Asymptotics for two-dimensional vectorial Allen-Cahn systems

Fabrice Bethuel

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Abstract

The formation of codimension-one interfaces for multi-well gradient-driven problems is well-known and established in the scalar case, where the equation is often referred to as the *Allen-Cahn equation*. The proofs rely for a large part on a monotonicity formula for the energy density, which is itself related to the vanishing of the so-called discrepancy function. The vectorial case in contrast is quite open. This lack of results and insight is to a large extent related to the absence of known appropriate monotonicity formula. In this paper, we focus on the *elliptic case in two dimensions*, and introduce methods, relying on the analysis of the partial differential equation, which allow to circumvent the lack of monotonicity formula for the energy density. In the last part of the paper, we recover a *new monotonicity formula* which relies on a *new discrepancy relation*. These tools allow to extend to the vectorial case in two dimensions most of the results obtained for the scalar case. We emphasize also some *specific features* of the vectorial case.

1 Introduction

1.1 Statement of the main result

Let Ω be a smooth bounded domain in $\mathbb{R}^2$. In the present paper, we investigate asymptotic properties of families of solutions $(u_\varepsilon)_{\varepsilon>0}$ of the systems of equations having the general form

$$-\Delta u_\varepsilon = -\varepsilon^{-2}\nabla_u V(u_\varepsilon) \text{ in } \Omega \subset \mathbb{R}^2, \tag{1}$$

as the parameter $\varepsilon > 0$ tends to zero. The function V, usually termed the *potential*, denotes a smooth scalar function on $\mathbb{R}^k$, where $k \in \mathbb{N}$ is a given integer. Given $\varepsilon > 0$, the function u_ε represents a function defined on the domain Ω with values into the *euclidian space* $\mathbb{R}^k$, so that equation (1) is a *system of k scalar partial differential equations* for each of the components of the map u_ε. The equation (1) and its parabolic version have been introduced as models in the physics and material literature (see e.g. [17] and the references therein, in particular [8]).

Equation (1) corresponds to the Euler-Lagrange equation of the energy functional E_ε which is defined for a function $u : \Omega \mapsto \mathbb{R}^k$ by the formula

$$\mathrm{E}_\varepsilon(u) = \int_\Omega e_\varepsilon(u) = \int_\Omega \varepsilon\frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon}V(u). \tag{2}$$

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We assume that the potential V is bounded below, so that we may impose, without loss of generality and changing possibly V by a suitable additive constant, that

$$\inf V = 0. \tag{3}$$

We introduce the set Σ of minimizers of V, sometimes called the *vacuum manifold*, that is the subset of $\mathbb{R}^k$ defined by

$$\Sigma \equiv \{y \in \mathbb{R}^k, V(y) = 0\}.$$

Properties of solutions to (1) crucially depend on the nature of Σ. In this paper, we will assume that the vacuum manifold is finite, with at least two distinct elements, so that

(H_1) $\quad \Sigma = \{\sigma_1, ..., \sigma_q\},\ q \geq 2,\ \sigma_i \in \mathbb{R}^k,\ \forall i = 1, ..., q.$

We impose furthermore a condition on the behavior of V near its zeroes, namely:

(H_2) *The matrix* $\nabla^2 V(\sigma_i)$ *is positive definite at each point* σ_i *of* Σ*, in other words, if* λ_i^- *denotes its smallest eigenvalue, then* $\lambda_i^- > 0$*. We denote by* λ_i^+ *its largest eigenvalue.*

Finally, we also impose a growth condition at infinity:

(H_3) *There exist constants* $\alpha_\infty > 0$ *and* $R_\infty > 0$ *such that*

$$\begin{cases} y \cdot \nabla V(y) \geq \alpha_\infty |y|^2, \ \textit{if } |y| > R_\infty \text{ and} \\ \quad V(x) \to +\infty \text{ as } |x| \to +\infty. \end{cases} \tag{4}$$

A potential V which fullfils conditions (H_1), (H_2) and (H_3) is termed throughout the paper a *potential with multiple equal depth wells* (see Figure 1).

A typical example is provided in the scalar case $k = 1$ by the potential, often termed *Allen-Cahn* or *Ginzburg-Landau* potential,

$$V(u) = \frac{(1-u^2)^2}{4}, \tag{5}$$

whose infimum equals 0 and whose minimizers are $+1$ and -1, so that $\Sigma = \{+1, -1\}$. It is used as an elementary model for *phase transitions* for materials with two equally preferred states, the minimizers $+1$ and -1 of the potential V.

Important efforts have been devoted so far to the study of solutions of the stationary *Allen-Cahn* equations, i.e. solutions to (1) for potentials similar to (5), or to the corresponding parabolic evolution equations, in the asymptotic limit $\varepsilon \to 0$, in arbitrary dimension N of the domain Ω. The mathematical theory for these questions is now well advanced and may be considered as satisfactory. The results found there provide a sound mathematical foundation to the intuitive idea that the domain Ω decomposes into regions where the solution takes values either close to $+1$ or close to -1, the regions being separated by interfaces of width of order ε. These interfaces are expected to converge to hypersurfaces of codimension 1, which are shown to be *generalized minimal surfaces* in the stationary case, or *moved by mean curvature* for the parabolic evolution equations.

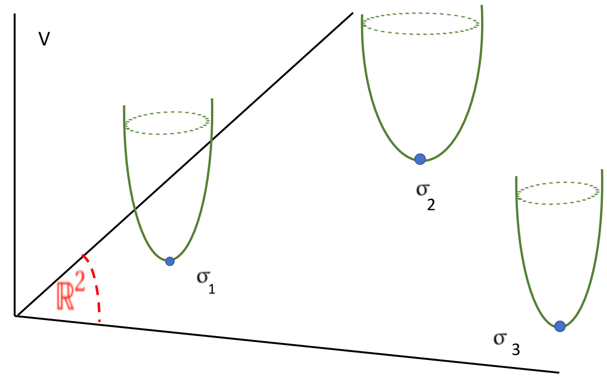

Figure 1: *Graph of a potential with several minimizers.*

Several of the arguments developed in the scalar case rely on *integral methods* and *energy estimates*. A central tool in the scalar case is the measure associated to the energy, defined on Ω by

$$\nu_\varepsilon \equiv e_\varepsilon(u_\varepsilon)\,\mathrm{d}x \text{ on } \Omega, \tag{6}$$

where $e_\varepsilon(u_\varepsilon)$ is defined in (2). The interfaces between the regions where u_ε takes approximately constants values close either to $+1$ or -1 (for the Allen-Cahn potential given in (5)) are then defined as *concentration sets* for the measure ν_ε. In [27], T.Ilmanen proved convergence of these interfaces to motion by mean curvature in the *weak sense of Brakke*, a notion relying on the language, concepts and methods of *geometric measure theory*. In the elliptic case considered in this paper, convergence to minimal surfaces was established by Modica and Mortola in their celebrated paper [31]. J.Hutchinson and Y.Tonegawa in [26] established related results for non-minimizing solutions. More references will be provided in Subsection 1.3.

Remark 1. The case of *minimizing solutions* was treated in the vectorial case by Baldo, on one hand (see [7]), and Fonseca and Tartar on the other (see [22]), where they obtained quite similar results to [31] (for the scalar case). The approaches rely on ideas from Gamma convergence, and *do not rely on monotonicity formulas*, as for general stationary solutions or solutions of the corresponding evolution equations in the scalar case.

The purpose of the present paper is to show that, to some extent, the results obtained in the scalar case can be transposed to the vectorial case for potentials V which fulfill conditions (H_1), (H_2) and (H_3), that is potentials with multiple equal depth wells, if we restrict ourselves to *two dimensional domains*. Let us emphasize that, prior to the present paper, *no monotonicity formula similar to* (36) *was known in the vectorial case*, so that different arguments have to be worked out. Several of them rely strongly on some specificities of dimension two.

We assume throughout the paper that we are given a constant $\mathrm{M}_0 > 0$ and a family $(u_\varepsilon)_{0<\varepsilon\leq 1}$ of solutions to the equation (1) for the corresponding value of the parameter ε, satisfying the natural energy bound

$$\mathrm{E}_\varepsilon(u_\varepsilon) \le \mathrm{M}_0, \quad \text{for any } 0 < \varepsilon \le 1. \tag{7}$$

Assumption (7) is rather standard in the field, since it corresponds to the energy magnitude required for the creation of $(N-1)$-dimensional interfaces. Our first main result is the following:

Theorem 1. *Let $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ be a sequence of solutions to* (1) *satisfying* (7)*. There exist a subset $\mathfrak{S}_\star$ of Ω and a subsequence of $(\varepsilon_n)_{n\in\mathbb{N}}$, still denoted $(\varepsilon_n)_{n\in\mathbb{N}}$ for sake of simplicity, such that the following properties hold:*

i) $\mathfrak{S}_\star$ *is a closed 1-dimensional* rectifiable *subset of* Ω *such that*

$$\mathcal{H}^1(\mathfrak{S}_\star) \le \mathrm{C_H}\,\mathrm{M}_0, \tag{8}$$

where $\mathrm{C_H}$ *is a constant depending only on the potential* V.

ii) Set $\mathfrak{U}_\star = \Omega \setminus \mathfrak{S}_\star$*, and let* $(\mathfrak{U}_\star^j)_{j\in I}$ *be the connected components of* $\mathfrak{U}_\star$*. For each* $j \in I$ *there exists an element* $\sigma_j \in \Sigma$ *such that*

$$u_{\varepsilon_n} \to \sigma_j, \text{ uniformly on every compact subset of } \mathfrak{U}_\star^j, \text{ as } n \to +\infty. \tag{9}$$

Similar to the results obtained for *the scalar case*, Theorem 1 expresses, *for the vectorial case in dimension two*, the fact that the domain can be decomposed into subdomains, where, for n large, the maps u_{ε_n} takes values close to an element of the vacuum set Σ (see Figure 2). These subdomains are separated by a closed one-dimensional set $\mathfrak{S}_\star$, on which the map u_{ε_n} might possibly undergo a transition from one element of Σ to another. Notice that Theorem 1 extends also to *non-minimizing* solutions the results[1] of [7, 22] (see Remark 1).

As in the scalar case, our proofs involve the energy measures ν_ε defined in (6), in particular in order to define the set $\mathfrak{S}_\star$. In view of (7), the total mass of the measure ν_ε is bounded by M_0, that is

$$\nu_\varepsilon(\Omega) \le \mathrm{M}_0,$$

so that by compactness, there exists a decreasing subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$ tending to 0 and a limiting measure $\nu_\star$ on Ω with $\nu_\star(\Omega) \le \mathrm{M}_0$, such that

$$\nu_{\varepsilon_n} \rightharpoonup \nu_\star \text{ in the sense of measures on } \Omega \text{ as } n \to +\infty. \tag{10}$$

The set $\mathfrak{S}_\star$ then corresponds to the concentration set for the measure $\nu_\star$: We will see that $\nu_\star$ vanishes on the complement of the $\mathfrak{S}_\star$, and that the one-dimensional density of $\nu_\star$ is bounded away from zero on $\mathfrak{S}_\star$. The proof of this result is established thanks to a *suitable clearing-out result*, a common method in the field : This result is stated in Theorem 7 and relies on corresponding results at the level of the PDE (1), which are stated in Theorem 6. The precise definition of $\mathfrak{S}_\star$ is given in (66). As we will also see below, our methods involve also other measures of interest concentrating on $\mathfrak{S}_\star$.

[1]This result holds however in arbitrary dimension and yields stronger, in particular minimizing, properties for $\mathfrak{S}_\star$.

An important property of the set $\mathfrak{S}_\star$ stated in Theorem 1 is its rectifiability. Recall that a Borel set $\mathcal{S} \subset \mathbb{R}^2$, is rectifiable of dimension 1 if its one-dimensional Hausdorff measure is locally finite, and if there is a countable family of C^1 one-dimensional submanifolds of $\mathbb{R}^2$ which cover $\mathcal{H}^1$-almost all of $\mathcal{S}$. Rectifiability of $\mathcal{S}$ implies in particular, that the set $\mathcal{S}$ has *an approximate tangent line* at $\mathcal{H}^1$-almost every point $x_0 \in \mathcal{S}$. More precisely, in our context, this means that there exists a set $\mathfrak{A}_\star \subset \mathfrak{S}_\star$, with $\mathcal{H}^1(\mathfrak{A}_\star) = 0$ such that, if $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{A}_\star$, then we have

$$\lim_{r\to 0} \frac{\mathcal{H}^1(\mathfrak{S}_\star \cap \mathbb{D}^2(x_0, r))}{2r} = 1, \tag{11}$$

and there exists a unit vector $\vec{e}_{x_0}$ (depending on the point $x_0 \in \mathfrak{S}_\star$) with the following property: For *any number* $\theta > 0$ we have

$$\lim_{r\to 0} \frac{\mathcal{H}^1\left(\mathfrak{S}_\star \cap \left(\mathbb{D}^2\left(x_0, r\right) \setminus \mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \theta\right)\right)\right)}{r} = 0, \tag{12}$$

where, for a unit vector $\vec{e}$ and $\theta > 0$, the set $\mathcal{C}_{\rm one}\left(x_0, \vec{e}, \theta\right)$ is the cone given by

$$\mathcal{C}_{\rm one}\left(x_0, \vec{e}, \theta\right) = \left\{ y \in \mathbb{R}^2, |\vec{e}^{\perp} \cdot (y - x_0)| \leq \tan\theta |\vec{e} \cdot (y - x_0)| \right\}, \tag{13}$$

$\vec{e}^{\perp}$ being a unit vector orthonormal to $\vec{e}$ (see e.g. [35]). A point $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{A}_\star$ is termed a *regular point* of $\mathfrak{S}_\star$.

In the minimizing case, it is established in [7, 22] that the interface $\mathfrak{S}_\star$ is a co-dimension one minimal surface, which hence reduces, in dimension two, to *a union of segments*. Our next result shows that, in dimension two, the same kind of result holds for *non-minimizing solutions*. In order to state the result, and since the *notion of minimality* is also related in our context to the presence of *densities* of measures, we specify first which other measures, besides $\nu_\star$, we have in mind. To that aim, we introduce a limiting measure for the potential term: Consider the positive measure ζ_ε defined on Ω by

$$\zeta_\varepsilon \equiv \frac{V(u_\varepsilon)}{\varepsilon} {\rm d}\, x, \text{ so that } \zeta_\varepsilon(\Omega) \leq {\rm M}_0. \tag{14}$$

Since the family $(\zeta_\varepsilon)_{\varepsilon>0}$ is uniformly bounded, passing possibly to a further subsequence, we have the convergence

$$\zeta_{\varepsilon_n} \equiv \frac{V(u_{\varepsilon_n})}{\varepsilon_n} {\rm d}\, x \rightharpoonup \zeta_\star, \text{ in the sense of measures on } \Omega, \text{ as } n \to +\infty. \tag{15}$$

Theorem 2. *There exists a set $\mathfrak{E}_\star \subset \mathfrak{S}_\star$ such that $\mathcal{H}^1(\mathfrak{E}_\star) = 0$, such that $\mathfrak{A}_\star \subset \mathfrak{E}_\star$ and such that, for $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, the set $\mathfrak{S}_\star$ is, locally near x_0, a* segment. *More precisely, there exists a unit vector $\vec{e}_{x_0}$ and a radius $r_0 > 0$, such that*

$$\mathfrak{S}_\star \cap \mathbb{D}^2(x_0, r_0) = (x_0 - r_0\vec{e}_{x_0}, x_0 + r_0\vec{e}_{x_0}). \tag{16}$$

Moreover the restriction of the measure $\zeta_\star$ to $\mathbb{D}^2(x_0, r_0)$ is proportional to the $\mathcal{H}^1$ measure of $(x_0 - r_0\vec{e}_{x_0}, x_0 + r_0\vec{e}_{x_0})$, that is, there exists a number $c_{x_0} > 0$, depending on x_0, such that

$$\zeta_\star \llcorner \mathbb{D}^2(x_0, r_0) = c_{x_0}\left(\mathcal{H}^1 \llcorner (x_0 - r_0\vec{e}_{x_0}, x_0 + r_0\vec{e}_{x_0})\right). \tag{17}$$

The number c_{x_0} is bounded below, that is, there exists a constant $\eta_0({\rm d}(x_0)) > 0$, depending only on V, M_0 and ${\rm d}(x_0) \equiv {\rm dist}(x_0, \partial\Omega)$ such that

$$c_{x_0} \geq \eta_0({\rm d}(x_0)), \text{ for any } x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star. \tag{18}$$

Notice that, as a consequence of (18), for any $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, the one-dimensional density $\Theta_\star$ defined by

$$\Theta_\star(x_0) = \liminf_{r\to 0} \frac{\zeta_\star\left(\mathbb{D}^2(x_0, r)\right)}{2r} \tag{19}$$

is bounded below by $\eta_0(\mathrm{d}(x_0))$, hence away from zero, and is locally constant, equal to $c_{x_0} = \Theta_\star(x_0)$.

Remark 2. It is known, in the scalar case (see e.g. [29]), that, the set $\mathfrak{S}_\star$ is orthogonal, in some appropriate sense, to the boundary, if the boundary is smooth, and if we impose furthermore *a Neumann type boundary condition*

$$\frac{\partial u_\varepsilon}{\partial n}(\sigma) = 0, \text{ for } \sigma \in \partial\Omega. \tag{20}$$

One might conjecture that the same holds true in the two-dimensional vectorial case, that is the segments composing the set $\mathfrak{S}_\star$ are orthogonal to the boundary.

Theorem 2 expresses *local stationarity properties* of the set $\mathfrak{S}_\star$ and the measure $\zeta_\star$. As we will discuss later, the set $\mathfrak{S}_\star$ may have singularities, and hence $\mathfrak{E}_\star$ *may be non empty.* However, more global stationary properties are also available, which encompass the presence of singularities. In order to state these properties, the abstract language of *varifolds* is the most appropriate. In order to use this language, an important preliminary step is to establish that the measure $\zeta_\star$ concentrates on the set $\mathfrak{S}_\star$, i.e. its restriction to the set $\Omega \setminus \mathfrak{S}_\star$ vanishes (see Theorem 1), and that it is *absolutely continuous* with respect to the $\mathcal{H}^1$-measure on $\mathfrak{S}_\star$ (see Theorem 4). In particular, this property implies that the measure $\zeta_\star$ is completely determined by the set $\mathfrak{S}_\star$ and the density $\Theta_\star$, and we have

$$\zeta_\star = \Theta_\star(\mathcal{H}^1 \llcorner \mathfrak{S}_\star) = \Theta_\star \mathrm{d}\lambda, \text{ where } \mathrm{d}\lambda = \mathcal{H}^1 \llcorner \mathfrak{S}_\star. \tag{21}$$

These properties will be discussed later (see Theorem 4 where they are established). We have:

Theorem 3. *The rectifiable one-dimensional varifold* $\mathbf{V}(\mathfrak{S}_\star, \Theta_\star)$ *corresponding to the measure* $\zeta_\star$ *is stationary.*

The theory of varifolds has been developed in the context of minimal surfaces, but it turns out to be also an important tool in the study of singular limits (see e.g. [35] for a general presentation of the theory of varifolds). The fact that $\mathbf{V}(\mathfrak{S}_\star, \Theta_\star)$ is a stationary varifold is equivalent to the following statement: Given any smooth vector field $\vec{X} \in C_c^\infty(\Omega, \mathbb{R}^2)$ on Ω with compact support, the following identity holds

$$\int_\Omega \mathrm{div}_{T_x\mathfrak{S}_\star} \vec{X} \mathrm{d}\zeta_\star = 0. \tag{22}$$

Here, for $x \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, the number $\mathrm{div}_{T_x\mathfrak{S}_\star} \vec{X}(x)$ is defined by

$$\mathrm{div}_{T_x\mathfrak{S}_\star} \vec{X}(x) = \left(\vec{e}_x \cdot \vec{\nabla}\vec{X}(x)\right) \cdot \vec{e}_x. \tag{23}$$

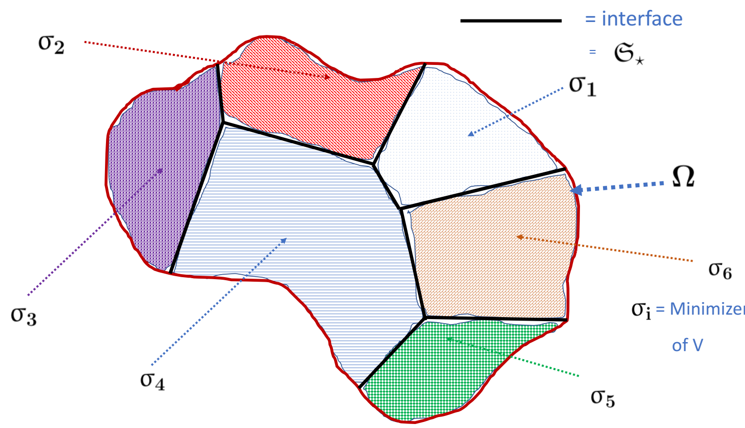

Figure 2: *The domain Ω is divided in subdomains where u_ε is nearly constant. The interfaces are union of segments. One might conjecture that the segments are orthogonal to the boundary if one imposes Neumann boundary conditions (see Remark 2).*

The structure on *one-dimensional stationary varifolds*, with *densities bounded away from zero*, was thoroughly investigated by Allard and Almgren in [5]. They showed that such varifolds have a graph structure and are the union of segments with densities. Theorem 2 *may therefore be deduced from Theorem 3* invoking the results of Section 3 in [5]. In the present paper, we provide however a simple self-contained proof, based on several results which are worked out independently.

One-dimensional stationary varifolds may have singularities, which are characterized by the fact that the density is not constant in their neighborhood. The simplest example of such a singular varifold in the whole plane with a singularity at 0 is provided by the union of a finite number of distinct half-lines, intersecting at the origin, with appropriate constant densities. More precisely, consider an integer $d > 2$, and let $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_d$ be d distinct unit vectors in $\mathbb{R}^2$. Set

$$\mathcal{S}_\star = \mathop{\cup}_{i=1}^{d} \mathbb{H}_i, \text{ where for } i = 1, \dots, d, \text{ we set } \mathbb{H}_i = \{t\vec{e}_i, t \geq 0\}, \tag{24}$$

and let $\theta_1, \dots, \theta_d$ be d positive numbers. If θ_i represents the density Θ of $\mathcal{S}_\star$ on $\mathbb{H}_i$ (which is hence constant there), then $\mathbf{V}(\mathcal{S}_\star, \Theta)$ is a stationary one-dimensional rectifiable varifold *if and only if*

$$\sum_{i=1}^{d} \theta_i \vec{e}_i = 0. \tag{25}$$

Singularities x_0 which behave *locally* as (24)-(25) are termed of *finite type*. It turns out that singularities of finite type appear in the asymptotics of the vectorial Allen-Cahn equation, *even in the minimizing case*, and are actually an intrinsic part in the problem. A first trivial example is provided by an uncoupled system of two scalar Allen-Cahn equation, taking for instance as a potential $V : \mathbb{R}^2 \to \mathbb{R}$ the potential $V(u_1, u_2) = \frac{1}{4}\left[(1-u_1^2)^2 + (1-u_2^2)^2\right]$. For this potential, the map u_ε defined on $\mathbb{R}^2$ by

$$u_\varepsilon(x_1, x_2) = \left(\tanh\left(\frac{x_1}{\sqrt{2}\varepsilon}\right), \tanh\left(\frac{x_2}{\sqrt{2}\varepsilon}\right)\right), \text{ for } (x_1, x_2) \in \mathbb{R}^2,$$

is a solution to (1) on the whole plane. The limiting interface $\mathfrak{S}_\star$ for $\varepsilon \to 0$ is then given as the union of the lines $x_1 = 0$ and $x_2 = 0$, so that 0 is a singularity where these lines cross with right angles. One may actually construct similar examples where the angle between the two lines is *arbitrary*.

A more involved example is constructed in [16], where a sequence of minimizing solutions is constructed on the entire plane, for a potential with three minimizers and equilateral symmetry. The set $\mathfrak{S}_\star$ then consists of three half lines with equal angles and equal densities, and yields a singularity at zero with *triple junction* (see Figure 3). The appearance of triple junctions in general minimizing problems is discussed in [37] and analyzed there through *Gamma-convergence methods*.

Remark 3. Singularities of finite type have also been constructed as limits of *scalar Allen-Cahn problems* (see [19, 18, 24]). In these constructions, the number d of half-lines in (24) is even.

Besides singularities with a locally finite sum of segments as in (24), an example of a singularity of a stationary varifold with an *infinite complexity* is produced in [5]. It is however shown in [5] that the occurrence of such singularities is ruled out if the set of densities is discrete. As we will see later, there are examples of potential such that the possible set of densities is infinite, so that singularities of *infinite type* cannot be ruled out a priori in the limits of solutions to (1).

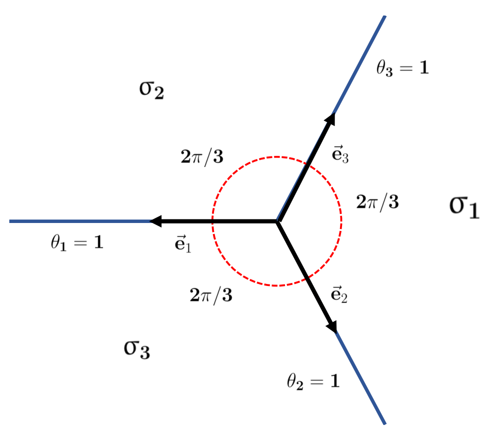

Figure 3: *Example of a triple junction, as in [16].*

1.2 Comparing results in the scalar and vectorial cases

Although the results stated in Theorems 1, 2 and 3 for the vectorial Allen-Cahn equation are somewhat parallel with the results obtained so far in the literature for the scalar case, it is worthwhile to stress some major differences between the scalar and the vectorial case.

The one-dimensional case
Distinct behaviors are already observed for the one-dimensional case. Indeed, for $\Omega = \mathbb{R}$, equation (1) reduces to the ordinary differential equation

$$-\frac{\mathrm{d}^2 w_\varepsilon}{\mathrm{d}s^2} = -\varepsilon^{-2}\nabla_w V(w_\varepsilon) \text{ on } \mathbb{R}. \tag{26}$$

Finite energy solutions to (26) necessarily connect at $\pm\infty$ two minimizers σ^- and σ^+: They are called *profiles* or *heteroclinic connections*, if $\sigma^- \neq \sigma^+$. Multiplying (26) by $\dot{w}_\varepsilon$, we are led to the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\varepsilon}V(w_\varepsilon) - \varepsilon\frac{|\dot{w}_\varepsilon|^2}{2}\right) = 0, \tag{27}$$

so that for *profiles* one derives the identity

$$\varepsilon|\dot{w}_\varepsilon| = \sqrt{2V(w_\varepsilon)} \text{ on } \mathbb{R}. \tag{28}$$

In the *scalar* case, the first order equation (28) is easily integrated by separation of variables, so that profiles connect only *nearby minimizers* σ^- and σ^+ of the potential, and are essentially unique, up to translations and symmetries. For instance, in the case of the *Allen-Cahn potential* (5), the solution is given up to translation and symmetry, by

$$w_\varepsilon(s) = \tanh\left(\frac{s}{\sqrt{2}\,\varepsilon}\right), \text{ for } s \in \mathbb{R}.$$

The situation is very different in the *vectorial case*, since relation (28) is less constraining: Under additional assumptions on the potential V, one may find several profiles connecting two minimizers of the potential (see e.g [3] and references therein). The search for such solution is still an active field of research (see for instance [4, 38, 32]). As we will see next, the genuine non-uniqueness of one-dimensional profiles is a first source of important difference also in the higher dimensional case, in particular concerning the conservation law (28).

The higher dimensional case
The higher-dimensional theory in the scalar case is rather advanced and a very satisfactory theory has been set up in any dimension $N \geq 2$. As mentioned, the existence of a $(N-1)$-dimensional set $\mathfrak{S}_\star$ is established in [27, 26]. Moreover, it is shown there that the $(N-1)$-rectifiable set $\mathfrak{S}_\star$, equipped with the energy density corresponding to the measure $\nu_\star$ defined in (10) is a *stationary rectifiable varifold.* The results in [26] embody the intuitive idea that locally, the equation reduces to a one-dimensional problem. More precisely, typically, in dimension two, the expected situation reduces, *locally near some point* x_0, to the case

$$u_\varepsilon(x) \underset{x\to x_0}{\simeq} w_\varepsilon(x_2), \text{ with } x = (x_1, x_2) \in \mathbb{R}^2, \tag{29}$$

where the coordinates are chosen so that the tangent to $\mathfrak{S}_\star$ at x_0 has equation $x_2 = 0$, and where w_ε stands for a solution to the one-dimension problem (26) (see Figure 4). Notice that the possibility of gluing of several such one-dimensional solutions is not excluded, but we will not discuss this here. Ultimately, the results in [27, 26] provide a rather simple picture of the solutions. They involve a minimal surface, the solution may be represented as one-dimensional profiles glued to the surface in the transversal direction, so that one is tempted to write the correspondance

$$\text{solutions to (1)} \sim \text{ minimal surface} + \text{glued profiles.} \tag{30}$$

The general structure of solution is hence fairly well understood (see Figure 4). As a matter of fact, the correspondance goes to some extent in either way, since, conversely, given a minimal surface, one may construct solutions to the scalar Allen-Cahn equation having the previous behavior (see [33]). This should be also connected with the famous De Giorgi conjecture ([20]) (see [34], and references therein).

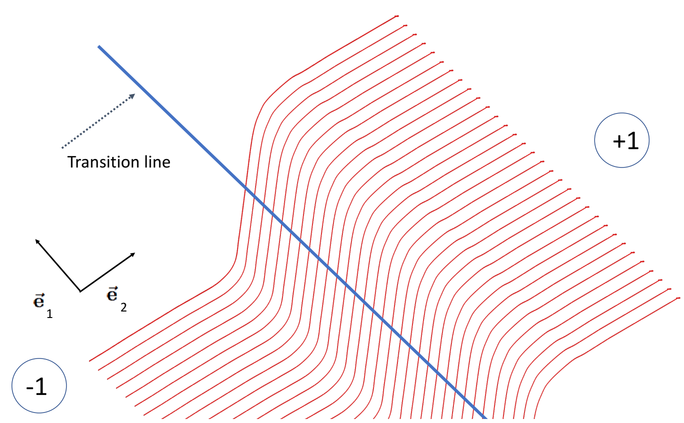

Figure 4: *Interface near a regular point x_0 in the scalar case, with an Allen-Cahn type potential.*

The picture in the vectorial case is *more complex*. Firstly, as we have already seen, the set of one-dimensional profiles is much larger, it could perhaps be even infinite. Besides this, there are solutions which *cannot be reduced to one dimensional profiles*, in view of results in [1] and [15], and are hence genuinely multi-dimensional, so that a property similar to (29) or (30) *cannot not be expected in full generality*.

In [15], it is shown that, under specific conditions on the potential V, one may construct *mountain-pass solutions* to $-\Delta u = \nabla_u V(u)$ on the cylinder $\Lambda_{\rm L} = [-{\rm L}, {\rm L}] \times \mathbb{R}$ provided $L > 0$ is sufficiently large, with periodic boundary conditions in the x_1 direction, namely such that

$$u(-{\rm L}, x_2) = u({\rm L}, x_2) \text{ and } \frac{\partial u}{\partial x_1}(-{\rm L}, x_2) = \frac{\partial u}{\partial x_1}({\rm L}, x_2), \text{ for any } x_2 \in \mathbb{R}. \tag{31}$$

The solution obtained in [15] is *not a one-dimensional profile*, since one may show that there are also *tangential contributions*: Indeed, we have

$$\frac{\partial u}{\partial x_1} \neq 0 \text{ on } \Lambda_{\rm L}. \tag{32}$$

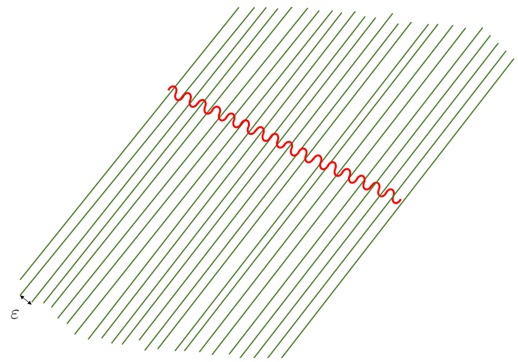

Figure 5: *Interface with a periodic pseudo-profile*

One then considers the scaled map on $\mathbb{R}^2$ defined for $x=(x_1,x_2)$ by

$$u_\varepsilon(x)=u\left(\frac{x-N\varepsilon\vec{\mathbf{e}}_1}{\varepsilon}\right),\ \text{if } x_1\in[N\varepsilon,(N+1)\varepsilon], \tag{33}$$

which solves (1) on $\mathbb{R}^2$ (see Figure 5). Moreover, it follows from (32), that for the transversal derivative, we have

$$\varepsilon\left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2 \rightharpoonup \upmu_{\star,1,1}\neq 0,\ \text{where } \upmu_{\star,1,1}=c\mathcal{H}^1(D) \text{ with } D=\left\{(x_1,0),x_1\in\mathbb{R}\right\}, \tag{34}$$

for some constant $c>0$. Finally it can be shown that the set of densities obtained for such solution is infinite, by choosing various values for the constant $\mathrm{L}>0$.

1.3 Comparing the methods in the scalar and vectorial cases

Monotonicity for the energy in the scalar case

A large part of the arguments developed for the scalar theory, as well as actually in the present paper, rely on integral estimates, starting with the energy, but also the integral of the potential. In the present context, we set for an arbitrary subdomain $G\subset\Omega$,

$$\mathrm{E}_\varepsilon\left(u_\varepsilon,G\right)=\int_G e_\varepsilon(u)\mathrm{d}x \text{ and } \mathbb{V}_\varepsilon(u,G)=\frac{1}{\varepsilon}\int_G V(u)\mathrm{d}x. \tag{35}$$

Monotonicity formulas play a distinguished role in the field. We recall that the monotonicity formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{N-2}}\mathrm{E}_\varepsilon\left(u_\varepsilon,\mathbb{B}^N(x_0,r)\right)\right)\geq 0,\ \text{for any } x_0\in\Omega,$$

holds for *arbitrary potentials*, and is relevant if one wants to establish concentration on $N-2$ dimensional sets, as it occurs in Ginzburg-Landau theory (see e.g [11, 14, 6]). If one wants instead to establish concentration on $N-1$ dimensional sets, then the stronger monotonicity formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{N-1}}\mathrm{E}_\varepsilon\left(u_\varepsilon,\mathbb{B}^N(x_0,r)\right)\right)\geq 0,\ \text{for any } x_0\in\Omega, \tag{36}$$

seems more appropriate. As a matter of fact, we have, in dimension $N = 2$, the identity (see Subsection 3.7)

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(r)\right)}{r}\right) = \frac{1}{r^2}\int_{\mathbb{D}^2(r)} \xi_\varepsilon(u_\varepsilon)\mathrm{d}x + \frac{\varepsilon}{r}\int_{\mathbb{S}^1(r)} |\frac{\partial u_\varepsilon}{\partial r}|^2 \mathrm{d}\ell, \tag{37}$$

where $\xi_\varepsilon(u_\varepsilon)$ denotes the *discrepancy function* given by

$$\xi_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon}V(u_\varepsilon) - \varepsilon\frac{|\nabla u_\varepsilon|^2}{2}. \tag{38}$$

Notice that, in view of (28), the discrepancy function vanishes for one-dimensional profiles, a property which allows to compute solution in the scalar case as seen before.

Formula (36) has been established in [27] in the *scalar case*. As identity (37) shows in dimension 2, if the discrepancy function is non-negative, then (36) holds. As the matter of fact, the proof provided in [27] relies strongly on *the non-negativity* of the *discrepancy* function ξ_ε, a property obtained there thanks to the maximum principle. The fact that ξ_ε is non-negative for *scalar solutions* of (1) was observed first by L. Modica in [30] for entire solutions. It is actually proved in [26] that the discrepancy ξ_ε vanishes asymptotically as $\varepsilon \to 0$.

Inequality (36) is the cornerstone of the scalar theory, as developed in [27, 26]. It yields both upper and lower bounds for the concentration of the energy. A large part of the arguments deals with properties of limiting measures, obtained as $\varepsilon \to 0$. As already mentioned, instead of the measure $\zeta_\star$ which appears both in Theorem 1 and Theorem 2, obtained as a limit of the potential (see (14) and (15)), the central tool in the scalar case is the corresponding measure for the full energy. More precisely, let $(\nu_\varepsilon)_{0<\varepsilon\leq 1}$ be the family of measures defined on Ω by (6), and $\nu_\star$ be the limiting measure obtained by compactness in (10). A first straightforward consequence of the monotonicity formula (36) for the energy is that the one-dimensional density of the measure $\nu_\star$ is bounded from above. This property then implies that the concentration set $\mathfrak{S}_\star$ of $\nu_\star$ has *at least dimension one*. Combining the monotonicity (36) with a weak form of the clearing-out property, similar to Proposition 6.1 in the present paper, the monotonicity formula yields also a *lower bound* on the density of $\nu_\star$ which is hence bounded away from zero. This property implies that the concentration set $\mathfrak{S}_\star$ of $\nu_\star$ has *at most dimension one*, hence its dimension is *exactly one*. The previous discussion therefore shows that the concentration property of $\nu_\star$ is a direct consequence of (36).

Notice also that the previous arguments show that the measure $\nu_\star$ is *absolutely continuous with respect to* $\mathrm{d}\lambda$, the $\mathcal{H}^{N-1}$ measure on $\mathfrak{S}_\star$, so that one may write $\nu_\star = \mathbb{e}_\star\,\mathrm{d}\lambda$, where $\mathbb{e}_\star$ is an integrable function on $\mathfrak{S}_\star$. Going to the limit $\varepsilon \to 0$ in (38), we obtain, since $\xi_\varepsilon \to 0$ as $\varepsilon \to 0$,

$$2\zeta_\star = \nu_\star, \tag{39}$$

a relation which in some sense extends (28) to the high-dimensional setting. We will see, in contrast, that relation (39) *does not extend* to the vectorial case.

Remark 4. As already mentioned, it has been proven in [27, 26] that, *in the scalar case*, the rectifiable varifolds $\mathbf{V}(\mathfrak{S}_\star, \mathbb{e}_\star)$ corresponding to the measure $\nu_\star$ is stationary. In view of relation (39) this implies that the rectifiable varifold $\mathbf{V}(\mathfrak{S}_\star, \Theta_\star)$ corresponding to the measure $\zeta_\star$ is also stationary: This is hence consistent with Theorem 3 of the present paper.

Circumventing lack of monotonicity for the energy in the two-dimensional vectorial case.

Concerning the vectorial case, non-negativity of the discrepancy as well as the monotonicity formula are known to fail for some solutions of the *Ginzburg-Landau system*, so that the question whether they might still hold under some possible additional conditions on the potential or the solution itself is widely open to our knowledge (see [2] for a discussion of these issues and for additional references).

In order to circumvent the lack of monotonicity formula for the energy, we have to work out new results on the level of solutions to PDE (termed in the paper the ε*-level*), which will be presented in Subsection 1.4. The clearing-out result given in Theorem 6 is central in our analysis: It implies, as in the scalar case, that the set $\mathfrak{S}_\star$ has dimension at most one. Combining with several other results for the PDE, we are able to deduce most of the properties developed in Theorem 1.

For the proofs of Theorems 2 and 3, the fact that the measures $\zeta_\star$ and $\nu_\star$ are absolutely continuous with respect to the $\mathcal{H}^1$ measure of $\mathfrak{S}_\star$ is *crucial*. We will show, in the last part of this paper:

Theorem 4. *The measures $\nu_\star$ and $\zeta_\star$ have support on the set $\mathfrak{S}_\star$ defined in Theorem 1, and are absolutely continuous with respect to $\mathrm{d}\lambda = \mathcal{H}^1 \llcorner \mathfrak{S}_\star$, the one-dimensional Hausdorff measure on $\mathfrak{S}_\star$. Let $\mathrm{e}_\star$ and $\Theta_\star$ denote the densities of $\nu_\star$ and $\zeta_\star$ with respect to $\mathrm{d}\lambda$ respectively, so that $\nu_\star = \mathrm{e}_\star \mathrm{d}\lambda$ and $\zeta_\star = \Theta_\star \mathrm{d}\lambda$. We have the inequalities, for $x \in \mathfrak{S}_\star$,*

$$\begin{cases} \eta_1 \leq \mathrm{e}_\star(x) \leq \mathrm{K}_{\mathrm{dens}}\,(\mathrm{d}(x))\,\Theta_\star(x), \text{ and} \\ \Theta_\star(x) \leq \dfrac{\mathrm{M}_0}{\mathrm{d}(x)}, \end{cases} \tag{40}$$

where $\eta_1 > 0$ is some constant depending only on V, $\mathrm{d}(x) = \mathrm{dist}(x, \partial\Omega)$ and where $\mathrm{K}_{\mathrm{dens}}\,(\mathrm{d}(x)) > 0$ denotes a constant depending only on V, M_0 and $\mathrm{d}(x)$.

Notice that we have also the straightforward inequality $\zeta_\star \leq \nu_\star$, so that $\Theta_\star \leq \mathrm{e}_\star$. It follows from the inequalities (40) that the densities $\mathrm{e}_\star$ and $\Theta_\star$ are locally bounded from above and away from zero.

A new discrepancy relation

Our arguments require to split the energy, in particular the gradient term, into its components, leading to several other measures. For a given orthonormal basis $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$, we consider, for $i, j = 1, 2$, the quadratic gradient terms $\varepsilon u_{\varepsilon x_i} \cdot u_{\varepsilon x_j}$, and pass to the limit $\varepsilon \to 0$, extracting possibly a further subsequence

$$\varepsilon_n u_{\varepsilon_n x_i} \cdot u_{\varepsilon_n x_j} \rightharpoonup \mu_{\star,i,j} \text{ in the sense of measures on } \Omega, \text{ as } n \to +\infty, \text{for } i, j = 1, 2, \tag{41}$$

where $\mu_{\star,i,j}$ denotes a bounded (signed) Radon measure on Ω. Notice that

$$-2\nu_\star \leq \mu_{\star,i,j} \leq 2\nu_\star \text{ and } \mu_{\star,j,i} = \mu_{\star,i,j}. \tag{42}$$

In the scalar case, the fact that solutions essentially reduce to the one-dimensional profile, with respect to the transversal direction, also implies the vanishing of the tangential contributions to the gradient terms. More precisely, we may write, in view of Theorem 4 since $\nu_\star$ is absolutely continuous with respect to $\mathrm{d}\lambda$

$$\mu_{\star,i,j} = \mathbb{m}_{\star,i,j}\mathrm{d}\lambda, \tag{43}$$

where $\mathbb{m}_{\star,i,j}$ is an integrable function on $\mathfrak{S}_\star$. The definition and values of $\mathbb{m}_{\star,i,j}$ strongly depend on the choice of orthonormal frame. In order to derive some more intrinsic objects, we may work in a *moving frame* associated to $\mathfrak{S}_\star$. More precisely if $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, and if the orthonormal frame $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$ is chosen so that $\vec{\mathbf{e}}_1 = \vec{e}_{x_0}$, then we set

$$\mathbb{m}_{\star,\perp,\perp}(x_0) = \mathbb{m}_{\star,2,2}(x_0),\ \ \mathbb{m}_{\star,\|,\|}(x_0) = \mathbb{m}_{\star,1,1}(x_0) \text{ and } \mathbb{m}_{\star,\perp,\|}(x_0) = \mathbb{m}_{\star,1,2}(x_0), \tag{44}$$

and define the measures

$$\mu_{\star,\perp,\perp} = \mathbb{m}_{\star,\perp,\perp}\mathrm{d}\lambda,\ \ \mu_{\star,\|,\|} = \mathbb{m}_{\star,\|,\|}\mathrm{d}\lambda, \ \text{ and } \mu_{\star,\perp,\|} = \mathbb{m}_{\star,\perp,\|}\mathrm{d}\lambda. \tag{45}$$

In the scalar case, the fact that the tangential contributions vanish (see [26]) can be expressed as

$$\begin{cases} \mu_{\star,\|,\|} = 0 \text{ when } k = 1 \text{ (i.e. in the scalar case) and} \\ \mu_{\star,\perp,\|} = 0 \text{ when } k = 1 \text{ (i.e. in the scalar case).} \end{cases} \tag{46}$$

On the other hand, vanishing of the discrepancy leads to (see [26] once more)

$$2\zeta_\star = \mu_{\star,\perp,\perp}, \text{ when } k = 1 \text{ (i.e. in the scalar case).} \tag{47}$$

It turns out that the relation (47) does not hold *in general for the vectorial case*. Indeed, for the map constructed in [15] and given in (33), we have $\mu_{\star,\|,\|} \neq 0$, so that the first relation in (46) is not *satisfied*. We will see later that the second one is always satisfied, whereas the discrepancy relation (47) *is not, in general*. Our next result provides a generalization of (47) for the vectorial case.

Theorem 5. *We have the identities*

$$2\zeta_\star = \mu_{\star,\perp,\perp} - \mu_{\star,\|,\|} \ \text{ and } \mu_{\star,\perp,\|} = 0. \tag{48}$$

Notice that, in view of identities (46), the discrepancy identity (47) appears as a special case of (48).

Recovering monotonicity

So far, we have introduced in Theorems 1, 2, 3, 4 and 5 the main results of this paper. As mentioned, many arguments have to be carried out without monotonicity formula, in particular Theorem 1. However, in order to obtain the proofs of Theorems 2 to 5, we rely ultimately on a *new monotonicity formula*, which we describe at the end of this subsection.

Before doing so, let us emphasize that, in order to prove Theorem 4 and several intermediate results, we rely in an essential way on Lebesgue's decomposition theorem for measures, a result which asserts that measures at hand can be decomposed into *an absolutely continuous part* and *a singular part* with respect to the one-dimensional Hausdorff measure $\mathrm{d}\lambda$ on $\mathfrak{S}_\star$. More precisely, we decompose the measures $\zeta_\star$ and $\nu_\star$ as

$$\nu_\star = \nu_\star^s + \nu_\star^{ac}, \text{ and } \zeta_\star = \zeta_\star^s + \zeta_\star^{ac}, \tag{49}$$

where the measures $\nu_\star^{ac}$ and $\zeta_\star^{ac}$ are absolutely continuous with respect to the measure $\mathcal{H}^1 \llcorner \mathfrak{S}_\star$, that is

$$\nu_\star^{ac} \ll \mathcal{H}^1 \llcorner \mathfrak{S}_\star \text{ and } \zeta_\star^{ac} \ll \mathcal{H}^1 \llcorner \mathfrak{S}_\star,$$

and

$$\nu_\star^{s} \perp \nu_\star^{ac} \text{ and } \zeta_\star^{s} \perp \zeta_\star^{ac}. \tag{50}$$

We are then in a position to write, prior to the proof of Theorem 4,

$$\nu_\star^{ac} = \mathbb{e}_\star \mathrm{d}\lambda \text{ and } \zeta_\star^{ac} = \Theta_\star \mathrm{d}\lambda.$$

An important intermediate step in the paper, is a preliminary version of Theorem 5 (see Proposition 5) established only for *the absolutely continuous* parts of the measures.

In order to show that $\nu_\star^s = \zeta_\star^s = 0$, the cornerstone of the argument is an *alternate differential inequality* for solutions of the equation (1). We have indeed, for any $x_0 \in \Omega$ such that $\mathbb{D}^2(x_0, r) \subset \Omega$ (see Subsection 3.6 for the proof), the differential relation

$$\frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathbb{V}\left(u_\varepsilon, \mathbb{D}^2(x_0,r)\right)}{r}\right) = \frac{1}{4r}\int_{\partial\mathbb{D}^2(x_0,r)}\left(\frac{2}{\varepsilon^2}V(u_\varepsilon) - \frac{1}{r^2}\left|\frac{\partial u_\varepsilon}{\partial\theta}\right|^2 + \left|\frac{\partial u_\varepsilon}{\partial r}\right|^2\right)\mathrm{d}\tau. \tag{51}$$

Although this does not transpire from the formula above, we will see that the right hand side has, in an asymptotic limit $\varepsilon \to 0$, an appropriate sign, yielding monotonicity for the measure $\zeta_\star$: As a matter of fact, it turns out that the function $r \mapsto \zeta_\star(\mathbb{D}^2(x_0,r))/r$ is non-decreasing (see Proposition 6). This yields an upper bound for the density of $\zeta_\star$, so that the singular part vanishes.

Remark 5. Let us emphasize once more that, at the ε level, we do not know that the right hand side of identity (51) is not negative or not.

In the next subsections, we provide more details on the structure of the proof.

1.4 Elements in the proof of Theorem 1: PDE analysis

As mentioned, many of our main results, dealing with the limiting measures, are derived from corresponding results at the ε-level for the map u_ε, for given $\varepsilon > 0$, which rely on PDE methods. We describe next these PDE results.

