad \Longrightarrow \qquad (QV)_{\text{bosonic}} = \sum_{\psi} |Q\psi|^2 \tag{5.11}$$

for a suitable definition of the conjugate $\overline{Q\psi}$ which ensures positivity of $(Q\psi)\overline{Q\psi}$ along the path integral contour.

Saddle-point calculation. Once such a term is chosen, we deform the action by tQV for $t \in [0, +\infty)$ and notice that the observable is unaffected since

$$\begin{aligned} \langle \mathcal{O} \rangle_t &= \int [\mathrm{D}\phi] e^{-S-tQV} \mathcal{O} \\ \Longrightarrow \partial_t \langle \mathcal{O} \rangle_t &= -\int [\mathrm{D}\phi] e^{-S-tQV} \mathcal{O}\, QV = -\int [\mathrm{D}\phi] Q\big(e^{-S-tQV} \mathcal{O}\, V\big) = 0. \end{aligned} \tag{5.12}$$

Here we used that $Q\mathcal{O} = 0 = Q(S + tQV)$ to write the integrand as Q of something.

The observable is t-independent, so we can take the limit $t \to \infty$, in which limit the saddle-point approximation becomes exact. In addition, any saddle with $(QV)_{\text{bosonic}} > 0$ is infinitely suppressed by e^{-tQV}. Since we assumed $QV \geq 0$, in this limit we are left with an integral over field configurations with $QV = 0$ and the Gaussian integral of quadratic fluctuations around it:

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{t=0} = \lim_{t \to \infty} \langle \mathcal{O} \rangle_t = \int_{QV=0} [\mathrm{D}\phi] e^{-S[\phi]} Z_{\text{one-loop}}[\phi] \mathcal{O}[\phi]. \tag{5.13}$$

Here we wrote schematically $\int_{QV=0}$, but this may also involve discrete sums if the space of zeros of QV is disconnected. Here $Z_{\text{one-loop}}[\phi]$ is the result of a Gaussian integral of $\exp(-tQV)$ around a field configuration ϕ that is a zero of QV.

Calculating the one-loop determinant $Z_{\text{one-loop}}$ is often the most technical step. It is determined by the quadratic terms $(QV)_2$ in QV, and more precisely is a ratio of determinants of the fermionic and bosonic parts of $(QV)_2$. In sufficiently simple geometries one can explicitly diagonalize these operators by listing all the modes and regularize the infinite products over modes using e.g. zeta-function regularization. The calculation involves huge cancellations, due to how the supercharge Q pairs up most bosonic and fermionic modes together. Roughly, only modes that are annihilated by Q contribute, and this observation helps extend the cases where one-loop determinants can be evaluated in a pedestrian manner. Finally, this latter calculation can be much simplified by using powerful index theorems. We refer to the review volume [125] for details.

Exercise 5.3. *1. Remind yourself that for a symmetric $n \times n$ matrix Δ we have the Gaussian integral $\int_{\mathbb{R}^n} \mathrm{d}^n x \exp(-x^T \Delta x) = \sqrt{\det(\pi/\Delta)}$.*

2. Consider next n pairs of Grassmann variables $\theta_1, \bar{\theta}_1, \dots, \theta_n, \bar{\theta}_n$ and an $n \times n$ matrix D, and evaluate the Berezin integral $\int \mathrm{d}^n\theta \mathrm{d}^n\bar{\theta} \exp(-\bar{\theta}.D.\theta) = \pm \det D$.

In concrete QFT *applications, the bosonic operator* Δ *is roughly the Laplacian and the fermionic operator* D *is roughly the Dirac operator, so that* $\Delta \sim D^2$ *and their one-loop determinants mostly cancel.*

Saddle-points on ellipsoid. We now apply localization to 4d $\mathcal{N}=2$ theories on the squashed sphere S^4_b. We take the standard deformation term (5.11) where the sum ranges over quarks ψ (hypermultiplet spinors) and gauginos λ (vector multiplet spinors). The resulting QV is pretty similar to the 4d $\mathcal{N}=2$ action of these multiplets, and we only mention what is needed to determine the space $(QV)_{\text{bosonic}}=0$.

Supersymmetric localization relies on the existence of a supercharge Q that is an *off-shell* symmetry. This requires the addition of some auxiliary fields K to the 4d $\mathcal{N}=2$ theory. For now we focus on the sphere [10], restoring the squashing only in the final expressions [11].

For the hypermultiplet we have

$$(QV_{\text{hyper}})_{\text{bosonic}} = |Dq|^2 + |D\tilde{q}|^2 + \cdots + \frac{R}{6}|q|^2 + \frac{R}{6}|\tilde{q}|^2 + |K_q|^2. \tag{5.14}$$

Here, R is the Ricci scalar, which is positive: this term arises upon conformally mapping $|Dq|^2$ from flat space to the sphere. The "…" are also a sum of squares, so the zero locus has the whole hypermultiplet set to zero:

$$q = \tilde{q} = K_q = 0, \tag{5.15}$$

and fermions as well since they are Grassmann variables.

For the vector multiplet we have similar terms with $q, \tilde{q}$ replaced by the vector multiplet scalar ϕ, but we also have terms like $|F_{\mu\nu}|^2$ and terms due to the supergravity background. Eventually (the bosonic part of) the deformation term can be massaged to a sum of squares of the form

$$\begin{aligned}(QV)_{\text{vector,bosonic}} = {} & \frac{r-x^0}{2r}\big(F^-_{\mu\nu} + w^-_{\mu\nu}\operatorname{Re}\phi\big)^2 + \frac{r+x^0}{2r}\big(F^+_{\mu\nu} + w^+_{\mu\nu}\operatorname{Re}\phi\big)^2 \\ & + |D\phi|^2 + [\phi,\phi^\dagger]^2 + |K_{\phi,i} + w_i \operatorname{Im}\phi|^2.\end{aligned} \tag{5.16}$$

Here F^- and F^+ are the (anti-)self-dual parts of the gauge field strength, $K_{\phi,i}$, $i=1,2,3$ are auxiliary fields (a triplet of $\mathfrak{su}(2)_{\mathrm{R}}$), and $w^\pm_{\mu\nu}$ and w_i are determined by the supergravity background.

Let us find zeros of (5.16). Each term must vanish, so in particular ϕ is covariantly constant ($D\phi=0$). Away from the poles $x^0=\pm r$, we have $F^\pm_{\mu\nu} = -w^\pm_{\mu\nu}\operatorname{Re}\phi$ so $F_{\mu\nu} = -w_{\mu\nu}\operatorname{Re}\phi$ where $w = w^+ + w^-$. Then the Bianchi identities imply

$$0 = D^\mu F_{\mu\nu} = D^\mu(-w_{\mu\nu}\operatorname{Re}\phi) = -(\partial^\mu w_{\mu\nu})\operatorname{Re}\phi - w_{\mu\nu}\operatorname{Re}(D^\mu\phi) \tag{5.17}$$

and the last term vanish since $D^\mu\phi=0$. In the specific supergravity background we have here, $\partial^\mu w_{\mu\nu}\neq 0$, so we learn that $\operatorname{Re}\phi=0$, hence $F_{\mu\nu}=0$. Thus, in a suitable gauge the gauge field vanishes and

$$A_\mu = 0, \quad \phi = a, \quad K_{\phi,i} = -w_i a \qquad \text{away from poles,} \tag{5.18}$$

for a constant a.

At the poles $x^0 = \pm r$, on the other hand, we only have one of the two equations $F^{\pm}_{\mu\nu} = -w^{\pm}_{\mu\nu} \operatorname{Re}\phi$, while the other part of $F_{\mu\nu}$ is unconstrained. This suggests to include point-like instanton configurations at the poles:

$$\text{instantons } (F^+ = 0) \text{ at } x^0 = r; \quad \text{anti-instantons } (F^- = 0) \text{ at } x^0 = -r. \tag{5.19}$$

Let us concentrate on the North pole $x^0 = r$. Instanton configurations are insensitive to matter, so that the instanton moduli space $\mathcal{M}_{\text{inst}}$ is a product, over simple gauge group factors, of a moduli space of instantons for each gauge group. This, in turn, splits as a union of infinitely many connected components, labeled by the instanton number $k = \# \int \operatorname{Tr}(F \wedge F) \in \mathbb{Z}_{\geq 0}$ (for some calculable constant $\#$), with one instanton number per gauge group. Altogether, denoting the gauge group by $\prod_I G_I$,

$$\mathcal{M}_{\text{inst}} = \prod_I \Bigl(\bigsqcup_{k_I \geq 0} \mathcal{M}_{G_I, k_I} \Bigr). \tag{5.20}$$

Reality of Coulomb branch parameter and masses. The condition $\operatorname{Re} a = 0$ simply means that $a \in \mathfrak{g}$ rather than the complexification thereof. We gauge-fix it so that it belongs to the Cartan subalgebra $\mathfrak{h}$ modulo the Weyl group. The reality condition is then that

$$\langle w, a \rangle \in i\mathbb{R}, \qquad \text{for every weight } w. \tag{5.21}$$

For instance, a is anti-Hermitian for $\mathfrak{g} = \mathfrak{su}(N)$.

This reality condition arose here from studying saddle-points of the deformation term, but the same condition also derives from requiring the configuration to be Q-invariant. As explained previously, hypermultiplet masses m simply amount to constant background values for vector multiplet scalars. Thus, the reality condition (5.21) is also required for masses in order for them to preserve the supercharge Q used by supersymmetric localization. In other words, curved-space supersymmetry on S^4_b requires masses to be imaginary.

Result for round and squashed sphere. Saddle-point configurations defined by (5.15), (5.18), (5.19) are thus characterized by a choice, for each gauge group, of an imaginary Coulomb branch parameter a and point-like (anti-)instanton configurations at the poles. For any such saddle-point we compute the classical action

$$S_{\text{cl}} = -\operatorname{Re}(2\pi i \tau) \operatorname{Tr}(ra)^2 + 2\pi i \tau n - 2\pi i \overline{\tau n}, \qquad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}, \tag{5.22}$$

where the radius r of the (squashed) sphere (5.8) makes the first term dimensionless.

We refer to [126] for the computation of one-loop determinants (Gaussian integral) capturing the effect of quadratic fluctuations around the saddle-points. For the zero (anti-)instanton saddle, one gets

$$Z_{\text{one-loop}} = Z^{\text{vector}}_{\text{one-loop}} Z^{\text{hyper}}_{\text{one-loop}} \quad \text{for } k = \overline{k} = 0. \tag{5.23}$$

Here, the vector multiplet one-loop determinant is a product over roots α of all gauge group factors (non-zero weights of the adjoint representation),[42]

$$Z^{\text{vector}}_{\text{one-loop}} = \prod_{\alpha\in\Delta} \Upsilon_b(\langle\alpha, ra\rangle), \tag{5.24}$$

where Υ_b is a special function defined in Appendix A. We emphasize that its argument here is purely imaginary. The hypermultiplet one-loop determinant is a product over weights w (with multiplicity) of the representation in which the hypermultiplet transforms,

$$Z^{\text{hyper}}_{\text{one-loop}} = \prod_{w\in R} \frac{1}{\Upsilon_b\big(\frac{b+1/b}{2} + \langle w, ra\rangle\big)}. \tag{5.25}$$

As explained previously, hypermultiplet masses are simply background values for vector multiplet scalars corresponding to flavour symmetries, so adding a mass m in (5.25) simply changes

$$\Upsilon_b\left(\frac{b+1/b}{2} + \langle w, ra\rangle\right) \to \Upsilon_b\left(\frac{b+1/b}{2} + \langle w, ra\rangle + rm\right). \tag{5.26}$$

Importantly, hypermultiplets in the representations R and $\overline{R}$ are equivalent (with mass $m \to -m$) and one checks that the symmetry $\Upsilon_b(b + 1/b - x) = \Upsilon_b(x)$ ensures that the one-loop determinant (5.25) computed with both presentations is the same. For a half-hypermultiplet in a pseudoreal representation $R \simeq \overline{R}$ one should keep only one factor for each pair of conjugate weights; thanks to the same symmetry of Υ_b it does not matter which weight one selects in each pair. As expected all factors are invariant under the $b \to 1/b$ symmetry of S^4_b thanks to $\Upsilon_b = \Upsilon_{1/b}$.

One-loop determinants can be further understood as products of contributions from both hemispheres, essentially by decomposing each Υ_b function as $\Upsilon_b(x) = 1/(\Gamma_b(x)\,\Gamma_b(b + 1/b - x))$. (Anti-)instantons at each pole only affect one-loop contributions from the corresponding hemisphere, which leads to a factorization property of the form

$$Z_{\text{one-loop}}(a, k, \xi, \overline{k}, \xi) = Z_{\text{one-loop}}(a) Z_{\text{one-loop,inst}}(a, k, \xi) Z_{\text{one-loop,inst}}(a, \overline{k}, \overline{\xi}) \tag{5.27}$$

where $\xi \in \mathcal{M}_{G,k}$ and $\overline{\xi} \in \mathcal{M}_{G,\overline{k}}$ parametrize the instanton configurations and $Z_{\text{one-loop}}(a)$ is the ratio of Υ_b written above.

Altogether, collecting all (anti-)instanton contributions together, including the classical contributions expressed in terms of $q = \exp(2\pi i\tau)$, the partition function reads

$$Z_{S^4_b} = \int \mathrm{d}a\, Z_{\text{cl}}(a, q\overline{q}) Z_{\text{one-loop}}(a) Z_{\text{inst}}(a, q) Z_{\text{inst}}(a, \overline{q}). \tag{5.28}$$

Here, the integral ranges over the Cartan algebras of all gauge groups, $Z_{\text{one-loop}}$ is given above, and the exponent in $Z_{\text{cl}} = \exp(-S_{\text{cl}}) = |q|^{r^2|a|^2}$ involves the Killing form

[42] To be precise, we have included here in $Z^{\text{vector}}_{\text{one-loop}}$ the Vandermonde determinant $\prod_{\alpha\in\Delta}(\langle\alpha, ra\rangle)$ that arises when converting from an integral over the whole gauge algebra $\mathfrak{g}$ to its Cartan subalgebra.

$|a|^2 = -\operatorname{Tr}(a^2)$ of the Lie algebra and is positive since a is imaginary. We collected into Z_{inst} the shift of S_{cl} due to instantons, and $Z_{\text{one-loop,inst}}$, suitably integrated over the instanton moduli space,

$$Z_{\text{inst}}(a,q) = \sum_{\text{all } k_I \geq 0} \biggl(\prod_I q_I^{k_I}\biggr) \int_{\prod_I \mathcal{M}_{G_I,k_I}} Z_{\text{one-loop,inst}}(a,k,\xi)\, d\xi. \tag{5.29}$$

There remains to compute this instanton partition function.

5.3 Instanton partition functions

Omega background. The point-like configurations in (5.19) are only sensitive to the leading expansion of the supergravity background around the poles. This supergravity background has a flat metric and a non-trivial graviphoton, and coincides with the Omega background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ discovered by Nekrasov [12], with parameters $\epsilon_1 = b/r$, $\epsilon_2 = 1/(rb)$. It was thus naturally conjectured in [10, 11] (and later works) that in the expression (5.28) of $Z_{S^4_b}$, the function $Z_{\text{inst}}(a,q) = 1 + O(q^1)$ to be included is the partition function of the 4d $\mathcal{N}=2$ theory on the Omega background $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$, called Nekrasov's instanton partition function [12, 13]. We refer to reviews [116, 127] for a more detailed introduction to Z_{inst}.

The Omega background tends to $\mathbb{R}^4$ as $\epsilon_1,\epsilon_2 \to 0$, and can be understood as a regulator for IR divergences due to non-compactness of $\mathbb{R}^4$. In fact, as explained in [12, 13] and the appendix of [193], the partition function gives the low-energy prepotential of the gauge theory:[43]

$$F(a,q) = F_{\text{pert}}(a,q) + \lim_{\epsilon_1,\epsilon_2\to 0}\Bigl(\epsilon_1\epsilon_2 \log Z_{\text{inst}}(a,q;\epsilon_1,\epsilon_2)\Bigr), \tag{5.30}$$

where F_{pert} results from a one-loop computation and can be extracted from $Z_{\text{one-loop}}$. In this way, the instanton partition function gives access to the low-energy dynamics of the 4d $\mathcal{N}=2$ theory at a point a along the Coulomb branch. The whole SW curve can then be rigorously derived, as done in [13, 194, 195]. The derivation relies on a link with the theory of random partitions.

The Omega background can also be obtained as the $\beta \to 0$ limit of a 5d background $S^1_\beta \times_{\epsilon_1,\epsilon_2} \mathbb{R}^4$ defined as the quotient of $\mathbb{R}\times\mathbb{C}\times\mathbb{C}$ under the identification $(x,z_1,z_2) \sim (x+\beta, e^{i\beta\epsilon_1}z_1, e^{i\beta\epsilon_2}z_2)$. Many 4d $\mathcal{N}=2$ Lagrangian theory can be lifted to a 5d $\mathcal{N}=1$ theory, in which case the instanton partition function Z_{inst} has a 5d analogue defined as the partition function on $S^1_\beta \times_{\epsilon_1,\epsilon_2} \mathbb{R}^4$. It is also worth mentioning developments in how to construct the Omega background in string theory and M-theory [196–200].

Computation methods. The instanton partition function is an integral over the moduli space of instantons with an integrand depending on matter. As mentioned above, this moduli space decomposes as a product over simple gauge group factors, and the

[43] The prepotential is the Lagrangian density in $\mathcal{N}=2$ superspace, and it encodes fully the low-energy dynamics of U(1) gauge fields at a generic point on the Coulomb branch. We point to reviews such as [115] for more discussion of this crucial function.

moduli space for each gauge group factor has one connected component for each instanton number $k \geq 0$, with dimensions growing with k. There are several methods to compute the instanton partition function [127].[44]

- **ADHM construction.** For classical gauge groups U (rather than SU), USp, SO the n-instanton moduli space can be realized as a symplectic quotient through the Atiyah–Drinfeld–Hitchin–Manin (ADHM) construction [201], while no such constructions are available in general for the exceptional groups E_6, E_7, E_8, F_4, G_2, nor for SU. Physically, the ADHM construction expresses the instanton moduli space as the moduli space of a supersymmetric matrix model, describing $D(-1)$ brane instantons in a background of D3 branes and O3 planes that realize 4d $\mathcal{N} = 2$ vector multiplets [202–204].

  The matrix model is known for hypermultiplets in suitable representation (e.g. bifundamental). Localization then reduces Z_{inst} from an integral over the whole moduli space down to a discrete sum of contributions from collections of point-like instantons respecting certain U(1) symmetries. We give the resulting formula for $U(N)$ gauge theories in (5.33), below. This Losev–Moore–Nekrasov–Shatashvili (LMNS) formula was obtained in [205–207], derived in [12], and extended to other classical groups in [208–210], to quivers in [194], and to supergroup theories in [211]. These methods apply to 5d $\mathcal{N} = 1$ lifts of these theories: the $S^1_\beta \times_{\epsilon_1,\epsilon_2} \mathbb{R}^4$ instanton partition function matches the supersymmetric index of a quantum mechanics analogue of the matrix model (see e.g. [212, 213]).

- **Pure 4d $\mathcal{N} = 2$ SYM.** For exceptional gauge groups there is no ADHM construction. The one-instanton moduli space is ($\mathbb{C}^2$ times) the orbit under G of the highest weight vector of its adjoint representation. This allows a group-theoretic calculation of one-instanton partition functions for arbitrary gauge groups in pure SYM [99, 214, 215]. This was extended to two instantons in [216].

  Instead of realizing the k-instanton moduli space as a Higgs branch as in the ADHM construction, it can be realized as the Coulomb branch of a 3d $\mathcal{N} = 4$ SCFT. The Coulomb branch Hilbert series (a specialization of the superconformal index) of this 3d theory then gives the k-instanton partition function of pure SYM on $S^1_\beta \times_{\epsilon_1,\epsilon_2} \mathbb{R}^4$ with an arbitrary gauge group [217]. The instanton partition function for pure E_n SYM theory can be determined from the Hall–Littlewood index of the E_n Minahan–Nemeschansky theory, calculated using the TQFT realization of this index discussed in subsection 9.2.

- **Mass-deformed 4d $\mathcal{N} = 4$ SYM.** For 4d $\mathcal{N} = 2^*$ SYM, which interpolates between $\mathcal{N} = 4$ and $\mathcal{N} = 2$ SYM as the mass is varied, the prepotential (or its Omega-deformation) is expected to obey a modular anomaly equation that describes how it transforms under S-duality [218, 219]. This led to a formula for the prepotential in terms of Eisenstein series for arbitrary gauge groups [220–223]. This technique extends somewhat, for instance to conformal SQCD [224, 225].

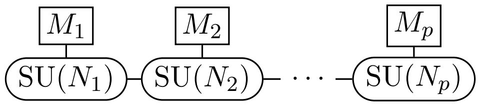
[44] I thank Jaewon Song for correspondence on some methods.

- **Recursion relations.** Comparing the instanton partition function to the partition function on the blow-up of $\mathbb{C}^2$ yields recursion relations (see [226–228] and references therein). For hypermultiplets (but not half-hypermultiplets) in a large class of representations of both classical and exceptional gauge groups, one can solve these recursion relations and deduce Z_{inst} from the perturbative (one-loop) partition function.

Some matter representations escape all the available methods: most notably, general Lagrangians constructed from SU(2) vector multiplets and trifundamental half-hypermultiplets, which are the Lagrangian descriptions of SU(2) class S theories. Among these theories, Z_{inst} is known whenever the theory can be written with gauge groups SU(2) and $\mathrm{SO}(4) = (\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2$ and bifundamental matter (of two SU(2) groups or of an SU(2) and an SO(4) group); see [96, 98]. A possible way forward relies on the topological vertex formalism [229].

Nekrasov partition functions have many generalizations, such as on ALE space [82, 230–235] and more general spaces [236–238], spiked instantons [239], and a proposed generalization to 4d $\mathcal{N}=1$ theories [240].

Explicit formula for linear quiver gauge theories. We focus now on the linear quiver gauge theory



(5.31)

which is a slight generalization (allowing N_i to be distinct) of those in subsection 4.4. This theory has gauge group $\prod_{i=1}^{p}\mathrm{SU}(N_i)$, one hypermultiplet in each bifundamental representation $N_i\otimes\overline{N_{i+1}}$, and M_i hypermultiplets transforming in the fundamental representation N_i of each group. If the theory is conformal or asymptotically free, namely the beta-function coefficient $b_j = 2N_j - N_{j-1} - N_{j+1} - M_j$ is non-negative, then it can be realized in class S using quiver tails, see (7.15).

Extending the gauge groups from $\mathrm{SU}(N_i)$ to $\mathrm{U}(N_i)$ allows the theory to be realized in IIA string theory, as the world-volume theory of groups of N_i parallel D4 branes stretched between consecutive NS5 branes, together with M_i transverse D6 branes that give rise to fundamental hypermultiplets, as depicted in Figure 3. Instantons are described as D0 branes also stretching between NS5 branes, and Z_{inst} is the partition function of their world-volume theory, summed over the number of D0 branes. Eventually, one must remove spurious factors caused by the additional U(1) gauge groups.

We denote the Coulomb branch parameter $a^i = \{a^i_1,\dots,a^i_{N_i}\}$ for $1\le i\le p$, and by k_i the number of instantons for $\mathrm{U}(N_i)$, namely the number of D0 branes in the i-th interval between NS5 branes. Then

$$Z_{\text{inst}}\Big[\prod \mathrm{U}(N_i)\Big] = \sum_{k_1\ge 0,\dots,k_p\ge 0} z_1^{k_1}\cdots z_p^{k_p}\, Z^{(k)}_{\text{inst}}, \tag{5.32}$$

where the counting parameters z_j encode the dynamical scale Λ_j or gauge coupling τ_j

of the j-th gauge group.[45] The k-instanton contribution to Z_{inst} is the matrix model contour integral

$$Z^{(k)}_{\text{inst}} = \int \mathrm{d}\phi \frac{\prod\limits_{i,F,I} (m^i_F - \phi^i_I) \prod\limits_{i=1}^{p-1} \Big[\prod\limits_{I,J} S(\phi^{i+1}_J - \phi^i_I) \prod\limits_{A,J} (\phi^{i+1}_J - a^i_A + \epsilon_1 + \epsilon_2) \prod\limits_{I,B} (a^{i+1}_B - \phi^i_I) \Big]}{\prod\limits_i \Big[(\frac{\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2})^{k_i} \prod\limits_{I\neq J} S(\phi^i_I - \phi^i_J) \prod\limits_{A,I} (\phi^i_I - a^i_A + \epsilon_1 + \epsilon_2)(a^i_A - \phi^i_I) \Big]} \tag{5.33}$$

where $S(\phi) = -(\phi+\epsilon_1)(\phi+\epsilon_2)/[\phi(-\phi-\epsilon_1-\epsilon_2)]$, indices run over the ranges that are natural given where they appear ($1 \leq i \leq p$, $1 \leq F \leq M_i$, $1 \leq I \leq k_i$, $1 \leq J \leq k_{i+1}$, $1 \leq A \leq N_i$, $1 \leq B \leq N_{i+1}$), and $\mathrm{d}\phi$ denotes a product of all $\mathrm{d}\phi^i_I$. The first factor in the numerator captures the effect of fundamental hypermultiplets, the rest of the numerator comes from bifundamental hypermultiplets, and the denominator from vector multiplets.

