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ON THE HOCHSCHILD HOMOLOGY OF SINGULARITY CATEGORIES

YU WANG, UMAMAHESWARAN ARUNACHALAM, AND BERNHARD KELLER

Abstract. Let \( k \) be an algebraically closed field and \( A \) a finite-dimensional \( k \)-algebra. In this note, we determine complexes which compute the Hochschild homology of the canonical dg enhancement of the bounded derived category of \( A \) and of the canonical dg enhancement of the singularity category of \( A \). As an application, we obtain a new approach to the computation of Hochschild homology of Leavitt path algebras.

1. Reminder on Hochschild homology of algebras and categories

Let \( k \) be a field. We write \( \otimes \) for \( \otimes_k \). Let \( A \) be a \( k \)-algebra (associative, with 1). We write \( \text{Mod} A \) for the category of all (right) \( A \)-modules and \( \mathcal{D} A = \mathcal{D}(\text{Mod} A) \) for its unbounded derived category. Let \( A^e = A \otimes A^{\text{op}} \) be the enveloping algebra of \( A \) so that \( A^e \)-modules identify with \( A \)-bimodules. The Hochschild homology of \( A \) is defined by

\[ \text{HH}_p(A) = \text{Tor}_{A^e}^p(A, A), \quad p \in \mathbb{Z}. \]

Alternatively, we may define it as the \( p \)th homology group of the Hochschild chain complex \( \text{HH}(A) \) of \( A \), i.e. the complex \( C_* A \) concentrated in homological degrees \( \geq 0 \)

\[ A \leftarrow A \otimes A \leftarrow \ldots \leftarrow A^\otimes p \leftarrow A^\otimes (p+1) \leftarrow \ldots \]

with \( C_p A = A^\otimes (p+1), p \geq 0, \) and differential given by

\[ d(a_0, \ldots, a_p) = \sum_{i=0}^{p-1} (-1)^i (a_0, \ldots, a_ia_{i+1}, \ldots, a_p) + (-1)^p (a_p a_0, \ldots, a_{p-1}), \]

where we write \((a_0, \ldots, a_p)\) for \( a_0 \otimes \cdots \otimes a_p \). Notice that the first differential takes \( a \otimes b \) to the commutator \( ab - ba \).

We see that \( \text{HH}_0(A) \) is the quotient \( A/[A, A] \) of the vector space \( A \) by its subspace generated by all commutators and that \( \text{HH}_p(A) \) and \( \text{HH}(A) \in \mathcal{D} k \) are functorial in the algebra \( A \). The definitions extend from \( k \)-algebras to small \( k \)-categories. For example, the Hochschild complex then becomes the complex

\[ \bigoplus \mathcal{A}(X_0, X_0) \leftarrow \bigoplus \mathcal{A}(X_1, X_0) \otimes \mathcal{A}(X_0, X_1) \leftarrow \ldots \]

whose \( p \)th term \((p \geq 0)\) is the sum

\[ \bigoplus \mathcal{A}(X_p, X_0) \otimes \mathcal{A}(X_{p-1}, X_p) \otimes \cdots \otimes \mathcal{A}(X_0, X_1) \]

taken over all sequences of objects \( X_0, X_1, \ldots, X_p \) of \( \mathcal{A} \) and whose horizontal differential is given by formula (1.0.1). One then shows that the inclusion \( A \to \text{proj} (A) \) of the one-object category given by \( A \) into the category \( \text{proj} (A) \) of finitely generated projective right

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$A$-modules induces a quasi-isomorphism
$$HH(A) \xrightarrow{\sim} HH(\text{proj } A).$$
In particular, this yields Morita invariance of Hochschild homology. The definitions further extend to small differential graded (=dg) categories $A$, for example the dg category $\mathcal{C}_{dg}^b(\text{proj } A)$ of bounded complexes over proj ($A$). We refer the reader to [9] for more information on this example and dg categories in general. The inclusion $\text{proj } (A) \to \mathcal{C}_{dg}^b(\text{proj } A)$ yields an isomorphism
$$HH(\text{proj } A) \xrightarrow{\sim} HH(\mathcal{C}_{dg}^b(\text{proj } A))$$
and this yields the invariance of Hochschild homology under derived equivalences. We will need the following localization theorem.

