Iterated Fractal Drums ∼ Some New Perspectives: Polyhedral Measures, Atomic Decompositions and Morse Theory

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Some New Perspectives: Polyhedral Measures, Atomic Decompositions and Morse Theory

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Abstract

We carry on our exploration of the connections between the Complex Fractal Dimensions of an iterated fractal drum (IFD) and the intrinsic properties of the fractal involved – in our present case, the Weierstrass Curve.
In order to gain a better understanding of the differential operators associated to this everywhere singular object, we identify the trace of the classical Sobolev spaces on this curve, by means of trace theorems which extend the results of Alf Jonsson and Hans Wallin obtained in the case of a d-set. For this purpose, we construct a specific polyhedral measure, which is done by means of a polygonal neighborhood of the Curve. We then obtain the order of the fractal Laplacian on the IFD.
We then lay out some of the foundations of an extension of Morse theory dedicated to fractals, where the Complex Fractal Dimensions appear to play a major role, by means of level sets connected to the successive prefractal approximations.
In the end, we envision the Weierstrass Curve as the projection of a 3-dimensional vertical comb, where each horizontal row is associated to the k^{th} cohomogical infinitesimal, the *fractal signature* of the k^{th} prefractal approximation, according to our previous results on fractal cohomology.

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Keywords: Weierstrass Curve, iterated fractal drum (IFD), Complex Dimensions of an IFD, box-counting (or Minkowski) dimension, cohomology infinitesimal, polyhedral measure, atomic decomposition, trace theorems, order of the fractal Laplacian, fractal Morse theory, fractal Morse indexes.

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Contents

1 Introduction

In [DL22a], [DL22c], we introduced the concept of *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs), by analogy with the *relative fractal drums* (RFDs) involved, for instance, in the case of the Cantor Staircase, in [LRŽ17a], Section 5.5.4, as well as in [LRŽ17b] and in [LRŽ18]. Those iterated fractal drums simply consist in a sequence of appropriate tubular neighborhoods of prefractal polygonal approximations of the Curve.

By exploring the connections between the Complex Dimensions of those IFDs and the cohomological properties of a fractal object, we showed, in [DL22c], that the functions belonging to the cohomology groups associated to the Curve are obtained, by induction, as (finite or infinite) sums indexed by the underlying Complex Dimensions. In particular, to each prefractal approximation of the given iterated fractal drum, we associate a *maximal* Complex Dimension. Contrary to fractal tube formulas, which are obtained for small values of a positive parameter ε, the aforementioned expansions are only valid for the value of the (multi-scales) *cohomology infinitesimal* ε associated to the scaling relationship obeyed by the Weierstrass Curve (or else, for a smaller positive infinitesimal). (See also [DL22a] for an exposition of these results, as well as of the computation of the Complex Dimensions via a fractal tube formula obtained in [DL22b].)

Our differentials δ and $\bar{\delta}$ (see [DL22c]) enable us to define the associated Laplacian $\delta\,\bar{\delta} + \bar{\delta}\,\delta$. This naturally raises questions as to the possible connections with the usual Laplacian (i.e., the one of classical analysis). Based on the seminal works of Alf Jonsson and Hans Wallin in [Wal89], [JW84], a hint was that it could involve the Complex Dimensions, insofar as, in the case of a d-set $\mathcal{F} \subset \mathbb{R}^n$, for $n \in \mathbb{N}^\star$ and $d > 0$, characterizations of the restrictions of classical Sobolev spaces to $\mathcal{F}$ can be obtained by means of trace theorems. More precisely, the restrictions to $\mathcal{F}$ of those Sobolev spaces are obtained as Besov spaces $B^{p,q}_\beta(\mathcal{F})$ on $\mathcal{F}$, where the index β depends explicitly on d.

We hereafter extend the results of Alf Jonsson and Hans Wallin in the case of our Weierstrass IFDs. This requires, in particular, the construction of a specific measure (called the polyhedral measure) on the Weierstrass IFD, which is done by means of a sequence of polygons – a polygonal neighborhood of the Curve. This enables us to define the Besov spaces on the Weierstrass Curve; more precisely, we characterize those spaces by means of an atomic decomposition, as is done in [Kab12] in the case of nested fractals. We are then able to connect explicitly the order of the restriction of the usual Laplacian on the Curve – or, rather, on the IFD – and the maximal real part of the Complex Dimensions, namely, the Minkowski Dimension of the Curve.

In [DL22c], we showed that, in the case of junction vertices, i.e., points belonging to consecutive prefractal graphs, the aforementioned *maximal* Complex Dimension, associated to the cohomology group involved, changes. Hence, a change of shape – when one switches from a prefractal approximation, to the consecutive polygonal approximation, also corresponds to the occurence of new polygons – is closely connected to a change of fractal dimensions. In this light, it was natural to explore further connections between the Complex Dimensions of a fractal object – the Weierstrass Curve, or the Weierstrass IFDs – and a suitable analog of Morse theory: given the Complex Dimensions and the fractal Morse indexes, can we, in some sense, reconstruct the fractal? Towards the end of the paper, we begin to lay the foundations for addressing this challenging and very interesting inverse topological and geometric problem.