1.4.1 Scaling invariance of the equation

As a first preliminary remark, we notice the invariance of the equation by translations as well as scale changes, an observation which plays an important role in our later arguments. Given any fixed $r > 0$ and $\varepsilon > 0$, we introduce the corresponding *scaled parameter* $\tilde{\varepsilon} = \dfrac{\varepsilon}{r}$. For a given map $u_\varepsilon : \mathbb{D}^2(x_0, r) \to \mathbb{R}^k$, we consider the *scaled (and translated) map* $\tilde{u}_{\tilde{\varepsilon}}$ defined on the unit disk $\mathbb{D}^2$ by

$$\tilde{u}_{\tilde{\varepsilon}}(x) = u_\varepsilon(rx + x_0)), \forall x \in \mathbb{D}^2. \tag{52}$$

If the map u_ε is a solution to (1), then the map $\tilde{u}_{\tilde{\varepsilon}}$ is a solution to (1) with the parameter ε changed into $\tilde{\varepsilon}$. The scale invariance of the energy is given by the relation

$$e_{\tilde{\varepsilon}}(\tilde{u}_{\tilde{\varepsilon}})(x) = r e_\varepsilon(u_\varepsilon)(rx + x_0), \ \forall x \in \mathbb{D}^2, \text{ with } \tilde{\varepsilon} = \frac{\varepsilon}{r}. \tag{53}$$

Integrating this identity, we obtain the integral relations

$$\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(r)\right) = r\mathrm{E}_{\tilde\varepsilon}\left(\tilde u_{\tilde\varepsilon}, \mathbb{D}^2(1)\right) \text{ and } \mathbb{V}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(r)\right) = r\mathbb{V}_{\tilde\varepsilon}\left(\tilde u_{\tilde\varepsilon}, \mathbb{D}^2(1)\right), \tag{54}$$

where we have made use of the notation (35). It follows from the previous discussion that the parameter ε as well as the energy E_ε behave, according to the previous scaling laws, essentially as *lengths*. If we emphasize the dependance on r by writing $\tilde\varepsilon = \tilde\varepsilon_r$, then, in a loose sense, identity (54) shows that the quantity $\tilde\varepsilon_r^{-1}\mathrm{E}_{\tilde\varepsilon_r}$ is scale invariant, since

$$\tilde\varepsilon_r^{-1}\mathrm{E}_{\tilde\varepsilon_r}\left(\tilde u_{\tilde\varepsilon_r}, \mathbb{D}^2(1)\right) = \varepsilon^{-1}\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(r)\right), \text{ for any } 0 < r \leq 1. \tag{55}$$

1.4.2 The ε-clearing-out Theorem

We next provide clearing-out results for solutions of the PDE (1). In view of the assumptions (H_1), (H_2) and (H_3) on the potential V, we may choose some constant $\mu_0 > 0$ sufficiently small so that

$$\begin{cases} \mathbb{B}^k(\sigma_i, 2\mu_0) \cap \mathbb{B}^k(\sigma_j, 2\mu_0) = \emptyset, \text{ for all } i \neq j \text{ in } \{1, \cdots, q\} \text{ and such that} \\ \qquad \dfrac{1}{2}\lambda_i^- \mathrm{Id} \leq \nabla^2 V(y) \leq 2\lambda_i^+ \mathrm{Id}, \quad \text{for all } i \in \{1, \cdots, q\} \text{ and } y \in \mathbb{B}(\sigma_i, 2\mu_0). \end{cases} \tag{56}$$

We then have:

Theorem 6. *Let $0 < \varepsilon \leq 1$ and u_ε be a solution of* (1) *on $\mathbb{D}^2$. There exist constants $\eta_1 > 0$ and $\mathrm{C}_{\mathrm{well}} > 0$, depending only on the potential V, such that if*

$$\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2) \leq 2\eta_1, \tag{57}$$

then there exists some $\sigma \in \Sigma$ such that

$$|u_\varepsilon(x) - \sigma| \leq \mathrm{C}_{\mathrm{well}}\left(\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2)\right)^{\frac{1}{6}} \leq \frac{\mu_0}{2}, \text{ for every } x \in \mathbb{D}^2(\frac{3}{4}), \tag{58}$$

where μ_0 is defined in (56)*. Moreover, there exists some constant $\mathrm{C}_{\mathrm{nrg}} > 0$ depending only on the potential V, such that we have the energy estimate*

$$\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2\left(\frac{5}{8}\right)\right) \leq \mathrm{C}_{\mathrm{nrg}}\, \varepsilon \mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2). \tag{59}$$

Theorem 6 is the main (new) PDE result of the present paper: It paves the way to the concentration of measures on the set $\mathfrak{S}_\star$, and will be used to show that its dimension is *at most one*. The main ingredient in the proof of Theorem 6 is provided by the following estimate:

Proposition 1. *Let $0 < \varepsilon \leq 1$ and u_ε be a solution of* (1) *on $\mathbb{D}^2$. There exists a constant $\mathrm{C}_{\mathrm{dec}} > 0$ depending only on V such that*

$$\int_{\mathbb{D}^2(\frac{9}{16})} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq \mathrm{C}_{\mathrm{dec}}\left[\left(\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)\mathrm{d}x\right)^{\frac{3}{2}} + \varepsilon\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)\mathrm{d}x\right]. \tag{60}$$

Proposition 1 provides a very fast decay of the energy on smaller balls, provided both $\mathrm{E}_\varepsilon(u_\varepsilon)$ and ε are sufficiently small. Combining the result (60) of Proposition 1 with the scale invariance properties of the equation given in subsection 1.4.1, we obtain corresponding results for arbitrary discs $\mathbb{D}^2(x_0, r) \subset \Omega$. Indeed, we first apply Proposition 1 to the scaled and translated map $\tilde{u}_{\tilde{\varepsilon}}$ defined on $\mathbb{D}^2(1)$ by (52) with parameter $\tilde{\varepsilon} = \varepsilon/r$: Expressing the corresponding inequality (60) we obtain, provided $\tilde{\varepsilon} \leq 1$, i.e. $\varepsilon \leq r$,

$$\mathrm{E}_{\tilde{\varepsilon}}\left(\tilde{u}_\varepsilon, \mathbb{D}^2\left(\frac{9}{16}\right)\right) \leq \mathrm{C}_{\mathrm{dec}}\left[\mathrm{E}_{\tilde{\varepsilon}}\left(\tilde{u}_\varepsilon\right)^{\frac{3}{2}} + \tilde{\varepsilon}\mathrm{E}_{\tilde{\varepsilon}}\left(\tilde{u}_{\tilde{\varepsilon}}\right)\right].$$

Since $\mathrm{E}_{\tilde{\varepsilon}}\left(\tilde{u}_{\tilde{\varepsilon}}\right) = r^{-1}\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(x_0, r)\right)$ and $\mathrm{E}_{\tilde{\varepsilon}}\left(\tilde{u}_{\tilde{\varepsilon}}, \mathbb{D}^2(9/16)\right) = r^{-1}\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(x_0, 9r/16)\right)$ we are led, provided $\varepsilon \leq r$, to the inequality

$$\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2\left(x_0, \frac{9r}{16}\right)\right) \leq \mathrm{C}_{\mathrm{dec}}\left[\frac{1}{\sqrt{r}}\left(\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(x_0, r)\right)\right)^{\frac{3}{2}} + \frac{\varepsilon}{r}\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(x_0, r)\right)\right]. \tag{61}$$

Iterating this decay estimate on concentric discs centered at x_0, and combining with elementary properties of the solution u_ε, we eventually obtain the proof of Theorem 6.

Invoking once more the scale invariance properties of the equation given in subsection 1.4.1, the scaled version of Theorem 6 writes then as follows:

Proposition 2. *Let $x_0 \in \Omega$ and $0 < \varepsilon \leq r$ be given, assume that $\mathbb{D}^2(x_0, r) \subset \Omega$ and let u_ε be a solution to (1) on Ω. If*

$$\frac{\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2\left(x_0, r\right)\right)}{r} \leq 2\eta_1, \tag{62}$$

then there exist some $\sigma \in \Sigma$ such that

$$\begin{cases} |u_\varepsilon(x) - \sigma| \leq \mathrm{C}_{\mathrm{well}}\left(\dfrac{\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2(x_0, r))}{r}\right)^{\frac{1}{6}} \leq \dfrac{\mu_0}{2}, \text{ for } x \in \mathbb{D}^2(x_0, \dfrac{3r}{4}) \\ \text{and} \\ \mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2\left(x_0, \dfrac{5r}{8}\right)\right) \leq \mathrm{C}_{\mathrm{nrg}}\dfrac{\varepsilon}{r}\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2\left(x_0, r\right)\right). \end{cases} \tag{63}$$

The proof of Proposition 2 is a straightforward consequence of Theorem 6 and the scaling properties given in subsection 1.4.1, in particular identities (54).

1.4.3 Other results at the ε-level

The analysis of the limiting measures requires some other ingredients, in particular concerning the interplay between the measures ζ_ε and ν_ε, leading to the relations (40) on the limiting densities. The connectedness of $\mathfrak{S}_\star$ also requires results at the ε-level, in particular we will rely on Proposition 4.6.

We next present the main tools for handling the measures and the concentration set $\mathfrak{S}_\star$.

1.5 Elements in the proof of Theorem 1: construction of $\mathfrak{S}_\star$ and clearing-out for the measure $\nu_\star$

As mentioned, the set $\mathfrak{S}_\star$ introduced in Theorem 1 is obtained as a concentration set of the energy measure $\nu_\star$. The properties stated in Theorem 1 are, for a large part, consequences of the two results we present next. These results are deduced from corresponding properties of solutions to (1), and presented in the previous subsection.

The first result represents a classical form of a clearing-out result for the measure $\nu_\star$ and leads directly to the fact that energy concentrates on sets which are *at most one-dimensional*.

Theorem 7. *Let $x_0 \in \Omega$ and $r > 0$ be given such that $\mathbb{D}^2(x_0, r) \subset \Omega$. There exists a constant $\eta_1 > 0$ such that, if we have*

$$\frac{\nu_\star\left(\overline{\mathbb{D}^2(x_0, r)}\right)}{r} < \eta_1, \text{ then it holds } \nu_\star\left(\overline{\mathbb{D}^2(x_0, \frac{r}{2})}\right) = 0. \tag{64}$$

The previous statement leads to consider the one-dimensional lower density of the measure $\nu_\star$ defined, for $x_0 \in \Omega$, by

$$\mathrm{e}_\star(x_0) = \liminf_{r \to 0} \frac{\nu_\star\left(\overline{\mathbb{D}^2(x_0, r)}\right)}{r}, \tag{65}$$

so that $\mathrm{e}_\star(x_0) \in [0, +\infty]$. We define the set $\mathfrak{S}_\star$ as the *concentration set* for the measure $\nu_\star$. More precisely, we set

$$\mathfrak{S}_\star = \{x \in \Omega, \mathrm{e}_\star(x_0) \geq \eta_1\}, \tag{66}$$

where $\eta_1 > 0$ is the constant provided by Theorem 7. The fact that $\mathfrak{S}_\star$ is closed of finite one-dimensional Hausdorff measure is then a rather direct consequence of the clearing-out property for the measure $\nu_\star$ stated in Theorem 7.

Remark 6. Let us emphasize once more that the previous definition of $\mathfrak{S}_\star$ directly leads, by construction and in view of Theorem 7, to concentration of the measure $\nu_\star$ and $\zeta_\star$ on the set $\mathfrak{S}_\star$ and also a *lower bound* on the density of $\nu_\star$. The *upper bounds on densities* require different arguments, in particular a monotonicity formula.

The connectedness properties of $\mathfrak{S}_\star$ stated in Theorem 1, part ii) require a different type of clearing-out result. Its statement involves general regular subdomains $\mathcal{U} \subset \Omega$, and, for $\delta > 0$, the related sets (see Figure 6)

$$\begin{cases} \mathcal{U}_\delta = \{x \in \Omega, \operatorname{dist}(x, \mathcal{U}) \leq \delta\} \supset \mathcal{U} \text{ and} \\ \mathcal{V}_\delta = \mathcal{U}_\delta \setminus \mathcal{U}. \end{cases} \tag{67}$$

Theorem 8. *Let $\mathcal{U} \subset \Omega$ be an open subset of Ω and $\delta > 0$ be given. If we have*

$$\nu_\star(\mathcal{V}_\delta) = 0, \text{ then it holds } \nu_\star\left(\overline{\mathcal{U}}\right) = 0. \tag{68}$$

In other terms, if the measure $\nu_\star$ vanishes in some neighborhood of the boundary $\partial\mathcal{U}$, then it vanishes on $\overline{\mathcal{U}}$. This result will allow us to establish connectedness properties of $\mathfrak{S}_\star$. For instance, we will prove the following *local connectedness property*:

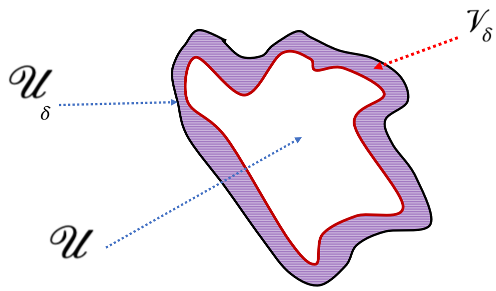

Figure 6: *The sets $\mathcal{U}_\delta$ and $\mathcal{V}_\delta$.*

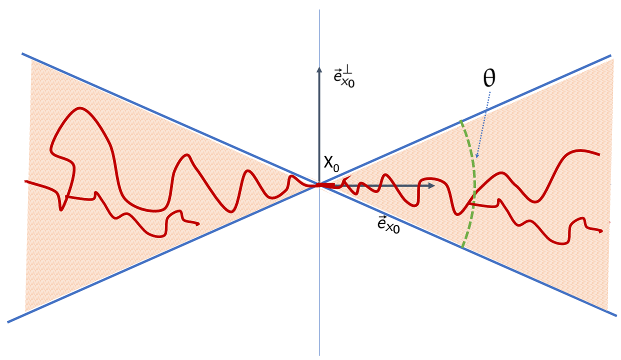

Figure 7: *The tangent cone property, as given in Proposition 4.*

Proposition 3. *Let $x_0 \in \Omega$, $r > 0$ such that $\mathbb{D}^2(x_0, 2r) \subset \Omega$. There exists a radius $\rho_0 \in (r, 2r)$ such that $\mathfrak{S}_\star \cap \mathbb{D}^2(x_0, \rho_0)$ is a finite union of path-connected components.*

The connectedness property provided by Proposition 3 implies the rectifiability of $\mathfrak{S}_\star$, invoking classical results on continua of bounded one-dimensional Hausdorff measure (see e.g [21]). The proof of Theorem 1 is then a combination of the results in Theorem 7 and Proposition 3.

For the set $\mathfrak{S}_\star$ given by Theorem 1, the approximate tangent line property (12) can actually be strengthened as follows (see Figure 7):

Proposition 4. *Let x_0 be a regular point of $\mathfrak{S}_\star$. Given any $\theta > 0$ there exists a radius $R_{\rm cone}(\theta, x_0)$ such that*

$$\mathfrak{S}_\star \cap \mathbb{D}^2\left(x_0, r\right) \subset \mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \theta\right), \text{ for any } 0 < r \leq R_{\rm cone}(\theta, x_0). \tag{69}$$

1.6 A useful tool: The limiting Hopf differential $\omega_\star$

We introduce the complex-valued measure referred to as the *limiting Hopf differential*

$$\omega_\star = (\mu_{\star,1,1} - \mu_{\star,2,2}) - 2i\mu_{\star,1,2}, \tag{70}$$

where the measures $\mu_{\star,i,j}$ have been defined in (41). Since the measures $\mu_{\star,i,j}$ depend on the choice of orthonormal basis, the expression of the Hopf differential also strongly depends on this choice. The measures $\zeta_\star$ and $\omega_\star$ are strongly related in view of our next result.

Lemma 1. *We have, in the sense of distributions,*

$$\frac{\partial \omega_\star}{\partial \bar{z}} = 2\frac{\partial \zeta_\star}{\partial z} \text{ in } \mathcal{D}'(\Omega). \tag{71}$$

Relation (71) is the two-dimensional analog of the conservation law (28) for the ordinary differential equation. It expresses the fact that the energy of the solution u_ε is stationary with respect to variations of the domain. Since the measures $\nu_\star$ and $\zeta_\star$ are supported by $\mathfrak{S}_\star$, identity (71) also expresses a stationary condition, when integrated against a test function, for the set $\mathfrak{S}_\star$ and the measures $\mu_{\star,i,j}$. As a matter of fact, identity (71) is *the starting point of the proofs of Theorems 2, 3, 4 and 5.*

Taking the real and imaginary parts of this relation, we obtain, in the sense of distributions, the *modified Cauchy-Riemann relations*

$$\begin{cases} \dfrac{\partial}{\partial x_2}(2\mu_{\star,1,2}) = \dfrac{\partial}{\partial x_1}\left(2\zeta_\star - \mu_{\star,1,1} + \mu_{\star,2,2}\right) \text{ and} \\ \dfrac{\partial}{\partial x_1}\left(2\mu_{\star,1,2}\right) = \dfrac{\partial}{\partial x_2}\left(2\zeta_\star + \mu_{\star,1,1} - \mu_{\star,2,2}\right), \end{cases} \tag{72}$$

the second relation being in some sense the closest to (28).

Our next results involve the decomposition of the measures into absolutely continuous parts with respect to $d\lambda = \mathcal{H}^1 \llcorner \mathfrak{S}_\star$ and singular parts, and describe properties of the absolutely continuous part. Besides (49), we may also decompose the measures $\mu_{\star,i,j}$, writing

$$\mu_{\star,i,j} = \mu^s_{\star,i,j} + \mu^{ac}_{\star,i,j} \text{ with } \mu^s_{\star,i,j} \perp \mu^{ac}_{\star,i,j}. \tag{73}$$

where the measures $\mu^{ac}_{\star,i,j}$ is absolutely continuous with respect to the measure $d\lambda = \mathcal{H}^1 \llcorner \mathfrak{S}_\star$. Relations (49) and (73) imply that there exists a set $\mathfrak{B}_\star \subset \mathfrak{S}_\star$ such that $\mathcal{H}^1 \llcorner \mathfrak{S}_\star(\mathfrak{B}_\star) = 0$ and

$$\nu^s_\star(\mathfrak{S}_\star \setminus \mathfrak{B}_\star) = 0, \ \zeta^s_\star(\mathfrak{S}_\star \setminus \mathfrak{B}_\star) = 0, \text{ and } \mu^s_{\star,i,j}(\mathfrak{S}_\star \setminus \mathfrak{B}_\star) = 0, \text{ for } i,j = 1,2. \tag{74}$$

Since, by construction, the measures $\zeta^{ac}_\star$, $\nu^{ac}_\star$ and $\mu^{ac}_{\star,i,j}$ are absolutely continuous with respect to $d\lambda$, there exist functions $\Theta_\star$, $\mathbb{e}_\star$ and $\mathbb{m}_{\star,i,j}$ defined on $\mathfrak{S}_\star$, such that we have

$$\zeta^{ac}_\star = \Theta_\star \, d\lambda, \nu^{ac}_\star = \mathbb{e}_\star \, d\lambda, \text{ and } \mu^{ac}_{\star,i,j} = \mathbb{m}_{\star,i,j} d\lambda, \tag{75}$$

Besides $\mathfrak{A}_\star$ and $\mathfrak{B}_\star$, we introduce a *third class of exceptional points*, the set $\mathfrak{C}_\star$, defined as the complementary of the set of Lebesgue points for the densities of the measures $\mu^{ac}_\star, \zeta^{ac}_\star, \mu^{ac}_{\star,i,j}$ with respect to $d\lambda = \mathcal{H}^1 \llcorner \mathfrak{S}_\star$. The set $\mathfrak{S}_\star \setminus \mathfrak{C}_\star$, then corresponds to the intersection of the set of Lebesgue's points of the functions $\Theta_\star, \mathbb{e}_\star$ and $\mathbb{m}_{\star,i,j}$. We consider the union of all exceptional points

$$\mathfrak{E}_\star = \mathfrak{A}_\star \cup \mathfrak{B}_\star \cup \mathfrak{C}_\star, \tag{76}$$

which is precisely the set appearing in Theorem 2.

Proposition 5. *Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$. Assume that the orthonormal frame $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$ is chosen so that $\vec{\mathbf{e}}_1 = \vec{e}_{x_0}$. We have the identities, for the functions $\Theta_\star$, $\mathbb{m}_{\star,i,j}$ defined in* (75),

$$\begin{cases} 2\Theta_\star(x_0) = \mathbb{m}_{\star,2,2}(x_0) - \mathbb{m}_{\star,1,1}(x_0) \text{ and} \\ \mathbb{m}_{\star,1,2}(x_0) = 0. \end{cases} \tag{77}$$

Next, let $\omega_\star^{ac} = \left(\mu_{\star,1,1}^{ac} - \mu_{\star,2,2}^{ac}\right) - 2i\mu_{\star,1,2}^{ac}$ denote the absolutely continuous part of $\omega_\star$ with respect to $\mathrm{d}\lambda$. The previous result yields, after change of orthonormal basis:

Lemma 2. *For a given orthonormal basis $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$, we have the identity*

$$\omega_\star^{ac} = -2\exp(-2i\gamma_\star)\zeta_\star^{ac} = -2(\cos 2\gamma_\star - i\sin 2\gamma_\star)\zeta_\star^{ac}, \tag{78}$$

where $\gamma_\star(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ denotes, for $x \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, the angle between $\vec{\mathbf{e}}_1$ and $\vec{e}_{x_0}$.

Remark 7. Changing possibly $\vec{e}_{x_0}$ into $-\vec{e}_{x_0}$, we may indeed always choose $\gamma_\star(x_0)$ in an interval of length π, here $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

We present some arguments involved in the proof of Proposition 5. We work near a *regular* point $x_0 = (x_{0,1}, x_{0,2}) \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, where $\mathfrak{E}_\star$ is defined in (76), and choose the orthonormal basis so that $\vec{\mathbf{e}}_1 = \vec{e}_{x_0}$. In the neighborhood of the point x_0, the measure $\nu_\star$ hence concentrates near the line $x_2 = x_{0,2}$, and we may follow the approach of [6], eliminating the derivatives according to the transversal direction, that is *eliminating the x_2-variable*, in order to obtain a one-dimensional problem: For that purpose, we integrate along "vertical" lines. The general idea would be to consider integrals of the form

$$I_{i,j}(s) = \int_{(x_{0,2}-3/4r)}^{(x_{0,2}+3/4r)} \mu_{\star,i,j}(s, x_{0,2})\, \mathrm{d}x_2 \text{ or } W(s) = \int_{(x_{0,2}-3/4r)}^{(x_{0,2}+3/4r)} \zeta_\star(s, x_{0,2})\mathrm{d}x_2.$$

However, since at this stage of our argument we don't know that the measures are absolutely continuous with respect to $\mathrm{d}\lambda$, one has to be a little more careful in order to define properly the previous integrals. To that aim, we introduce for $s > 0$, the segment $\mathcal{I}_r(s) = [s-r, s+r] = \mathbb{B}^1(s,r)$ and the square $Q_r(x_0) = \mathcal{I}_r(x_{0,1}) \times \mathcal{I}_r(x_{0,2})$, and consider the *localized* measures

$$\tilde{\mu}_{\star,i,j} = \mathbf{1}_{Q_r(x_0)}\mu_{\star,i,j} \text{ and } \tilde{\zeta}_\star = \mathbf{1}_{Q_r(x_0)}\zeta_\star.$$

We introduce also the orthogonal projection $\mathbb{P}$ onto the tangent line $\mathrm{D}^1_{x_0} = \{x_0 + s\vec{\mathbf{e}}_1, s \in \mathbb{R}\}$, and the *pushforward measures* on $\mathrm{D}^1_{x_0}$ of the localized measures we have introduced so far, namely the measures on $\mathrm{D}^1_{x_0}$ given by

$$\tilde{\mu}_{\star,i,j}^{x_1} = \mathbb{P}_\sharp\left(\tilde{\mu}_{\star,i,j}\right) \text{ and } \tilde{\zeta}_\star^{x_1} = \mathbb{P}_\sharp\left(\tilde{\zeta}_\star\right), \tag{79}$$

defined for every Borel set A of $\mathrm{D}^1_{x_0}$ by

$$\begin{cases} \tilde{\mu}_{\star,i,j}^{x_1}(A) = \mu_{\star,i,j}\left(\mathbb{P}^{-1}(A) \cap Q_r(x_0)\right) = \mu_{\star,i,j}\left((A \times \mathbb{R}) \cap Q_r(x_0)\right) \text{ and} \\ \tilde{\zeta}_\star^{x_1}(A) = \zeta_\star\left((A \times \mathbb{R}) \cap Q_r(x_0)\right). \end{cases}$$

We then introduce the measures $\mathbb{L}_{x_0,r}$ and $\mathbb{N}_{x_0,r}$ defined on $\mathcal{I}_r(x_{0,1})$ by

$$\begin{cases} \mathbb{L}_{x_0,r} = \mathbb{P}_\sharp\left(2\tilde{\zeta}_\star - \tilde{\mu}_{\star,1,1} + \tilde{\mu}_{\star,2,2}\right) = 2\tilde{\zeta}_\star^{x_1} - \tilde{\mu}_{\star,1,1}^{x_1} + \tilde{\mu}_{\star,2,2}^{x_1} \text{ and} \\ \mathbb{N}_{x_0,r} = \mathbb{P}_\sharp\left(2\tilde{\zeta}_\star + \tilde{\mu}_{\star,1,1} - \tilde{\mu}_{\star,2,2}\right) = 2\tilde{\zeta}_\star^{x_1} + \tilde{\mu}_{\star,1,1}^{x_1} - \tilde{\mu}_{\star,2,2}^{x_1}. \end{cases} \tag{80}$$

Mutiplying (1) by appropriate test functions and integrating, we are led to the somewhat remarkable properties of these measures, expressed in Propositions 8.1, 8.4, 8.5 and 8.6, leading to the completion of the proof of Proposition 5.

1.7 Monotonicity for $\zeta_\star$ and its consequences

The next important step in the proofs of Theorem 2, 3, 4 and 5 is to show that the singular part of all measures introduced so far vanish. We first establish this statement for the measure $\zeta_\star$. Our argument involves a new ingredient, the monotonicity formula for $\zeta_\star$, which actually directly yields the absolute continuity of $\zeta_\star$ with respect to $\mathcal{H}^1 \llcorner \mathfrak{S}_\star$.

Proposition 6. *Let $x_0 \in \Omega$, let $\rho > 0$ be such that $\mathbb{D}^2(x_0, \rho) \subset \Omega$. If $0 < r_0 \le r_1 \le \rho$, then we have the inequality*

$$\frac{\zeta_\star(\mathbb{D}^2(x_0, r_1))}{r_1} \ge \frac{\zeta_\star(\mathbb{D}^2(x_0, r_0))}{r_0}. \tag{81}$$

For every $x_0 \in \Omega$ the quantity $\dfrac{\zeta_\star(\mathbb{D}^2(x_0, r))}{r}$ has a limit when $r \to 0$ and we have the estimate

$$\Theta_\star(x_0) = \lim_{r\to 0} \frac{\zeta_\star(\mathbb{D}^2(x_0, r))}{r} \le \frac{\zeta_\star(\mathbb{D}^2(x_0, \rho))}{\rho} \le \frac{M_0}{d(x_0, \partial\Omega)}. \tag{82}$$

The measure $\zeta_\star$ is hence absolutely continuous with respect to the $\mathcal{H}^1$-measure on $\mathfrak{S}_\star$.

The starting point of the proof of Proposition 6 is the monotonicity formula (51) for the potential term V, which may be written, after integration, for a solution u_ε of (1) on Ω and $0 < r_0 < r_1 \le \rho$ such that $\mathbb{D}^2(x_0, \rho) \subset \Omega$

$$\frac{\zeta_\varepsilon(\mathbb{D}^2(x_0, r_1))}{r_1} - \frac{\zeta_\varepsilon(\mathbb{D}^2(x_0, r_0))}{r_0} = \int_{\mathbb{D}^2(x_0,r_1)\setminus\mathbb{D}^2(x_0,r_0)} \frac{1}{4r} \mathrm{d}\mathscr{N}_{x_0,\varepsilon}, \tag{83}$$

with $r = |x - x_0|$, and where we have set

$$\mathscr{N}_{x_0,\varepsilon} = \left(\frac{2}{\varepsilon} V(u_\varepsilon) - \varepsilon r^{-2} \left|\frac{\partial u_\varepsilon}{\partial \theta}\right|^2 + \varepsilon \left|\frac{\partial u_\varepsilon}{\partial r}\right|^2\right) \mathrm{d}x.$$

Here (r, θ) denote radial coordinates, so that $x_1 - x_{0,1} = r\cos\theta$ and $x_2 - x_{0,2} = r\sin\theta$. Passing to the limit $\varepsilon \to 0$ in identity (83), we are led to :

Lemma 3. *Let $x_0 \in \Omega$, let $\rho > 0$ and assume that $\mathbb{D}^2(x_0, \rho) \subset \Omega$. For almost every radii $0 < r_0 < r_1 \le \rho$, we have the identity*

$$\frac{\zeta_\star(\mathbb{D}^2(x_0, r_1))}{r_1} - \frac{\zeta_\star(\mathbb{D}^2(x_0, r_0))}{r_0} = \int_{\mathbb{D}^2(x_0,r_1)\setminus\mathbb{D}^2(x_0,r_0)} \frac{1}{4r} \mathrm{d}\mathscr{N}_{x_0,\star} \tag{84}$$

where $\mathscr{N}_{x_0,\star} = 2\zeta_\star - r^{-2}\mu_{\star,\theta,\theta} + \mu_{\star,r,r}$, *with*

$$\begin{cases} \mu_{\star,r,r} = \cos^2\theta\mu_{\star,1,1} + \sin^2\theta\mu_{\star,2,2} + 2\sin\theta\cos\theta\mu_{\star,1,2} \text{ and} \\ r^{-2}\mu_{\star,\theta,\theta} = \sin^2\theta\mu_{\star,1,1} + \cos^2\theta\mu_{\star,2,2} - 2\sin\theta\cos\theta\mu_{\star,1,2}. \end{cases} \tag{85}$$

Notice that we may verify that

$$\varepsilon_n \left|\frac{\partial u_{\varepsilon_n}}{\partial r}\right|^2 \underset{n\to+\infty}{\rightarrow} \mu_{\star,r,r} \text{ and } \varepsilon_n \left|\frac{\partial u_{\varepsilon_n}}{\partial \theta}\right|^2 \underset{n\to+\infty}{\rightarrow} \mu_{\star,\theta,\theta} \text{ as measures.}$$

The next step in the proof of Proposition 6 is the fact that, as a consequence of Proposition 5, the absolutely continuous part of $\mathcal{N}_\star$ is non-negative. We then perform a few manipulations which allow to get rid of the singular part in (81), and lead to the completion of the proof of Proposition 6.

In order to prove that $\nu_\star$ is also absolutely continuous with respect to $d\lambda$, we will invoke the fact that $\nu_\star$ is "dominated" by the measure $\zeta_\star$. Whereas the reverse statement is straightforward, since we have the inequality $\zeta_\star \leq \nu_\star$, the fact that $\nu_\star$ is "dominated" by the measure $\zeta_\star$ is a consequence of several estimates at the ε-level, requiring some PDE analysis (in particular Proposition 4.5). Theorem 5 is then a direct consequence of Theorem 4 and Proposition 5.

1.8 On the proofs of Theorems 2 and 3

The proof of Theorem 3 is a direct consequence of Lemma 1 combined with Theorem 5. Theorem 2 could be deduced from Theorem 3 combined with the result of [5], but we provide in this paper a self contained and perhaps more elementary proof.

1.9 Open questions and conclusion

As already mentioned, one of the main unsolved open problems in the present paper, i.e. in the two dimensional elliptic context, is the existence or not of singularities of *"infinite type"* in the limiting varifold. If such singularities do exist, their actual construction may turn out to be extremely difficult.

Although the paper focuses exclusively on the two-dimensional case, it is quite tempting to believe that a large part of the results might extend to higher dimensions. However, it is not clear how the arguments presented in this paper, in particular concerning properties of solutions to the PDE (1), can be adapted in higher dimensions. Indeed, as the previous presentation hopefully shows, many of our arguments rely on the fact that we work in two dimensions, and do not seem to have natural extensions in higher dimensions.

Another challenging problem is the related parabolic two-dimensional equation, which has already attracted attention (see e.g.[17] or more recently [28]). One might express the hope that some of the methods introduced in this paper extend also to this case.

1.10 Plan of the paper

The outline of the paper merely follows the description given in Subsections 1.4 to 1.8. As a matter of fact, the presentation of the arguments is divided into three parts. Part I is a preliminary part which presents various properties of the energy functional and consists of a single section, Section 2. It presents some consequences of the energy bound, starting with estimates on one-dimensional sets, as well as consequences of the co-area formula. Part II, which runs from Section 3 to Section 6, gathers all properties of solutions to the PDE (1), including standard one. For a large part, in both parts, special emphasis is put on energy estimates on *level sets of some appropriate simple scalar functions* (see (2.8)). Section 5 presents the proof of Proposition 1. The last part, Part III, describes the properties of the limiting set $\mathfrak{S}_\star$ and the limiting measures, and contains therefore the proofs to the main results of the paper.

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Contents

Part I : Properties of the functional E_ε

2 First consequences of the energy bounds

The results in this section are based on variants of an idea of Modica and Mortola (see [31]), adapted to the vectorial case in [7, 22]. We also present some applications of the co-area formula in connection with the functional. The results in this section apply to maps having a suitable bound on their energy E_ε, of the type of the bound (7). We stress in particular *BV* type bounds obtained under these energy bounds. None of the results in this section *involves the PDE* (1). We start with simple consequences of assumptions (H_1), (H_2) and (H_3) for the potential with multiple equal depth wells (see Figure 1).

2.1 Properties of the potential

It follows from the definition of μ_0 and property (56) that we have the following behavior near the points of the vacuum manifold Σ:

Proposition 2.1. *For any $i = 1, \dots, q$ and any $y \in \mathbb{B}^k(\sigma_i, 2\mu_0)$, we have the local bound*

$$\begin{cases} \frac{1}{4}\lambda_i^-|y-\sigma_i|^2 \le V(y) \le \lambda_i^+|y-\sigma_i|^2, \\ \frac{1}{2}\lambda_i^-|y-\sigma_i|^2 \le \nabla V(y)\cdot(y-\sigma_i) \le 2\lambda_i^+|y-\sigma_i|^2. \end{cases} \tag{2.1}$$

Choosing possibly an even smaller constant μ_0, we have

$$V(y) \ge \alpha_0 \equiv \frac{1}{2}\lambda_0\mu_0^2 \text{ on } \mathbb{R}^k \setminus \bigcup_{i=1}^{q} \mathbb{B}^k(\sigma_i, \frac{\mu_0}{4}), \tag{2.2}$$

where we have set $\lambda_0 = \inf\{\lambda_i^-, i = 1, \dots, q\}$.

The proof relies on a straightforward integration of (56) and we therefore omit it. Proposition 2.1 shows that the potential V essentially behaves as a *positive definite quadratic function near points of the vacuum manifolds* Σ. This observation will be used throughout as a guiding thread. Proposition 2.1 leads to a first elementary observation:

Lemma 2.1. *Let $y \in \mathbb{R}^k$ be such that $V(y) < \alpha_0$, where α_0 is defined in* (2.2)*. Then there exists some point $\sigma \in \Sigma$ such that*

$$|y-\sigma| \le \mu_0.$$

Moreover, we have the upper bound

$$|y-\sigma| \le \sqrt{4\lambda_0^{-1}V(y)}. \tag{2.3}$$

We next turn to the behavior at infinity. For that purpose, we introduce the radius

$$\mathrm{R}_0 = \sup\{|\sigma|, \sigma \in \Sigma\}, \tag{2.4}$$

and study the properties of V on the set $\mathbb{R}^k \setminus \mathbb{B}^k(2\mathrm{R}_0)$.

Proposition 2.2. *There exists a constant $\beta_\infty > 0$ such that*

$$V(y) \ge \beta_\infty|y|^2, \text{ for any } y \text{ such that } |y| \ge 2\mathrm{R}_0. \tag{2.5}$$

Proof. Integrating assumption (H_3) along a line joining y to the origin, we obtain that, for some constant $C_\infty > 0$, we have

$$V(y) \ge \frac{\alpha_\infty|y|^2}{2} - C_\infty, \text{ for any } y \in \mathbb{R}^k. \tag{2.6}$$

It follows that

$$V(y) \ge \frac{\alpha_\infty|y|^2}{4}, \text{ provided } |y| \ge \mathrm{R}'_0 \equiv \sup\left\{2\sqrt{\frac{C_\infty}{\alpha_\infty}}, 4\mathrm{R}_0\right\}. \tag{2.7}$$

On the other hand, by assumption, we have

$$\frac{V(y)}{|y|^2} > 0 \text{ for } y \in \overline{\mathbb{B}^k(\mathrm{R}'_0) \setminus \mathbb{B}^k(2\mathrm{R}_0)},$$

so that, by compactness, we deduce that there exist some constant $\alpha'_\infty > 0$, such that

$$V(y) \geq \alpha'_\infty |y|^2 \text{ for } y \in \overline{\mathbb{B}^k(2\mathrm{R}'_0) \setminus \mathbb{B}^k(2\mathrm{R}_0)}.$$

Combining the last inequality with (2.7), the conclusion follows, by choosing $\beta_\infty = \inf\{\frac{\alpha_\infty}{4}, \alpha'_\infty\}$. □

2.2 Modica-Mortola type inequalities

Let σ_i be an arbitrary element in Σ. We consider the function $\chi_i : \mathbb{R}^k \to \mathbb{R}^+$ defined by

$$\chi_i(y) = \varphi(|y - \sigma_i|), \text{ for } y \in \mathbb{R}^k,$$

where φ denotes a function $\varphi : [0, +\infty) \to \mathbb{R}^+$ such that $0 \leq \varphi' \leq 1$ and

$$\varphi(t) = t, \text{ if } 0 \leq t \leq \frac{\mu_0}{2}, \text{ and } \varphi(t) = \frac{3\mu_0}{4}, \text{ if } t \geq \mu_0.$$

Given a function $u : \Omega \to \mathbb{R}^k$ we finally define the *scalar* function w_i on Ω as

$$w_i(x) = \chi_i(u(x)), \forall x \in \Omega. \tag{2.8}$$

First properties of the map w_i are summarized in the next Lemma.

Lemma 2.2. *Let $w_i : \Omega \to \mathbb{R}$ be defined in (2.8). We have*

$$\begin{cases} w_i(x) = |u(x) - \sigma_i|, \text{ if } |u(x) - \sigma_i| \leq \frac{\mu_0}{2}, \\ w_i(x) = \frac{3\mu_0}{4}, \text{ hence } \nabla w_i = 0 \text{ if } |u(x) - \sigma_i| \geq \mu_0, \\ |\nabla w_i| \leq |\nabla u| \text{ on } \Omega, \end{cases} \tag{2.9}$$

and

$$|\nabla (w_i)^2| \leq 4\sqrt{\lambda_0}^{-1} J(u)(x), \tag{2.10}$$

where we have set

$$J(u) = |\nabla u|\sqrt{V(u)}. \tag{2.11}$$

Proof. Properties (2.9) are straightforward consequences of the definition (2.8). For (2.10), we notice that, in view of the second line in (2.9), we may restrict ourselves to the case $u(x) \in \mathbb{B}^k(\sigma_i, \mu_0)$, since otherwise $\nabla w_i(x) = 0$, and inequality (2.10) is hence straightforwardly satisfied. In that case, it follows from Lemma 2.1 and the fact that $0 \leq \varphi' \leq 1$, that we have

$$|w_i(x)| \leq |u(x) - \sigma_i| \leq \sqrt{4\lambda_0^{-1} V(u(x))}, \text{ for all } x \text{ such that } u(x) \in \mathbb{B}^k(\sigma_i, \mu_0),$$

so that

$$|\nabla (w_i)^2(x)| = 2\,|w_i(x)| \cdot |\nabla w_i(x)| \leq 2|\nabla u|\sqrt{4\lambda_0^{-1} V(u(x))} \leq 4\sqrt{\lambda_0}^{-1} J(u)(x), \tag{2.12}$$

and the proof is complete. □

Lemma 2.3. *We have, for any $x \in \Omega$, the inequality*

$$J(u(x)) \leq e_\varepsilon(u)(x). \tag{2.13}$$

Proof. We have, in view of the definition of J given in (2.11),

$$J(u(x)) = (\sqrt{\varepsilon}|\nabla u(x)|).\sqrt{\varepsilon^{-1}V(u(x))}. \tag{2.14}$$

We invoke next the inequality $ab \leq \frac{1}{2}(a^2+b^2)$ to obtain

$$J(u(x)) \leq \frac{1}{2}\left(\varepsilon|\nabla u(x)|^2 + \varepsilon^{-1}V(u(x))\right) \leq e_\varepsilon(u)(x),$$

which yields the desired result. □

2.3 The one-dimensional case

In dimension 1 estimate (2.10) directly leads to uniform bound on w_i, as expressed in our next result. For that purpose, we consider, for $r > 0$, the circle $\mathbb{S}^1(r) = \{x \in \mathbb{R}^2, |x| = r\}$ and maps $u : \mathbb{S}^1(r) \to \mathbb{R}^k$.

Lemma 2.4. *Let $0 < \varepsilon \leq 1$ and $\varepsilon < r \leq 1$ be given. There exists a constant $\mathrm{C_{unf}} > 0$, depending only on V, such that, for any given map $u : \mathbb{S}^1(r) \to \mathbb{R}^k$, there exists an element $\sigma_{\rm main} \in \Sigma$ such that*

$$|u(\ell) - \sigma_{\rm main}| \leq \mathrm{C_{unf}}\sqrt{\int_{\mathbb{S}^1(r)} \frac{1}{2}(J(u(\tau)) + r^{-1}V(u(\tau)))\mathrm{d}\tau}, \quad \text{for all } \ell \in \mathbb{S}^1(r). \tag{2.15}$$

Hence, we have

$$|u(\ell) - \sigma_{\rm main}| \leq \mathrm{C_{unf}}\sqrt{\int_{\mathbb{S}^1(r)} e_\varepsilon(u)(\tau)\mathrm{d}\tau}, \text{ for } \ell \in \mathbb{S}^1(r). \tag{2.16}$$

Proof. By the mean-value formula, there exists some point $\ell_0 \in \mathbb{S}^1(r)$ such that

$$V(u(\ell_0)) = \frac{1}{2\pi r}\int_{\mathbb{S}^1(r)} V(u(\tau))\mathrm{d}\tau. \tag{2.17}$$

We introduce the quantity

$$\mathrm{Z}_r(u) = \frac{1}{2}\int_{\mathbb{S}^1(r)} (J(u(\tau)) + r^{-1}V(u(\tau)))\mathrm{d}\tau, \tag{2.18}$$

and distinguish next two cases.

Case 1. *The function u satisfies additionally the smallness condition*

$$\mathrm{Z}_r(u) \leq \alpha_1, \text{ where } \alpha_1 = \inf\{\sqrt{\lambda_0}\frac{\mu_0^2}{64}, \frac{\pi\mu_0\lambda_0}{16}, \alpha_0\}, \tag{2.19}$$

α_0 denoting the constant introduced in Proposition 2.1.

We deduce from inequality (2.19) and (2.3) that

$$V(u(\ell_0)) \leq \frac{1}{2\pi r}\int_{\mathbb{S}^1(r)} V(u(\tau))\,\mathrm{d}\tau \leq \frac{\mathrm{Z}_r(u)}{\pi} \leq \mathrm{Z}_r(u) \leq \alpha_1 \leq \alpha_0.$$

It follows from Lemma 2.1 that there exists some $\sigma_{\mathrm{main}} \in \Sigma$ such that

$$\begin{aligned}|u(\ell_0) - \sigma_{\mathrm{main}}|^2 &\leq 4\lambda_0^{-1} V(u(\ell_0)) \leq \frac{2\lambda_0^{-1}}{\pi r}\int_{\mathbb{S}^1(r)} V(u(\tau))\mathrm{d}\tau \leq \frac{4\lambda_0^{-1}}{\pi}\mathrm{Z}_r(u)\\ &\leq \frac{4\lambda_0^{-1}}{\pi}\alpha_1 \leq \frac{\mu_0}{4}, \text{ since } \alpha_1 \leq \frac{\pi\mu_0\lambda_0}{16}.\end{aligned} \tag{2.20}$$

Next we claim that for any $\ell \in \mathbb{S}^1(r)$, we have the bound

$$|u(\ell) - \sigma_{\mathrm{main}}| \leq \frac{\mu_0}{2}. \tag{2.21}$$

Proof of the claim (2.21). We first notice that inequality (2.21) is already satisfied for $\ell = \ell_0$. We argue next by contradiction and assume that there exists some $\ell_1 \in \mathbb{S}^1(r)$ such that $|u(\ell_1) - \sigma_{\mathrm{main}}| > \frac{\mu_0}{2}$. It follows by continuity that there exists some $\ell_2 \in \mathbb{S}^1(r)$ such that

$$\begin{cases} |u(\ell) - \sigma_{\mathrm{main}}| \leq \dfrac{\mu_0}{2} \text{ for any } \ell \in \mathcal{C}(\ell_0, \ell_2),\\ |u(\ell_2) - \sigma_{\mathrm{main}}| = \dfrac{\mu_0}{2}, \end{cases} \tag{2.22}$$

where $\mathcal{C}(\ell_0, \ell_2)$ denotes the arc on $\mathbb{S}^1(r)$ joining counterclockwise the points ℓ_0 and ℓ_2. We obtain, by integration on $\mathcal{C}(\ell_0, \ell_2)$ and using the bound (2.10), the assumption (2.19), inequality (2.20) and the definition of the constant α_1,

$$\begin{aligned}\frac{3\mu_0^2}{16} &\leq |u - \sigma_{\mathrm{main}}|^2(\ell_2) - |u - \sigma_{\mathrm{main}}|^2(\ell_0) \leq \int_{\mathcal{C}(\ell_0,\ell_2)} \left|\nabla |u(\tau) - \sigma_{\mathrm{main}}|^2\right|\\ &\leq \int_{\mathcal{C}(\ell_0,\ell_2)} |\nabla w_{\mathrm{main}}^2| \leq 4\sqrt{\lambda_0^{-1}}\int_{\mathbb{S}^1(r)} J(u)(\tau)\mathrm{d}\tau,\\ &\leq 8\sqrt{\lambda_0^{-1}}\alpha_1 \leq \frac{\mu_0^2}{8},\end{aligned} \tag{2.23}$$

where we have set $w_{\mathrm{main}} = \varphi(|u(\cdot) - \sigma_{\mathrm{main}}|)$. This is a contradiction and hence establishes the claim (2.21).

In view of (2.21) and arguing as for (2.23), we deduce, integrating as above the bound (2.10), that, for any $\ell \in \mathbb{S}^1(r)$, we have

$$\left|\,|u - \sigma_{\mathrm{main}}|^2(\ell) - |u - \sigma_{\mathrm{main}}|^2(\ell_0)\right| \leq 4\sqrt{\lambda_0^{-1}}\int_{\mathbb{S}^1(r)} J(u) \leq 8\sqrt{\lambda_0^{-1}}\mathrm{Z}_r(u). \tag{2.24}$$

Combining (2.24) with the second inequality in (2.20), we obtain

$$|u - \sigma_{\mathrm{main}}|^2(\ell) \leq \left(8\sqrt{\lambda_0^{-1}} + 4\lambda_0^{-1}\right)\mathrm{Z}_r(u), \text{ for any } \ell \in \mathbb{S}^1(r). \tag{2.25}$$

Inequality (2.25) yields the desired result (2.15) in case 1, using the fact that $\varepsilon \leq 1$ and provided the constant $\mathrm{C_{unf}}$ satisfies the bound

$$\mathrm{C}^2_{\mathrm{unf}} \geq 8\sqrt{\lambda_0^{-1}} + 4\lambda_0^{-1}.$$

Case 2. *Inequality* (2.19) *does not hold.* In that case, we have hence

$$\mathrm{Z}_r(u) \geq \alpha_1. \tag{2.26}$$

We consider the number $\mathrm{R}_0 = \sup\{|\sigma|, \sigma \in \Sigma\}$, introduced in definition (2.4) and discuss next three subcases.

Subcase 2a: For any $\ell \in \mathbb{S}^1(r)$, we have $u(\ell) \in \mathbb{B}^k(2\mathrm{R}_0)$.
Then, in this case, for any $\sigma \in \Sigma$, we have, in view of assumption (2.26)

$$|u(\ell) - \sigma|^2 \leq 9\mathrm{R}_0^2 = \left(\frac{9\mathrm{R}_0^2}{\alpha_1}\right)\alpha_1 \leq \left(\frac{9\mathrm{R}_0^2}{\alpha_1}\right)\mathrm{Z}_r(u), \text{ for any } \ell \in \mathbb{S}^1(r). \tag{2.27}$$

Hence, inequality (2.15) is immediately satisfied, whatever the choice of σ_{main}, provided we impose the additional condition on the constant $\mathrm{C_{unf}}$

$$\mathrm{C}^2_{\mathrm{unf}} \geq \frac{9\mathrm{R}_0^2}{\alpha_1}. \tag{2.28}$$

Subcase 2b : There exists some $\ell_1 \in \mathbb{S}^1(r)$, and some $\ell_2 \in \mathbb{S}^1(r)$ such that, we have

$$u(\ell_1) \in \mathbb{B}^k(2\mathrm{R}_0) \text{ and } u(\ell_2) \notin \mathbb{B}^k(2\mathrm{R}_0).$$

Let $\ell \in \mathbb{S}^1(r)$. If $u(\ell) \in \mathbb{B}^k(2\mathrm{R}_0)$, then we argue as in subcase 2a, so that we obtain inequality (2.27) as before, hence (2.15) holds for ℓ, and we are done. Otherwise, by continuity, there exists some $\ell' \in \mathbb{S}^1(r)$ such that $u(\ell') \in \partial\mathbb{B}^k(2\mathrm{R}_0)$ and such that, for any point $a \in \mathcal{C}(\ell, \ell')$, we have $u(a) \notin \mathbb{B}^k(2\mathrm{R}_0)$. We have, by integration,

$$|u(\ell)|^2 - |u(\ell')|^2 \leq 2\int_{\mathcal{C}(\ell,\ell')} |u(a)| \cdot |\nabla u(a)| \,\mathrm{d}a.$$

Using the fact that $|u(a)| \geq 2\mathrm{R}_0$, for $a \in \mathcal{C}(\ell, \ell')$, and inequality (2.5), we obtain

$$|u(a)| \leq \sqrt{\frac{V(u(a))}{\beta_\infty}}, \text{ for } a \in \mathcal{C}(\ell, \ell'),$$

so that, combining the two previous inequalities, we are led to

$$\begin{aligned}|u(\ell)|^2 - |u(\ell')|^2 &\leq \frac{2}{\sqrt{\beta_\infty}}\int_{\mathcal{C}(\ell,\ell')} \sqrt{V(u(a))} \cdot |\nabla u(a)| \,\mathrm{d}a \\ &\leq \frac{2}{\sqrt{\beta_\infty}}\int_{\mathbb{S}^1(r)} J(u(a))\mathrm{d}a.\end{aligned}$$

Since $|u(\ell')| = 2\mathrm{R}_0$, we obtain, for any $\sigma \in \Sigma$,

$$
\begin{aligned}
|u(\ell) - \sigma|^2 &\leq 2\left(|u(\ell)|^2 + |\sigma|^2\right) \leq 2\left(|u(\ell)|^2 + \mathrm{R}_0^2\right) \\
&\leq 2\left(\frac{2}{\sqrt{\beta_\infty}}\int_{\mathbb{S}^1(r)} J(u(a))\mathrm{d}a + \mathrm{R}_0^2 + |u(\ell')|^2\right) \\
&\leq \left(\frac{4}{\sqrt{\beta_\infty}}\int_{\mathbb{S}^1(r)} J(u(a))\mathrm{d}a + 10\mathrm{R}_0^2\right), \text{ since } |u(\ell')| \leq 2\mathrm{R}_0, \\
&\leq \left(\frac{4}{\sqrt{\beta_\infty}}\int_{\mathbb{S}^1(r)} J(u(a))\mathrm{d}a + \frac{10\mathrm{R}_0^2}{\alpha_1}\alpha_1\right) \\
&\leq \left(\frac{8}{\sqrt{\beta_\infty}}\mathrm{Z}_r(u) + \frac{10\mathrm{R}_0^2}{\alpha_1}\mathrm{Z}_r(u)\right) = \left(\frac{8}{\sqrt{\beta_\infty}} + \frac{10\mathrm{R}_0^2}{\alpha_1}\right)\mathrm{Z}_r(u),
\end{aligned}
$$

so that the conclusion (2.15) follows for any choice of $\sigma_{\text{main}} \in \Sigma$, imposing again an appropriate lower bound on the constant C_{unf}.