**Theorem 1.1 ([8]).** Let
$$A \xrightarrow{F} B \xrightarrow{G} C$$
be a sequence of dg categories such that the induced sequence of derived categories
$$0 \to \mathcal{D}A \xrightarrow{F^*} \mathcal{D}B \xrightarrow{G^*} \mathcal{D}C \to 0$$
is exact. Then there is a canonical triangle
$$HH(A) \xrightarrow{HH(F)} HH(B) \xrightarrow{HH(G)} HH(C) \to \Sigma HH(A)$$
in $\mathcal{D}k$ and hence long exact sequences in Hochschild (and cyclic) homology.

Let $Q$ be a finite quiver and $I$ an admissible ideal in $kQ$, i.e. a two-sided ideal contained in the square of the ideal generated by the arrows and such that the quotient $kQ/I$ is finite-dimensional. Let $R$ be the quotient of $A$ by its radical. Thus, as an $A$-module, the algebra $R$ is the direct sum of the simple $A$-modules. Following [7], we define the Koszul dual of $A$ to be the dg algebra
$$A^! = \text{RHom}_A(R, R).$$
Thus, if $P$ is a projective resolution of the $A$-module $R$, then the Koszul dual is quasi-isomorphic to the dg endomorphism algebra $\text{Hom}_A(P, P)$ of $P$. The following theorem is a special case of Corollary D.2 of Van den Bergh’s [12]. We write $D$ for the dual $\text{Hom}_k(?, k)$ over the ground field.

**Theorem 1.2 (Van den Bergh).** We have a canonical isomorphism
$$HH(A^!) \xrightarrow{\sim} DHH(A).$$

We refer to [6] for a comparison taking into account much more structure.

2. Hochschild homology of derived categories and singularity categories

Let $Q$ be a finite quiver and $I$ an admissible ideal in $kQ$. Let mod $A$ be the category of $k$-finite-dimensional right $A$-modules. Denote by $\mathcal{D}^b(A) = \mathcal{D}^b(\text{mod } A)$ the bounded derived category of $A$ and by per ($A$) the perfect derived category, i.e. the thick subcategory generated by the free $A$-module of rank 1. Following Buchweitz [2] and Orlov [10], one defines the singularity category of $A$ as the Verdier quotient
$$\text{sg}(A) = \mathcal{D}^b(A)/\text{per } (A).$$
Using the canonical dg enhancements of $\mathcal{D}^b(A)$ and per ($A$), cf. [9], we obtain a canonical exact sequence of dg categories
$$0 \to \text{per}_{dg}(A) \to \mathcal{D}^b_{dg}(A) \to \text{sg}_{dg}(A) \to 0 ,$$
where the dg quotient \( \text{sg}_{dg}(A) \) yields a canonical dg enhancement for \( \text{sg}(A) \). It is not hard to see that, in the homotopy category of dg categories, it is functorial with respect to bimodule complexes \( X \in \mathcal{D}(A^{op} \otimes B) \) such that \( XB \) is perfect over \( B \) and \( A \) is perfect over \( A \). From the localization theorem 1.1, we deduce a triangle

\[
(2.0.1) \quad HH(\text{per}_{dg}(A)) \rightarrow HH(\mathcal{D}^b_{dg}(A)) \rightarrow HH(\text{sg}_{dg}(A)) \rightarrow \Sigma HH(\text{per}_{dg}(A))
\]

in the derived category of vector spaces.

**Theorem 2.1.** We have a canonical isomorphism \( HH(\mathcal{D}^b_{dg}(A)) \cong \text{DHH}(A) \).

**Proof.** Recall that we have defined \( R \) to be the quotient of \( A \) by its radical and the Koszul dual \( A^! \) as \( \text{RHom}_A(R, R) \). Since the module \( R \) is a classical generator of the bounded derived category \( \mathcal{D}^b(A) \), we deduce from the results of [7] that we have a triangle equivalence

\[
\text{RHom}_A(R, ?) : \mathcal{D}^b(A) \cong \text{per}(A^!).
\]

This lifts to a quasi-equivalence

\[
\mathcal{D}^b_{dg}(A) \cong \text{per}_{dg}(A^!).
\]

By Morita invariance of Hochschild homology, we have

\[
HH(A^!) \cong HH(\text{per}_{dg}(A^!)).
\]

By Van den Bergh’s theorem 1.2, we have

\[
HH(A^!) \cong \text{DHH}(A).
\]

The claim follows if we combine these isomorphisms.