Our main results in the present setting can be found in the following places:

i. In Section 3, Definition 3.6, where we introduce the polyhedral measure on the Weierstrass IFD. In particular, we prove that this polyhedral measure is well defined, as well as nontrivial, and is a bounded and singular Borel measure on the Weierstrass Curve (see Theorem 3.7).

i. In Section 4, where our polyhedral measure enables us to extend the aforementioned results of Alf Jonsson and Hans Wallin to the case of the Weierstrass IFD. More precisely, we define *the atomic decomposition* of a function defined on the IFD; see Definition 4.4. We can then define and characterize the Besov spaces on the IFD; see Definition 4.6, along with Property 4.5. In the end, we establish a trace theorem in this context (Theorem 4.8), and (in Corollary 4.9) deduce from it the order of the fractal Laplacian on the IFD, which is in agreement with previous results of Robert S. Strichartz in the case of the Sierpiński Gasket $\mathcal{SG}$ in [Str03].

iii. In Section 5, where we use the Complex Dimensions, along with the fractal cohomology, of the Weierstrass IFD, obtained and developed by the authors in [DL22b], [DL22c] (see also [DL22a]), in order to propose an extension of Morse theory devoted to fractals. In particular, a *maximal Complex Dimension* is associated to each prefractal approximation (in Definition 5.4), along with *cohomological vertex integers*, associated, this time, to each vertex of the prefractal approximation; see Definition 5.5. We also define new Morse indexes – applicable to fractal curves such as the Weierstrass Curve; see Definition 5.10.

iv. In Section 6, we explain how the Weierstrass Curve can be viewed as the projection of a 3-dimensional vertical comb, the teeth of which are directly connected with the cohomological vertex integers. This provides us with new research directions for future work, in connection with our fractal Morse theory.

For a thorough discussion of the theory of Complex Dimensions, we refer the interested reader to [LvF13] and to [LRŽ17a], in the case of fractal strings and of (relative) higher-dimensional fractal drums, respectively; see also the survey article on the subject, [Lap19].

In closing this introduction, we mention that, for clarity and by necessity of concision, we work throughout this paper with the important example of the Weierstrass Curve and the associated Weierstrass IFD. We stress, however, that we expect that our main results will extend to a large class of fractal curves and their associated IFDs – as well as, eventually, to a large class of higher-dimensional fractal manifolds obtained by means of polyhedral prefractal approximations.

2 Geometry of the Weierstrass Curve

We begin by reviewing the main geometric properties of the Weierstrass Curve (and of the associated IFD), which will be key to our work in the rest of this paper.

Henceforth, we place ourselves in the Euclidean plane, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by (x,y). The horizontal and vertical axes will be respectively referred to as $(x'x)$ and $(y'y)$.

Notation 1 (Set of all Natural Numbers, and Intervals).

As in Bourbaki [Bou04] (Appendix E. 143), we denote by $\mathbb{N} = \{0, 1, 2, \cdots\}$ the set of all natural numbers, and set $\mathbb{N}^\star = \mathbb{N} \setminus \{0\}$.

Given a, b with $-\infty \leqslant a \leqslant b \leqslant \infty$, $]a,b[= (a,b)$ denotes an open interval, while, for example, $]a,b] = (a,b]$ denotes a half-open, half-closed interval.

Notation 2 (Wave Inequality Symbol).

Given two positive numbers a and b, we will use the notation $a \lesssim b$ when there exists a strictly positive constant C such that $a \leqslant C\, b$.

Notation 3 (Weierstrass Parameters).

In the sequel, λ and N_b are two real numbers such that

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^\star \quad \text{and} \quad \lambda\, N_b > 1 \cdot \qquad (\clubsuit)$$

Note that this implies that $N_b > 1$; i.e., $N_b \geqslant 2$, if $N_b \in \mathbb{N}^\star$, as will be the case in this paper.

As explained in [Dav19], we deliberately made the choice to introduce the notation N_b which replaces the initial number b, in so far as, in Hardy's paper [Har16] (in contrast to Weierstrass' original article [Wei75]), b is any positive real number satisfying $\lambda\, b > 1$, whereas we deal here with the specific case of a natural integer, which accounts for the natural notation N_b.

Definition 2.1 (Weierstrass Function, Weierstrass Curve).

We consider the *Weierstrass function* $\mathcal{W}$ (also called, in short, the *$\mathcal{W}$-function*) defined, for any real number x, by

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) \cdot$$

We call the associated graph the *Weierstrass Curve*, and denote it by $\Gamma_{\mathcal{W}}$.

Due to the one-periodicity of the $\mathcal{W}$-function (since $N_b \in \mathbb{N}^\star$), from now on, and without loss of generality, we restrict our study to the interval $[0,1[= [0,1)$. Note that $\mathcal{W}$ is continuous, and hence, bounded on all of $\mathbb{R}$. In particular, $\Gamma_{\mathcal{W}}$ is a (nonempty) compact subset of $\mathbb{R}^2$.

Property 2.1 (**Scaling Properties of the Weierstrass Function, and Consequences [DL22b]**).

Since, for any real number x, $\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right)$, *one also has*

$$\mathcal{W}(N_b\, x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^{n+1}\, x\right) = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) = \frac{1}{\lambda} \left(\mathcal{W}(x) - \cos\left(2\,\pi\, x\right)\right) ,$$

which yields, for any strictly positive integer m *and any* j *in* $\{0, \cdots, \#V_m\}$,

$$\mathcal{W}\left(\frac{j}{(N_b - 1)\, N_b^m}\right) = \lambda\, \mathcal{W}\left(\frac{j}{(N_b - 1)\, N_b^{m-1}}\right) + \cos\left(\frac{2\,\pi\, j}{(N_b - 1)\, N_b^m}\right) .$$

By induction, one then obtains that

$$\mathcal{W}\left(\frac{j}{(N_b - 1)\, N_b^m}\right) = \lambda^m\, \mathcal{W}\left(\frac{j}{(N_b - 1)}\right) + \sum_{k=0}^{m-1} \lambda^k \cos\left(\frac{2\,\pi\, N_b^k\, j}{(N_b - 1)\, N_b^m}\right) .$$

Definition 2.2 (**Weierstrass Complexified Function**).