The integrand can be understood from the IIA string theory construction depicted in Figure 3. The vector multiplet contribution (denominator) arises from strings stretching between different D0 branes in a given interval ($1/S$ factors) and strings stretching between D0 and D4 branes (the remaining factors). The fundamental hypermultiplet contribution arises from strings connecting D0 and D6 branes. Strings joining D0 or D4 branes on two sides of an NS5 brane yield the bifundamental hypermultiplet contribution, in which one omits the D4-D4 interactions because they are already taken into account in $Z_{\text{one-loop}}$.

The same formula can be obtained from first principles using the mathematically rigorous ADHM construction. The contour in (5.33) is such that it enclose poles at

$$\{\phi^i_I \mid 1 \leq I \leq k_i\} = \{a^i_A + (r-1)\epsilon_1 + (s-1)\epsilon_2 \mid (r,s) \in \lambda^i_A\} \tag{5.34}$$

for each collection of Young diagrams λ^i_A, $1 \leq i \leq p$, $1 \leq A \leq N_i$ with a total number of boxes equal to the instanton number, $\sum_{A=1}^{N_i} |\lambda^i_A| = k_i$. The set of poles (5.34) arises from a prescription called the Jeffrey–Kirwan (JK) residue prescription, which follows from equivariant localization [210]. This prescription was also obtained from localization of the aforementioned ADHM supersymmetric matrix model [241–244].

Exercise 5.4. *Specialize these formulas to* U(2) *SQCD with* $N_f = 4$ *flavours* ($p = 1$, $N_1 = 2$, $M_1 = 4$). *Note that the four mass parameters* m^1_B, $B = 1, \dots, 4$ *can all be shifted by shifting the Coulomb branch parameters* a^1_A, $A = 1, 2$. *This is a shadow of the fact that in the* U(2) *theory we have only* $\mathrm{SU}(N_f)$ *flavour symmetry, not* $\mathrm{SO}(2N_f)$ *like for* SU(2) *SQCD. Compute the one-instanton contribution in* Z_{inst}. *Start computing the two-instanton contribution to get an idea of the complexity: write the integrand and list the poles.*

U(1) **factor.** The brane realization of the theory required the gauging of U(1) symmetries to complete $\mathrm{SU}(N_i)$ into $\mathrm{U}(N_i)$ gauge groups. In this process, U(1) mass parameters

[45] If $b_j = 2N_j - N_{j-1} - N_{j+1} - M_j$ is positive then $z_j = \Lambda_j^{b_j}$ while if $b_j = 0$ then z_j is essentially $e^{2\pi i \tau_j}$. To be precise there is some renormalization scheme ambiguity in what we mean by τ_j, as we discuss momentarily on page 56. The logic is rather to extract from Z_{inst} the prepotential as a function of z_j, deduce how IR couplings relate to z_j, and finally invert the map if we want to express z_j in terms of physically meaningful quantities.

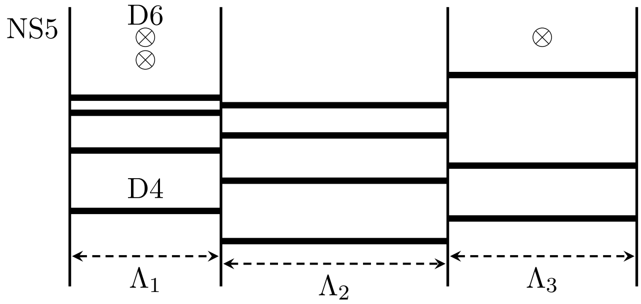


Figure 3: Brane construction of a U(4) × U(4) × U(3) gauge theory. The leftmost U(4) factor has two fundamental hypermultiplets (inserted by transverse D6 branes depicted as ⊗) and the rightmost U(3) factor has one fundamental hypermultiplet. In this example each gauge group is asymptotically free.

become dynamical vector multiplet scalars. Thankfully, Z_{inst} is calculated for constant values of these scalars, so that we can simply set these Coulomb branch parameters to the U(1) mass parameters we started with.

The major problematic effect of the gauging is to introduce spurious instanton contributions from the U(1) gauge factor. After spending a week puzzled about a mismatch with 2d CFT conformal blocks, the authors of [5] proposed to divide out this spurious U(1) instanton contribution, schematically[46]

$$Z_{\text{inst}}\Big[\prod \mathrm{SU}(N_i)\Big] = Z_{\text{inst}}\Big[\prod \mathrm{U}(N_i)\Big] \Big/ (\mathrm{U}(1)\text{ factor}). \tag{5.35}$$

This U(1) factor is more readily singled out on the CFT side. Instanton partition functions for linear quivers of SU(N) gauge groups are expected to match conformal blocks of the W_N algebra (Virasoro for $N = 2$), while U(N) gauge groups correspond to conformal blocks of W_N times the Heisenberg chiral algebra. The difference (or rather ratio) of these two situations is thus given by a conformal block of the Heisenberg chiral algebra, namely a free field correlation function of chiral vertex operators. This point of view was understood in [14] for $N = 2$ and later extended in [245]. For the linear quiver considered here, the factor is schematically

$$(\mathrm{U}(1)\text{ factor}) \simeq \prod_{1\leq i\leq j\leq p} (1 - z_i \cdots z_j)^{c_{ij}} \tag{5.36}$$

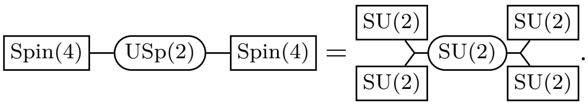
for some exponents c_{ij} that are quadratic in the mass parameters.

Renormalization schemes: an example. Given the issue of extracting the U(1) factor, and additional restrictions on which theories can be treated by brane constructions,

[46]References welcome: it would be good to lay down the procedure nicely on the instanton partition function side, and I likely missed the correct reference.

people have sought other methods to determine Z_{inst}. We present one method [96] that exemplifies a subtlety in comparing different results.

Brane setups with orientifold planes gives access to Z_{inst} for linear quivers with alternating USp and Spin groups. We return to such quivers and their class S construction at the end of subsection 7.2. One of the simplest examples is in fact a different description of SU(2) SQCD with $N_f = 4$ flavors:



(5.37)

More generally, linear quivers of $\text{USp}(2) = \text{SU}(2)$ and $\text{Spin}(4) = \text{SU}(2) \times \text{SU}(2)$ gauge groups are equivalent to generalized SU(2) quivers. In simple enough cases such as (5.37) the instanton partition function can also be computed through the U(2) ADHM construction and removing U(1) factors (5.35). Surprisingly, the two methods yield different series in powers of the exponentiated gauge coupling $q_{\text{Lag}} = e^{2\pi i \tau_{\text{Lag}}}$, which would naively suggest an inconsistency!

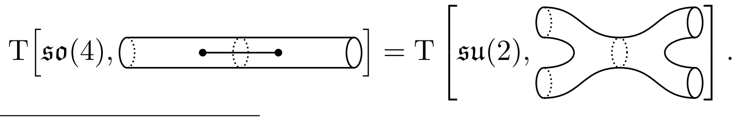
The same issue showed up a long time ago when comparing SW solutions coming from different constructions of 4d $\mathcal{N} = 2$ theories. It was resolved [246] by noting a renormalization scheme ambiguity in the definition of the couplings. We have already encountered this ambiguity in (4.20) when comparing the gauge coupling τ_{Lag} appearing in the Lagrangian of SU(2) SQCD to the position $q = e^{2\pi i \tau}$ arising in the class S construction of this same theory and found a relation $\tau = \tau_{\text{Lag}} + \cdots$ with a constant shift and an infinite tower of instanton corrections.[47]

Let us concentrate on (mass-deformed) SCFTs such as SU(2) $N_f = 4$ SQCD, whose gauge couplings do not run, and let us simplify expressions by assuming there is a single gauge coupling. In two renormalization schemes that agree at leading order, the gauge coupling constants $\tau, \tilde{\tau}$ are related as

$$\tilde{\tau} = \tau + \sum_{n \geq 0} c_n e^{2\pi i n \tau}, \tag{5.38}$$

where the τ-independent coefficients c_n are dimensionless. In concrete cases these coefficients are independent of Coulomb branch and mass parameters; assuming that the schemes agree at leading order throughout the parameter space, a possible argument is that the coefficients c_n should be bounded functions of the Coulomb branch and mass parameters, hence must be constants by the Liouville theorem on analytic functions.

The U(2) and USp(2)–Spin(4) approaches to Z_{inst} are related in precisely this way, as understood in [96]. Consider SU(2) SQCD with $N_f = 4$ massless hypermultiplets. The quiver descriptions (5.37) correspond to seeing SQCD either as a reduction of $\mathcal{X}(\mathfrak{so}(4)) = \mathcal{X}(\mathfrak{su}(2)) \otimes \mathcal{X}(\mathfrak{su}(2))$ with $\mathbb{Z}_2$-automorphism twist defects, or a reduction of $\mathcal{X}(\mathfrak{su}(2))$:



(5.39)

[47] In the AGT context, an early reference pointing out these instanton corrections is [247], see also [248] for an F-theory derivation.

The double-cover of the $\mathfrak{so}(4)$ curve (cylinder with a branch cut) is the $\mathfrak{su}(2)$ curve (four-punctured sphere). The relation between the modular parameter $q_{\mathfrak{so}(4)}$ of the cut cylinder and the IR coupling is fixed through the same considerations as for the modular parameter $q_{\mathfrak{su}(2)}$ of the four-punctured sphere (4.20). This allows to relate them to each other,

$$\begin{aligned} q_{\mathfrak{so}(4)} &= \sqrt{16\lambda(2\tau_{\mathrm{IR}})} = 16e^{\pi i\tau_{\mathrm{IR}}} + O(e^{3\pi i\tau_{\mathrm{IR}}}), \\ q_{\mathfrak{su}(2)} &= \lambda(\tau_{\mathrm{IR}}) = q_{\mathfrak{so}(4)}\Big(1 + \frac{q_{\mathfrak{so}(4)}}{4}\Big)^{-2} = 16e^{\pi i\tau_{\mathrm{IR}}} - 128e^{2\pi i\tau_{\mathrm{IR}}} + O(e^{3\pi i\tau_{\mathrm{IR}}}). \end{aligned} \tag{5.40}$$

When comparing Nekrasov instanton partition functions, the non-trivial map between the counting parameters $q_{\mathfrak{su}(2)}$ and $q_{\mathfrak{so}(4)}$ must be taken into account.

5.4 Cutting by localization

Supersymmetric localization without a Lagrangian. We have presented supersymmetric localization so far for Lagrangian theories, as the technique relies on a path-integral formulation. In the absence of a path-integral formulation we cannot deform the action $S \to S + tQV$ as before, but we can still deform an expectation value $\langle \mathcal{O} \rangle$ to $\langle \mathcal{O} e^{-tQV} \rangle$. In the same way as in (5.12) this expectation value is t-independent provided the supercharge Q is not spontaneously broken in the state of interest, and $Q\mathcal{O} = 0$. In the $t \to +\infty$ limit the observable $\langle \mathcal{O} \rangle$ should reduce in some sense to "zeros of $(QV)_{\text{bosonic}}$", whenever such a notion can be defined.

In the following,[48] using a deformation term for vector multiplets only, we decompose the partition function of a general class S theory into simple building blocks for any pants decomposition of C. On an orthogonal note, using a deformation term restricted to $S_b^3 \subset S_b^4$, we factorize the partition function as the integral of a product of (anti-)holomorphic functions of q as in (5.28), for any theory whose 3d restriction is Lagrangian.

Cutting C by localizing vector multiplets. Higher-rank class S theories are typically non-Lagrangian, obtained by gauging common flavour symmetries of certain isolated SCFTs called tinkertoys (most notably T_N). Gauging symmetries, however, is performed using standard vector multiplets that are described by a standard path integral, with a gauge coupling τ just as in the Lagrangian case. This raises the prospect of applying supersymmetric localization to these vector multiplets.

Let us make more explicit what we mean by gauging a flavour symmetry group G of some theory T (a product of tinkertoys). The G symmetry current multiplet in T is first coupled to a non-dynamical vector multiplet (A, λ, ϕ). The partition function $Z_{\mathrm{T}}[A, \lambda, \phi]$ depends on this background vector multiplet, whose components may be given any values (constant or not). Gauging consists of performing the path integral over the vector multiplet fields, so that the partition function of the gauged theory is

$$Z_{\mathrm{T,gauged}} = \int [\mathrm{D}A\, \mathrm{D}\lambda\, \mathrm{D}\phi]\, e^{-S_{\mathrm{SYM}}[A,\lambda,\phi]} Z_{\mathrm{T}}[A, \lambda, \phi], \tag{5.41}$$

[48]The ideas mostly appear in the literature, but may not have been collected previously in this way.

where the SYM action includes the gauge coupling and theta term, and the path integral measure implicitly accounts for gauge fixing. Similar expressions are available for any other observable of the gauged theory, in terms of observables of the non-gauged theory.

The idea then is to add the deformation term (5.11) for the vector multiplet, without adding any deformation terms for the tinkertoys. Supersymmetric localization restricts the vector multiplet path integral as

$$\begin{aligned} Z_{\mathrm{T,gauged}} &= \lim_{t\to+\infty} \int [\mathrm{D}A\,\mathrm{D}\lambda\,\mathrm{D}\phi]\, e^{-S_{\mathrm{SYM}}-tQV_{\mathrm{vector}}} Z_{\mathrm{T}}[A,\lambda,\phi] \\ &= \int_{QV_{\mathrm{vector}}=0} [\mathrm{D}A\,\mathrm{D}\phi] e^{-S_{\mathrm{SYM}}[A,\phi]} Z^{\mathrm{vector}}_{\text{one-loop}}[A,\phi] Z_{\mathrm{T}}[A,\phi]. \end{aligned} \tag{5.42}$$

The locus $QV_{\mathrm{vector}} = 0$ cannot in principle have the fermion λ turned on, which is why λ disappears in the last line. The locus is described by (5.18) away from the poles and (5.19) at poles. Namely, zeros of the vector multiplet deformation term are characterized by a constant imaginary value $\phi = a \in \mathfrak{g}$ and by (anti-)instantons at the poles.

In our applications T splits into decoupled sectors $\mathrm{T} = \prod_L \mathrm{T}_L$. Decomposing $G = \prod_I G_I$ into simple factors and denoting by $\mathfrak{h}_I$ their Cartan algebra, we find

$$\begin{aligned} Z_{\mathrm{T,gauged}} = \prod_I \Bigg[\sum_{k_I,\overline{k}_I\geq 0} \int_{\mathfrak{h}_I\times\mathcal{M}_{G_I,k_I}\times\mathcal{M}_{G_I,\overline{k}_I}} \mathrm{d}a_I \mathrm{d}\xi_I \mathrm{d}\overline{\xi}_I \Bigg] \Bigg\{ \\ \prod_I \Big[|q_I|^{r^2|a_I|^2} q_I^{k_I} \overline{q}_I^{\overline{k}_I} Z^{\mathrm{vector}}_{\text{one-loop}}[G_I; a_I, k_I, \xi_I, \overline{k}_I, \overline{\xi}_I] \Big] \prod_L Z_{\mathrm{T}_L}[a,k,\xi,\overline{k},\overline{\xi}] \Bigg\}. \end{aligned} \tag{5.43}$$

For class S theories, this matches precisely how a correlator on C would be calculated by inserting, on each tube of a pants decomposition of C, a complete set of states labeled by $a_I, k_I, \xi_I, \overline{k}_I, \overline{\xi}_I$. Indeed, we recognize the sum over states $\sum_{k,\overline{k}} \int_{a,\xi,\overline{\xi}}$ for each tube I, the inverse norm of that state, and the remaining three-point functions Z_{T_L} for each three-punctured sphere L. We know that Z_{T_L} is only sensitive to $a_I, k_I, \xi_I, \overline{k}_I, \overline{\xi}_I$ for groups under which T_L is charged; in class S these correspond to the tubes that connect to the given tinkertoy.

Holomorphic factorization by localizing on S^3_b. We now turn to a method to cut and glue partition functions using supersymmetric localization. This was most deeply explored in [249, 250] where many previously observed factorisation properties were derived (see also [91] for an earlier application of the idea). Our aim here is to explain the holomorphic dependence in q by cutting the sphere in halves along the equator $S^3_b \subset S^4_b$. This is nicely complementary to how we have cut the Riemann surface C above.

As a warm-up we revise the Lagrangian case. The keys for factorization in that case are that

- the q and $\overline{q}$ dependence arises from instantons at the poles;
- these instantons do not interact through Z_{T_L}.

To be precise, T consists of hypermultiplets and supersymmetric localization with the hypermultiplet deformation term sets all fields of T to zero, and quadratic fluctuations yield the product of a function of a, k, ξ and a function of $a, \overline{k}, \overline{\xi}$, as for the vector multiplet (5.27). Thus, the q, k, ξ and $\overline{q}, \overline{k}, \overline{\xi}$ dependence completely decouple from each other in (5.43). Collecting together the sum-integral over k, ξ into an instanton partition function, we retrieve the expected factorization as an integral over a of a holomorphic times an antiholomorphic function of q, as stated in (5.28).

We now rephrase this in terms of localization. The S^3_b restriction of the 4d fields are 3d $\mathcal{N} = 2$ vector and hypermultiplets, and the full 4d theory is alternatively described as two (identical) 4d theories on the hemispheres of S^4_b, coupled together through their boundary condition: for instance the partition function reads

$$Z_{S^4_b} = \int [\mathrm{D}\Phi_{3\mathrm{d}}] e^{-S_{3\mathrm{d}}[\Phi_{3\mathrm{d}}]} Z_{HS^4_b,\mathrm{North}}[\Phi_{3\mathrm{d}}, q, \overline{q}] Z_{HS^4_b,\mathrm{South}}[\Phi_{3\mathrm{d}}, q, \overline{q}], \tag{5.44}$$

where $\Phi_{3\mathrm{d}}$ denotes collectively all fields of the 3d $\mathcal{N} = 2$ theory on the equator and $Z_{HS\ldots}$ denote the path integrals over fields on each of the two hemispheres. This expression can be seen as the expectation value of (very complicated) observables $Z_{HS\ldots}$ of the 3d theory.

The equator theory can then be localized by adding to it the usual deformation term that is used for localization on S^3_b. The 3d hypermultiplets get localized to zero as in 4d (5.15). The 3d vector multiplets get localized to constant values $\phi = a \in \mathfrak{g}$. As for any other observable, the $Z_{HS\ldots}$ factors are evaluated on the localization locus so that

$$Z_{S^4_b} = \int \mathrm{d}a\, Z_{\mathrm{cl},3\mathrm{d}}[a] Z_{\text{one-loop},3\mathrm{d}}[a] Z_{HS^4_b,\mathrm{North}}[a, q] Z_{HS^4_b,\mathrm{South}}[a, \overline{q}]. \tag{5.45}$$

The hemisphere partition functions are evaluated here with Dirichlet boundary conditions with a constant boundary value a for the vector multiplet scalars. The absence of gauge-coupling dependence in 3d is standard: YM terms in 3d are Q-exact. More generally, YM and theta terms are Q-exact everywhere away from the North/South poles where their sum/difference is not Q-exact in a smooth way. This explains why the hemisphere contributions only depend on q or $\overline{q}$.

This idea of cutting the theory along the equator generalizes beyond Lagrangian theories. Even though Z_{T_L} may not have a path integral description, general axioms of QFT still apply and one can insert a complete set of states along the hypersurface $S^3_b \subset S^4_b$. The partition function is then seen as the overlap of two states prepared by the path integral over hemispheres,

$$Z_{S^4_b} = \sum_{|\Phi\rangle \in \mathcal{H}[S^3_b]} \langle HS^4_b \text{ North} \mid \Phi \rangle \langle \Phi \mid HS^4_b \text{ South} \rangle. \tag{5.46}$$

Then, Q-invariance of the set-up implies that the sum restricts to Q-invariant states,

$$Z_{S^4_b} = \sum_{|a\rangle \in \mathcal{H}[S^3_b], Q|a\rangle = 0} \langle HS^4_b \text{ North} \mid a \rangle \langle a \mid HS^4_b \text{ South} \rangle, \tag{5.47}$$

and Q-exactness of the YM ± theta terms on each hemisphere shows that the two factors are (anti-)holomorphic functions of q as wanted.

We observe that the Coulomb branch integral over $a \in \mathfrak{h}$ is in general replaced by a sum/integral over Q-invariant states in the S^3_b Hilbert space of the 4d theory. This is typically a larger space, especially in the presence of non-Lagrangian tinkertoys.

6 AGT for SU(2) quivers

The AGT relation (5.1) relates observables of two different dimensional reductions of the 6d (2,0) theory $\mathcal{X}(\mathfrak{su}(2))$. We have explained in section 4 how reducing $\mathcal{X}(\mathfrak{su}(2))$ on a Riemann surface $C = \overline{C} \setminus \{z_1, \ldots, z_n\}$ yields a 4d $\mathcal{N} = 2$ $\mathfrak{su}(2)$ Sicilian[49] quiver gauge theory that depends on C, with gauge couplings related to the complex structure of C. Likewise, we expect that reducing $\mathcal{X}(\mathfrak{su}(2))$ on S^4_b should yield a 2d theory with a coupling constant b, and the codimension 2 defects of $\mathcal{X}(\mathfrak{su}(2))$ inserted at punctures z_i of C should become local operators in 2d. Since this 2d dimensional reduction is somewhat technical, we postpone it to subsection 6.3, explaining there briefly why one should expect 6d observables to be computable both on the 4d and 2d sides.

Before that we determine the relevant 2d theory in a more historically accurate way in subsection 6.2: its correlators should reproduce the S^4_b partition functions of class S theories. These partition functions only depend on the complex structure of C, hence only on the conformal class of the metric on C. This means that the 2d theory we seek should be a CFT, and it turns out to be Liouville CFT. As a result, we begin by reviewing Liouville CFT and 2d CFT basics in subsection 6.1. We summarize aspects of the correspondence in Table 3.

Table 3: Basics of AGT. In Liouville CFT conventions, change Q to $Q/2$.

	Gauge theory	Toda/Liouville CFT
	Partition function on S^4_b	Toda correlator on C
Parameters	Squashing parameter $b = \sqrt{\epsilon_1/\epsilon_2}$	Toda coupling constant b
	Gauge couplings in 4d	Complex structure of C
	Dual gauge theory descriptions	Pants decompositions of C
	Full tame puncture of mass m	Vertex operator $\widehat{V}_{Q+rm}$
Building blocks	Coulomb branch parameter a	Internal momentum $\alpha = Q + ra$
	Gauge one-loop determinant	Inverse two-point function
	Matter one-loop determinant	Three-point structure constant
	Classical action contribution	Conformal block leading term
	Instanton partition function	Conformal block power series

[49]Thus named because of a resemblance between trinions and the triskelion featuring prominently on the flag of Sicily.

6.1 2d CFT and Liouville CFT

We refer to reviews such as [128, 129] (and references therein) for an introduction to 2d CFT and in particular to Liouville CFT. The Young Researchers Integrability School and Workshop (YRISW) course [251] also provides a brief introduction to 2d CFT and their symmetry algebras (chiral algebras), which curiously also show up independently as a protected subsector of 4d $\mathcal{N} = 2$ theories.

Virasoro algebra. In contrast to other dimensions, the conformal symmetry algebra in 2d is the product $\mathrm{Vir} \times \overline{\mathrm{Vir}}$ of two infinite-dimensional algebras. The Virasoro algebra Vir is spanned by L_n, $n \in \mathbb{Z}$ and a central element C, subject to

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}\delta_{m+n=0}(m^3 - m)C. \tag{6.1}$$

This algebra, and the other copy spanned by $\overline{L}_n$, $n \in \mathbb{Z}$ and $\overline{C}$, acts on the Hilbert space of the theory on a circle. For any given 2d CFT the elements $C, \overline{C}$ of the conformal algebra act as multiplications by constants called central charges and denoted $c, \overline{c}$ or c_L, c_R.

In a CFT, radial quantization identifies such states to their infinite-radial-past limit, which is a local operator $\mathcal{O}$ at the center of radial quantization. The state corresponding to $\mathcal{O}$ under this state-operator correspondence is denoted by $|\mathcal{O}\rangle$. Under this correspondence, the action of $\mathrm{Vir} \times \overline{\mathrm{Vir}}$ on states translates to an action on local operators by commutator.

Exercise 6.1. *Using* (6.1), *check* $[L_m, [L_n, L_p]] + [L_n, [L_p, L_m]] + [L_p, [L_m, L_n]] = 0$. *This is the Jacobi identity, essential for consistency of the Lie algebra* Vir. *Check that the factor* $m^3 - m$ *is fixed by the Jacobi identity up to mixing* L_n *with* C *and scaling* C.

Check that $\tilde{C} = kC$ *and* $\tilde{L}_n = L_{kn}/k + \delta_{n=0}(k - 1/k)C/24$ *obey the Virasoro commutation relations for any* $k \in \mathbb{Z} \setminus \{0\}$, *so that the Virasoro algebra contains infinitely many Virasoro subalgebras. Conversely, the Virasoro algebra can be embedded as the integer modes of a larger Virasoro algebra with fractional modes, useful in symmetric product orbifolds [252].*

Conformal dimensions. Dilations and rotations around the center of radial quantizations are generated by $L_0 \pm \overline{L}_0$, hence the dimension $\Delta = h_{\mathcal{O}} + \overline{h}_{\mathcal{O}}$ and spin $h_{\mathcal{O}} - \overline{h}_{\mathcal{O}} \in \frac{1}{2}\mathbb{Z}$ of a local operator are given by the action of L_0 and $\overline{L}_0$:

$$[L_0, \mathcal{O}] = h_{\mathcal{O}}\mathcal{O}, \qquad [\overline{L}_0, \mathcal{O}] = \overline{h}_{\mathcal{O}}\mathcal{O}. \tag{6.2}$$

Despite the notation, the conformal dimensions $h_{\mathcal{O}}$ and $\overline{h}_{\mathcal{O}}$ are independent numbers. In a unitary CFT they are both real and non-negative. We find it more convenient to mostly work with states, in which case $\Delta_{\mathcal{O}}$ and $h_{\mathcal{O}} - \overline{h}_{\mathcal{O}}$ are the energy and momentum of the state.