Define a linear map \( \tau : A \rightarrow DA \) by sending an element \( a \in A \) to the linear form which takes \( b \in A \) to the trace of the linear map

\[
\lambda_a \rho_b : A \rightarrow A, \quad x \mapsto axb,
\]

where \( \lambda_a \) is left multiplication by \( a \) and \( \rho_b \) right multiplication by \( b \). Notice that since \( A \) is finite-dimensional, this is well-defined. Moreover, the value of \( \langle a, b \rangle = (\tau(a))(b) \) only depends on the classes of \( a \) and \( b \) in \( HH_0(A) \), which is canonically isomorphic to \( R \). It is not hard to check that in the basis formed by the \( e_i \), the matrix of the induced bilinear form

\[
HH_0(A) \times HH_0(A) \rightarrow k
\]

is the Cartan matrix of \( A \), whose \((i, j)\)-entry is the dimension of \( e_i A e_j \). Define the double Hochschild complex of \( A \) to be the complex

\[
\ldots \rightarrow A \otimes A \xrightarrow{\tau} DA \xrightarrow{Db} D(A \otimes A) \xrightarrow{Db} \ldots,
\]

where \( DA \) sits in degree 0, the differentials \( b \) are those of the Hochschild complex and the \( Db \) their duals.

Let us abbreviate \( S = \text{sg}_{dg}(A) \).

**Theorem 2.2.** In \( \mathcal{D}k \), we have a canonical isomorphism between \( HH(S) \) and the double Hochschild complex of \( A \).

Notice that this implies in particular that \( HH_n(S) \) is finite-dimensional for all \( n \). This is surprising since the singularity category \( \text{sg}(A) \) is usually not Hom-finite (except if \( A \) is Gorenstein), cf. for example [3].
Proof. We use the triangle
\[ HH(\text{per}_{dg}(A)) \rightarrow HH(D^b_{dg}(A)) \rightarrow HH(S) \rightarrow \Sigma HH(\text{per}_{dg}(A)) \]

obtained from the localization theorem 1.1. We have already seen that it is isomorphic to a triangle
\[ HH(A) \rightarrow HH(A^!1) \rightarrow HH(S) \rightarrow \Sigma HH(A), \]
where the first morphism is induced by the inclusion per_{dg}(A) \rightarrow D^b_{dg}(A). Thus, the complex \( HH(S) \) identifies with the mapping cone over the morphism \( HH(A) \rightarrow HH(A^!1) \).

Let us determine this morphism explicitly. Recall that the functor \( HH \), considered as a functor on the homotopy category of small dg categories with values in the derived category \( Dk \), commutes with tensor products. We have the following commutative square
\[
\begin{array}{ccc}
\text{per}_{dg}(A^{op}) \otimes \text{per}_{dg}(A) & \rightarrow & \text{per}_{dg}(k) \\
\downarrow & & \downarrow \\
\text{per}_{dg}(A)^{op} \otimes D^b_{dg}(A) & \rightarrow & \text{per}_{dg}(k)
\end{array}
\]

Here, a pair \((P_1, P_2)\), \( P_1 \in \text{proj}(A^{op}) \), \( P_2 \in \text{proj}(A) \) is taken to \( P_2 \otimes_A P_1 \) by the top arrow and to \((\text{Hom}_A(P_1, A), P_2)\) by the left vertical arrow. It follows from Appendix D in [12] that the lower horizontal arrow induces a non degenerate pairing
\[ HH(A) \otimes HH(D^b_{dg}(A)) \rightarrow HH(k) = k. \]

A direct computation now shows that the morphism
\[ HH(A) \rightarrow DHH(A) \]
is the composition
\[ HH(A) \rightarrow HH_0(A) \rightarrow DHH_0(A) \rightarrow DHH(A) \]
where the middle morphism is induced by the map \( \tau \).

Corollary 2.3. For \( n \geq 2 \), we have canonical isomorphisms
\[ \text{HH}_n(S) \cong \text{HH}_{n-1}(A) \cong DHH_{1-n}(S). \]
Moreover, we have
\[ \text{HH}_1(S) \cong \ker(HH_0(A) \xrightarrow{\tau} DHH_0(A)) \cong DHH_0(S). \]