We introduce the *Weierstrass Complexified function* $\mathcal{W}_{comp}$, defined, for any real number x, by

$$\mathcal{W}_{comp}(x) = \sum_{n=0}^{\infty} \lambda^n\, e^{2\,i\,\pi\, N_b^n\, x} .$$

Clearly, $\mathcal{W}_{comp}$ is also a continuous and 1-periodic function on $\mathbb{R}$.

Notation 4 (**Logarithm**).

Given $y > 0$, $\ln y$ denotes the natural logarithm of y, while, given $a > 0$, $a \neq 1$, $\ln_a y = \dfrac{\ln y}{\ln a}$ denotes the logarithm of y in base a; so that, in particular, $\ln = \ln_e$.

Notation 5 (**Minkowski Dimension and Hölder Exponent**).

For the parameters λ and N_b satisfying condition (♣) (see Notation 3), we denote by

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b} = 2 - \ln_{N_b} \frac{1}{\lambda} \in \,]1, 2[$$

the box-counting dimension (or Minkowski dimension) of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, which happens to be equal to its Hausdorff dimension [KMPY84], [BBR14], [She18], [Kel17]. We point out that the results in our previous paper [DL22b] also provide a direct geometric proof of the fact that $D_{\mathcal{W}}$, the Minkowski dimension (or box-counting dimension) of $\Gamma_{\mathcal{W}}$, exists and takes the above values, as well as of the fact that $\mathcal{W}$ is Hölder continuous with *optimal* Hölder exponent

$$2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b} = \ln_{N_b} \frac{1}{\lambda} .$$

Convention (The Weierstrass Curve as a Cyclic Curve).

In the sequel, we identify the points $(0, \mathcal{W}(0))$ and $(1, \mathcal{W}(1)) = (1, \mathcal{W}(0))$. This is justified by the fact that the Weierstrass function $\mathcal{W}$ is 1-periodic, since N_b is an integer.

Remark 2.1. The above convention makes sense, because, in addition to the periodicity property of the $\mathcal{W}$-function, the points $(0, \mathcal{W}(0))$ and $(1, \mathcal{W}(1))$ have the same vertical coordinate.

Property 2.2 (Symmetry with Respect to the Vertical Line $x = \frac{1}{2}$).

Since, for any $x \in [0,1]$,

$$\mathcal{W}(1-x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n - 2\,\pi\, N_b^n\, x\right) = \mathcal{W}(x)\,,$$

the Weierstrass Curve is symmetric with respect to the vertical straight line $x = \frac{1}{2}$.

Proposition 2.3 (Nonlinear and Noncontractive Iterated Function System (IFS)).

Following our previous work [Dav18], *we approximate the restriction $\Gamma_{\mathcal{W}}$ to $[0,1[\times\mathbb{R}$, of the Weierstrass Curve, by a sequence of graphs, built via an iterative process. For this purpose, we use the nonlinear iterated function system (IFS) consisting of a finite family of C^∞ maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ and denoted by*

$$\mathcal{T}_{\mathcal{W}} = \{T_0, \cdots, T_{N_b-1}\}\,,$$

where, for any integer i belonging to $\{0, \cdots, N_b-1\}$ and any point (x,y) of $\mathbb{R}^2$,

$$T_i(x,y) = \left(\frac{x+i}{N_b}, \lambda\, y + \cos\left(2\,\pi\left(\frac{x+i}{N_b}\right)\right)\right).$$

Note that unlike in the classical situation, these maps are not contractions. Nevertheless, $\Gamma_{\mathcal{W}}$ can be recovered from this IFS in the usual way, as we next explain.

Property 2.4 (Attractor of the IFS [Dav18], [Dav19]).

The Weierstrass Curve $\Gamma_{\mathcal{W}}$ is the attractor of the IFS $\mathcal{T}_{\mathcal{W}}$, and hence, is the unique nonempty compact subset $\mathcal{K}$ of $\mathbb{R}^2$ satisfying $\mathcal{K} = \bigcup_{i=0}^{N_b-1} T_i(\mathcal{K})$; in particular, we have that $\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$.

Notation 6 (Fixed Points).

For any integer i belonging to $\{0, \cdots, N_b-1\}$, we denote by

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b-1}, \frac{1}{1-\lambda}\cos\left(\frac{2\,\pi\, i}{N_b-1}\right)\right)$$

the unique fixed point of the map T_i; see [Dav19].

Definition 2.3 (**Sets of Vertices, Prefractals**).

We denote by V_0 the ordered set (according to increasing abscissae) of the points

$$\{P_0, \cdots, P_{N_b-1}\} \cdot$$

The set of points V_0 – where, for any integer i in $\{0, \cdots, N_b - 2\}$, the point P_i is linked to the point P_{i+1} – constitutes an oriented finite graph, ordered according to increasing abscissae, which we will denote by $\Gamma_{\mathcal{W}_0}$. Then, V_0 is called *the set of vertices* of the graph $\Gamma_{\mathcal{W}_0}$.

For any nonnegative integer m, i.e., for $m \in \mathbb{N}$, we set $V_m = \bigcup_{i=0}^{N_b-1} T_i\,(V_{m-1})$.

The set of points V_m, where two consecutive points are linked, is an oriented finite graph, ordered according to increasing abscissae, called the $\boldsymbol{m^{th}}$**-order** $\boldsymbol{\mathcal{W}}$**-prefractal**. Then, V_m is called *the set of vertices* of the prefractal $\Gamma_{\mathcal{W}_m}$; see Figure 2.

Property 2.5 (**Density of the Set** $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ **in the Weierstrass Curve** [**DL22b**]).