Subcase 2c : For any $\ell \in \mathbb{S}^1(r)$, we have

$$|u(\ell)| \geq 2\mathrm{R}_0.$$

Let ℓ_0 satisfy (2.17), so that, in view of Proposition 2.2

$$|u(\ell_0)|^2 \leq \frac{1}{\beta_\infty}V(u(\ell_0)) = \frac{1}{\beta_\infty}\left(\frac{1}{2\pi r}\int_{\mathbb{S}^1(r)} V(u(\ell))\mathrm{d}\ell\right).$$

We obtain hence, for any arbitrary $\sigma \in \Sigma$

$$
\begin{aligned}
|u(\ell_0) - \sigma|^2 &\leq 2\left(|u(\ell_0)|^2 + |\sigma|^2\right) \leq \frac{2}{\beta_\infty}\left(\frac{1}{2\pi r}\int_{\mathbb{S}^1(r)} V(u(\ell))\mathrm{d}\ell + \mathrm{R}_0^2\beta_\infty\right) \\
&\leq \frac{2}{\beta_\infty}\left(\frac{1}{\pi}\mathrm{Z}_r(u) + \alpha_1\left(\frac{\mathrm{R}_0^2\beta_\infty}{\alpha_1}\right)\right) \\
&\leq \frac{2}{\pi\beta_\infty}\left(1 + \left(\frac{2\pi\mathrm{R}_0^2\beta_\infty}{\alpha_1}\right)\right)\mathrm{Z}_r(u).
\end{aligned}
\tag{2.29}
$$

This yields again (2.15) for an arbitrary choice of $\sigma_{\text{main}} \in \Sigma$ and imposing an additional suitable lower bound on C_{unf}.

We have hence established for upper bound (2.15) in all three possible cases $2a, 2b$ and $2c$, for an arbitrary choice of $\sigma_{\text{main}} \in \Sigma$ and imposing an additional suitable lower bound on C_{unf}. It is hence established in case 2. Since we already established it in Case 1, the proof of (2.15) is complete.

Turning to inequality (2.16), we first observe that, since by assumption $r \geq \varepsilon$, we have

$$r^{-1}\int_{\mathbb{S}^1(r)} V(u(\ell))\mathrm{d}\ell \leq \int_{\mathbb{S}^1(r)} \varepsilon^{-1}V(u(\ell))\mathrm{d}\ell \leq \int_{\mathbb{S}^1(r)} e_\varepsilon(u)(\ell)\mathrm{d}\ell. \tag{2.30}$$

Combining (2.15) with (2.13) and (2.30), we obtain the desired result (2.16). □

2.4 Controlling energy and uniform bound on "good" circles

When working on two-dimensional disks, the tools developed in the previous section allow to choose radii with appropriate control on the energy, invoking a standard mean-value argument. More precisely, we have:

Lemma 2.5. *Let $\varepsilon \leq r_0 < r_1 \leq 1$ and $u : \mathbb{D}^2 \to \mathbb{R}^k$ be given. There exists a radius $\mathfrak{r}_\varepsilon \in [r_0, r_1]$ such that*

$$\int_{\mathbb{S}^1(\mathfrak{r}_\varepsilon)} e_\varepsilon(u)(\ell)\mathrm{d}\ell \leq \frac{1}{r_1 - r_0}\,\mathrm{E}_\varepsilon(u, \mathbb{D}^2(r_1)).$$

The proof is based on a classical mean-value argument, therefore we omit it. □

In the sequel, we will often make use of Lemma 2.5 combined with the uniform bounds obtained in dimension one. For instance, it follows from Lemma 2.4 that there exists some point $\sigma_{\mathfrak{r}_\varepsilon} \in \Sigma$, *depending on* $\mathfrak{r}_\varepsilon$, such that

$$|u(\ell) - \sigma_{\mathfrak{r}_\varepsilon}| \leq \frac{\mathrm{C}_{\mathrm{unf}}}{\sqrt{r_1 - r_0}}\sqrt{\mathrm{E}_\varepsilon(u, \mathbb{D}^2(r_1))}, \quad \text{for all } \ell \in \mathbb{S}^1(\mathfrak{r}_\varepsilon). \tag{2.31}$$

Moreover, it follows from (2.13) that

$$\int_{\mathbb{S}^1(\mathfrak{r}_\varepsilon)} |J(u)| \leq \frac{1}{r_1 - r_0}\int_{\mathbb{D}^2(r_1)} e_\varepsilon(u)\mathrm{d}x. \tag{2.32}$$

2.5 BV estimates and the coarea formula

The right-hand side of estimate (2.15), in particular the term involving $J(u)$, may be interpreted as a *BV* estimate (as in [31]). In dimension 1, as expected, it yields uniform estimates on u. In higher dimensions of course, this is no longer true. Nevertheless our *BV*-estimates lead to useful estimates for the measure of specific level sets. In order to state the kind of results we have in mind, we consider more generally an arbitrary smooth function $\phi : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^N$ is an arbitrary N-dimensional domain, and introduce, for a given number $s \in \mathbb{R}$, the level set

$$\phi^{-1}(s) = \{x \in \Omega, \text{ such that } \phi(x) = s\}.$$

If ϕ is assumed to be sufficiently smooth, then Sard's theorem asserts that $\phi^{-1}(s)$ is a regular submanifold of dimension $(N-1)$, for almost every $s \in \mathbb{R}$, and the coarea formula relates the integral of the total measures of these level sets to the *BV*-norm through the formula

$$\int_{\mathbb{R}} \mathcal{H}^{N-1}\left(\phi^{-1}(s)\right)\mathrm{d}s = \int_\Omega |\nabla\phi(x)|\mathrm{d}x. \tag{2.33}$$

We specify this formula to our needs in the case $N = 2$, $\Omega = \mathbb{D}^2(r)$, for some $r > \varepsilon$, and $\phi = (w_i)^2 : \Omega \to \mathbb{R}^+$, where $i \in \{1, \dots, q\}$ and where $w_i : \Omega \to \mathbb{R}$ is the map constructed in (2.8) for a given $u : \Omega \to \mathbb{R}^k$. Combining (2.33) with (2.10) and (2.13), we are led to the inequality, for the level sets $(w_i^2)^{-1}(s) \subset \Omega = \mathbb{D}^2(r)$,

$$\begin{aligned}\int_{\mathbb{R}^+} \mathcal{L}\left((w_i^2)^{-1}(s)\right)\mathrm{d}s &\leq 4\sqrt{\lambda_0}^{-1}\int_{\mathbb{D}^2(r)} J(u(x))\mathrm{d}x \\ &\leq 4\sqrt{\lambda_0}^{-1}\int_{\mathbb{D}^2(r)} e_\varepsilon(u)\mathrm{d}x = 4\sqrt{\lambda_0}^{-1}\mathrm{E}_\varepsilon\left(u, \mathbb{D}^2(r)\right),\end{aligned} \tag{2.34}$$

where $\mathcal{L} = \mathcal{H}^1$ denotes length. In several places, we will invoke this inequality jointly with a mean value argument. This kind of argument yields:

Lemma 2.6. *Let u, w_i and $r > \varepsilon$ be as above. Given any number $A > 0$, there exists some number $A_0 \in [\frac{A}{2}, A]$ such that $w_i^{-1}(A_0)$ is a regular curve in $\mathbb{D}^2(r)$ and such that*

$$\mathcal{L}\left(w_i^{-1}(A_0)\right) \leq \frac{8}{\sqrt{\lambda_0}A^2}\int_{\mathbb{D}^2(r)} e_\varepsilon(u)\mathrm{d}x \leq \frac{8\,\mathrm{E}_\varepsilon\left(u, \mathbb{D}^2(r)\right)}{\sqrt{\lambda_0}A^2}. \tag{2.35}$$

Proof. In view of the definition (2.8), the map w_i takes values in the interval $[0, \frac{3\mu_0}{4}]$, so that $w_i^{-1}(s) = \emptyset$, if $s > \frac{3\mu_0}{4}$. Hence, it suffices only to consider the case $A \leq \frac{3\mu_0}{4}$. We introduce to that aim the domain $\Omega_{i,A} = \{x \in \mathbb{D}^2(r), \frac{A}{2} \leq |u(x) - \sigma_i| \leq A\}$. Using formula (2.34) on this domain, we are led to the inequality

$$\int_{\frac{A^2}{4}}^{A^2} \mathcal{L}((w_i^2)^{-1}(s))\mathrm{d}s \leq 4\sqrt{\lambda_0}^{-1}\int_{\Omega_{i,A}} e_\varepsilon(u)\mathrm{d}x \leq 4\sqrt{\lambda_0}^{-1}\mathrm{E}_\varepsilon\left(u, \mathbb{D}^2(r)\right).$$

The conclusion follows by a mean-value argument. □

2.6 Controlling uniform bounds on good circles

Whereas in Subsection 2.4 we have selected radii with controlled energy for the map u, in this subsection, we select radii with appropriate *uniform bounds* on u. We assume throughout this subsection that we are given a radius $\varrho \in [\frac{1}{2}, 1]$, a number $0 < \kappa < \frac{\mu_0}{2}$, a smooth map $u : \overline{\mathbb{D}^2(\varrho)} \to \mathbb{R}^k$ and an element $\sigma \in \Sigma$ such that

$$|u - \sigma| < \frac{\kappa}{2} \text{ on } \partial\mathbb{D}^2(\varrho). \tag{2.36}$$

We introduce the subset $\mathcal{I}(u, \kappa)$ of radii $r \in [\frac{1}{2}, \varrho]$ such that

$$\mathcal{I}(u, \kappa) = \left\{ r \in [\frac{1}{2}, \varrho] \text{ such that } |u(\ell) - \sigma| \leq \kappa, \forall \ell \in \mathbb{S}^1(r) \right\}. \tag{2.37}$$

Notice that we have, for $\kappa' \leq \kappa$,

$$\mathcal{I}(u, \kappa') \subset \mathcal{I}(u, \kappa), \text{ and hence } |\mathcal{I}(u, \kappa)| \geq |\mathcal{I}(u, \kappa')|. \tag{2.38}$$

We have:

Proposition 2.3. *Assume that* (2.36) *holds. Then, the following lower bound on the measure of $\mathcal{I}(u, \kappa)$ holds*

$$|\mathcal{I}(u, \kappa)| \geq \varrho - \frac{9}{16}, \tag{2.39}$$

provided we have the lower bound on κ

$$\kappa \geq \mathrm{C_{lev}}\sqrt{\mathrm{E}_\varepsilon(u, \mathbb{D}^2(\varrho))}, \text{ where } \mathrm{C_{lev}} = \sqrt{\frac{32}{\sqrt{\lambda_0}}}. \tag{2.40}$$

Proof. Set $w = \varphi(|\cdot - \sigma|)$. We apply Lemma 2.6 with the choice $r = \varrho$, $A = \kappa$ and $\sigma_i = \sigma$, so that $w = w_i$. This yields a number $A_0 \in [\frac{\kappa}{2}, \kappa]$ such that $w^{-1}(A_0) \subset \mathbb{D}^2(\varrho)$ is smooth and verifies

$$\mathcal{L}(w^{-1}(A_0)) \leq \frac{8\mathrm{E}_\varepsilon(u, \mathbb{D}^2(\varrho))}{4\sqrt{\lambda_0}\kappa^2} = \frac{2\mathrm{E}_\varepsilon(u, \mathbb{D}^2(\varrho))}{\sqrt{\lambda_0}\kappa^2}.$$

If moreover (2.40) is satisfied, then we have

$$\mathcal{L}(w^{-1}(A_0)) < \frac{1}{16}. \tag{2.41}$$

Since, by definition, $A_0 \leq \kappa$, it follows from (2.38) with $\kappa' = A_0$, that $|\mathcal{I}(u, A_0)| \leq |\mathcal{I}(u, \kappa)|$, so that it suffices to establish the lower bound

$$|\mathcal{I}(u, A_0)| \geq \varrho - \frac{9}{16}. \tag{2.42}$$

For that purpose, we introduce the auxiliary set

$$\mathcal{Z}(u, A_0) = \{r \in [\frac{1}{2}, \varrho], \text{ such that } |u(\ell) - \sigma| > A_0, \forall \ell \in \mathbb{S}^1(r)\}.$$

We first claim that

$$\mathcal{Z}(u, A_0) = \emptyset. \tag{2.43}$$

Indeed, assume by contradiction that (2.43) does not hold: In that case, there exists some radius $\frac{1}{2} \leq r_0 \leq \varrho$ in $\mathcal{Z}(u, A_0)$. In view of the definition of $\mathcal{Z}(u, A_0)$, we have therefore

$$|u(\ell) - \sigma| > A_0, \text{ for any } \ell \in \partial\mathbb{D}^2(r_0). \tag{2.44}$$

On the other hand, in view of assumption (2.36), we have

$$|u(\ell) - \sigma| < \frac{\kappa}{2} \leq A_0, \text{ for } \ell \in \partial\mathbb{D}^2(\varrho) = \mathbb{S}^1(\varrho),$$

so that

$$w^{-1}(A_0) \cap \big(\mathbb{S}^1(\varrho) \cup \mathbb{S}^1(r_0)\big) = \emptyset.$$

Combining (2.44) and (2.36), it follows from the intermediate value theorem that there exists some smooth domain V such that $|u(x) - \sigma| = A_0$ for $x \in \partial V$, so that $\partial V \subset w^{-1}(A_0)$, and hence is smooth, and such that

$$\mathbb{D}^2(r_0) \subset V \subset \mathbb{D}^2(\varrho). \tag{2.45}$$

We deduce from (2.45) that, since by assumption $1/2 \leq r_0 \leq \varrho$,

$$\partial V \subset w^{-1}(A_0) \text{ and } \mathcal{L}(\partial V) \geq 2\pi r_0 \geq \pi,$$

so that

$$\mathcal{L}(w^{-1}(A_0)) \geq \pi > \frac{1}{16}.$$

This however contradicts inequality (2.41), and hence establishes the claim (2.43). We next establish (2.42). For that purpose, consider an arbitrary radius $\frac{1}{2} \leq r \leq \varrho$ such that $r \notin \mathcal{I}(u, A_0)$ (see Figure 8). It follows from the definition of $\mathcal{I}(u, A_0)$ that there exists some

$\ell_r \in \mathbb{S}^1(r)$ such that $|u(\ell_r) - \sigma| > A_0$. On the other hand, we deduce from (2.43), that there exists some $\ell'_r \in \mathbb{S}^1(r)$ such that

$$|u(\ell'_r) - \sigma| < A_0.$$

Hence, by the intermediate value theorem, there exists some point $\check{\ell}_r \in \mathbb{S}^1(r)$ such that $|u(\check{\ell}_r) - \sigma| = A_0$, so that $\check{\ell}_r \in w^{-1}(A_0)$. Hence, we have

$$w^{-1}(A_0) \cap \mathbb{S}^1(r) \neq \emptyset, \ \forall r \in [\frac{1}{2}, \varrho] \setminus \mathcal{I}(u, A_0). \tag{2.46}$$

Notice that $\left|[\frac{1}{2}, \varrho] \setminus \mathcal{I}(u, A_0)\right| = \left(\varrho - \frac{1}{2}\right) - |\mathcal{I}(u, A_0)|$. Hence, relation (2.46) implies, by Fubini's theorem, that

$$\mathcal{L}(w^{-1}(A_0)) \geq \left(\varrho - \frac{1}{2}\right) - |\mathcal{I}(u, A_0)|,$$

so that

$$|\mathcal{I}(u, A_0)| \geq \left(\varrho - \frac{1}{2}\right) - \mathcal{L}(w^{-1}(A_0)) \geq \varrho - \frac{9}{16}, \tag{2.47}$$

where we made use of estimate (2.41). This establishes (2.42), and hence completes the proof. □

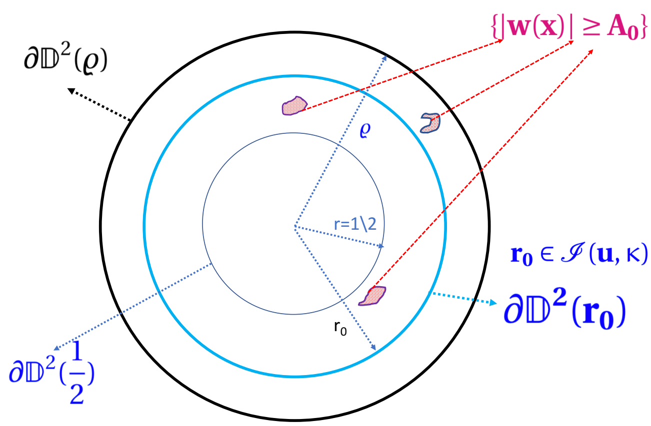

Figure 8: *The circle $\partial\mathbb{D}^2(r_0)$ does not interset the level set $\mathcal{L}(w^{-1}(A_0))$.*

2.7 Revisiting the control of the energy on concentric circles

Using the results of the previous section, we work out variants of the Lemma 2.5. For that purpose, given a radius $\varrho \in [\frac{3}{4}, 1]$, a number $0 < \kappa \leq \dfrac{\mu_0}{2}$, a smooth map $u : \overline{\mathbb{D}^2(\varrho)} \to \mathbb{R}^k$ and an element $\sigma \in \Sigma$ such that (2.36) holds, we introduce the set

$$\Upsilon_\sigma(u, \varrho, \kappa) = \left\{x \in \mathbb{D}^2(\varrho), \text{ such that } |u(x) - \sigma| \leq \kappa\right\}. \tag{2.48}$$

The following result is a major tool in the proof of our main results:

Lemma 2.7. *Let u, ϱ and κ be as above, assume that* (2.36) *holds on the boundary $\partial\mathbb{D}^2(\varrho)$ and assume that the bound* (2.40) *on κ holds also. Assume moreover that $\varrho \geq \frac{3}{4}$. Then there exists a radius $\tau_\varepsilon \in [\frac{5}{8}, \varrho]$ such that $\mathbb{S}^1(\tau_\varepsilon) \subset \Upsilon_\sigma(u, \varrho, \kappa)$, i.e.*

$$|u(\ell) - \sigma| \leq \kappa, \text{ for any } \ell \in \mathbb{S}^1(\tau_\varepsilon),$$

and such that

$$\int_{\mathbb{S}^1(\tau_\varepsilon)} e_\varepsilon(u)\mathrm{d}\ell \leq \frac{1}{\varrho - \frac{11}{16}} \mathrm{E}_\varepsilon(u, \Upsilon_\sigma(u, \varrho, \kappa)).$$

Proof. In view of definition (2.48) of $\Upsilon_\sigma(u, \varrho, \kappa)$ and the definition (2.37) of $\mathcal{I}(u, \kappa)$, we have $\mathbb{S}^1(r) \subset \Upsilon_\sigma(u, \varrho, \kappa)$ for any $r \in \mathcal{I}(u, \kappa)$, so that, by Fubini's theorem, we have

$$\int_{\mathcal{I}(u,\kappa)} \left(\int_{\mathbb{S}^1(\varrho)} e_\varepsilon(u)\mathrm{d}\ell \right) \mathrm{d}\varrho \leq \int_{\Upsilon_\sigma(u,\varrho,\kappa)} e_\varepsilon(u)\mathrm{d}x = \mathrm{E}_\varepsilon(u, \Upsilon_\sigma(u, \varrho, \kappa)).$$

Since we assume that the bound (2.40) holds, it follows from Proposition 2.3 that

$$\begin{cases} |\mathcal{I}(u, \kappa)| \geq \varrho - \dfrac{9}{16} \text{ and hence} \\ |\mathcal{I}(u, \kappa) \cap [\dfrac{5}{8}, \varrho]| \geq \varrho - \dfrac{11}{16}. \end{cases} \tag{2.49}$$

The second inequality in (2.49) follows from the fact that, by definition, we have $\mathcal{I}(u, \kappa) \subset [1/2, \varrho]$ and the identity $|[1/2, \varrho] \cap [5/8, \varrho]| = |[1/2, 5/8]| = 1/8$. Hence by a mean value argument, we deduce that there exists some radius $\tau_\varepsilon \in [\frac{5}{8}, \varrho] \cap \mathcal{I}(u, \kappa)$ such that

$$\int_{\mathbb{S}^1(\tau_\varepsilon)} e_\varepsilon(u)\mathrm{d}\ell \leq \frac{1}{\varrho - \frac{11}{16}} \int_{\Upsilon_\sigma(u,\varrho,\kappa)} e_\varepsilon(u)\mathrm{d}x,$$

which is precisely the conclusion. □

Comment. The result above will be used in connection with the estimates for u when $u = u_\varepsilon$ is the solution to (1). Thanks to the equation, we will be able to estimate the growth of $\mathrm{E}_\varepsilon(u, \Upsilon_\sigma(u, \varrho, \kappa))$ with κ. We will choose κ as small as possible in order to satisfy (2.40). This merely amounts to choose it of the magnitude of $\sqrt{\mathrm{E}_\varepsilon(u)}$, as we will see in (5.1).

2.8 Gradient estimates on level sets

We go back first to the general setting introduced in Subsection 2.5. Given an arbitrary smooth function $\phi : \Omega \to \mathbb{R}$, where Ω denotes an open subset of $\mathbb{R}^N$, and an arbitrary integrable function $f : \Omega \to \mathbb{R}$, the coarea formula (2.33) generalized as

$$\int_{\mathbb{R}} \left(\int_{\phi^{-1}(s)} f(\ell)\mathrm{d}\ell \right) \mathrm{d}s = \int_\Omega |\nabla\phi(x)| f(x)\mathrm{d}x. \tag{2.50}$$

For the vectorial case, given a smooth function $u : \Omega \to \mathbb{R}^k$, we specify identity (2.50) with choices $\phi = |u|$ and $f = |\nabla u|$: We are hence led to the identity

$$\begin{aligned} \int_{\mathbb{R}} \left(\int_{|u|^{-1}(s)} |\nabla u|(\ell)\mathrm{d}\ell \right) \mathrm{d}s &= \int_\Omega |\nabla u(x)|.|\nabla |u||\mathrm{d}x, \\ &\leq \int_\Omega |\nabla u(x)|^2 \mathrm{d}x. \end{aligned} \tag{2.51}$$

We specify furthermore this formula, as in Subsection 2.5, for a given map u defined on a disk $\mathbb{D}^2(r)$ and w_i being the corresponding maps w_i defined on $\mathbb{D}^2(r)$ by formula (2.8). We introduce the subdomain

$$\begin{aligned}\Theta(u,r) &= \left\{x \in \mathbb{D}^2(r), \text{ such that } u(x) \in \mathbb{R}^k \setminus \bigcup_{i=1}^{q} \mathbb{B}^k(\sigma_i, \frac{\mu_0}{4})\right\} \\ &= u^{-1}\left(\mathbb{R}^k \setminus \bigcup_{i=1}^{q} \mathbb{B}^k(\sigma_i, \frac{\mu_0}{4})\right) = \mathbb{D}^2(r) \setminus \bigcup_{i=1}^{q} \mathring{\Upsilon}_{\sigma_i}(u, r, \frac{\mu_0}{4}).\end{aligned} \tag{2.52}$$

We have:

Lemma 2.8. *Let u be as above. There exists some number $\tilde{\mu} \in [\frac{\mu_0}{4}, \frac{\mu_0}{2}]$, where μ_0 denotes the constant introduced in* (56)*, such that*

$$\sum_{i=1}^{q} \int_{w_i^{-1}(\tilde{\mu})} |\nabla u|(\ell)\mathrm{d}\ell \leq \frac{4}{\mu_0} \int_{\Theta(u,r)} |\nabla u|^2 \leq \frac{8}{\mu_0 \varepsilon} \mathrm{E}_\varepsilon(u, \Theta(u,r)). \tag{2.53}$$

Proof. It follows from identity (2.51), applied to $w_i = \varphi(u - \sigma_i)$, that

$$\begin{aligned}\sum_{i=1}^{q} \int_{\frac{\mu_0}{4}}^{\frac{\mu_0}{2}} \left(\int_{w_i^{-1}(s)} |\nabla u|(\ell)\mathrm{d}\ell\right)\mathrm{d}s &= \int_{\frac{\mu_0}{4}}^{\frac{\mu_0}{2}} \sum_{i=1}^{q} \left(\int_{w_i^{-1}(s)} |\nabla u|(\ell)\mathrm{d}\ell\right)\mathrm{d}s \\ &\leq \int_{\Theta(u,r)} |\nabla u|^2 \mathrm{d}x.\end{aligned} \tag{2.54}$$

We conclude once more by a mean-value argument. □

Part II : PDE Analysis

3 Some properties of the PDE

In this section, we recall first several classical properties of the solutions to the equation (1). We then provide some energy and potential estimates (see e.g. [11]).

3.1 Uniform bound through the maximum principle

The following uniform upper bound is standard:

Proposition 3.1. *Let $u_\varepsilon \in H^1(\Omega)$ be a solution of* (1)*. Then we have the uniform bound bound, for $x \in \Omega$*

$$|u_\varepsilon(x)|^2 \leq \frac{4\mathrm{C}_{\mathrm{unf}}}{\mathrm{dist}(x, \partial\Omega)} \mathrm{E}_\varepsilon(u_\varepsilon) + \mathrm{K}_{\mathrm{unf}}, \text{ where } \mathrm{K}_{\mathrm{unf}} = 2\mathrm{R}_0^2 + \frac{C_\infty}{\alpha_\infty}, \tag{3.1}$$

where $\mathrm{R}_0 = \sup\{|\sigma|, \sigma \in \Sigma\}$ is defined in (2.4)*, and where $\mathrm{C}_{\mathrm{unf}}$ is defined in Lemma 2.4.*

Proof. We argue as in [10]. We compute, using equation (1)

$$\begin{aligned}\frac{1}{2}\Delta|u_\varepsilon|^2 &= u_\varepsilon\cdot\Delta u_\varepsilon + |\nabla u_\varepsilon|^2 = \varepsilon^{-2}u_\varepsilon\cdot\nabla_u V(u_\varepsilon) + |\nabla u_\varepsilon|^2 \\ &\geq \varepsilon^{-2}u_\varepsilon\cdot\nabla_u V(u_\varepsilon), \text{ on } \Omega.\end{aligned} \tag{3.2}$$

On the other hand, it follows from assumption (4), that there exists some constant $C_\infty \geq 0$ such that

$$y.\nabla V(y) \geq \alpha_\infty|y|^2 - C_\infty, \text{ for any } y\in\mathbb{R}^k, \tag{3.3}$$

see inequality (2.6). Hence, combining (3.2) and (3.3) we obtain the inequality

$$-\frac{1}{2}\Delta|u_\varepsilon|^2 + \alpha_\infty\varepsilon^{-2}\left(|u_\varepsilon|^2 - \frac{C_\infty}{\alpha_\infty}\right) \leq 0 \text{ on } \Omega.$$

We introduce next the function $W_\varepsilon = |u_\varepsilon|^2 - \frac{C_\infty}{\alpha_\infty}$. We are led to the differential inequality for W_ε

$$-\frac{1}{2}\Delta W_\varepsilon + \alpha_\infty\varepsilon^{-2}W_\varepsilon \leq 0 \text{ on } \Omega. \tag{3.4}$$

Let $x\in\Omega$ and set $R_x = \mathrm{dist}(x,\partial\Omega)$, so that $\mathbb{D}^2(x,R_x)\subset\Omega$. It follows from Lemma 2.5 and inequality (2.31) that there exists some radius $\tau\in[\frac{R_x}{2}, R_x]$ and some element $\sigma\in\Sigma$ such that

$$|u_\varepsilon(\ell)-\sigma| \leq \frac{\sqrt{2}\mathrm{C}_{\mathrm{unf}}}{\sqrt{R_x}}\sqrt{\mathrm{E}_\varepsilon(u_\varepsilon,\mathbb{D}^2(R_x))} \leq \frac{\sqrt{2}\mathrm{C}_{\mathrm{unf}}}{\sqrt{R_x}}\sqrt{\mathrm{E}_\varepsilon(u_\varepsilon)}), \quad \text{for all } \ell\in\mathbb{S}^1(x,\tau),$$

and hence

$$|u_\varepsilon(\ell)|^2 \leq \frac{4\mathrm{C}_{\mathrm{unf}}}{R_x}\mathrm{E}_\varepsilon(u_\varepsilon) + 2\mathrm{R}_0^2, \quad \text{for all } \ell\in\mathbb{S}^1(x,\tau), \tag{3.5}$$

where $\mathbb{S}^1(x,\tau) = \{\ell\in\mathbb{R}^2, |\ell-x| = \tau\}$. We consider the function

$$\tilde{W}_\varepsilon = W_\varepsilon - \mathrm{N}(u_\varepsilon), \text{ where } \mathrm{N}(u_\varepsilon) = \left(\frac{4\mathrm{C}_{\mathrm{unf}}}{R_x}\mathrm{E}_\varepsilon(u_\varepsilon) + 2\mathrm{R}_0^2\right) \geq 0,$$

and notice that, in view of (3.4) and (3.5), we have

$$\begin{cases} -\frac{1}{2}\Delta\tilde{W}_\varepsilon + \alpha_\infty\varepsilon^{-2}\tilde{W}_\varepsilon = -\Delta W_\varepsilon + \alpha_\infty\varepsilon^{-2}W_\varepsilon - \alpha_\infty\varepsilon^{-2}\mathrm{N}(u_\varepsilon) \leq 0, \text{ on } \mathbb{D}^2(x,\tau) \\ \tilde{W}_\varepsilon(\ell) \leq 0 \text{ for } \ell\in\partial\mathbb{D}^2(x,\tau) = \mathbb{S}^1(x,\tau), \end{cases} \tag{3.6}$$

We may hence apply the maximum principle to $\tilde{W}_\varepsilon$, to assert that

$$\tilde{W}_\varepsilon(y) = |u_\varepsilon(y)|^2 - \mathrm{N}(u_\varepsilon) - \frac{C_\infty}{\alpha_\infty} \leq 0, \text{ for } y\in\mathbb{D}^2(x,\tau).$$

Choosing $y = x$, the conclusion follows from the definition of $\mathrm{N}(u_\varepsilon)$. □

3.2 Regularity and gradient bounds

The next result is a standard consequence of the smoothness of the potential, the regularity theory for the Laplacian and the maximum principle.

Proposition 3.2. *Let $u_\varepsilon \in H^1(\Omega)$ be a solution of* (1) *and $\delta > 2\varepsilon$. Set*

$$\mathcal{O}_\delta = \{x \in \Omega, \mathrm{dist}(x, \partial\Omega) \geq \delta\}.$$

Then u_ε is smooth on Ω and there exists a constant $C_{\mathrm{gd}}\left(\|u_\varepsilon\|_{L^\infty(\mathcal{O}_{\delta/2})}, \delta\right)$, depending only on V, $\|u_\varepsilon\|_{L^\infty(\mathcal{O}_{\delta/2})}$ and δ such that

$$|\nabla u_\varepsilon|(x) \leq \frac{C_{\mathrm{gd}}\left(\|u_\varepsilon\|_{L^\infty(\mathcal{O}_{\delta/2})}, \delta\right)}{\varepsilon}, \text{ if } \mathrm{dist}(x, \partial\Omega) \geq \delta. \tag{3.7}$$

Proof. Estimate (3.7) is a consequence of Lemma A.1 of [10]. It asserts that, if v is a solution on some domain $\mathcal{U}$ of $\mathbb{R}^n$ of $-\Delta v = f$, then we have the inequality

$$|\nabla v|^2(x) \leq C\left(\|f\|_{L^\infty(\mathcal{U})}\|v\|_{L^\infty(\mathcal{U})} + \frac{1}{\mathrm{dist}(x, \partial\mathcal{U})^2}\|v\|^2_{L^\infty(\mathcal{U})}\right), \text{ for all } x \in \mathcal{U}. \tag{3.8}$$

We apply inequality (3.8) to the solution u_ε, with source term $f = \varepsilon^{-2}\nabla_u V(u_\varepsilon)$ on the domain $\mathcal{U} = \mathcal{O}_{\frac{\delta}{2}}$: This yields (3.7). □

Whereas the result of Proposition 3.2 involves the uniform norm of u_ε, the next results provides a related results, involving the energy $\mathrm{E}_\varepsilon(u_\varepsilon)$.

Proposition 3.3. *Let $u_\varepsilon \in H^1(\Omega)$ be a solution of* (1)*, $\delta > 2\varepsilon$, $M > 0$, and assume that that $\mathrm{E}_\varepsilon(u_\varepsilon) \leq M$. There exists some constant $K_{\mathrm{dr}}(M, \delta) > 0$, depending only on the potential V, M and δ, such that,*

$$|\nabla u_\varepsilon|(x) \leq \frac{K_{\mathrm{dr}}(M, \delta)}{\varepsilon}, \text{ if } \mathrm{dist}(x, \partial\Omega) \geq \delta. \tag{3.9}$$

Proof. We invoke the uniform estimates provided by Proposition 3.1. We have, indeed, in view of (3.1), the uniform upper bound, for u_ε

$$|u_\varepsilon(x)|^2 \leq C\left(\frac{M}{\delta} + 1\right), \text{ for } x \in \mathcal{O}_{\frac{\delta}{2}}.$$

Combining this bound with (3.7) we derive the conclusion. □

3.3 Gradient term versus potential term: First estimates

Major ingredients in the proof of our main PDE result, namely Proposition 1, are provided in Proposition 4.2 and Proposition 4.4, which we will state below and prove a little later. They roughly states that the *total energy*, which involves both a gradient term and a potential terms, can "essentially" be bounded by *the integral of the sole potential term.* In order to derive these results, we are led to divide domains into two regions:

- the region where the solution is near the potential wells Σ,
- the region where it is far from Σ.

Whereas the region where the solution is *near the potential wells* requires some further analysis, the region where the solution is far from the wells can be handled thanks to the results of the previous subsection, in particular the gradient bound described in Proposition 3.2. Restricting ourselves to the case u_ε is defined on $\Omega = \mathbb{D}^2$, we consider, for $r > 0$, the set

$$\begin{aligned}\Theta(u_\varepsilon, r) &= \left\{ x \in \mathbb{D}^2(r), \text{ such that } u_\varepsilon(x) \in \mathbb{R}^k \setminus \overset{q}{\underset{i=1}{\cup}} \mathbb{B}^k(\sigma_i, \frac{\mu_0}{4}) \right\} \\ &= \left(u_{\varepsilon_{|\mathbb{D}^2(r)}} \right)^{-1} \left(\mathbb{R}^k \setminus \overset{q}{\underset{i=1}{\cup}} \mathbb{B}^k(\sigma_i, \frac{\mu_0}{4}) \right) = \mathbb{D}^2(r) \setminus \overset{q}{\underset{i=1}{\cup}} \mathring{\Upsilon}_{\sigma_i}(u_\varepsilon, r, \frac{\mu_0}{4}),\end{aligned} \tag{3.10}$$

where $\Theta(\cdot, r)$ has already been defined in (2.52) and describes the region where the solution is *far from* Σ. Indeed, we have, by definition

$$\operatorname{dist}(u_\varepsilon(x), \Sigma) \geq \frac{\mu_0}{4}, \text{ for } x \in \Theta(u_\varepsilon, r). \tag{3.11}$$

The integral of the energy on the set $\Theta(u_\varepsilon, 3/4)$ can be estimated by the integral of the potential as follows:

Lemma 3.1. *Let $u_\varepsilon \in H^1(\mathbb{D}^2)$ be a solution of* (1). *There exist a constant $C_{\rm pt}\left(\|u\|_{L^\infty(\mathbb{D}^2(7/8))}\right)$ depending only on V and $\|u\|_{L^\infty(\mathbb{D}^2(\frac{7}{8})}$ such that*

$$e_\varepsilon(u_\varepsilon) \leq C_{\rm pt}\left(\|u\|_{L^\infty(\mathbb{D}^2(\frac{7}{8})}\right) \frac{V(u_\varepsilon)}{\varepsilon}, \quad \text{on } \Theta\left(u_\varepsilon, \frac{3}{4}\right). \tag{3.12}$$

Let $M > 0$ and assume that $\mathrm{E}_\varepsilon(u_\varepsilon) \leq M$. There exists a constant $C_{\rm T}$ depending only on the potential V and on M such that we have the pointwise inequality

$$e_\varepsilon(u_\varepsilon) \leq C_{\rm T}(M) \frac{V(u_\varepsilon)}{\varepsilon} \quad \text{on } \Theta\left(u_\varepsilon, \frac{3}{4}\right). \tag{3.13}$$

Proof. It follows from the definition of Θ and in view of inequalities (2.2) or (2.3) that

$$V(u_\varepsilon(x)) \geq \frac{\alpha_0}{16}, \text{ for } x \in \Theta(u_\varepsilon, \frac{3}{4}). \tag{3.14}$$

Since, by definition $\Theta\left(u_\varepsilon, \frac{3}{4}\right) \subset \mathbb{D}^2(3/4)$, we have $\operatorname{dist}(x, \partial\mathbb{D}^2) \geq 1/4$, for $x \in \Theta\left(u_\varepsilon, \frac{3}{4}\right)$. We may therefore invoke inequality (3.7) of Proposition 3.2 with $\delta = 1/4$. We obtain

$$\begin{aligned}\varepsilon|\nabla u_\varepsilon|^2(x) &\leq C_{\rm gd}^2\left(\|u\|_{L^\infty(\mathbb{D}^2(7/8))}, 1/4\right)\varepsilon^{-1} = \frac{\alpha_0}{16\varepsilon}\left(\frac{16C_{\rm gd}^2}{\alpha_0}\right) \\ &\leq \left(\frac{16C_{\rm gd}^2}{\alpha_0}\right)\frac{V(u_\varepsilon(x))}{\varepsilon}, \text{ for } x \in \Theta\left(u_\varepsilon, \frac{3}{4}\right),\end{aligned} \tag{3.15}$$

where we have used (3.14) for the last inequality. Set $\mathrm{L} = \|u\|_{L^\infty(\mathbb{D}^2(7/8))}$. Inequality (3.15) yields

$$e_\varepsilon(u_\varepsilon)(x) \leq \left(\frac{2C_{\rm gd}^2(\mathrm{L}, 1/4)}{\alpha_0} + 1\right)\frac{V(u_\varepsilon)}{\varepsilon}(x), \text{ for } x \in \Theta\left(u_\varepsilon, \frac{3}{4}\right).$$

The conclusion (3.12) follows choosing the constant $C_{\rm pt}$ as $C_{\rm pt} = \left(\dfrac{4C_{\rm gd}^2(\mathrm{L}, 1/4)}{\alpha_0}\right)$. For (3.13), we combine (3.12) with the uniform bound (3.1) for $x \in \mathbb{D}^2(7/8)$. □

3.4 The stress-energy tensor

The stress-energy tensor is an important tool in the analysis of singularly perturbed gradient-type problems. In dimension two, its expression is simplified thanks to complex analysis.

Lemma 3.2. *Let u_ε be a solution of* (1) *on* Ω. *Given any vector field $\vec{X} \in \mathcal{D}(\Omega, \mathbb{R}^2)$ we have*

$$\int_\Omega A_\varepsilon(u_\varepsilon)_{i,j} \cdot \frac{\partial X_i}{\partial x_j}\, dx = 0 \text{ where } A_\varepsilon(u_\varepsilon) = e_\varepsilon(u_\varepsilon)\delta_{ij} - \varepsilon \frac{\partial u_\varepsilon}{\partial x_i} \cdot \frac{\partial u_\varepsilon}{\partial x_j}. \tag{3.16}$$

The proof is standard (see [13] and references therein): It is derived multiplying the equation (1) by the function $v = \sum X_i \partial_i u_\varepsilon$ and integrating by parts on Ω. The 2×2 stress-energy matrix A_ε may be decomposed as

$$A_\varepsilon \equiv A_\varepsilon(u_\varepsilon) = T_\varepsilon(u_\varepsilon) + \frac{V(u_\varepsilon)}{\varepsilon}\, \mathrm{I}_2\,, \tag{3.17}$$

where the matrix $T_\varepsilon(u)$ is defined, for a map $u : \Omega \to \mathbb{R}^k$, by

$$T_\varepsilon(u) = \frac{\varepsilon}{2} \begin{pmatrix} |u_{x_2}|^2 - |u_{x_1}|^2 & -2u_{x_1} \cdot u_{x_2} \\ -2u_{x_1} \cdot u_{x_2} & |u_{x_1}|^2 - |u_{x_2}|^2 \end{pmatrix}. \tag{3.18}$$

Remark 3.1. Formula (3.16) corresponds to the first variation of the energy when one performs deformations of the domain induced by the diffeomorphism related to the vector field $\vec{X}$. More precisely, it can be derived from the fact that

$$\frac{d}{dt} \mathrm{E}_\varepsilon(u_\varepsilon \circ \Phi_t) = 0,$$

where, for $t \in \mathbb{R}$, the map $\Phi_t : \Omega \to \Omega$ is a diffeomorphism such that

$$\frac{d}{dt}\Phi_t(x) = \vec{X}(\Phi_t(x)), \forall x \in \Omega.$$

In dimension two, one may use complex notation to obtain a simpler expression of $T_{ij}\, \dfrac{\partial X_i}{\partial x_j}$. Setting $X = X_1 + iX_2$ we consider the complex function $\omega_\varepsilon : \Omega \to \mathbb{C}$ defined by

$$\omega_\varepsilon = \varepsilon \left(|u_{\varepsilon_{x_1}}|^2 - |u_{\varepsilon_{x_2}}|^2 - 2i u_{\varepsilon_{x_1}} \cdot u_{\varepsilon_{x_2}} \right), \tag{3.19}$$

the quantity ω_ε being usually termed the *Hopf differential* of u_ε. We obtain the identities

$$T_{ij}(u_\varepsilon)\frac{\partial X_i}{\partial x_j} = \mathrm{Re}\left(-\omega_\varepsilon \frac{\partial X}{\partial \bar{z}}\right) \text{ and } \delta_{i,j}\frac{\partial X_i}{\partial x_j} = 2\mathrm{Re}\left(\frac{\partial X}{\partial z}\right).$$

Identity (3.16) is turned into

$$\int_\Omega \mathrm{Re}\left(\omega_\varepsilon \frac{\partial X}{\partial \bar{z}}\right) = \frac{2}{\varepsilon}\int_\Omega V(u_\varepsilon)\,\mathrm{Re}\left(\frac{\partial X}{\partial z}\right) = \frac{1}{\varepsilon}\int_\Omega V(u_\varepsilon)\,\mathrm{div}\,\vec{X}. \tag{3.20}$$

Remark 3.2. Recall that the Dirichlet energy is invariant by conformal transformation. Such transformation are locally obtained through vector-fields $\vec{X}$ which are holomorphic.

3.5 Pohozaev's identity on disks

Identity (3.20) allows to derive integral estimates of the potential $V(u_\varepsilon)$ using a suitable choice of test vector fields. We restrict ourselves to the special case the domain is $\Omega = \mathbb{D}^2(r)$, for some $r > 0$. We notice that for the vector field $X = z$, we have

$$\frac{\partial X}{\partial \bar{z}} = 0 \text{ and } \frac{\partial X}{\partial z} = 1.$$

However $X = z$ is not a test vector field, since in does not have compact support, so that we consider instead vector fields X_δ of the form

$$X_\delta = z\varphi_\delta(|z|)),$$

where $0 < \delta < \frac{1}{2}$ is a small parameter and φ_δ is a scalar function defined on $[0, r]$ such that

$$\varphi_\delta(s) = 1 \text{ for } s \in [0, r-\delta), \ |\varphi'(s)| \leq \frac{2}{\delta}, \text{ for } s \in [r-\delta, r], \text{ and } \varphi(s) = 0 \text{ on } [r-\delta/4, r], \quad (3.21)$$

so that $\varphi_\delta(r) = 0$. A short computation shows that

$$\frac{\partial \varphi_\delta(|z|)}{\partial \bar{z}} = \frac{z}{2|z|}\varphi'_\delta(|z|) \text{ and } \frac{\partial \varphi_\delta(|z|)}{\partial z} = \frac{\bar{z}}{2|z|}\varphi'_\delta(|z|),$$

so that

$$\frac{\partial X_\delta}{\partial \bar{z}} = \frac{z^2}{2|z|}\varphi'_\delta(|z|) \text{ and } \frac{\partial X_\delta}{\partial z} = \frac{|z|}{2}\varphi'_\delta(|z|) + \varphi_\delta(|z|) \in \mathbb{R}.$$

We drop the subscript ε and simply write $u = u_\varepsilon$. Using polar coordinates (r, θ) such that $(x_1, x_2) = (r\cos\theta, r\sin\theta)$, we have $u_{x_1} = \cos\theta\, u_r - r^{-1}\sin\theta\, u_\theta$ and $u_{x_2} = \sin\theta\, u_r + r^{-1}\cos\theta\, u_\theta$. After some computations, this leads to the formula

$$\begin{aligned} \omega_\varepsilon &= \varepsilon(\cos 2\theta - i\sin 2\theta)\left[(|u_r|^2 - r^{-2}|u_\theta|^2) - 2ir^{-1}u_r.u_\theta)\right] \\ &= \varepsilon\frac{\bar{z}^2}{|z|^2}\left[(|u_r|^2 - r^{-2}|u_\theta|^2) - 2ir^{-1}u_r.u_\theta)\right]. \end{aligned}$$

Combining the previous computations, we obtain

$$\begin{cases} \operatorname{Re}\left(\omega_\varepsilon \dfrac{\partial X_\delta}{\partial \bar{z}}\right) = \dfrac{\varepsilon}{2}\left(|u_r|^2 - r^{-2}|u_\theta|^2\right)|z|\varphi'_\delta(|z|) \text{ and} \\ \operatorname{Re}\left(\dfrac{\partial X_\delta}{\partial z}\right) = \dfrac{1}{2}|z|\varphi'_\delta(|z|) + \varphi_\delta(|z|) \text{ on } \mathbb{D}^2(r). \end{cases} \quad (3.22)$$

We check that, as expected, we have

$$\frac{\partial X_\delta}{\partial \bar{z}} = 0 \text{ and } \frac{\partial X_\delta}{\partial z} = 1 \text{ on } \mathbb{D}^2(r-\delta).$$

Inserting these relations into (3.20) and passing to the limit $\delta \to 0$ yields the following identity, usually termed Pohozaev's identity:

Lemma 3.3. *Let u_ε be a solution of* (1) *on $\mathbb{D}^2$. We have, for any radius $0 < r \leq 1$*

$$\frac{1}{\varepsilon^2}\int_{\mathbb{D}^2(r)} V(u_\varepsilon) = \frac{r}{4}\int_{\partial\mathbb{D}^2(r)} \left(\left|\frac{\partial u_\varepsilon}{\partial \tau}\right|^2 - \left|\frac{\partial u_\varepsilon}{\partial r}\right|^2 + \frac{2}{\varepsilon^2}V(u_\varepsilon)\right) \mathrm{d}\tau, \tag{3.23}$$

where τ denotes arclength on $\partial\mathbb{D}^2(r)$, so that $\mathrm{d}\tau = r\mathrm{d}\theta$ corresponds to the $\mathcal{H}^1$-measure on $\mathbb{S}^1(r)$ and $\dfrac{\partial}{\partial\tau} = \dfrac{1}{r}\dfrac{\partial}{\partial\theta}$.

Proof. Using the vector field X_δ in (3.20), we obtain, in view of identities (3.22)

$$\frac{2}{\varepsilon^2}\int_{\mathbb{D}^2(r)} V(u_\varepsilon)\left[\frac{1}{2}|x|\varphi_\delta'(|x|) + \varphi_\delta(|x|)\right] \mathrm{d}x = \int_{\mathbb{D}^2(r)} \frac{1}{2}\left(|u_r|^2 - r^{-2}|u_\theta|^2\right)|x|\varphi_\delta'(|x|)\mathrm{d}x.$$

so that

$$\frac{2}{\varepsilon}\int_{\mathbb{D}^2(r)} V(u_\varepsilon)\varphi_\delta(|z|)\mathrm{d}x = \frac{1}{2}\int_{\mathbb{D}^2(r)}\left(|u_r|^2 - r^{-2}|u_\theta|^2 - \frac{2}{\varepsilon}V(u_\varepsilon)\right)|x|\varphi_\delta'(|x|)\mathrm{d}x. \tag{3.24}$$

Next we observe that

$$\begin{cases} \varphi_\delta(|\cdot|) \to \mathbf{1}_{\mathbb{D}^2(r)} \text{ as } \delta \to 0 \ \text{ in the sense of measures, and} \\ |\cdot|\varphi_\delta'(|\cdot|) \to -r\mathrm{d}\tau \text{ as } \delta \to 0 \text{ in } \mathcal{D}'(\mathbb{R}^2), \end{cases}$$

The conclusion follows.

□

A straightforward consequence of Lemma 3.3 is the estimate:

Proposition 3.4. *Let u_ε be a solution of* (1) *on $\mathbb{D}^2$. We have, for any $0 < r \leq 1$*

$$\frac{1}{\varepsilon}\int_{\mathbb{D}^2(r)} V(u_\varepsilon) \leq \frac{r}{2}\int_{\mathbb{S}^1(r)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell. \tag{3.25}$$

Proposition 3.4 follows immediately from Lemma 3.3 noticing that the absolute value of the integrand on the left hand side is bounded by $2\varepsilon^{-1}e_\varepsilon(u_\varepsilon)$.

Identities (3.23) and (3.25) are central in the paper, in particular (3.23) leads to the monotonicity for $\zeta_\star$. Identity (3.25) has the remarkable property that its yields a bound of the integral of the potential *inside* the disk involving only energy terms on the *boundary*. We will see later that the energy (on smaller disks) can be bounded by the integral of the potential (see Proposition 4.3), so that ultimately, we will end up with an interior estimate of the energy by the integral of the energy on the boundary. We will show that the latter is "small", for a suitable radius, and considering level sets.

Besides Proposition 3.4, we notice that Pohozaev's identity leads directly to remarkable consequences: For instance, all solutions which are constant with values in Σ on $\partial\mathbb{D}^2(r)$ are necessarily constant.

Remark 3.3. The previous results are specific to dimension 2, however the use of the stress-energy tensor yields other results in higher dimensions (for instance monotonicity formulas).