3. Application: Hochschild homology of dg Leavitt path algebras

Let \( Q \) be a finite quiver, for example a quiver with one vertex and a unique loop \( \alpha \). Let \( A \) be the associated radical square zero algebra, i.e. the quotient of \( kQ \) by the square of the ideal generated by the arrows. So for the one-loop quiver, we have \( A = k[\varepsilon]/(\varepsilon^2) \). Let \( Q^* \) be the graded quiver obtained from the opposite quiver of \( Q \) by assigning each arrow \( \alpha^* : j \rightarrow i \) corresponding to an arrow \( \alpha : i \rightarrow j \) of \( Q \) the degree +1. For each vertex \( i \) of \( Q \), consider the arrows \( \alpha_s^* : i \rightarrow t(\alpha_s^*) \), \( 1 \leq s \leq t_i \), starting in \( Q^* \) at \( i \). Let
\[ \varphi_i : P_i \rightarrow \bigoplus_{s=1}^{t_i} \Sigma P_{t(\alpha_s^*)} \]
be the morphism with components \( \alpha_s^* \), where \( P_i = e_i kQ^* \). For example, for the one-loop quiver, we just have \( \varphi(1) = \alpha^* : P_1 \rightarrow \Sigma P_1 \). Note that if \( i \) is a sink of \( Q \), then

\[
\bigoplus_{s=1}^{t_i} P_{t(\alpha_s^*)} = 0.
\]

For each vertex \( i \in Q_0 \), let \( \varphi(i) = [\beta_{i,1}, \ldots, \beta_{i,t_i}] : \bigoplus_{s=1}^{t_i} \Sigma P_{t(\alpha_s^*)} \rightarrow P_i \) be the formal inverse of \( \varphi(i) \). The graded Leavitt path algebra of \( Q \) is obtained from \( kQ^* \) by adjoining all coefficients \( \beta_{ij} \) of all formal inverses \( \varphi(i)^{-1} \), \( i \in Q_0 \). We endow \( L_Q \) with the grading inherited from \( Q^* \) and with \( d = 0 \).

**Theorem 3.1** (Smith [11], Chen–Yang [5]). We have a triangle equivalence \( \text{per } (L_Q) \sim - \rightarrow \text{sg}(A) \) taking \( e_i L_Q \) to the simple \( S_i \).

**Corollary 3.2.** The Hochschild homology \( HH_*(L_Q) \) of the Leavitt path algebra is computed by the double Hochschild complex

\[
\ldots \xrightarrow{b} A \otimes A \xrightarrow{b} A \xrightarrow{\tau} DA \xrightarrow{Dh} D(A \otimes A) \xrightarrow{Dh} \ldots ,
\]

(with \( DA \) in degree 0). In particular, we have

\[
\dim HH_p(L_Q) = 0 < \infty
\]

for all \( p \in \mathbb{Z} \).

A different description of the Hochschild homology of Leavitt path algebras is due to Ara–Cortiñas [1].

4. Beyond radical square zero

Let \( Q \) be a finite quiver and \( A = kQ/I \) the quotient of its path algebra by an admissible ideal. Let \( J \) be the radical of \( A \) and \( R = kQ^0 \) so that we have \( A = R \oplus J \) as \( R \)-bimodules. Let \( A_0 = (TR)/(J \otimes R) \) be the radical square zero algebra associated with \( A \). Thus, we have \( A_0 = R \oplus J = A \) as \( R \)-bimodules but we have \( xy = 0 \) in \( A_0 \) for any two elements of \( J \). We view \( A_0 \) as a degeneration of \( A \) and \( A \) as a deformation of \( A_0 \). As pointed out by Chen–Wang [4], this suggests that the singularity category \( \text{sg}(A) \) is a deformation of the singularity category \( \text{sg}(A_0) \), which is equivalent to the perfect derived category \( \text{per } (L_{A_0}) \) of the graded Leavitt path algebra \( L_{A_0} \). Hence we can hope for the existence of a dg algebra \( L_A \) obtained from \( L_{A_0} \) by deformation such that \( \text{per } (L_{A_0}) \) is equivalent to \( \text{sg}(A) \). We sum up the situation in the following diagram

\[
A_0 \xrightarrow{\text{deformation}} A \xrightarrow{?} \text{sg}(A) \xrightarrow{?} \text{per } (L_{A_0}) \xrightarrow{\text{deformation}} \text{per } (L_A) \xrightarrow{?} L_{A_0} \xrightarrow{\text{deformation}} L_A \xrightarrow{?}
\]

The following theorem confirms this hope.
Theorem 4.1 (Chen–Wang [4]). The graded algebra $L_{A_0}$ admits a canonical differential $d_A$ such that for $L_A = (L_{A_0}, d_A)$, we have a triangle equivalence

$$\text{per}(L_A) \cong \text{sg}(A).$$

Corollary 4.2. The Hochschild homology of the dg Leavitt path algebra $L_A$ is computed by the double Hochschild complex of $A$.

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