The set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ *is dense in the Weierstrass Curve* $\Gamma_{\mathcal{W}}$.

Definition 2.4 (**Adjacent Vertices, Edge Relation**).

For any $m \in \mathbb{N}$, the prefractal graph $\Gamma_{\mathcal{W}_m}$ is equipped with an edge relation $\underset{m}{\sim}$, as follows: two vertices X and Y of $\Gamma_{\mathcal{W}_m}$ (i.e., two points belonging to V_m) will be said to be *adjacent* (i.e., *neighboring* or *junction points*) if and only if the line segment $[X, Y]$ is an edge of $\Gamma_{\mathcal{W}_m}$; we then write $X \underset{m}{\sim} Y$. Note that this edge relation depends on m, which means that points adjacent in V_m might not remain adjacent in V_{m+1}.

We refer to part *iv.* of Property 2.6, along with Figure 1, for the definition of the polygons $\mathcal{P}_{m,k}$ and $\mathcal{Q}_{m,k}$ associated with the Weierstrass Curve.

Property 2.6. [Dav18] *For any* $m \in \mathbb{N}$*, the following statements hold*:

i. $V_m \subset V_{m+1}$ $\cdot$

ii. $\#V_m = (N_b - 1)\; N_b^m + 1$*, where* $\#V_m$ *denotes the number of elements in the finite set* V_m.

iii. The prefractal graph $\Gamma_{\mathcal{W}_m}$ *has exactly* $(N_b - 1)\; N_b^m$ *edges.*

iv. *The consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$ are the vertices of N_b^m simple nonregular polygons $\mathcal{P}_{m,k}$ with N_b sides. For any strictly positive integer m, the junction point between two consecutive polygons $\mathcal{P}_{m,k}$ and $\mathcal{P}_{m,k+1}$ is the point*

$$\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 1 \leqslant k \leqslant N_b^m - 1\cdot$$

Hence, the total number of junction points is $N_b^m - 1$. For instance, in the case $N_b = 3$, the polygons are all triangles; see Figure 1.

We call extreme vertices *of the polygon $\mathcal{P}_{m,k}$ the junction points*

$$\mathcal{V}_{initial}\left(\mathcal{P}_{m,k}\right) = \left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 0 \leqslant k \leqslant N_b^m - 1\,,$$

and

$$\mathcal{V}_{end}\left(\mathcal{P}_{m,k}\right) = \left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 0 \leqslant k \leqslant N_b^m - 2\cdot$$

*In the sequel, we will denote by $\mathcal{P}_0$ **the initial polygon**, whose vertices are the fixed points of the maps T_i, $0 \leqslant i \leqslant N_b - 1$, introduced in Notation 6 and Definition 2.3, i.e., $\{P_0, \cdots, P_{N_b-1}\}$; see, again, Figure 1.*

In the same way, the consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$, distinct from the fixed points P_0 and P_{N_b-1} (see Notation 6), are also the vertices of $N_b^m - 1$ simple nonregular polygons $\mathcal{Q}_{m,j}$, for $1 \leqslant j \leqslant N_b^m - 2$, again with N_b sides. For any integer j such that $1 \leqslant j \leqslant N_b^m - 2$, one obtains each polygon $\mathcal{Q}_{m,j}$ by connecting the point number j to the point number $j+1$ if $j \equiv i \bmod N_b$, for $1 \leqslant i \leqslant N_b - 1$, and the point number j to the point number $j - N_b + 1$ if $j \equiv 0 \bmod N_b$.

As previously, we call extreme vertices *of the polygon $\mathcal{Q}_{m,k}$ the junction points*

$$\mathcal{V}_{initial}\left(\mathcal{Q}_{m,k}\right) = \left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 1 \leqslant k \leqslant N_b^m - 1\,,$$

and

$$\mathcal{V}_{end}\left(\mathcal{Q}_{m,k}\right) = \left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 1 \leqslant k \leqslant N_b^m - 2\cdot$$

Definition 2.5 (**Polygonal Sets**).

For any $m \in \mathbb{N}$, we introduce the following polygonal sets

$$\mathcal{P}_m = \left\{\mathcal{P}_{m,k}\,,\, 0 \leqslant k \leqslant N_b^m - 1\right\} \quad \text{and} \quad \mathcal{Q}_m = \left\{\mathcal{Q}_{m,k}\,,\, 0 \leqslant k \leqslant N_b^m - 2\right\}\cdot$$

