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Abstract. Let k be an algebraically closed field and A a finite-dimensional k-algebra. In this note, we determine complexes which compute the Hochschild homology of the canonical dg enhancement of the bounded derived category of A and of the canonical dg enhancement of the singularity category of A. As an application, we obtain a new approach to the computation of Hochschild homology of Leavitt path algebras.

1. Reminder on Hochschild homology of algebras and categories

Let k be a field. We write $\otimes$ for $\otimes_k$. Let A be a k-algebra (associative, with 1). We write $\operatorname{Mod} A$ for the category of all (right) A-modules and $\mathcal{D}A = \mathcal{D}(\operatorname{Mod} A)$ for its unbounded derived category. Let $A^e = A \otimes A^{op}$ be the *enveloping algebra* of A so that A^e-modules identify with A-bimodules. The *Hochschild homology* of A is defined by

$$HH_p(A) = \operatorname{Tor}_p^{A^e}(A, A)\ ,\ p \in \mathbb{Z}.$$

Alternatively, we may define it as the pth homology group of the *Hochschild chain complex* $HH(A)$ *of* A, i.e. the complex C_*A concentrated in homological degrees ≥ 0

$$A \longleftarrow A \otimes A \longleftarrow \dots \longleftarrow A^{\otimes p} \longleftarrow A^{\otimes(p+1)} \longleftarrow \dots$$

with $C_pA = A^{\otimes(p+1)}$, $p \geq 0$, and differential given by

$$d(a_0, \dots, a_p) = \sum_{i=0}^{p-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots, a_p) + (-1)^p (a_p a_0, \dots, a_{p-1})\ , \qquad (1.0.1)$$

where we write $(a_0, \dots, a_p)$ for $a_0 \otimes \dots \otimes a_p$. Notice that the first differential takes $a \otimes b$ to the commutator $ab - ba$.

We see that $HH_0(A)$ is the quotient $A/[A, A]$ of the vector space A by its subspace generated by all commutators and that $HH_p(A)$ and $HH(A) \in \mathcal{D}k$ are functorial in the algebra A. The definitions extend from k-algebras to small k-categories. For example, the Hochschild complex then becomes the complex

$$\bigoplus \mathcal{A}(X_0, X_0) \longleftarrow \bigoplus \mathcal{A}(X_1, X_0) \otimes \mathcal{A}(X_0, X_1) \longleftarrow \dots$$

whose pth term ($p \geq 0$) is the sum

$$\bigoplus \mathcal{A}(X_p, X_0) \otimes \mathcal{A}(X_{p-1}, X_p) \otimes \dots \otimes \mathcal{A}(X_0, X_1)$$

taken over all sequences of objects X_0, X_1, …, X_p of $\mathcal{A}$ and whose horizontal differential is given by formula (1.0.1). One then shows that the inclusion $A \to \operatorname{proj}(A)$ of the one-object category given by A into the category $\operatorname{proj}(A)$ of finitely generated projective right

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A-modules induces a quasi-isomorphism

$$HH(A) \xrightarrow{\sim} HH(\text{proj}\, A).$$

In particular, this yields *Morita invariance* of Hochschild homology. The definitions further extend to small differential graded (=dg) categories $\mathcal{A}$, for example the dg category $\mathcal{C}^b_{dg}(\text{proj}\, A)$ of bounded complexes over $\text{proj}\,(A)$. We refer the reader to [9] for more information on this example and dg categories in general. The inclusion $\text{proj}\,(A) \to \mathcal{C}^b_{dg}(\text{proj}\, A)$ yields an isomorphism

$$HH(\text{proj}\, A) \xrightarrow{\sim} HH(\mathcal{C}^b_{dg}(\text{proj}\, A))$$

and this yields the invariance of Hochschild homology under *derived equivalences*. We will need the following localization theorem.

Theorem 1.1 ([8])**.** *Let*

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$$

be a sequence of dg categories such that the induced sequence of derived categories

$$0 \longrightarrow \mathcal{D}\mathcal{A} \xrightarrow{F^*} \mathcal{D}\mathcal{B} \xrightarrow{G^*} \mathcal{D}\mathcal{C} \longrightarrow 0$$

is exact. Then there is a canonical triangle

$$HH(\mathcal{A}) \xrightarrow{HH(F)} HH(\mathcal{B}) \xrightarrow{HH(G)} HH(\mathcal{C}) \longrightarrow \Sigma HH(\mathcal{A})$$

in $\mathcal{D}k$ and hence long exact sequences in Hochschild (and cyclic) homology.