3.6 Proofs of the "monotonicity" formula for ζ_ε

We provide here a proof of formula (51), which is actually not a real monotonicity, since there is no evidence that the right hand side is non negative (only the *asymptotic version* turns out to be, in the last part of this paper, a monotonicity formula). The proof relies on Lemma 3.3, identity (3.23). We have indeed, by Leibnitz rule

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathbb{V}_\varepsilon\left(u_\varepsilon,\mathbb{D}^2(r)\right)}{r}\right)=-\frac{1}{r^2}\mathbb{V}_\varepsilon\left(u_\varepsilon,\mathbb{D}^2(r)\right)+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathbb{V}_\varepsilon\left(u_\varepsilon,\mathbb{D}^2(r)\right)\right),$$

where $\mathbb{V}_\varepsilon$ is defined in (35), so that $\mathbb{V}_\varepsilon(u_\varepsilon,\mathbb{D}^2(r))=\varepsilon^{-1}\int_{\mathbb{D}^2(r)}V(u_\varepsilon)\mathrm{d}x$. By Fubini's theorem, we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\mathbb{V}_\varepsilon\left(u_\varepsilon,\mathbb{D}^2(r)\right)\right)=\frac{1}{\varepsilon}\int_{\mathbb{S}^1(r)}V(u_\varepsilon)\mathrm{d}\tau,$$

so that, combining the previous identities, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathbb{V}_\varepsilon\left(u_\varepsilon,\mathbb{D}^2(r)\right)}{r}\right)&=-\frac{1}{r^2}\int_{\mathbb{D}^2(r)}\varepsilon^{-1}V(u_\varepsilon)\mathrm{d}x+\frac{1}{r}\int_{\mathbb{S}^1(r)}\varepsilon^{-1}V(u_\varepsilon)\mathrm{d}\tau\\&=\frac{1}{4r}\int_{\mathbb{S}^1(r)}(\varepsilon|(u_\varepsilon)_r|^2-\varepsilon|(u_\varepsilon)_\tau|^2-2\varepsilon^{-1}V(u_\varepsilon))\mathrm{d}\tau+\frac{1}{r}\int_{\mathbb{S}^1(r)}\varepsilon^{-1}V(u_\varepsilon)\mathrm{d}\tau\\&=\frac{1}{4r}\int_{\mathbb{S}^1(r)}(\varepsilon|(u_\varepsilon)_r|^2-\varepsilon|(u_\varepsilon)_\tau|^2+2\varepsilon^{-1}V(u_\varepsilon))\mathrm{d}\tau\end{aligned}$$

where we have used (3.23) for the second line. Hence, identity (51) is established.

3.7 Proof of formula (37)

For the identity (37), we have similarly

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{E}_\varepsilon\left(u_\varepsilon,\mathbb{D}^2(r)\right)}{r}\right)&=-\frac{1}{r^2}\int_{\mathbb{D}^2(r)}e_\varepsilon(u_\varepsilon)\mathrm{d}x+\frac{1}{r}\int_{\mathbb{S}^1(r)}e_\varepsilon(u_\varepsilon)\mathrm{d}\tau\\&=-\frac{1}{2r^2}\int_{\mathbb{D}^2(r)}\varepsilon|\nabla u_\varepsilon|^2\mathrm{d}x-\frac{1}{r^2}\int_{\mathbb{D}^2(r)}\varepsilon^{-1}V(u_\varepsilon)\mathrm{d}x\\&\quad+\frac{1}{2r}\int_{\mathbb{S}^1(r)}(\varepsilon(|(u_\varepsilon)_\tau|^2+|(u_\varepsilon)_r|^2)+2\varepsilon^{-1}V(u_\varepsilon))\mathrm{d}\tau\end{aligned}\tag{3.26}$$

We may decompose $\varepsilon|\nabla u_\varepsilon|^2$ as $\varepsilon|\nabla u_\varepsilon|^2=2\varepsilon^{-1}V(u_\varepsilon)-2\xi_\varepsilon(u_\varepsilon)$, where the discrepancy $\xi_\varepsilon(u_\varepsilon)$ is defined in (38). The second line in (3.26) may hence be written as

$$-\frac{1}{2r^2}\int_{\mathbb{D}^2(r)}\varepsilon|\nabla u_\varepsilon|^2\mathrm{d}x-\frac{1}{r^2}\int_{\mathbb{D}^2(r)}\varepsilon^{-1}V(u_\varepsilon)\mathrm{d}x=\frac{1}{r^2}\int_{\mathbb{D}^2(r)}\xi_\varepsilon(u_\varepsilon)-\frac{2}{r^2}\int_{\mathbb{D}^2(r)}\varepsilon^{-1}V(u_\varepsilon)\mathrm{d}x.$$

Combining this identity with (3.26) and (3.23), we obtain a nice cancelation which yields (37).

3.8 Pohozaev's type inequalities on general subdomain

We present in this subsection a tool similar to Proposition 3.4, which will be of interest in the proof of Theorem 8. We consider a solution u_ε of (1) on a general domain Ω, a subdomain $\mathcal{U}$ of Ω and for $\delta > 0$ the domain $\mathcal{U}_\delta$ introduced in (67). As a variant of Proposition 3.4, we have:

Proposition 3.5. *Let u_ε be a solution of* (1) *on Ω. We have, for any* $0 < \delta$

$$\frac{1}{\varepsilon}\int_{\mathcal{U}_{\frac{\delta}{2}}} V(u_\varepsilon)\mathrm{d}x \leq C(\mathcal{U},\delta)\int_{\mathcal{V}_\delta} e_\varepsilon(u_\varepsilon)\mathrm{d}x, \tag{3.27}$$

where the constant $C(\mathcal{U},\delta) > 0$ depends on $\mathcal{U}$, δ and V.

The main difference with Proposition 3.4 is that, in the case of a disk, the form of the $C(\mathcal{U},\delta) > 0$ is determined more accurately.

Proof of Proposition 3.5. Turning back to identity (3.20), we choose once more a test vector field $\vec{X}_\delta$ of the form $X_\delta(z) = z\chi_\delta(z)$, where the function χ_δ is a smooth scalar positive function such that

$$\chi_\delta(z) = 1 \text{ for } z \in \mathcal{U}_{\frac{\delta}{2}} \text{ and } \chi_\delta(z) = 0 \text{ for } z \in \mathbb{R}^2 \setminus \mathcal{U}_\delta$$

so that $\nabla\chi_\delta = 0$ on the set $\mathcal{U}_{\frac{\delta}{2}}$ and hence

$$\frac{\partial X_\delta}{\partial \bar{z}} = 0 \text{ and } \frac{\partial X_\delta}{\partial z} = 1 \text{ on } \mathcal{U}_{\frac{\delta}{2}}.$$

Inserting these relations into (3.20), we are led to inequality (3.27). □

4 Energy estimates

4.1 First energy estimates on levels sets close to Σ

In this subsection, we estimate the energy on domains where the solution is close to one of minimizers of the potential $\sigma \in \Sigma$. Near such a point, the potential is locally convex, close to a quadratic potential. In such a situation, solutions to the equation behave, at first order, as solution to the *linear equation* of the type

$$-\Delta v + \varepsilon^{-2}\nabla^2 V(\sigma)\cdot v \simeq 0,$$

for which energy estimates can be obtained directly by multiplying the equation by the solution itself and integration by parts, provided estimates on the boundary are available. More precisely, we consider again for given $0 < \varepsilon \leq 1$ a solution $u_\varepsilon : \mathbb{D}^2 \to \mathbb{R}^k$ to (1) and assume that we are given a radius $\varrho_\varepsilon \in [\frac{1}{2}, \frac{3}{4}]$, a number $0 < \kappa < \mu_0/2$, where $\mu_0 > 0$ is the constant provided in (56). We introduce the subdomain $\Upsilon_\varepsilon(\varrho_\varepsilon,\kappa)$ defined by

$$\begin{aligned}\Upsilon_\varepsilon(\varrho_\varepsilon,\kappa) &= \left\{x \in \mathbb{D}^2(\varrho_\varepsilon) \text{ such that } |u_\varepsilon(x) - \sigma_i| \leq \kappa, \quad \text{for some } i = 1\dots q\right\}\\ &= \bigcup_{i=1}^{q}\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa),\end{aligned} \tag{4.1}$$

where we have set

$$\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)=w_i^{-1}([0,\kappa]\cap\mathbb{D}^2(\varrho_\varepsilon))=\Upsilon_{\sigma_i}(u_\varepsilon,\varrho_\varepsilon,\kappa)=\{x\in\mathbb{D}^2(\varrho_\varepsilon),|u_\varepsilon(x)-\sigma_i|\leq\kappa\}.$$

The sets $\Upsilon_\sigma(u,\varrho,\kappa)$ of the above form have already been introduced in (2.48) for general maps u. The set $\Upsilon_\varepsilon(\varrho_\varepsilon,\kappa)$ corresponds hence to a truncation of the domain $\mathbb{D}^2(\varrho_\varepsilon)$, where points with values by u_ε far from the set Σ have been removed, whereas the set $\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)$ corresponds to a truncation of the domain $\mathbb{D}^2(\varrho_\varepsilon)$ where points with values far from the point $\sigma_i\in\Sigma$ have been removed.

The main result of the present section is to establish an estimate on the integral of the energy on the domain $\Upsilon_\varepsilon(\varrho_\varepsilon,\kappa)$ in terms of the integral of the potential as well as boundary integrals. As a matter of fact, in many results of this part, we choose a fixed value of κ, namely

$$\begin{cases}\kappa=\mu_1=\dfrac{\mu_0}{4},\text{so that}\\ \Upsilon_\varepsilon(\varrho_\varepsilon,\mu_1)\cup\Theta(u_\varepsilon,\varrho_\varepsilon)=\mathbb{D}^2(\varrho_\varepsilon).\end{cases}\tag{4.2}$$

However, several intermediate results carry out for a full range of values of κ, and will be used later in Subsection 4.2.

Proposition 4.1. *Let u_ε be a solution of* (1) *on $\mathbb{D}^2$, let $\mathrm{L}>0$ be given and assume that*

$$\|u_\varepsilon\|_{L^\infty}\leq\mathrm{L}.\tag{4.3}$$

Let $\varrho_\varepsilon\in[\frac12,\frac34]$. We have, for some constant $\mathrm{K}_\Upsilon(\mathrm{L})>0$, depending only on the potential V and L*, the inequality*

$$\int_{\Upsilon_\varepsilon(\varrho_\varepsilon,\mu_1)}e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x\leq\mathrm{K}_\Upsilon(\mathrm{L})\left[\int_{\mathbb{D}^2(\varrho_\varepsilon)}\frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x+\varepsilon\int_{\partial\mathbb{D}^2(\varrho_\varepsilon)}e_\varepsilon(u_\varepsilon)\mathrm{d}\ell\right].\tag{4.4}$$

The proof will be divided in several results of independent interest. Firstly, since u_ε is smooth and in view of Sard's Lemma, the boundary $\partial\Upsilon_\varepsilon(\varrho_\varepsilon,\kappa)$ is smooth and a finite union of smooth curves *for almost every* κ, which we will assume throughout. Hence, for $i=i,\dots,q$ the set $\partial\Upsilon_{\varepsilon,i}$ is a union of smooth curves intersecting the boundary $\partial\mathbb{D}^2(\varrho_\varepsilon)$ transversally. For $i=i,\dots,q$, we define the curves Γ_ε^i and Π_ε^i as

$$\begin{cases}\Gamma_\varepsilon^i(\varrho_\varepsilon,\kappa)\equiv\partial\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)\cap\mathbb{D}^2(\varrho_\varepsilon)=w_i^{-1}(\kappa)\cap\mathbb{D}^2(\varrho_\varepsilon)\ \text{ for } i=1\dots q,\\ \Pi_\varepsilon^i(\varrho_\varepsilon,\kappa)\equiv\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)\cap\partial\mathbb{D}^2(\varrho_\varepsilon)=w_1^{-1}([0,\kappa])\cap\partial\mathbb{D}^2(\varrho_\varepsilon),\end{cases}\tag{4.5}$$

so that

$$\partial\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)=\Gamma_\varepsilon^i(\varrho_\varepsilon,\kappa)\cup\Pi_\varepsilon^i(\varrho_\varepsilon,\kappa).\tag{4.6}$$

In view of (4.1), we introduce, for $i=1,\dots,q$, the integral quantities

$$\mathfrak{Q}_\varepsilon^i(\varrho_\varepsilon,\kappa)=\int_{\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)}\varepsilon|\nabla u_\varepsilon|^2+\varepsilon^{-1}\nabla_u V(u_\varepsilon)\cdot(u_\varepsilon-\sigma_i).\tag{4.7}$$

We first notice that:

Lemma 4.1. *We have, for every* $\kappa \in [0, \mu_0]$, *the inequality*

$$\int_{\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq \frac{2\lambda_{\max}}{\lambda_0}\mathfrak{Q}^i_\varepsilon(\varrho_\varepsilon,\kappa). \tag{4.8}$$

Proof. Since, by the definition of $\Upsilon_{\varepsilon,i}$, we have $|u-\sigma_i| \leq \kappa \leq \mu_0$, we are in position to invoke estimates (2.1), which yields, for $i \in \{1,\dots,q\}$,

$$\frac{\lambda_0}{2\lambda_{\max}}V(u_\varepsilon) \leq \frac{1}{2}\lambda_0|u_\varepsilon-\sigma_i|^2 \leq \nabla V(u_\varepsilon)\cdot(u_\varepsilon-\sigma_i) \text{ on } \Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa), \tag{4.9}$$

where $\lambda_{\max} = \sup\{\lambda_i^+, i = 1,\dots,q_i\}$. Multiplying the previous inequality by $2\lambda_{\max}/\lambda_0$ and integrating on $\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)$, we are led to

$$\int_{\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)} \varepsilon^{-1}V(u_\varepsilon)\mathrm{d}x \leq \frac{2\lambda_{\max}}{\lambda_0}\int_{\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)} \varepsilon^{-1}\nabla_u V(u_\varepsilon)\cdot(u_\varepsilon-\sigma_i). \tag{4.10}$$

The conclusion then follows from the definitions of e_ε and $\mathfrak{Q}^i_\varepsilon(\varrho_\varepsilon,\kappa)$. □

A simple integration by parts yields the following:

Lemma 4.2. *Assume that* $0 < \varepsilon \leq 1$ *and that* u_ε *is a solution to* (1) *on* $\mathbb{D}^2(1)$. *Let* ϱ_ε *be in* $[1/2, 3/4]$. *We have, for every* $\kappa \in [0,\mu_0]$, *the identity, for every* $i = 1,\dots,q$

$$\mathfrak{Q}^i_\varepsilon(\varrho_\varepsilon,\kappa) = \varepsilon\left[\int_{\Gamma^i_\varepsilon(\varrho_\varepsilon,\kappa)} \kappa\frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}}\mathrm{d}\ell + \int_{\Pi^i_\varepsilon(\varrho_\varepsilon,\kappa)} |u-\sigma_i|\left|\frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}}\right|\mathrm{d}\ell\right]. \tag{4.11}$$

Proof. For $i = 1,\dots,q$, we multiply equation (1) by $\varepsilon(u_\varepsilon-\sigma_i)$ and integrate by parts on the domain $\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)$. This yields, for $i = 1,\dots,q$

$$\begin{aligned}\mathfrak{Q}^i_\varepsilon(\varrho_\varepsilon,\kappa) &= \int_{\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)} \varepsilon|\nabla u_\varepsilon|^2 + \varepsilon^{-1}\nabla_u V(u_\varepsilon)\cdot(u_\varepsilon-\sigma_i) \\ &= \int_{\partial\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)} \varepsilon\frac{\partial u_\varepsilon}{\partial\vec{n}}\cdot(u_\varepsilon-\sigma_i) \\ &= \frac{\varepsilon}{2}\int_{\Gamma^i_\varepsilon(\varrho_\varepsilon,\kappa)} \frac{\partial|u_\varepsilon-\sigma_i|^2}{\partial\vec{n}} + \frac{\varepsilon}{2}\int_{\Pi^i_\varepsilon(\varrho_\varepsilon,\kappa)} \frac{\partial|u_\varepsilon-\sigma_i|^2}{\partial\vec{n}},\end{aligned} \tag{4.12}$$

which yields the desired result, since $\dfrac{\partial|u_\varepsilon-\sigma_i|^2}{\partial\vec{n}} = 2|u_\varepsilon-\sigma_i|\dfrac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}}$, so that

$$\frac{\partial|u_\varepsilon-\sigma_i|^2}{\partial\vec{n}} = 2\kappa\frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}} \text{ on } \Gamma^i_\varepsilon(\varrho_\varepsilon,\kappa). \tag{4.13}$$

□

Remark 4.1. Notice that we have the inequality

$$\frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}} \geq 0 \text{ on } \Gamma^i_\varepsilon(\varrho_\varepsilon,\kappa). \tag{4.14}$$

Indeed, by definition $|u_\varepsilon-\sigma_i| = \kappa$ on $\Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\kappa)$, so that we are on a level set and the normal derivative $\vec{n}(\ell)$ is pointing towards the outside.

The next result will also be used extensively in Subsection 4.2:

Lemma 4.3. *Assume that* $0 < \varepsilon \leq 1$ *and that* u_ε *is a solution to* (1) *on* $\mathbb{D}^2(1)$. *Let* ϱ_ε *be in* $[1/2, 3/4]$. *We have, for every* $\kappa \in [0, \mu_0]$, *the inequality*

$$\int_{\Upsilon_\varepsilon(\varrho_\varepsilon,\kappa)} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq C\varepsilon\left[\kappa\sum_{i=1}^{q}\int_{\Gamma^i_\varepsilon(\varrho_\varepsilon,\kappa)}\frac{\partial|u_\varepsilon(\ell)-\sigma_i|}{\partial\vec{n}(\ell)}\mathrm{d}\ell + \int_{\partial\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon(\ell))\mathrm{d}\ell\right]. \tag{4.15}$$

where $C > 0$ *is some constant depending only on the potential* V *and where* $\vec{n}(\ell)$ *denotes the unit vector normal to* $\Gamma_{\varepsilon,i} \cup \Pi_{\varepsilon,i}$ *pointing in the direction increasing* $|u_\varepsilon - \sigma_i|$.

Remark 4.2. Let us emphasize that in this statement, κ is not constrained by (4.2) and may actually take arbitrary small values.

Proof. The proof relies on a combination of the results of Lemmas 4.2 and 4.1. We first estimate the second term on the r.h.s of (4.11). Since by definition, we have the inclusion $\Pi^i_\varepsilon(\varrho_\varepsilon,\kappa) \subset \mathbb{S}^1(\varrho_\varepsilon)$, it follows that $\vec{n}(\ell) = \vec{e}_r$ on $\Pi^i_\varepsilon(\varrho_\varepsilon,\kappa)$, so that

$$\left|\frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}}\right| = \left|\frac{\partial|u_\varepsilon-\sigma_i|}{\partial r}\right| \leq \left|\frac{\partial u_\varepsilon}{\partial r}\right| \leq |\nabla u_\varepsilon|, \text{ on } \Pi^i_\varepsilon(\varrho_\varepsilon,\kappa). \tag{4.16}$$

On the other hand, in view of Proposition 2.1, as well as the fact that $|u_\varepsilon(\ell) - \sigma_i| \leq \kappa \leq \mu_0$ for $\ell \in \Pi^i_\varepsilon(\varrho_\varepsilon,\kappa)$, we have

$$|u_\varepsilon - \sigma_i| \leq \frac{2}{\sqrt{\lambda_0}}\sqrt{V(u_\varepsilon)} \text{ on } \Pi^i_\varepsilon(\varrho_\varepsilon,\kappa). \tag{4.17}$$

Combining (4.16) with (4.17) and integrating on $\Pi_\varepsilon(\varrho_\varepsilon,\kappa)$, we obtain the estimate

$$\begin{aligned}\int_{\Pi^i_\varepsilon(\varrho_\varepsilon,\kappa)}|u_\varepsilon-\sigma_i|.|\frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}}|\mathrm{d}\ell &\leq \frac{2}{\sqrt{\lambda_0}}\int_{\Pi^i_\varepsilon(\varrho_\varepsilon,\kappa)}\sqrt{V(u_\varepsilon)}.|\nabla u_\varepsilon|\mathrm{d}\ell\\ &\leq \frac{2}{\sqrt{\lambda_0}}\int_{\mathbb{S}^1(\varrho_\varepsilon)} e_\varepsilon(u)\mathrm{d}\ell,\end{aligned} \tag{4.18}$$

where, for the second inequality, we used Lemma 2.3 and the fact that $\Pi^i_\varepsilon(\varrho_\varepsilon,\kappa) \subset \mathbb{S}^1(\varrho_\varepsilon)$. Combining (4.18) with (4.11) and (4.8), we obtain the desired conclusion (4.15) for the choice of constant $C = \frac{2\lambda_{\max}}{\lambda_0}(1 + \frac{2}{\sqrt{\lambda_0}})$.

□

Our next results allows to obtain, for a suitable choice of κ, a bound on the first term on the right hand side of (4.15):

Lemma 4.4. *Assume that* $0 < \varepsilon \leq 1$ *and that* u_ε *is a solution to* (1) *on* $\mathbb{D}^2(1)$. *Let* $\varrho_\varepsilon \in [\frac{1}{2}, \frac{3}{4}]$. *There exists some number* $\tilde{\mu}_\varepsilon \in [\frac{\mu_0}{4}, \frac{\mu_0}{2}]$ *such that*

$$\varepsilon\int_{\Gamma_{\varepsilon,i}(\varrho,\tilde{\mu}_\varepsilon)}\frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}(\ell)}\mathrm{d}\ell \leq \varepsilon\int_{\Gamma_{\varepsilon,i}(\varrho,\tilde{\mu}_\varepsilon)}|\nabla u_\varepsilon|\mathrm{d}\ell \leq \frac{8}{\mu_0}\mathrm{E}_\varepsilon(u,\Theta(u_\varepsilon,\varrho_\varepsilon)), \tag{4.19}$$

where $\Theta(u_\varepsilon, \varrho_\varepsilon)$ *is defined in* (3.10).

Proof. We invoke Lemma 2.8 with the choices $r = \varrho_\varepsilon$ and $u = u_\varepsilon$. This yields directly a number $\tilde{\mu}_\varepsilon \in [\frac{\mu_0}{4}, \frac{\mu_0}{2}]$ such that (4.19) is satisfied, so that the proof is complete. □

Proof of Proposition 4.1 completed. We combine (4.15) for $\kappa = \tilde{\mu}_\varepsilon$ with (4.19). This yields

$$\int_{\Upsilon_\varepsilon(\varrho_\varepsilon, \tilde{\mu}_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq C \left[\frac{8\tilde{\mu}_\varepsilon}{\mu_0} \mathrm{E}_\varepsilon(u_\varepsilon, \Theta(u_\varepsilon, \varrho_\varepsilon)) + \varepsilon \int_{\partial\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon(\ell))\mathrm{d}\ell \right]. \tag{4.20}$$

On the other hand, it follows from assumption (4.3) and Lemma 3.1 that

$$e_\varepsilon(u_\varepsilon) \leq C_{\mathrm{pt}}\left(\mathrm{L}\right) \frac{V(u_\varepsilon)}{\varepsilon}, \quad \text{on } \Theta\left(u_\varepsilon, \frac{3}{4}\right) \supset \Theta\left(u_\varepsilon, \varrho_\varepsilon\right), \tag{4.21}$$

so that

$$\begin{aligned} \mathrm{E}_\varepsilon(u_\varepsilon, \Theta(u_\varepsilon, \varrho_\varepsilon)) = \int_{\Theta(u_\varepsilon, \varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}x &\leq C_{\mathrm{pt}}\left(\mathrm{L}\right) \int_{\Theta(u_\varepsilon, \varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x \\ &\leq C_{\mathrm{pt}}\left(\mathrm{L}\right) \int_{\mathbb{D}^2(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x. \end{aligned} \tag{4.22}$$

Combining (4.22) with (4.20), together with the fact that $\Theta(u_\varepsilon) \cup \Upsilon_\varepsilon(\varrho_\varepsilon, \tilde{\mu}_\varepsilon) = \mathbb{D}^2(\varrho_\varepsilon)$, which follows from (4.2), since $\tilde{\mu}_\varepsilon \geq \frac{\mu_0}{4}$, we obtain (4.4) for $\mathrm{K}_\Upsilon(\mathrm{L}) = 8C.C_{\mathrm{pt}}\left(\mathrm{L}\right)$.

4.2 Refined estimates on level sets close to Σ

Whereas we obtained in Proposition 4.1 an energy estimate on a *fixed level* set $\Upsilon_\varepsilon(\varrho_\varepsilon, \mu_1)$, we derive here an energy estimate on the set $\Upsilon_\varepsilon(\varrho_\varepsilon, \kappa)$ allowing the value of κ to vary and in *particular to be small*. This will be completed at the cost of an additional assumption. Indeed, we will assume that there exists an element $\sigma_{\mathrm{main}} \in \Sigma$ such that

$$|u_\varepsilon - \sigma_{\mathrm{main}}| < \kappa \text{ on } \partial\mathbb{D}^2(\varrho_\varepsilon). \tag{4.23}$$

The main result of this subsection is:

Proposition 4.2. *Let u_ε be a solution of* (1) *on $\mathbb{D}^2$, $M > 0$, $0 < \kappa < \frac{\mu_0}{4}$ and $\varrho_\varepsilon \in [\frac{1}{2}, \frac{3}{4}]$. Assume that* (4.23) *is satisfied and that*

$$\mathrm{E}_\varepsilon(u_\varepsilon) \leq M. \tag{4.24}$$

We have, for some constant $\mathrm{C}_\Upsilon(M) > 0$, depending only on the potential V and on M,

$$\int_{\Upsilon_\varepsilon(\varrho_\varepsilon, \kappa)} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq \mathrm{C}_\Upsilon(M) \left[\kappa \int_{\mathbb{D}^2(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \varepsilon \int_{\partial\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell \right]. \tag{4.25}$$

Of major importance in estimate (4.25) is the *presence of the term κ in front of the integral of the potential*, so that the energy on $\Upsilon_\varepsilon(\varrho_\varepsilon, \kappa)$ grows essentially at most linearly with respect to κ. Proposition 4.2 will be used in the proof of the clearing-out result, so that we will use it for *small M*.

We may assume without loss of generality that $\sigma_{\rm main} = \sigma_1$, so that it follows from assumption (4.23) that

$$|u_\varepsilon(\ell) - \sigma_1| < \kappa \text{ for } \ell \in \partial\mathbb{D}^2(\varrho_\varepsilon). \tag{4.26}$$

We deduce from inequality (4.26) that $\partial\mathbb{D}^2(\varrho_\varepsilon) \subset \overline{\Upsilon_{\varepsilon,1}(\varrho_\varepsilon, \kappa)}$, and that, for $i = 2, \dots, q$, we have

$$\partial\mathbb{D}^2(\varrho_\varepsilon) \cap \partial\Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa) = \emptyset.$$

In particular, we notice the identities

$$\begin{cases} \Gamma^i_\varepsilon(\varrho_\varepsilon, \kappa) \equiv \partial\Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa) = w_i^{-1}(\kappa) \cap \mathbb{D}^2(\varrho_\varepsilon) \ \text{ for } i = 2 \dots q, \\ \Pi^i_\varepsilon(\varrho_\varepsilon, \kappa) = \emptyset, \ \text{ for } i = 2 \dots q, \\ \Gamma^1_\varepsilon(\varrho_\varepsilon, \kappa) \equiv \partial\Upsilon_{\varepsilon,1}(\varrho_\varepsilon, \kappa) \setminus \partial\mathbb{D}^2(\varrho_\varepsilon) = \big(\big[w_1^{-1}(\kappa) \cap \mathbb{D}^2(\varrho_\varepsilon)\big]\big) \setminus \partial\mathbb{D}^2(\varrho_\varepsilon) \text{ and} \\ \Pi^1_\varepsilon(\varrho_\varepsilon, \kappa) = \partial\mathbb{D}^2(\varrho_\varepsilon). \end{cases} \tag{4.27}$$

As for Proposition 4.1, we will deduce Proposition 4.2 from Lemma 4.3. For that purpose, we will make use of a *new ingredient*, given by the following monotonicity formula:

Lemma 4.5. *Let $\mu_0 \geq \kappa_1 \geq \kappa_0 \geq \kappa$ be given. If u_ε satisfies condition (4.23), then we have, for $i = 1, \dots, q$, the inequality*

$$0 \leq \int_{\Gamma^i_\varepsilon(\varrho_\varepsilon, \kappa_0)} \frac{\partial|u_\varepsilon(\ell) - \sigma_i|}{\partial\vec{n}(\ell)} {\rm d}\ell \leq \int_{\Gamma^i_\varepsilon(\varrho_\varepsilon, \kappa_1)} \frac{\partial|u_\varepsilon(\ell) - \sigma_i|}{\partial\vec{n}(\ell)} {\rm d}\ell. \tag{4.28}$$

Proof. The proof involves again Stokes formula, now on the domain

$$\mathcal{C}_i(\kappa_0, \kappa_1) = \Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa_1) \setminus \Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa_0).$$

It follows from assumption (4.23) that, for any $i = 1, \dots, q$

$$\overline{\mathcal{C}_i(\kappa_0, \kappa_1)} \cap \partial\mathbb{D}^2(\varrho_\varepsilon) = \emptyset,$$

so that

$$\partial\mathcal{C}_i(\kappa_0, \kappa_1) = \partial\Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa_1) \cup \partial\Upsilon_{\varepsilon,i}(\varrho_\varepsilon, \kappa_0).$$

We multiply the equation (1) by $\dfrac{u_\varepsilon - \sigma_i}{|u_\varepsilon - \sigma_i|}$, which is well defined on $\mathcal{C}_i(\kappa_0, \kappa_1)$, and integrate by parts. Since, on $\Gamma_{\varepsilon,i}(\varrho_\varepsilon, \kappa)$, we have

$$\frac{\partial u_\varepsilon}{\partial\vec{n}} \cdot \frac{u_\varepsilon - \sigma_i}{|u_\varepsilon - \sigma_i|} = \frac{\partial(u_\varepsilon - \sigma_i)}{\partial\vec{n}} \cdot \frac{u_\varepsilon - \sigma_i}{|u_\varepsilon - \sigma_i|} = \frac{\partial|u_\varepsilon - \sigma_i|}{\partial\vec{n}},$$

whereas on $\mathcal{C}_i(\kappa_0, \kappa_1)$, we have

$$\begin{aligned} \nabla u_\varepsilon \cdot \nabla\left(\frac{u_\varepsilon - \sigma_i}{|u_\varepsilon - \sigma_i|}\right) &= \nabla(u_\varepsilon - \sigma_i) \cdot \nabla\left(\frac{u_\varepsilon - \sigma_i}{|u_\varepsilon - \sigma_i|}\right) \\ &= \frac{1}{|u_\varepsilon - \sigma_i|}|\nabla(u_\varepsilon - \sigma_i)|^2 + [\nabla(u_\varepsilon - \sigma_i) \cdot (u_\varepsilon - \sigma_i)] \cdot \nabla(\frac{1}{|u_\varepsilon - \sigma_i|}) \\ &= \frac{1}{|u_\varepsilon - \sigma_i|}\left[|\nabla(u_\varepsilon - \sigma_i)|^2 - |\nabla|u_\varepsilon - \sigma_i||^2\right], \end{aligned}$$

integration by parts thus yields

$$\int_{\Gamma_{\varepsilon,i}(\varrho_\varepsilon,\kappa_1)} \frac{\partial|u_\varepsilon - \sigma_i|}{\partial\vec{n}} - \int_{\Gamma_{\varepsilon,i}(\varrho_\varepsilon,\kappa_0)} \frac{\partial|u_\varepsilon - \sigma_i|}{\partial\vec{n}} = \int_{\mathcal{C}_i(\kappa_0,\kappa_1)} \frac{1}{|u-\sigma_i|}\left[|\nabla u_\varepsilon|^2 - |\nabla|u_\varepsilon - \sigma_i||^2\right] + \int_{\mathcal{C}_i(\kappa_0,\kappa_1)} \varepsilon^{-2}\nabla_u V(u_\varepsilon)\cdot\frac{(u_\varepsilon-\sigma_i)}{|u-\sigma_i|}. \tag{4.29}$$

Since

$$\begin{cases} |\nabla u_\varepsilon|^2 - |\nabla|u_\varepsilon - \sigma_i||^2 = |\nabla(u_\varepsilon-\sigma_i)|^2 - |\nabla|u_\varepsilon-\sigma_i||^2 \geq 0, \text{ on } \mathbb{D}^2(\varrho_\varepsilon), \text{ and} \\ \nabla_u V(u_\varepsilon)\cdot\dfrac{(u_\varepsilon-\sigma_i)}{|u-\sigma_i|} \geq 0, \text{ on } \Upsilon_{\varepsilon,i}(\varrho_\varepsilon,\mu_0) \supset \mathcal{C}_i(\kappa_0,\kappa_1), \end{cases}$$

it follows that the r.h.s of inequality (4.29) is positive. Hence, we deduce (4.28).

□

Lemma 4.6. *Assume that* $0 < \varepsilon \leq 1$ *and that* u_ε *is a solution to* (1) *which satisfies* (4.23) *and* (4.24). *Then, there exists a constant* $\mathrm{C}(\mathrm{M}) > 0$ *depending only on* V *and* M *such that have*

$$0 \leq \varepsilon\int_{\Gamma_{\varepsilon,i}(\varrho_\varepsilon,\kappa)} \frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}(\ell)}\mathrm{d}\ell \leq C(M)\int_{\mathbb{D}^2(\varrho_\varepsilon)} \frac{V(u)}{\varepsilon}\mathrm{d}x \leq C(M)\mathbb{V}(u_\varepsilon, \mathbb{D}^2(\frac{3}{4})), \tag{4.30}$$

where, for a point $\ell \in \Gamma_\varepsilon$, $\vec{n}(\ell)$ *denotes the unit vector perpendicular to* Γ_ε *and oriented in the direction which increases* $|u-\sigma_i|$.

Proof. By Lemma 4.4, there exists a number $\tilde{\mu}_\varepsilon \in [\frac{\mu_0}{4}, \frac{\mu_0}{2}]$ such that

$$\varepsilon\int_{\Gamma_{\varepsilon,i}(\varrho,\tilde{\mu}_\varepsilon)} \frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}(\ell)}\mathrm{d}\ell \leq \varepsilon\int_{\Gamma_{\varepsilon,i}(\varrho,\tilde{\mu}_\varepsilon)} |\nabla u_\varepsilon|\mathrm{d}\ell \leq \frac{8}{\mu_0}\mathrm{E}_\varepsilon(u,\Theta(u_\varepsilon,\varrho_\varepsilon)) \tag{4.31}$$

On the level set $\Theta(u_\varepsilon,\varrho_\varepsilon)$, we may however bound point-wise the energy in terms of the potential, as stated in Lemma 3.1, inequality (3.13). This yields by integration

$$\mathrm{E}_\varepsilon(u,\Theta(u_\varepsilon,\varrho_\varepsilon)) \leq \mathrm{C_T}(M)\mathbb{V}(u_\varepsilon,\Theta(u_\varepsilon,\varrho_\varepsilon)).$$

Combining the two previous inequalities, we obtain

$$\varepsilon\int_{\Gamma_{\varepsilon,i}(\varrho,\tilde{\mu}_\varepsilon)} \frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}(\ell)}\mathrm{d}\ell \leq \frac{\mathrm{C_T}(\mathrm{M})}{\mu_0}\mathbb{V}(u_\varepsilon,\Theta(u_\varepsilon,\varrho_\varepsilon)) \leq \frac{\mathrm{C_T}(\mathrm{M})}{\mu_0}\mathbb{V}(u_\varepsilon,\mathbb{D}^2(\frac{3}{4})). \tag{4.32}$$

On the other hand, we invoke to Lemma 4.5 with the choice $\kappa_1 = \tilde{\mu}_\varepsilon$ and $\kappa_0 = \kappa$ to deduce that

$$\int_{\Gamma_{\varepsilon,i}(\varrho,\kappa)} \frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}(\ell)}\mathrm{d}\ell \leq \int_{\Gamma_{\varepsilon,i}(\varrho,\tilde{\mu}_\varepsilon)} \frac{\partial|u_\varepsilon-\sigma_i|}{\partial\vec{n}(\ell)}\mathrm{d}\ell,$$

which together with (4.32) leads to the desired result (4.30), with $C(M) = \dfrac{\mathrm{C_T}(\mathrm{M})}{\mu_0}$. □

Proof of Proposition 4.2 completed. We go back to Lemma 4.3 and combine (4.15) with (4.30): This yields the desired inequality (4.25). □

4.3 Bounding the total energy by the integral of the potential

The main result of the present paragraph is the following result, which will be used both in the proof of the clearing-out results as in the proof of Theorem 4:

Proposition 4.3. *Let u_ε be a solution of* (1) *on $\mathbb{D}^2$ and let $\mathrm{L} > 0$ be given and assume that*

$$\|u_\varepsilon\|_{L^\infty(\mathbb{D}^2(\frac{4}{5}))} \leq \mathrm{L}. \tag{4.33}$$

There exists some constant $\mathrm{K_{pot}}(\mathrm{L})$ depending only on V and L such that

$$\int_{\mathbb{D}^2(\frac{1}{2})} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq \mathrm{K_{pot}}(\mathrm{L}) \left[\int_{\mathbb{D}^2(\frac{3}{4})} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \varepsilon \int_{\mathbb{D}^2\setminus\mathbb{D}^2(\frac{1}{2})} e_\varepsilon(u_\varepsilon)\mathrm{d}x.\right]. \tag{4.34}$$

In the context of the present paper, the main contribution of the r.h.s of inequality (4.34) is given by the potential terms, so that Proposition 4.3 yields an estimate of the energy by the integral of potential, provided the solution is bounded on a small domain, according to assumption (4.33).

Before turning to the proof of Proposition 4.3, we observe, as a preliminary remark, that the result of proposition 4.3 is, at first sight, rather close to the result of Proposition 4.1. However, let us emphasize that estimate (4.25) yields only an energy bound only for the domain where the value of u_ε *is close to one of the potential wells*, whereas (4.34) yields an estimate for the full domain $\mathbb{D}^2(1/2)$.

The first step in the proof of Proposition 4.3 is:

Lemma 4.7. *Let $\varrho_\varepsilon \in [\frac{1}{2}, \frac{3}{4}]$, let u_ε be a solution of* (1) *on $\mathbb{D}^2$ and assume that* (4.33) *is satisfied. We have, for some constant $\mathrm{C_{pot}}(\mathrm{L}) > 0$, depending only on the potential V and the value of L, such that*

$$\int_{\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq \mathrm{C_{pot}}(\mathrm{L}) \left[\int_{\mathbb{D}^2(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \frac{\varepsilon}{4} \int_{\partial\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell\right]. \tag{4.35}$$

Proof. We observe first that

$$\mathbb{D}^2(\varrho_\varepsilon) = \Theta(u_\varepsilon, \varrho_\varepsilon) \cup \Upsilon_\varepsilon(\varrho_\varepsilon, \frac{\mu_0}{4}). \tag{4.36}$$

In view of Lemma 3.1, we have

$$\int_{\Theta(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq \mathrm{C_T}(\mathrm{L}) \int_{\Theta(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x,$$

whereas Proposition 4.1 yields

$$\int_{\Upsilon_\varepsilon(\varrho_\varepsilon, \frac{\mu_0}{4})} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq \mathrm{K}_\Upsilon(L) \left[\int_{\mathbb{D}^2(u_\varepsilon,\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \varepsilon \int_{\partial\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell\right].$$

The proof of (4.35) then follows straightforwardly from our first observation (4.36). □

Proof of Proposition 4.3 completed. As usual, a mean-value argument allows us to choose some radius $\varrho_\varepsilon \in [\frac{1}{2}, \frac{3}{4}]$ such that

$$\int_{\partial\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell \le 8 \int_{\mathbb{D}^2(\frac{3}{4})\setminus\mathbb{D}^2(\frac{1}{2})} e_\varepsilon(u_\varepsilon)\mathrm{d}x. \tag{4.37}$$

Combining with Lemma 4.7, we are led to

$$\begin{aligned}\int_{\mathbb{D}^2(\frac{1}{2})} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x &\le \int_{\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \\ &\le \mathrm{C}_{\mathrm{pot}}\left[\int_{\mathbb{D}^2(\varrho_\varepsilon)} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \frac{\varepsilon}{4}\int_{\partial\mathbb{D}^2(\varrho_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell\right] \\ &\le \mathrm{C}_{\mathrm{pot}}\left[\int_{\mathbb{D}^2(\frac{3}{4})} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \varepsilon\int_{\mathbb{D}^2(\frac{3}{4})} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell\right].\end{aligned} \tag{4.38}$$

The proof of Proposition 4.3 is hence complete. □

We will also invoke the following variant of Proposition 4.3:

Proposition 4.4. *Let u_ε be a solution of* (1) *on $\mathbb{D}^2$, let $\mathrm{M} > 0$ be given and assume that* (4.24) *holds. There exists some constant $\mathrm{C}_{\mathrm{pot}}(M)$ depending only on V and M such that*

$$\int_{\mathbb{D}^2(\frac{1}{2})} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \le \mathrm{C}_{\mathrm{pot}}(\mathrm{M})\left[\int_{\mathbb{D}^2(\frac{3}{4})} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \varepsilon\int_{\mathbb{D}^2\setminus\mathbb{D}^2(\frac{1}{2})} e_\varepsilon(u_\varepsilon)\mathrm{d}x.\right]. \tag{4.39}$$

Proof. If u_ε satisfies (4.24), then it follows from Proposition 3.1

$$\|u_\varepsilon\|_{L^\infty(\mathbb{D}^2(\frac{4}{5}))} \le \mathrm{L}_{\mathrm{M}} \equiv 5\mathrm{C}_{\mathrm{unf}}\mathrm{M} + \mathrm{K}_{\mathrm{unf}}. \tag{4.40}$$

Invoking Proposition 4.3, inequality (4.39) follows with

$$\mathrm{C}_{\mathrm{pot}}(\mathrm{M}) = \mathrm{K}_{\mathrm{pot}}(\mathrm{L}_{\mathrm{M}}) = \mathrm{K}_{\mathrm{pot}}\left(5\mathrm{C}_{\mathrm{unf}}\mathrm{M} + \mathrm{K}_{\mathrm{unf}}\right).$$

□

In the course of the paper, we will invoke the scaled versions of Proposition 4.3 and 4.4. Given $\varrho > \varepsilon > 0$ and $x_0 \in \Omega$, we consider a solution u_ε on Ω and assume it satisfies the bound (4.33) or the bound

$$\mathrm{E}(u_\varepsilon, \mathbb{D}^2(x_0, \varrho)) \le M\varrho, \tag{4.41}$$

then, thanks to the relations (54), we have the scaled version of (4.34) or (4.39) respectively, namely

$$\int_{\mathbb{D}^2(x_0,\frac{\varrho}{2})} e_\varepsilon(u_\varepsilon)\mathrm{d}x \le \mathrm{K}_{\mathrm{pot}}(L)\left[\int_{\mathbb{D}^2(x_0,\frac{3\varrho}{4})} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \frac{\varepsilon}{\varrho}\int_{\mathbb{D}^2(x_0,\varrho)\setminus\mathbb{D}^2(x_0,\frac{\varrho}{2})} e_\varepsilon(u_\varepsilon)\mathrm{d}x\right], \tag{4.42}$$

and

$$\int_{\mathbb{D}^2(x_0,\frac{\varrho}{2})} e_\varepsilon(u_\varepsilon)\mathrm{d}x \le \mathrm{C}_{\mathrm{pot}}(M)\left[\int_{\mathbb{D}^2(x_0,\frac{3\varrho}{4})} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \frac{\varepsilon}{\varrho}\int_{\mathbb{D}^2(x_0,\varrho)\setminus\mathbb{D}^2(x_0,\frac{\varrho}{2})} e_\varepsilon(u_\varepsilon)\mathrm{d}x\right]. \tag{4.43}$$

These relations lead to:

Proposition 4.5. *Let $M_0 > 0$ and $\varepsilon > 0$ be given. Let u_ε be a solution of* (1) *on Ω such that $\mathrm{E}_\varepsilon(u_\varepsilon) \leq M_0$, and $x_0 \in \Omega$ and $\varrho > \varepsilon > 0$ such that $\mathbb{D}^2(x_0, \varrho) \subset \Omega$. Then, we have*

$$\int_{\mathbb{D}^2(x_0,\frac{\varrho}{2})} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq \mathrm{K}_V\left(\mathrm{dist}(x_0, \partial\Omega)\right)\left[\int_{\mathbb{D}^2(x_0,\frac{3\varrho}{4})} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \frac{\varepsilon}{\varrho}\int_{\mathbb{D}^2(x_0,\varrho)\setminus\mathbb{D}^2(x_0,\frac{\varrho}{2})} e_\varepsilon(u_\varepsilon)\mathrm{d}x,\right].$$

where the constant $\mathrm{K}_V\left(\mathrm{dist}(x_0, \partial\Omega)\right)$ depends only on V, M_0 and $\mathrm{dist}(x_0, \partial\Omega)$,

Proof. Since $\mathbb{D}^2(x_0, \varrho) \subset \Omega$, we have $\mathrm{dist}\left(\mathbb{D}^2(x_0, \frac{4\varrho}{5}), \partial\Omega\right) \geq \frac{\varrho}{5}$. It therefore follows from Proposition 3.1 that

$$\|u\|_{L^\infty(\mathbb{D}^2(x_0,\frac{4\varrho}{5}))} \leq \mathrm{L}_0 \equiv \frac{20\mathrm{C}_{\mathrm{unf}} M_0}{\mathrm{dist}(x_0, \partial\Omega)} + \mathrm{K}_{\mathrm{unf}}.$$

The conclusion then follows directly from (4.43) with the choice $\mathrm{K}_V\left(\mathrm{dist}(x_0, \partial\Omega)\right) = \mathrm{K}_{\mathrm{pot}}(\mathrm{L}_0)$. □

4.4 Energy bounds by integrals on external domains

Our next result paves the way for the proof of Theorem 8. As there, we consider an open subset $\mathcal{U}$ of Ω and define $\mathcal{U}_\delta$ and $\mathcal{V}_\delta$ according to (67).

Proposition 4.6. *let u_ε be a solution of* (1) *on Ω, $\mathcal{U}$ be an open bounded subset of Ω and $1 > \delta > \varepsilon > 1 > 0$ be given such that $\mathcal{U}_\delta \subset \Omega$. Assume that*

$$\int_{\mathcal{V}_\delta} e_\varepsilon(u_\varepsilon)\,\mathrm{d}x \leq \mathrm{K}_{\mathrm{ext}}(\mathcal{U}, \delta), \tag{4.44}$$

where $\mathrm{K}_{\mathrm{ext}}(\mathcal{U}, \delta) > 0$ denotes some constant depending possibly on $\mathcal{U}$ and δ. Then, we have the bound, for some constant $\mathrm{C}_{\mathrm{ext}}(\mathcal{U}, \delta)$ depending possibly on $\mathcal{U}$ and δ

$$\int_{\mathcal{U}_{\frac{\delta}{4}}} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq \mathrm{C}_{\mathrm{ext}}(\mathcal{U}, \delta)\left(\int_{\mathcal{V}_\delta} e_\varepsilon(u_\varepsilon) + \varepsilon\int_{\mathcal{U}_\delta} e_\varepsilon(u_\varepsilon)\mathrm{d}x.\right) \tag{4.45}$$

Proof. The proof combines Proposition 4.4, Proposition 3.5 with a standard covering by disks. We first bound the potential on the set $\mathcal{U}_{\frac{\delta}{2}}$ thanks to Proposition 3.5, which yields

$$\frac{1}{\varepsilon}\int_{\mathcal{U}_{\frac{\delta}{2}}} V(u_\varepsilon)\mathrm{d}x \leq C(U, \delta)\int_{\mathcal{V}_\delta} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq C(U, \delta)\mathrm{K}_{\mathrm{ext}}(\mathcal{U}, \delta). \tag{4.46}$$

In inequality (4.46), we have assumed that the bound (4.44) is fullfilled for some constant $\mathrm{K}_{\mathrm{ext}}(\mathcal{U}, \delta)$, which we choose now as

$$\mathrm{K}_{\mathrm{ext}}(\mathcal{U}, \delta) = \frac{\mathrm{K}_{\mathrm{pot}}(M_0)\delta}{8C(\mathcal{U}, \delta)}. \tag{4.47}$$

Inequality (4.46) then yields

$$\frac{1}{\varepsilon}\int_{\mathcal{U}_{\frac{\delta}{2}}} V(u_\varepsilon)\mathrm{d}x \leq \frac{\delta}{8}K_{\mathrm{pot}}(M_0). \tag{4.48}$$

This bound will allow us to apply inequality (4.41) on disks of radius $\frac{\delta}{8}$ covering $\mathcal{U}_{\frac{\delta}{4}}$. In this direction, we claim that there exists a finite collections of disks $\left\{\mathbb{D}^2\left(x_i, \frac{\delta}{8}\right)\right\}_{i\in I}$ such that

$$\mathcal{U}_{\frac{\delta}{4}} \subset \underset{i\in I}{\cup} \mathbb{D}^2\left(x_i, \frac{\delta}{8}\right) \text{ and } x_i \in \overline{\mathcal{U}_{\frac{\delta}{4}}}, \text{ for any } i \in I. \tag{4.49}$$

Indeed, such a collections may be obtained invoking the collection of disks $\left\{\mathbb{D}^2\left(x, \frac{\delta}{8}\right)\right\}$ with $x \in \overline{\mathcal{U}_{\frac{\delta}{4}}}$ and then extracting a finite subcover thanks to Lebesgue's Theorem. Notice that we also have

$$\underset{i\in I}{\cup} \mathbb{D}^2\left(x_i, \frac{\delta}{4}\right) \subset \mathcal{U}_{\frac{\delta}{2}}. \tag{4.50}$$

On each of the disks $\mathbb{D}^2\left(x_i, \frac{\delta}{4}\right)$, we have, thanks to (4.48)

$$\frac{1}{\varepsilon}\int_{\mathbb{D}^2(x_i, \frac{\delta}{4})} V(u_\varepsilon)\mathrm{d}x \leq \frac{\delta}{8} K_{\mathrm{pot}}(M_0),$$

so that we may apply the scaled version (4.43) of Proposition 4.4 on the disk $\mathbb{D}^2(x_i, \frac{1}{4}\delta)$: This yields the estimate

$$\int_{\mathbb{D}^2(x_i, \frac{1}{8}\delta)} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq \mathrm{C}_{\mathrm{pot}}\left[\int_{\mathbb{D}^2(x_i, \frac{3}{16}\delta)} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \frac{\varepsilon}{\delta}\int_{\mathbb{D}^2(x_i, \frac{\delta}{4})} e_\varepsilon(u_\varepsilon)\mathrm{d}x\right].$$

Adding these relations for $i \in I$ and invoking relations (4.49) and (4.50) we are led to

$$\int_{\mathcal{U}_{\frac{\delta}{4}}} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq \sharp(I)\mathrm{C}_{\mathrm{pot}}\left[\int_{\mathcal{U}_{\frac{\delta}{2}}} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \frac{\varepsilon}{\delta}\int_{\mathcal{U}_{\frac{\delta}{2}}} e_\varepsilon(u_\varepsilon)\mathrm{d}x\right]. \tag{4.51}$$

Invoking again the first inequality in (4.46) we may bound the potential term on the right hand side, so that we obtain

$$\int_{\mathcal{U}_{\frac{\delta}{4}}} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq \sharp(I)\mathrm{C}_{\mathrm{pot}}\left[C(U,\delta)\int_{\mathcal{V}_\delta} e_\varepsilon(u_\varepsilon)\mathrm{d}x + \frac{\varepsilon}{\delta}\int_{\mathcal{U}_{\frac{\delta}{2}}} e_\varepsilon(u_\varepsilon)\mathrm{d}x\right].$$

This inequality finally leads to the conclusion (4.45). □

5 Proof of the energy decreasing property

The purpose of this section is to provide a proof to Proposition 1, which is a major step in the proofs of the main theorems of the paper.