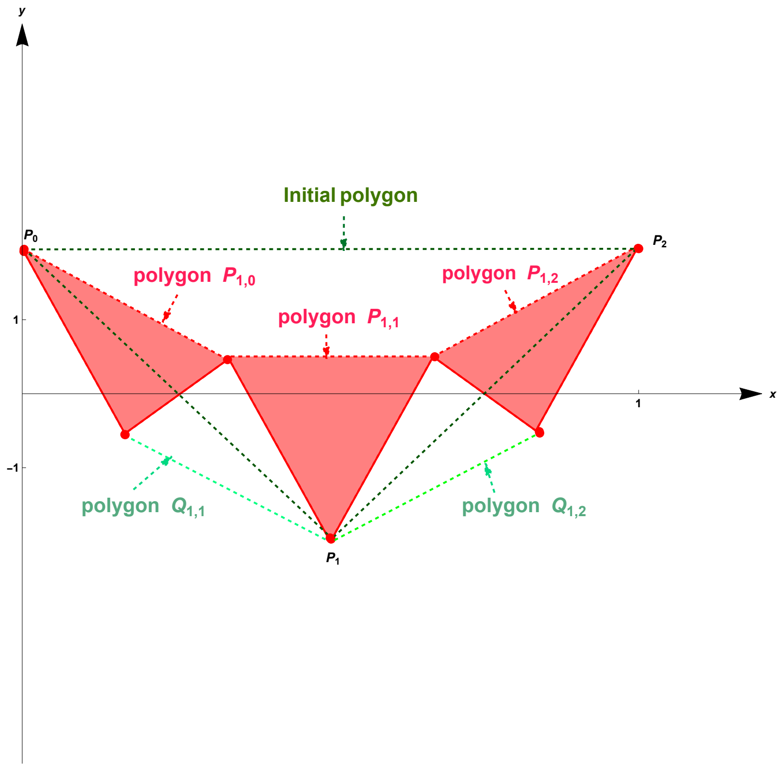


Figure 1: **The initial polygon $\mathcal{P}_0$, and the respective polygons $\mathcal{P}_{0,1}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, $\mathcal{Q}_{1,1}$, $\mathcal{Q}_{1,2}$, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$. (See also Figure 2.)**

Notation 7. For any $m \in \mathbb{N}$, we denote by:

ii. $X \in \mathcal{P}_m$ (resp., $X \in \mathcal{Q}_m$) a vertex of a polygon $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$ (resp., a vertex of a polygon $\mathcal{Q}_{m,k}$, with $1 \leqslant k \leqslant N_b^m - 2$).

ii. $\mathcal{P}_m \bigcup \mathcal{Q}_m$ the reunion of the polygonal sets $\mathcal{P}_m$ and $\mathcal{Q}_m$, which consists in the set of all the vertices of the polygons $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$, along with the vertices of the polygons $\mathcal{Q}_{m,k}$, with $1 \leqslant k \leqslant N_b^m - 2$. In particular, $X \in \mathcal{P}_m \bigcup \mathcal{Q}_m$ simply denotes a vertex in $\mathcal{P}_m$ or $\mathcal{Q}_m$.

iii. $\mathcal{P}_m \bigcap \mathcal{Q}_m$ the intersection of the polygonal sets $\mathcal{P}_m$ and $\mathcal{Q}_m$, which consists in the set of all the vertices of both a polygon $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$, and a polygon $\mathcal{Q}_{m,k'}$, with $1 \leqslant' k \leqslant N_b^m - 2$.