Interestingly, the commutator $[L_0, L_n] = -nL_n$ implies that if $|\mathcal{O}\rangle$ has conformal dimensions $(h_{\mathcal{O}}, \overline{h}_{\mathcal{O}})$ then $L_n|\mathcal{O}\rangle$ has conformal dimensions $(h_{\mathcal{O}} - n, \overline{h}_{\mathcal{O}})$. For this reason, L_n, $n \geq 1$ are called lowering operators, and L_n, $n \leq -1$ are called raising operators. The same applies to $\overline{L}_n$.

Primary operators. The Hilbert space of a given theory organizes into conformal families, namely representations of $\mathrm{Vir}\times\overline{\mathrm{Vir}}$. In a unitary CFT, states have a non-negative energy so each conformal family has a state $|V\rangle$ of minimal dimension. Such a state is annihilated by all lowering operators L_n, $\overline{L}_n$, $n\geq 1$ and is called a *primary state* (the corresponding operator V is called a *primary operator*):

$$L_n|V\rangle = \overline{L}_n|V\rangle = 0, \quad n\geq 1. \tag{6.3}$$

From this state of conformal dimensions $(h,\overline{h})$ one can construct a tower of states of higher dimensions by acting with L_{-n} and $\overline{L}_{-n}$, $n\geq 1$. Using the Virasoro commutator (6.1) these raising operators can be ordered, and the set of descendants is spanned by the following states

$$\mathbb{L}_{-Y}\overline{\mathbb{L}}_{-\overline{Y}}|V\rangle \coloneqq L_{-m_1}\dots L_{-m_k}\overline{L}_{-n_1}\dots\overline{L}_{-n_l}|V\rangle, \tag{6.4}$$

where $Y,\overline{Y}$ are two Young diagrams, $m_1\geq m_2\geq\cdots\geq m_k\geq 1$ are the successive lengths of rows of Y, and likewise $n_1\geq\cdots\geq 1$ the rows of $\overline{Y}$. The conformal dimensions of (6.4) are $\bigl(h+m_1+\cdots+m_k,\overline{h}+n_1+\cdots+n_l\bigr)$. These states are called *descendants* of $|V\rangle$. In fact, the whole representation of $\mathrm{Vir}\times\overline{\mathrm{Vir}}$ is spanned (as a vector space) by (6.4), and generically these states are linearly independent.

Exercise 6.2. *1. For $|V\rangle$ a primary state, and for $m,n\in\mathbb{Z}$, rewrite $L_mL_n|V\rangle$ as a linear combination of terms* (6.4) *with properly sorted indices.*

2. Check that for any $Y,\overline{Y}$, acting with any L_n or $\overline{L}_n$ on $\mathbb{L}_{-Y}\overline{\mathbb{L}}_{-\overline{Y}}|V\rangle$ yields a linear combination of such descendants.

Two and three-point functions. Conformal symmetry constrains correlators of local operators. Denoting by $z_{ij}=z_i-z_j$, the two- and three-point functions of primary operators V_i of conformal dimensions $(h_i,\overline{h}_i)$ take the form

$$\begin{aligned}\langle V_1(z_1,\overline{z}_1)\,V_2(z_2,\overline{z}_2)\rangle &= g_{12}\,\delta_{h_1=h_2}\delta_{\overline{h}_1=\overline{h}_2}\,z_{12}^{-2h_1}\overline{z}_{12}^{-2\overline{h}_1},\\ \langle V_1(z_1,\overline{z}_1)\,V_2(z_2,\overline{z}_2)\,V_3(z_3,\overline{z}_3)\rangle &= C_{123}\,z_{23}^{h_1-h_2-h_3}\,z_{31}^{h_2-h_3-h_1}\,z_{12}^{h_3-h_1-h_2}\\ &\qquad\times\overline{z}_{23}^{\overline{h}_1-\overline{h}_2-\overline{h}_3}\,\overline{z}_{31}^{\overline{h}_2-\overline{h}_3-\overline{h}_1}\,\overline{z}_{12}^{\overline{h}_3-\overline{h}_1-\overline{h}_2},\end{aligned} \tag{6.5}$$

where g_{12},C_{123} are constants depending on the primary operators involved. Descendant operators can be written as Virasoro generators $L_n,\overline{L}_n$ acting on primary operators, and these generators act on (6.5) as various differential operators.

If the CFT has a single primary operator of each conformal dimension $(h,\overline{h})$, which is the case for Liouville CFT, then g_{12} can be absorbed in a normalization of that operator. Despite this possibility, the standard normalization of primary operators V_α has non-trivial g_{12}, and our AGT-friendly normalization $\widehat{V}_\alpha$ given below also does. Once this normalization is chosen, the theory is characterized by its spectrum of primary operators (their conformal dimensions) and by three-point functions C_{123} as explained next.

Correlators and conformal blocks. The $n \geq 4$ point functions of primary operators are then fixed by conformal invariance [253]. For concreteness, consider a 4-point function, and insert a complete set of states, which we separate according to representations of $\mathrm{Vir} \times \overline{\mathrm{Vir}}$:

$$\langle V_1 V_2 V_3 V_4 \rangle = \sum_{\substack{V,V' \text{ primaries} \\ Y,\overline{Y},Y',\overline{Y}' \text{ Young diagrams}}} \langle V_1 V_2 \mathbb{L}_{-Y} \overline{\mathbb{L}}_{-\overline{Y}} | V \rangle \, g^{-1}(Y, \overline{Y}, V; Y', \overline{Y}', V') \, \langle V' | \mathbb{L}_{Y'} \overline{\mathbb{L}}_{\overline{Y}'} V_3 V_4 \rangle, \tag{6.6}$$

where $g^{-1}(Y, \overline{Y}, V; Y', \overline{Y}', V')$ denotes components of the (matrix) inverse of the "matrix" with components $\langle V' | \mathbb{L}_{Y'} \overline{\mathbb{L}}_{\overline{Y}'} \mathbb{L}_{-Y} \overline{\mathbb{L}}_{-\overline{Y}} | V \rangle$. Translating all Virasoro generators to differential operators acting on the position dependence in (6.5) gives

$$\langle V_1 V_2 V_3 V_4 \rangle = \sum_{V_5, V_6 \text{ primaries}} C_{125} g_{56}^{-1} C_{634} \mathcal{F}(h_1, \ldots, h_5; z_1, \ldots, z_4) \mathcal{F}(\overline{h}_1, \ldots, \overline{h}_5; \overline{z}_1, \ldots, \overline{z}_4). \tag{6.7}$$

Up to unimportant factors, the (locally) holomorphic factor $\mathcal{F}$ is called a *conformal block*. It is entirely determined by dimensions $h_1, \ldots, h_4$ of the *external operators*, and the dimension $h_5 = h_6$ of the *internal operator* inserted as part of the complete set of states.

We could have inserted a complete set of states with a different choice of which pair of operators V_i lies on the two sides of the inserted states. More generally, the possible ways to insert complete set of states to reduce a sphere n-point function (down to the constants g, C, and conformal blocks) correspond to the ways of decomposing the n-punctured sphere into three-punctured spheres: a complete set of states is inserted along each closed loop cutting the sphere into pieces. In all cases, conformal blocks are purely representation-theoretic objects; they depend on dimensions of the n external operators and of $n-3$ internal operators inserted along cuts.

Liouville theory. Liouville theory describes a single scalar field subject to the action

$$S[\phi] = \frac{1}{4\pi} \int \mathrm{d}^2 z \, \sqrt{g} \, (\partial_\nu \phi \partial^\nu \phi + Q R \phi + 4\pi\mu \, e^{2b\phi}) \tag{6.8}$$

where R is the Ricci scalar. Provided $Q = b + 1/b$ this theory is conformal, with holomorphic stress-tensor $T = (\partial\phi)^2 + Q\partial^2\phi$ and central charges $c = \overline{c} = 1 + 6Q^2 \geq 25$.

While it looks like the cosmological constant μ is a coupling constant, it turns out to only appears in trivial ways in correlators: instead there is interesting dependence on $b > 0$, with $b \to 0$ being the semiclassical limit. The Liouville CFT admits a (non-manifest) duality $b \to 1/b$ while keeping $\lambda = \big(\frac{\pi\Gamma(b^2)}{\Gamma(1-b^2)}\mu\big)^{1/b}$ fixed.

One can check that $V_\alpha = {:}e^{2\alpha\varphi}{:}$ are conformal primary operators of left/right-moving dimension $h(\alpha) = \alpha(Q-\alpha) = Q^2/4 + P^2$, for $\alpha = (b+1/b)/2 + iP$, $P \in \mathbb{R}$. The invariance $h(Q-\alpha) = h(\alpha)$ suggests the identification $V_\alpha = R(\alpha) V_{Q-\alpha}$. The *reflection coefficient* can be determined (using conformal bootstrap) to be

$$R(\alpha) = -\lambda^{Q-2\alpha} \frac{\Gamma(b(2\alpha - Q))\Gamma(\frac{1}{b}(2\alpha - Q))}{\Gamma(b(Q - 2\alpha))\Gamma(\frac{1}{b}(Q - 2\alpha))}. \tag{6.9}$$

The two-point function is then

$$\langle V_{\alpha_1} V_{\alpha_2} \rangle = \delta_{\alpha_1+\alpha_2=Q} + R(\alpha_1)\delta_{\alpha_1=\alpha_2}. \tag{6.10}$$

The three-point function is known to be given by the Dorn–Otto–Zamolodchikov–Zamolodchikov (DOZZ) formula [254, 255]

$$\begin{aligned} C_{\alpha_1\alpha_2\alpha_3} &= \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \rangle \\ &= \frac{(b^{2/b-2b}\lambda)^{Q-\alpha_1-\alpha_2-\alpha_3}\Upsilon_b'(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}. \end{aligned} \tag{6.11}$$

Four-point functions for instance read

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(q)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle = \frac{1}{2}\int_{Q/2+i\mathbb{R}} \mathrm{d}\alpha_{\mathrm{s}}\, C_{\alpha_1\alpha_2\alpha_{\mathrm{s}}} C_{(Q-\alpha_{\mathrm{s}})\alpha_3\alpha_4} \Big| q^{h_{\mathrm{s}}-h_1-h_2}(1+O(q))\Big|^2 \tag{6.12}$$

where the factor of $1/2$ cancels the double-counting from the identification $\alpha_{\mathrm{s}} \sim Q - \alpha_{\mathrm{s}}$, and $1 + O(q)$ denotes an infinite series in positive integer power of q, the normalized conformal block.

Exercise 6.3. *1. Check that the* DOZZ *formula* (6.11) *respects the expected $b \to 1/b$ duality, and the symmetries $\alpha_i \to Q - \alpha_i$ for any of the i, up to the appropriate reflection coefficient.*

2. Using properties of Υ_b listed in Appendix A, show that at fixed generic α_1, α_2, the $\alpha_3 \to 0$ limit of $C_{\alpha_1\alpha_2\alpha_3}$ vanishes. Show that for $\alpha_1 = \alpha_2$ the limit is infinite, while $(\alpha_3/2)C_{\alpha\alpha\alpha_3} \to g_{\alpha\alpha} = R(\alpha)$.

6.2 Finding the AGT dictionary

We expect a relation of the form

$$Z_{S_b^4}(\mathrm{T}(\mathfrak{su}(2), C, m)) = \Big\langle \widehat{V}_{\alpha_1}(z_1) \dots \widehat{V}_{\alpha_n}(z_n) \Big\rangle_{\overline{C}} \tag{6.13}$$

for any number n of puncture, where $\widehat{V}_{\alpha_i}(z_i)$ are the reductions of codimension 2 operators of the 6d theory down to points. In this section we use known S_b^4 partition function to determine that the relevant 2d CFT is Liouville CFT described above, and that $\widehat{V}_\alpha$ are suitable rescalings of vertex operators V_α.

Three-point functions and normalization. A 2d CFT is characterized by its spectrum (left and right conformal dimensions of primary operators) and OPE structure constants (equivalently, three-point functions of conformal primary operators). When constructing class S theories from $\mathcal{X}(\mathfrak{su}(2))$, the data associated to a puncture is a mass parameter $m \in i\mathbb{R}/\mathbb{Z}_2$. We thus want local operators V with a continuous parameter. For consistency with earlier notation we denote this (dimensionless) parameter as $\alpha = Q/2 + rm$, where $Q = b + 1/b$.

Determining the conformal dimension of $\widehat{V}_\alpha$ will have to wait; let us begin with three-point functions. We know that the theory associated to a three-punctured sphere is a trifundamental half-hypermultiplet. Its partition function is a hypermultiplet one-loop determinant (5.25), so that the three-point function is

$$\begin{aligned}\widehat{C}_{\alpha_1\alpha_2\alpha_3} \coloneqq \langle \widehat{V}_{\alpha_1}\widehat{V}_{\alpha_2}\widehat{V}_{\alpha_3}\rangle = \prod_{\pm\pm} \frac{1}{\Upsilon_b(\alpha_1 \pm (\alpha_2 - Q/2) \pm (\alpha_3 - Q/2))} \\ = \frac{1}{\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)},\end{aligned} \tag{6.14}$$

in which we used the invariance $\Upsilon_b(x) = \Upsilon_b(Q-x)$. This matches precisely the denominator of the DOZZ formula (6.11), and the numerator can be absorbed (except for an α-independent factor) by the normalization

$$\widehat{V}_\alpha = \frac{(b^{2/b-2b}\lambda)^{\alpha-Q/2}}{\Upsilon_b(2\alpha)} V_\alpha. \tag{6.15}$$

With this normalization one can check that $\widehat{V}_\alpha = \widehat{V}_{Q-\alpha}$ and that the two-point function reads

$$\widehat{g}_{\alpha\alpha'} = \langle \widehat{V}_\alpha \widehat{V}_{\alpha'}\rangle = \frac{\delta_{\alpha+\alpha'=Q} + \delta_{\alpha=\alpha'}}{\Upsilon_b(Q-2\alpha)\Upsilon_b(2\alpha-Q)}. \tag{6.16}$$

Four-point functions and dimensions. To determine the conformal dimension of $\widehat{V}_\alpha$ we consider a four-punctured sphere and cut it in a channel suitable for the $q \to 0$ limit, where q is the cross-ratio of the four punctures. The gauge theory corresponding to a four-punctured sphere is $\mathfrak{su}(2)$ $N_f = 4$ SQCD, and its partition function, computed using supersymmetric localization, takes the form (5.28)

$$Z_{S^4_b} = \int_{i\mathbb{R}/\mathbb{Z}_2} \mathrm{d}a\, |q|^{r^2|a|^2} Z_{\text{one-loop}}(a) Z_{\text{inst}}(a,q) \overline{Z_{\text{inst}}}(a,\overline{q}). \tag{6.17}$$

In the $q \to 0$ limit, $Z_{\text{inst}} \to 1$. This expression should be compared to the decomposition of a four-point function in 2d CFT,

$$\begin{aligned}&\langle \widehat{V}_{\alpha_1}(0)\widehat{V}_{\alpha_2}(q)\widehat{V}_{\alpha_3}(1)\widehat{V}_{\alpha_4}(\infty)\rangle \\ &= \int_{Q/2+i\mathbb{R}/\mathbb{Z}_2} \mathrm{d}\alpha\, q^{h(\alpha)}\overline{q}^{\overline{h}(\alpha)} \frac{\widehat{C}_{\alpha_1\alpha_2\alpha}\widehat{C}_{\alpha\alpha_3\alpha_4}}{\widehat{g}_{\alpha\alpha}} \mathcal{F}(\alpha_i,\alpha;q)\mathcal{F}(\alpha_i,\alpha;\overline{q})\end{aligned} \tag{6.18}$$

in which $\mathcal{F}$ are conformal blocks that depend (anti-)holomorphically on the cross-ratio q, and tend to 1 as $q \to 0$.

We have already identified the three-point functions C to hypermultiplet one-loop determinants. In turn, the inverse two-point function $\widehat{g}_{\alpha\alpha}^{-1}$ is equal to the vector multiplet one-loop determinant. It is thus natural to expect the conformal blocks to match instanton partition functions, and to identify the powers of q, namely $h(\alpha) = \overline{h}(\alpha) = Q^2/4 + P^2$ and $r^2|a|^2$, up to a harmless shift by $Q^2/4$.

Conformal blocks and proofs of AGT. The key remaining piece to check the AGT dictionary is to verify that conformal blocks do indeed match instanton partition functions,[50] as tested at low orders (in powers of q) in [5, 93, 257–263]. There have been many approaches to this (see for instance [124, section 5.3] for a short review).

One set of approaches relies on exhibiting an action of the Virasoro algebra (and many generalizations) on the instanton moduli space. See [264, 265] for an early example, and generalizations in [14, 29, 82, 266–284]. In particular, one can construct [14, 29, 245, 285, 286] (see also [287–289]) an orthonormal basis of conformal descendants of $|\widehat{V}_\alpha\rangle$ such that inserting these states in a four-point function as in (6.6) yields term by term the expression of Nekrasov instanton partition functions as sums over U(1)-invariant point-like instanton configurations. The Virasoro algebra and W-algebras also appear in a 6d context in [290–293]. See also our discussion of more elaborate symmetry algebras on page 101 in section 10.

Recursion relations are studied in [227, 271, 294–296]. One difficulty is for instance the presence of spurious poles in terms of the instanton expansion, which disappear when summing all contributions [297]. The large c limit is investigated in [298, 299]. A free-field approach based on Dotsenko–Fateev representations of CFT correlators is given in [289, 300–306]. A string-theory derivation of the AGT dictionary (from a 5d generalization) is given in [67, 307] and reviewed in [308]. A rather different approach is based on characterizing both conformal blocks and instanton partition functions as solutions to Riemann–Hilbert problems [309].

6.3 Liouville from 6d

We have argued that the relevant 2d theory for the AGT correspondence is Liouville CFT, and numerous checks of the AGT correspondence validate this. Could we see it directly from 6d?

The approach. Deformations of the metric of C that preserve its conformal class (or equivalently complex structure) are Q-exact with respect to the supercharge Q that we used for supersymmetric localization [18]. Thus, such deformations do not affect the partition function of the 6d theory, which can be computed in the limits where C is infinitely smaller or larger than S_b^4. Importantly, this argument holds also in the presence of any Q-closed observables such as the loops, surfaces, or walls that we consider in section 8. Remembering that the 6d theory is conformally invariant, these limits are equivalent to dimensionally reducing on either one of the factors. We should thus expect to obtain Liouville CFT (or its higher-rank generalization, Toda CFT discussed further in subsection 7.1) by dimensionally reducing the 6d theory $\mathcal{X}(\mathfrak{su}(2))$ (or $\mathcal{X}(\mathfrak{su}(N))$) along S_b^4.

In [18, 310], Córdova and Jafferis have performed this reduction in three steps:

[50] Interestingly, the case of $\mathcal{N}=4$ SYM caused some early confusion in the literature, clarified in [256]: this theory has $Z_{\text{inst}}=1$, and it corresponds to a torus with a non-trivial vertex operator insertion $\alpha=Q/2$.

$\mathcal{X}(\mathfrak{g})$ reduced on S^1 yields 5d $\mathcal{N}=2$ SYM; $\mathcal{X}(\mathfrak{g})$ reduced on S^3_b (or quotients thereof[51]) yields 3d complex Chern–Simons theory; $\mathcal{X}(\mathfrak{g})$ reduced on S^4_b yields 2d complex Toda CFT. They conjectured that this complexified version of Toda CFT is dual to ordinary Toda CFT. The derivation was extended in [313] to include orbifold surface operators (see also [314] for another approach).

Reduction to complex Chern–Simons theory. The reduction to 3d is relevant for the 3d/3d analogue of the AGT correspondence that we will describe in subsection 9.3. We place the 6d theory on $S^3_b \times C_3$, where the squashed sphere is described for instance by its isometric embedding into $\mathbb{R}^4$ as $S^3_b = \{b^2(y_1^2+y_2^2) + b^{-2}(y_3^2+y_4^2) = r^2\} \subset \mathbb{R}^4$. Preserving supersymmetry requires a partial topological twist, which amounts to including suitable background values for supergravity fields, determined in [315]. The approach in [310] was to work with a different squashing of the sphere S^3 that preserves $\mathrm{U}(1)\times\mathrm{SU}(2)$ isometries instead of $\mathrm{U}(1)\times\mathrm{U}(1)$. We will gloss over this, as the backgrounds differ by suitably Q-exact terms that do not affect partition functions eventually.

The Hopf fibration of S^3, namely an S^1 fibration over S^2, is compatible with the squashing. Thus, $\mathcal{X}(\mathfrak{g})$ can be reduced first on the S^1 fibers, obtaining 5d $\mathcal{N}=2$ SYM theory. Thankfully, the non-abelian 5d theory has a Lagrangian description, hence can be further dimensionally reduced explicitly, in contrast to $\mathcal{X}(\mathfrak{g})$, which has no known Lagrangian description.

A further reduction on the S^2 base of the Hopf fibration gives the following light fields, all valued in the adjoint representation of $\mathfrak{g}$.

- One 3d gauge field A arising from components of the 5d gauge field along C_3. It has a 3d Chern–Simons term at level $k=1$, which arises because the 5d graviphoton has one unit of flux through S^2 in the supergravity background. This in turn stems from the Hopf fibration; when reducing on $S^2\times S^1$ instead, $k=0$.

- Zero modes of the five vector multiplet scalars of 5d $\mathcal{N}=2$ SYM. Because the twist identifies an $\mathfrak{so}(3)\subset\mathfrak{so}(5)$ subgroup of R-symmetry with 3d rotations, these zero modes combine into a one-form X and a pair of scalars Y_i.

- Four fermions λ with a two-derivative Lagrangian $-\lambda(\nabla^A)^2\lambda + [X,\lambda]^2$. In terms of $\Delta = (\nabla^A)^2 + (\mathrm{ad}_X)^2$, the quadratic path integral over λ yields a factor of $(\det\Delta)^2$, while the pair of scalars Y_i yields $1/\det\Delta$ since their Lagrangian is $-Y\Delta Y$.

Altogether, Y and λ give a factor of $\det\Delta$, which matches the Faddeev–Popov determinant for gauge fixing $(\nabla^A)_\mu X^\mu = 0$ the "imaginary" gauge transformation $(A_\mu, X_\mu)\mapsto (A_\mu - [X_\mu, g], X_\mu + \nabla^A_\mu g)$ for a local gauge parameter g.

The final 3d theory has a pair of one-forms, hence a complex one-form $\mathcal{A} = A + iX \in \mathfrak{g}_{\mathbb{C}}$ with action

$$S = \frac{q}{8\pi}\int_{C_3}\mathrm{Tr}\Big(\mathcal{A}\wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big) + \frac{\tilde{q}}{8\pi}\int_{C_3}\mathrm{Tr}\Big(\overline{\mathcal{A}}\wedge \mathrm{d}\overline{\mathcal{A}} + \frac{2}{3}\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}\wedge\overline{\mathcal{A}}\Big) \quad (6.19)$$

[51]The reduction of $\mathcal{X}(\mathfrak{g})$ on $S^1\times S^2$ also yields complex Chern–Simons, at a different level [311, 312].

subject to $G_{\mathbb{C}}$ gauge invariance stemming from the standard and the imaginary gauge invariances. The action for X arises from numerous supergravity fields necessary to ensure supersymmetry. The coupling constants $q = k + is$ and $\tilde{q} = k - is$ encode the geometry as

$$k = 1, \qquad s = \frac{1 - b^2}{1 + b^2}. \tag{6.20}$$

More generally the theory makes sense for $k \in \mathbb{Z}$ and $s \in \mathbb{R} \cup i\mathbb{R}$. The $\mathrm{SU}(2) \times \mathrm{U}(1)$ preserving squashing of S^3 used in [310] has a parameter $\ell \in (0, +\infty)$ and $s = \sqrt{1 - \ell^2}$ can also take imaginary values. Other values of k arise from changing S_b^3 to $S^2 \times S^1$ for $k = 0$, or to the (squashed) Lens space $L(k,1)_b = S_b^3/\mathbb{Z}_k$. The 3d/3d correspondence is discussed in subsection 9.3.