5.1 An improved estimate of the energy on level sets

In this paragraph, we consider again for given $0 < \varepsilon \leq 1$ a solution $u_\varepsilon : \mathbb{D}^2 \to \mathbb{R}^k$ to (1) and specify the result of Proposition 4.2 for special choices of κ and ϱ_ε. More precisely, we choose

$$\varrho_\varepsilon = \mathfrak{r}_\varepsilon \text{ and } \kappa_\varepsilon = \mathrm{C}_{\mathrm{bd}}\sqrt{\mathrm{E}_\varepsilon(u_\varepsilon)}, \tag{5.1}$$

where $\frac{3}{4} \leq \mathfrak{r}_\varepsilon \leq 1$ is the radius introduced in subsection 2.4, Lemma 2.5 for the choice $r_1 = 1, r_0 = \dfrac{3}{4}$ and where the constant C_{bd} is choosen as

$$\mathrm{C}_{\mathrm{bd}} = \sup\left\{4\mathrm{C}_{\mathrm{unf}}, \mathrm{C}_{\mathrm{lev}}\right\}, \tag{5.2}$$

$\mathrm{C}_{\mathrm{unf}}$ being the constant provided in Lemma 2.4 whereas $\mathrm{C}_{\mathrm{lev}}$ is the the constant introduced in Lemma 2.3. With the choice (5.2), the lower bound (2.40) is automatically satisfied for $\kappa = \kappa_\varepsilon$. We notice, in view of (2.31), the construction of $\mathfrak{r}_\varepsilon$ by Lemma 2.5, the definition (5.1) of κ_ε, and the definition (5.2) of C_{bd}, that there exists some element $\sigma_{\mathrm{main}} \in \Sigma$ such that

$$|u(\ell) - \sigma_{\mathrm{main}}| \leq 2\mathrm{C}_{\mathrm{unf}}\sqrt{\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2))} \leq \frac{\mathrm{C}_{\mathrm{bd}}}{2}\sqrt{\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2))} \leq \frac{\kappa_\varepsilon}{2}, \quad \text{for all } \ell \in \mathbb{S}^1(\mathfrak{r}_\varepsilon). \tag{5.3}$$

Hence, condition (4.23) is also automatically fulfilled in view of our choices of parameters. The main result of this subsection is the following:

Proposition 5.1. *Assume that $0 < \varepsilon \leq 1$ and that u_ε is a solution of* (1) *on $\mathbb{D}^2$. There exists a constant $\mathrm{K}_\Upsilon > 0$ such*

$$\int_{\Upsilon_\varepsilon(\mathfrak{r}_\varepsilon, \kappa_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq \mathrm{K}_\Upsilon\left[\left(\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x\right)^{\frac{3}{2}} + \varepsilon\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x\right]. \tag{5.4}$$

Proof. Notice first that the result (5.4) is non trivial only when the energy is small, otherwise it is obvious, for a suitable choice of constant. We introduce therefore the smallness condition on the energy given by

$$\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq \nu_1 \equiv \frac{\mu_0^2}{4\mathrm{C}_{\mathrm{bd}}^2}, \tag{5.5}$$

and distinguish two cases.

Case 1: *Inequality* (5.5) *does not hold, that is* $\mathrm{E}_\varepsilon(u_\varepsilon) \geq \nu_1$. In this case (5.4) is straightforwardly satisfied, provided we choose the constant K_Υ sufficiently large so that

$$\mathrm{K}_\Upsilon \geq \frac{1}{\sqrt{\nu_1}}.$$

Indeed, we obtain, since (5.5) is not satisfied,

$$\begin{aligned}\mathrm{K}_\Upsilon\left(\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x\right)^{\frac{3}{2}} &\geq \mathrm{K}_\Upsilon(\nu_1)^{\frac{1}{2}}\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \\ &\geq \int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \geq \int_{\Upsilon_\varepsilon(\mathfrak{r}_\varepsilon,\kappa_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x.\end{aligned} \tag{5.6}$$

Case 2: *Inequality* (5.5) *does hold.* Since assumption (4.23) is satisfied for $\varrho_\varepsilon = \mathfrak{r}_\varepsilon$ thanks to (5.3), we are in position to apply Proposition 4.2. It yields

$$\int_{\Upsilon_\varepsilon(\mathfrak{r}_\varepsilon,\kappa_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \le \mathrm{C}_\Upsilon(\nu_1)\left[\kappa_\varepsilon\int_{\mathbb{D}^2(\mathfrak{r}_\varepsilon)}\frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \varepsilon\int_{\partial\mathbb{D}^2(\mathfrak{r}_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell\right]. \tag{5.7}$$

We choose the constant K_Υ so that

$$\mathrm{K}_\Upsilon \ge \sup\{\mathrm{C}_\Upsilon(\nu_1)C_{\mathrm{bd}}, \frac{1}{\sqrt{\nu_1}}, 1\}.$$

Inequality (5.4) then follows directly from (5.7) in view of the definition $\kappa_\varepsilon = C_{\mathrm{bd}}\sqrt{\mathrm{E}_\varepsilon(u_\varepsilon)}$ of κ_ε and the fact that, by definition of the energy, we have the point-wise inequality $\dfrac{V(u_\varepsilon)}{\varepsilon} \le e_\varepsilon(u_\varepsilon)$, so that

$$\kappa_\varepsilon\int_{\mathbb{D}^2(\mathfrak{r}_\varepsilon)}\frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x \le \mathrm{C}_{\mathrm{bd}}\sqrt{\mathrm{E}_\varepsilon(u_\varepsilon)}\int_{\mathbb{D}^2(\mathfrak{r}_\varepsilon)}\frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x \le \mathrm{C}_{\mathrm{bd}}\left(\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)\mathrm{d}x\right)^{\frac{3}{2}}.$$

□

At this stage, we have already derived an inequality very close to (60) of Proposition 1, namely inequality (5.4) of Proposition 5.1. However it holds only on a domain where points with large values of $|u_\varepsilon - \sigma_i|$, in sense appropriate sense, have been removed. To go further and obtain an estimate on a full disk, we invoke improved estimates on the potential V which are derived in the next subsection.

5.2 Improved potential estimates

Proposition 5.2. *Assume that* $0 < \varepsilon \le 1$ *and that* u_ε *is a solution of* (1) *on* $\mathbb{D}^2$. *There exists a constant* $C_{\mathrm{V}} > 0$ *such that*

$$\frac{1}{\varepsilon}\int_{\mathbb{D}^2(\frac{5}{8})} V(u_\varepsilon)\mathrm{d}x \le C_{\mathrm{V}}\left[\left(\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x\right)^{\frac{3}{2}} + \varepsilon\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x\right]. \tag{5.8}$$

Proof. The proof combines the energy estimates of Proposition 5.1, the avering argument of Lemma 2.7, together with the Pohozaev type potential estimate provided in Proposition 3.4.

We first observe that, in view of inequality (5.3), the bound (2.36) is satisfied for the solution u_ε, the radius $\varrho = \mathfrak{r}_\varepsilon$ and the choice of parameter $\kappa = \kappa_\varepsilon$, where $\mathfrak{r}_\varepsilon$ and κ_ε have been defined in (5.1). Moreover, the lower-bound (2.40) is verified for κ_ε, as the definition (5.2) shows. We are therefore in position to apply Lemma 2.7 with the choice $\varrho = \mathfrak{r}_\varepsilon$ and $\kappa = \kappa_\varepsilon$. This yields a *new radius* $\tau_\varepsilon \in [\dfrac{5}{8}, \mathfrak{r}_\varepsilon]$ such that

$$\int_{\mathbb{S}^1(\tau_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell \le \frac{1}{\mathfrak{r}_\varepsilon - \frac{11}{16}}\,\mathrm{E}_\varepsilon(u_\varepsilon, \Upsilon(\mathfrak{r}_\varepsilon,\kappa_\varepsilon)) \le 16\,\mathrm{E}_\varepsilon(u_\varepsilon, \Upsilon_\varepsilon(\mathfrak{r}_\varepsilon,\kappa_\varepsilon)),$$

where, for the last inequality, we have used the fact that $\mathfrak{r}_\varepsilon \ge 3/4$, so that $\mathfrak{r}_\varepsilon - 11/16 \ge 1/16$. Invoking inequality (5.4) of Proposition 5.1, we are led to

$$\int_{\mathbb{S}^1(\tau_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell \le 16\mathrm{K}_\Upsilon\left[\left(\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x\right)^{\frac{3}{2}} + \varepsilon\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x\right]. \tag{5.9}$$

On the other hand, thanks to Proposition 3.4, we have

$$\frac{1}{\varepsilon}\int_{\mathbb{D}^2(\tau_\varepsilon)} V(u_\varepsilon)\mathrm{d}x \leq 2\tau_\varepsilon \int_{\mathbb{S}^1(\tau_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell \leq 2\int_{\mathbb{S}^1(\tau_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell. \tag{5.10}$$

Combining (5.9) and (5.10) with the fact that $\tau_\varepsilon \geq \frac{5}{8}$, we derive (5.8) with

$$C_{\mathrm{V}} = 32\mathrm{K}_\Upsilon.$$

The proof is hence complete. □

5.3 Proof of Proposition 1 completed

We introduce first a new radius $\tilde{\mathfrak{r}}_\varepsilon \in [\frac{9}{16}, \frac{5}{8}]$ corresponding to the intermediate radius defined in Lemma 2.5 for the choice $r_1 = \frac{9}{16}, r_0 = \frac{7}{8}$, so that it satisfies

$$\int_{\mathbb{S}^1(\tilde{\mathfrak{r}}_\varepsilon)} e_\varepsilon(u)\mathrm{d}\ell \leq 16\,\mathrm{E}_\varepsilon(u, \mathbb{D}^2(\frac{5}{8})). \tag{5.11}$$

It follows as above from Lemma 2.4 that there exists some element $\sigma_{\mathrm{bis}} \in \Sigma$, possibly different from σ_{main} defined in (5.3), such that

$$|u(\ell) - \sigma_{\mathrm{bis}}| \leq 4\mathrm{C}_{\mathrm{unf}}\sqrt{\mathrm{E}_\varepsilon\left(u, \mathbb{D}^2(\frac{5}{8})\right)}, \quad \text{for all } \ell \in \mathbb{S}^1(\tilde{\mathfrak{r}}_\varepsilon). \tag{5.12}$$

In order to apply Proposition 4.7, we introduce once more a smallness condition on the energy, namely

$$\mathrm{E}_\varepsilon(u_\varepsilon) \leq \eta_2 \equiv \frac{\mu_0^2}{256\mathrm{C}_{\mathrm{unf}}^2}. \tag{5.13}$$

We then distinguish two cases:

Case 1: *The smallness condition* (5.13) *holds.* In this case, we have, in view of (5.12)

$$|u(\ell) - \sigma_{\mathrm{bis}}| \leq 4\mathrm{C}_{\mathrm{unf}}\sqrt{\eta_2} = \frac{\mu_0}{4}, \quad \text{for all } \ell \in \mathbb{S}^1(\tilde{\mathfrak{r}}_\varepsilon),$$

so that condition (4.23) holds fo $\varrho_\varepsilon = \tilde{\mathfrak{r}}_\varepsilon$ (with σ_{main} replaced by σ_{bis}). We are therefore in position to apply Lemma 4.7 on the disk $\mathbb{D}^2(\tilde{\mathfrak{r}}_\varepsilon)$, which yields

$$\int_{\mathbb{D}^2(\tilde{\mathfrak{r}}_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq \mathrm{C}_{\mathrm{pot}}(\mathrm{L}_M)\left[\int_{\mathbb{D}^2(\frac{5}{8})} \frac{V(u_\varepsilon)}{\varepsilon}\mathrm{d}x + \varepsilon\int_{\mathbb{S}^1(\tilde{\mathfrak{r}}_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell\right], \tag{5.14}$$

where L_M is defined in (4.40), so that $\|u_\varepsilon\|_{L^\infty(\mathbb{D}^2(\frac{4}{5}))} \leq \mathrm{L}_\mathrm{M}$. Invoking Proposition 5.2 and inequality (5.11), we are hence led to

$$\begin{aligned}\int_{\mathbb{D}^2(\tilde{\mathfrak{r}}_\varepsilon)} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x \leq\ & \mathrm{C}_{\mathrm{pot}}((\mathrm{L}_M)C_{\mathrm{V}}\left(\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x\right)^{\frac{3}{2}} \\ & + 16\mathrm{C}_{\mathrm{pot}}(\mathrm{L}_M)\mathrm{C}_{\mathrm{V}}\varepsilon\int_{\mathbb{D}^2} e_\varepsilon(u_\varepsilon)(x)\mathrm{d}x,\end{aligned}$$

which, since $5/8 \geq \tilde{\mathfrak{r}}_\varepsilon \geq 9/16$, yields (60), for a suitable choice of the constant $\mathrm{C}_{\mathrm{dec}}$.

Case 2: *The smallness condition* (5.13) does not *hold.* In this case, inequality (60) is straightforwardly fullfilled, provided we choose

$$\mathrm{C}_{\mathrm{dec}} \geq \eta_2^{-\frac{1}{2}}.$$

The proof is hence complete in both cases. □

6 Proof of the Clearing-out theorem

The purpose of this section is to provide the proof of the clearing-out property stated in Theorem 6, a main step being the uniform bound (58). We first introduce a *very weak form of the clearing-out theorem.*

6.1 A very weak form of the clearing-out

The following result is classical in the field (see e.g. [27, 11]).

Proposition 6.1. *Let u_ε be a solution of* (1) *on $\mathbb{D}^2$ with $0 < \varepsilon \leq 10$. There exists a constant $\eta_3 > 0$ such that, if*

$$\mathrm{E}_\varepsilon(u_\varepsilon) \leq \eta_3 \varepsilon,$$

then we have, for some $\sigma \in \Sigma$, the bound

$$|u_\varepsilon(x) - \sigma| \leq \mathrm{C}_{\mathrm{wk}} \left(\frac{\mathrm{E}_\varepsilon(u_\varepsilon)}{\varepsilon} \right)^{\frac{1}{6}} \leq \frac{\mu_0}{2}, \text{ for } x \in \mathbb{D}^2\left(\frac{3}{4}\right), \tag{6.1}$$

where C_{wk} denotes some positive constant depending only on the potential V.

Remark 6.1. In the scalar case, Proposition 6.1 combined with the monotonicity formula for the energy directly yields the proof of Theorem 6.

Proof. Assume that the bound $\mathrm{E}_\varepsilon(u) \leq \eta_3 \varepsilon$ holds, for some constant η_3 to be determined later. Imposing first $\eta_3 \leq 1$, it follows from Proposition 3.1 applied with $\Omega = \mathbb{D}^2$ that there exists a constant $C_0 > 0$ depending only on V such that

$$|u_\varepsilon(x)| \leq C_0, \text{ for } x \in \mathbb{D}^2(\frac{7}{8}).$$

Since the potential V is smooth, and hence its gradient is bounded on the disc $\mathbb{B}^k(C_0)$, we deduce from Proposition 3.2 that there exists a constant C_1 such that

$$|\nabla V(u_\varepsilon)(x)| \leq \frac{C_1}{\varepsilon}, \text{ for } x \in \mathbb{D}^2(\frac{7}{8}). \tag{6.2}$$

On the other hand, since we assume $\mathrm{E}_\varepsilon(u_\varepsilon) \leq \eta_3 \varepsilon$, we deduce from the definition of the energy that

$$\int_{\mathbb{D}^2(\frac{7}{8})} V(u_\varepsilon(x)) \mathrm{d}x \leq \int_{\mathbb{D}^2} V(u_\varepsilon(x)) \mathrm{d}x \leq \varepsilon \mathrm{E}_\varepsilon(u_\varepsilon) \leq \eta_3 \varepsilon^2. \tag{6.3}$$

We introduce next the number $\alpha = \left(\frac{16C_1^2 \mathrm{E}_\varepsilon(u_\varepsilon)}{\pi\varepsilon}\right)^{\frac{1}{3}}$. We impose a second upper bound on η_3 given by

$$\eta_3 \leq \pi \frac{\alpha_0^3}{16C_1^2}, \tag{6.4}$$

where α_0 is the constant introduced in (2.2). It follows from the above definitions, that, if $\mathrm{E}_\varepsilon(u) \leq \eta_3\varepsilon$, then we have $\alpha \leq \alpha_0$. We claim that, if η_3 is chosen so that (6.4) is satisfied, then, we have

$$V(u_\varepsilon(x)) \leq \alpha \leq \alpha_0, \text{ for any } x \in \mathbb{D}^2(\frac{3}{4}). \tag{6.5}$$

Indeed, assume by contradiction that there exists some $x_0 \in \mathbb{D}^2(3/4)$ such that $V(u(x_0)) > \alpha$. Invoking the gradient bound (6.2), we deduce that

$$V(u_\varepsilon(x)) \geq \frac{\alpha}{2} \text{ for } x \in \mathbb{D}^2\left(x_0, \frac{\alpha\varepsilon}{2C_1}\right). \tag{6.6}$$

Without loss of generality, we may assume that C_1 is chosen sufficiently large so that $\frac{4\alpha_0}{2C_1} \leq \frac{1}{8}$ and hence

$$\mathbb{D}^2\left(x_0, \frac{\alpha\varepsilon}{2C_1}\right) \subset \mathbb{D}^2\left(x_0, \frac{\alpha_0\varepsilon}{2C_1}\right) \subset \mathbb{D}^2(\frac{7}{8}), \text{ since } |x_0| \leq 3/4.$$

Integrating (6.6) on the disk $\mathbb{D}^2\left(x_0, \frac{\alpha\varepsilon}{2C_1}\right)$, we are led to the lower bound

$$\int_{\mathbb{D}^2(\frac{7}{8})} V(u_\varepsilon(x))\mathrm{d}x \geq \int_{\mathbb{D}^2(x_0, \frac{\alpha\varepsilon}{2C_1})} V(u_\varepsilon(x))\mathrm{d}x \geq \pi\frac{\alpha^3}{8C_1^2}\varepsilon^2 = 2\varepsilon\mathrm{E}_\varepsilon(u_\varepsilon).$$

This yields a contradiction with (6.3) and hence establishes the claim (6.5). Combining (6.5) and Lemma 2.1 together with the continuity of the map u_ε, we may assert that there exists some $\sigma \in \Sigma$ such that, for any $x \in \mathbb{D}^2(\frac{3}{4})$, we have

$$\begin{aligned} |u_\varepsilon(x) - \sigma| &\leq \inf\{\mu_0, \sqrt{4\lambda_0^{-1}\alpha}\} = \inf\left\{\mu_0, \left(\frac{1024\lambda_0^{-3}C_1^2\mathrm{E}_\varepsilon(u_\varepsilon)}{\pi\varepsilon}\right)^{\frac{1}{6}}\right\} \\ &\leq \inf\left\{\mu_0, \mathrm{C_{wk}}\left(\frac{\mathrm{E}_\varepsilon(u_\varepsilon)}{\varepsilon}\right)^{\frac{1}{6}}\right\}, \text{ where } \mathrm{C_{wk}} \equiv \left(\frac{1024\lambda_0^{-3}C_1^2}{\pi}\right)^{\frac{1}{6}}. \end{aligned} \tag{6.7}$$

To complete the proof of (6.1), we impose an additional upper bound on η_3, namely

$$\eta_3 \leq \left(\frac{\mu_0}{2\mathrm{C_{wk}}}\right)^6, \tag{6.8}$$

So that, if $\mathrm{E}_\varepsilon(u_\varepsilon) \leq \eta_3\varepsilon$, then we have $\mathrm{C_{wk}}\left(\frac{\mathrm{E}_\varepsilon(u_\varepsilon)}{\varepsilon}\right)^{\frac{1}{6}} \leq \frac{\mu_0}{2}$. Combining with (6.7), we obtain (6.1), which hence completes the proof of the Lemma. □

Remark 6.2. In some place, in particular in the proof of Proposition 6.2 below, our arguments require clearing-out results, for values of ε which are not necessarily *very small.* In such a situation, we will make use of the following immediate consequence of Proposition 6.1:

Corollary 6.1. *Let* $0 < \check{\epsilon}_0 \leq 10$ *be given. Assume that* $\varepsilon \geq \check{\epsilon}_0$ *and*

$$\mathrm{E}_\varepsilon(u_\varepsilon) \leq \eta_3 \check{\epsilon}_0. \tag{6.9}$$

Then (6.1) *holds.*

The proof is straightforward, since (6.9) and $\varepsilon \geq \check{\epsilon}_0$ imply $\mathrm{E}_\varepsilon(u_\varepsilon) \leq \eta_3 \varepsilon$, so that Proposition 6.1 applies.

6.2 Confinement near a well of the potential

Our next result is the main step in the proof of Theorem 6. It shows that, if the energy is sufficiently small, then $u_\varepsilon(0)$ takes its values inside a well of the potential.

Proposition 6.2. *Let* $0 < \varepsilon \leq 1$ *and* u_ε *be a solution of* (1) *on* $\mathbb{D}^2$*. There exists a constant* $\eta_4 > 0$ *such that if*

$$\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2) \leq \eta_4, \tag{6.10}$$

then, we have, for some $\sigma \in \Sigma$*, the bound*

$$|u_\varepsilon(0) - \sigma| \leq \frac{\mu_0}{2}. \tag{6.11}$$

Proof. We consider first the case that ε is *not so small*, more precisely the case where

$$1 \geq \varepsilon \geq \eta_3, \tag{6.12}$$

where η_3 denotes the constant introduced in Proposition 6.1. In this case, we apply Corollary 6.1 with $\check{\epsilon}_0 = \eta_3$. We therefore impose a first condition on η_4, given by

$$\eta_4 \leq \eta_3 \check{\epsilon}_0 = \eta_3^2. \tag{6.13}$$

Assumption (6.9) then follows from (6.10), so that Corollary 6.1 yields the desired results and we are done. It remains therefore to establish the result in *the case* ε *is small*, that is, under the assumption

$$0 \leq \varepsilon \leq \eta_3. \tag{6.14}$$

Under assumption (6.14), we distinguish again two cases, first the *easy* case, where

$$\mathrm{E}_\varepsilon(u_\varepsilon) \leq \varepsilon^2. \tag{6.15}$$

In that case, we have, on view of (6.14), the upper bound $\mathrm{E}_\varepsilon(u_\varepsilon) \leq \eta_3 \varepsilon$, and we are hence in position to apply the result of Proposition 6.1, which leads again to the desired conclusion (6.11). We therefore restrict ourselves throughout the end of the proof to the remaining case, namely

$$\mathrm{E}_\varepsilon(u_\varepsilon) > \varepsilon^2, \text{ and } 0 < \varepsilon < \eta_3. \tag{6.16}$$

The proof of the Proposition 6.2 under assumption (6.16) then relies on inequality (60) of Proposition 1, a standard scaling argument combined with an iteration procedure. We divide the rather lengthy argument into several steps.

Step 1: A scaled version of inequality (61). *Set, for* $0 < r \leq 1$, $\mathrm{E}_\varepsilon(r) = \mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(r)\right)$, *and assume that*

$$\mathrm{E}_\varepsilon(r) \geq \frac{\varepsilon^2}{r}. \tag{6.17}$$

Then, we have

$$\mathrm{E}_\varepsilon(\frac{r}{2}) \leq \mathrm{K}_{\mathrm{dec}} \frac{\mathrm{E}_\varepsilon(r)^{\frac{3}{2}}}{\sqrt{r}}, \quad \text{provided } r \geq \varepsilon, \text{ and where } \mathrm{K}_{\mathrm{dec}} = \sup\{2\mathrm{C}_{\mathrm{dec}}, 1\}. \tag{6.18}$$

Indeed, scaling inequality (61), we are led to

$$\mathrm{E}_\varepsilon(\frac{r}{2}) \leq \mathrm{C}_{\mathrm{dec}} \left[\frac{1}{\sqrt{r}} \mathrm{E}_\varepsilon(r)^{\frac{3}{2}} + \frac{\varepsilon}{r} \mathrm{E}_\varepsilon(r)\right], \text{ provided } r \geq \varepsilon, \tag{6.19}$$

Since, by assumption (6.17), we have $\frac{\varepsilon}{r} \leq \sqrt{\frac{\mathrm{E}_\varepsilon(r)}{r}}$, we deduce from (6.19) that $\mathrm{E}_\varepsilon(\frac{r}{2}) \leq 2\mathrm{C}_{\mathrm{dec}} \frac{\mathrm{E}_\varepsilon(r)^{\frac{3}{2}}}{\sqrt{r}}$, and (6.18) follows.

Step 2: The iteration procedure. We consider the sequence $(r_n)_{n\in\mathbb{N}}$ of decreasing radii r_n defined as $r_n = \frac{1}{2^n}$, for $n \in \mathbb{N}$, and set $\mathrm{E}_n^\varepsilon = \mathrm{E}_\varepsilon(r_n) = \mathrm{E}_\varepsilon(\frac{1}{2^n})$, dropping the superscript in case this induces no ambiguity. We introduce the number

$$n_\varepsilon = \sup\left\{n \in \mathbb{N}, \text{ such that } \mathrm{E}_n^\varepsilon \geq 2^n\varepsilon^2 = \frac{\varepsilon^2}{r_n}, \text{ and } r_n = \frac{1}{2^n} \geq \varepsilon\right\}. \tag{6.20}$$

We first notice that, under assumptions (6.16), the number n_ε is well defined. Indeed, we have, in view of (6.16),

$$\mathrm{E}_0^\varepsilon > \varepsilon^2 = 2^0\varepsilon^2 \text{ and } r_0 = 1 \geq \varepsilon,$$

so that the number 0 belongs to the set of the r.h.s of (6.20), which is hence not empty. On the other hand, since 2^n tends to infinity as n tends to infinity, and since the sequence $(\mathrm{E}_n)_{n\in\mathbb{N}}$ is non-increasing, hence bounded by E_0^ε, the set of the r.h.s of (6.20) is a finite set of sequential number and the number n_ε is hence a well-defined integer. In view of the definition of n_ε, inequality (6.17) is straightforwardly satisfied for every $r_n < r_{n_\varepsilon}$. We deduce therefore from Step 1, inequality (6.18), and the definition of r_n, that we have the inequality

$$\mathrm{E}_{n+1} \leq \sqrt{2}^n \mathrm{K}_{\mathrm{dec}} \left(\mathrm{E}_n\right)^{\frac{3}{2}}, \text{ for } n = 0, \dots n_\varepsilon. \tag{6.21}$$

We introduce, for $n \in \mathbb{N}$, the number $A_n = -\log \mathrm{E}_n$. Inequality (6.21) for E_n is turned into the inequality for A_n given by

$$A_{n+1} \geq \frac{3}{2}A_n - \frac{(\log 2)}{2} n - \log(\mathrm{K}_{\mathrm{dec}}), \quad \text{for } n = 0, \dots n_\varepsilon. \tag{6.22}$$

In order to study the sequence $(A_n)_{n\in\mathbb{N}}$, we will invoke the next elementary result.

Lemma 6.1. *Let $n_\star \in \mathbb{N}^*$, $(a_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ be two sequences of numbers such that*

$$a_{n+1} \geq \mathrm{c}_0\, a_n - f_n, \text{ for all } n \in \mathbb{N}, n \leq n_\star, \tag{6.23}$$

where $\mathrm{c}_0 > 1$ represents a given constant. Then we have the inequality,

$$a_n \geq \mathrm{c}_0^n \left(a_0 - \sum_{k=0}^{n} \frac{1}{\mathrm{c}_0^{k+1}} f_k \right), \text{ for } n \in \mathbb{N}^*,\ n \leq n_\star. \tag{6.24}$$

We postpone the proof of Lemma 6.1 and complete first the proof of Proposition 6.2, under assumption (6.16).

Step 3: Imposing a new constraint on η_4 and energy decay estimates. We apply Lemma 6.1 to the sequences $(a_n)_{n\in\mathbb{N}} = (A_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ given, in view of (6.22), by

$$f_n = \frac{(\log 2)}{2} n + \log(\mathrm{K}_{\mathrm{dec}}), \text{ for any } n \in \mathbb{N}.$$

We notice, in view of the definition of $\mathrm{K}_{\mathrm{dec}}$ given in (6.18), that $f_n \geq 0$, $\forall n \in \mathbb{N}$. It follows from these definitions that inequality (6.23) is satisfied with

$$\mathrm{c}_0 = \frac{3}{2},\ a_0 = -\log \mathrm{E}_0 = -\log(\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2)), \text{ and } n_\star = n_\varepsilon.$$

Inequality (6.24) then yields, for $n = 0, \dots n_\varepsilon$,

$$\begin{aligned} A_n = -\log \mathrm{E}_n &\geq \left(\frac{3}{2}\right)^n \left[\log\left(\frac{1}{\mathrm{E}_\varepsilon(u_\varepsilon)}\right) - \gamma_n \right], \text{ where } \gamma_n = \sum_{k=0}^{n} \left(\frac{2}{3}\right)^{k+1} f_k, \\ &\geq \left(\frac{3}{2}\right)^n \left[-\log \eta_4 - \gamma_0 \right]. \end{aligned} \tag{6.25}$$

Here we have used, for the second inequality, assumption (6.10) and we have set

$$\gamma_0 = \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^{k+1} \left(\frac{(\log 2)}{2} k + \log(\mathrm{K}_{\mathrm{dec}}) \right) < +\infty.$$

Inequality (6.25) leads us to impose another constraint on the constant η_4, besides (6.13), namely we impose

$$\eta_4 \leq \exp\left[-(1+\gamma_0)\right], \text{ so that } -\log \eta_4 \geq 1 + \gamma_0, \tag{6.26}$$

It follows that inequality (6.25) yields, provided inequality (6.10) holds,

$$\mathrm{E}_n \leq \exp\left[-\left(\frac{3}{2}\right)^n \right], \quad \text{for } n = 0, \dots n_\varepsilon - 1. \tag{6.27}$$

Step 4: Estimating n_ε and r_{n_ε}. It follows from (6.27) and the definition of n_ε that

$$\varepsilon^2 = \exp(2\log\varepsilon) \leq r_n \mathrm{E}_n = 2^{-n}\mathrm{E}_n \leq \exp\left[-\left(\frac{3}{2}\right)^n - n(\log 2) \right], \quad \text{for } n = 0, \dots n_\varepsilon,$$

so that we are led to the inequality

$$\left(\frac{3}{2}\right)^{n_\varepsilon} + n_\varepsilon(\log 2) \leq 2|\log \varepsilon|,$$

Hence, since $n_\varepsilon(\log 2) > 0$, we have

$$\left(\frac{3}{2}\right)^{n_\varepsilon} \leq 2|\log \varepsilon|.$$

Taking the logarithm of both sides, we obtain *the upper bound* for n_ε

$$n_\varepsilon \leq \frac{\log(2|\log \varepsilon|)}{\log 3 - \log 2}.$$

This upper bound yields a *lower bound* for r_{n_ε}, namely

$$\begin{aligned} r_{n_\varepsilon} = 2^{-n_\varepsilon} = \exp(-(\log 2)\, n_\varepsilon) &\geq \exp\left(-\log(2|\log \varepsilon|)\frac{\log 2}{\log 3 - \log 2}\right) \\ &\geq \exp\left(-\gamma_1 \log(2|\log \varepsilon|)\right) \\ &\geq (2|\log \varepsilon|)^{-\gamma_1}. \end{aligned} \tag{6.28}$$

Here we have set

$$\gamma_1 = \frac{\log 2}{\log 3 - \log 2} \simeq 1,7, \text{ so that } 1 < \gamma_1 < 2.$$

We notice that $(2|\log \varepsilon|)^{-\gamma_1} \underset{\varepsilon \to 0}{\gg} \varepsilon$, so that there exists some universal constant $0 < \varepsilon_1 \leq 1$ such that

$$r_{n_\varepsilon} \geq 2\,\varepsilon, \text{ provided } 0 < \varepsilon \leq \varepsilon_1. \tag{6.29}$$

Going back to the definition of n_ε, we deduce from (6.29) and (6.28) that

$$\mathrm{E}^\varepsilon_{n_\varepsilon+1} \leq \varepsilon^2 r^{-1}_{n_\varepsilon+1} = 2^{n_\varepsilon+1}\varepsilon^2 \leq 8|\log \varepsilon|^{\gamma_1}\varepsilon^2, \text{ if } 0 < \varepsilon \leq \varepsilon_1. \tag{6.30}$$

Step 5: Change of scale. We introduce the scaled parameter $\tilde{\varepsilon}$ defined by

$$\tilde{\varepsilon} = r^{-1}_{n_\varepsilon+1}\varepsilon = 2r^{-1}_{n_\varepsilon}\varepsilon, \text{ so that by (6.30) we have } \tilde{\varepsilon} \geq \varepsilon, \text{ if } \varepsilon \leq \varepsilon_1,$$

Invoking (6.29), we have moreover

$$\tilde{\varepsilon} \leq 1, \text{ provided } 0 < \varepsilon \leq \varepsilon_1. \tag{6.31}$$

We consider once more the scaled map $\tilde{u}_{\tilde{\varepsilon}}$ defined on $\mathbb{D}^2$ by

$$\tilde{u}_{\tilde{\varepsilon}}(x) = u_\varepsilon(r_{(n_\varepsilon+1)}\, x), \text{ for } x \in \mathbb{D}^2.$$

Using the scaling properties (54), we are led to the identity, for the energy

$$\mathrm{E}_{\tilde{\varepsilon}}(\tilde{u}_{\tilde{\varepsilon}}) = r^{-1}_{n_\varepsilon+1}\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2(r_{n_\varepsilon+1})) = r^{-1}_{n_\varepsilon+1}\mathrm{E}^\varepsilon_{n_\varepsilon+1},$$

so that, in view of (6.28) and (6.30), we have

$$\begin{cases} \mathrm{E}_{\tilde\varepsilon}(\tilde u_{\tilde\varepsilon}) \le 16|\log\varepsilon|)^{2\gamma_1}\,\varepsilon^2, \text{ if } \varepsilon\le\varepsilon_1, \text{ and} \\ \dfrac{\mathrm{E}_{\tilde\varepsilon}(\tilde u_{\tilde\varepsilon})}{\tilde\varepsilon} = \dfrac{\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2(r_{n_\varepsilon+1}))}{\varepsilon} \le 8|\log\varepsilon|^{\gamma_1}\varepsilon, \text{ if } \varepsilon\le\varepsilon_1. \end{cases} \tag{6.32}$$

The derivative of map $s \to \varphi(s) \equiv 8|\log s|^{\gamma_1}s$ is given, on $(0,1)$ by $\varphi'(s) = 8(|\log s|^{\gamma_1} - (\gamma_1 - 1)|\log s|^{\gamma_1-1})$, so that it is non-negative on the intervalle $I = (0, e^{1-\gamma_1}) \subset (0,1)$. It follows that φ is non-decreasing on the interval I and tends to 0 as s tends to 0. Hence, there exists some universal constant $\varepsilon_2 \in I$ such that

$$\varphi(\varepsilon_2) = 8|\log\varepsilon_2|^{\gamma_1}\varepsilon_2 \le \eta_3, \text{ and } \varepsilon_2\le\varepsilon_1, \tag{6.33}$$

where η_3 denotes the constants introduced in Proposition 6.1. We have therefore, by monotonicity of φ

$$\varphi(\varepsilon) = 8|\log\varepsilon|^{\gamma_1}\varepsilon \le \eta_3, \text{ for } 0<\varepsilon\le\varepsilon_2.$$

Going back to (6.32) and (6.31) we obtain, for $0\le\varepsilon\le\varepsilon_2$

$$\mathrm{E}_{\tilde\varepsilon}(\tilde u_{\tilde\varepsilon}) \le \eta_3\,\tilde\varepsilon \text{ and } \ \tilde\varepsilon\le 1. \tag{6.34}$$

Step 6: Proof of Proposition 6.2 completed. We conclude invoking the weak clearing-out property stated in Proposition 6.1. For that purpose, we distinguish two cases:

Case 1: $0<\varepsilon\le\varepsilon_2$. In view of (6.34), we are in position to apply Proposition 6.1 to the map $\tilde u_\varepsilon$ with parameter $\tilde\varepsilon$: Hence there exists some point $\sigma\in\Sigma$ such that

$$|\tilde u_{\tilde\varepsilon}(0) - \sigma| \le \frac{\mu_0}{2}.$$

since $u_\varepsilon(0) = \tilde u_{\tilde\varepsilon}(0)$ the conclusion of Proposition 6.2 follows.

Case 2: $1\ge\varepsilon>\varepsilon_2$. Here we apply directly Corollary 6.1 to u_ε, choosing $\check\epsilon_0 = \varepsilon_2$. Besides (6.13), (6.26) we impose a last constraint on η_4 given by

$$\eta_4 \le \eta_3\check\epsilon_0 = \eta_3\varepsilon_2.$$

Hence, if u_ε satisfies (6.10), then it fulfills assumption (6.9) of Corollary 6.1, so that its conclusion yields again the existence of an element $\sigma\in\Sigma$ such that $|u_\varepsilon(0)-\sigma| \le \dfrac{\mu_0}{2}$.

In both cases, we have hence established the conclusion of Proposition 6.2 so that the proof is complete. □

In the course of the proof, we have used Lemma 6.1, which has not been proved yet.

Proof of Lemma 6.1. We introduce, inspired by the method of variation of constant, the sequence $(b_n)_{n\in\mathbb{N}}$ defined by $a_n = \mathrm{c}_0^n\, b_n$, for any $n\in\mathbb{N}$. Substituting into (6.23), we obtain

$$\mathrm{c}_0^{k+1} b_{k+1} \ge \mathrm{c}_0^{k+1} b_k - f_k, \text{ for all } k\in\{0,\dots,n_\star\},$$

so that

$$b_{k+1} - b_k \ge -\frac{1}{\mathrm{c}_0^{k+1}} f_k, \text{ for all } k\in\{0,\dots,n_\star\}.$$

Let $n \in \mathbb{N}$, $n \leq n_\star$. Summing these relations for $k = 0$ to $k = n-1$, we are led to

$$b_n \geq b_0 - \sum_{k=0}^{n} \frac{1}{\mathrm{c}_0^{k+1}} f_k = a_0 - \sum_{k=0}^{n} \frac{1}{\mathrm{c}_0^{k+1}} f_k,$$

which, in view of the definition of b_n, yields the desired conclusion (6.24). □

A direct consequence of Proposition 6.2 is the following:

Corollary 6.2. *Let* $0 < \varepsilon \leq 1$ *and* u_ε *be a solution of* (1). *Set* $\eta_5 = \inf\{\frac{1}{8}\eta_3, \frac{1}{8}\eta_4\}$ *and assume that*

$$\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2) \leq \eta_5. \tag{6.35}$$

then, there exists some $\sigma \in \Sigma$ *such that*

$$|u_\varepsilon(x) - \sigma| \leq \frac{\mu_0}{2}, \text{ for any } x \in \mathbb{D}^2\left(\frac{7}{8}\right). \tag{6.36}$$

Proof. Let $x_0 \in \mathbb{D}^2(\frac{7}{8})$ be an arbitrary point. We consider the scaled parameter $\tilde{\varepsilon} = 8\varepsilon$ and the scaled and translated map $\tilde{u}_\varepsilon$ defined on $\mathbb{D}^2$ by

$$\tilde{u}_{\tilde{\varepsilon}}(x) = u_\varepsilon\left(x_0 + \frac{1}{8}x\right), \text{ for every } x \in \mathbb{D}^2,$$

so that

$$\mathrm{E}_{\tilde{\varepsilon}}(\tilde{u}_{\tilde{\varepsilon}}) = 8\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(x_0, \frac{1}{8})\right) \leq 8\mathrm{E}_\varepsilon(u_\varepsilon) \leq 8\eta_5 \leq \eta_4, \tag{6.37}$$

where we have used assumption (6.35) and the definition of η_5 for the last inequality. As above, we distinguish two cases.

Case 1: $\varepsilon \leq \frac{1}{8}$. In this case $\tilde{\varepsilon} \leq 1$, so that, in view of (6.37), we are in position to apply Proposition 6.2 to $\tilde{u}_{\tilde{\varepsilon}}$: It yields an element $\sigma_{x_0} \in \Sigma$, depending possibly on the point x_0, such that

$$|\tilde{u}_{\tilde{\varepsilon}}(0) - \sigma_{x_0}| = |u_\varepsilon(x_0) - \sigma_{x_0}| \leq \frac{\mu_0}{2}, \tag{6.38}$$

where we used the fact that $\tilde{u}_{\tilde{\varepsilon}}(0) = u_\varepsilon(x_0)$. Since inequality (6.38) holds for *any point* $x_0 \in \mathbb{D}^2(7/8)$, a continuity argument shows that the element σ_{x_0} of Σ does not depend on x_0, so that the proof of Corollary 6.2 is complete in Case 1.

Case 2: $1 \geq \varepsilon \geq \frac{1}{8}$. In this case $1 \leq \tilde{\varepsilon} \leq 8$. In view of the definition of η_5, we have $8\eta_5 \leq \eta_3$, It then follows from assumption (6.35) that

$$\mathrm{E}_{\tilde{\varepsilon}}(\tilde{u}_{\tilde{\varepsilon}}) = 8\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2\left(x_0, \frac{1}{8}\right)\right) \leq 8\mathrm{E}_\varepsilon(u_\varepsilon) \leq 8\eta_5 \leq \eta_3 \leq \eta_3\tilde{\varepsilon}. \tag{6.39}$$

Hence, we are in position to apply Proposition 6.1, so that there exists an element $\sigma_{x_0} \in \Sigma$, depending possibly on the point x_0 such that $|\tilde{u}_{\tilde{\varepsilon}}(0) - \sigma_{x_0}| \leq \frac{\mu_0}{2}$. Since $\tilde{u}_{\tilde{\varepsilon}}(0) = u_\varepsilon(x_0)$, we conclude that

$$|u_\varepsilon(x_0) - \sigma_{x_0}| \leq \frac{\mu_0}{2}.$$

The proof of Corollary 6.2 is hence complete.

□

6.3 Energy estimates for solutions near potential wells

We turn in this section to energy estimates, for solutions having their image near a well, a condition which replaces the smallness assumption on the energy.

Proposition 6.3. *Let u_ε be a solution of* (1) *on* $\mathbb{D}^2$ *with* $0 < \varepsilon \leq 10$ *such that, for some* $\sigma \in \Sigma$*, we have*

$$|u(x) - \sigma| \leq \frac{\mu_0}{2}, \text{ for } x \in \mathbb{D}^2(1). \tag{6.40}$$

Then, we have the energy estimate

$$\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2\left(\frac{3}{4}\right)\right) \leq 8\lambda_0^{-\frac{3}{2}}\lambda_{\max}\,\varepsilon\mathrm{E}_\varepsilon(u, \mathbb{D}^2)). \tag{6.41}$$

Proof. The proof is parallel and actually much easier then our earlier energy estimate. We first invoke Lemma 2.5 with $r_0 = \frac{5}{8}$ and $r_1 = \frac{3}{4}$: This yields a radius $\mathfrak{r}_\varepsilon \in [\frac{5}{8}, \frac{3}{4}]$ and an element $\sigma \in \Sigma$ such that

$$\int_{\mathbb{S}^1(\mathfrak{r}_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}\ell \leq 8\,\mathrm{E}_\varepsilon(u, \mathbb{D}^2)) \text{ and } \int_{\mathbb{S}^1(\mathfrak{r}_\varepsilon)} |u_\varepsilon - \sigma||\nabla u_\varepsilon| \leq 16\sqrt{\lambda_0^{-1}}\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2)), \tag{6.42}$$

where we have used (2.12) and (2.13) for the second inequality. We multiply the equation (1) by $(u_\varepsilon - \sigma)$ and integrate on the disk $\mathbb{D}^2(\mathfrak{r}_\varepsilon)$ which yields, as in (4.29)

$$\int_{\mathbb{D}^2(\mathfrak{r}_\varepsilon)} \varepsilon|\nabla u_\varepsilon|^2 + \varepsilon^{-1}\nabla_u V(u_\varepsilon)\cdot(u_\varepsilon - \sigma) = \varepsilon\int_{\mathbb{S}^1(\mathfrak{r}_\varepsilon)} \frac{\partial u_\varepsilon}{\partial r}\cdot(u_\varepsilon - \sigma). \tag{6.43}$$

We deduce from (6.42) that

$$\int_{\mathbb{S}^1(\mathfrak{r}_\varepsilon)} \frac{\partial u_\varepsilon}{\partial r}\cdot(u_\varepsilon - \sigma) \leq \int_{\mathbb{S}^1(\mathfrak{r}_\varepsilon)} |u_\varepsilon - \sigma||\nabla u_\varepsilon| \leq 16\sqrt{\lambda_0^{-1}}\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2)). \tag{6.44}$$

We use next the fact that, in view of assertion (58), we have $|u_\varepsilon - \sigma| \leq \frac{\mu_0}{2}$ on the disk $\mathbb{D}^2(\mathfrak{r}_\varepsilon)$. Arguing as in (4.9), we have the point-wise inequality

$$\varepsilon|\nabla u_\varepsilon|^2 + \varepsilon^{-1}\nabla_u V(u_\varepsilon)\cdot(u_\varepsilon - \sigma) \geq \frac{\lambda_0}{2\lambda_{\max}}e_\varepsilon(u). \tag{6.45}$$

Combining (6.43) with (6.45) and (6.44), we obtain

$$\int_{\mathbb{D}^2(\mathfrak{r}_\varepsilon)} e_\varepsilon(u_\varepsilon)\mathrm{d}x \leq 8\lambda_0^{-\frac{3}{2}}\lambda_{\max}\,\varepsilon\mathrm{E}_\varepsilon(u, \mathbb{D}^2)),$$

Which yields the energy estimate (6.41). □

6.4 Improved uniform bounds

Combining Proposition 6.3 with Proposition 6.1 and Proposition 6.2 leads to an improvement of uniform bound provided by Corollary 6.2, under the assumption that the energy is sufficiently small.

Proposition 6.4. *Let $0 < \varepsilon \leq 1$ and u_ε be a solution of* (1) *on $\mathbb{D}^2$. There exist constants $\eta_6 > 0$ and $\mathrm{C}_{\rm well} > 0$, depending possibly on V, such that, if*

$$\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2) \leq \eta_6, \tag{6.46}$$

then there exists some $\sigma \in \Sigma$ such that we have the uniform bound

$$|u(x) - \sigma| \leq \mathrm{C}_{\rm well} \left(\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2)\right)^{\frac{1}{6}} \leq \frac{\mu_0}{2}, \text{ for } x \in \mathbb{D}^2\left(\frac{3}{4}\right). \tag{6.47}$$

Proof. We assume that the bound (6.46) holds, for a constant $\eta_6 > 0$, whose value will be determined in the course of the proof. We first impose that $\eta_6 \leq \eta_5$, so that (6.46) implies (6.35), and we are in position to apply Corollary 6.2 and assert that, for some $\sigma \in \Sigma$,

$$|u_\varepsilon(x) - \sigma| \leq \frac{\mu_0}{2}, \text{ for } x \in \mathbb{D}^2(\frac{7}{8}). \tag{6.48}$$

Next let $x_0 \in \mathbb{D}^2(3/4)$ be an arbitrary point. We consider the scaled parameter $\tilde{\varepsilon} = 8\varepsilon$ and the scaled and translated map $\tilde{u}_{\tilde{\varepsilon}}$ defined on $\mathbb{D}^2$ by

$$\tilde{u}_{\tilde{\varepsilon}}(x) = u_\varepsilon\left(x_0 + \frac{1}{8}x\right) \text{ for every } x \in \mathbb{D}^2,$$

so that $\tilde{u}_{\tilde{\varepsilon}}$ solves (1) on $\mathbb{D}^2$, with ε replaced by $\tilde{\varepsilon}$. For $x \in \mathbb{D}^2$, the point $x_0 + 1/8x$ belongs to $\mathbb{D}^2(\frac{7}{8})$, so that if follows from (6.48) that

$$|\tilde{u}_{\tilde{\varepsilon}}(x) - \sigma| \leq \frac{\mu_0}{2}, \text{ for } x \in \mathbb{D}^2(1). \tag{6.49}$$

Moreover, we have, using (55),

$$\mathrm{E}_{\tilde{\varepsilon}}(\tilde{u}_{\tilde{\varepsilon}}) = 8\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2(x_0, \frac{1}{8})\right) \leq 8\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2) \leq 8\eta_6. \tag{6.50}$$

In view of (6.49) and the fact that $\tilde{\varepsilon} \leq 8$, we are in position to apply Proposition 6.3 to $\tilde{u}_{\tilde{\varepsilon}}$, so that we derive that

$$\mathrm{E}_{\tilde{\varepsilon}}\left(u_{\tilde{\varepsilon}}, \mathbb{D}^2\left(\frac{3}{4}\right)\right) \leq 8\lambda_0^{-\frac{3}{2}}\lambda_{\max}\,\tilde{\varepsilon}\mathrm{E}_{\tilde{\varepsilon}}(u_{\tilde{\varepsilon}}, \mathbb{D}^2)) \leq 8^3\lambda_0^{-\frac{3}{2}}\lambda_{\max}\,\varepsilon\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2\right). \tag{6.51}$$

We perform now another change of coordinates, introducing the parameter $\epsilon = \dfrac{4}{3}\tilde{\varepsilon} = \dfrac{32}{3}\varepsilon$, so that $0 < \epsilon < 10$. We consider the scaled map $\check{u}_\epsilon$ defined on $\mathbb{D}^2$ by

$$\check{u}_\epsilon(x) = \tilde{u}_{\tilde{\varepsilon}}\left(\frac{3x}{4}\right), \text{ for every } x \in \mathbb{D}^2,$$

so that $\check{u}_\epsilon$ solves (1) on $\mathbb{D}^2$, with ε replaced by ϵ and, in view of (6.51)

$$\mathrm{E}_\epsilon(\check{u}_\epsilon) = \frac{4}{3}\mathrm{E}_{\tilde{\varepsilon}}\left(\tilde{u}_{\tilde{\varepsilon}}, \mathbb{D}^2(\frac{3}{4})\right) \leq \frac{2048}{3}\lambda_0^{-\frac{3}{2}}\lambda_{\max}\,\varepsilon\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2\right) \leq \left(64\lambda_0^{-\frac{3}{2}}\lambda_{\max}\eta_6\right)\epsilon. \tag{6.52}$$

We impose next the following additional condition on η_6

$$64\lambda_0^{-\frac{3}{2}}\lambda_{\max}\eta_6 \leq \eta_3, \tag{6.53}$$

so that, if (6.53) is satisfied, then we deduce from (6.50) that

$$\mathrm{E}_\epsilon(\check{u}_\epsilon) \leq \eta_3 \epsilon.$$

We are hence in position to apply Proposition 6.1 to the map $\check{u}_\epsilon$: It follows, since $\check{u}_\epsilon(0) = u_\varepsilon(x_0)$, that

$$|u_\varepsilon(x_0) - \sigma| = |\check{u}_\epsilon(0) - \sigma| \leq \mathrm{C}_{\mathrm{wk}} \left(\frac{\mathrm{E}_\epsilon(\check{u}_\epsilon)}{\epsilon}\right)^{\frac{1}{6}} \leq \frac{\mu_0}{2}. \tag{6.54}$$

In view of (6.52), we have

$$\frac{\mathrm{E}_\epsilon(\check{u}_\epsilon)}{\epsilon} = \frac{3\mathrm{E}_\epsilon(\check{u}_\epsilon)}{32\varepsilon} \leq 32\lambda_0^{-\frac{3}{2}}\lambda_{\max}\mathrm{E}_\varepsilon\left(u_\varepsilon, \mathbb{D}^2\right),$$

so that (6.54) becomes

$$|u_\varepsilon(x_0) - \sigma| \leq \mathrm{C}_{\mathrm{wl}}\left(\mathrm{E}_\varepsilon(u_\varepsilon, \mathbb{D}^2)\right)^{\frac{1}{6}},$$

with $\mathrm{C}_{\mathrm{wl}} = \mathrm{C}_{\mathrm{wk}}\left(32\lambda_0^{-\frac{3}{2}}\lambda_{\max}\right)^{\frac{1}{6}}$. This yields (6.47) and completes the proof of the Proposition 6.4. □

6.5 Proof of Theorem 6 completed

We choose

$$\eta_1 = \frac{\eta_6}{2}.$$

With this choice of constant η_1, Proposition 6.4 directly yields (58), whereas the energy estimate (59), follows directly from inequality (6.41) of Proposition 6.3 , choosing $\mathrm{C}_{\mathrm{nrg}} = 16\lambda_0^{-\frac{3}{2}}\lambda_{\max}$. The proof of Theorem 6 is hence complete.