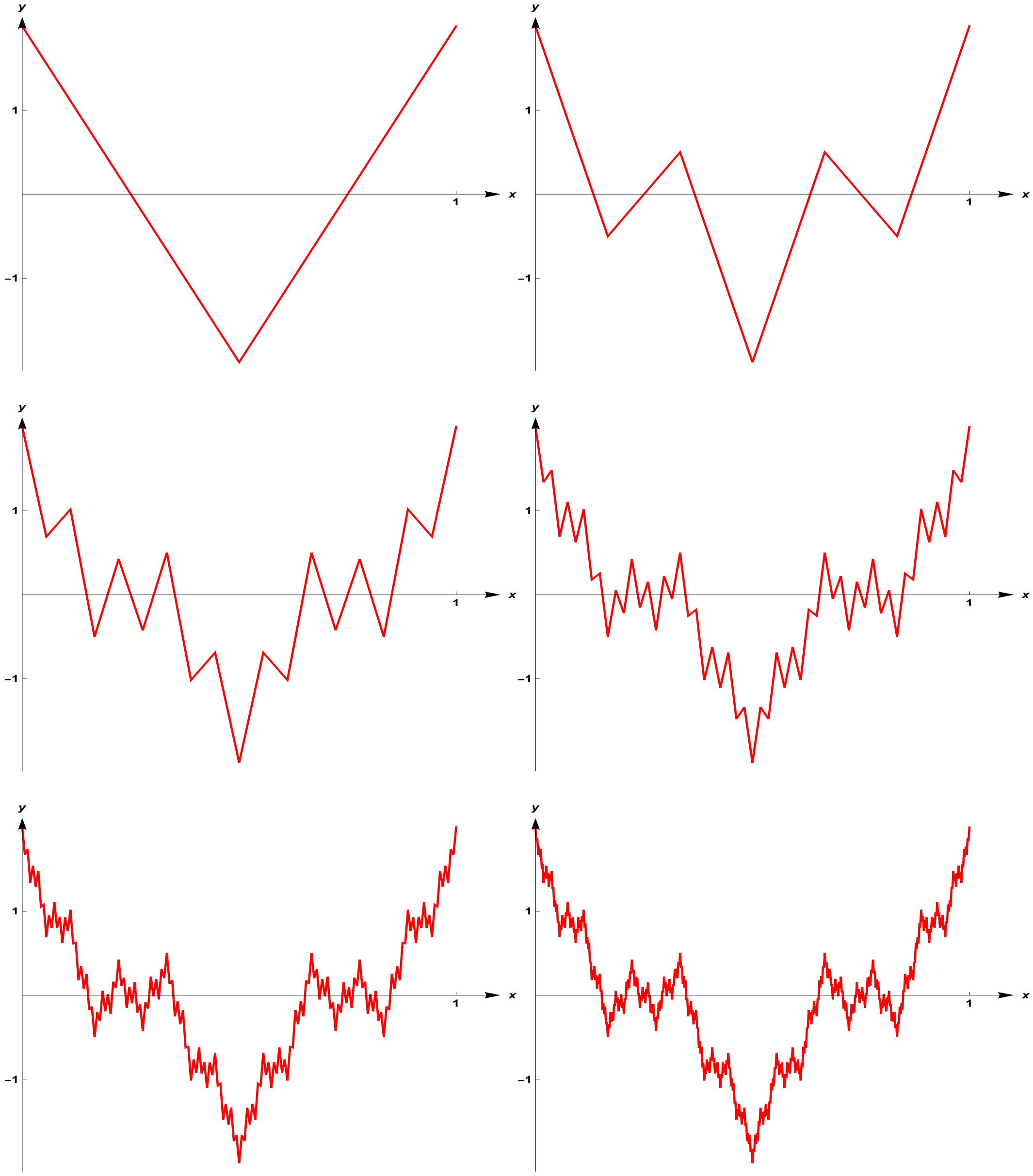


Figure 2: **The prefractal graphs $\Gamma_{\mathcal{W}_0}$, $\Gamma_{\mathcal{W}_1}$, $\Gamma_{\mathcal{W}_2}$, $\Gamma_{\mathcal{W}_3}$, $\Gamma_{\mathcal{W}_4}$, $\Gamma_{\mathcal{W}_5}$, in the case where $\lambda = \frac{1}{2}$ and $N_b = 3$. For example, $\Gamma_{\mathcal{W}_1}$ is on the right side of the top row, while $\Gamma_{\mathcal{W}_4}$ is on the left side of the bottom row.**

Definition 2.6 (**Vertices of the Prefractals, Elementary Lengths, Heights and Angles**).

Given a strictly positive integer m, we denote by $\left(M_{j,m}\right)_{0\leqslant j\leqslant (N_b-1)\,N_b^m}$ *the set of vertices* of the prefractal graph $\Gamma_{\mathcal{W}_m}$. One thus has, for any integer j in $\left\{0,\cdots,(N_b-1)\,N_b^m\right\}$:

$$M_{j,m} = \left(\frac{j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right) \cdot$$

We also introduce, for any integer j in $\left\{0,\cdots,(N_b-1)\,N_b^m-1\right\}$:

i. the elementary horizontal lengths:

$$L_m = \frac{j}{(N_b-1)\,N_b^m}\;;$$

ii. the elementary lengths:

$$\ell_{j,j+1,m} = d\left(M_{j,m},M_{j+1,m}\right) = \sqrt{L_m^2+h_{j,j+1,m}^2}\,,$$

where $h_{j,j+1,m}$ is defined in *iii.* just below.

iii. the elementary heights:

$$h_{j,j+1,m} = \left|\mathcal{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right|\;;$$

iv. the minimal height:

$$h_m^{inf} = \inf_{0\leqslant j\leqslant (N_b-1)\,N_b^m-1} h_{j,j+1,m}\,,$$

along with the the maximal height:

$$h_m = \sup_{0\leqslant j\leqslant (N_b-1)\,N_b^m-1} h_{j,j+1,m}\,,$$

v. the geometric angles:

$$\theta_{j-1,j,m} = \left((y'y),\left(\widehat{M_{j-1,m}M_{j,m}}\right)\right)\quad,\quad \theta_{j,j+1,m} = \left((y'y),\left(\widehat{M_{j,m}M_{j+1,m}}\right)\right),$$

which yield **the value of the geometric angle between consecutive edges**, namely, $\left[M_{j-1,m}\,M_{j,m},M_{j,m}\,M_{j+1,m}\right]$:

$$\theta_{j-1,j,m}+\theta_{j,j+1,m} = \arctan\frac{L_m}{\left|h_{j-1,j,m}\right|}+\arctan\frac{L_m}{\left|h_{j,j+1,m}\right|}\,.$$

Property 2.7. *For the geometric angle $\theta_{j-1,j,m}$, $0\leqslant j\leqslant (N_b-1)\,N_b^m$, $m\in\mathbb{N}$, we have the following relation:*

$$\tan\theta_{j-1,j,m} = \frac{h_{j-1,j,m}}{L_m}\,.$$

Property 2.8 (**A Consequence of the Symmetry with Respect to the Vertical Line $x=\frac{1}{2}$**).

For any strictly positive integer m and any j in $\{0,\cdots,\#V_m\}$, we have that

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \mathcal{W}\left(\frac{(N_b-1)\,N_b^m-j}{(N_b-1)\,N_b^m}\right),$$

which means that the points

$$\left(\frac{(N_b-1)\,N_b^m-j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{(N_b-1)\,N_b^m-j}{(N_b-1)\,N_b^m}\right)\right) \quad \textit{and} \quad \left(\frac{j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right)$$

are symmetric with respect to the vertical line $x=\frac{1}{2}$*; see Figure 3.*

Definition 2.7 (Left-Side and Right-Side Vertices).

Given natural integers m, k such that $0 \leqslant k \leqslant N_b^m - 1$, and a polygon $\mathcal{P}_{m,k}$, we define:

i. The set of its *left-side vertices* as the set of the first $\left[\frac{N_b-1}{2}\right]$ vertices, where $[y]$ denotes the integer part of the real number y.

ii. The set of its *right-side vertices* as the set of the last $\left[\frac{N_b-1}{2}\right]$ vertices.

When the integer N_b is odd, we define the bottom vertex as the $\left(\frac{N_b-1}{2}\right)^{th}$ one; see Figure 4.