Let Q be a finite quiver and I an admissible ideal in kQ, *i.e.* a two-sided ideal contained in the square of the ideal generated by the arrows and such that the quotient kQ/I is finite-dimensional. Let R be the quotient of A by its radical. Thus, as an A-module, the algebra R is the direct sum of the simple A-modules. Following [7], we define the Koszul dual of A to be the dg algebra

$$A^! = \text{RHom}_A(R, R).$$

Thus, if P is a projective resolution of the A-module R, then the Koszul dual is quasi-isomorphic to the dg endomorphism algebra $\text{Hom}_A(P, P)$ of P. The following theorem is a special case of Corollary D.2 of Van den Bergh's [12]. We write D for the dual $\text{Hom}_k(?, k)$ over the ground field.

Theorem 1.2 (Van den Bergh)**.** *We have a canonical isomorphism*

$$HH(A^!) \xrightarrow{\sim} DHH(A).$$

We refer to [6] for a comparison taking into account much more structure.

2. Hochschild homology of derived categories and singularity categories

Let Q be a finite quiver and I an admissible ideal in kQ. Let $\text{mod}\, A$ be the category of k-finite-dimensional right A-modules. Denote by $\mathcal{D}^b(A) = \mathcal{D}^b(\text{mod}\, A)$ the bounded derived category of A and by $\text{per}\,(A)$ the perfect derived category, *i.e.* the thick subcategory generated by the free A-module of rank 1. Following Buchweitz [2] and Orlov [10], one defines the *singularity category of A* as the Verdier quotient

$$\text{sg}(A) = \mathcal{D}^b(A)/\text{per}\,(A).$$

Using the canonical dg enhancements of $\mathcal{D}^b(A)$ and $\text{per}\,(A)$, *cf.* [9], we obtain a canonical exact sequence of dg categories

$$0 \longrightarrow \text{per}_{dg}(A) \longrightarrow \mathcal{D}^b_{dg}(A) \longrightarrow \text{sg}_{dg}(A) \longrightarrow 0 \ ,$$

where the dg quotient $\mathrm{sg}_{dg}(A)$ yields a canonical dg enhancement for $\mathrm{sg}(A)$. It is not hard to see that, in the homotopy category of dg categories, it is functorial with respect to bimodule complexes $X \in \mathcal{D}(A^{op} \otimes B)$ such that X_B is perfect over B and ${}_A X$ is perfect over A. From the localization theorem 1.1, we deduce a triangle

$$HH(\mathrm{per}_{dg}(A)) \longrightarrow HH(\mathcal{D}^b_{dg}(A)) \longrightarrow HH(\mathrm{sg}_{dg}(A)) \longrightarrow \Sigma HH(\mathrm{per}_{dg}(A)) \tag{2.0.1}$$

in the derived category of vector spaces.

Theorem 2.1. *We have a canonical isomorphism* $HH(\mathcal{D}^b_{dg}(A)) \xrightarrow{\sim} DHH(A)$.

Proof. Recall that we have defined R to be the quotient of A by its radical and the Koszul dual $A^!$ as $\mathrm{RHom}_A(R,R)$. Since the module R is a classical generator of the bounded derived category $\mathcal{D}^b(A)$, we deduce from the results of [7] that we have a triangle equivalence

$$\mathrm{RHom}_A(R,?) : \mathcal{D}^b(A) \xrightarrow{\sim} \mathrm{per}\,(A^!).$$

This lifts to a quasi-equivalence

$$\mathcal{D}^b_{dg}(A) \xrightarrow{\sim} \mathrm{per}_{dg}(A^!).$$

By Morita invariance of Hochschild homology, we have

$$HH(A^!) \xrightarrow{\sim} HH(\mathrm{per}_{dg}(A^!)).$$

By Van den Bergh's theorem 1.2, we have

$$HH(A^!) \xrightarrow{\sim} DHH(A).$$

The claim follows if we combine these isomorphisms. √

Define a linear map $\tau : A \to DA$ by sending an element $a \in A$ to the linear form which takes $b \in A$ to the trace of the linear map