Reduction to complex Toda theory. The idea in [18] is to treat S_b^4 as a squashed three-sphere S_b^3 fibered over an interval. In the notation of (5.8) the interval is parametrized by $y_5 \in [-r, r]$ and the S_b^3 has squared radius $r^2 - y_5^2$, namely it degenerates to a point at both ends. The product metric g on $S_b^4 \times C$ is mapped by a Weyl transformation to $S_b^3 \times C_3$, where C_3 is a warped product of C with an interval:

$$g = dy_5^2 + (r^2 - y_5^2)\, g_{S_b^3} + g_C, \qquad \frac{1}{r^2 - y_5^2}\, g = g_{S_b^3} + \frac{dy_5^2 + g_C}{r^2 - y_5^2}. \tag{6.21}$$

The resulting metric is singular at $y_5 = \pm r$, which leads to boundary conditions for the theory on $C_3 = [-r, r] \times_w C$. The edge modes coming from each extremity are then understood to be described by chiral complex Toda theory. Combining these two chiral theories gives complex Toda CFT on C. This theory describes a complex boson Φ in the complexification of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, with an exponential potential,

$$S_{\mathfrak{g}_{\mathbb{C}}\ \mathrm{Toda}} = \frac{q}{8\pi} \int_C \biggl(\langle \partial\Phi, \bar{\partial}\Phi \rangle + \sum_{j=1}^{r} e^{\langle e_j, \Phi \rangle} \biggr) \mathrm{d}^2 z + \frac{\tilde{q}}{8\pi} \int_C \biggl(\langle \partial\overline{\Phi}, \bar{\partial}\overline{\Phi} \rangle + \sum_{j=1}^{r} e^{\langle e_j, \overline{\Phi} \rangle} \biggr) \mathrm{d}^2 z, \tag{6.22}$$

where e_j are the simple roots of $\mathfrak{g}$, $r = \operatorname{rank} \mathfrak{g}$, and the Killing form and the pairing of $\mathfrak{h}^*$ and $\mathfrak{h}$ are both denoted $\langle\ ,\ \rangle$. The coupling constants $q = k + is$, $\tilde{q} = k - is$ encode the geometry as in (6.20).

Relation to ordinary Toda theory. The conjecture is then that complex Toda CFT is related to an earlier proposal of [22] (based on [20] in the $k = 2$ case) for the AGT correspondence on $S_b^4/\mathbb{Z}_k$. For $\mathfrak{g} = \mathfrak{su}(N)$, the 2d CFT proposed in [22] consists of two decoupled theories: an $\hat{\mathfrak{su}}(k)_N/\mathfrak{u}(1)^{k-1}$ coset, and real parafermionic Toda CFT with parameters N, k, and $b = \sqrt{\tilde{q}/q}$ (coinciding with the squashing parameter). The latter theory describes parafermions and real bosons $\psi, \varphi \in \mathfrak{h}$, where the parafermions are described by another coset model $\hat{\mathfrak{su}}(N)_k/\hat{\mathfrak{u}}(1)^{N-1}$ and are coupled through dimension $1 - 1/k$ operators $\psi_j \bar{\psi}_j$ to the real bosons,

$$S_{\text{para-Toda}} = S\biggl(\frac{\hat{\mathfrak{su}}(N)_k}{\hat{\mathfrak{u}}(1)^{N-1}} \biggr) + \int_C \biggl(\langle \partial\varphi, \bar{\partial}\varphi \rangle + \sum_{j=1}^{N-1} \psi_j \bar{\psi}_j e^{(b/\sqrt{k}) \langle e_j, \varphi \rangle} \biggr) \mathrm{d}^2 z. \tag{6.23}$$

For $k = 1$ both the decoupled coset and the parafermions trivialize and we are left with ordinary Toda CFT with coupling b, as stated by the standard AGT correspondence.

The conjectured duality between complex Toda CFT and coset plus para-Toda CFT has only been checked explicitly for the simplest case of $\mathfrak{g} = \mathfrak{su}(2)$ with $k = 1$ in [316], and in [29, 317] for the case $N = k = 2$ where it essentially boils down to bosonization. See page 83 for a further discussion of the orbifold case.

Part III
Extensions of AGT

7 General class S theories

In this section we enter the realm of non-Lagrangian theories: while all class S theories arising from $\mathcal{X}(\mathfrak{su}(2))$ with tame punctures can be realized by coupling vector and hypermultiplets, we now extend the story in two ways.

The 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{g})$ is labeled by an arbitrary simply-laced simple Lie algebra $\mathfrak{g}$, so it is no wonder that the AGT correspondence [5] extends beyond $\mathfrak{su}(2)$ to $\mathfrak{su}(N)$ [93] and general gauge algebras [96, 98, 99, 318], with Liouville CFT generalizing to the Toda CFT. After reviewing this CFT in subsection 7.1 with an eye towards its connections to gauge theory, we describe an example of AGT correspondence and important considerations about punctures in subsection 7.2.

Second, in subsection 7.3, we consider interesting limits where two punctures collide while the parameters describing the defects are appropriately scaled. The resulting wild punctures allow to realize asymptotically-free gauge theories (such as SU(2) SQCD with $N_f < 4$), and AD theories [2] as part of class S.

- Most higher-rank class S theories are non-Lagrangian.
- Partial Higgsing gives a hierarchy of tame punctures. Some are described by quiver tails of SU groups.
- Collisions of tame punctures give wild punctures. This often results in AD theories.
- Tame punctures map to Toda semi-degenerate primaries.
- Wild punctures map to Toda CFT irregular operators.

7.1 Toda CFT

Lagrangian and symmetries. We cannot do justice to the fifty year history of Toda theory, starting from the Toda lattice [319, 320] in 1967, which consists of particles with nearest-neighbor exponential interactions (see [321] for an early review). Its QFT version was defined in 1982 in [322] and found to be conformal. Given a simply-laced[52] Lie algebra $\mathfrak{g}$ and its Cartan Lie algebra $\mathfrak{h}$, Toda CFT describes a real scalar field $\phi \in \mathfrak{h}$

[52]Toda CFT is defined for an arbitrary simple Lie algebra, but we only present the simply-laced case because only that case is relevant for the AGT correspondence. Correlators in non-simply-laced Toda CFT presumably correspond to correlators of suitable outer automorphism twist operators in the gauge theory on S^4_b.

subject to the Lagrangian density

$$\mathcal{L} = \frac{1}{8\pi}(\hat{g}^{ab}\langle\partial_a\phi, \partial_b\phi\rangle + 2\langle Q, \varphi\rangle\hat{R}) + \mu\sum_{i=1}^{\operatorname{rank}\mathfrak{g}} e^{b\langle e_i,\phi\rangle}. \tag{7.1}$$

Here, e_i are the simple roots of $\mathfrak{g}$ and $\langle\ ,\ \rangle$ denotes both the pairing of $\mathfrak{h}^*$ and $\mathfrak{h}$ and the Killing form. The background charge vector Q that multiplies the scalar curvature $\hat{R}$ of the background metric $\hat{g}$ is set to $Q = (b + \frac{1}{b})\rho$ where ρ is the Weyl vector, namely the half-sum of positive roots, equivalently the sum of all fundamental weights ϖ_j. Vertex operators $V_\alpha = e^{\langle\alpha,\phi\rangle}$ (we suppress normal ordering in this notation) are labeled by $\alpha \in \mathfrak{h}^*_{\mathbb{C}}$ and have holomorphic conformal dimension[53]

$$h(\alpha) = \frac{1}{2}\langle\alpha, 2Q - \alpha\rangle. \tag{7.2}$$

In particular $h(be_i) = 1$, which ensures that the exponential potential terms are exactly marginal. Their coupling μ is redundant and amounts to shifting ϕ by a multiple of ρ. The coupling $b > 0$ however plays an essential role: for instance the central charge $c = \operatorname{rank}\mathfrak{g} + 12\langle Q, Q\rangle$ depends on it. For simply-laced $\mathfrak{g}$ the theory is (expected to be) dual under $b \mapsto 1/b$, while keeping $\lambda = [\pi\Gamma(b^2)\mu/\Gamma(1-b^2)]^{1/b}$ fixed. This is quite satisfactory for the AGT correspondence since S^4_b and $S^4_{1/b}$ are isometric.

Beyond the infinite-dimensional Virasoro symmetries of 2d CFT, Toda CFT has (anti-)holomorphic $W_{\mathfrak{g}}$ symmetries. This chiral algebra was uncovered in [323] for $\mathfrak{g} = \mathfrak{su}(3)$, and more broadly in [324–326] as a symmetry of minimal models. See [327] for an early review.[54] It can be realized by quantum Drinfeld–Sokolov reduction of an affine Lie algebra [131, 132, 329, 330]. (See [331, 332] for applications to W-strings.) In the $\mathfrak{g} = \mathfrak{su}(N)$ case it can be realized as a truncation of a more general chiral algebra W_∞ generated by infinitely many conserved currents, as reviewed in [134]. As a chiral algebra, $W_{\mathfrak{g}}$ is generated by $\operatorname{rank}\mathfrak{g}$ conserved currents $W^{(k)}(z)$ whose spins k are the degrees of Casimir invariants of $\mathfrak{g}$. The quadratic Casimir invariant yields the holomorphic stress-tensor $T(z) = W^{(2)}(z)$ which generates the Virasoro subalgebra of $W_{\mathfrak{g}}$.

Primary operators and normalization. The vertex operators $V_\alpha = e^{\langle\alpha,\phi\rangle}$ have definite quantum numbers $w^{(k)}(\alpha)$ under zero-modes of all $W^{(k)}$, namely the OPE starts as

$$W^{(k)}(z)V_\alpha(0) = \frac{w^{(k)}(\alpha)}{z^k}V_\alpha(0) + \cdots \tag{7.3}$$

with for instance $w^{(2)}(\alpha) = h(\alpha)$ given in (7.2). The conserved current $W^{(k)}(z)$ translates on the gauge theory side to the degree k differential ϕ_k that shows up in the construction (1.4) of the SW curve. The momenta $\operatorname{Im}\alpha$ are r times the (diagonalized) mass parameter $m \in \mathfrak{g}$ at a given tame puncture. In the classical limit $r \to \infty$ the quantum

[53]The standard conventions for Q in Liouville and Toda CFT differ by a factor of 2, so that typical Liouville momenta take the form $\alpha = Q/2 + iP$ while typical Toda momenta are $\alpha = Q + iP$.

[54]The contemporary [328] by the same authors is supposedly relevant, but not available to me.

numbers $w^{(k)}(\alpha)$ simplify to Casimir invariants of $\mathfrak{g}$ and the OPE (7.3) becomes the singularities of ϕ_k near tame punctures [19].

The quantum numbers $w^{(k)}(\alpha)$ are invariant under the Weyl group action $\alpha \mapsto Q+w(\alpha-Q)$ for any Weyl group element $w\colon \mathfrak{h}^* \to \mathfrak{h}^*$. Thus, V_α and $V_{Q+w(\alpha-Q)}$ have the same quantum numbers; in Toda CFT they are the same operator up to a normalization called reflection amplitude and determined in [333–335]. For generic α, the expressions can be recast as the statement that [93][55]

$$\widehat{V}_\alpha = \frac{\lambda^{\langle\rho,\alpha-Q\rangle}}{\prod_{e>0}\Upsilon_b(\langle Q-\alpha,e\rangle)} V_\alpha(z) \tag{7.4}$$

is invariant under Weyl reflections of α, where the product ranges over all positive roots e and we recall $\lambda = [\pi\Gamma(b^2)\mu/\Gamma(1-b^2)]^{1/b}$. While often convenient, the normalization (7.4) does not make sense for values of α where the denominator blows up.

The operator spectrum of Toda CFT consists of vertex operators V_{Q+a} with $a \in \mathfrak{h}$ (purely imaginary in our conventions), modulo the Weyl group. Each of them is the highest-weight of a Verma module of the $W_\mathfrak{g}$ algebra, with no null states. As for the Virasoro algebra, there are some values of momenta (away from this line) for which the vertex operators have null descendants. The precise condition[56] is that V_α has null descendants if $\langle\alpha-Q,e\rangle = -n_1 b - n_2/b$ for any root e and positive integers $n_1, n_2 > 0$.

Correlators in Toda CFT. The $W_\mathfrak{g}$ symmetry severely constrains two and three-point functions in Toda CFT. A two-point function of primary operators V_{α_1} and V_{α_2} can only be non-vanishing if their quantum numbers obey $w^{(k)}(\alpha_1) = (-1)^k w^{(k)}(\alpha_2)$, hence $\alpha_1 = 2Q - \alpha_2$ modulo the Weyl group. Taking into account our preferred normalization (7.4) one has

$$\langle \widehat{V}_\alpha(z,\bar{z})\widehat{V}_{\alpha'}(0)\rangle = |z|^{-4h(\alpha)} \frac{\sum_{w\in\text{Weyl}} \delta_{Q-\alpha'=w(\alpha-Q)}}{\prod_{\text{roots } e}\Upsilon_b(\langle Q-\alpha,e\rangle)}, \tag{7.5}$$

where the sum of delta functions simply ensures Weyl invariance and working with the unnormalized V_α (which is necessary to treat partially degenerate momenta) would simply introduce some reflection amplitudes in this sum. The Shapovalov matrix of two-point functions of $W_\mathfrak{g}$-descendants follows in the standard way by commuting the W-algebra modes $W_n^{(k)}$. Our normalization choice is pleasant because, as in the $\mathfrak{su}(2)$ case, the inverse two-point function $\prod_e \Upsilon_b(\langle Q-\alpha,e\rangle)$ matches the one-loop determinant of a vector multiplet for the gauge algebra $\mathfrak{g}$.

The three-point functions of primaries are encoded in coefficients $\widehat{C}_{123} = \widehat{C}(\alpha_1,\alpha_2,\alpha_3)$

$$\langle \widehat{V}_{\alpha_1}(z_1,\bar{z}_1)\widehat{V}_{\alpha_2}(z_2,\bar{z}_2)\widehat{V}_{\alpha_3}(z_3,\bar{z}_3)\rangle = \frac{\widehat{C}(\alpha_1,\alpha_2,\alpha_3)}{|z_1-z_2|^{2h_{12}}|z_1-z_3|^{2h_{13}}|z_2-z_3|^{2h_{23}}}, \tag{7.6}$$

[55] The reflection amplitude in [333–335] and in later work [136] (by one of the same authors) seems to differ by a sign. I take the latter sign to be correct as it seems to agree with Liouville CFT.

[56] This is obtained using screening charges, see e.g. [336]. Reading modern references, I think that the null descendants start at level $n_1 n_2$, but I have not found the suitable references in old Toda CFT literature. Help welcome.

where $h_{ij} = h(\alpha_i) + h(\alpha_j) - h(\alpha_k)$. The general three-point function is not known, and in addition three-point functions of most $W_{\mathfrak{g}}$-descendants cannot be expressed in terms of $\widehat{C}_{123}$, unlike the standard case of Virasoro descendants. Only certain special cases [136, 337, 338] discussed below have been determined.

Under-determined conformal blocks. This has a knock-on effect on higher-point Toda CFT correlators, as the conformal blocks describing how $W_{\mathfrak{g}}$ descendants contribute are not fixed by symmetry. Consider for instance

$$\langle \widehat{V}_{\alpha_1} \widehat{V}_{\alpha_2} \widehat{V}_{\alpha_3} \widehat{V}_{\alpha_4} \rangle = \int_{a\in\mathfrak{h}/\mathrm{Weyl}} \mathrm{d}a \sum_{Y} \langle \widehat{V}_{\alpha_1} \widehat{V}_{\alpha_2} \widehat{V}^{[Y]}_{Q-a} \rangle \frac{1}{\langle \widehat{V}_{Q-a} \widehat{V}_{Q+a} \rangle} \langle \widehat{V}^{[Y]}_{Q+a} \widehat{V}_{\alpha_3} \widehat{V}_{\alpha_4} \rangle, \tag{7.7}$$

where we suppressed the spatial dependence, the integral ranges over primary operators $\widehat{V}_\alpha$ in the spectrum, and the sum ranges over their descendants, orthogonalized and normalized to have the same norm $\langle \widehat{V}_{Q-a} \widehat{V}_{Q+a} \rangle$ as primaries. For generic α_i, the only three-point functions $\langle \widehat{V}_{\alpha_1} \widehat{V}_{\alpha_2} \widehat{V}^{[Y]}_{Q-a} \rangle$ that are determined by primary three-point functions are those where $\widehat{V}^{[Y]}_{Q-a}$ is in fact a Virasoro descendant of $\widehat{V}_{Q-a}$.

The class S theory coming from a three-punctured sphere with full tame punctures is the non-Lagrangian tinkertoy $T_{\mathfrak{g}}$, and supersymmetric localization has nothing to say on its sphere partition function. A very powerful roundabout way is to consider the 5d lift and work out the limit of $S^4_b \times S^1$ partition function when the circle radius β shrinks [339–342]. In principle this provides conjectural expressions for $\widehat{C}_{123}$ and all descendant three-point functions [343], but it is not clear that the $\beta \to 0$ limits exist, and it is not clear how to relate parameters of the topological vertex formalism to bases of descendants, as explained in detail in [229]. Higher-point correlators correspond to theories obtained by gauging together copies of $T_{\mathfrak{g}}$. Since the gauge group is simply one factor G per tube, the instanton moduli space is known, and the instanton partition function is some integral over this space. The integrand, however, depends on the matter theories $T_{\mathfrak{g}}$, whose reaction to instantons is not known. This is exactly analogous to how the sum over descendants is known but the requisite three-point functions do not derive from $\widehat{C}_{123}$.

7.2 Higher rank AGT correspondence

The main building block of class S theories is the tinkertoy $T_{\mathfrak{g}}$ for three full tame punctures, reviewed in [117]. This tinkertoy is non-Lagrangian[57] for $\mathfrak{g} \neq \mathfrak{su}(2)$. As a result, another type of tame punctures (called simple punctures) is needed for the simplest examples of higher-rank AGT correspondence, such as the matching of an $\mathrm{SU}(N)$ SQCD partition function with an $\mathfrak{su}(N)$ Toda CFT four-point function.

Besides full and simple tame punctures, there are other tame punctures (and of course a host of wild punctures) studied in [1, 6, 261, 318, 336, 345, 346] A large program to

[57] Interestingly, its 3d $\mathcal{N} = 4$ dimensional reduction is mirror to a Lagrangian theory described by a star-shaped quiver [344].

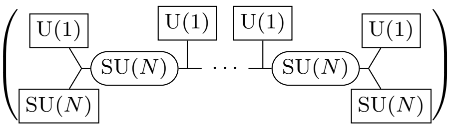
classify tinkertoys has been carried out by Chacaltana and Distler and collaborators in [95, 97, 100–106, 108] (a warning though, their use of “irregular” is non-standard in this context). The punctures that can arise in a limit where one of the tubes in the Riemann surface becomes pinched were studied in [8, 347].

The various tame punctures correspond to Toda CFT vertex operators V_α that are partially degenerate, as we explain for $\mathfrak{g} = \mathfrak{su}(N)$. We also describe how punctures can be “partially closed” by tuning their parameters, which on the gauge theory side corresponds to a partial Higgsing. Finally, we outline how to include non-simply-laced gauge groups.

Wyllard relation: SU(N) linear quiver and a $\mathfrak{su}(N)$ Toda CFT correlator. For simplicity we now focus on the $\mathfrak{g} = \mathfrak{su}(N)$ case, which is understood best. The chiral algebra is denoted variously $W_{\mathfrak{su}(N)} = WA_{N-1} = W_N$.

In subsection 4.4 we considered a linear quiver gauge theory whose hypermultiplets transform in bifundamental representations of $n-1$ successive SU(N) groups, of which the middle $n-3$ are gauged. Besides the two SU(N) flavour symmetries at the ends of the quiver, each of the $n-2$ hypermultiplets has a U(1) flavour symmetry. As stated in (4.26), this theory is realized by reducing $\mathcal{X}(\mathfrak{su}(N))$ on a sphere with n punctures corresponding to these n flavour symmetry factors. The SU(N) flavour symmetries correspond to full tame punctures, at which each differential ϕ_k has a pole of order $k-1$ in the massless case, or k when SU(N) masses are turned on. The U(1) flavour symmetries correspond to simple tame punctures where each ϕ_k has a simple pole in the massless case (the massive case is more complicated).

The AGT correspondence proposed in [93] takes the form

$$Z_{S^4_b}$$


$$= \left\langle \widehat{V}_{\alpha_1} \widehat{V}_{\mu_2} \widehat{V}_{\mu_3} \cdots \widehat{V}_{\mu_{n-1}} \widehat{V}_{\alpha_n} \right\rangle^{\mathrm{Toda}(\mathfrak{su}(N))} \qquad (7.8)$$

where α_j ($j = 1, n$) encode the imaginary SU(N) mass parameters of the two full punctures, $m_j = (m_{j1}, \dots, m_{jN})$ with $\sum_p m_{jp} = 0$, while μ_j ($j = 2, \dots, n-1$) encodes the j-th U(1) mass parameter $m_j \in i\mathbb{R}$ as

$$\begin{aligned} \alpha_j &= Q + r m_j, && j = 1, n, \\ \mu_j &= \left(\frac{N}{2}\left(b + \frac{1}{b}\right) + r m_j\right)\varpi_1, && j = 2, \cdots n-1. \end{aligned} \qquad (7.9)$$

For instance in the case $n = 4$ this identifies a Toda CFT four-point function to the partition function of SU(N) SQCD with $N_f = 2N$ flavours.

To understand these momenta, we discuss the corresponding vertex operators V_α and their W_N descendants. We have already encountered momenta in $Q + \mathfrak{h}$ which describe normalizable states that occur in the spectrum of Toda CFT. Momenta proportional to the first weight ϖ_1 are called semi-degenerate momenta: as uncovered starting in [348, 349] they have null W_N descendants at level 1. Consider a three-point function $\langle V_\alpha V_{\varkappa\varpi_1} V_{\alpha'} \rangle$ with a semi-degenerate vertex operator. Thanks to null vectors, the action of arbitrary W_N generators can be converted to Virasoro generators, hence to differential operators acting on the known coordinate-dependence. This means that three-point functions of descendants of V_α, $V_{\varkappa\varpi_1}$, and $V_{\alpha'}$ are uniquely fixed as a multiple of the three-point function of primaries.

Checking the Wyllard relation. As in the $\mathfrak{su}(2)$ case, the matching (7.8) is most directly checked in the S-duality frame corresponding to the s-channel decomposition of the sphere correlator. Each three-punctured sphere piece has one simple puncture and two full punctures (and corresponds on the gauge theory side to a bifundamental hypermultiplet). Expanding the correlator in this channel, rewriting descendant three-point functions in terms of the primaries, and collecting the descendant's contributions into a conformal block, we have

$$\begin{aligned}&\langle \widehat{V}_{\alpha_1}\widehat{V}_{\mu_2}\cdots\widehat{V}_{\mu_{n-1}}\widehat{V}_{\alpha_n}\rangle\\ &\quad=\int_{\substack{a_j\in\mathfrak{h}/\mathrm{Weyl}\\ 2\leq j\leq n-2}}\mathrm{d}a\,\frac{\langle\widehat{V}_{\alpha_1}\widehat{V}_{\mu_2}\widehat{V}_{Q+a_2}\rangle\langle\widehat{V}_{Q-a_2}\widehat{V}_{\mu_3}\widehat{V}_{Q+a_3}\rangle\cdots\langle\widehat{V}_{Q-a_{n-2}}\widehat{V}_{\mu_{n-1}}\widehat{V}_{\alpha_n}\rangle}{\langle\widehat{V}_{Q+a_2}\widehat{V}_{Q-a_2}\rangle\cdots\langle\widehat{V}_{Q+a_{n-2}}\widehat{V}_{Q-a_{n-2}}\rangle}\mathcal{F}(z)\mathcal{F}(\overline{z}),\end{aligned}\tag{7.10}$$

for some conformal blocks $\mathcal{F}(z)$ that are (in principle) calculable as series in powers of complex structure parameters of C, and that depend on all external and internal momenta. The very fact that conformal blocks are calculable (for these momenta) matches nicely with the fact that the 4d theory is Lagrangian hence its partition function is calculable by supersymmetric localization.

One key part of the matching is that the $n-2$ three-point functions in (7.10) should match with the one-loop determinants of the $n-2$ bifundamental hypermultiplets in the quiver. Thankfully, the three-point functions $\langle V_{\alpha_1}V_{\varkappa\varpi_1}V_\alpha\rangle$ of two non-degenerate and one semi-degenerate vertex operators were worked out in [136, 337, 338] by inserting a fully degenerate vertex operator $V_{-b\varpi_1}$ into the correlator and solving a differential equation that results.[58] Consider the normalization (7.4) of non-degenerate operators and an ad-hoc normalization

$$\widehat{V}_{\varkappa\varpi_1}=\frac{\lambda^{\langle\varkappa\varpi_1,\rho\rangle}}{(\Upsilon_b(b))^{N-1}\Upsilon_b(\varkappa)}V_{\varkappa\varpi_1}\tag{7.11}$$

which is Weyl-invariant but is an abuse of notation since $\widehat{V}$ does not relate to V in the same way as in (7.4). Then the three-point function is [136]

$$\widehat{C}(Q+a_1,\varkappa\varpi_1,Q+a_2)=\frac{1}{\prod_{i,j=1}^N\Upsilon_b(\varkappa/N-\langle a_1,h_i\rangle-\langle a_2,h_j\rangle)}\tag{7.12}$$

where h_i are the weights of the fundamental representation of $\mathfrak{su}(N)$. The product of Upsilon functions correctly coincides with the one-loop determinants of N^2 hypermultiplets

[58] This bootstrap technique was introduced by Teschner to solve Liouville CFT. See also [68, 350–352] for further explorations in the Toda CFT context with more general fully degenerate vertex operators. See [353, 354] for other related correlators.

on S_b^4 with suitable masses. The inverse two-point functions match with vector multiplet one-loop determinants, see below (7.5). The classical action also reproduces correctly the leading z-dependence of the Toda CFT correlator. Finally, one can tediously match conformal blocks with instanton partition functions order by order, to confirm (7.8).