Part III: Analysis of the limiting sets and measures

7 Properties of the concentration set $\mathfrak{S}_\star$

The purpose of this section is to provide the proof of assertion i) of Theorem 1. We start with the proof of Theorem 7, that is the clearing-out property for the measure $\nu_\star$.

7.1 Proof of Theorem 7

Recall that $\nu_\star$ is the weak limit of the measure ν_{ε_n} defined in (6) by $\nu_\varepsilon = e_\varepsilon(u_\varepsilon)\mathrm{d}x$, so that

$$\mathrm{E}_\varepsilon(u, \mathbb{D}^2(x_0, r)) = \nu_\varepsilon(\mathbb{D}^2(x_0, r)) = \nu_\varepsilon(\overline{\mathbb{D}^2(x_0, r)}).$$

Let $x_0 \in \Omega$ and $r > \rho > 0$ be such that $\mathbb{D}^2(x_0, r) \subset \Omega$. Since $\overline{\mathbb{D}^2(x_0, \rho)}$ is a closed set, we have, by standard properties of weak convergence of measures

$$\limsup_{n\to+\infty} \nu_{\varepsilon_n}(\mathbb{D}^2(x_0, \rho)) \leq \nu_\star(\overline{\mathbb{D}^2(x_0, \rho)}) \leq \nu_\star(\mathbb{D}^2(x_0, r)). \tag{7.1}$$

Next, let x_0 and $r > 0$ be such that

$$\nu_\star(\mathbb{D}^2(x_0, r)) < \eta_1 r.$$

It follows from (7.1) that, for given $\rho < r$, there exists some $n(\rho) \in \mathbb{N}$ such that, if $n \geq n(\rho)$, then we have

$$\nu_{\varepsilon_n}(\mathbb{D}^2(x_0, \rho)) \leq \frac{5}{4}\eta_1 r. \tag{7.2}$$

We choose $\rho = \dfrac{8r}{9}$. We obtain, inserting in (7.2),

$$\nu_{\varepsilon_n}(\mathbb{D}^2(x_0, \rho)) = \nu_{\varepsilon_n}(\mathbb{D}^2(x_0, \frac{8r}{9})) \leq \frac{5}{4}.\frac{9\rho}{8}\eta_1 = \frac{45}{32}\rho\eta_1 < 2\eta_1\rho. \tag{7.3}$$

Hence, for sufficiently large n, we are in position to apply Proposition 2 on the disk $\mathbb{D}^2(x_0, \rho)$ so that (63) yields

$$\begin{aligned} \nu_{\varepsilon_n}\left(\mathbb{D}^2\left(x_0, \frac{5r}{9}\right)\right) = \nu_{\varepsilon_n}\left(\mathbb{D}^2\left(x_0, \frac{5\rho}{8}\right)\right) &\leq \mathrm{C}_{\mathrm{nrg}}\frac{\varepsilon_n}{\rho}\mathrm{E}_{\varepsilon_n}\left(u_{\varepsilon_n}, \mathbb{D}^2\left(x_0, \rho\right)\right) \\ &\leq \varepsilon_n \eta_1 \frac{r}{\rho} = \frac{9}{8}\varepsilon\eta_1 \to 0 \text{ as } n \to +\infty. \end{aligned} \tag{7.4}$$

Letting $n \to +\infty$, it follows that $\nu_\star\left(\mathbb{D}^2(x_0, \frac{r}{2})\right) = 0$ and the proof is complete.

7.2 Elementary consequences of the clearing-out property

We present here some simple consequences of the definition of $\mathfrak{S}_\star$, as well as of the clearing-out property stated in Theorem 7. For $x \in \Omega$, we set

$$\mathbb{e}^\star(x) = \limsup_{r\to 0} \frac{\nu_\star\left(\overline{\mathbb{D}^2(x, r)}\right)}{r} \in [0, +\infty], \tag{7.5}$$

so that $\mathbb{e}_\star(x) \leq \mathbb{e}^\star(x)$, where $\mathbb{e}_\star$ is defined in (65).

Lemma 7.1. *i) Let $x_0 \in \mathfrak{U}_\star = \Omega \setminus \mathfrak{S}_\star$. There exists some radius $r_{x_0} > 0$ such that $\mathbb{D}^2(x_0, r_{x_0}) \subset \Omega$ and*

$$\nu_\star(\overline{\mathbb{D}^2(x_0, r_{x_0})}) = 0. \tag{7.6}$$

In particular

$$\mathbb{e}_\star(x) = \mathbb{e}^\star(x) = 0, \text{ for any } x \in \mathbb{D}^2(x_0, r_{x_0}). \tag{7.7}$$

ii) Let $x_0 \in \mathfrak{S}_\star$, and $r_0 > 0$ such that $\mathbb{D}^2(x_0, r_0) \subset \Omega$. We have

$$\nu_\star(\mathbb{D}^2(x_0, r)) \geq \eta_1 r, \text{ for any } 0 < r \leq r_0. \tag{7.8}$$

Proof. It follows from the definition (66) of $\mathfrak{S}_\star$ that, if $x_0 \in \mathfrak{U}_\star$, then we have $\mathbb{e}_\star(x_0) < \eta_1$. Hence there exists some radius $\tilde{r}_0 > 0$ such that $\mathbb{D}^2(x_0, \tilde{r}_0) \subset \Omega$ and such that

$$\nu_\star(\mathbb{D}^2(x_0, \tilde{r}_0)) < r_0 \eta_1.$$

In view of Theorem 7, we deduce that $\nu_\star(\mathbb{D}^2(x_0, \frac{\tilde{r}_0}{2})) = 0$. Choosing $r_{x_0} = \frac{\tilde{r}_0}{4}$, we obtain (7.6). Identity (7.7) is then a straightforward consequence of the definition (7.5). Turning to assertion ii), we argue by contradiction: If (7.8) were not true for some $0 < r \leq r_0$, then we would be in position to apply Theorem 7 to the ball $\mathbb{D}^2(x_0, r)$, which would imply that $\nu_\star(\mathbb{D}^2(x_0, \frac{r}{2})) = 0$, and hence that $\mathbb{e}_\star(x_0) = 0$, a contradiction with the definition of $\mathfrak{S}_\star$. □

Proposition 7.1. *The set $\mathfrak{S}_\star$ is a closed subset of Ω.*

Proof. It suffices to prove that its complement, the set $\mathfrak{U}_\star = \Omega \setminus \mathfrak{S}_\star$ is an *open subset* of Ω. Let x_0 be an arbitrary point in $\mathfrak{U}_\star$. It follows from Lemma 7.1 that

$$\mathbb{e}_\star(x) = 0 \text{ for } x \in \mathbb{D}^2(x_0, r_{x_0}),$$

so that $\mathbb{D}^2(x_0, r_{x_0}) \subset \mathfrak{U}_\star$. Hence, $\mathfrak{U}_\star$ is an open set. □

Lemma 7.1 leads also immediately to:

Proposition 7.2. *The set $\mathfrak{S}_\star$ has finite one-dimensional Hausdorff measure. There exists a constant $\mathrm{C_H} > 0$ depending only on the potential V such that*

$$\mathcal{H}^1(\mathfrak{S}_\star) \leq \mathrm{C_H} M_0.$$

Proof. The proof relies on a standard covering argument. Let $0 < \rho < \frac{1}{4}$ be given, and consider the set

$$\Omega_\rho = \{x \in \Omega, \mathrm{dist}(x, \partial\Omega) \geq \rho\}.$$

Next let $0 < \delta < \rho/4$ be given. Consider the points x_i on a uniform square lattice of $\mathbb{R}^2$, with nearest neighbour at distance $\frac{\delta}{2}$. We obtain for a subfamily I a standard finite covering of Ω_ρ of size δ, that is such that

$$\Omega_\rho \subseteq \underset{j \in I}{\cup} \mathbb{D}^2(x_j, \delta) \ \text{ and } \mathbb{D}^2\left(x_i, \frac{\delta}{2}\right) \cap \mathbb{D}^2\left(x_j, \frac{\delta}{2}\right) = \emptyset, \text{ for } i \neq j \in I.$$

We introduce then the set of indices

$$I_\delta = \left\{ i \in I, \text{ such that } \mathbb{D}^2(x_i, \delta) \cap \mathfrak{S}_\star \neq \emptyset \right\},$$

so that given any arbitrary index $i \in I_\delta$, there exists a point $y_i \in \mathfrak{S}_\star \cap \mathbb{D}^2(x_i, \delta)$. It follows from the definition of $\mathfrak{S}_\star$ that

$$\mathbb{e}_\star(y_i) \geq \eta_1. \tag{7.9}$$

Invoking Lemma 7.1, inequality (7.8), we see that, for any $0 < r \leq \delta$, we have

$$\nu_\star(\mathbb{D}^2(y_i, r)) \geq \eta_1 r. \tag{7.10}$$

Since $\mathbb{D}^2(y_i, \delta) \subset \mathbb{D}^2(x_i, 2\delta)$, we deduce from (7.10) that

$$\nu_\star(\mathbb{D}^2(x_i, 2\delta)) \geq \eta_1 \, \delta. \tag{7.11}$$

Since the points x_i are on a uniform grid, we notice that a given point $x \in \mathbb{R}^2$ belongs to at most 25 distinct balls of the collection $\mathbb{D}^2(x_i, 2\delta)$. We have therefore

$$\sharp(I_\delta)\eta_1\delta \leq \sum_{i\in I_\delta} \nu_\star \left(\mathbb{D}^2(x_i, 2\delta)\right) \leq 25\nu_\star(\Omega) \leq 25M_0. \tag{7.12}$$

It follows therefore that

$$\sharp(I_\delta)\delta \leq \frac{25M_0}{\eta_1}.$$

Therefore, letting $\delta \to 0$, we deduce, as a consequence of the definition of the one-dimensional Hausdorff measure that

$$\mathcal{H}^1(\mathfrak{S}_\star \cap \Omega_\rho) \leq \liminf_{\delta\to 0} 2\,\sharp(I_\delta)\delta \leq \frac{50M_0}{\eta_1}.$$

We conclude letting $\rho \to 0$, choosing $\mathrm{C_H} = \dfrac{50}{\eta_1}$. □

7.3 Proof of Theorem 8

Theorem 8 is a direct consequence of Proposition 4.6 which has actually been taylored for this purpose. Indeed, since $\nu_\star(\mathcal{V}_\delta) = 0$, we have the convergence

$$\int_{\mathcal{V}_\delta} e_{\varepsilon_n}(u_{\varepsilon_n}) \,\mathrm{d}\, x \to 0 \text{ as } n \to +\infty,$$

so that condition (4.44) is fulfilled for $\varepsilon = \varepsilon_n$ and the map u_{ε_n}, provided n is sufficiently large, say larger than some given value n_0. We are therefore in position to conclude, thanks to Proposition 4.6, provided $n \geq n_0$ is sufficiently large, that

$$\begin{aligned}\int_{\mathcal{U}_{\frac{\delta}{4}}} e_{\varepsilon_n}(u_{\varepsilon_n})\mathrm{d}x &\leq \mathrm{C_{ext}}(\mathcal{U}, \delta)\left(\int_{\mathcal{V}_\delta} e_{\varepsilon_n}(u_{\varepsilon_n})\mathrm{d}x + \varepsilon_n \int_{\mathcal{U}_\delta} e_{\varepsilon_n}(u_{\varepsilon_n})\mathrm{d}x\right)\\ &\leq \mathrm{C_{ext}}(\mathcal{U}, \delta)\left(\int_{\mathcal{V}_\delta} e_{\varepsilon_n}(u_{\varepsilon_n})\mathrm{d}x + \varepsilon_n \mathrm{M}_0\right).\end{aligned}$$

It follows that

$$\int_{\mathcal{U}_{\frac{\delta}{4}}} e_{\varepsilon_n}(u_{\varepsilon_n})\mathrm{d}x \to 0 \text{ as } n \to +\infty,$$

so that the proof is complete. □

7.4 Connectedness properties of $\mathfrak{S}_\star$

The purpose of the present section is, among other things, to provide the proof of Proposition 3. Given $r > 0$ and $x_0 \in \Omega$ such that $\mathbb{D}^2(x_0, 2r) \subset \Omega$, we consider the closed set

$$\mathfrak{S}_{\star,\varrho} = \mathfrak{S}_{\star,\varrho}(x_0) \equiv \mathfrak{S}_\star \cap \overline{\mathbb{D}^2(x_0, \varrho)} \text{ for } \varrho \in [0, 2r).$$

The proof of Proposition 3 relies on several intermediate properties we present next.

Proposition 7.3. *Let $r > 0$ and $x_0 \in \Omega$ be as above. The closed set*

$$\mathfrak{Q}_{\star,r}(x_0) = \mathfrak{S}_{\star,r}(x_0) \cup \mathbb{S}^2(x_0, r) \tag{7.13}$$

is a continuum, that is, it is compact and connected.

Proof. The proof of compactness of $\mathfrak{Q}_{\star,r}(x_0)$ is a straightforward consequence of Proposition 7.1, since both sets composing the union (7.13) are compact. The proof of connectedness of $\mathfrak{Q}_{\star,r}(x_0)$ is more involved, and strongly relies on Theorem 8, as we will see next. In order to invoke Theorem 8, a first step is to approximate $\mathfrak{S}_{\star,r}$ by sets $\mathfrak{S}_{\delta,r}$ with a simpler structure.

Definition of the approximating sets $\mathfrak{S}_{\delta,r}$. These sets are defined using a *Besicovitch covering* of $\mathfrak{S}_{\star,r}$. Let

$$\delta_{x_0,r} = \text{dist}(\mathbb{D}^2(x_0, r), \partial\Omega) > 0.$$

For given $0 < \delta < \delta_{x_0,r}$, we consider the covering of $\mathfrak{S}_{\star,r}$ by the collection of open disks $\{\mathbb{D}^2(x_0, \delta)\}_{x \in \mathfrak{S}_{\star,r}}$, which is obviously a covering of $\mathfrak{S}_{\star,r}$, and actually a Besicovitch covering. We may therefore invoke Besicovitch covering theorem to assert that there exists a universal constant $\mathfrak{p}$, depending only on the dimension $N = 2$, and $\mathfrak{p}$ families of points $\{x_{i_1}\}_{i_1 \in A_1}$, $\{x_{i_2}\}_{i_2 \in A_1}, \ldots, \{x_{i_\mathfrak{p}}\}_{i_\mathfrak{p} \in A_\mathfrak{p}}$, such that $x_i \in \mathfrak{S}_{\star,r}(x_0)$, for any $i \in A \equiv A_1 \cup A_2 \ldots \cup A_\mathfrak{p}$,

$$\mathfrak{S}_{\star,r} \subset \mathfrak{V}_{\delta,r} \equiv \overset{\mathfrak{p}}{\underset{\ell=1}{\cup}} \left(\underset{i_\ell \in A_\ell}{\cup} \mathbb{D}^2(x_{i_\ell}, \delta) \right) = \underset{i \in A}{\cup} \mathbb{D}^2(x_i, \delta), \tag{7.14}$$

and such that the balls in each collection $\{\mathbb{D}^2(x_i, \delta)\}_{i \in A_\ell}$ are disjoint, that is, for any $\ell = 1, \ldots, \mathfrak{p}$, we have

$$\mathbb{D}^2(x_i, \delta) \cap \mathbb{D}^2(x_j, \delta) = \emptyset \text{ for } i \neq j \text{ with } i, j \in A_\ell. \tag{7.15}$$

As a consequence of the above constructions, a point $x \in \mathfrak{V}_{\delta,r}$, where $\mathfrak{V}_{\delta,r}$ is defined in (7.14), belongs to at most $\mathfrak{p}$ distinct disks of the collection $\{\overline{\mathbb{D}^2(x_i, \delta)}\}_{i \in A}$. We define the set $\mathfrak{S}_{\delta,r}$ as the closure of the set $\mathfrak{V}_{\delta,r}$ that is

$$\mathfrak{S}_{\delta,r} \equiv \overline{\mathfrak{V}_{\delta,r}} = \overset{\mathfrak{p}}{\underset{\ell=1}{\cup}} \underset{i_\ell \in A_\ell}{\cup} \overline{\mathbb{D}^2(x_{i_\ell}, \delta)},$$

Notice that, by construction, the total number $\sharp(A)$ of distinct disks is finite. Actually, we have the bound

$$\sharp(A) \leq \frac{4\mathfrak{p}r^2}{\delta^2}. \tag{7.16}$$

Indeed, since the family of balls $\{\mathbb{D}^2(x_{i_\ell}, \delta)\}_{i \in A_\ell}$ are disjoint disks of radius δ which are included in a ball of radius $2r$, we have

$$\sharp(A_\ell) \leq \frac{4r^2}{\delta^2} \text{ for } \ell = 1, \ldots, \mathfrak{p},$$

so that (7.16) follows by summation.

We next consider the set

$$\mathfrak{Q}_{\delta,r} = \mathfrak{S}_{\delta,r} \cup \mathbb{S}^2(x_0, r)$$

and its distinct connected components $\{\mathfrak{Q}^k_{\delta,r}\}_{k \in \mathcal{J}_\delta}$. In view of the structure of $\mathfrak{Q}_{\delta,r}$, which is a union of $\sharp(A)$ disks with a circle, the total number of connected components $\sharp\mathcal{J}_\delta$ is finite and actually bounded by $\sharp(A)+1$, hence the number on the right hand side of inequality (7.16) plus one. As a matter of fact, we claim

$$\textit{The set } \mathfrak{Q}_{\delta,r} \textit{ is simply connected, so that } \sharp(\mathcal{J}_\delta)=1. \tag{7.17}$$

Proof of the claim (7.17). We assume by contradiction that $\mathfrak{Q}_{\delta,r}$ has at least two distinct connected components and denote by $\mathfrak{Q}^1_{\delta,r}$ the connected component which contains the circle $\mathbb{S}^1(x_0,r)$. Let $\mathfrak{Q}^2_{\delta,r}$ be a connected component distinct from $\mathfrak{Q}^1_{\delta,r}$, and set

$$\beta \equiv \inf\left\{\mathrm{dist}(\mathfrak{Q}^2_{\delta,r},\mathfrak{Q}^j_{\delta,r}), j\in\mathcal{J}_\delta, j\neq 2\right\}>0.$$

We consider the open set

$$\mathcal{U}=\left\{x\in\mathbb{R}^2, \mathrm{dist}\left(x,\mathfrak{Q}^2_{\delta,r}\right)<\frac{\beta}{4}\right\}\subset\mathbb{D}^2(x_0,r)\setminus\underset{j\in\mathcal{J}_\delta\setminus\{2\}}{\cup}\mathfrak{Q}^j_{\delta,r},$$

so that using the notation (67), we have

$$\mathcal{U}_{\frac{\beta}{4}}=\left\{x\in\mathbb{R}^2, \mathrm{dist}\left(x,\mathcal{U}\right)<\frac{\beta}{4}\right\}\subset\mathbb{D}^2(x_0,r)\setminus\underset{j\in\mathcal{J}_\delta\setminus\{2\}}{\cup}\mathfrak{Q}^j_{\delta,r}$$

and

$$\mathcal{V}_{\frac{\beta}{4}}\equiv\mathcal{U}_{\frac{\beta}{4}}\setminus\mathcal{U}\subset\left\{x\in\mathbb{R}^2,\frac{\beta}{4}\le\mathrm{dist}\left(x,\mathfrak{Q}^2_{\delta,r}\right)\le\frac{\beta}{2};\right\} \tag{7.18}$$

combining (7.18) with the definition of β, we obtain

$$\mathcal{V}_{\frac{\beta}{4}}\cap\mathfrak{S}_\star=\emptyset \text{ and } \nu_\star\left(\mathcal{V}_{\frac{\beta}{4}}\right)=0. \tag{7.19}$$

We are therefore in position to apply Theorem 8 to assert that $\nu_\star(\mathcal{U})=0$. However, since by definition $\mathfrak{Q}^2_{\delta,r}\subset\mathcal{U}$, it follows that $\mathcal{U}\cap\mathfrak{S}_\star\neq\emptyset$, so that $\nu_\star(\mathcal{U})>0$. We have hence reached a contradiction, which establishes the claim (7.17).

Proof of Proposition 7.3 completed. It follows from the definition of $\mathfrak{S}_{\delta,r}$ that

$$\mathrm{dist}(\mathfrak{Q}_{\delta,r},\mathfrak{Q}_{\star,r})\le\delta, \text{ where } \mathfrak{Q}_{\star,r}=\mathfrak{S}_{\delta,r}\cup\mathbb{S}^2(x_0,r),$$

so that $\mathfrak{Q}_{\delta,r}$ converges as $\delta\to 0$ to $\mathfrak{Q}_{\star,r}$ in the Hausdorff metric. Since for every δ, the set $\mathfrak{S}_{\delta,r}$ is a continuum, it then follows (see e.g. [21], Theorem 3.18) that the Hausdorff limit $\mathfrak{Q}_{\star,r}$ is also a continuum and the proof is complete. □

We deduce as a consequence of Proposition 7.3:

Corollary 7.1. *The set $\mathfrak{Q}_{\star,r}$ is arcwise connected.*

Proof. Indeed, any continuum with finite one-dimensional Hausdorff dimension is arcwise connected, see e.g [21], Lemma 3.12, p 34. □

7.4.1 Proof of Proposition 3

Invoking Fubini's theorem together with a mean value argument, we may choose some radius $r_0 \in [r, 2r)$ such that the number of points in $\mathfrak{S}_\star \cap \partial\mathbb{D}^2(x_0, r_0)$ is finite, more precisely

$$m_0 \equiv \sharp\left(\mathfrak{S}_\star \cap \partial\mathbb{D}^2(x_0, r_0)\right) \leq \frac{\mathrm{C_H}}{r} M_0,$$

where we have used estimate (8) of the $\mathcal{H}^1$ measure of $\mathfrak{S}_\star$. We may hence write

$$\mathfrak{S}_\star \cap \partial\mathbb{D}^2(x_0, r_0) = \{a_1, \ldots, a_{m_0}\}. \tag{7.20}$$

Next, we claim that for any point $y \in \mathfrak{S}_{\star,r_0}$, there exists a continuous path $p : [0,1] \mapsto \mathfrak{S}_{\star,r_0}$ connecting the point y to one of the points $a_1, \ldots, a_{m_0}$, that is such that

$$p(0) = y \text{ and } p(1) \in \{a_1, \ldots, a_{m_0}\}. \tag{7.21}$$

Proof of the claim (7.21). If $|y - x_0| = r_0$, then $y \in \mathfrak{S}_\star \cap \partial\mathbb{D}^2(x_0, r_0)$, and it therefore suffices to choose $p(s) = y$, for all $s \in [0,1]$. Otherwise, since, in view of Corollary 7.1 applied at x_0 with radius r_0, the set $\mathfrak{S}_{\star,r_0} \cup \partial\mathbb{D}^2(x_0, r_0)$ is path-connected, there exists a continuous path $\tilde{p} : [0,1] \to \mathfrak{S}_{\star,r_0} \cup \partial\mathbb{D}^2(x_0, r_0)$ such that

$$\tilde{p}(0) = y \text{ and } \tilde{p}(1) \in \partial\mathbb{D}^2(x_0, r_0).$$

By continuity, there exists some number $s_0 \in [0,1]$ such that

$$|\tilde{p}(s)| < r_0, \text{ for } 0 \leq s < s_0 \text{ and } |\tilde{p}(s_0)| = r_0.$$

It follows that

$$\tilde{p}(s_0) \in \mathfrak{S}_\star \cap \partial\mathbb{D}^2(x_0, r_0) = \{a_1, \ldots, a_{m_0}\}.$$

We then set

$$p(s) = \tilde{p}(s), \text{ for } 0 \leq s < s_0, \text{ and } p(s) = \tilde{p}(s_0), \text{ for } s_0 \leq s \leq 1,$$

and verify that p has the desired property, so that the proof of the claim is complete.

Proof of Proposition 3 completed. It follows from the claim (7.21) that any point $y \in \mathfrak{S}_{\star,r_0}$ is connected to one of the points $a_1, \ldots, a_{m_0}$ given in (7.20). Hence $\mathfrak{S}_{\star,r_0}$ has at most m_0 connected components and the proof is complete. □

7.5 Rectifiability of $\mathfrak{S}_\star$

In this section, we prove:

Theorem 7.1. *The set $\mathfrak{S}_\star$ is rectifiable.*

Proof. The result is actually an immediate consequence of Proposition 7.3 and the fact that any 1-dimensional continuum is rectifiable, a result due to Wazewski and independently Besicovitch (see e.g [21], Lemma 3.13). Indeed, given any $x_0 \in \Omega$, $r > 0$ such that $\mathbb{D}^2(x_0, r) \subset \Omega$, the set $\mathfrak{S}_{\star,r} \cup \mathbb{S}^2(x_0, r)$ is a continuum, hence rectifiable in view of the result quoted above, and hence so is the set $\mathfrak{S}_{\star,\frac{r}{2}}$. Since rectifiability is a local property, the conclusion follows. □

7.6 Uniform convergence of $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ off the set $\mathfrak{S}_\star$

We go back in this Subsection at the level of the solutions u_{ε_n} and establish uniform convergence *off the set* $\mathfrak{S}_\star$. Our results will rely on the following main tool.

Proposition 7.4. *Let $x_0 \in \Omega \setminus \mathfrak{S}_\star$ and $r_{x_0} > 0$ be given by Lemma 7.1, Assertion i). There exists a sequence $(\sigma_{x_0,n})_{n\in\mathbb{N}}$ with $\sigma_{x_0,n} \in \Sigma$, for $n \in \mathbb{N}$, such that*

$$\|u_{\varepsilon_n} - \sigma_{x_0,n}\|_{L^\infty(\mathbb{D}^2(x_0,\tilde{r}_{x_0}))} \to 0 \text{ as } n \to +\infty, \text{ with } \tilde{r}_{x_0} = \frac{1}{2} r_{x_0}. \tag{7.22}$$

Proof. It follows from (7.6) that $\nu_\star\left(\overline{\mathbb{D}^2(x_0, r_{x_0})}\right) = 0$, and hence

$$\limsup_{n\to+\infty} \mathrm{E}_{\varepsilon_n}(u_{\varepsilon_n}, \mathbb{D}^2(x_0, r_{x_0})) = \limsup_{n\to+\infty} \nu_{\varepsilon_n}(\mathbb{D}^2(x_0, r_{x_0})) \leq \nu_\star(\overline{\mathbb{D}^2(x_0, r_{x_0})}) = 0. \tag{7.23}$$

Hence, there exists some $n_0 \in \mathbb{N}$, such that for $n \geq n_0$, we have

$$\mathrm{E}_{\varepsilon_n}(u_{\varepsilon_n}, \mathbb{D}^2(x_0, r_{x_0})) \leq \eta_1 r_{x_0}. \tag{7.24}$$

We are therefore in position to apply Proposition 2 to u_{ε_n}, x_0 and $r = r_{x_0}$: The first inequality in (63) yields the existence of some $\sigma_{x_0,n} \in \Sigma$ such that

$$|u_{\varepsilon_n}(x) - \sigma_{x_0,n}| \leq \mathrm{C}_{\mathrm{well}} \left(\frac{\mathrm{E}_{\varepsilon_n}\left(u_{\varepsilon_n}, \mathbb{D}^2(x_0, r_{x_0})\right)}{r_{x_0}} \right)^{\frac{1}{6}} \underset{n\to 0}{\to} 0, \text{ for } x \in \mathbb{D}^2(x_0, \frac{r_{x_0}}{2}). \tag{7.25}$$

This yields (7.22) and completes the proof of Lemma 7.4. □

The previous result can be extended to more general domains by covering as follows.

Corollary 7.2. *Let $\mathrm{K} \subset \mathfrak{U}_\star = \Omega \setminus \mathfrak{S}_\star$ be compact and connected. Then there exists a sequence $(\sigma_{K,n})_{n\in\mathbb{N}}$ with $\sigma_{\mathrm{K},n} \in \Sigma$, for $n \in \mathbb{N}$, such that*

$$\|u_{\varepsilon_n} - \sigma_{\mathrm{K},n}\|_{L^\infty(\mathrm{K})} \to 0 \text{ as } n \to +\infty. \tag{7.26}$$

Proof. Let $\delta = \mathrm{dist}(\mathrm{K}, \mathfrak{S}_\star) > 0$. We consider the covering of $\mathrm{K} \subset \underset{x\in\mathrm{K}}{\cup} \mathbb{D}^2(x, \check{r}_x)$, where $\check{r}_x = \inf\{\tilde{r}_x, \frac{\delta}{2}\}$. By Lebesgue covering theorem, we extract a finite covering, so that

$$\mathrm{K} \subset \check{\mathrm{K}}_{\mathrm{Leb}} = \underset{x\in A}{\cup} \mathbb{D}^2(x, \check{r}_x), \text{ with } A = \{x_1, \dots, x_\ell\}, \ell < +\infty. \tag{7.27}$$

Since K is assumed to be connected, we may assume likewise that $\check{\mathrm{K}}_{\mathrm{Leb}}$ is connected. Applying Proposition 7.4 to each of the points $x_1, \dots, x_\ell$, we obtain the existence of ℓ points $\sigma_{x_i,n} \in \Sigma$ such that

$$|u_{\varepsilon_n}(x) - \sigma_{x_i,n}| \underset{n\to 0}{\to} 0, \text{ for } x \in \mathbb{D}^2(x_i, \check{r}_{x_i}), i = 1, \dots, \ell. \tag{7.28}$$

Hence, there exists $n_0 \in \mathbb{N}$ such that $u_{\varepsilon_n}(x) \in \Sigma + \mathbb{B}^k(\mu_0/2)$, for $n \geq n_0$ and $x \in \check{\mathrm{K}}_{\mathrm{Leb}}$. Since the set $\Sigma + \mathbb{B}^k(\mu_0/2)$ has q distinct connected components, each of them containing a single point of Σ, we deduce from the continuity of u_{ε_n} and the connectedness of $\check{\mathrm{K}}_{\mathrm{Leb}}$ that, for $n \geq n_0$, all points $\sigma_{x_i,n}$ coincide, for $i = 1, \dots, \ell$, i.e. there exists $\sigma_{\mathrm{K},n} \in \Sigma$, such that $\sigma_{x_i,n} = \sigma_{\mathrm{K},n}$, for $i = 1, \dots, \ell$, and $n \geq n_0$. Combining with (7.27) and (7.28), we derive the conclusion. □

7.7 Proof of Theorem 1 completed

The statements in Theorem 1 , assertions i) have been obtained so far : they follow combining results in Section 7, namely Proposition 7.1 (closedness of $\mathfrak{S}_\star$), Proposition 7.2 (upper bound on the one-dimensional Hausdorff dimension of $\mathfrak{S}_\star$), Proposition 3 (its connectedness) and Theorem 7.1(rectifiability of $\mathfrak{S}_\star$).

The proof of assertions ii) of Theorem 1 is based on Corollary 7.2 and requires some additional arguments, and extracting more subsequences. For $m \in \mathbb{N}$, we consider the set

$$\mathfrak{U}_m = \left\{x \in \Omega, \operatorname{dist}(x, \mathfrak{S}_\star) > \frac{1}{m}\right\} \cap \mathbb{D}^2(m), \tag{7.29}$$

so that $\overline{\mathfrak{U}}_m$ is a compact subset of $\mathfrak{U}_\star$, and such that

$$\overline{\mathfrak{U}}_m \subset \mathfrak{U}_{m+1} \text{ and } \bigcup_{m=1}^{+\infty} \mathfrak{U}_m = \mathfrak{U}_\star.$$

We denote by $\mathfrak{U}_m^j$, $j \in J_m$ the connected components of $\mathfrak{U}_m$. We claim that the set of indices J_m is countable. Indeed, since $\mathfrak{U}_m^j$ is an open set, it contains a disk $\mathbb{D}_{m,j}$ of radius $r_{m,j} > 0$, and since the sets $\mathfrak{U}_{m,j}$ do not intersect and are, by definition (7.29), included in the disk $\mathbb{D}^2(m)$, the same holds for the disks $\mathbb{D}_{m,j}$ and hence $\sum_{j\in J} r_{m,j}^2 \leq m^2$, which implies that J_m is countable, for any $m \in \mathbb{N}$. Hence the set $\bigcup_{m\in M} \{m\} \times J_m$, is countable. Invoking a diagonal argument together with Corollary 7.2 applied with $K = \overline{\mathfrak{U}}_m^j$, $j \in J_m$, we may extract a further subsequence, still denoted for sake of simplicity $(\varepsilon_n)_{n\in\mathbb{N}}$, such that, for any $m \in \mathbb{N}$, and $j \in J_m$, we have

$$u_{\varepsilon_n} \to \sigma_{m,j}, \text{ as } n \to +\infty, \text{ uniformly on } \mathfrak{U}_m^j, \tag{7.30}$$

where $\sigma_{m,j} \in \Sigma$. Hence, given any $x \in \mathfrak{U}_\star$, the limit $\mathcal{O}(x) = \lim_{n\to+\infty} u_{\varepsilon_n}(x)$ exists, and is constant and equal to $\sigma_{m,j}$ on $\mathfrak{U}_m^j$. It follows that $\mathcal{O}$ is continuous on $\mathfrak{U}_\star$, with values in a discrete set. Hence, $\mathcal{O}$ is constant, equal to some $\sigma_i \in \Sigma$, on each connected component $\mathfrak{U}_\star^i$ of $\mathfrak{U}_\star$. Invoking again Corollary 7.2 for an arbitrary compact subset of $\mathfrak{U}_\star^i$, we derive that

$$u_{\varepsilon_n} \to \sigma_i, \text{ as } n \to +\infty, \text{ uniformly on } K,$$

so that the proof is complete.

7.8 On the tangent line at regular points of $\mathfrak{S}_\star$

In this subsection, we provide the proof to Proposition 4. It relies on the following Lemma, which is actually a weaker statement:

Lemma 7.2. *Let x_0 be a regular point of $\mathfrak{S}_\star$ and $\vec{e}_{x_0}$ be a unit tangent vector to $\mathfrak{S}_\star$ at x_0. Given any $\theta > 0$ there exists a radius $R_{\rm cone}(\theta, x_0)$ such that*

$$\mathfrak{S}_\star \cap \left(\mathbb{D}^2(x_0, \tau) \setminus \mathbb{D}^2\left(x_0, \frac{\tau}{2}\right)\right) \subset \mathcal{C}_{\rm one}(x_0, \vec{e}_{x_0}, \theta), \text{ for any } 0 < \tau \leq R_{\rm cone}(\theta, x_0). \tag{7.31}$$

Proof. Since we have the inclusion

$$\mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \theta\right) \subset \mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \theta'\right),$$

for $< 0 \leq \theta \leq \theta'$, it suffices to establish the statement for θ arbitrary small. For a given regular point x_0 of $\mathfrak{S}_\star$, we may invoke the convergence (12) to assert that there exists some $r_1 > 0$ such that for $0 < \tau \leq r_1$ we have

$$\mathcal{H}^1\left(\mathfrak{S}_\star \cap \mathbb{D}^2\left(x_0, 2\tau\right) \setminus \mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \frac{\theta}{2}\right)\right) \leq \frac{\theta\tau}{8}. \tag{7.32}$$

We set

$$A(x_0, \tau, \theta) = \left(\mathfrak{S}_\star \cap \mathbb{D}^2\left(x_0, \tau\right)\right) \setminus \left(\mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \theta\right) \cup \mathbb{D}^2\left(x_0, \frac{\tau}{2}\right)\right),$$

so that we have to prove that $A(x_0, \tau, \theta)$ is empty, if τ is sufficiently small. We assume by contradiction that $A(x_0, \tau, \theta) \neq \emptyset$ for some small τ, and will show that we obtain a contradiction. We have, in view of the definition of $A(x_0, \tau, \theta)$ and (7.32)

$$A(x_0, \tau, \theta) \cap \mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \frac{\theta}{2}\right) = \emptyset \text{ and } \mathcal{H}^1\left(A(x_0, \tau, \theta)\right) \leq \frac{\theta\tau}{8}. \tag{7.33}$$

we notice that, if $A(x_0, \tau, \theta)$ is not empty, then we have

$$\begin{cases} \operatorname{dist}\left(A(x_0, \tau, \theta), \mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \frac{\theta}{2}\right)\right) \geq \frac{\tau}{2} \sin\left(\frac{\theta}{2}\right) \\ \operatorname{dist}\left(A(x_0, \tau, \theta), \partial\mathbb{D}^2(x_0, 2\tau)\right) \geq \tau, \end{cases}$$

so that, if $\theta > 0$ is sufficiently small,

$$\operatorname{dist}\left(A(x_0, \tau, \theta), \mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \frac{\theta}{2}\right) \cup \partial\mathbb{D}^2(x_0, 2\tau)\right) \geq \frac{\tau}{2} \sin\left(\frac{\theta}{2}\right). \tag{7.34}$$

Since we assume, by contradiction that the set $A(x_0, \tau, \theta)$ is not empty, there exists some point $x_1 \in A(x_0, \tau, \theta)$. We consider the set $\mathfrak{Q}_{\star, 2\tau}(x_0) \equiv \mathfrak{S}_\star \cup \partial\mathbb{D}^2\left(x_0, 2\tau\right)$ introduced in (7.13). In view of Proposition 7.3 and Corollary 7.1, the set $\mathfrak{Q}_{\star, 2\tau}(x_0)$ is path-connected: Hence, there exists a continuous path p joining x_1 to some point $x_2 \in \partial\mathbb{D}^2(x_0, 2\tau)$ which stays inside $\mathfrak{S}_{\star, 2\tau}(x_0)$. On the other hand, since $x_1 \in \mathbb{D}^2(x_0, \tau)$ the length $\mathcal{H}^1(p)$ of this path is larger than τ. We claim that

$$p \cap \mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \frac{\theta}{2}\right) \neq \emptyset. \tag{7.35}$$

Indeed, otherwise p would be a path inside $\mathfrak{S}_\star \cap \mathbb{D}^2\left(x_0, 2\tau\right) \setminus \mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \frac{\theta}{2}\right)$. Since its length is larger then τ, this would contradict (7.32). Next, combining (7.35) and (7.34), we obtain

$$\mathcal{H}^1\left(p \cap \mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \frac{\theta}{2}\right)\right) \geq \frac{\tau}{2} \sin\left(\frac{\theta}{2}\right) \underset{\theta \to 0}{\sim} \frac{\tau\theta}{4}.$$

Since p is a path inside $\mathfrak{S}_{\star, 2\tau}(x_0)$ this contradicts (7.32), provided θ is chosen sufficiently small. This completes the proof of the Lemma, choosing $R_{\rm cone}(\theta, x_0) = r_1$. $\square$

7.8.1 Proof of Proposition 4 completed

Given $\tau < R_1$, we apply Lemma 7.2, the sequence of radii $(\tau_k)_{k\in\mathbb{N}}$ given by

$$\tau_k = \frac{\tau}{2^k} \text{ for } k \in \mathbb{N},$$

so that

$$\mathfrak{S}_\star \cap \left(\mathbb{D}^2(x_0, \tau_k) \setminus \mathbb{D}^2(x_0, \tau_{k+1})\right) \subset \mathcal{C}_{\rm one}(x_0, \vec{e}_{x_0}, \theta), \text{ for any } k \in \mathbb{N}.$$

We take the union of the sets on the left-hand side, so that we obtain

$$\mathfrak{S}_\star \setminus \{x_0\} = \bigcup_{k\in\mathbb{N}} \mathfrak{S}_\star \cap \left(\mathbb{D}^2(x_0, \tau_k) \setminus \mathbb{D}^2(x_0, \tau_{k+1})\right) \subset \mathcal{C}_{\rm one}(x_0, \vec{e}_{x_0}, \theta).$$

This yields the result.

8 Behavior near points in $\mathfrak{S}_\star \setminus \mathfrak{E}_\star$

In this section, we analyze more precisely the behavior of the measures $\zeta_\star$ and $\mu_{\star,i,j}$ in the vicinity of *good* points, that is points x_0 in $\mathfrak{S}_\star \setminus \mathfrak{E}_\star$, in particular points having the Lebesgue property for the absolutely continuous part of the measure. One of our main goals is to provide the proof to Proposition 5 and Lemma 2. The results in this section also pave the way to the proof of Theorem 2 provided in Section 10.

8.1 The limiting Hopf differential

The Hopf differential

$$\omega_\varepsilon \equiv \varepsilon\left(|(u_\varepsilon)_{x_1}|^2 - |(u_\varepsilon)_{x_2}|^2 - 2i(u_\varepsilon)_{x_1}\cdot(u_\varepsilon)_{x_2}\right)$$

defined in (3.19) has turned out to be a central tool in our analysis so far. We combine it in the present subsection with *the rectifiability* properties and Proposition 4 to derive new properties near good points. Recall that we have defined $\omega_\star$ in (70) as

$$\omega_\star = (\mu_{\star,1,1} - \mu_{\star,2,2}) - 2i\mu_{\star,1,2}.$$

So that, in view of the definition (41) of the measures $\mu_{\star,i,j}$, we have

$$\omega_{\varepsilon_n} \rightharpoonup \omega_\star, \text{ in the sense of measures on } \Omega, \text{ as } n \to +\infty. \tag{8.1}$$

8.2 The limiting differential relation for $\omega_\star$ and $\zeta_\star$

In this paragraph, we provide a proof to Lemma 1. First, passing to the limit in (3.20), we are led to:

Lemma 8.1. *Let $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ be a sequence of solutions to* (1) *on Ω with $\varepsilon_n \to 0$ as $n \to +\infty$ and assume that* (7) *holds. Let $\omega_\star$ and $\zeta_\star$ be the bounded measures on Ω given by* (8.1) *and* (15) *respectively. Then, we have, in the sense of distributions*

$$\mathrm{Re}\left(\left\langle \omega_\star, \frac{\partial X}{\partial \bar{z}}\right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}\right) = \left\langle 2\zeta_\star, \mathrm{Re}\left(\frac{\partial X}{\partial z}\right)\right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}, \text{ for any } X \in C_0^\infty(\Omega, \mathbb{C}). \tag{8.2}$$

Lemma 8.1 is actually our main tool in the rest of the discussion, and will be used with vector fields X of various types.

Proof of Lemma 1. Using iX as test function in (8.2) and the fact that $\mathrm{Re}(iz) = -\mathrm{Im}(z)$ for any complex number $z \in \mathbb{C}$, we obtain likewise

$$\mathrm{Im}\left(\left\langle\left(\omega_\star, \frac{\partial X}{\partial \bar{z}}\right)\right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}\right) = 2\left\langle \zeta_\star, \mathrm{Im}\left(\frac{\partial X}{\partial z}\right)\right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \quad \text{for any } X \in C_0^\infty(\Omega, \mathbb{C}). \quad (8.3)$$

Combining (8.2) and (8.3), we are hence led to the simple identity

$$\left\langle \omega_\star, \frac{\partial X}{\partial \bar{z}}\right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 2\left\langle \zeta_\star, \frac{\partial X}{\partial z}\right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}, \text{ for any } X \in C_0^\infty(\Omega, \mathbb{C}), \quad (8.4)$$

which yields (71) in the sense of distributions. □

We describe next some additional properties of the measures $\omega_\star$ et $\zeta_\star$, mostly based on Lemma 8.1, choosing various kinds of test vector fields $\vec{X}$. Whereas we have used so far mainly vector fields yielding dilatations of the domain (see e.g. Lemma 3.3), we consider also vector fields of different nature. Given a point $x_0 = (x_{0,1}, x_{0,2}) \in \Omega$, $\rho > 0$ such that $\mathbb{D}^2(x_0, 2\rho) \subset \Omega$, the fields we will consider in the next paragraphs are of the form

$$\vec{X}_f(x_1, x_2) = f_1(x_1) f_2(x_2) \vec{\mathbf{e}}_j, \text{ with } j = 1, 2, \quad (8.5)$$

where f_i represents, for $i = 1, 2$, an arbitrary function in $C_c^\infty((x_{0,i} - \rho, x_{0,i} + \rho))$. These vector fields have hence support on the square $Q_\rho(x_0)$, defined by

$$Q_\rho(x_0) = \mathcal{I}_\rho(x_{0,1}) \times \mathcal{I}_\rho(x_{0,2}), \text{ where } \mathcal{I}_\rho(s) = [s - \rho, s + \rho] = \mathbb{B}^1(s, \rho), \text{ for } s > 0. \quad (8.6)$$

We consider also the subset $\mathcal{R}_\rho(x_0)$ of $Q_\rho(x_0)$ given by

$$\mathcal{R}_\rho(x_0) \equiv \mathcal{I}_\rho(x_{0,1}) \times \mathcal{I}_{\frac{3\rho}{4}}(x_{0,2}) \subset Q_\rho(x_0), \quad (8.7)$$

so that $Q_\rho(x_0) \setminus \mathcal{R}_\rho(x_0)$ is the union of two disjoint rectangles

$$Q_\rho(x_0) \setminus \mathcal{R}_\rho(x_0) = \left(\mathcal{I}_\rho(x_{0,1}) \times (x_{0,2} + \frac{3\rho}{4}, x_{0,2} + \rho)\right) \cup \left(\mathcal{I}_\rho(x_{0,1}) \times (x_{0,2} - \rho, x_{0,2} - \frac{3\rho}{4})\right).$$

In several places, we will assume that the following conditions holds

$$\nu_\star(\overline{Q_\rho(x_0) \setminus \mathcal{R}_\rho(x_0)}) = 0, \quad (8.8)$$

which means that the measure $\nu_\star$ concentrates, locally near x_0, in a neighborhood of the segment $(x_0 - \rho\vec{\mathbf{e}}_1, x_0 + \rho\vec{\mathbf{e}}_1)$.

8.3 Projecting the measures on the tangent line

In the above framework, the $\vec{\mathbf{e}}_1$ direction plays a distinguished role: Integrating various quantities with respect to the x_2-variable, we obtain one-dimensional quantities, treated as measures on the interval $\mathcal{I}_{\rho_0}(x_{0,1}) = (x_{0,1} - \rho_0, x_{0,1} + \rho_0)$. Using appropriate test functions, relation (8.2) is then turned into a differential equation.

Given a Radon measure υ on $Q_\rho(x_0)$, and a test function $\varphi \in C_c(Q_\rho(x_0), \mathbb{C})$, we define the Radon measure $(\varphi\upsilon)^{x_1} = \mathbb{P}_\sharp(\varphi\upsilon)$ defined on $\mathcal{I}_\rho(x_{0,1})$ as follows: For any Borel set A of $\mathcal{I}_\rho(x_{0,1})$, we have

$$(\varphi\upsilon)^{x_1}(A) = (\varphi\upsilon)\left(\mathbb{P}^{-1}(A) \cap Q_\rho(x_0)\right) = \varphi\upsilon\left((A \times \mathbb{R}) \cap Q_\rho(x_0)\right).$$

so that

$$\langle \upsilon, \varphi \rangle = (\varphi\upsilon)(Q_\rho(x_0)) = \int_{Q_\rho(x_0)} \varphi \mathrm{d}\upsilon = \int_{\mathcal{I}_\rho(x_0)} \mathrm{d}(\varphi\upsilon)^{x_1}. \tag{8.9}$$

We mainly will make use of test functions φ of the form

$$\varphi(x_1, x_2) = g_1(x_1) g_2(x_2), \tag{8.10}$$

where g_1 and g_2 are defined on the intervals $\mathcal{I}_\rho(x_{0,1})$ and $\mathcal{I}_\rho(x_{0,2})$ respectively. If φ is of the form (8.10), then (8.9) becomes

$$\begin{aligned} \langle \upsilon, \varphi \rangle_{\mathcal{D}'(Q_\rho(x_0)), \mathcal{D}(Q_\rho(x_0))} &= \int_{\mathcal{I}_\rho(x_0)} g_1(x_1) \mathrm{d}(g_2(x_2)\upsilon)^{x_1} \\ &= \langle (\mathrm{d}g_2(x_2)\upsilon)^{x_1}, g_1 \rangle_{\mathcal{D}'(\mathcal{I}_\rho(x_{0,1})), \mathcal{D}(\mathcal{I}_\rho(x_{0,1}))}. \end{aligned} \tag{8.11}$$

In the case where $\upsilon(\overline{Q_\rho(x_0) \setminus \mathcal{R}_\rho(x_0)}) = 0$ and $g_2(s) = 1$ for $s \in \mathcal{I}_{\frac{3}{4}\rho}(x_{0,2})$, then we have $g(x_2)\upsilon = \upsilon$, so that identity (8.11) becomes

$$\langle \upsilon, \varphi \rangle = \int_{\mathcal{I}_\rho(x_{0,1})} g_1(x_1) \mathrm{d}\upsilon^{x_1}. \tag{8.12}$$

We will make use of these formulas in several places for the Radon measures $\tilde{\mu}_{\star,i,j}$, for $i = 1, 2$, $\tilde{\nu}_\star$, and $\tilde{\zeta}_\star$ and also related measures, obtained by multiplication and sums of the previous ones.