$$\lambda_a \rho_b : A \to A\,,\ x \mapsto axb\,,$$

where λ_a is left multiplication by a and ρ_b right multiplication by b. Notice that since A is finite-dimensional, this is well-defined. Moreover, the value of $\langle a,b\rangle = (\tau(a))(b)$ only depends on the classes of a and b in $HH_0(A)$, which is canonically isomorphic to R. It is not hard to check that in the basis formed by the e_i, the matrix of the induced bilinear form

$$HH_0(A) \times HH_0(A) \to k$$

is the Cartan matrix of A, whose (i,j)-entry is the dimension of $e_i A e_j$. Define the *double Hochschild complex of* A to be the complex

$$\ldots \xrightarrow{b} A \otimes A \xrightarrow{b} A \xrightarrow{\tau} DA \xrightarrow{Db} D(A \otimes A) \xrightarrow{Db} \ldots\ ,$$

where DA sits in degree 0, the differentials b are those of the Hochschild complex and the Db their duals.

Let us abbreviate $\mathcal{S} = \mathrm{sg}_{dg}(A)$.

Theorem 2.2. *In* $\mathcal{D}k$, *we have a canonical isomorphism between* $HH(\mathcal{S})$ *and the double Hochschild complex of* A.

Notice that this implies in particular that $HH_n(\mathcal{S})$ is finite-dimensional for all n. This is surprising since the singularity category $\mathrm{sg}(A)$ is usually not Hom-finite (except if A is Gorenstein), *cf.* for example [3].

Proof. We use the triangle

$$HH(\mathrm{per}_{dg}(A)) \longrightarrow HH(\mathcal{D}^b_{dg}(A)) \longrightarrow HH(\mathcal{S}) \longrightarrow \Sigma HH(\mathrm{per}_{dg}(A))$$

obtained from the localization theorem 1.1. We have already seen that it is isomorphic to a triangle

$$HH(A) \to HH(A^!) \to HH(\mathcal{S}) \to \Sigma HH(A)\,,$$

where the first morphism is induced by the inclusion $\mathrm{per}_{dg}(A) \to \mathcal{D}^b_{dg}(A)$. Thus, the complex $HH(\mathcal{S})$ identifies with the mapping cone over the morphism $HH(A) \to HH(A^!)$. Let us determine this morphism explicitly. Recall that the functor HH, considered as a functor on the homotopy category of small dg categories with values in the derived category $\mathcal{D}k$, commutes with tensor products. We have the following commutative square

$$\begin{array}{ccc} \mathrm{per}_{dg}(A^{op}) \otimes \mathrm{per}_{dg}(A) & \longrightarrow & \mathrm{per}_{dg}(k) \\ \downarrow & & \| \\ \mathrm{per}_{dg}(A)^{op} \otimes \mathcal{D}^b_{dg}(A) & \longrightarrow & \mathrm{per}_{dg}(k) \end{array}$$

Here, a pair (P_1, P_2), $P_1 \in \mathrm{proj}\,(A^{op})$, $P_2 \in \mathrm{proj}\,(A)$ is taken to $P_2 \otimes_A P_1$ by the top arrow and to $(\mathrm{Hom}_A(P_1, A), P_2)$ by the left vertical arrow. It follows from Appendix D in [12] that the lower horizontal arrow induces a non degenerate pairing

$$HH(A) \otimes HH(\mathcal{D}^b_{dg}(A)) \to HH(k) = k.$$

A direct computation now shows that the morphism

$$HH(A) \to DHH(A)$$

is the composition

$$HH(A) \to HH_0(A) \to DHH_0(A) \to DHH(A)$$

where the middle morphism is induced by the map τ. √

Corollary 2.3. *For $n \geq 2$, we have canonical isomorphisms*

$$HH_n(\mathcal{S}) \xrightarrow{\sim} HH_{n-1}(A) \xrightarrow{\sim} DHH_{1-n}(\mathcal{S}).$$

Moreover, we have

$$HH_1(\mathcal{S}) \xrightarrow{\sim} \ker(HH_0(A) \xrightarrow{\tau} DHH_0(A)) \xrightarrow{\sim} DHH_0(\mathcal{S}).$$