The Toda correlator in (7.8) can be decomposed in principle in many other channels. For instance taking the OPE of the semi-degenerate vertex operators $\widehat{V}_{\mu_1}$ and $\widehat{V}_{\mu_2}$ in the four-point function ($n = 4$) yields a t-channel decomposition. In the $N = 3$ case, the internal momenta produced by this fusion are non-degenerate, so that the decomposition involves general three-point functions. The corresponding S-duality frame of SU(3) SQCD consists of the $T_{\mathfrak{su}(3)}$ tinkertoy (the $\mathfrak{e}_6$ Minahan–Nemeschansky SCFT) coupled to some hypermultiplets by a gauge group that turns out to be SU(2). For general N, such fusions lead to numerous types of punctures intermediate between the full and the simple puncture.

Partial Higgsing. Consider the linear quiver in (7.8). Its Higgs branch consists of supersymmetric vacua where hypermultiplet scalars get a VEV. Upon moving to any given point on the Higgs branch, the hypermultiplet VEV may break gauge symmetry to a smaller group, thus reducing the quiver to a smaller one.

We denote scalars in the bifundamental hypermultiplets as $(Q_j, \tilde{Q}_j)$ for $j = 2, \dots, n-1$. An important class of vacua are obtained by imposing a nilpotent VEV to the moment map (see e.g. the appendix of [355])

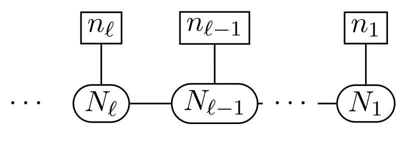
$$\mu_1 = \tilde{Q}_2 Q_2 - \frac{1}{N}\operatorname{Tr}(\tilde{Q}_2 Q_2) \tag{7.13}$$

of the leftmost SU(N) flavour symmetry group. Nilpotent matrices in $\mathfrak{su}(N)_{\mathbb{C}} = \mathfrak{sl}(N, \mathbb{C})$ are classified up to conjugation by a partition of N, or equivalently a Young diagram Y with N boxes. Denoting by n_k the number of columns of length k in Y, the nilpotent VEV we consider takes the block-diagonal form

$$\langle \mu_1 \rangle = J_1^{\oplus n_1} \oplus J_2^{\oplus n_2} \oplus \cdots \oplus J_\ell^{\oplus n_\ell} \tag{7.14}$$

with n_k Jordan blocks J_k of size $k \times k$ (for instance $J_1 = (0)$). The VEV $\langle \mu_1 \rangle$ cannot be imposed in isolation, as the F-term relations lead to non-vanishing values for the hypermultiplets $(Q_j, \widetilde{Q}_j)$ for $j = 2, \cdots, \ell$.

The Higgs mechanism thus breaks multiple gauge symmetries. Specifically, it reduces the first few groups of the quiver to a *quiver tail*



(7.15)

where round nodes are SU(N_j) gauge groups, rectangles count fundamental hypermultiplets, and N_j is determined by $N_0 = 0$ and $N_j - N_{j-1} = n_j + n_{j+1} + \cdots + n_\ell$ (which is the length of the j-th row). In other words, N_j counts all boxes in rows $1, \cdots, j$ and in particular $N_\ell = N$. The flavour symmetry is

$$\Big[\prod_{k=1}^{\ell} U(n_k)\Big]/U(1)_{\text{diag}}. \tag{7.16}$$

The puncture has $n_1 + \cdots + n_\ell - 1 = N_1 - 1$ mass parameters. The nilpotent matrix (7.14) can also be understood as the image of the raising operator under an embedding $\rho\colon \mathrm{SU}(2) \to \mathrm{SU}(N)$. In that equivalent description, the flavour symmetry (7.16) arises as the commutant of ρ.

Quiver tails and punctures. Starting from a quiver tail (7.15) and its SW solution obtained from M-theory [178], Gaiotto understood in [1] the relevant patterns of pole orders for the differentials ϕ_k. This helped determine that tame punctures are labeled by partitions of N. Linear quivers with arbitrary quiver tails are realized in class S as the reduction of $\mathcal{X}(\mathfrak{su}(N))$ on a sphere with arbitrarily many simple tame punctures and with two (general) tame punctures.

In fact, the partial Higgsing procedure we described replaces the full tame puncture (that we started with) by precisely the tame puncture labeled by Y. Just as full tame punctures carry $\mathrm{SU}(N)$ flavour symmetry, the puncture carries flavour symmetry (7.16). The dictionary between quiver tails, punctures, and the order of poles, is nicely written in [6].

Punctures can be closed entirely. This is most easily seen for the simple punctures in (7.8): on the gauge theory side, two neighboring $\mathrm{SU}(N)$ groups are reduced to their diagonal subgroup, and we are left with a shorter linear quiver.

Partially degenerate vertex operators. Partial Higgsing translates on the Toda CFT side to changing a non-degenerate vertex operator to a partially degenerate one.

According to footnote 56, level 1 null W_N descendants of a vertex operator V_α are characterized by roots e for which $\langle \alpha - Q, e\rangle = -b - 1/b$. Up to a Weyl group transformation of $\alpha - Q$, we choose these roots be simple roots e_j only. The condition then reduces to $\langle \alpha, e_j\rangle = 0$, namely $\langle \alpha, h_j\rangle = \langle \alpha, h_{j+1}\rangle$ in terms of the weights h_i of the defining representation of $\mathfrak{su}(N)$. The components $\langle \alpha, h_i\rangle$ of α organize as

$$\alpha = \big(\underbrace{\alpha_{(1)}, \cdots, \alpha_{(1)}}_{l_1}, \cdots, \underbrace{\alpha_{(r)}, \cdots, \alpha_{(r)}}_{l_r}\big) \tag{7.17}$$

where l_k, $1 \leq k \leq r$ denote the number of equal components, and we can reorder the components such that $l_1 \geq l_2 \geq \cdots \geq l_r \geq 0$ defines lengths of columns of some Young diagram Y. This is the same Young diagram classification as for the punctures. For instance the momentum has $r - 1$ parameters (because of tracelessness), which is the length of the first row of Y (minus one) thus matches the counting in (7.16).

The precise proposal in [336] is that a tame puncture labeled by Y corresponds to a vertex operator with momentum

$$\alpha = \Big(b + \frac{1}{b}\Big)\rho_Y + m_Y, \tag{7.18}$$

with m_Y the mass parameters for the flavour symmetry (7.16) and ρ_Y the projection of the Weyl vector ρ onto the subspace with the multiplicities (7.17). For the full puncture case $Y = 1^N$ we have $\rho_Y = \rho$, while for the simple puncture $Y = (N-1)+1$ one finds $\rho_Y = (N/2)\varpi_1$. In both cases the proposal reproduces (7.9).

Punctures: nilpotent orbits, Nahm, Hitchin, and Toda. For the A-type case $\mathfrak{g} = \mathfrak{su}(N)$ we have seen that tame punctures are labeled by partitions of N. A broader perspective is that tame codimension 2 defects of $\mathcal{X}(\mathfrak{su}(N))$ are labeled by such a partition. The D-type case and USp–SO quiver tails are considered in [6].

For general 6d $(2,0)$ theories $\mathcal{X}(\mathfrak{g})$, there are (at least) three sets of data that equivalently characterize the defect.[59]

- **Nahm data.** A nilpotent orbit $\mathcal{O}_\mathrm{N} \subset \mathfrak{g}_\mathbb{C}$ that describes a Nahm pole boundary condition for SYM with gauge group G. This arises by considering $\mathcal{X}(\mathfrak{g})$ on $\mathbb{R}^{2,1}\times$ cigar $\times\, S^1$ with the defect at the tip of the cigar. Reducing first on the circle direction of the cigar gives 5d $\mathcal{N}=2$ SYM with gauge group G, with a Nahm pole boundary condition.

- **Hitchin data.** A nilpotent orbit $\mathcal{O}_\mathrm{H} \subset \mathfrak{g}_\mathbb{C}$ with additional discrete data. Within the same $\mathbb{R}^{2,1}\times\text{cigar}\times S^1$ geometric setup, reducing first on S^1 gives a codimension 2 defect in 5d, then reducing the cigar to a half-line gives the S-dual of the Nahm pole boundary condition. This data was studied early on in [356].

- **Toda data.** A partially degenerate primary operator of $\mathfrak{g}$ Toda CFT specified by its null $W_\mathfrak{g}$ descendants.

Nahm and Hitchin data were related in [100]. The relation to Toda data was understood in [318] for the case where e is principal nilpotent inside some Levi subalgebra of $\mathfrak{g}$, see also [357, 358].

The same classification holds for codimension 2 operators in the 6d $(2,0)$ little string theory with Lie algebra $\mathfrak{g}$ (which implies the classification for $\mathcal{X}(\mathfrak{g})$). The approach in [342, 359, 360] is to realize little strings as IIB strings on a $\mathbb{C}^2/\Gamma$ singularity, where the defect consists of D5 branes wrapping certain *non-compact* two-cycles in $\mathbb{C}^2/\Gamma$.

Non-simply-laced gauge groups. Reducing $\mathcal{X}(\mathfrak{g})$ on a circle yields 5d $\mathcal{N}=2$ SYM with gauge algebra $\mathfrak{g}$, which is simply-laced. Non-simply-laced gauge groups are achieved by twisting, namely changing the periodicity of fields,[60] by an outer automorphism of $\mathfrak{g}$.

[59] Help welcome to complete the references.

[60] This notion of twist is unrelated to the partial topological twist used to preserve supersymmetry when reducing $\mathcal{X}(\mathfrak{g})$ on C.

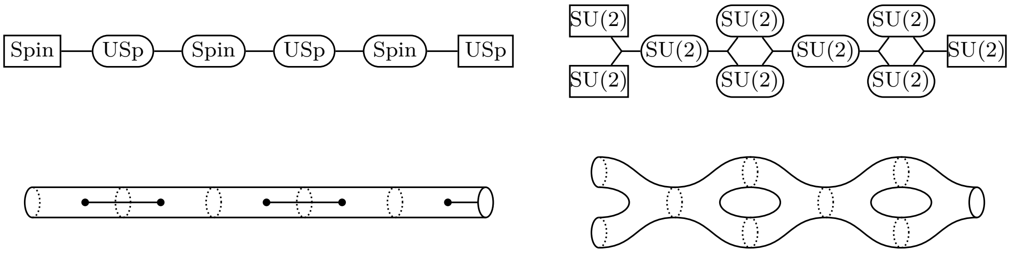


Figure 4: Left: linear quiver with alternating $\mathrm{USp}(2N-2)$ and $\mathrm{Spin}(2N)$ groups and its $\mathfrak{so}(2N)$ class S curve with branch cuts. Right: for $N=2$, same theory as a SU(2) generalized quiver, and its $\mathfrak{su}(2)$ class S curve, which is a double-cover of the $\mathfrak{so}(4)$ curve on the left.

Likewise, class S includes 4d $\mathcal{N}=2$ theories with arbitrary gauge groups: these are obtained by including (outer automorphism) twist lines that may end on punctures. This direction was explored in [96, 98, 99, 318] (see also [361]). See Figure 4 for an example with $\mathfrak{g}=\mathfrak{so}(2N)$ and its reformulation as a $\mathfrak{su}(2)$ class S theory for $N=2$ since $\mathfrak{so}(4)=\mathfrak{su}(2)^2$.

7.3 Wild punctures and AD theories

Our investigations so far only involved so-called "tame" codimension 2 defects of the 6d $(2,0)$ theory. They admit a broad generalization to "wild" defects, introduced by Witten [362] in the context of surface operators in 4d $\mathcal{N}=4$ SYM. These defects impose a stronger blow-up near their support for the "fields" of the 6d theory.

Wild punctures from collisions. We recall the massive tame puncture (4.2)

$$\varphi(z) \sim \biggl(\frac{\operatorname{diag}(m,-m)}{z-z_i}+O(1)\biggr)\mathrm{d}z \implies \phi_2(z)=\biggl(\frac{m^2}{(z-z_i)^2}+O\Bigl(\frac{1}{z-z_i}\Bigr)\biggr)\mathrm{d}z^2 \quad (7.19)$$

and its massless version (4.4). Colliding l such simple poles of φ, while scaling appropriately the mass parameters, leads to a pole of order l, hence generically to $\phi_2 \sim \mathrm{d}z^2/(z-z_i)^{2l}$. Just as their tame counterparts, the resulting *wild punctures* of level l can be partially closed by imposing some relations between eigenvalues in the series expansion of Φ_z, so that the pole of ϕ_2 has an order lower than $2l$.

The collision limits can have two main effects on the 4d gauge theory: decoupling some hypermultiplets by making them massive while keeping the dynamical scale Λ fixed, or tuning the theory to an AD point on the Coulomb branch [2, 363–365] at which point the theory becomes a strongly-coupled isolated SCFT.

For the case $\mathfrak{g}=\mathfrak{su}(2)$ that we consider for now, wild punctures are labeled by the order of the pole of ϕ_2 (which is 2 for a tame puncture), and of course by coefficients of the expansion at these poles. By cutting the Riemann surface along circles as in the tame case, $\mathfrak{su}(2)$ class S theories can be constructed by gauging SU(2) flavour symmetries of

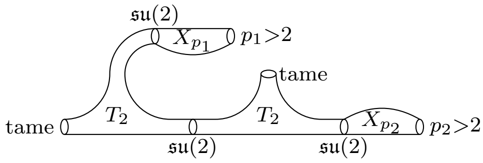


Figure 5: Decomposition of a typical $\mathfrak{su}(2)$ class S theory (associated to a sphere with two tame and two wild punctures) into T_2 and X_p building blocks coupled together by gauging diagonal $\mathfrak{su}(2)$ flavour symmetries.

the trifundamental half-hypermultiplet T_2 (corresponding to a sphere with three tame punctures) and of theories X_p corresponding to a sphere with a tame puncture and a wild puncture at which ϕ_2 has a pole of order $p > 2$. See Figure 5. Spheres with a single wild puncture cannot be cut into these building blocks and lead to other interesting theories Y_p. This exhausts $\mathfrak{su}(2)$ class S.

Examples of theories with wild punctures. Just for this explanation we denote by $(p_1\, p_2 \dots p_k)$ the class S theory obtained for a sphere with k punctures at which ϕ_2 has poles of order $p_1, \dots, p_k$, respectively. Let us exemplify both effects above starting from SU(2) $N_f = 4$ SQCD, realized as $(2\,2\,2\,2)$ in this notation, namely by taking C to be a sphere with four tame punctures. We first decouple hypermultiplets.

- SU(2) $N_f = 3$ SQCD arises from $(2\,2\,4)$, a sphere with two tame punctures and one wild puncture of order 4, obtained as a collision of two tame punctures.
- SU(2) $N_f = 2$ SQCD appears in two ways in class S. First, as $(4\,4)$ obtained from $(2\,2\,4)$ by colliding the two tame punctures. Alternatively, as $(2\,2\,3)$: one can decouple the hypermultiplet by tuning a mass parameter of the wild puncture in the $(2\,2\,4)$ description of the $N_f = 3$ theory, and this reduces the pole of ϕ_2 at the wild puncture from order 4 to order 3. A consistency check is that the two constructions lead to equivalent SW geometry.
- SU(2) $N_f = 1$ SQCD then appears as $(4\,3)$.
- Pure SU(2) SYM appears as $(3\,3)$ with two minimally wild punctures.

There are further collision limits, which turn out to realize AD theories. By colliding the two wild punctures in the $(4\,3)$ realization of SU(2) $N_f = 1$ SQCD we get a single wild puncture of rather high order (7): this is the Y_7 theory mentioned above. The AD point (most singular point) of the Coulomb branch of SU(2) $N_f = 2$ is obtained by colliding punctures $(4\,4) \to (8)$ or $(2\,2\,3) \to (2\,5)$, both punctured curves C turning out to give the same 4d SCFT (namely $Y_8 \simeq X_5$). For SU(2) $N_f = 3$ we find the collision $(2\,2\,4) \to (2\,6)$, which is X_6. Of course, these limits all translate to tuning parameters on the gauge theory side and were thus found a long time ago [2, 363], but the class S realization embeds them in a broader setting.

Exercise 7.1. *Recall the SW curve (4.13) of the $\mathfrak{su}(2)$ class S theory for a four-punctured sphere: $x^2 = u_2(z)$ with*

$$u_2(z) = \frac{\frac{q}{z}m_1^2 + \frac{q(q-1)}{z-q}m_2^2 + \frac{z-q}{z-1}m_3^2 + zm_4^2 - u}{z(z-q)(z-1)}. \tag{7.20}$$

This theory has a description as SU(2) SQCD with gauge coupling $\tau = (\log q)/(2\pi i)$ and $N_f = 4$ flavours of masses $m_1 \pm m_2$ and $m_3 \pm m_4$.

1. Decouple one hypermultiplet: take $m_1 + m_2 \to \infty$, keeping $m_1 - m_2$ and $m_3 \pm m_4$ and $\Lambda = q(m_1 + m_2)$ fixed. You should get $u_2 = P(z)/(z^4(z-1)^2)$ for some quartic polynomial P.

2. Decouple a second hypermultiplet in two ways. First, take $m_1 - m_2 \to \infty$, keeping $\Lambda'^2 = \Lambda(m_1 - m_2)$ and other masses fixed. Second, instead, take $m_3 + m_4 \to \infty$, keeping $\tilde{z} = z(m_3 + m_4)$ and $\tilde{x} = x/(m_3 + m_4)$ and $\Lambda'^2 = \Lambda(m_3 + m_4)$ and other masses fixed. Map one SW curve to the other and check the difference of SW differentials λ is inessential (residues are masses, no Coulomb branch dependence).

3. Decouple a third and a fourth hypermultiplet and rescale $z \to z\Lambda^2$ to get the well-known curve of pure SU(2) SYM: $z^2x^2 = u + \Lambda^2(z + 1/z)$ with $\lambda = x\mathrm{d}z$.

CFT side. On the 2d CFT side, the limits that produce wild punctures correspond to colliding primary vertex operators $\widehat{V}_\alpha$. The collision of $n \geq 2$ vertex operators yields irregular operators denoted I_{n-1}, which depend on the original n momenta (suitably rescaled). Their Ward identities with the stress tensor involve poles of the same order $2n$ as the pole of ϕ_2 on the gauge theory side that arises in the same collision. This matching is consistent with the fact that ϕ_2 can be understood as the semiclassical limit of T:

$$T(z)I_{n-1}(0) = \sum_{k=n-1}^{2n-2} \frac{\Lambda_k}{z^{k+2}} I_{n-1}(0) + \cdots \underset{r\to\infty}{\simeq} r^2\phi_2(z)I_{n-1}(0) + \ldots. \tag{7.21}$$

By the state-operator correspondence, these operators give coherent states of the Virasoro algebra [94] (alternatively called Whittaker vectors or Gaiotto states) and generalizations thereof called irregular states (sometimes Bonelli–Maruyoshi–Tanzini (BMT) states) [366, 367].

Wild punctures and AD theories. The decoupling of hypermultiplets starting from SU(2) $N_f = 4$ SQCD or $\mathcal{N} = 2^*$ SYM, and its effect in the AGT corespondence, were studied in [94, 368–375]. Higher-level irregular states (BMT states) of the Virasoro algebra were investigated in [366, 367, 376–380], and generalizations to W-algebras in [99, 378, 381–383]. On the CFT side, collisions of primary operators and direct definitions of irregular states were made by Rim and collaborators [377, 379, 380, 384–391], and others [68, 392–401].

Further directions. Some AD theories were found at particular points on the Coulomb branch of Lagrangian gauge theories, and as appearing in S-dual descriptions in [2, 169, 363–365, 402]. Class S constructions of a variety of AD theories and related topics are in [6, 8, 403–410] and in Xie's work with collaborators [411–423]. It is not expected that class S theories exhaust all possible 4d $\mathcal{N} = 2$ theories (see e.g. [424]). Besides a classification of Lagrangian field theories [425, 426], and a program to classify theories according to their Coulomb branch geometry [427–435], other constructions of 4d $\mathcal{N} = 2$ theories have been explored [406, 414, 416, 417, 419, 436–447].

An interesting tool to check the AGT correspondence even in the absence of Lagrangian descriptions of the class S theories is to compute central charges and anomalies. The central charge of Toda CFT (and generalizations) was matched with a reduction to 2d of the anomaly 8-form of the 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{g})$ in [19, 22, 448, 449]. Reducing instead on C gives the a and c conformal anomalies of the 4d class S theories [100, 117, 120, 450–453].

8 Operators of various dimensions

Wilson [454] and 't Hooft [455] loop operators, and dyonic loops combining them [456], play an important role in studying phases of 4d gauge theories. Surface operators are less studied yet very rich; for instance the moduli space of some surface operators is the UV curve of the class S theory. Finally, domain walls describe interfaces between two 4d theories, or boundary phenomena. A large source of such operators in the AGT correspondence are the half-BPS codimension 2 and codimension 4 defects of the 6d $(2,0)$ theory $\mathcal{X}(\mathfrak{g})$. Another source is to orbifold the 6d setup, with an orbifold group that must respect orientation since $\mathcal{X}(\mathfrak{g})$ is a chiral theory. These defects can be inserted into the AGT correspondence with various orientations relative to the product spacetime $M_4 \times C$. We have already covered at length the case where a codimension 2 defect inserted at a point in C wraps the whole 4d spacetime: indeed these are simply the punctures and twist operators described throughout this review, especially in section 7.

Main examples of gauge theory and CFT operators

- Codimension 2 case: coupling to a tinkertoy matches with inserting a vertex operator; symmetry-breaking walls match with Verlinde loops; Gukov–Witten surface operators change the Toda CFT.
- Codimension 4 case: vortex string surface operators match with degenerate vertex operators; Wilson–'t Hooft loops match with degenerate Verlinde loops; orbifold singularities at poles of S_b^4 change the Toda CFT.

We refer to Table 1 in the introduction for a full list of possibilities that have been investigated, and references. Here, we organize our discussion by increasing dimension on the 4d side, starting with a discussion of point-like operators in subsection 8.1, then line and loop operators in subsection 8.2 (see the review [457]), 2d operators in subsection 8.3 (see the review [458]), and 3d walls and interfaces in subsection 8.4.

8.1 Local operators in 4d

Codimension 4 operators of $\mathcal{X}(\mathfrak{g})$ are labeled by representations of $\mathfrak{g}$ and the effect of wrapping such an operator over all of C has not been fully understood. In fact, for most of the geometries we describe in the following, the main ideas have been understood in certain theories such as for $\mathfrak{g} = \mathfrak{su}(2)$, but not in full generality.

Coulomb branch operators. The order k holomorphic differentials $\phi_k(z) = u_k(z)\mathrm{d}z^k$ that define the SW curve can be calculated from the classical limit of Toda CFT correlators. In this limit, where the radius of S_b^4 is large, or $\epsilon_1, \epsilon_2 \to 0$, or equivalently 2d CFT conformal dimensions are large, the quadratic differential u_2 is given by the insertion of the energy-momentum tensor, and more generally u_k by the spin k current W_k:[61]

$$u_k(z) \propto \frac{\langle W_k(z)\widehat{V}_{\mu_1} \dots \widehat{V}_{\mu_n}\rangle}{\langle \widehat{V}_{\mu_1} \dots \widehat{V}_{\mu_n}\rangle} \qquad \text{as } \epsilon_1, \epsilon_2 \to 0. \tag{8.1}$$

For $\mathfrak{su}(2)$ see the original AGT paper [5] or the more explicit [459] for instance.

Consider now a pants decomposition of C, and the corresponding description of $\mathrm{T}(\mathfrak{g}, C, D)$ as a collection of tinkertoys and vector multiplets gauging some symmetries. Let φ be the scalar in one of the vector multiplets (corresponding to a tube), and consider gauge-invariants such as $\operatorname{Tr}\varphi^l$ in the A-type case, and more generally all Casimirs of $\mathfrak{g}$. Classically, they appear as coefficients of the differentials ϕ_k and can thus be retrieved as certain weighted integrals of ϕ_k. Going back to general ϵ_1, ϵ_2, the *operator* $\operatorname{Tr}\varphi^l$ on the 4d side corresponds to a suitable weighted integral of currents $\tilde{W}_l$ [19].[62] For instance, inserting $\operatorname{Tr}\varphi^2$ takes a derivative of $Z_{S_b^4}$ with respect to gauge couplings [460–462], namely to the shape of C, which indeed translates to an integrated insertion of the holomorphic stress-tensor $T = \tilde{W}_2$.

Correlation functions on S_b^4 with (products of) $\operatorname{Tr}\varphi^j$ inserted at one pole and $\operatorname{Tr}\bar{\varphi}^k$ at the other can be computed by supersymmetric localization, although the operators complicate instanton counting. By a conformal transformation the round case $b = 1$ leads to results on flat space correlators with exactly one antichiral operator [192, 463–467], which provide detailed checks of various field theory ideas such as resurgence [192, 468], large-charge expansions [469–476], and more [477]. These specific correlators have not been pursued on the 2d CFT side of the correspondence.

Orbifold $\mathbb{C}^2/\mathbb{Z}_M$. Next we consider another operation whose effect is to make one point singular inside $\mathbb{R}^4$, or both poles of S_b^4: orbifolding by a group $\mathbb{Z}_M$ acting as $(\exp(2\pi i/M), \exp(-2\pi i/M))$ on $\mathbb{C}^2$. Supersymmetric localization still works: one must simply restrict all modes to $\mathbb{Z}_M$-invariant ones, and instanton counting to $\mathbb{Z}_M$-invariant

[61]The numerical coefficient depends on conventions.