8.4 Some quantities of interest

The measures $\mathbb{L}_{x_0,\rho}$, $\mathbb{N}_{x_0,\rho}$, defined on $\mathcal{I}_\rho(x_0)$ as well as the measures $\tilde{\mu}^{x_1}_{\star,i,j}$ have already been introduced in the introduction in (80) and correspond to the description provided in the previous paragraph. Our computations will also involve some auxiliary "moment " measures, defined for, $k \in \mathbb{N}$, by

$$\begin{cases} \mathbb{J}_{k,x_0,\rho} \equiv \mathbb{J}_{k,\rho} = \mathbb{P}_\sharp\left((x_2 - x_{0,2})^k \tilde{\mu}_{\star,1,2}\right) \\ \mathbb{L}_{k,x_0,\rho} \equiv \mathbb{L}_{k,\rho} = \mathbb{P}_\sharp\left((x_2 - x_{0,2})^k \left[2\tilde{\zeta}_\star - \tilde{\mu}_{\star,1,1} + \tilde{\mu}_{\star,2,2}\right]\right) \\ \mathbb{N}_{k,x_0,\rho} \equiv \mathbb{N}_{k,\rho} = \mathbb{P}_\sharp\left((x_2 - x_{0,2})^k \left[2\tilde{\zeta}_\star + \tilde{\mu}_{\star,1,1} - \tilde{\mu}_{\star,2,2}\right]\right). \end{cases} \tag{8.13}$$

With this notation, we have $\tilde{\mu}^{x_1}_{\star,1,2} = \mathbb{J}_{0,x_0,\rho}$, $\mathbb{L}_{x_0,\rho} = \mathbb{L}_{0,x_0,\rho}$ and $\mathsf{N}_{x_0,\rho} = \mathsf{N}_{0,x_0,\rho}$. We will drop the subscript x_0 when there is no ambiguity (that is, in most places). We also consider the measures, for $k \in \mathbb{N}$,

$$\mathbb{H}_{k,x_0,\rho}(s) = \frac{1}{4}(\mathsf{N}_{k,\rho} + \mathbb{L}_{k,\rho}) = \mathbb{P}_\sharp\left((x_2 - x_{0,2})^k \tilde{\zeta}_\star\right). \tag{8.14}$$

The main result of this section is:

Proposition 8.1. *Assume that* (8.8) *holds. Then, the measures* $\mathbb{L}_{x_0,\rho}$ *and* $\mathbb{J}_{x_0,\rho}$ *are proportional to the Lebesgue measure on* $\mathcal{I}_\rho(x_{0,1})$*, that is, there exist constants* $L_{0,\rho}$ *and* $J_{0,\rho}$ *such that*

$$\mathbb{L}_{x_0,\rho} = L_{0,\rho}\,\mathrm{d}x_1 \text{ and } \mathbb{J}_{x_0,\rho} = J_{0,\rho}\,\mathrm{d}x_1, \text{ where } L_{0,\rho} \in \mathbb{R} \text{ and } J_{0,\rho} \in \mathbb{R}.$$

Moreover, we have the differential relations, for $k \in \mathbb{N}^*$*,*

$$\begin{cases} -2\dfrac{\mathrm{d}}{\mathrm{d}s}\mathbb{J}_{k,\rho} = k\mathsf{N}_{k-1,\rho} \text{ in } \mathcal{D}'((x_{0,1}-\rho, x_{0,1}+\rho)), \\ -\dfrac{\mathrm{d}}{\mathrm{d}s}\mathbb{L}_{k,\rho} = 2k\mathbb{J}_{k-1,\rho} \text{ in } \mathcal{D}'((x_{0,1}-\rho, x_{0,1}+\rho)). \end{cases} \tag{8.15}$$

In the case $k = 1$, we obtain hence the relations

$$\begin{cases} -2\dfrac{\mathrm{d}}{\mathrm{d}s}\mathbb{J}_{1,\rho} = \mathsf{N}_\rho, \text{ in } \mathcal{D}'((x_{0,1}-\rho, x_{0,1}+\rho)) \text{ and} \\ -\dfrac{\mathrm{d}}{\mathrm{d}s}\mathbb{L}_{1,\rho} = 2\mathbb{J}_\rho \text{ in } \mathcal{D}'((x_{0,1}-\rho, x_{0,1}+\rho)). \end{cases} \tag{8.16}$$

Notice the following consequence of Proposition 8.1:

Corollary 8.1. *For any* $k \in \mathbb{N}^\star$*, the measures* $\mathbb{J}_{k,\rho}$ *and* $\mathbb{L}_{k,\rho}$ *are absolutely continuous with respect to the Lebesgue measure* $\mathrm{d}x_1$*. Hence there exist measurable functions* $J_{k,\rho}$ *and* $L_{k,\rho}$ *on* $\mathcal{I}_\rho(x_{0,1})$ *such that*

$$\mathbb{J}_{k,\rho} = J_{k,\rho}\mathrm{d}x_1 \text{ and } \mathbb{L}_{k,\rho} = L_{k,\rho}\mathrm{d}x_1. \tag{8.17}$$

Moreover, the functions $J_{k,\rho}$ *and* $L_{k,\rho}$ *are bounded on* $\mathcal{I}_\rho(x_{0,1})$*.*

Proof of Corollary 8.1. The result is an immediate consequence of the fact that the measures $\mathsf{N}_{k-1,\rho}$ and $\mathbb{J}_{k-1,\rho}$ are bounded, so that, $\mathbb{J}_{k,\rho}$ and $\mathbb{L}_{k,\rho}$ represent BV functions on $\mathcal{I}_\rho(x_0)$, and hence are bounded. □

The proof of Proposition 8.1 involves the use of various kinds of vector fields of the form (8.5) in (8.2), that we will describe next in details in Subsections 8.5 and 8.6. The proof of Proposition 8.1 is then completed in Subsection 8.7.

8.5 Shear vector fields

We use in this section vector fields of the form (8.5), specifying $j = 2$. More precisely, we consider here vector fields of the form

$$\vec{X}_f(x_1, x_2) = f_1(x_1)f_2(x_2)\vec{\mathbf{e}}_2 = if_1(x_1)f_2(x_2), \tag{8.18}$$

where, for the last identity, we have identified $i = \vec{\mathbf{e}}_2$. A short computation shows that

$$\begin{cases} \dfrac{\partial X_f}{\partial z} = \dfrac{1}{2} f_1(x_1) f_2'(x_2) + \dfrac{i}{2} f_1'(x_1) f_2(x_2), \\ \dfrac{\partial X_f}{\partial \bar{z}} = -\dfrac{1}{2} f_1(x_1) f_2'(x_2) + \dfrac{i}{2} f_1'(x_1) f_2(x_2), \end{cases} \tag{8.19}$$

and hence

$$\begin{cases} \zeta_\star \operatorname{Re}\left(\dfrac{\partial X_f}{\partial z}\right) = \dfrac{1}{2} f_1(x_1) f_2'(x_2) \zeta_\star \text{ and} \\ \operatorname{Re}\left(\omega_\star \dfrac{\partial X_f}{\partial \bar{z}}\right) = -\dfrac{\operatorname{Re}(\omega_\star)}{2} f_1(x_1) f_2'(x_2) - \dfrac{\operatorname{Im}(\omega_\star)}{2} f_1'(x_1) f(x_2). \end{cases} \tag{8.20}$$

Identity (8.2) then becomes

$$\big\langle (\operatorname{Re}(\omega_\star) + 2\zeta_\star),\, f_2'(x_2) f_1(x_1) \big\rangle + \operatorname{Im} \langle \omega_\star, f(x_2)\, f_1'(x_1) \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = 0. \tag{8.21}$$

In view of (8.11) and the fact that $\operatorname{Im} \omega_\star = -2\mu_{\star,1,2}$, we may write

$$\begin{cases} \big\langle (\operatorname{Re}(\omega_\star) + 2\zeta_\star),\, f_2'(x_2) f_1(x_1) \big\rangle = \displaystyle\int_{\mathcal{I}_\rho(x_{0,1})} f_1(s) \mathrm{d} \left[f_2'(x_2) \left(\operatorname{Re}(\tilde{\omega}_\star) + 2\tilde{\zeta}_\star \right) \right]^{x_1} \\ \qquad\qquad \text{and} \\ \operatorname{Im} \langle \omega_\star, f(x_2)\, f_1'(x_1) \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = -2 \displaystyle\int_{\mathcal{I}_\rho(x_{0,1})} f_1'(s) \mathrm{d}[f(x_2) \tilde{\mu}_{\star,1,2}]^{x_1}. \end{cases} \tag{8.22}$$

8.5.1 A first choice for the function f_2

We choose, in this subsection as functions f_1, f_2 in (8.18) $f_1 = f$, where f is an arbitrary function in $C_c^\infty(\mathcal{I}_\rho(x_0))$ and, for f_2, a function of the form

$$f_2(x_2) = \chi(\frac{x_2 - x_{0,2}}{\rho}),$$

where χ is a non-negative given smooth plateau function such that

$$\chi(s) = 1, \text{ for } s \in [-\frac{3}{4}, \frac{3}{4}], \text{ and } \varphi(s) = 0, \text{ for } |s| \geq 1. \tag{8.23}$$

In particular, we have $f_2(x_{0,2}) = 1$ and

$$f_2'(x_2) = 0, \text{ if } |x_2 - x_{0,2}| \leq \frac{3\rho}{4}, \tag{8.24}$$

Such a vector field corresponds somewhat to a *shear vector field.* Using these shear vector fields, as test vector fields in (8.2), we obtain:

Proposition 8.2. *Assume that* (8.8) *holds. Then the measure* $\mathbb{J}_\rho$ *defined on* $\mathring{\mathcal{I}}_\rho(x_0)$ *by* (8.13) *is proportional to the Lebesgue measure, that is* $\mathbb{J}_\rho = J_{0,\rho} \mathrm{d}x$*, for some number* $J_{0,\rho} \in \mathbb{R}$*.*

Proof. We first claim that

$$\mathrm{Im}\left\langle \omega_\star, f(x_2)\, f_1'(x_1)\right\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0. \tag{8.25}$$

The proof of (8.25) relies on identity (8.21) and formulas (8.22). Indeed, since we assume that $\nu_\star(\overline{Q_\rho(x_0)\setminus\mathcal{R}_\rho(x_0)}) = 0$, it follows from (8.24) that f_2' vanishes on the support of $\mathrm{Re}(\omega_\star)+2\zeta_\star$. Hence, we are led to the identity

$$f_2'(x_2)\left(\mathrm{Re}(\tilde\omega_\star)+2\tilde\zeta_\star\right) = 0 \text{ and therefore } \left(f_2'(x_2)\left(\mathrm{Re}(\tilde\omega_\star)+2\tilde\zeta_\star\right)\right)^{x_1} = 0.$$

The first term on the left hand side of (8.21) hence vanishes, which yields the claim (8.25). We next notice that

$$\mathrm{Im}(\omega_\star f(x_2)) = -2\mu_{\star,1,2}\mathbf{1}_{Q_\rho(x_0)} = -2\tilde\mu_{\star,1,2},$$

so that (8.25) leads to the identity

$$\langle\tilde\mu_{\star,1,2},\ f_2(x_2)f'(x_1)\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0. \tag{8.26}$$

We invoke now the second identity in (8.22), together with the fact that $[f(x_2)\tilde\mu_{\star,1,2}]^{x_1} = \tilde\mu^{x_1}_{\star,1,2} = \mathbb{J}_\rho$, to deduce from (8.26) that

$$\langle\mathbb{J}_\rho, f'\rangle_{\mathcal{D}'(\mathcal{I}_\rho(x_{0,1})),\mathcal{D}(\mathcal{I}_\rho(x_{0,1}))} = \int_{\mathcal{I}_\rho(x_{0,1})} f'(s)\mathrm{d}\mathbb{J}_\rho = 0. \tag{8.27}$$

We have hence, in the sense of distributions

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{J}_\rho = 0, \text{ in } \mathcal{D}'(\mathcal{I}_\rho(x_{0,1})).$$

A classical result in distribution theory then shows that $\mathbb{J}_\rho$ is proportional to the uniform Lebesgue measure, so that the proof of Proposition 8.2 is complete. □

8.5.2 Another choice for f_2 : Transversal stretching vector fields

In this subsection, we assume that $f_1 = f$, where f is an arbitrary function in $C_c^\infty(\mathcal{I}_\rho(x_0))$ as above, and, that f_2 is given, for $k\in\mathbb{N}^\star$, by

$$f_2(x_2) = (x-x_{0,2})^k\chi(\frac{x_2-x_{0,2}}{\rho}),$$

where χ is a non-negative given smooth plateau function such that (8.23) holds. With this choice, we have now $f_2(x_{0,2}) = 0$

$$f'(x_2) = k(x-x_{0,2})^{k-1}, \text{ if } |x_2-x_{0,2}|\le\frac{3\rho}{4}. \tag{8.28}$$

Combining as above (8.21) and (8.28), we obtain:

Lemma 8.2. *Assume that* (8.8) *holds. We have, for $k\ge1$ and for any function f in $C_c^\infty(\mathcal{I}_\rho(x_0))$*

$$\left\langle\mathbf{1}_{Q_\rho}k(x_2-x_{0,2})^{k-1}((\mathrm{Re}(\omega_\star)+2\zeta_\star)\,, f(x_1)\right\rangle + \left\langle\mathbf{1}_{Q_\rho}(x_2-x_{0,2})^k\mathrm{Im}(\omega_\star), f'(x_1)\right\rangle = 0, \tag{8.29}$$

in $\mathcal{D}'(\Omega)$.

Identity (8.29) of Lemma 8.2 can be rephrased in terms of one-dimensional distributions using definitions (8.13). Arguing as in the proof of Proposition 8.2 and using (8.11), we obtain, if (8.8) holds

$$\langle k\mathbb{N}_{k-1,\rho}, f\rangle - \langle 2\mathbb{J}_{k,\rho}, f'\rangle = 0, \text{ for } f \in C_c^\infty((x_{0,1}-\rho, x_{0,1}+\rho),$$

so that, in the sense of distributions, we have

$$-2\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{J}_{k,\rho} = k\mathbb{N}_{k-1,\rho} \text{ in } \mathcal{D}'((x_{0,1}-\rho, x_{0,1}+\rho)), \text{ for } k \in \mathbb{N}^\star. \tag{8.30}$$

8.6 Dilation vector fields

We use here as test vector fields in (8.2), vector fields of the form

$$\vec{X}_d(x_1,x_2) = f_1(x_1)f_2(x_2)\vec{\mathbf{e}}_1 = f_1(x_1)f_2(x_2). \tag{8.31}$$

Computations similar to (8.19) yield

$$\begin{cases} \dfrac{\partial X_d}{\partial z} = \dfrac{1}{2}f_1'(x_1)f_2(x_2) - \dfrac{i}{2}f_1(x_1)f_2'(x_2), \\ \dfrac{\partial X_d}{\partial \bar{z}} = \dfrac{1}{2}f_1'(x_1)f_2(x_2) + \dfrac{i}{2}f_1(x_1)f_2'(x_2), \end{cases}$$

Relation (8.2) then becomes

$$\big\langle (\mathrm{Re}(\omega_\star) - 2\zeta_\star)\,, f_2(x_2)\, f_1'(x_1)\big\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} + \big\langle \mathrm{Im}\,(\omega_\star)\,, f_2'(x_2)\, f_1(x_1)\big\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = 0, \tag{8.32}$$

Arguing as for (8.22), we obtain the relations

$$\begin{cases} \big\langle (\mathrm{Re}(\omega_\star) - 2\zeta_\star)\,, f_2(x_2)f_1'(x_1)\big\rangle = \displaystyle\int_{\mathcal{I}_\rho(x_{0,1})} f_1'(s)\mathrm{d}\left[f_2(x_2)\left(\mathrm{Re}(\tilde{\omega}_\star) - 2\tilde{\zeta}_\star\right)\right]^{x_1} \\ \qquad\qquad\text{and} \\ \mathrm{Im}\,\langle\omega_\star, f'(x_2)\, f_1(x_1)\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = -2\displaystyle\int_{\mathcal{I}_\rho(x_{0,1})} f_1(s)\mathrm{d}[f_2'(x_2)\tilde{\upmu}_{\star,1,2}]^{x_1}. \end{cases} \tag{8.33}$$

We next choose test vectors fields $\vec{X}_d$ as in (8.31), with f_2 as in Subsection 8.5.1, that is of the form

$$f_2(x_2) = \chi(\frac{x_2 - x_{0,2}}{\rho}),$$

so that (8.24) holds. With this choice, we obtain, if (8.8) holds

$$[f_2'(x_2)\tilde{\upmu}_{\star,1,2}]^{x_1} = 0 \text{ and } \left(f_2(x_2)\left(\mathrm{Re}(\tilde{\omega}_\star) - 2\tilde{\zeta}_\star\right)\right)^{x_1} = -\mathbb{L}_\rho.$$

Inserting into (8.32), we derive the relation

$$\langle \mathbb{L}_\rho, f'\rangle = 0, \text{ for any } f_1 \in C_c^\infty(\mathcal{I}_\rho(x_{0,1})). \tag{8.34}$$

We have hence, in the sense of distributions

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{L}_\rho = 0, \text{ in } \mathcal{D}'(\mathcal{I}_\rho(x_{0,1})).$$

Arguing as in the proof of Proposition 8.2, we derive from (8.32) and (8.24) that:

Proposition 8.3. *Assume that* (8.8) *holds. Then, the measure* $\mathbb{L}_{x_0,\rho}$ *defined on* $\mathcal{I}_\rho(x_0)$ *by* (8.13) *is proportional to the uniform Lebesgue measure, that is* $\mathbb{L}_\rho = L_{0,\rho}\,\mathrm{d}x$*, for some number* $L_{0,\rho} \in \mathbb{R}$*.*

We finally use, as in Subsection 8.5.2, test vectors fields $\vec{X}_d$ given by (8.31), with f_2 of the form

$$f_2(x_2) = (x_2 - x_{0,2})^k \varphi(\frac{x_2 - x_{0,2}}{\rho}), k \in \mathbb{N}^\star,$$

so that (8.28) holds. Inserting into (8.32), and setting $f = f_1$, we are led to

$$-2 \left\langle k\mathbb{J}_{k-1}, f\right\rangle_{\mathcal{D}'(\mathcal{I}_\rho),\mathcal{D}'(\mathcal{I}_\rho)} + \left\langle \mathbb{L}_k(s), f'\right\rangle_{\mathcal{D}'(\mathcal{I}_\rho),\mathcal{D}(\mathcal{I}_\rho)} = 0, \text{ for } f \in C_c^\infty((x_{0,1} - \rho, x_{0,1} + \rho).$$

Hence, we have, in the sense of distributions, for $k \in \mathbb{N}^\star$

$$-\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{L}_{k,\rho} = 2k\mathbb{J}_{k-1,\rho} \text{ in } \mathcal{D}'((x_{0,1} - \rho, x_{0,1} + \rho)). \tag{8.35}$$

8.7 Proof of Proposition 8.1 completed

The proof of Proposition 8.1 follows combining Proposition 8.2, Proposition 8.3, together with identities (8.30) and (8.35).

8.8 Behavior near regular points

We specify in this part the consequences of Proposition 8.1 to regular points. More precisely, we consider a $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{A}_\star$, so that a unit tangent vector $\vec{e}_{x_0}$ to $\mathfrak{S}_\star$ exists at x_0. Throughout Subsection 8.8 we may choose therefore the orthonormal basis $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$ so that

$$\vec{\mathbf{e}}_1 = \vec{e}_{x_0} \tag{8.36}$$

is a unit tangent vector to $\mathfrak{S}_\star$ at x_0.

8.8.1 Property (8.8) is satisfied near regular points

The analysis carried out so far in Section 8 was mainly constrained by condition (8.8). We next show that this condition is satisfied near regular points.

Proposition 8.4. *Assume that* $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{A}_\star$*. Then, there exists* $\rho_0 > 0$ *such that property* (8.8) *is satisfied for any* $0 < \rho \leq \rho_0$*. Consequently, for any* $0 < \rho \leq \rho_0$*, the measures* $\mathbb{L}_{x_0,\rho}$ *and* $\mathbb{J}_{x_0,\rho}$ *are proportional to the Lebesgue measure on* $\mathcal{I}_\rho(x_{0,1})$ *and the differential relations* (8.15) *hold.*

Proof. Let $r > 0$ be such that $\mathbb{D}^2(x_0, r) \subset \Omega$. Since we assume that x_0 is a regular point of $\mathfrak{S}_\star$, we may choose a the orthonormal basis so that (8.36) holds. In view of Proposition 4, we have, for any $\theta \in [0, \frac{\pi}{2}]$ and $0 < \varrho \leq R_{\mathrm{cone}}(\theta, x_0)$

$$\mathfrak{S}_\star \cap \mathbb{D}^2\left(x_0, \varrho\right) \subset \mathcal{C}_{\mathrm{one}}\left(x_0, \vec{e}_{x_0}, \theta\right) = \mathcal{C}_{\mathrm{one}}\left(x_0, \vec{\mathbf{e}}_1, \theta\right), \tag{8.37}$$

Since we have $Q_{\frac{\varrho}{\sqrt{2}}}(x_0) \subset \mathbb{D}^2\left(x_0, \varrho\right)$, we obtain, for $0 \leq \rho \leq \rho_0 \equiv \sqrt{2}^{-1} R_{\mathrm{cone}}(\theta, x_0)$

$$\mathfrak{S}_\star \cap Q_\rho(x_0) \subset \mathcal{C}_{\mathrm{one}}\left(x_0, \vec{\mathbf{e}}_1, \theta\right). \tag{8.38}$$

Specifying (8.38) with $\theta = \dfrac{\pi}{8}$, we obtain, for $0 \le \rho \le \rho_0 \equiv \sqrt{2}^{-1} R_{\rm cone}(\frac{\pi}{8}, x_0)$

$$\mathfrak{S}_\star \cap Q_\rho(x_0) \subset \mathcal{C}_{\rm one}\left(x_0, \vec{e}_{x_0}, \frac{\pi}{8}\right) \cap Q_\rho(x_0) \subset \mathcal{R}_\rho(x_0). \tag{8.39}$$

It follows that, if $\rho \le \rho_0$, then (8.8) holds. In particular, we are in position to apply Proposition 8.1 at the point x_0. This yields immediately, for $0 < \rho \le \rho_0$, the fact that the functions $L_{x_0,\rho}$ and $J_{x_0,\rho}$ are constant on the interval $(x_{0,1} - \rho_0, x_{0,1} + \rho_0)$, and relations (8.15) hold. The proof of the proposition is hence complete. □

Remark 8.1. The argument actually shows also that, for $0 < \rho \le \rho_0$, the measures $\mathbb{L}_{x_0,\rho}$ and $\mathbb{J}_{x_0,\rho}$ correspond to the restriction to $\mathcal{I}_\rho(x_{0,1})$ of the measures $\mathbb{L}_{x_0,\rho_0}$ and $\mathbb{J}_{x_0,\rho_0}$ respectively.

8.8.2 Some additional properties near regular points

We derive next some additional properties for regular points, in connection with the singular part of the measures. We introduce therefore the set

$$B_\star = \{s \in \mathbb{R} \text{ such that } (\{s\} \times \mathbb{R}) \cap \mathfrak{B}_\star \neq \emptyset\} = \mathbb{P}(\mathfrak{B}_\star).$$

where $\mathfrak{B}_\star$ is defined in (74) and represents the set where the singular part of the measures concentrates. Notice that, since $\mathcal{H}^1(\mathfrak{B}_\star) = 0$, the Lebesgue measure of the set $B_\star$ vanishes likewise. Recall, in view of Corollary 8.1, that we have, for any $0 < r \le \rho_0$ (where ρ_0 is provided by Proposition 8.4) $\mathbb{J}_{1,r} = J_{1,r} {\rm d}x_1$ and $\mathbb{L}_{1,r} = L_{1,r} {\rm d}x_1$, where the function $L_{1,r}$ and $J_{1,r}$ are bounded. We have first:

Lemma 8.3. *Let $x_0 = (x_{0,1}, x_{0,2})$ be a regular point in $\mathfrak{S}_\star \setminus \mathfrak{A}_\star$ and let ρ_0 be given by Proposition 8.4. Let $\theta \in [0, \frac{\pi}{8}]$. We have, for any $r \le \frac{1}{2} \inf\{R_{\rm cone}(\theta, x_0), \rho_0\}$,*

$$\begin{cases} \displaystyle\int_{x_{0,1}-2r}^{x_{0,1}+2r} |J_{1,\rho_0}(s)| {\rm d}s \ \le 4r \sin\theta\, \nu_\star^{ac}\left(\mathbb{D}^2(x_0, 2r)\right) \text{ and} \\ \displaystyle\int_{x_{0,1}-2r}^{x_{0,1}+2r} |L_{1,\rho_0}(s)| {\rm d}s \le 8r \sin\theta\, \nu_\star^{ac}\left(\mathbb{D}^2(x_0, 2r)\right). \end{cases} \tag{8.40}$$

Proof. If $2r \le R_{\rm cone}(\theta, x_0)$, it follows from (69) that we have $\nu_\star(\mathcal{R}_{2r}(x_0) \backslash \mathcal{C}_{\rm one}(x_0, \vec{e}_{x_0}, \theta)) = 0$. On the other hand, we have

$$|x_2 - x_{0,2}| \le 2r \sin\theta, \text{ for } x = (x_1, x_2) \in \mathcal{R}_r(x_0) \cap \mathcal{C}_{\rm one}(x_0, \vec{e}_{x_0}, \theta).$$

Multipling by $\mu_{\star,1,2}$ and integrating on the set $\mathcal{R}_{2r}(x_0) \setminus B_\star \times \mathbb{R}$, we are led to

$$\begin{aligned} \int_{\mathcal{R}_{2r}(x_0) \backslash B_\star \times \mathbb{R}} {\rm d}\, |\mu_{\star,1,2}(x_2 - x_{0,2})| &\le 4r \sin\theta\, \nu_\star\left(\mathbb{D}^2(x_0, 2r) \setminus B_\star \times \mathbb{R}\right) \\ &\le 4r \sin\theta\, \nu_\star^{ac}\left(\mathbb{D}^2(x_0, 2r)\right). \end{aligned} \tag{8.41}$$

For the last inequality, we invoke the fact that we have the inclusion $\mathbb{D}^2(x_0, 2r) \setminus B_\star \times \mathbb{R} \subset \mathbb{D}^2(x_0, 2r) \setminus \mathfrak{B}_\star$, so that

$$\nu_\star\left(\mathbb{D}^2(x_0, 2r) \setminus B_\star \times \mathbb{R}\right) \le \nu_\star\left(\mathbb{D}^2(x_0, 2r) \setminus \mathfrak{B}_\star\right) = \nu_\star^{ac}\left(\mathbb{D}^2(x_0, 2r)\right),$$

the last identity being a consequence of the decomposition (73) and (74). Since, by definition $\mathbb{J}_{1,r} = \mathbb{P}_\sharp(\tilde{\upmu}_{\star,1,2}(x_2 - x_{0,2}))$, we have hence, in view of(42)

$$\int_{(x_{0,1}-2r,x_{0,1}+2r)\setminus B_\star} |J_{1,\rho_0}(s)|\mathrm{d}s \leq \int_{\mathcal{R}_{2r}(x_0)\setminus B_\star\times\mathbb{R}} \mathrm{d}\left|\upmu_{\star,1,2}\left(x_2 - x_{0,2}\right)\right|. \tag{8.42}$$

Combining (8.41), (8.42) together with the fact that $B_\star$ has zero Lebesgue measure and the function J_{1,ρ_0} is bounded, thus integrable, we deduce the first inequality in (8.40). The second is established invoking similar arguments. □

Lemma 8.4. *Let $x_0 = (x_{0,1}, x_{0,2})$ be a regular point $\mathfrak{S}_\star \setminus \mathfrak{A}_\star$, and and let ρ_0 be given by Proposition 8.4. Let $\uptheta \in [0, \frac{\pi}{8}]$. For any $0 < r < \dfrac{1}{2}\inf\{R_{\rm cone}(\theta, x_0), \rho_0)\}$, there exists some $\varrho_r \in [r, 2r]$ such that*

$$\begin{cases} \left|\displaystyle\int_{x_{0,1}-\varrho_r}^{x_{0,1}+\varrho_r} J_{\rho_0}(s)\mathrm{d}s\right| \leq 8\sin\uptheta\, \upnu_\star^{ac}\left(\mathbb{D}^2(x_0, 2r)\right) \text{ and} \\ \left|\displaystyle\int_{x_{0,1}-\varrho_r}^{x_{0,1}+\varrho_r} \mathrm{d}\mathsf{N}_{\rho_0}\right| \leq 16\sin\uptheta\, \upnu_\star^{ac}\left(\mathbb{D}^2(x_0, 2r)\right). \end{cases} \tag{8.43}$$

Proof. The proof of (8.43) follows from (8.40) integrating the differential equations (8.15) for $k = 1$. Indeed, for almost every $\varrho \in [r, 2r]$, $x_{0,1} - \varrho$ and $x_{0,1} + \varrho$ are Lebesgue points of J_{1,ρ_0}, L_{1,ρ_0}, J_{ρ_0} and the absolutely continuous part of N_{ρ_0}. We choose next a sequence of smooth, compactly supported test functions $\{\psi_m\}_{m\in\mathbb{N}}$ such that such that $0 \leq \psi_m \leq 1$, for any $m \in \mathbb{N}$, and

$$\psi_m \underset{m\to+\infty}{\to} \mathbf{1}_{(x_{0,1}-\varrho,x_{0,1}+\varrho)} \text{ in } L^1(\mathcal{I}_r(x_{0,1})). \tag{8.44}$$

Using ψ_m as test function for the differential equation (8.16), we obtain, by integration by parts, for any $m \in \mathbb{N}$

$$\begin{cases} 2\displaystyle\int_{x_{0,1}-\varrho}^{x_{0,1}+\varrho} \psi_m(s)J_{\rho_0}(s)\mathrm{d}s = \int_{x_{0,1}-\varrho}^{x_{0,1}+\varrho} \psi_m'(s)L_{1,\rho_0}(s)\mathrm{d}s \text{ and} \\ \displaystyle\int_{x_{0,1}-\varrho}^{x_{0,1}+\varrho} \psi_m \mathrm{d}\mathsf{N}_{\rho_0} = 2\int_{x_{0,1}-\varrho}^{x_{0,1}+\varrho} \psi_m'(s)J_{1,\rho_0}(s)\mathrm{d}s. \end{cases}$$

Passing to the limit $m \to +\infty$, we obtain, using the Lebesgue properties of the points $x_{0,1} - \varrho$ and $x_{0,1} + \varrho$

$$\begin{aligned} \int_{x_{0,1}-\varrho}^{x_{0,1}+\varrho} \mathrm{d}\mathsf{N}_{\rho_0} &= 2\left(J_{1,\rho_0}(x_{0,1} + \varrho) - J_{1,\rho_0}(x_{0,1} - \varrho)\right) \text{ and} \\ \int_{x_{0,1}-\varrho}^{x_{0,1}+\varrho} J_{\rho_0}(s)\mathrm{d}s &= \frac{1}{2}\left(L_{1,\rho_0}(x_{0,1} - \varrho) - L_{1,\rho_0}(x_{0,1} + \varrho)\right). \end{aligned} \tag{8.45}$$

Next, we use a mean value argument to deduce that there exists some number $\varrho_r \in [r, 2r]$, such that $x_{0,1} - \varrho_r$ and $x_{0,1} + \varrho_r$ are Lebesgue points of J_{1,ρ_0}, L_{1,ρ_0}, J_{ρ_0} and the absolutely continuous part of $\mathbb{N}_{\rho_0}$ and such that

$$\begin{cases} |J_{1,\rho_0}(x_{0,1}+\varrho_r)| + |J_{1,\rho_0}(x_{0,1}-\varrho_r)| \leq \dfrac{2}{r}\displaystyle\int_{x_{0,1}-2r}^{x_{0,1}+2r} |J_{1,\rho_0}(s)|\mathrm{d}s \\ |L_{1,\rho_0}(x_{0,1}+\varrho_r)| + |L_{1,\rho_0}(x_{0,1}-\varrho_r)| \leq \dfrac{2}{r}\displaystyle\int_{x_{0,1}-2r}^{x_{0,1}+2r} |L_{1,\rho_0}(s)|\mathrm{d}s \end{cases} \tag{8.46}$$

Combining (8.45), (8.46) with (8.40), we obtain the desired result. □

8.9 Behavior near Lebesgue points: Proofs to Proposition 5 and Lemma 2

Recall that, at this stage we already know that, if $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{A}_\star$, in view of Propositions 8.2 and 8.3, we have

$$\mathbb{L}_{x_0,\rho_0} = L_{0,\rho_0}\mathrm{d}x_1 \text{ and } \mathbb{P}_\sharp(\tilde{\mu}_{\star,i,2}) = J_{0,\rho_0}\mathrm{d}x_1,$$

where $L_{0,\rho_0} \in \mathbb{R}$ and $J_{0,\rho_0} \in \mathbb{R}$. We derive here additional properties in the case $x_0 \notin \mathfrak{E}_\star$, leading eventually to the proof of Proposition 5.

8.9.1 Additional properties of J_{x_0,ρ_0} and $\mathbb{N}_{x_0,\rho_0}$ at Lebesgue points

Let $x_0 \in \mathfrak{S}_\star$ and $\rho_0 > 0$. We impose in this paragraph the additional condition that $x_0 \notin \mathfrak{E}_\star$, i.e. x_0 is a regular point, which is not on the support of the singular part, and is moreover a Lebesgue point for the densities of the absolutely continuous part for all measures of interest. More precisely, this means that

$$\begin{cases} \lim\limits_{r\to 0} \dfrac{1}{r}\displaystyle\int_{\mathfrak{S}_\star\cap\mathbb{D}^2(x_0,r)} |\Theta_\star(\tau) - \Theta_\star(x_0)|\,\mathrm{d}\tau = 0 \\ \lim\limits_{r\to 0} \dfrac{1}{r}\displaystyle\int_{\mathfrak{S}_\star\cap\mathbb{D}^2(x_0,r)} |\mathbb{e}_\star(\tau) - \mathbb{e}_\star(x_0)|\,\mathrm{d}\tau = 0, \text{ and} \\ \lim\limits_{r\to 0} \dfrac{1}{r}\displaystyle\int_{\mathfrak{S}_\star\cap\mathbb{D}^2(x_0,r)} |\mathbb{m}_{\star,i,j}(\tau) - \mathbb{m}_{\star,i,j}(x_0)|\,\mathrm{d}\tau = 0, \text{ for } i,j = 1,2. \end{cases} \tag{8.47}$$

As a first direct consequence, we deduce that, for some constant $K = K(x_0) > 0$ depending on x_0, we have

$$\nu_\star^{ac}\left(\mathbb{D}^2(x_0,r)\right) \leq Kr \text{ for any } 0 < r < R, \tag{8.48}$$

and also that

$$\begin{cases} \lim\limits_{r\to 0} \dfrac{1}{r}\displaystyle\int_{\mathfrak{S}_\star\cap\mathbb{D}^2(x_0,r)} \Theta_\star(\tau)\mathrm{d}\tau = \Theta_\star(x_0), \\ \lim\limits_{r\to 0} \dfrac{1}{r}\displaystyle\int_{\mathfrak{S}_\star\cap\mathbb{D}^2(x_0,r)} \mathbb{e}_\star(\tau)\mathrm{d}\tau = \mathbb{e}_\star(x_0), \text{ and} \\ \lim\limits_{r\to 0} \dfrac{1}{r}\displaystyle\int_{\mathfrak{S}_\star\cap\mathbb{D}^2(x_0,r)} \mathbb{m}_{\star,i,j}(\tau)\mathrm{d}\tau = \mathbb{m}_{\star,i,j}(x_0). \end{cases} \tag{8.49}$$

At this stage, we already know that J_{ρ_0} is a constant map. Concerning $\mathbb{N}_{\rho_0}$ we may decompose this measure on $\mathcal{I}_{\rho_0}(x_{0,1})$ as a sum of an absolutely continuous part and a singular part

$$\mathbb{N}_{\rho_0} = \mathbb{N}_{\rho_0}^{ac} + \mathbb{N}_{\rho_0}^{s} \text{ with } \mathbb{N}_{\rho_0}^{ac} \ll \mathrm{d}x_1 \text{ and } \mathbb{N}_{\rho_0}^{s} \perp \mathbb{N}_{\rho_0}^{ac},$$

so that there exists a set $F_{\rho_0} \subset \mathcal{I}_{\rho_0}(x_{0,1})$ supporting the singular part, that is, such that

$$\mathcal{H}^1(F_{\rho_0}) = 0 \text{ and } \mathbb{N}^s_{\rho_0}(\mathcal{I}_{\rho_0}(x_{0,1}) \setminus F_{\rho_0}) = 0, \tag{8.50}$$

and a measurable function N_{ρ_0} defined on $\mathcal{I}_{\rho_0}(x_{0,1})$ such that $\mathbb{N}^{ac}_{\rho_0} = N_{\rho_0} \mathrm{d}x_1$. In this setting, the functions L_{ρ_0}, N_{ρ_0} and J_{ρ_0} on $\mathcal{I}_{\rho_0}(x_{0,1})$ are related to the functions $\Theta_\star$ and $\mathbb{m}_{\star,i,j}$, for $i,j = 1,2$ defined on $\mathfrak{S}_\star$ by (43) by the following result.

Proposition 8.5. *Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$ and $\rho_0 > 0$ be given by Proposition 8.4 so that* (8.8) *holds for $\rho = \rho_0$. Choose the orthonormal basis so that $\vec{\mathbf{e}}_1 = \vec{e}_{x_0}$ is a unit tangent vector to $\mathfrak{S}_\star$ at x_0. Then, $x_{0,1} \notin F_{\rho_0}$ and is a Lebesgue point for N_{ρ_0} and $J_{\rho_0}(x_0)$. We have the identities, at the point x_0,*

$$\begin{cases} N_{\rho_0}(x_{0,1}) = 2\Theta_\star(x_0) - \mathbb{m}_{\star,2,2}(x_0) + \mathbb{m}_{\star,1,1}(x_0), \\ J_{\rho_0}(x_{0,1}) = \mathbb{m}_{\star,1,2}(x_0) \text{ and} \\ L_{\rho_0}(x_{0,1}) = 2\Theta_\star(x_0) - \mathbb{m}_{\star,1,1}(x_0) + \mathbb{m}_{\star,2,2}(x_0). \end{cases} \tag{8.51}$$

Proof. We go back to the definition (76) of $\mathfrak{E}_\star$. Since $x_0 \notin \mathfrak{E}_\star$, and hence $x_0 \notin \mathfrak{B}_\star$ (see (74)), we have by definition of the set $\mathfrak{B}_\star$

$$D_\lambda(\nu_\star)(x_0) = \lim_{r\to 0} \frac{\nu_\star\left(\mathbb{D}^2(x_0,r)\right)}{\lambda\left(\mathbb{D}^2(x_0,r)\right)} < +\infty \text{ and } D_\lambda(\nu^s_\star)(x_0) = \lim_{r\to 0} \frac{\nu^s_\star\left(\mathbb{D}^2(x_0,r)\right)}{\lambda\left(\mathbb{D}^2(x_0,r)\right)} = 0, \tag{8.52}$$

where λ represents the one-dimensional Hausdorff measure on $\mathfrak{S}_\star$. On the other hand, since x_0 is a regular point, we have, in view of (11)

$$\lim_{r\to 0} \frac{\lambda\left(\mathbb{D}^2(x_0,r)\right)}{2r} = 1,$$

so that

$$D_\lambda(\nu_\star)(x_0) = D_\lambda(\nu^{ac}_\star)(x_0) = \lim_{r\to 0} \frac{\nu_\star\left(\mathbb{D}^2(x_0,r)\right)}{2r} < +\infty. \tag{8.53}$$

Turning to the measure $\tilde{\nu}^{x_1}_\star$, we have $\nu^{x_1}_\star(\mathcal{I}_r(x_{0,1})) = \tilde{\nu}_\star\left(\mathcal{I}_r(x_{0,1}) \times \mathcal{I}_r(x_{0,2})\right)$. In view of Proposition 4, given $\theta > 0$, we have, for $r \le R_{\text{cone}}(\theta, x_0)$, the inclusion

$$\mathfrak{S}_\star \cap \mathbb{D}^2\left(x_0, r\right) \subset \mathcal{C}_{\text{one}}\left(x_0, \vec{e}_{x_0}, \theta\right).$$

On the other hand, we have also the chain of inclusions

$$\mathbb{D}^2(x_0,r) \subset \left(\mathcal{I}_r(x_{0,1}) \times \mathcal{I}_r(x_{0,2})\right) \subset \mathbb{D}^2(x_0, \frac{r}{\cos\theta}), \tag{8.54}$$

so that combining the previous relations, we are led to the bounds

$$\nu_\star\left(\mathbb{D}^2(x_0,r)\right) \le \nu^{x_1}_\star(\mathcal{I}_r(x_{0,1})) \le \nu_\star\left(\mathbb{D}^2(x_0, \frac{r}{\cos\theta})\right). \tag{8.55}$$

Letting θ and r go to zero, we deduce from (8.52) and (8.55) the identity

$$\lim_{r\to 0} \frac{\nu^{x_1}_\star(\mathcal{I}_r(x_{0,1}))}{2r} = D_\lambda(\nu_\star)(x_0) = D_\lambda(\nu^{ac}_\star)(x_0) < +\infty,$$

and similarily, for $i, j = 1, 2$

$$\begin{cases} \lim\limits_{r\to 0} \dfrac{\zeta_\star^{x_1}(\mathcal{I}_r(x_{0,1}))}{2r} = D_\lambda(\zeta_\star^{ac})(x_0) = \Theta_\star(x_0) \text{ and} \\ \lim\limits_{r\to 0} \dfrac{\mu_{\star,i,j}^{x_1}(\mathcal{I}_r(x_{0,1}))}{2r} = D_\lambda(\mu_{\star,i,j}^{ac})(x_0) = \mathrm{m}_{\star,i,j}(x_0) < +\infty. \end{cases}$$

It follows that, in view of the definition (80) of $\mathbb{N}_{\rho_0}$, we have

$$\begin{aligned} \lim_{r\to 0} \frac{\mathbb{N}_{\rho_0}(\mathcal{I}_r(x_{0,1}))}{2r} &= 2D_\lambda(\zeta_\star^{ac})(x_0) - D_\lambda(\mu_{2,2}^{ac})(x_0) + D_\lambda(\mu_{1,1}^{ac})(x_0) \in \mathbb{R} \\ &= 2\Theta_\star(x_0) - \mathrm{m}_{\star,2,2}(x_0) + \mathrm{m}_{\star,1,1}(x_0). \end{aligned}$$

We deduce that $x_{0,1} \notin F_{\rho_0}$, where F_{ρ_0} is defined in (8.50), and that we have

$$\lim_{r\to 0} \frac{\mathbb{N}_{\rho_0}^{ac}(\mathcal{I}_r(x_{0,1}))}{2r} = \lim_{r\to 0} \frac{\mathbb{N}_{\rho_0}(\mathcal{I}_r(x_{0,1}))}{2r} = 2\Theta_\star(x_0) - \mathrm{m}_{\star,2,2}(x_0) + \mathrm{m}_{\star,1,1}(x_0)..$$

We prove using similar arguments that $x_{0,1}$ is a Lebesgue point for the map N_{ρ_0}, so that the first identity in (8.51) is established. Turning to the maps J_{ρ_0} and L_{ρ_0} we observe that, since these maps are constant, $x_{0,1}$ is obviously a Lebesgue point for them. The two last identities in (8.51) are established using the same arguments. □

We compute next $J_{\rho_0}(x_0)$ and $N_{\rho_0}(x_0)$ in a different way.

Proposition 8.6. *Let x_0 and $\rho_0 > 0$ be as in Proposition 8.5. We have*

$$\begin{cases} J_{x_0,\rho_0}(s) = 0 \text{ for } s \in (x_{0,1} - \rho_0, x_{0,1} + \rho_0) \text{ and} \\ N_{x_0,\rho_0}(x_{0,1}) = 0. \end{cases} \tag{8.56}$$

In order to proof Proposition 8.6, we rely on an intermediate result:

Lemma 8.5. *Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$ and $\rho_0 > 0$ be given by Proposition 8.4. Choose the orthonormal basis so that $\vec{\mathbf{e}}_1 = \vec{e}_{x_0}$ is a unit tangent vector to $\mathfrak{S}_\star$ at x_0. For $< r < \rho_0$, let $\varrho_r > 0$ be given by Lemma 8.4. Then, we have*

$$\lim_{r\to 0} \frac{1}{2\varrho_r} \int_{x_{0,1}-\varrho_r}^{x_{0,1}+\varrho_r} \mathrm{d}\mathbb{N}_r(s) = 0 \text{ and } \lim_{r\to 0} \frac{1}{2\varrho_r} \int_{x_{0,1}-\varrho_r}^{x_{0,1}+\varrho_r} J_r(s)\mathrm{d}s = 0. \tag{8.57}$$

Proof. For any given $\theta \in [0, \frac{\pi}{8}]$, and $0 < r \le \inf\{\rho_0, \frac{1}{2}R_{\mathrm{cone}}(\theta, x_0)\}$, we deduce, combining (8.48) with (8.43), that

$$\left| \int_{x_{0,1}-\varrho_r}^{x_{0,1}+\varrho_r} \mathrm{d}\mathbb{N}_r \right| + \left| \int_{x_{0,1}-\varrho_r}^{x_{0,1}+\varrho_r} J_r(s)\mathrm{d}s \right| \le 24 \sin\theta\, \nu_\star^{ac}\left(\mathbb{D}^2(x_0, 2r)\right) \le 48Kr \sin\theta, \tag{8.58}$$

so that, since $\varrho_r \ge r$,

$$\left| \frac{1}{2\varrho_r} \int_{x_{0,1}-\varrho_r}^{x_{0,1}+\varrho_r} \mathrm{d}\mathbb{N}_r \right| + \left| \frac{1}{2\varrho_r} \int_{x_{0,1}-\varrho_r}^{x_{0,1}+\varrho_r} J_r(s)\mathrm{d}s \right| \le 48K \sin\theta. \tag{8.59}$$

We first let $r \to 0$, so that $\varrho_r \to 0$ as $r \to 0$, and then let $\theta \to 0$ in (8.59), which yields (8.57). □

Proof of Proposition 8.6 completed. We first consider J_{ρ_0}. We already know that the function J_{ρ_0} is constant on $\mathcal{I}_{\rho_0}(x_{0,1})$, so that

$$\frac{1}{2\varrho_r}\int_{x_{0,1}-\varrho_r}^{x_{0,1}+\varrho_r} J_r(s)\mathrm{d}s = J_r(x_{0,1}),$$

we deduce therefore from the second relation in (8.57) that $J_{\rho_0}(x_{0,1}) = 0$. We now turn to N_{ρ_0}. Since $x_0 \notin F_{\rho_0}$, we have $D_\lambda(\mathbb{N}^s_{\rho_0})(x_{0,1}) = 0$, that is

$$\lim_{r\to 0} \frac{\mathbb{N}^s_{\rho_0}(\mathcal{I}_{\varrho_r}(x_{0,1}))}{2\varrho_r} = 0.$$

Combining with the first identity in (8.57), we are led to

$$\lim_{r\to 0} \frac{1}{2\varrho_r}\int_{x_{0,1}-\varrho_r}^{x_{0,1}+\varrho_r} N_{\rho_0}(s)\mathrm{d}s = \lim_{r\to 0} \frac{1}{2\varrho_r}\int_{x_{0,1}-\varrho_r}^{x_{0,1}+\varrho_r} \mathrm{d}\mathbb{N}^{ac}_r(s) = 0. \tag{8.60}$$

Since $x_{0,1}$ is a Lebesgue point for N_{ρ_0}, we derive that $N_{\rho_0}(x_{0,1}) = 0$, so that the proof is complete.

8.9.2 Proof of Proposition 5 completed

Since $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, we are in position to apply Propositions 8.5 and 8.6. Combining (8.51) with (8.56), we obtain (77) and the proof is complete. □

8.9.3 Change of orthonormal basis for the Hopf differential

Recall that we have assumed in Proposition 5 that the orthonormal basis is chosen so that $\vec{\mathbf{e}}_1$ is tangent to $\mathfrak{S}_\star$ at x_0. However, the definition of the Hopf differential clearly depends on the choice of coordinates, and we will need to compute it in various bases, for instance a moving frame on $\mathfrak{S}_\star$ or a frame related to polar coordinates. For that purpose, and for a given angle $\theta \in \mathbb{R}$, let $(\vec{\mathbf{e}}_1^\theta, \vec{\mathbf{e}}_2^\theta)$ be a new orthonormal basis deduced from $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$ by a rotaion of angle θ, that is

$$\begin{cases} \vec{\mathbf{e}}_1^\theta = \cos\theta\, \vec{\mathbf{e}}_1 + \sin\theta\, \vec{\mathbf{e}}_2 \\ \vec{\mathbf{e}}_2^\theta = -\sin\theta\, \vec{\mathbf{e}}_1 + \cos\theta\, \vec{\mathbf{e}}_2. \end{cases} \tag{8.61}$$

Let $(x_{\theta,1}, x_{\theta,2}) = (\cos\theta\, x_1 + \sin\theta\, x_2, -\sin\theta\, x_1 + \cos\theta\, x_2)$ denote the coordinates related to the new basis and $\omega_{\varepsilon,\theta}$ and $\omega_{\star,\theta}$ the corresponding Hopf differentials. Then, for any map $u : \Omega \to \mathbb{R}^2$, we have the identities $u_{x_{\theta,1}} = u_{x_1}\cos\theta + u_{x_2}\sin\theta$ and $u_{x_{\theta,2}} = -u_{x_1}\sin\theta + u_{x_2}\cos\theta$, so that

$$\begin{cases} |u_{x_{\theta,1}}|^2 - |u_{x_{\theta,2}}|^2 = \cos 2\theta\left(|u_{x_1}|^2 - |u_{x_2}|^2\right) + 2\sin 2\theta\, u_{x_1}\cdot u_{x_2} \\ \qquad 2u_{x_{\theta,1}}\cdot u_{x_{\theta,2}} = -\sin 2\theta\left(|u_{x_1}|^2 - |u_{x_2}|^2\right) + 2\cos 2\theta u_{x_1}\cdot u_{x_2}. \end{cases} \tag{8.62}$$

We are hence led to the transformation law

$$\begin{cases} \omega_{\varepsilon,\theta} = (\cos 2\theta + i\sin 2\theta)\omega_\varepsilon = \exp(2i\theta)\omega_\varepsilon \text{ and} \\ \omega_{\star,\theta} = (\cos 2\theta + i\sin 2\theta)\omega_\star = \exp(2i\theta)\omega_\star. \end{cases} \tag{8.63}$$

It follows in particular from the above relations that, if the limits (8.1) and (41) exist for a given orthonormal basis, then they exist also for any other one.