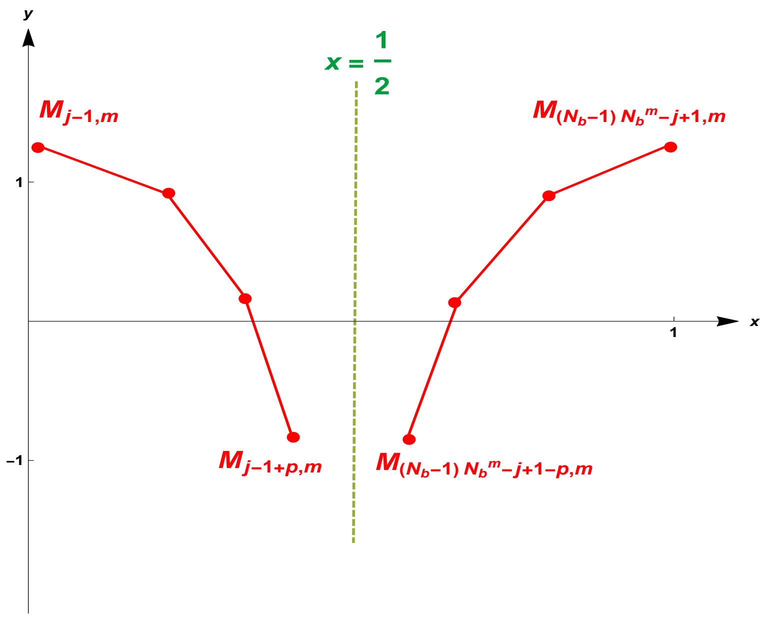
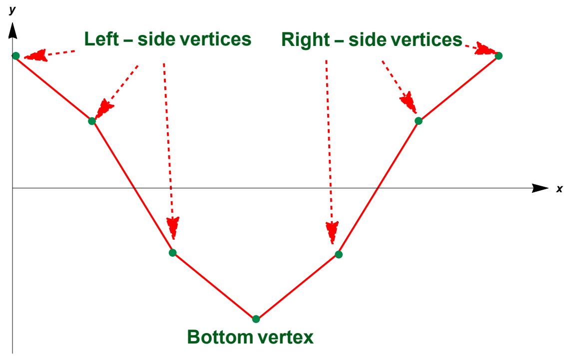


Figure 3: **Symmetric points with respect to the vertical line** $\boldsymbol{x=\frac{1}{2}}$**.**



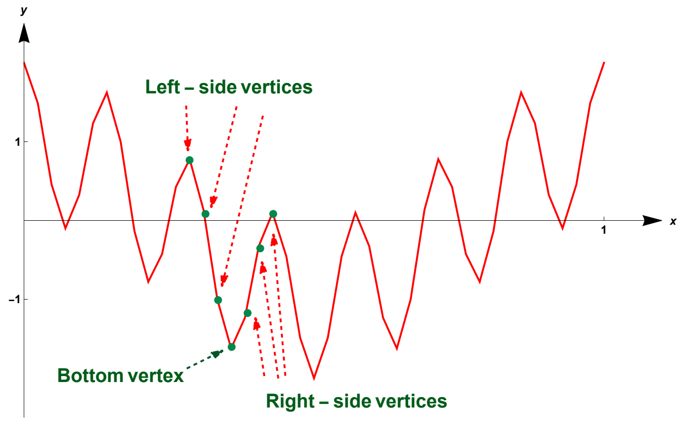


Figure 4: **The Left and Right-Side Vertices.**

Property 2.9 ([DL22b]).

For any integer j in $\{0, \cdots, N_b - 1\}$:

$$\mathcal{W}\left(\frac{j}{N_b-1}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\,N_b^n\,\frac{j}{(N_b-1)}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(\frac{2\,\pi\,j}{N_b-1}\right) = \frac{1}{1-\lambda}\cos\left(\frac{2\,\pi\,j}{N_b-1}\right).$$

Property 2.10 ([DL22b]).

For $0 \leqslant j \leqslant \frac{(N_b-1)}{2}$ *(resp., for* $\frac{(N_b-1)}{2} \leqslant j \leqslant N_b - 1$*), we have that*

$$\mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \leqslant 0 \quad \left(\text{resp., } \mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \geqslant 0\right).$$

Notation 8 (**Signum Function**).

The *signum function* of a real number x is defined by

$$\text{sgn}\,(x) = \begin{cases} -1, & \text{if} \quad x < 0, \\ 0, & \text{if} \quad x = 0, \\ +1, & \text{if} \quad x > 0. \end{cases}$$

Property 2.11 ([DL22b]).

Given any strictly positive integer m, we have the following properties:

i. For any j in $\{0, \cdots, \#V_m\}$, the point

$$\left(\frac{j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right)$$

is the image of the point

$$\left(\frac{j}{(N_b-1)\,N_b^{m-1}}-i,\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}-i\right)\right)=\left(\frac{j-i\,(N_b-1)\,N_b^{m-1}}{(N_b-1)\,N_b^{m-1}},\mathcal{W}\left(\frac{j-i\,(N_b-1)\,N_b^{m-1}}{(N_b-1)\,N_b^{m-1}}\right)\right)$$

under the map T_i, where $i \in \{0,\cdots,N_b-1\}$ is arbitrary.

*Consequently, for $0 \leqslant j \leqslant N_b-1$, **the j^{th} vertex of the polygon** $\mathcal{P}_{m,k}$, $0 \leqslant k \leqslant N_b^m-1$, i.e., the point*

$$\left(\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^{m}},\mathcal{W}\left(\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^{m}}\right)\right),$$

is the image of the point

$$\left(\frac{(N_b-1)\,\left(k-i\,(N_b-1)\,N_b^{m-1}\right)+j}{(N_b-1)\,N_b^{m-1}},\mathcal{W}\left(\frac{(N_b-1)\,\left(k-i\,(N_b-1)\,N_b^{m-1}\right)+j}{(N_b-1)\,N_b^{m-1}}\right)\right);$$

*under the map T_i, where $i \in \{0,\cdots,N_b-1\}$ is again arbitrary. It is also **the j^{th} vertex of the polygon** $\mathcal{P}_{m-1,k-i\,(N_b-1)\,N_b^{m-1}}$. Therefore, there is an exact correspondance between vertices of the polygons at consecutive steps $m-1$, m.*

ii. *Given j in $\{0,\cdots,N_b-2\}$ and k in $\left\{0,\cdots,N_b^m-1\right\}$, we have that*

$$sgn\left(\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^{m}}\right)-\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^{m}}\right)\right)=sgn\left(\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right).$$