[62]The spin l current $\tilde{W}_l$ is polynomial in the W_k to account for the difference between Casimirs $\operatorname{Tr}\varphi^l$ and coefficients in a characteristic polynomial $\det(x - \varphi)$.

instanton counting. This appears to correspond to the coset CFT

$$\frac{\widehat{\mathfrak{su}}(N)_k \times \widehat{\mathfrak{su}}(N)_M}{\widehat{\mathfrak{su}}(N)_{k+M}} \times \frac{\widehat{\mathfrak{su}}(M)_N}{\widehat{\mathfrak{u}}(1)^{M-1}}, \quad k = -N - \frac{Mb^2}{1+b^2}, \tag{8.2}$$

with a fractional level k, which for $M = 1$ reduces nontrivially to the usual Toda CFT. The case $N = M = 2$, essentially super-Liouville CFT, is studied in [20, 23–25, 27, 29, 30, 32, 478, 479], see also [63] with a surface operator. Instanton counting on $\mathbb{C}^2/\mathbb{Z}_2$ and on its blow up, where one has instantons at two fixed point of an U(1) isometry, are related [230, 233, 480, 481]. This leads to a decomposition of super-Liouville CFT into a Liouville and time-like Liouville pieces [317, 482–484]. The general N, M extension is partially worked out in [22, 26, 28, 31, 34] and the coset (8.2) studied further in [485–487]. Another perspective is to realize $\mathbb{R}^4/\mathbb{Z}_M$ by dimensional reduction of $\mathbb{R}^4 \times S^1$, which corresponds to taking q to a root of unity [21, 33, 35–37] in the q-deformed AGT correspondence of subsection 9.1.

8.2 Line operators

We now place a codimension 4 operator of the 6d theory along $L \times \gamma$, where L is one of the circles $\{y_3 = y_4 = 0, y_5 = \text{const}\}$ or $\{y_1 = y_2 = 0, y_5 = \text{const}\}$ in S^4_b allowed by supersymmetry, while γ is a closed loop in C with no self-intersection. Upon dimensional reduction, the operator inserts loop operators in the AGT correspondence: a loop operator labeled by γ and placed on $L \subset S^4_b$ in the partition function, and a loop operator on γ in the Toda CFT correlator. This setup is studied for $\mathfrak{su}(2)$ in [38–43], for more general $\mathfrak{g}$ in [45–52], for networks of such operators [44, 51–54], and for the algebra of line operators [343, 488, 489], see also [490, 491]. Line and loop operators play an important role in characterizing phases of 4d gauge theories [454–456], and refining the global structure of the gauge group [492]; for class S see [38, 50–52, 493–500]. Exact expectation values of loop operators in 4d $\mathcal{N} = 2$ gauge theories are calculated using supersymmetric localization in [10, 501, 502] and reviewed in [457], with important subtleties being clarified later in [503–508]. Other considerations about line defect observables in 4d theories include [509–515].

Wilson loop operators. Since $\gamma \subset C$ has no self-intersection we can cut C along it and get a (possibly disconnected) surface C' with two additional punctures (with some data, say D_1, D_2). As discussed near (1.2), the corresponding class S theory $\mathrm{T}(\mathfrak{g}, C, D)$ is obtained from the theory $\mathrm{T}\big(\mathfrak{g}, C', \{D, D_1, D_2\}\big)$ corresponding to C' by gauging a diagonal subgroup of the flavour symmetries F_1, F_2 associated to D_1, D_2 as in (1.2). In this way each non-self-intersecting loop γ is associated to a gauge group $G_\gamma = (F_1 \times F_2)_{\text{diag}}$ in some description of $\mathrm{T}(\mathfrak{g}, C, D)$.

In the limit of weak coupling for G_γ, the loop γ wraps a thin tube. The reduction of $\mathcal{X}(\mathfrak{g})$ on this tube gives 5d $\mathcal{N} = 2$ SYM on an interval, and the codimension 4 defect gives a line operator, specifically a Wilson loop, which depends additionally on a choice of representation R of $\mathfrak{g}$. In fact this is the clearest way to see that codimension 4 operators of $\mathcal{X}(\mathfrak{g})$ should depend on such a representation of $\mathfrak{g}$. Further reduction to 4d gives a half-BPS Wilson loop measuring the holonomy of the corresponding gauge field A_γ along L, plus some contribution from scalar superpartners to ensure supersymmetry:

$$W_{\gamma,R} = \mathrm{Tr}_R\Big(\mathrm{Pexp}\int_L (A_\gamma + \mathrm{scalars}_\gamma)\Big). \tag{8.3}$$

The gauge group G_γ and the Wilson loop only depend on the homotopy class of γ.

On the 2d CFT side the corresponding object is a certain 1d topological defect along γ called a degenerate Verlinde loop operator. Verlinde loops are constructed as monodromies of a vertex operator V_ω. The specific choice corresponding to $W_{\gamma,R}$ is to take a momentum $\omega = -b^{\pm 1}\Omega_R$, where $\pm$ depends on the choice of circle L, while Ω_R is the highest weight[63] of the representation R. For this choice of ω, the vertex operator V_ω is degenerate in the sense that it is annihilated by various combinations of W-algebra generators. Incidentally, the most general degenerate momentum $\omega = -b\Omega - b^{-1}\Omega'$ corresponds to inserting Wilson loops along both allowed circles.

Concrete checks of the correspondence are straightforward. The Wilson loop is compatible with supersymmetric localization [10] and inserts a simple a-dependent factor in (1.7). The Verlinde loop $\mathcal{L}_\gamma$ acts diagonally on a complete set of states inserted along γ hence appears in the correlator (1.9) simply as a function of the internal momentum α (related to a). They match:

$$\begin{aligned}\langle W_{\gamma,R}\rangle^{\mathrm{T}(\mathfrak{g},C,D)}_{S^4_b} &= \int \mathrm{d}a\ \mathrm{Tr}_R(e^{a_\gamma})\, Z_{\mathrm{cl}}(a,q,\overline{q})\, Z_{\text{one-loop}}(a) Z_{\mathrm{inst}}(a,q) Z_{\mathrm{inst}}(a,\overline{q}) \\ &= \int \mathrm{d}\alpha\, f(\alpha)\, C(\alpha)\mathcal{F}(\alpha,q)\mathcal{F}(\alpha,\overline{q}) \\ &= \Big\langle \widehat{V}_{\mu_1}\dots\widehat{V}_{\mu_n}\mathcal{L}_\gamma\Big\rangle^{\mathrm{Toda}(\mathfrak{g})}_{\overline{C}}.\end{aligned} \tag{8.4}$$

Dyonic loop operators. Now consider a pants decomposition of C that does not include γ among its cuts. The 2d CFT side is still given by a Verlinde loop along γ, but its expression in the given basis of conformal blocks is no longer diagonal: it is

$$\Big\langle \widehat{V}_{\mu_1}\dots\widehat{V}_{\mu_n}\mathcal{L}_\gamma\Big\rangle^{\mathrm{Toda}(\mathfrak{g})}_{\overline{C}} = \int \mathrm{d}\alpha\, C(\alpha)\mathcal{F}(\alpha,q)\sum_h \mathcal{L}_\gamma(\alpha,\alpha+h)\mathcal{F}(\alpha+h,\overline{q}) \tag{8.5}$$

where h ranges over a finite collection[64] of momenta related to the weights of R.

The corresponding loop operator in 4d is described in this S-duality frame as a 't Hooft or dyonic loop instead of a Wilson loop. Rather than being defined by an insertion in the path integral like the Wilson loop (8.3), a 't Hooft loop on L is defined by imposing a singular boundary condition on the gauge field that prescribes a non-zero monopole charge $\frac{1}{2\pi}\int F$ on a two-sphere S^2 surrounding L. Dyonic loops involve additionally a Wilson loop insertion along the same circle L. The path integral ranges over such singular

[63]Weights are in $\mathfrak{h}^*$, which we identify with the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ using the Killing form.

[64]The fusion of a degenerate operator V_ω with another vertex operator has a finite number of terms.

field configurations instead of the usual smooth ones, and supersymmetric localization still applies [501] and reproduces (8.5), provided one correctly accounts for monopole bubbling [503–508].

Interestingly, Verlinde loops must be generalized to Verlinde networks (involving fusion of degenerate vertex operators) to reproduce half-BPS dyonic loops with the most general electric and magnetic charges [46, 51, 52, 516].

Algebra of line operators. In light of (8.5), dyonic loops or Verlinde networks can be understood as difference operators acting on functions of internal momenta α, or equivalently acting on functions on the Coulomb branch $\mathcal{B}$. Inserting dyonic loops at different latitudes y^5 yields a product of difference operators, which is non-commutative because the loops cannot be reordered while preserving supersymmetry. The OPE of loop operators provides skein relations that express these products as linear combinations of dyonic loops and defines an algebra $\mathfrak{A}_{\epsilon_1,\epsilon_2}$ of loop operators (we recall $\epsilon_1 = b/r$ and $\epsilon_2 = 1/(rb)$). Teschner [309, 517] emphasized early on the importance of this algebra in the AGT correspondence; see the reviews [489, 518]. The earlier work [519] concentrated on 4d $\mathcal{N} = 4$ SYM.

Each pants decomposition of C gives a representation of $\mathfrak{A}_{\epsilon_1,\epsilon_2}$ as difference operators, in which certain loops are diagonalized (8.4) while others remain nondiagonal (8.5). The modular kernels, which relate conformal blocks in different pants decompositions, simply map between eigenbases of various loop operators, and they can be worked out from $\mathfrak{A}_{\epsilon_1,\epsilon_2}$. The eigenbases themselves (instanton partition functions) are then solutions of a Riemann–Hilbert type problem that can be solved. Their $\epsilon_1, \epsilon_2 \to 0$ limit is then the low-energy prepotential (5.30). All of this rich content hidden in $\mathfrak{A}_{\epsilon_1,\epsilon_2}$ led the authors of [309] to call it the "non-perturbative skeleton" of $\mathrm{T}(\mathfrak{g}, C, D)$.

In the Nekrasov–Shatashvili (NS) limit $\epsilon_2 \to 0$ of the algebra $\mathfrak{A}_{\epsilon_1,\epsilon_2}$, the difference operators become differential operators on $\mathcal{B}$, which further become coordinates on a torus fibration over $\mathcal{B}$, in the classical limit $\epsilon_1, \epsilon_2 \to 0$.[65] The geometry simplifies in the NS limit, where S^4_b degenerates to $\mathbb{R}^2 \times S^2_b$. Near the loops along the equator of S^2_b the geometry is $\mathbb{R}^3 \times_b S^1$ with a twisted periodicity around S^1, and the twist is removed in the classical limit. Precisely this geometry was considered in [167, 502, 510, 520–523] (on "framed BPS states"). On untwisted $\mathbb{R}^3 \times S^1$, vacuum expectation values of the loop operators define coordinates on the Coulomb branch $\mathcal{M}$ of the class S theory on $\mathbb{R}^3 \times S^1$, which is the aforementioned torus fibration over $\mathcal{B}$. The twisted periodicity of $\mathbb{R}^3 \times_b S^1$ quantizes this algebra of coordinates into an algebra of differential operators, the NS limit of $\mathfrak{A}_{\epsilon_1,\epsilon_2}$.

As described near (3.14) the $\mathbb{R}^3 \times S^1$ Coulomb branch $\mathcal{M}$ can be seen alternatively as the Hitchin moduli space on C, or as the moduli space of flat $G_\mathbb{C}$ connections on C (where the Lie algebra of $G_\mathbb{C}$ is the complexification of $\mathfrak{g}$). The last point of view fits nicely with the 3d/3d correspondence discussed in subsection 9.3, which involves $G_\mathbb{C}$ Chern–Simons theory. Further works on the Hitchin system, opers, and Darboux coordinates on $\mathcal{M}$ include [50, 343, 356, 366, 415, 488, 524–535] (some are reviewed in [536]); a different

[65] Similar to how momentum $p \sim i\partial_x$ in quantum mechanics is a coordinate in classical mechanics.

technique is based on spectral networks, which abelianize flat connections on C [537–549]; see also [49, 550–552].

8.3 Surface operators

Surface operators are reviewed in [458] (for early references, see [553, 554]). Surface operators compatible with supersymmetric localization on S^4_b can be inserted along two squashed spheres intersecting at the poles or some two-tori, expressed in Cartesian coordinates of an $\mathbb{R}^5$ as follows:

$$\begin{aligned} S^4_b &\coloneqq \{y_5^2 + b^2(y_1^2+y_2^2) + b^{-2}(y_3^2+y_4^2) = r^2\}, \\ S^2_b &\coloneqq \{y_1 = y_2 = 0, y_5^2 + b^{-2}(y_3^2+y_4^2) = r^2\}, \\ S^2_{1/b} &\coloneqq \{y_5^2 + b^2(y_1^2+y_2^2) = r^2, y_3 = y_4 = 0\}, \\ T^2_{\theta,\varphi} &\coloneqq \{y_5 = r\cos\theta, y_1^2+y_2^2 = (rb^{-1}\sin\theta\cos\varphi)^2, y_3^2+y_4^2 = (rb\sin\theta\sin\varphi)^2\}. \end{aligned} \tag{8.6}$$

The latter case has not been studied so we concentrate here on the spheres, on which the surface operators preserve a 2d $\mathcal{N}=(2,2)$ subalgebra of 4d $\mathcal{N}=2$. Such operators arise either from a codimension 4 operator of $\mathcal{X}(\mathfrak{g})$ at a point $z\in C$ or from a codimension 2 operator wrapping all of C. (See also [516] on foams of surface operators, [555, 556] on duality defects, [557] for a holographic approach.) The space of couplings of the first type of surface operators is exactly the UV curve C, thus it gives a definition of C directly from the 4d $\mathcal{N}=2$ theory.

Vortex string operators. We begin with surface operators arising from a codimension 4 operator of the 6d theory placed at a point $z\in C$ and one of the two possible spheres S^2_b (sign "+" below) or $S^2_{1/b}$ (sign "−" below). This class of operators is sometimes called M2-brane surface operators because it arises from the addition of M2 branes in the M-theory construction of class S theories. They can also arise via Higgsing a larger 4d theory if the Higgsed field has a non-trivial space-dependent profile [558–560]. Their AGT interpretation is explored in [39, 55–63, 66–71, 561, 562], in part based on their exact partition functions, studied in [69, 563–576]. The same operators are important in the 5d version and $S^1\times S^3_b$ version of the correspondence [64, 65, 190, 570, 571, 577–580]; see also [537, 539, 581–590] for other considerations on this class of surface operators.

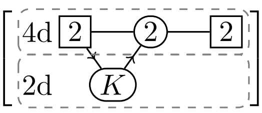
As we have learned from studying loops, codimension 4 operators carry a choice of representation R of $\mathfrak{g}$. On the 2d CFT side we thus want a point operator labeled by R: the natural guess is a degenerate vertex operator V_ω with $\omega = -b^{\pm1}\Omega_R$, the sign $\pm$ being determined by which squashed two-sphere we use on the 4d side. This suggests an equality

$$\langle\text{surface operator}\rangle^{\mathrm{T}(\mathfrak{g},C,D)}_{S^4_b} = \Big\langle \widehat{V}_{\mu_1}\dots\widehat{V}_{\mu_n} V_{-b^{\pm1}\Omega_R}\Big\rangle^{\mathrm{Toda}(\mathfrak{g})}_{\overline{C}}. \tag{8.7}$$

The right-hand side can be written as an analytic continuation of an $(n+1)$-point correlator $\langle\widehat{V}_{\mu_1}\dots\widehat{V}_{\mu_{n+1}}\rangle$ of non-degenerate vertex operators. The analytic continuation in the corresponding class S theory T' was first understood in [558] in Lagrangian cases[66]: it amounts to considering a supersymmetric "vortex string" configuration in T' in which certain hypermultiplet scalars acquire space-dependent VEVs concentrated in codimension 2. In the low-energy limit, the non-zero scalars Higgs some gauge symmetries of T' down to those of T, and the configuration is effectively described by a surface operator in the theory T.

Description as a 4d-2d coupled system. Besides this vortex string construction of surface operators obtained from codimension 4 operators of $\mathcal{X}(\mathfrak{g})$, these surface operators can be described by coupling to the 4d theory a 2d $\mathcal{N} = (2,2)$ gauge theory living on the defect. In this context the left-hand side of (8.7) is the partition function of the 4d-2d coupled system on squashed spheres. A simple example is that of SU(2) SQCD with $N_f = 4$ and a defect labeled by the K-th symmetric representation. The 2d theory then consists of chiral multiplets in doublet representations of 4d flavour and gauge groups, and in fundamental and antifundamental representations of a 2d U(K) gauge group:

$$Z_{S_b^2 \subset S_b^4} \left[\text{quiver} \right] = \left\langle \widehat{V}_{\mu_1} \widehat{V}_{\mu_2} \widehat{V}_{\mu_3} \widehat{V}_{\mu_4} V_{-Kb/2} \right\rangle_{S^2}^{\text{Liouville}}. \tag{8.8}$$



The position z of $V_{-Kb/2}$ matches the Fayet–Iliopoulos (FI) parameter of the 2d U(K) gauge group. Such a 2d description of the most general R in $\mathfrak{su}(N)$ Lagrangian theories is proposed in [68] and checked by comparing limits $z \to z_i$ in gauge theory to the known OPE of $V_{-b^{\pm 1}\Omega_R}$ and $\widehat{V}_{\mu_i}$. More general degenerate insertions $V_{-b\Omega-\Omega'/b}$ translate to intersecting defects with extra 0d fields living at the poles where the defects intersect [69]. An important difficulty in checking equalities like (8.8) is to compute contributions $Z_{\text{inst,vort}}$ from the poles of S_b^4, which involve both instantons of the 4d theory and vortices of the 2d theory [574]. Incidentally, in an $\epsilon_1, \epsilon_2 \to 0$ limit this 4d-2d analogue of Nekrasov's partition function gives both the 4d theory's effective prepotential F and the 2d theory's effective twisted superpotential $\mathcal{W}$, obtained earlier in [55, 565]:

$$\log Z_{\text{inst,vort}} = \frac{F}{\epsilon_1 \epsilon_2} + \frac{\mathcal{W}}{\epsilon_1} + O(1). \tag{8.9}$$

Gukov–Witten operators: monodromy defects and orbifolds. Next we discuss surface operators called M5-brane surface operators, or codimension 2 operators, or orbifold surface defects, studied in [72–85, 120, 318, 358, 362, 553, 570, 576, 591–598] and in [359, 360] from the little string theory viewpoint. We have already encountered codimension 2 defects of $\mathcal{X}(\mathfrak{g})$, since they are the origin of tame punctures that impose certain boundary conditions on the differentials ϕ_k. Wrapping these codimension 2 operators on C thus gives surface operators that impose certain singular boundary conditions on the 4d fields. Specifically, this yields $\mathcal{N} = 2$ versions of Gukov–Witten (GW) surface operators [553], which impose that 4d gauge fields A behave as $A \sim \alpha d\theta$ as $r \to 0$

[66] It would be good to clarify the situation for the most general class S theories.

with a prescribed $\alpha \in \mathfrak{t}$ in the Cartan algebra of $\mathfrak{g}$, where (r, θ) are polar coordinates transverse to the defect.

If A is an $\mathrm{SU}(N)$ gauge field, say, let us denote eigenvalues of $\alpha = \mathrm{diag}(\alpha_1, \dots)$ as α_i with multiplicities N_i, $i = 1, \dots, M$ so that $\sum_i N_i \alpha_i = 0$ and $\sum_i N_i = N$. Then the 4d gauge group breaks to $(\prod_i \mathrm{U}(N_i))/\mathrm{U}(1)$ at the defect. The instanton moduli space with such a monodromy defect is equivalent as a complex manifold to the moduli space of instantons on an orbifold $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$ [599]. Here, $\mathbb{Z}_M$ embeds into both rotations of $\mathbb{C}$ with charge $+1$ and the gauge group $\mathrm{SU}(N)$ with charges i with multiplicity N_i, thus reproduces the expected symmetry breaking. The Nekrasov partition function Z_{inst} is obtained from the usual one by restricting to $\mathbb{Z}_M$-invariant instantons. It matches conformal blocks of the affine $\mathrm{SL}(N)$ algebra (for the full defect that has all $N_i = 1$) [72, 73], or its Drinfeld–Sokolov (DS) reductions [74] more generally.

The GW defects can also be described by coupling suitable 2d $\mathcal{N} = (2, 2)$ gauge theories to the 4d theory. For pure 4d $\mathcal{N} = 2$ SYM,

$$Z_{S_b^2 \subset S_b^4}\left[\, \text{4d}\,\textcircled{N} \leftarrow \textcircled{K_{n-1}} \leftarrow \cdots \leftarrow \textcircled{K_1}\,\text{2d} \,\right] = \left\langle \widehat{V}_{\mu_1} \widehat{V}_{\mu_2} \widehat{V}_{\mu_3} \widehat{V}_{\mu_4} \right\rangle_{S^2}^{\text{generalization of Toda CFT}} \tag{8.10}$$

where $K_i = N_1 + \dots + N_i$. While the symmetry algebra is understood, the relevant (non-chiral) CFTs generalizing Toda CFT is not.

Interestingly, conformal blocks of the affine $\mathrm{SL}(2)$ algebra are related to conformal blocks of the Virasoro algebra with additional degenerate vertex operators, as pointed out early on in an AGT setting in [600]. This, and its $N > 2$ analogues, leads to some identifications between the two types of surface operators up to a suitable integral kernel [532, 601].

One should be careful in reading some early 2010's literature on surface operators in the AGT context, as the two types of surface operators are hard to distinguish in the $\mathfrak{su}(2)$ case. The "codimension 2" orbifold $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$ considered here should also not be confused with the "codimension 4" orbifold $\mathbb{C}^2/\mathbb{Z}_M$ discussed on page 83, for which $\mathbb{Z}_M$ rotates both factors. Just as for the $\mathbb{C}^2/\mathbb{Z}_M$ orbifold, the $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_M)$ orbifold ought to arise as a limit of a 5d gauge theory on $\mathbb{C}^2 \times_q S^1$ for a suitable root of unity limit of q, t.

8.4 Domain walls

The AGT correspondence also allows for half-BPS 3d operators that separate the 4d spacetime into two parts or give it a boundary. A good starting point is [43].

Symmetry-breaking wall. A first construction [43] is to place a (tame) codimension 2 defect of $\mathcal{X}(\mathfrak{g})$ on the equator of S_b^4, times a closed loop $\gamma \subset C$. The gauge theory description is understood in terms of the gauge group G_γ defined on page 84. The boundary conditions for the 4d vector multiplet are similar to those near a GW surface defect, and they define a symmetry-breaking wall where gauge symmetries reduce to a subgroup $H \subset G_\gamma$. On the CFT side the situation is similar to loop operators, and one gets the Verlinde loop on γ constructed by inserting the vertex operator V_α associated to the defect and moving it around γ.

The continuous parameters of α (of the codimension 2 defect) are FI parameters of the unbroken gauge symmetry H on the wall. General vertex operators can also include a discrete part, and correspondingly tame codimension 2 defects that are not full can be dressed with additional codimension 4 defects living on their world-volume. In the present construction this leads to Wilson loop operators in representations R_1 and R_2 of H on the two circles of subsection 8.2. These loops are stuck on the domain wall unless R_1 (resp. R_2) are representations of G itself.

Janus wall. Our second construction does not involve codimension 2 or 4 operators. Instead, we place $\mathcal{X}(\mathfrak{g})$ on $S_b^4 \times C$ with the complex structure of C varying with the latitude of S_b^4 [43]. This preserves half of the supersymmetry and in the limit where the variation happens sharply at the equator (or a parallel) we get a so-called Janus domain wall [602–607] in the 4d theory. This is a half-BPS interface between class S theories with different gauge couplings. The partition function with this interface has the usual factorized form (1.7) with holomorphic and antiholomorphic contributions from the poles, but the gauge couplings used in each factor are not complex conjugates. Correspondingly, the CFT correlator (1.9) changes to using different complex structures for the holomorphic and antiholomorphic factors.

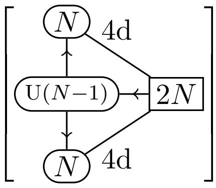
S-duality wall. Tuning gauge couplings we can get theories that are S-dual. By switching to the same S-duality frame on both sides we get a 3d operator called the S-duality wall [608] that has the same theory (and same gauge couplings) on both sides. Inserting an S-duality wall in a 4d theory amounts on the 2d side to performing a modular transformation (fusion, braiding, S-move) on holomorphic (or antiholomorphic) conformal blocks. This is related to special cases [87, 89, 609, 610] of the 3d/3d correspondence we discuss in subsection 9.3. The S-duality wall has an interplay with loop operators: it translates in a suitable sense from Wilson loops on one side of the wall to 't Hooft loops on the other side.

For instance, the S-duality wall of 4d $\mathcal{N} = 4$ SYM is equivalent to coupling a 3d $\mathcal{N} = 4$ theory $T[G]$ to SYM on both sides of the wall [608], and the S-move kernel is expected [86] to match the S_b^3 partition function of $T[G]$. For $\mathfrak{g} = \mathfrak{su}(N)$, the wall theory is a 3d $\mathcal{N} = 4$ linear quiver,

$$Z_{S_b^3}\Big[\boxed{N}\!-\!\!(N\!-\!1)\!-\cdots-\!(1)\Big] = (W_N \text{ algebra S-move kernel}). \tag{8.11}$$

The known braiding kernel [611, 612] for Virasoro four-point conformal blocks led to a description [88] of the S-duality wall of 4d $\mathcal{N} = 2$ SU(2) SQCD with $N_f = 4$, as 3d $\mathcal{N} = 2$ SU(2) SQCD with $N_f = 6$, suitably coupled to the 4d theories on both sides.