8.9.4 Proof of Lemma 2 completed

In view of (43), we may write, in the basis $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$

$$\omega_\star^{ac} = ((\mathbb{m}_{\star,1,1} - \mathbb{m}_{\star,2,2}) - 2i\mathbb{m}_{\star,1,2})\, \mathrm{d}\lambda. \tag{8.64}$$

Next let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, $\vec{e}_{x_0}$ be a tangent vector at x_0 to $\mathfrak{S}_\star$, so that the angle of $\vec{\mathbf{e}}_1$ with $\vec{e}_{x_0}$ is given by $\gamma_\star(x_0) \in [-\pi/2, \pi/2]$. In view of the notation (8.61), we have $\vec{e}_{x_0} = \vec{\mathbf{e}}_1^{\gamma_\star(x_0)}$. It follows from (8.63) that

$$\omega_{\star,\gamma_\star(x_0)}^{ac} = \exp(2i\gamma_\star(x_0))\omega_\star^{ac} = \exp(2i\gamma_\star(x_0))\,((\mathbb{m}_{\star,1,1} - \mathbb{m}_{\star,2,2}) - 2i\mathbb{m}_{\star,1,2})\, \mathrm{d}\lambda. \tag{8.65}$$

Appying Proposition 5 at x_0 in the basis $\left(\vec{\mathbf{e}}_1^{\gamma_\star(x_0)}, \vec{\mathbf{e}}_2^{\gamma_\star(x_0)}\right)$, we are led to the identity

$$\exp(2i\gamma_\star(x_0))\,((\mathbb{m}_{\star,1,1}(x_0) - \mathbb{m}_{\star,2,2}(x_0)) - 2i\mathbb{m}_{\star,1,2}(x_0)) = -2\Theta_\star(x_0),$$

so that, for any $x \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, we have the identity

$$(\mathbb{m}_{\star,1,1}(x) - \mathbb{m}_{\star,2,2}(x)) - 2i\mathbb{m}_{\star,1,2}(x) = -2\exp(-2i\gamma_\star(x))\Theta_\star(x).$$

Going back to (8.64), we obtain hence that

$$\omega_\star^{ac} = -2\exp(-2i\gamma_\star(x))\Theta_\star \mathrm{d}\lambda = -2\exp(-2i\gamma_\star)\zeta_\star^{ac}.$$

The proof is hence complete.

9 Monotonicity for $\zeta_\star$ and its consequence

The purpose of present section is to establish Proposition 6.

9.1 Proof of Lemma 3

Since $\zeta_{\varepsilon_n} \rightharpoonup \zeta_\star$, as $n \to +\infty$, weakly in the sense of measures, we have, for any Borel set A such that $\zeta_\star(\partial A) = 0$, the convergence $\zeta_{\varepsilon_n}(A) \to \zeta_\star(A)$, as $n \to +\infty$. Since $\nu_\star(\partial \mathbb{D}^2(x_0, r)) = 0$ for almost every $r \in (0, \rho)$, we have hence, for almost every $r_i \in (0, \rho)$,

$$\begin{cases} \zeta_{\varepsilon_n}(\mathbb{D}^2(x_0, r_i)) \underset{n\to+\infty}{\to} \zeta_\star(\mathbb{D}^2(x_0, r_i)) \text{ and} \\ \displaystyle\int_{D^2(x_0,r_1)\setminus\mathbb{D}^2(x_0,r_0)} \frac{1}{r} \mathrm{d}\mathscr{N}_{x_0,\varepsilon} \underset{n\to+\infty}{\to} \int_{\mathbb{D}^2(x_0,r_1)\setminus\mathbb{D}^2(x_0,r_0)} \frac{1}{r} \mathrm{d}\mathscr{N}_{x_0,\star}. \end{cases} \tag{9.1}$$

Passing to the limit $n \to +\infty$ in (83) and combining with (9.1), we obtain the identity (84).

9.2 First properties of $\mathscr{N}_{x_0,\star}$

Let $\mu_{\star,\theta,\theta}$ and $\mu_{\star,r,r}$ be defined by (85) on $\mathbb{D}^2(x_0, \rho)$, where (r, θ) denote the polar coordinates of $x = (x_1, x_2)$ with x_0 as the origin, so that $x_1 - x_{0,1} = r\cos\theta$ and $x_2 - x_{0,2} = r\sin\theta$. We denote by $\mu_{\star,\theta,\theta}^{ac}$ and $\mu_{\star,r,r}^{ac}$ the absolutely continuous parts of these measures with respect to the $\mathcal{H}^1$-Hausdorff measure $\mathrm{d}\lambda$ on $\mathfrak{S}_\star \cap \mathbb{D}^2(x_0, \rho)$. We prove in this subsection:

Lemma 9.1. *We have the relations*

$$\mathscr{N}^{ac}_{x_0,\star} = \left(2\zeta^{ac}_\star - r^{-2}\upmu^{ac}_{\star,\theta,\theta} + \upmu^{ac}_{\star,r,r}\right) \llcorner \mathbb{D}^2(x_0,\rho) = 4\sin^2\left(\upgamma_\star - \theta\right)\left(\zeta^{ac}_\star \llcorner \mathbb{D}^2(x_0,\rho)\right) \geq 0. \quad (9.2)$$

Remark 9.1. Let ∇r denote the gradient of the function

$$r : (x_1,x_2) \mapsto \sqrt{(x_1 - x_{0,1})^2 + (x_2 - x_{0,2})^2}, \text{ so that } \nabla r(x) = ((x_1 - x_{0,1})/r, (x_2 - x_{0,2})/r).$$

For given $x \in (\mathfrak{S}_\star \setminus \mathfrak{E}_\star) \cap \mathbb{D}^2(x_0,\rho)$, we denote by $\nabla^\perp r(x)$, the projection of $\nabla r(x)$ onto the line orthogonal to the tangent to $\mathfrak{S}_\star$ at the point x. We compute

$$|\nabla^\perp r(x)| = |\sin\left(\upgamma_\star(x) - \theta\right)|.$$

Formula (9.2) can therefore be rewritten as

$$\mathscr{N}^{ac}_{x_0,\star} = 4|\nabla^\perp r|^2 \zeta^{ac}_\star \geq 0. \quad (9.3)$$

Proof of Lemma 9.1. We may write

$$\upmu^{ac}_{\star,r,r} = \mathbb{m}_{\star,r,r} d\lambda \text{ and } r^{-2}\upmu^{ac}_{\star,\theta,\theta} = r^{-2}\mathbb{m}_{\star,\theta,\theta} d\lambda, \quad (9.4)$$

where, similar to (85), we have set, for $x \in (\mathfrak{S}_\star \setminus \mathfrak{E}_\star) \cap \mathbb{D}^2(x_0,\rho)$,

$$\begin{cases} \mathbb{m}_{\star,r,r}(x) = \cos^2\theta(x)\mathbb{m}_{\star,1,1}(x) + \sin^2\theta(x)\mathbb{m}_{\star,2,2}(x) + 2\sin\theta\cos\theta(x)\mathbb{m}_{\star,1,2}(x) \\ r^{-2}\mathbb{m}_{\star,\theta,\theta}(x) = \sin^2\theta(x)\mathbb{m}_{\star,1,1}(x) + \cos^2\theta(x)\mathbb{m}_{\star,2,2}(x) - 2\sin\theta(x)\cos\theta(x)\mathbb{m}_{1,2}(x). \end{cases} \quad (9.5)$$

We have, in view of Lemma 2 and relations (8.63)

$$\omega^{ac}_{\star,\theta} = -2\exp(2i(\upgamma_\star(x) - \theta))\zeta^{ac}_\star, \text{ on } \mathbb{D}^2(x_0,\rho). \quad (9.6)$$

Since $\omega^{ac}_{\star,\theta}$ is absolutely continuous with respect to $d\lambda$, we may write $\omega^{ac}_{\star,\theta} = \mathbb{w}_{\star,\theta} d\lambda$, where $\mathbb{w}_{\star,\theta}$ is a function on $\mathfrak{S}_\star \cap \mathbb{D}^2(x_0,\rho)$. Concerning the measure $\mathscr{N}^{ac}_{x_0,\star}$, we have

$$\mathscr{N}^{ac}_{x_0,\star} = \left(2\Theta_\star - r^{-2}\mathbb{m}_{\star,\theta,\theta} + \mathbb{m}_{\star,r,r}\right) d\lambda. \quad (9.7)$$

It follows from the definitions (9.5) and (9.4), that we have the identity

$$\mathbb{w}_{\star,\theta}(x) = \left(\mathbb{m}_{\star,r,r}(x) - r^{-2}\mathbb{m}_{\star,\theta,\theta}(x)\right) - 2ir^{-1}\mathbb{m}_{\star,r,\theta}(x). \quad (9.8)$$

Combining (9.6) and (9.8), we are hence led to

$$\mathbb{m}_{\star,r,r}(x) - r^{-2}\mathbb{m}_{\star,\theta,\theta}(x) = -2\cos\left(2(\upgamma_\star(x) - \theta)\right)\Theta_\star(x),$$

so that

$$\begin{aligned} \left(2\Theta_\star(x) - r^2\mathbb{m}_{\star,\theta,\theta}(x) + \mathbb{m}_{\star,r,r}(x)\right) &= 2(1 - \cos\left(2(\upgamma_\star(x) - \theta)\right)\Theta_\star(x) \\ &= 4\sin^2\left(\upgamma_\star(x) - \theta\right)\Theta_\star(x). \end{aligned}$$

Going back to (9.7), we deduce that

$$\mathscr{N}^{ac}_{x_0,\star} = 4\sin^2\left(\upgamma_\star - \theta\right)\zeta^{ac}_\star,$$

so that (9.2) is established. $\square$

9.3 Integrating on growing disks

Let $\rho > \delta > 0$ be given. We introduce and study in this section the scalar functions Z, F and G_δ defined on the interval $[\delta, \rho]$, by

$$\begin{cases} Z(r) = \zeta_\star(\mathbb{D}^2(x_0, r)), \\ F(r) = \dfrac{Z(r)}{r} \text{ and} \\ G_\delta(r) = \displaystyle\int_{D^2(x_0,r)\setminus\mathbb{D}^2(x_0,\delta)} \frac{1}{|x - x_0|} \mathrm{d}\mathscr{N}_{x_0,\star}, \text{ for } r \in [\delta, \rho]. \end{cases} \tag{9.9}$$

The three functions defined in (9.9) are clearly bounded on the interval $[\delta, \rho]$, since, for any $\delta < r \leq \rho$, we have

$$\begin{cases} 0 \leq Z(r) \leq Z(\rho) \leq \nu_\star(\mathbb{D}^2(x_0, \rho)),\ 0 \leq F(r) \leq Z(\rho)/\delta, \text{ and} \\ |G_\delta(r)| \leq 2\nu_\star(\mathbb{D}^2(x_0, \rho))/\delta. \end{cases}$$

Moreover, the function Z is clearly *non-decreasing*. We will show below that these functions have bounded variation. In order to relate these functions and their derivatives to the measures on $\mathbb{D}^2(x_0, \rho)$ introduced so far, we have to eliminate the polar angle θ. For that purpose, we consider the map $\Pi : \mathbb{D}^2(x_0, \rho) \setminus \{0\} \to (0, \rho)$ defined by

$$\Pi(x) = r = \sqrt{(x_1 - x_{0,1})^2 + (x_2 - x_{0,2})^2}, \text{ for } x = (x_1, x_2) \in \mathbb{D}^2(x_0, \rho).$$

Hence, we have $\Pi^{-1}(\varrho) = \mathbb{S}^1(x_0, \varrho)$, for any $\varrho \in (0, r]$. We define the measures $\check{\zeta}_\star$ and $\check{\mathscr{N}}_\star$ on $[\delta, \rho)$ by

$$\check{\zeta}_\star = \Pi_\sharp(\zeta_\star) \text{ and } \check{\mathscr{N}}_{x_0,\star} = \Pi_\sharp(\mathscr{N}_{x_0,\star}),$$

so that, for any Borel subset of (δ, ρ), we have

$$\check{\zeta}_\star(A) = \zeta_\star\left(\Pi^{-1}(A)\right) \text{ and } \check{\mathscr{N}}_{x_0,\star}(A) = \left(\Pi^{-1}(A)\right). \tag{9.10}$$

We first have:

Lemma 9.2. *The function Z and G_δ have bounded variation. We have*

$$\frac{\mathrm{d}}{\mathrm{d}r} Z = \check{\zeta}_\star \geq 0, \text{ and } \frac{\mathrm{d}}{\mathrm{d}r} G_\delta = r^{-1}\check{\mathscr{N}}_\star, \text{ in the sense of distributions } \mathcal{D}'(\delta, \rho). \tag{9.11}$$

Proof. We first observe that, as a consequence of the definitions (9.9) and (9.10), we have the identities

$$Z(r) = \check{\zeta}_\star([0, r)) = \int_0^r \mathrm{d}\check{\zeta}_\star = \int_0^\rho \mathbf{1}_{[0,r)} \mathrm{d}\check{\zeta}_\star.$$

The desired result (9.11) is then a direct consequence of Fubini's Theorem. Indeed, let $\varphi \in C_c(\delta, \rho)$. We have

$$\begin{aligned} \langle Z, \varphi' \rangle_{\mathcal{D}(\delta,\rho),\mathcal{D}'(\delta,\rho)} &= \int_\delta^\rho \varphi'(r) Z(r) \mathrm{d}r \\ &= \int_0^\rho \varphi'(r) \left[\int_0^r \mathrm{d}\check{\zeta}_\star\right] \mathrm{d}r = \iint_{(0,\rho)\times(0,\rho)} \varphi'(r) \mathbf{1}_{[0,r)} \mathrm{d}\check{\zeta}_\star \mathrm{d}r \\ &= \int_{[0,\rho)} \left[\int_0^\rho \varphi'(r) \mathbf{1}_{[0,r)} \mathrm{d}r\right] \mathrm{d}\check{\zeta}_\star = -\int_{[0,\rho)} \varphi(r) \mathrm{d}\check{\zeta}_\star \\ &= -\langle \zeta_\star, \varphi \rangle_{\mathcal{D}(\delta,\rho),\mathcal{D}'(\delta,\rho)}. \end{aligned} \tag{9.12}$$

Which establishes the first identity in (9.11). The second is proved using the same argument. Finally, since $\check{\zeta}_\star$ and $r^{-1}\check{\mathscr{N}}_{x_0,\star}$ are bounded measures on $[\delta, \rho]$, it follows that the functions Z and G_δ have bounded variation. □

For the proof of Proposition 6, we will make use of the fact that the derivative of F may be written in two different ways, as stated in the next Lemma.

Lemma 9.3. *The function F has bounded variation. We have the identities*

$$\frac{\mathrm{d}}{\mathrm{d}r}F = \frac{1}{r}\check{\zeta}_\star - \frac{1}{r^2}Z = \frac{1}{4r}\check{\mathscr{N}}_{x_0,\star}, \text{ in the sense of distributions } \mathcal{D}'(\delta, \rho). \tag{9.13}$$

Proof. The first identity in (9.13) corresponds to the Leibniz rule applied to the product $F = \dfrac{Z}{r}$ of the *measure* Z, handled as a *distribution* on (δ, ρ), by the *smooth function* $r \mapsto \dfrac{1}{r}$. It yields

$$\frac{\mathrm{d}}{\mathrm{d}r}F = -\frac{Z}{r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}Z, \text{ in the sense of distributions,}$$

so that the first identity in (9.13) follows, in view of the first identity in (9.11).

For the second identity, we invoke Lemma 3, which asserts that, for almost every $r \in (\delta, \rho)$, we have

$$F(r) - F(\delta) = \int_{\mathbb{D}^2(x_0,r)\setminus\mathbb{D}^2(x_0,\delta)} \frac{1}{4|x - x_0|}\mathrm{d}\mathscr{N}_{x_0,\star} = \frac{G_\delta(r)}{4}. \tag{9.14}$$

Taking the derivative, in the sense of distributions, of this identity, the second identity in (9.13) then follows from the second identity in (9.11). □

9.4 Refined analysis of the derivative of F: Proof of Proposition 6

In this subsection, we take advantage of the two different forms of the derivative F', namely $F' = r^{-1}\check{\mathscr{N}}_{x_0,\star}$ and $F' = \check{\zeta}_\star/r - Z/r^2$ provided by Lemma 9.3, in order to show that this distribution is actually a *non-negative measure*, yielding the main statement in Proposition 6. We first have:

Lemma 9.4. *Set $\mathbb{B}_\rho = \Pi^{-1}\left(\mathfrak{E}_\star \cap \mathbb{D}^2(x_0, \rho)\right)$. We have $\mathcal{H}^1(\mathbb{B}_\rho) = 0$ and*

$$\check{\mathscr{N}}_{x_0,\star} \llcorner ((0, \rho) \setminus \mathbb{B}_\rho) \geq 0. \tag{9.15}$$

Proof. Since $\mathcal{H}^1(\mathfrak{E}_\star) = 0$, we deduce that $\mathcal{H}^1(\mathbb{B}_\rho) = 0$. Recall that

$$\mathscr{N}_{x_0,\star} = \mathscr{N}^{ac}_{x_0,\star} \text{ on } \mathbb{D}^2(\rho) \setminus \mathfrak{E}_\star, \tag{9.16}$$

whereas in view of Lemma 9.1, we have $\mathscr{N}^{ac}_{x_0,\star} \geq 0$. Combining this inequality with (9.16) we obtain

$$\mathscr{N}_{x_0,\star} \llcorner \left(\mathbb{D}^2(x_0, \rho) \setminus \mathfrak{E}_\star\right) \geq 0. \tag{9.17}$$

In view of the definition of $\check{\mathscr{N}}_{x_0,\star}$, we obtain hence (9.15). □

It remains to study $\check{\mathscr{N}}_{x_0,\star} \llcorner \mathbb{B}_\rho$. We have:

Lemma 9.5. *The restriction of* $\check{\mathscr{N}}_{x_0,\star}$ *to* $\mathbb{B}_\rho$ *is non-negative, i.e.*

$$\check{\mathscr{N}}_{x_0,\star} \llcorner \mathbb{B}_\rho \geq 0. \tag{9.18}$$

Proof. Recall, that, in view of Lemma 9.3, we have in the sense of distributions

$$\check{\mathscr{N}}_{x_0,\star} = 4\left(\check{\zeta}_\star - \frac{Z}{r}\right), \text{ in } \mathcal{D}'(\delta, \rho). \tag{9.19}$$

Since both sides of (9.19) are bounded measures, the identity in (9.19) is also an identity of measures. Since Z is a bounded function, it follows from the fact that $\mathbb{B}_\rho$ has vanishing one-dimensional Lebesgue measure that

$$\frac{Z}{r} \llcorner \mathbb{B}_\rho = 0 \text{ and hence } \check{\mathscr{N}}_{x_0,\star} \llcorner \mathbb{B}_\rho = \check{\zeta}_\star \llcorner \mathbb{B}_\rho \geq 0.$$

□

Proof of Proposition 6 completed. Combining (9.15) and (9.18), we obtain that

$$\check{\mathscr{N}}_{x_0,\star} \geq 0 \text{ on } (0, \rho).$$

Since $F' = r^{-1}\check{\mathscr{N}}_{x_0,\star}$, we deduce that $F' \geq 0$ on $(0, \rho)$, so that F is non-decreasing. Inequality (81) follows. The other statements of Proposition 6 are then straightforward consequence of the former, so that the proof is complete. □

9.5 Proofs of Theorems 4 and 5

Recall that at this stage we already know, thanks to Proposition 6 that the measure $\zeta_\star$ is absolutely continuous with respect to the measure $\mathrm{d}\lambda$. We next derive the same statement for the measure $\nu_\star$, thanks to a comparison with the measure $\zeta_\star$ relying on our PDE analysis developed in Part II.

9.5.1 An upper bound for the measure $\nu_\star$

It follows from the very definition of the measures $\zeta_\star$ and $\nu_\star$ that we have the inequality $\zeta_\star \leq \nu_\star$. Indeed, we have, for every $\varepsilon > 0$, the straightforward inequality $\zeta_\varepsilon \leq \nu_\varepsilon$. We next present a reverse inequality:

Lemma 9.6. *Let* $x_0 \in \Omega$ *and* $r > 0$ *be such that* $\mathbb{D}^2(x_0, r) \subset \Omega$. *Then we have*

$$\nu_\star\left(\mathbb{D}^2(x_0, \frac{r}{2}\right) \leq \mathrm{K_V}(\mathrm{d}(x_0))\zeta_\star\left(\mathbb{D}^2(x_0, r)\right), \tag{9.20}$$

where $\mathrm{d}(x_0) = \mathrm{dist}(x_0, \partial\Omega)$ *and where the constant* $\mathrm{K}_V > 0$, *depending only on* V, M_0 *and* $\mathrm{d}(x_0)$, *has been introduced in Proposition 4.5.*

Proof. The result is an immediate consequence of Proposition 4.5 for the solution u_ε. Indeed, for $n \in \mathbb{N}$, we have the inequality

$$\nu_{\varepsilon_n}\left(\mathbb{D}^2(x_0, \frac{r}{2})\right) \leq \mathrm{K}_V\left(\mathrm{dist}(x_0, \partial\Omega)\right)\left[\zeta_{\varepsilon_n}\left(\mathbb{D}^2(x_0, \frac{3r}{4})\right) + \frac{\varepsilon_n}{r}\nu_{\varepsilon_n}\left(\mathbb{D}^2(x_0, \frac{r}{2})\right)\right].$$

Letting $n \to +\infty$, we are led to the inequality

$$\nu_\star\left(\mathbb{D}^2(x_0, \frac{r}{2})\right) \leq \mathrm{K}_V\left(\mathrm{dist}(x_0, \partial\Omega)\right) \zeta_\star\left(\overline{\mathbb{D}^2(x_0, \frac{3r}{4})}\right),$$

which yields (9.20). □

An immediate consequence is:

Corollary 9.1. *The measure $\nu_\star$ is absolutely continuous with respect to the measure $\mathrm{d}\lambda = \mathcal{H}^1 \llcorner \mathfrak{S}_\star$. Moreover, we have, writing $\nu_\star = \mathbb{e}_\star \mathrm{d}\lambda$,*

$$\mathbb{e}_\star(x_0) \leq \mathrm{K}_V\left(\mathrm{dist}(x_0, \partial\Omega)\right) \Theta_\star(x_0), \text{ for } \lambda\text{−almost every } x_0 \in \mathfrak{S}_\star. \tag{9.21}$$

Proof. We have, for every $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, the identity

$$\begin{aligned}\overline{D}_\lambda(\nu_\star)(x_0) &\equiv \limsup_{r\to 0} \frac{\nu_\star\left(\mathbb{D}^2(x_0, r)\right)}{2r} = \limsup_{r\to 0} \frac{\nu_\star\left(\mathbb{D}^2(x_0, \frac{r}{2})\right)}{r} \\ &\leq \mathrm{K_V}(\mathrm{d}(x_0)) \limsup_{r\to 0} \frac{\zeta_\star\left(\mathbb{D}^2(x_0, r)\right)}{r} = 2\mathrm{K_V}(\mathrm{d}(x_0))\Theta_\star(x_0),\end{aligned} \tag{9.22}$$

where we used Lemma 9.6 for the second line. It follows that $\overline{D}_\lambda(\nu_\star)(x_0)$ is locally bounded for λ-almost every $x_0 \in \Omega$, so that $\nu_\star$ is absolutely continuous with respect to λ. Since

$$\mathbb{e}_\star(x_0) = \overline{D}_\lambda(\nu_\star)(x_0),$$

for λ-almost every $x_0 \in \mathfrak{S}_\star$, inequality (9.21) follows from (9.22). □

9.5.2 Proof of Theorem 4

In view of Proposition 6, we know that $\zeta_\star$ is absolutely continous with respect to λ, whereas the same conclusion holds for $\nu_\star$, in view of Corollary 9.1. All inequalities in (40) follow from either (9.21) or (82), choosing $\mathrm{K_{dens}} = K_V$, except the first one, namely $\eta_1 \leq \mathbb{e}_\star(x)$. The latter inequality is a consequence of the clearing-out theorem (Theorem 7), and the definition (66) of $\mathfrak{S}_\star$.

9.5.3 Proof of Theorem 5

Theorem 5 is an immediate consequence of Proposition 5 combined with the fact that all measures are absolutely continuous with respect to the measure $\mathcal{H}^1 \llcorner \mathfrak{S}_\star$ (so that the singular parts actually vanish).

10 Proof of Theorem 2

The argument consists, for a large part, in revisiting the analysis provided in Section 8, taking however now into account the results obtained in Section 9, in particular the fact that all measures at stake are absolutely continuous with respect to $\mathcal{H}^1 \llcorner \mathfrak{S}_\star$. We present several observations which are relevant for the proof as separate subsections. Given an orthonormal basis $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$, we introduce the measure

$$\mathbf{N}_\star = 2\zeta_\star - \mu_{\star,2,2} + \mu_{\star,1,1} \text{ on } \Omega. \tag{10.1}$$

This measure plays in cartesian coordinates the same role as $\mathscr{N}_{x_0,\star}$ in radial coordinates, and as mentioned, depends on the choice of the basis. Properties of $\mathbf{N}_\star$ are the starting point of the proof.

10.1 Properties of $\mathbf{N}_\star$ and their consequences

Similar to inequality (9.17), we have:

Lemma 10.1. *Given any orthonormal basis $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$, we have the inequality*

$$\mathbf{N}_\star = 4\sin^2\gamma_\star\zeta_\star \geq 0. \tag{10.2}$$

Proof. Combining Lemma 2 with Theorem 4, we obtain, for $\omega_\star = (\mu_{\star,1,1} - \mu_{\star,2,2}) - 2i\mu_{\star,1,2}$

$$\omega_\star = -2\exp(-2i\gamma_\star)\zeta_\star = -2(\cos 2\gamma_\star - i\sin 2\gamma_\star)\zeta_\star,$$

so that

$$\mu_{\star,1,1} - \mu_{\star,2,2} = -2(\cos 2\gamma_\star)\zeta_\star \text{ and } \mu_{\star,1,2} = -(\sin 2\gamma_\star)\zeta_\star. \tag{10.3}$$

It follows from (10.3) that

$$\mathbf{N}_\star = 2\zeta_\star - \mu_{\star,2,2} + \mu_{\star,1,1} = 2\left(1 - \cos(2\gamma_\star)\right)\zeta_\star,$$

so that the proof is complete. □

Next, consider a point $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$, so that a tangent exists. We assume moreover that the orthonormal basis $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$ is chosen so that $\vec{\mathbf{e}}_1 = \vec{e}_{x_0}$. An immediate consequence of Lemma 10.1 is the following.

Corollary 10.1. *Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$ and $\rho_0 > 0$ be the number provided by Proposition 8.4. Then we have*

$$\mathbb{N}_{x_0,\rho_0} \geq 0 \text{ and, more generally, } \mathbb{N}_{x_0,k,\rho_0} \geq 0, \text{ if } k \text{ is even.} \tag{10.4}$$

Proof. We set

$$\tilde{\mathbf{N}}_\star = \mathbf{1}_{Q_{\rho_0}(x_0)}\mathbf{N}_\star = 2\tilde{\zeta}_\star + \tilde{\mu}_{\star,1,1} - \tilde{\mu}_{\star,2,2},$$

Recall that, if $\mathbb{P}$ denotes the orthogonal projection onto the line the tangent line $\mathrm{D}^1_{x_0} = \{x_0 + s\vec{\mathbf{e}}_1, s \in \mathbb{R}\}$, then, we have, in view of the definition (80),

$$\mathbb{N}_{x_0,\rho_0} = \mathbb{P}_\sharp(2\tilde{\zeta}_\star + \tilde{\mu}_{\star,1,1} - \tilde{\mu}_{\star,2,2}) = \mathbb{P}_\sharp(\tilde{\mathbf{N}}_\star), \text{ and } \mathbb{N}_{x_0,k,\rho_0} = \mathbb{P}_\sharp\left((x_2 - x_{0,2})^k\tilde{\mathbf{N}}_\star\right).$$

If k is even, then if follows from (10.2) that $(x_2 - x_{0,2})^k\tilde{\mathbf{N}}_\star \geq 0$, hence (10.4) follows. □

For $k \geq 0$, let J_{k,ρ_0} be defined on $\mathcal{I}_{\rho_0}(x_{0,1})$ by identity (8.17) in Corollary 8.1. As a consequence of Corollary 10.1, we have:

Lemma 10.2. *Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$ and $\rho_0 > 0$ be the number provided by Proposition 8.4. Assume that k is odd. Then, the function J_{k,ρ_0} is monotone, non-increasing on $\mathcal{I}_{\rho_0}(x_{0,1})$.*

Indeed, we have, in view of (8.15) and (8.17), in the case k is even,

$$-2\frac{\mathrm{d}}{\mathrm{d}s}J_{k,\rho_0} = k\mathbb{N}_{k-1,\rho} \geq 0 \text{ in } \mathcal{D}'((x_{0,1}-\rho_0, x_{0,1}+\rho_0)), \tag{10.5}$$

And the conclusion follows. □

10.2 Additional properties of the functions J_{k,ρ_0}

For $s \in \mathcal{I}_{\rho_0}(x_{0,1})$, we introduce the subset $\Lambda(s)$ of $\mathfrak{S}_\star$ defined by

$$\begin{cases} \Lambda(s) = \mathbb{P}^{-1}(s) \cap \mathfrak{S}_\star \cap Q_{\rho_0}(x_0) = (\{(s, x_{0,2})\} + [-\rho_0\vec{\mathbf{e}}_2, \rho_0\vec{\mathbf{e}}_2]) \cap \mathfrak{S}_\star, \text{ and set} \\ \mathcal{Z}(s) = \sharp(\Lambda(s)). \end{cases}$$

$\Lambda(s)$ represents the set of points in $Q_{\rho_0}(x_0) \cap \mathfrak{S}_\star$ whose orthogonal projection onto the line $x_0 + \mathbb{R}\vec{\mathbf{e}}_1$ is the point $(s, x_{0,2})$. Since $\mathfrak{S}_\star$ is connected, we have

$$\Lambda(s) \neq \emptyset \text{ and hence } \mathcal{Z}(s) \geq 1 \text{ for } s \in \mathcal{I}_{\rho_0}(x_{0,1}). \tag{10.6}$$

Lemma 10.3. *Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$ and $\rho_0 > 0$ be the number provided by Proposition 8.4. For almost every $s \in \mathcal{I}_{\rho_0}(x_{0,1})$, the number $\mathcal{Z}(s)$ is finite. If $\Lambda(s)$ is finite, then we have, for $k \in \mathbb{N}$,*

$$J_{k,\rho_0}(s) = -2 \sum_{a(s)=(s,a_2(s))\in\Lambda(s)} (x_{0,2} - a_2(s))^k \sin(\gamma_\star(a(s))\Theta_\star(a(s)). \tag{10.7}$$

Proof. It follows from (10.3) that

$$\tilde{\mu}_{\star,1,2} = -\sin(2\gamma_\star)\tilde{\zeta}_\star = -\mathbf{1}_{Q_{\rho_0}(x_0)} \sin(2\gamma_\star)\Theta_\star \mathrm{d}\lambda.$$

In view of the definition of J_{k,ρ_0}, we obtain therefore, since $J_{k,\rho_0}\mathrm{d}s = \mathbb{J}_{k,\rho_0}$

$$J_{k,\rho_0}\mathrm{d}s = -\mathbb{P}_\sharp\left((x_{0,2}-x_2)^k \,\tilde{\mu}_{\star,1,2}\right) = -\mathbb{P}_\sharp\left((x_{0,2}-x_2)^k \,\mathbf{1}_{Q_{\rho_0}(x_0)} \sin(2\gamma_\star)\Theta_\star \mathrm{d}\lambda\right). \tag{10.8}$$

Next assume that $\Lambda(s)$ is finite: Given any point $a(s) \in \Lambda(s)$, we may find some arbitrary small number $\delta > 0$ such find $(\mathfrak{S}_\star \cap \mathbb{D}^2(x_0, \delta)) \cap \Lambda(s) = \{a(s)\}$. If $a(s) \notin \mathfrak{E}_\star$, then the angle of the tangent to $\mathfrak{S}_\star$ at the point $a(s)$ with the vector $\vec{\mathbf{e}}_1$ is $\gamma_\star(a(s))$ so that, if $\gamma_\star(a(s)) \neq \pm\pi/2$, then we have

$$\frac{\mathrm{d}\mathbb{P}_\sharp\left(\mathbf{1}_{\mathbb{D}^2(a(s))}\mathrm{d}\lambda\right)}{\mathrm{d}s} = \frac{1}{\cos(\gamma_\star(a(s))}. \tag{10.9}$$

Since $\sin(2\gamma_\star(a(s))) = 2\sin(\gamma_\star(a(s))).\cos(\gamma_\star(a(s)))$, the conclusion follows combining (10.8) and (10.9). □

As a consequence, we obtain:

Lemma 10.4. *Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$ and $\rho_0 > 0$ be the number provided by Proposition 8.4. Let $s \in \mathcal{I}_{\rho_0}(x_{0,1})$ be such that $\mathcal{Z}(s) = 1$. Then, we have $J_{k,\rho_0}(s) = 0$, for any $k \in \mathbb{N}$.*

Proof. We already know, by Proposition 8.6, that $J_{0,\rho_0}(s) = 0$, so that it remains to establish the property for $k \geq 1$. Since, by assumption, $\Lambda(s)$ contains a unique element $a(s) = (s, a_2(s))$, we have, by (10.7)

$$J_{k,\rho_0}(s) = -2\left(x_{0,2} - a_2(s)\right)^k \sin(\gamma_\star(a(s))\Theta_\star(a(s)).$$

Applying this formula to the case $k = 0$, we obtain $J_{0,\rho_0}(s) = -2\sin(\gamma_\star(a(s))\Theta_\star(a(s))$, so that

$$J_{k,\rho_0}(s) = -2\left(x_{0,2} - a_2(s)\right)^k J_{0,\rho_0}(s) = 0, \forall s \in \mathcal{I}_\rho(x_{0,1}).$$

The proof is hence complete. □

Lemma 10.4 motivates to introduces, for $0 < \rho \leq \rho_0$, the set

$$\mathcal{G}(\rho) = \{s \in [x_{0,1} - \rho, x_{0,1} + \rho], \text{ such that } \mathcal{Z}(s) = 1\}$$

It plays a distinguished role in the proof of Theorem 2, as well as our next result shows.

10.3 Properties of the set $\mathcal{G}(\rho_0)$

We first show that, for $0 < \rho \leq \rho_0$, the set $\mathcal{G}(\rho)$, if not empty, is an interval.

Proposition 10.1. *Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$ and $\rho_0 > 0$ be the number provided by Proposition 8.4. Let $0 < \rho \leq \rho_0$, and s_1 and s_2 satisfying $s_1 < x_{0,1} < s_2$ be in $\mathcal{G}(\rho)$. Then $[s_1, s_2] \subset \mathcal{G}(\rho)$, and we have*

$$\mathfrak{S}_\star \cap ([s_1, s_2] \times \mathcal{I}_{\rho_0}(x_{0,2})) = [s_1, s_2] \times \{x_{0,2}\}. \tag{10.10}$$

Moreover, there exists a number $L_{0,\rho} > 0$ such that, we have

$$\zeta_\star \llcorner ((s_1, s_2) \times \mathcal{I}_{\rho_0}(x_{0,2})) = L_{0,\rho} \mathrm{d}\lambda, \tag{10.11}$$

with

$$\frac{\eta_1}{\mathrm{K}_{\mathrm{dens}}(\mathrm{d}(x_0))} \leq L_{0,\rho} < \frac{M_0}{\mathrm{dist}(x_0, \partial\Omega)} \tag{10.12}$$

The proof relies on the following more technical result:

Lemma 10.5. *Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$ and $\rho_0 > 0$ be the number provided by Proposition 8.4. Let $0 < \rho \leq \rho_0$, and s_1 and s_2 satisfying $s_1 < x_{0,1} < s_2$ be in $\mathcal{G}(\rho)$. Then, we have*

$$\begin{cases} \mathbb{N}_{k,\rho_0} \llcorner (s_1, s_2) = 0, \text{ for } k \in \mathbb{N}, \text{ and} \\ J_{k,\rho_0}(s) = 0 \text{ and } L_{k,\rho_0}(s) = 0, \text{ for every } s \in [s_1, s_2] \text{ and } k \geq 1. \end{cases} \tag{10.13}$$

Proof of Lemma 10.5. Since $\mathcal{Z}(s_1) = \mathcal{Z}(s_2) = 1$, we may apply Lemma 10.4 to s_1 et s_2, with $k = 1$, to assert that

$$J_{1,\rho_0}(s_1) = J_{1,\rho_0}(s_2) = 0.$$

On the other hand, since, in view of Lemma 10.2, the function J_{1,ρ_0} is *monotone* on $\mathcal{I}_{\rho_0}$, we deduce that

$$J_{1,\rho_0}(s) = 0 \text{ for } s \in [s_1, s_2], \text{ and hence } \mathbb{N}_{\rho_0} = -2\frac{\mathrm{d}}{\mathrm{d}s} J_{1,\rho_0} = 0 \text{ in } \mathcal{D}'((s_1, s_2)). \tag{10.14}$$

It follows from the second identity in (10.14), the definition (80) of $\mathbb{N}_{\rho_0}$ and (10.2), that the restriction of the measure $\tilde{\mathbf{N}}_\star$ to $(s_1, s_2) \times \mathcal{I}_{\rho_0}(x_{0,2})$ vanishes. This implies the first identity in (10.13), where $\mathbb{N}_{k,\rho_0}$ is defined in (8.13). For the second, we notice that, in view of the first differential equation in (8.15), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{J}_{k,\rho_0} = 0 \text{ on } (s_1, s_2), \text{ for } k \geq 1.$$

Since $J_{k,\rho_0}(x_{0,1}) = 0$, for $k \geq 1$, it follows that $J_{k,\rho_0}(s) = 0$, for every $s \in (s_1, s_2)$. Similarly, invoking the second relation in (8.15), that is $-\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{L}_{k,\rho_0} = 2k\mathbb{J}_{k-1,\rho_0}$,for $k \geq 1$, and the fact that $L_{k,\rho_0}(x_{0,1}) = 0$, we deduce that $L_{k,\rho_0}(s) = 0$ for every $s \in (s_1, s_2)$, for $k \in \mathbb{N}^*$. □

Proof of Proposition 10.1 completed. Combining the first and the third identities in (10.13) with (8.14) we deduce that

$$\frac{1}{4}(\mathbb{N}_{k,\rho} + \mathbb{L}_{k,\rho}) \llcorner (s_1, s_2) = \mathbb{P}_\sharp\left((x_2 - x_{0,2})^k\, \tilde{\zeta}_\star\right) \llcorner (s_1, s_2) = 0, \text{ for } k \in \mathbb{N}^*. \tag{10.15}$$

Choosing $k = 2$ in (10.15), we obtain hence that $(x_2 - x_{0,2})^2\, \zeta_\star = 0$ on $(s_1, s_2) \times \mathcal{I}_{\rho_0}(x_{0,2})$, so that

$$\zeta_\star \llcorner (s_1, s_2) \times (\mathcal{I}_{\rho_0}(x_{0,2}) \setminus \{x_{0,2}\}) = 0,$$

Since, in view of Theorem 4, $\zeta_\star = \Theta_\star \mathrm{d}\lambda \geq \eta_1 \mathrm{d}\lambda$, we deduce that

$$\mathrm{d}\lambda \llcorner (s_1, s_2) \times (\mathcal{I}_{\rho_0}(x_{0,2}) \setminus \{x_{0,2}\}) = 0,$$

so that (10.10) follows. Next, we use (8.14) with $k = 0$, so that

$$\mathbb{P}_\sharp\left(\tilde{\zeta}_\star\right) \llcorner (s_1, s_2) = \frac{1}{4}(\mathbb{N}_{0,\rho} + \mathbb{L}_{0,\rho}) \llcorner (s_1, s_2) = L_{0,\rho}\mathrm{d}x_1 \llcorner (s_1, s_2), L_{0,\rho} \in \mathbb{R}, \tag{10.16}$$

where for the last identity, we used the first identity in (10.13) and Proposition 8.4. Combining (10.16) with (10.10), we obtain (10.11). Since $\zeta_\star = \Theta_\star \mathrm{d}\lambda$, it follows from (10.11) that

$$L_{0,\rho} = \Theta_\star(x_0),$$

so that (10.12) follows from Theorem 4. Finally, the fact the $(s_1, s_2) \subset \mathcal{G}(\rho)$ is a direct consequence of (10.10) and the definition of $\mathcal{G}(\rho)$. □

10.4 On the measure of the set $\mathcal{G}(\rho)$

We show here that the set $\mathcal{G}(\rho)$ contains elements.

Lemma 10.6. *Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$ and $\rho_0 > 0$ be the number provided by Proposition 8.4. There exists $0 < \rho_1 \leq \rho_0$, such that we have the lower bound*

$$|\mathcal{G}(\rho)| \geq \frac{5\rho}{3}, \text{ for any } 0 < \rho \leq \rho_1.$$

Proof. We first notice that, since $\mathbb{P}_\star$ is a contraction, that for any $\rho \leq \rho_0$, we have

$$\begin{aligned} \int_{x_{0,1}-\rho}^{x_{0,1}+\rho} \mathcal{Z}(s)\mathrm{d}s &\leq \mathcal{H}^1(\mathfrak{S}_\star \cap Q_\rho(x_0)) \leq \mathcal{H}^1\left(\mathfrak{S}_\star \cap \mathbb{D}^2\left(x_0, \frac{\rho}{\cos\frac{\pi}{8}}\right)\right) \\ &\leq \mathcal{H}^1\left(\mathfrak{S}_\star \cap \mathbb{D}^2\left(x_0, \frac{10\rho}{9}\right)\right), \end{aligned} \tag{10.17}$$

where we used (8.54). On the other hand, in view of (11), there exists some $0 < \tilde{\rho}_1 \leq \rho_0$, such that, for $\rho \leq \tilde{\rho}_1$, we have

$$\mathcal{H}^1(\mathfrak{S}_\star \cap (\mathbb{D}^2(x_0, \rho)) \leq \frac{21\rho}{10}. \tag{10.18}$$

Combining (10.17) and (10.18), we obtain hence, for $\rho \leq \rho_1 \equiv \frac{9}{10}\tilde{\rho}_1$,

$$\int_{x_{0,1}-\rho}^{x_{0,1}+\rho} \mathcal{Z}(s)\mathrm{d}s \leq \frac{21}{10} \cdot \left(\frac{10\rho}{9}\right) = \frac{21\rho}{9} = \frac{7\rho}{3}. \tag{10.19}$$

We introduce the set $\mathcal{K}(\rho) = \{s \in [x_{0,1} - \rho, x_{0,1} + \rho], \text{ such that } \mathcal{Z}(s) \geq 2\}$. We have

$$\int_{x_{0,1}-\rho}^{x_{0,1}+\rho} \mathcal{Z}(s)\mathrm{d}s = \int_{\mathcal{G}(\rho)} \mathcal{Z}(s)\mathrm{d}s + \int_{\mathcal{K}(\rho)} \mathcal{Z}(s)\mathrm{d}s \geq |\mathcal{G}(\rho)| + 2|\mathcal{K}(\rho)| = 2\rho + |\mathcal{K}(\rho)|. \tag{10.20}$$

Combining (10.19) and (10.20), we deduce that $|\mathcal{K}(\rho)| \leq \frac{\rho}{3}$ and the conclusion follows. □

10.5 Proof of Theorem 2 completed

Let $x_0 \in \mathfrak{S}_\star \setminus \mathfrak{E}_\star$ and $\rho_0 > 0$ be the number provided by Proposition 8.4. We first invoke Lemma 10.6, so that there exists $0 < \rho_1 \leq \rho_0$ such that $|\mathcal{G}(\rho_1)| \geq 5\rho_1/3$. Hence there exists two numbers s_1 and s_2 such that

$$x_{0,1} - \rho_1 \leq s_1 < x_{0,1} - \frac{\rho_1}{3} \text{ and } x_{0,1} + \frac{\rho_1}{3} < s_2 \leq x_{0,1} + \rho_1, \text{ and such that } \{s_1, s_2\} \subset \mathcal{G}(\rho).$$

Setting $r_0 = \rho_1/3$, we obtain, in view of Proposition 10.1

$$\mathcal{I}_{r_0}(x_0) \subset (s_1, s_2) \subset \mathcal{G}(\rho).$$

Identity (10.10) of Proposition 10.1 then leads directly to identity (16), whereas identity (10.11) yields (17) and (10.12) yields the lower bound (18), with the choice

$$\eta_1(\mathrm{d}(x_0) = \frac{\eta_1}{\mathrm{K_{dens}}(\mathrm{d}(x_0))}.$$

The proof of Theorem 2 is hence complete.

11 Proof of Theorem 3

Inserting identities (10.3) into the system (72), we are led to the system of first-order equations

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left[(\sin 2\gamma_\star)\,\zeta_\star\right] = \dfrac{\partial}{\partial x_1}\left[(1+\cos 2\gamma_\star)\,\zeta_\star\right] \text{ and} \\ -\dfrac{\partial}{\partial x_1}\left[(\sin 2\gamma_\star)\zeta_\star\right] = \dfrac{\partial}{\partial x_2}\left[(1-\cos 2\gamma_\star)\,\zeta_\star\right]. \end{cases} \tag{11.1}$$

This system involves only the measure $\zeta_\star$ and the set $\mathfrak{S}_\star$ (through the angle $\gamma_\star$). We are going to show next that these relations are equivalent, in the sense of distributions, to (22). For that purpose, let $\vec{X} = (X_1, X_2)$ be a vector-field in $C_c^\infty(\Omega, \mathbb{R}^2)$. We have, for any $x \in \mathfrak{S} \setminus \mathfrak{E}_\star$, since by definition $\vec{e}_{x_0} = \cos\gamma(x_0)\vec{\mathbf{e}}_1 + \sin\gamma(x_0)\vec{\mathbf{e}}_2$

$$\begin{aligned} \operatorname{div}_{T_x\mathfrak{S}_\star}\vec{X}(x) &= \left(\vec{e}_x \cdot \vec{\nabla}\vec{X}(x)\right)\cdot \vec{e}_x \\ &= \left(\cos\gamma_\star(x)\frac{\partial \vec{X}}{\partial x_1}(x) + \sin\gamma_\star(x)\frac{\partial \vec{X}}{\partial x_2}(x)\right)\cdot\left(\cos\gamma_\star(x)\vec{\mathbf{e}}_1 + \sin\gamma_\star(x)\vec{\mathbf{e}}_2\right) \\ &= \cos^2\gamma_\star(x)\frac{\partial X_1}{\partial x_1}(x) + \sin^2\gamma_\star(x))\frac{\partial X_2}{\partial x_2}(x) \\ &+ \sin\gamma_\star(x)\cos\gamma_\star(x)\left[\frac{\partial X_2}{\partial x_1}(x) + + \frac{\partial X_1}{\partial x_2}(x)\right]. \end{aligned}$$

Using this computation, we may expand relation (22) as

$$\left\langle \zeta_\star, \cos^2\gamma_\star\frac{\partial X_1}{\partial x_1} + \sin^2\gamma_\star\frac{\partial X_2}{\partial x_2} + \sin\gamma_\star\cos\gamma_\star\left[\frac{\partial X_2}{\partial x_1} + \frac{\partial X_1}{\partial x_2}\right]\right\rangle = 0 \tag{11.2}$$

Integrating by parts in the sense of distributions, we obtain hence, for every $X_1 \in C_c^\infty(\Omega, \mathbb{R})$ and any $X_2 \in C_c^\infty(\Omega, \mathbb{R})$, the relation

$$\left\langle \frac{\partial}{\partial x_1}(\cos^2\gamma_\star\zeta_\star) + \frac{\partial}{\partial x_2}(\sin\gamma_\star\cos\gamma_\star\,\zeta_\star), X_1\right\rangle + \left\langle \frac{\partial}{\partial x_2}(\sin^2\gamma_\star\zeta_\star) + \frac{\partial}{\partial x_1}(\sin\gamma_\star\cos\gamma_\star\zeta_\star), X_2\right\rangle = 0.$$

Since X_1 and X_2 can be chosen independently, we are led to the system, in the sense of distributions,

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left[(\sin\gamma_\star\cos\gamma_\star, \zeta_\star\right] = \dfrac{\partial}{\partial x_1}\left[\left(\cos^2\gamma_\star\right)\zeta_\star\right] \text{ and} \\ -\dfrac{\partial}{\partial x_1}\left[(\sin\gamma_\star\cos\gamma_\star)\zeta_\star\right] = \dfrac{\partial}{\partial x_2}\left[\left(\sin^2\gamma_\star\right)\zeta_\star\right]. \end{cases} \tag{11.3}$$

Since $2\sin\gamma_\star\cos\gamma_\star = \sin 2\gamma_\star$, $1+\cos(2\gamma_\star) = 2\cos^2\gamma_\star$ and $1-\cos(2\gamma_\star) = 2\sin^2\gamma_\star$, we verify that (11.3) is equivalent to (11.1), so that the system (72) is equivalent to (22). The varifold $\mathbf{V}(\mathfrak{S}_\star, \Theta_\star)$ is hence stationary. The proof of Theorem 3 is complete.

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