3. Application: Hochschild homology of dg Leavitt path algebras

Let Q be a finite quiver, for example a quiver with one vertex and a unique loop α. Let A be the associated radical square zero algebra, *i.e.* the quotient of kQ by the square of the ideal generated by the arrows. So for the one-loop quiver, we have $A = k[\varepsilon]/(\varepsilon^2)$. Let Q^* be the graded quiver obtained from the opposite quiver of Q by assigning each arrow $\alpha^* : j \to i$ corresponding to an arrow $\alpha : i \to j$ of Q the degree $+1$. For each vertex i of Q, consider the arrows $\alpha^*_s : i \to t(\alpha^*_s)$, $1 \leq s \leq t_i$, starting in Q^* at i. Let

$$\varphi_i : P_i \to \bigoplus_{s=1}^{t_i} \Sigma P_{t(\alpha^*_s)}$$

be the morphism with components α_s^*, where $P_i = e_i kQ^*$. For example, for the one-loop quiver, we just have $\varphi(1) = \alpha^* : P_1 \to \Sigma P_1$. Note that if i is a sink of Q, then

$$\bigoplus_{s=1}^{t_i} P_{t(\alpha_s^*)} = 0.$$

For each vertex $i \in Q_0$, let

$$\varphi(i)^{-1} = [\beta_{i,1}, \dots, \beta_{i,t_i}] : \bigoplus_{s=1}^{t_i} \Sigma P_{t(\alpha_s^*)} \to P_i$$

be the formal inverse of $\varphi(i)$. The *graded Leavitt path algebra of* Q is obtained from kQ^* by adjoining all coefficients β_{ij} of all formal inverses $\varphi(i)^{-1}$, $i \in Q_0$. We endow L_Q with the grading inherited from Q^* and with $d = 0$.

Theorem 3.1 (Smith [11] , Chen–Yang [5])**.** *We have a triangle equivalence* $\mathrm{per}\,(L_Q) \xrightarrow{\sim} \mathrm{sg}(A)$ *taking* $e_i L_Q$ *to the simple* S_i*.*

Corollary 3.2. *The Hochschild homology* $HH_*(L_Q)$ *of the Leavitt path algebra is computed by the double Hochschild complex*

$$\dots \xrightarrow{b} A \otimes A \xrightarrow{b} A \xrightarrow{\tau} DA \xrightarrow{Db} D(A \otimes A) \xrightarrow{Db} \dots ,$$

(with DA *in degree* 0*). In particular, we have*

$$\dim HH_p(L_Q) = 0 < \infty$$

for all $p \in \mathbb{Z}$*.*

A different description of the Hochschild homology of Leavitt path algebras is due to Ara–Cortiñas [1].

4. Beyond radical square zero

Let Q be a finite quiver and $A = kQ/I$ the quotient of its path algebra by an admissible ideal. Let J be the radical of A and $R = kQ_0$ so that we have $A = R \oplus J$ as R-bimodules. Let $A_0 = (T_R J)/(J \otimes_R J)$ be the radical square zero algebra associated with A. Thus, we have $A_0 = R \oplus J = A$ as R-bimodules but we have $xy = 0$ in A_0 for any two elements of J. We view A_0 as a degeneration of A and A as a deformation of A_0. As pointed out by Chen–Wang [4], this suggests that the singularity category $\mathrm{sg}(A)$ is a deformation of the singularity category $\mathrm{sg}(A_0)$, which is equivalent to the perfect derived category $\mathrm{per}\,(L_{A_0})$ of the graded Leavitt path algebra L_{A_0}. Hence we can hope for the existence of a dg algebra L_A obtained from L_{A_0} by deformation such that $\mathrm{per}\,(L_A)$ is equivalent to $\mathrm{sg}(A)$. We sum up the situation in the following diagram

$$\begin{array}{ccc}
A_0 & \xrightarrow[\text{deformation}]{} & A \\
\mathrm{sg}(A_0) & \xrightarrow[\text{deformation}]{} & \mathrm{sg}(A) \\
\Big\| \wr & & \Big\| \wr \quad ? \\
\mathrm{per}\,(L_{A_0}) & \xrightarrow[\text{deformation?}]{} & \mathrm{per}\,(L_A) \\
 & & \\
L_{A_0} & \xrightarrow[\text{deformation?}]{} & L_A \quad ?
\end{array}$$

The following theorem confirms this hope.

Theorem 4.1 (Chen–Wang [4])**.** *The graded algebra L_{A_0} admits a canonical differential d_A such that for $L_A = (L_{A_0}, d_A)$, we have a triangle equivalence*

$$\mathrm{per}\,(L_A) \xrightarrow{\sim} \mathrm{sg}(A).$$

Corollary 4.2. *The Hochschild homology of the dg Leavitt path algebra L_A is computed by the double Hochschild complex of A.*

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