Bootstrapping the braiding kernel of W_N four-point conformal blocks led to a description [90] for the S-duality wall of SU(N) SQCD as U($N-1$) SQCD (with a 3d monopole superpotential understood in [613]), coupled by cubic superpotentials to the 4d theories on both sides of the wall,

Z  $= (W_N \text{ algebra braiding kernel}).$ (8.12)

Just like the $T[G]$ theories, this 3d $\mathcal{N}=2$ theory (in isolation, after decoupling the 4d fields) admits numerous dualities [614]. Duality walls of 5d theories were studied in [615, 616].

Boundary CFT. We mentioned orbifolds earlier. Instead of orbifolding the 4d space one can orbifold by a $\mathbb{Z}_2$ symmetry that acts as a reflection with respect to the equator of S_b^4 and a reflection on the Riemann surface (so as to preserve chirality of $\mathcal{X}(\mathfrak{g})$). This leads to an AGT correspondence for Riemann surfaces with boundaries and for non-orientable surfaces [91, 92]. On the gauge theory side, one obtains simultaneously some gauge fields defined on the squashed hemisphere HS_b^4 and others on the squashed projective space $\mathbb{RP}_b^4$. The hemisphere partition function was evaluated in [617] while the projective space partition function is obtained in [91] by the gluing technique explored further in [249, 250].

The inclusion of boundary operators has not been fully elucidated. It would also be interesting to go beyond the $\mathfrak{g}=\mathfrak{su}(N)$ case treated so far, for instance understanding how the double-cover construction discussed below (5.38) interacts with the $\mathbb{Z}_2$ quotient that produces the Riemann surface boundaries.

9 Other geometries

AGT correspondence in some other geometries

- For 5d $\mathcal{N}=1$ lifts of class S theories, Z_{inst} and Z_{S^5} match q-deformed Toda conformal blocks and correlators. A lift to 6d matches (q,t)-deformed Toda theory.
- The $S^3\times S^1$ partition function (superconformal index) of $\mathrm{T}(\mathfrak{g},C,D)$ matches a TQFT correlator on C.
- Twisted reductions of $\mathcal{X}(\mathfrak{g})$ on C_3 are 3d $\mathcal{N}=2$ theories whose $Z_{S^2\times S^1}$, Z_{S^3} or lens space partition functions match complex $\mathfrak{g}_{\mathbb{C}}$ Chern–Simons theory on C_3 at level 0, 1 or k.
- The 6d $(1,0)$ SCFT of M5 branes probing a $\mathbb{Z}_k$ singularity defines class S_k 4d $\mathcal{N}=1$ theories similar to class S.

Class S theories $\mathrm{T}(\mathfrak{g},C,D)$ are obtained by compactifying the 6d $(2,0)$ theory of type $\mathfrak{g}$ on $M_4\times C$ with C a Riemann surface with punctures which extra data D. So far we have extensively discussed the case $M_4=S_b^4$ and its building block $M_4=\mathbb{R}^4_{\epsilon_1,\epsilon_2}$, for which the partition function is equal to a 2d CFT correlator or conformal block, respectively.

We first discuss 5d lifts of these 4d observables to (deformations of) $\mathbb{R}^4\times S^1$, $S^4\times S^1$, and S^5, which are connected to q-deformations[67] (subsection 9.1). We then change the geometry, first relating the

[67] The parameter q appearing in the 5d lift is unrelated to the gauge couplings parameters describing the complex structure of C. We will actually not need a notation for this gauge coupling any longer.

supersymmetric index, which is the partition function on $M_4 = S^3 \times S^1$, to (a generalization of) a q-Yang–Mills TQFT correlator (subsection 9.2), then compactifying instead on products $M_3 \times C_3$ and $M_2 \times C_4$ in which the "internal" manifold C is a hyperbolic three-manifold (subsection 9.3) or a four-manifold (subsection 9.4). In this last subsection we also mention generalizations with less supersymmetry and a few geometric setups that have been less fruitful.

9.1 Lift to 5d and q-Toda

Here we briefly survey how lifting the 4d $\mathcal{N} = 2$ theories to 5d $\mathcal{N} = 1$ amounts to a q-deformation of the 2d theories. For a review, see [151].[68]

Instanton partition functions. The 4d $\mathcal{N} = 2$ Omega background used to define Nekrasov's instanton partition function [12, 13] is conveniently expressed in terms of a 5d $\mathcal{N} = 1$ lift: placing the theory on $\mathbb{R}^4 \times S^1$ with twisted boundary conditions around S^1 such that $\mathbb{R}^4$ rotates by q and t in two two-planes, and with an additional twist by an R-symmetry to preserve some supersymmetry. More precisely, this definition for $|q| = |t| = 1$ can be extended to complex q, t by turning on additional supergravity fields. The 5d lift deforms all factors in Z_{inst} from rational functions to trigonometric functions of masses and Coulomb branch parameters. It is natural to ask how the Toda CFT side of the AGT correspondence can be deformed to accomodate for this.

One finds that the 5d (also called K-theoretic) Z_{inst} is a chiral block for a q-deformed W-algebra [35, 113, 229, 340, 341, 618–620]: the relevant deformations of the Virasoro algebra and of W-algebras were constructed long ago [621–624] (see [625] for a modern construction).[69] When mass and Coulomb branch parameters are suitably quantized the equality can be proven using Dotsenko–Fateev integral representations of qW_N conformal blocks [67, 304, 307, 342, 626] (also used in [627]). See also [272, 628].

The 5d $\mathcal{N} = 1$ quiver gauge theories admit realizations in terms of webs of (p, q) fivebranes in IIB string theory [7]. Applying S-duality exchanges the role of D5 and NS5 branes, thus equating Z_{inst} for a $\mathrm{SU}(N)^{M-1}$ linear quiver gauge theory to an $\mathrm{SU}(M)^{N-1}$ one (see [629] for a proof for $M = N = 2$). This 5d spectral duality (also called fiber-base duality [630]) relates in general chiral blocks of different qW_N theories [631, 632], it implies certain instances of 3d mirror symmetry [585], and relations between spin chains [633]. When reading the literature, one should keep in mind which of the two spectral duality frames is adopted, see [634] for a nice explanation.

The 4d case is retrieved as the limit $q \to 1$ with $t = q^{-\beta}$ and fixed $-\beta = b^2 = \epsilon_1/\epsilon_2$ giving the 4d deformation parameters. Other interesting limits than $q, t \to 1$ exist, especially taking q and t a k-th roots of unity one obtains Nekrasov partition functions on $\mathbb{C}^2/\mathbb{Z}_k$ ALE space studied in [29, 33, 35–37, 85, 635]. Another simplifying limit is the Hall-Littlewood limit $q \to 0$ [281, 282].

[68] I thank Fabrizio Nieri for answering my questions thoroughly.

[69] As in 4d, working with $\mathrm{U}(N)$ rather than $\mathrm{SU}(N)$ gauge groups makes Z_{inst} more tractable; correspondingly one works with the Ding–Iohara algebra, a slight extension of (the universal envelopping algebra of) the qW-algebra of type $\mathfrak{g}$ [267].

An unrelated application of Z_{inst} and qW_N conformal blocks is to construct solutions of q-Painlevé equations [636–639], as in the 4d case.

Compact partition functions. Let us now glue instanton partition functions together. While the partition function on S^4 involves a pair of instanton contributions from the two fixed point of the supercharge squared, the partition function of 5d $\mathcal{N}=1$ theories on S^5 combines three K-theoretic instanton partition functions because the supercharge has three fixed circles [640]. Schematically,

$$Z_{S^5} = \int \mathrm{d}a\, Z_{\text{pert}} Z_{\text{inst},1} Z_{\text{inst},2} Z_{\text{inst},3}. \tag{9.1}$$

The squashed S^5 has three axis lengths $\omega_1, \omega_2, \omega_3$ and here the different $Z_{\text{inst},i}$ are computed in the Ω background with parameters (q,t) given by $(\frac{\omega_2}{\omega_1}, \frac{\omega_3}{\omega_1})$, $(\frac{\omega_1}{\omega_2}, \frac{\omega_3}{\omega_2})$, $(\frac{\omega_1}{\omega_3}, \frac{\omega_2}{\omega_3})$, respectively.

The picture that emerges [64, 339, 577] is that there exists a q-deformed version of Toda CFT, called q-Toda theory,[70] that has qW_N symmetry and whose correlators should match with S^5 partition functions. The fact that three chiral factors need to be combined leads to a remarkable "modular triple" of q-Virasoro algebras [114] (for $N=2$), similar to the modular double combining $U_q(\mathfrak{sl}(2))$ with $q = e^{2\pi i b^2}$ and $q = e^{2\pi i/b^2}$ in 2d CFT. A non-local Lagrangian for q-Liouville is proposed in [114].

Half-BPS operators with 3d $\mathcal{N}=2$ supersymmetry played an important early role right from the start. On the squashed S^5 they can be inserted along three distinct S^3 that intersect pairwise along S^1. The first explorations of the correspondence for Z_{S^5} concerned the case of a single 3d operator in a simple 5d bulk theory, which can be obtained by Higgsing a larger 5d theory [64, 577, 641]. Just as the analogous surface operators in the standard AGT correspondence, these 3d operators correspond to degenerate q-Toda CFT vertex operators. They are useful to bootstrap structure constants of q-Toda, and show up in a Higgs branch localization expression of the instanton and S^5 partition functions [189, 190, 642], again completely analogous to the 4d story [188, 571], albeit more technically involved. A mathematical take on this is in [284].

Codimension 4 operators of the 5d theory, specifically Wilson loops, are studied in [643]; they translate in q-Toda to stress tensor and higher-spin operator insertions.

Geometric setup. The correspondence is partially understood geometrically through 6d $(2,0)$ little string theories. Contrarily to the SCFTs $\mathcal{X}(\mathfrak{g})$, little string theories can only be reduced on zero-curvature surfaces, so the choice of Riemann surface and punctures is restricted. When reducing on a cylinder with full punctures at the two ends, one would expect a 4d reduction but the string winding modes give instead a 5d $\mathcal{N}=1$ theory on the T-dual circle times $\mathbb{R}^4$. The correct limits to reproduce the 4d/2d AGT correspondence and a dual version were discussed in [644, 645].

[70] One should be careful that many papers talk about q-Liouville or q-Toda theory even when they only consider chiral blocks, which only involve the q-Virasoro and qW_N symmetry algebras.

It is not clear at this point what q-Toda theory really is, in particular whether it is fundamentally a 2d theory that can only be placed on zero-curvature surfaces, as suggested by the little string construction, or whether it should be thought as a 1d theory in order to obtain an effective Lagrangian [114].

Elliptic lift. Lifting one dimension up, 6d partition functions on $\mathbb{R}^4 \times T^2$ (twisted) and $S^5 \times S^1$ (superconformal index) [646] are related to the elliptic deformation (q,t)-Toda: see [272, 306, 647–660].

9.2 Superconformal index and 2d q-YM

We now move on to partition functions of 4d $\mathcal{N}=2$ class S theories on $M_4 = S^3 \times S^1$.

Supersymmetric index. See the review [661] for superconformal class S theories and [662] for general 4d $\mathcal{N}=1$ theories. We shall not write too much here, but we rather point to another course in this school [663]. The AGT relation to q-YM is also surveyed briefly in [118].

The $S^3 \times S^1$ partition function is defined and computable for 4d $\mathcal{N}=1$ theories with an anomaly-free U(1) R-symmetry. Up to a factor involving the Casimir energy of the theory, expressible in terms of a and c anomalies, the partition function coincides with the supersymmetric index, defined to be the Witten index of the theory quantized on $S^3 \times \mathbb{R}$. Once refined by fugacities u_i for mutually commuting rotations, flavour, and R-symmetries (with charges K_i), the index is written as

$$\mathcal{I}(u) = \operatorname{Tr}\Big[(-1)^F e^{-\beta\tilde{H}} \prod_i u_i^{K_i}\Big] \tag{9.2}$$

where $(-1)^F$ counts bosonic and fermionic states with opposite signs, $\tilde{H} = \{Q, Q^\dagger\}$ for some supercharge Q, and $\mathcal{I}$ is β-independent thanks to cancellations between bosonic and fermionic states when $\tilde{H} \neq 0$. This simplification means that $\mathcal{I}(u)$ counts (with signs) short representations of the supersymmetry algebra. Fugacities u_i are encoded in the $S^3 \times S^1$ partition function as holonomies around the S^1 for background gauge fields coupled to the given symmetry: in particular, fugacities (p,q) for two combinations of rotations and R-symmetries can be understood as a non-trivial fibration of S^3 over S^1.

The index formally does not depend on any continuous parameter beyond these:[71] it is an renormalization group (RG) flow invariant and is independent of gauge couplings for instance, thus can be easily computed in any weakly-coupled Lagrangian description. In this way $\mathcal{I}(u)$ reduces to a simple signed count of local gauge-invariant operators built from the elementary fields in any given Lagrangian description (in other dimensions nonperturbative objects must be included). Being an eminently computable RG flow invariant makes the index a powerful window into nonperturbative physics of 4d $\mathcal{N}=1$ gauge theories, especially their IR dualities.

[71] In some contexts, this formal invariance of the index fails, which gives wall-crossing phenomena.

Computing the index is much harder if we have no Lagrangian description, but part of the structure remains: if a theory T is defined by gauging a common flavour symmetry G of two theories T_1, T_2, then the indices are related schematically as

$$\mathcal{I}[T](a_1, a_2) = \int [\mathrm{d}z]_G \, \mathcal{I}_{\mathrm{vec}}(z) \, \mathcal{I}[T_1](a_1, z) \, \mathcal{I}[T_2](a_2, z), \tag{9.3}$$

where we hid the p, q dependence but kept explicit the fugacities a_1 and a_2 for flavour symmetries of T_1 and T_2 commuting with G, which become flavour symmetries of T. The integral over the fugacity z for the symmetry G is done with a suitable measure $\mathcal{I}_{\mathrm{vec}}$, which from the localization point of view is the vector multiplet one-loop determinant. In fact, (9.3) gives a way to compute the index of a non-Lagrangian theory: embed it into a larger theory that is dual to a Lagrangian gauge theory, whose index is computable [664].

Class S. We now specialize to class S theories, and specifically to superconformal ones. Since the index cannot depend on gauge couplings, it only depends on the topology of the Riemann surface C and the type of punctures. Thus, compared to the standard AGT correspondence, the 2d CFT side should be replaced by a TQFT, as worked out in [665]. Consider the theory $\mathrm{T}(\mathfrak{g}, C, D)$. A flavour symmetry is associated to each puncture z_i, $i = 1, \dots, n$, and we turn on corresponding fugacities a_i. For any pants decomposition of C we can express $\mathrm{T}(\mathfrak{g}, C, D)$ as the result of gauging flavour symmetries of a collection of tinkertoys (isolated SCFTs) associated to three-punctured spheres. Through (9.3), the index then reduces to an integral of superconformal indices of tinkertoys. This precisely mimics the structure of correlators in a TQFT:

$$\mathcal{I}[\mathrm{T}(\mathfrak{g}, C, D)](a_i) = \langle \mathcal{O}_{D_1}(a_1) \dots \mathcal{O}_{D_n}(a_n) \rangle_{\text{some TQFT}} \tag{9.4}$$

for suitable operators $\mathcal{O}_D$ that depend on the type of puncture.

In analogy to Liouville CFT bootstrap, which relied on using degenerate vertex operators that correspond in gauge theory to surface operators, one can bootstrap the index of all tinkertoy building blocks using surface operators [558]. Adding a surface operator to the index corresponds to acting with a difference operator Θ on fugacities associated to any one of the punctures, and by topological invariance it does not matter which puncture. Expressing the result in an eigenbasis of Θ labeled by representations λ of $\mathfrak{g}$ eventually gives

$$\mathcal{I}[\mathrm{T}(\mathfrak{g}, C, D)](a_i) = \sum_{\lambda} (C_\lambda)^{2g-2} \phi_\lambda^{D_1}(a_1) \dots \phi_\lambda^{D_n}(a_n) \tag{9.5}$$

for some structure constants $C_\lambda(p, q, t)$ and wave functions $\phi_\lambda^D(p, q, t; a)$.[72] Wave functions for arbitrary punctures are related to those for full punctures by taking suitable residues in flavour fugacities [666]. The sum may diverge if there are too few punctures or if they are too "small", signalling either that the given class S theory does not exist or that the index is not sufficiently refined because there are additional flavour symmetries not associated to any of the punctures.

[72] Here we introduced a fugacity t for the additional R-symmetry of 4d $\mathcal{N} = 2$ theories.

The wave functions can be computed order by order in p, q, t, but are not known in closed form. In the Schur limit $q = t$ correlators are functions of q only (p-dependent terms are Q-exact), wavefunctions are proportional to Schur polynomials, and the corresponding TQFT is q-deformed 2d Yang–Mills theory [110]. In the more general Macdonald limit $p = 0$, wavefunctions are essentially Macdonald polynomials in q, t and the TQFT must be deformed by changing the measure in the path integral of q-YM theory [667]. The Hall-Littlewood limit $p = q = 0$ turns Macdonald polynomials to the easier Hall-Littlewood polynomials, which only depend on t. The Coulomb limit ($t, p \to 0$ with fixed p/t and q) is also interesting. The 2d TQFT description, which remains quite implicit for general p, q, t, was derived starting from the 6d theory $\mathcal{X}(\mathfrak{g})$ in [668] (earlier derivations in [669–671] did not account for instanton corrections).

The correspondence is tested and extended in natural ways: inserting Wilson–'t Hooft loops at the poles of S^3 and correspondingly loops in q-YM [53, 54, 672, 673], inserting general surface operators [65, 567, 578, 590, 674], replacing S^3 by the Lens space $L(p, 1) = S^3/\mathbb{Z}_p$ [675–677], taking C to have non-zero area [678], generalizing to D-type gauge groups and non-simply-laced ones (using outer automorphism twists) [679, 680]. The relation with Hilbert series of instanton moduli spaces is explored in [216, 226]. The superconformal index of many AD type theories is also known by now in the Macdonald limit [681–685]. The key open question in this direction seems to be getting a handle on the full parameter space (p, q, t) rather than its $p = 0$ Macdonald slice.

9.3 3d/3d correspondence

So far we have reduced the 6d $(2, 0)$ theory of type $\mathfrak{g}$ with a partial topological twist along a Riemann surface. Reducing it instead on $M_3 \times C_3$, with a twist along a three-manifold C_3, gives a 3d $\mathcal{N} = 2$ gauge theory on M_3. One can add codimension 2 operators of the 6d theory to get analogues of punctures: boundary conditions along knots $K_1 \subset C_3$ (which we leave implicit in our notation). This defines a large class of 3d $\mathcal{N} = 2$ gauge theories[73] $\mathrm{T}(\mathfrak{g}, C_3, K_1)$. Their supersymmetric partition functions match $K_1 \subset C_3$ partition functions of complexified Chern–Simons theory. See [150, 686, 687] for reviews and [89, 111, 310, 316, 525, 527, 609, 610, 688–727] for works on this correspondence and its applications.

Natural building blocks for $C_3 \setminus K_1$ are tetrahedra, and each triangulation of $C_3 \setminus K_1$ yields an explicit gauge theory description of the 3d $\mathcal{N} = 2$ theory as the IR limit of an abelian Chern–Simons theory (and deformations by masses or other parameters). More precisely, this description misses parts of the theory $\mathrm{T}(\mathfrak{g}, C_3, K_1)$, as pointed out in [698, 699], and these additional branches are still under investigation (see e.g. [725]). Similarly to how changing pants decompositions in the 4d/2d correspondence amounts to S-duality, Pachner's 2-3 move for triangulated 3-manifolds, which trades two neighboring tetrahedra for three tetrahedra covering the same part of the manifold, yields 3d $\mathcal{N} = 2$ dualities. Contrarily to S-duality, these are not all-scale dualities but only IR dualities.

[73]This is not a standard notation; sometimes $\mathrm{T}(\mathfrak{su}(N), C_3, K_1)$ is denoted $T_N[C_3 \setminus K_1]$.

Statement of the correspondence. The 3d/3d AGT correspondence was formulated in [111, 525], after several papers treating less general geometries: either reducing $T(\mathfrak{g}, C_3, K_1)$ further on S^1 [728] (getting a 2d $\mathcal{N} = (2, 2)$ theory), or taking C_3 to be a mapping cylinder or torus (Riemann surface fibered over an interval or circle) [87, 610, 729, 730]. The partition function of $T(\mathfrak{g}, C_3, K_1)$ on certain manifolds M_3 is equal to the partition function on $C_3 \setminus K_1$ of Chern–Simons theory with a gauge group $G_{\mathbb{C}}$ whose Lie algebra is the complexification of $\mathfrak{g}$. This, in turn, provides invariants of knots and of 3-manifolds. Complex Chern–Simons theory depends on levels (k, σ), one quantized $k \in \mathbb{Z}$ and one continuous $\sigma \in \mathbb{R} \cup i\mathbb{R}$, related to the choice of M_3. Its action is straightforward,

$$S = \frac{k + i\sigma}{8\pi} \int_{C_3} \mathrm{Tr}\Big(\mathcal{A} \wedge \mathrm{d}\mathcal{A} + \tfrac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big) + \frac{k - i\sigma}{8\pi} \int_{C_3} \mathrm{Tr}\Big(\overline{\mathcal{A}} \wedge \mathrm{d}\overline{\mathcal{A}} + \tfrac{2}{3}\overline{\mathcal{A}} \wedge \overline{\mathcal{A}} \wedge \overline{\mathcal{A}}\Big), \tag{9.6}$$

where $\mathcal{A} = A + i\Phi$ is a complex gauge field. However, defining Chern–Simons theory completely is subtle when the gauge group is noncompact [731], and in fact the 3d/3d correspondence helps define it for complex gauge group $G_{\mathbb{C}}$ [150, 316, 732, 733]. See also [734, 735] for a few applications of complexified Chern–Simons theory.

The squashed sphere partition function ($M_3 = S^3_b$) corresponds to Chern–Simons at level $k = 1$ [111, 310]. The supersymmetric index ($M_3 = S^2 \times_q S^1$) corresponds to Chern–Simons at level $k = 0$ [311, 312, 689].[74] The partition function on a squashed lens space $M_3 = L(k, 1)_b$ corresponds to a general Chern–Simons level k [316, 732]. (These supersymmetric partition functions and more are reviewed in [736].)

$$\begin{aligned} Z_{S^2 \times_q S^1}[T(\mathfrak{g}, C_3, K_1)] &= Z_{C_3}\big[G_{\mathbb{C}} \text{ at levels } (0, \sigma)\big], \ q = e^{2\pi/\sigma} \\ Z_{S^3_b}[T(\mathfrak{g}, C_3, K_1)] &= Z_{C_3}\big[G_{\mathbb{C}} \text{ at levels } (1, \tfrac{1-b^2}{1+b^2})\big] \\ Z_{L(k,1)_b}[T(\mathfrak{g}, C_3, K_1)] &= Z_{C_3}\big[G_{\mathbb{C}} \text{ at levels } (k, k\tfrac{1-b^2}{1+b^2})\big] \end{aligned} \tag{9.7}$$

All three can be decomposed into holomorphic blocks [691, 700, 737], which are partition functions on the Omega background $\mathbb{R}^2 \times_q S^1$ or equivalently the cigar $D^2 \times_q S^1$, and are wave functions on the Chern–Simons side. A semi-classical version of this is that the set of supersymmetric vacua on $\mathbb{R}^2 \times S^1$ matches the space of flat $G_{\mathbb{C}}$ connections on $C_3 \setminus K_1$ with suitable boundary conditions along K_1.

Boundaries and generalizations. When C_3 has 2d boundaries (on which the knot K_1 can end), the 3d $\mathcal{N} = 2$ theory lives at the boundary of (and is coupled to) the 4d $\mathcal{N} = 2$ class S theory associated to ∂C_3 [43, 89, 111, 673]. In particular, when C_3 is a cobordism, namely ∂C_3 consists of two disconnected components, it is more natural to think of the 3d $\mathcal{N} = 2$ theory as a domain wall between the two corresponding 4d $\mathcal{N} = 2$ class S theories which are only coupled through their common 3d boundary. The construction is thus functorial with respect to gluing. One particularly simple example of the setup was described in [43]: consider $C_3 = \mathbb{R} \times C$ where the complex structure of C varies along the $\mathbb{R}$ direction; then on the gauge theory side we have the 4d $\mathcal{N} = 2$ theory

[74] I don't know if the differences between [311, 312] have been resolved.

$\mathrm{T}(\mathfrak{g}, C)$ with a Janus domain wall defined by varying the 4d gauge couplings along one direction. Further works in this direction include [698, 711].

As known since Witten's [738], many knot invariants can be expressed as partition functions or other observables of gauge theories. For a sample of references, see [692, 739–746] and the review [747], as well as calculations of knot invariants through the study of families of superpolynomials in [748–762].

A different topological twist realizes homological invariants of knots and three-manifolds (monopole/Heegaard Floer and Khovanov–Rozansky homology) in terms of 3d $\mathcal{N}=2$ theory $\mathrm{T}(\mathfrak{g}, C_3, K_1)$ partially topologically twisted on a Riemann surface [698, 706, 716]. Holographic calculations [717, 719–722, 763–766] probe or use the correspondence at large N. Dimensional reduction from the 4d $\mathcal{N}=2$ superconformal index to the 3d $\mathcal{N}=2$ sphere partition function translates to dimensional oxydation from 2d q-YM to a hyperbolic manifold [675, 767]. The 3d/3d correspondence can also be refined using higher-form symmetries [724]. Half-BPS 2d $\mathcal{N}=(0,2)$ boundary conditions and domain walls of 3d $\mathcal{N}=2$ theories were studied in [609] and subsequent papers, and one offshoot is the 2d/4d correspondence [112] discussed next.

9.4 Some more geometric setups

2d/4d correspondence. Reducing the 6d $(2,0)$ theory on $T^2 \times C_4$ with a partial topological twist along the four-manifold C_4 gives 2d $\mathcal{N}=(0,2)$ supersymmetric gauge theories. This setting has been somewhat less studied, owing to how the topology of four-manifolds is more complicated than for the 4d/2d and 3d/3d correspondences. The relevant twist of the M5 brane action was constructed explicitly in [768, 769].

A dictionary à la AGT is proposed in [112]: the Vafa–Witten partition function on C_4 [770] is the superconformal index of the 2d $(0,2)$ theory, while 4d Kirby calculus translates to dualities of 2d $\mathcal{N}=(0,2)$ theories such as [771]. Four-manifolds with a boundary $\partial(C_4)$ correspond to domain walls between the 3d $\mathcal{N}=2$ theories associated to $\partial(C_4)$ by the 3d/3d correspondence [609] (see also [772]). Four-manifolds of the form $\mathbb{CP}^1 \times C$ are considered in [773] and provide an analogue of class S theories. This can be enriched further by inserting defects of the 6d $(2,0)$ theory. There are several variants: compactifying on $S^2 \times C_4$ [774], generalizing to 6d $(1,0)$ theories [775], and using a different twist to connect 4-manifold invariants to 2d $\mathcal{N}=(0,2)$ chiral correlators [776, 777]. The abelian case is studied in detail in [778]. See also [779, 780].

Less supersymmetry. One can learn properties of strongly-coupled 4d $\mathcal{N}=1$ theories, for instance analogues of Seiberg dualities, from supersymmetry-breaking deformations of 4d $\mathcal{N}=2$ theories and S-duality [139, 170, 781–787]. These developments have led to finding 4d $\mathcal{N}=1$ Lagrangian descriptions for 4d $\mathcal{N}=2$ AD theories that admit no 4d $\mathcal{N}=2$ Lagrangian description, as done for instance in [410, 788–801] (see also [802] with more supersymmetry). They also lead to new 4d $\mathcal{N}=2$ that may be "minimal" in the sense of having the smallest central charges (a, c) [797].

Another approach to getting 4d $\mathcal{N}=1$ theories is to consider more general compactifications of 6d $\mathcal{N}=(2,0)$ SCFTs that amount to placing M5 branes on a complex curve inside a Calabi–Yau three-fold [355, 361, 528, 784, 791, 803–809]. Generalizations of SW geometry appear to exist [810]. A further reduction yields 3d $\mathcal{N}=2$ theories [811].

A particularly natural path to lower supersymmetry arises from orbifolds of the M-theory setup. The 6d $(1,0)$ theory of M5 branes at a $\mathbb{Z}_k$ singularity has very interesting reductions to 4d $\mathcal{N}=1$ theories called class S_k [615, 674, 812–824], see [825–827] for M5 branes probing more general ADE singularities. In principle this leads to an analogue of the AGT correspondence, but the Z_{S^4} partition function of $\mathcal{N}=1$ theories suffers some ambiguities, and the instanton partition function is not known (see however a very interesting proposal [240]).

Beyond these orbifolded M5 branes, there exists a zoo of 6d $(1,0)$ theories constructed from F-theory [828, 829], reviewed in [119, 830]. Compactifying them further on a torus gives 4d $\mathcal{N}=2$ supersymmetry, reproducing many class S theories [814]. The set-up has also been studied on a Riemann surface [831–834] or on a torus with fluxes turned on [820, 825–827] to get 4d $\mathcal{N}=1$ theories, and with surface operators [579, 833, 835]. The reduction from 6d to 5d is also interesting [836].

Two other rich families of 4d $\mathcal{N}=1$ SCFTs are bipartite quivers [837, 838], and D3 branes probing orientifolds of toric singularities [839–843].

Miscellaneous. By fine-tuning (and analytically continuing) parameters, one gets the AGT correspondence for minimal models [277, 487, 844–848] and a "finite" version of AGT on the mathematical side [849–851].

Class S 4d $\mathcal{N}=2$ theories have also been localized on other geometries: $S^2\times S^2$ corresponds to Liouville gravity [852], $S^2\times S^1\times I$ to complex Toda CFT [853], $S^2\times T^2$ in [657].

Instead of reducing M5 branes on product geometries, reducing D3 branes yields a 2d/2d correspondence [854–857], while M2 branes give a 1d/2d correspondence [858, 859]. A 3d/3d $\mathcal{N}=1$ correspondence was also proposed in [715].

10 Conclusions

There is plenty more to be said about the AGT correspondence. Most obviously we have not placed this correspondence in the wider context of the BPS/CFT correspondence between 4d $\mathcal{N}=2$ gauge theories and integrable models underlying SW geometry. We have also omitted the connections to refined topological strings and matrix models.

- **BPS/CFT correspondence:** SW curves of quite general 4d $\mathcal{N}=2$ theories are spectral curves of integrable systems.
- **Topological strings** give Z_{inst} for theories obtained by reducing (p,q) 5-brane webs, or IIB geometric engineering.
- **Matrix models** with logarithmic potentials yield Z_{inst} of Lagrangian 4d $\mathcal{N}=2$ theories.

Integrable systems. The SW solutions of many 4d $\mathcal{N}=2$ theories can be realized as the spectral curve of known integrable systems [860–863], such as the periodic Toda spin chain, Calogero-Moser, Ruijsenaars, sine-Gordon etc. As a quite general example, for 4d $\mathcal{N}=2$ theories of class S it is the Hitchin integrable system [167]. Placing the 4d theory on the Omega background with $\epsilon_2=0$ (the Nekrasov–Shatashvili limit) corresponds to quantizing the integrable system, and turning on ϵ_2 yields a further refinement [864]. It is also understood how to include surface operators in these discussions.

Nekrasov has advocated for seeing these considerations as a BPS/CFT correspondence, reviewed in [71, 280, 865–867] (see also [70, 189, 190, 239, 628, 645, 868–870]), which relates supersymmetric gauge theories with 8 supercharges (e.g. 4d $\mathcal{N}=2$) to integrable models and 2d CFT. Since this applies beyond class S theories, one can consider the AGT correspondence as merely an instance of it in which one can make further progress.

Another instance is the Bethe/gauge correspondence, which roughly speaking arises in the Nekrasov–Shatashvili limit. In this limit, the Omega-deformed 4d $\mathcal{N}=2$ theories reduce to a 2d $\mathcal{N}=(2,2)$ theory whose properties match with those of quantum integrable systems. For instance the twisted chiral ring of the 2d theory gives quantum Hamiltonian, supersymmetric vacua correspond to Bethe states, and the 2d twisted superpotential is the Yang–Yang function of the integrable system. The limit was studied in [864, 871–891] among others.

For further unsorted references regarding integrable models and class S theories, see [57, 211, 279, 292, 400, 524, 530, 561, 584, 593, 609, 633, 634, 748, 864, 883, 892–949].

Topological strings. Topological strings and their relation with the AGT correspondence are reviewed in [950].

For 4d $\mathcal{N}=2$ theories realized from IIB geometric engineering [630, 951, 952] or as dimensional reductions of 5d $\mathcal{N}=1$ theories living on (p,q) fivebrane webs, instanton partition functions can be expressed as partition functions Z_{top} of topological strings, as reviewed in [953]. As advocated in [954] to explain the AGT correspondence, Z_{top} can be further expressed in terms of Penner-like matrix models with logarithmic potentials [955–961], which match with Dotsenko–Fateev integral representations of conformal blocks. We return to these matrix models shortly.

The topological string partition function Z_{top} is computed through the topological vertex formalism, developped in [962–965] in the unrefined case $\epsilon_1=-\epsilon_2$, and for general ϵ_1,ϵ_2 in two formulations in [966, 967] and [968, 969], whose equivalence is explained in [970] by realizing the refined topological vertex as an intertwiner of the Ding–Iohara–Miki (DIM) algebra. See also [911, 971–974] for further tests and subtleties, [975, 976] for a world-sheet perspective, [977, 978] for a discussion of dualities that ensure that Z_{top} is independent of the so-called preferred direction, and [979, 980] for a generalization beyond A-type quivers by introducing new topological vertices. Sometimes, Z_{top} can instead be bootstrapped using holomorphic anomaly equations [879, 953, 981–986], blowup equations [987], or the quantum curve [988, 989].

The calculation in [339–342] of the partition function of T_N theories, hence the three-point function of Toda CFT, is particularly interesting. Surface operators and their relation to geometric transition and qq-characters are discussed in [56, 58, 59, 562, 660, 934, 971, 990–992]. Other AGT developments based on the refined topological vertex include [229, 846, 988, 993–1006].

Symmetries and special polynomials. The renewed interest in refined topological strings in the context of AGT led to developping many families of special polynomials generalizing Jack and Macdonald polynomials, including Macdonald-Kerov functions, generalized Schur functions, elliptic generalized Macdonald polynomials and more: see [598, 762, 1007–1020] for recent developments.

These developments are based on various underlying symmetry algebras that generalize W-algebras [131, 330] and would deserve a review of their own by someone more qualified. My survey of the literature suggests the following main players. I found the references [268, 1021] useful starting points.

- The (centrally-extended) *elliptic Hall algebra* $\mathcal{E}_{\sigma,\overline{\sigma}}$ introduced in [1022] is an associative algebra generated by $u_{m,n}$ for $(m,n) \in \mathbb{Z}^2$ with $u_{0,0} = 0$, and by commuting generators $\sigma, \overline{\sigma}$. These are subject to commutation relations expressing $[u_y, u_x]$ (for some $x, y \in \mathbb{Z}^2$) as a polynomial in σ, $\overline{\sigma}$, and the generators u_z for $z \in \mathbb{Z}^2$ in the segment joining $x+y$ to the origin. The algebra may be thought as the stable limit [1023] $\mathcal{E}_{\sigma,\overline{\sigma}} = \mathrm{S\ddot{H}}_\infty$ of *spherical double affine Hecke algebra (DAHA)* $\mathrm{S\ddot{H}}_n$. The latter, also called *Cherednik algebras*, were introduced in [1024] and reviewed in [1025].

  The algebra $\mathcal{E}_{\sigma,\overline{\sigma}}$ admits an action of $\mathrm{SL}(2,\mathbb{Z})$ by automorphisms, induced by the action on $\mathbb{Z}^2$. For any coprime a, b, the subalgebra generated by $u_{ma,mb}$, $m \in \mathbb{Z}$, forms a copy of the quantum group $U_q(\mathfrak{gl}_1)$, and these copies are interchanged by the $\mathrm{SL}(2,\mathbb{Z})$ action.

  Let $\mathcal{M}_{\mathrm{U}(N)}$ be the moduli space of non-commutative $\mathrm{U}(N)$ instantons on $\mathbb{C}^2$ (Gieseker framed moduli space). Its equivariant K-theory admits an action of $\mathcal{E}_{\sigma,\overline{\sigma}}$ [1021, 1026, 1027].

- The *DIM algebra* (discovered independently by Ding–Iohara [1028], Miki [1029], and others [1022, 1026]) is an associative algebra depending on complex parameters with $q_1 q_2 q_3 = 1$. It is generated by ψ_0^{-1} and e_i, f_i, ψ_i for $i \in \mathbb{Z}$, with quadratic relations such as $[e_i, f_j] = \frac{1}{(1-q_1)(1-q_2)(1-q_3)}\big((\delta_{i+j>0} - \delta_{i+j<0})\psi_{i+j} + \delta_{i+j=0}(\psi_0 - \psi_0^{-1})\big)$.

- The *quantum toroidal algebra* $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, also called *quantum continuous* $\mathfrak{gl}_\infty$, is a quotient of DIM by cubic Serre relations $[e_0,[e_1,e_{-1}]] = 0$ and $[f_0,[f_1,f_{-1}]] = 0$. It appeared already in [1029] and is explored in [1030–1032]. It is also a specialization of the elliptic Hall algebra $\mathcal{E}_{\sigma,\overline{\sigma}}$ at $\overline{\sigma} = 1$ [1032], and as such, it acts on the aforementioned equivariant K-theory.

- The *affine Yangian of* $\mathfrak{gl}_1$ [269, 1033], denoted $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ for $h_1 + h_2 + h_3 = 0$ is generated by e_i, f_i, ψ_i for $i \geq 0$ and relations deriving from those of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$. It is a stable limit of spherical degenerate DAHA. The equivariant cohomology of $\mathcal{M}_{\mathrm{U}(N)}$ admits an action of $\ddot{Y}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ [268, 1021].

- The *algebra denoted* SH^c [268, 269] is isomorphic to the specialization of $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{gl}_1)$ at $h_1 = 1$ [1033]. It has various constructions, such as the spherical degenerate DAHA of $\mathrm{GL}(\infty)$, or the spherical cohomological Hall algebra (COHA) of the quiver with one vertex and one loop.

  Up to topological completions, SH^c at particular values of the parameters matches the universal envelopping algebra of the W_N *chiral algebra* [268]. This leads to an action of W_N on the equivariant cohomology of the instanton moduli space $\mathcal{M}_{\mathrm{U}(N)}$, which explains on the 4d gauge theory side the appearance of the W_N symmetry of Toda CFT.

The elliptic Hall algebra, DIM algebra, and their numerous degeneration limits were used for AGT-related applications in [36, 281, 282, 385, 642, 648, 651, 929, 944, 970, 978, 980, 1012, 1013, 1017, 1034–1043]. As these algebras are intimately related and almost equivalent for physics applications, it is hard to disentangle which precise algebra is relevant to any given physics work. The choice is often based on which generators of the algebra are the most physically meaningful: translating from one presentation of the algebras to another is highly involved, see for instance [1033, 1044].

Further algebras have also been considered.

- The quantum toroidal algebra $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{sl}_k)$ was introduced in [1045–1047]. The relation between different presentations of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{sl}_k)$ and its analogue $\ddot{Y}_{h_1,h_2,h_3}(\mathfrak{sl}_k)$, and a further classical limit, were explored in [1048–1051]. I do not know if there is are analogues of the DIM algebra or of the elliptic Hall algebra for this setting. This class of algebras is relevant to the AGT correspondence in the presence of a $\mathbb{Z}_k$ orbifold [1052–1054], see also perhaps [37].
- Just as the W_N chiral algebra embeds into SH^c, suitable specializations of $\mathcal{E}_{\sigma,\overline{\sigma}}$ contain [1055] qW-algebras and quiver W-algebras, themselves explored further in [84, 135, 625, 634, 645, 654, 659, 777, 868, 870, 1056–1073].
- A general point of view on BPS algebras is given by COHAs [1066, 1074–1077].
- The Skylanin algebra also appears in related literature. It is a one-parameter deformation of the quantum group $U_q(\mathfrak{sl}_2)$.

Based on these generalized symmetry algebras there exists an elliptic version of the refined topological vertex (an intertwiner of the elliptic DIM algebra) [927, 1078–1080] for use in the 6d lift of the AGT correspondence, as well as a Macdonald refined topological vertex [656, 660] and an analogue when the 4d spacetime is orbifolded [1053].

Matrix models. The AGT correspondence (including its q-deformed version) can be explored by studying matrix models [954] since both instanton partition functions and conformal blocks are Penner-like matrix model integrals with logarithmic potentials. See the reviews [140, 308].

On the 2d CFT side the matrix model representation arises as Dotsenko–Fateev free-field representations of conformal blocks, which are available provided the sum of

Liouville/Toda momenta in each three-punctured sphere is suitably quantized. Internal momenta in the conformal block translate to choices of contours in the matrix model integral [289, 1081, 1082]. Moving away from these quantized slices in parameter space requires analytic continuation, which is only completely under control in the $\mathfrak{g} = \mathfrak{su}(2)$ case since coefficients in various expansions are known to be rational functions of all parameters in this case.

The link between 4d gauge theory and matrix models shows up as an equality of Z_{inst} with the matrix model partition functions [1083, 1084] (directly without going through the topological string Z_{top}), a matching of the SW curve with the matrix model's spectral curve and of the SW differential with the 1-point resolvent [60, 370, 894, 1085, 1086]. Both this link and the one with 2d CFT generalize to $b^2 = \epsilon_1/\epsilon_2 \neq -1$ in terms of β-deformed matrix models [1083, 1086], to 5d [618, 1083, 1087], to asymptotically free theories [370, 373, 1088–1091], to $\mathfrak{su}(N)$ theories [1087, 1092], to quiver gauge theories [1093] and generalized quivers [1094] (higher genus C), and to an orbifold of $\mathbb{R}^4$ [21].

The relations are tested in various limits in [373, 1095–1099] and proven in some cases [67, 307]. Since the Nekrasov–Shatashvili limit $\epsilon_2 \to 0$ of Z_{inst} quantizes the integrable models underlying a 4d $\mathcal{N} = 2$ theory's SW solution, matrix models give useful information about integrable models, see for instance [729, 919, 1100–1103]. Matrix models have also been studied for applications to wild punctures and AD theories, especially in the classical limit $c \to \infty$ (Nekrasov–Shatashvili limit on the gauge theory side) in [371, 372, 377, 379, 380, 384, 386–391, 1104–1108].

In another direction, modular properties of (properly normalized) instanton partition functions under S-duality are studied in [218, 219, 224, 225, 573, 996, 997, 1109–1120] through matrix model and other techniques. Other works and reviews abound [301, 629, 639, 1121–1134], as well as PhD theses [1135–1137].

Other connected topics. We list disparate subjects that are connected in various ways with the AGT correspondence. Reference lists are both less complete and less properly filtered here than elsewhere in the review.

Holographic duals of the 6d $\mathcal{N} = (2,0)$ theory, of 4d $\mathcal{N} = 2$ class S theories, and in the presence of extended operators are explored in [448, 557, 587, 763, 764, 766, 1052, 1138–1156].

Supersymmetric localization applies to many background geometries, and for a sample of interesting cases see [240, 503, 822, 848, 1157–1167] and reviews [1160, 1163]. Resurgence and Borel summability of various expansions of Z_{inst} (and of other exact results from supersymmetric localization) are studied in [468, 514, 1168–1173]. These give some insight on how applicable resurgence techniques are in QFT.

There are numerous interplays with other properties of 4d $\mathcal{N} = 2$ theories.

- Topological anti-topological fusion (tt^* equations) [999, 1174–1176].
- The chiral algebra that appears as a protected subsector of 4d $\mathcal{N} = 2$ theories [172, 423, 580, 681, 683, 792, 795, 949, 1067, 1177–1217] (see also [293, 1218–1222] and references thereto).

- Gauge/Yang–Baxter equation (YBE) [674, 713, 1223–1229] reviewed in [1230]. See also [1231].
- The way class S theories are built by combining building blocks through gauging suggests to introduce a notion of theory space [1232, 1233].
- Conformal bootstrap: besides the constructions discussed in this review, another interesting method to find QFTs, specifically unitary CFTs, is the conformal bootstrap program started in [1234] and applied to 4d $\mathcal{N} = 2$ theories in [1235]. Some AD theories in particular are located at corners of the regions of parameter space allowed by the bootstrap.

The AGT correspondence has increased the interest in several old questions about 2d CFT.

- Computing conformal blocks, correlators, and fusion matrices, either through recursion relations [227, 294, 371, 619, 1118, 1236–1244], using holography in the large c limit [298, 479, 891, 1245–1262], or Chern–Simons theory [1263].
- Studying variants of Toda CFT, parafermionic Liouville CFT etc. [1264–1266].
- Some (disputed) links to the fractional quantum Hall effect [1141, 1267–1272].
- Isomonodromy problems, as it is now known that conformal blocks (and hence Nekrasov partition functions), Fourier transformed with respect to internal momenta, give solutions to Painlevé equations arising in isomonodromy problems for Fuchsian connections [233, 394, 398, 401, 591, 910, 925, 946, 989, 1002, 1106, 1120, 1130, 1131, 1169, 1173, 1246, 1259, 1273–1305]; likewise the chiral blocks of the q-deformed Virasoro algebra and qW-algebras give solutions of q-Painlevé equations [636–638].

Incidentally, the Liouville CFT has finally been defined mathematically from its path integral: see [1306, 1307] and references therein. Other mathematical references include the study of 6j symbols of (the modular double of) $U_q(\mathfrak{sl}_2)$ [88], relations to the geometric Langlands correspondence or deformations thereof [517, 644, 1063, 1064, 1308, 1309].

Final thoughts. The construction of new theories by dimensionally reduction in various geometrical setups has proven very fruitful. It has led to many new quantum field theories that can be used as building blocks for yet more discoveries. The large number of dualities uncovered in this way can be further enriched by considering extended operators in their various incarnations. I hope that readers will participate in this exciting journey charting the space of quantum field theories!

Acknowledgments

First, many thanks to Jaume Gomis for guiding me through basics of the AGT correspondence starting a decade ago, and to my other coauthors on AGT-related papers for sharing insights throughout the years: Alexander Gorsky, Alexey Milekhin, Yiwen Pan, Wolfger Peelaers, Nikita Sopenko and Gustavo Turiaci. I wish to thank Antoine Bourget, Ioana Coman, Taro Kimura, Fabrizio Nieri, Wolfger Peelaers, Shlomo Razamat and Jaewon Song who helpfully provided references or comments, as well as Elli Pomoni and Alessandro Sfondrini for organizing YRISW 2020. Lastly, I am grateful to my wife and kids for their patience while I prepared and later revised the lectures.

Table of acronyms

ABJM	Aharony–Bergman–Jafferis–Maldacena (M2 brane worldvolume theory)
AD	Argyres–Douglas (strongly coupled 4d $\mathcal{N}=2$ theories)
ADHM	Atiyah–Drinfeld–Hitchin–Manin (construction of instantons)
AGT	Alday–Gaiotto–Tachikawa (4d/2d correspondence)
ALE	asymptotically locally Euclidean space (resolution of $\mathbb{C}^2/\Gamma$)
BMT	Bonelli–Maruyoshi–Tanzini (irregular states in 2d CFT)
BPS	Bogomol'nyi–Prasad–Sommerfield (supersymmetric)
CFT	conformal field theory
COHA	cohomological Hall algebra
DAHA	double affine Hecke algebra
DIM	Ding–Iohara–Miki (algebra)
DOZZ	Dorn–Otto–Zamolodchikov–Zamolodchikov (three-point function in Liouville CFT)
DS	Drinfeld–Sokolov (reduction of W-algebras)
FI	Fayet–Iliopoulos (parameter in supersymmetric action)
GW	Gukov–Witten (surface defect)
IR	infra-red (low energy/long distance)
JK	Jeffrey–Kirwan (residue prescription)
KK	Kaluza–Klein (reduction on circle)
LMNS	Losev–Moore–Nekrasov–Shatashvili (formula for the instanton partition function)
NS	Nekrasov–Shatashvili ($b\to 0$ limit, i.e., $\epsilon_1\to 0$)
OPE	operator product expansion
QFT	quantum field theory
RG	renormalization group
SCFT	superconformal field theory
SQCD	super-QCD (SYM plus matter)
SW	Seiberg–Witten (curve Σ and differential λ giving IR description and prepotential F of 4d $\mathcal{N}=2$ theories)
SYM	super-Yang–Mills (for us, 4d $\mathcal{N}=2$ vector multiplet)
TQFT	topological quantum field theory
UV	ultra-violet (high energy/short distance)
VEV	vacuum expectation value
YBE	Yang–Baxter equation
YM	Yang–Mills (non-supersymmetric)
YRISW	Young Researchers Integrability School and Workshop

A Special functions

In the main text we use the following special functions.

- The Gamma function $\Gamma(x)=\prod_{n\geq 0}^{\text{reg}}\frac{1}{x+n}$ has poles at $-\mathbb{Z}_{\geq 0}$ and obeys the shift formula $\Gamma(x+1)=x\Gamma(x)$.

- The Barnes Gamma function $\Gamma_b(x) = \prod_{m,n\geq 0}^{\text{reg}} \frac{1}{x+mb+n/b}$ has poles at $-b\mathbb{Z}_{\geq 0} - b^{-1}Z_{\geq 0}$ and obeys the shift formula $\Gamma_b(x+b)/\Gamma_b(x) = \sqrt{2\pi} b^{xb-1/2}/\Gamma(xb)$.
- The double-sine function $S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(b+1/b-x)}$ has poles at $-b\mathbb{Z}_{\geq 0} - b^{-1}Z_{\geq 0}$, zeros at $b\mathbb{Z}_{\geq 1} + b^{-1}Z_{\geq 1}$, and obeys the shift formula $S_b(x+b)/S_b(x) = 2\sin(\pi bx)$.
- The Upsilon function $\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(b+1/b-x)}$ has zeros at $-b\mathbb{Z}_{\geq 0} - b^{-1}Z_{\geq 0}$ and $b\mathbb{Z}_{\geq 1} + b^{-1}Z_{\geq 1}$, and obeys the shift relation $\Upsilon_b(x+b)/\Upsilon_b(x) = b^{1-2bx}\Gamma(bx)/\Gamma(1-bx)$.

Note that $\Gamma_b, S_b, \Upsilon_b$ are invariant under $b \to 1/b$.

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