Proof.

i. Given $m \in \mathbb{N}^\star$, let us consider $i \in \{0,\cdots,N_b-1\}$. The image of the point

$$\left(\frac{j}{(N_b-1)\,N_b^{m-1}}-i,\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}-i\right)\right)$$

under the map T_i is obtained by applying the analytic expression given in Property 2.3 to the coordinates of this point, which, thanks to Property 2.1 above, yields the expected result, namely

$$\left(\frac{j}{(N_b-1)\,N_b^{m}},\lambda\,\underbrace{\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}-i\right)}_{\substack{\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}\right)\\ \text{(by 1-periodicity)}}}+\cos\frac{2\,\pi\,j}{(N_b-1)\,N_b^{m}}\right)=\left(\frac{j}{(N_b-1)\,N_b^{m}},\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m}}\right)\right).$$

ii. See [DL22b].

□

Property 2.12 (**Lower Bound and Upper Bound for the Elementary Heights [DL22b]**).

For any strictly positive integer m *and any* j *in* $\left\{0,\cdots,(N_b-1)\,N_b^m\right\}$*, we have the following estimates, where* L_m *is the elementary horizontal length introduced in part* i*. of Definition 2.6:*

$$C_{inf}\,L_m^{2-D_{\mathcal{W}}} \leqslant \underbrace{\left|\mathcal{W}\left((j+1)\,L_m\right)-\mathcal{W}\left(j\,L_m\right)\right|}_{h_{j,j+1,m}} \leqslant C_{sup}\,L_m^{2-D_{\mathcal{W}}} \quad , \quad m\in\mathbb{N},\, 0\leqslant j\leqslant (N_b-1)\,N_b^m, \qquad (\maltese)$$

where the finite and positive constants C_{inf} *and* C_{sup} *are given by*

$$C_{inf} = (N_b-1)^{2-D_{\mathcal{W}}} \min_{0\leqslant j\leqslant N_b-1}\left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right|$$

and

$$C_{sup} = (N_b-1)^{2-D_{\mathcal{W}}}\left(\max_{0\leqslant j\leqslant N_b-1}\left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right|+\frac{2\,\pi}{(N_b-1)\,(\lambda\,N_b-1)}\right).$$

One should note, in addition, that these constants C_{inf} *and* C_{sup} *depend on the initial polygon* $\mathcal{P}_0$*.*

As a consequence, we also have that

$$C_{inf}\,L_m^{2-D_{\mathcal{W}}} \leqslant h_m^{inf} \leqslant C_{sup}\,L_m^{2-D_{\mathcal{W}}} \quad and \quad C_{inf}\,L_m^{2-D_{\mathcal{W}}} \leqslant h_m \leqslant C_{sup}\,L_m^{2-D_{\mathcal{W}}}\,,$$

where h_m^{inf} *and* h_m *respectively denote the minimal and maximal heights introduced in part* iv*. of Definition 2.6.*

Theorem 2.13 (**Sharp Local Discrete Reverse Hölder Properties of the Weierstrass Function [DL22b]**).

For any natural integer m *(i.e., for any* $m\in\mathbb{N}$*), let us consider a pair of real numbers* (x,x') *such that*

$$x=\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m}=\left((N_b-1)\,k+j\right)\,L_m \quad , \quad x'=\frac{(N_b-1)\,k+j+\ell}{(N_b-1)\,N_b^m}=\left((N_b-1)\,k+j+\ell\right)\,L_m\,,$$

where $0\leqslant k\leqslant N_b-1^m-1$*, we then have the following (discrete, local)* reverse-Hölder inequality*, with sharp Hölder exponent* $-\dfrac{\ln\lambda}{\ln N_b}=2-D_{\mathcal{W}}$*:*

$$C_{inf}\,|x'-x|^{2-D_{\mathcal{W}}} \leqslant \left|\mathcal{W}(x')-\mathcal{W}(x)\right|\,,$$

where $(x,\mathcal{W}(x))$ *and* $\left(x',\mathcal{W}(x')\right)$ *are adjacent vertices of the same* m^{th} *prefractal approximation,* $\Gamma_{\mathcal{W}_m}$*, with* $m\in\mathbb{N}$ *arbitrary. Here,* C_{inf} *is given as in Property 2.12 just above.*

Corollary 2.14 (**Optimal Hölder Exponent for the Weierstrass Function (see [DL22b])**).

The local reverse Hölder property of Theorem 2.13 just above – in conjunction with the Hölder condition satisfied by the Weierstrass function (see also [Zyg02], Chapter II, Theorem 4.9, page 47) – shows that the Codimension $2 - D_{\mathcal{W}} = -\dfrac{\ln \lambda}{\ln N_b} \in]0,1[$ *is the best (i.e., optimal) Hölder exponent for the Weierstrass function (as was originally shown, by a completely different method, by G. H. Hardy in [Har16]).*

Note that, as a consequence, since the Hölder exponent is strictly smaller than one, it follows that the Weierstrass function $\mathcal{W}$ *is nowhere differentiable.*

Corollary 2.15 (**of Property 2.12 (see [DL22b])**).

Thanks to Property 2.12, one may now write, for any strictly positive integer m *and any integer* j *in* $\left\{0, \cdots, (N_b-1)\,N_b^m-1\right\}$*, and with* C_{inf} *and* C_{supf} *defined as in Property 2.12:*

i. for the elementary heights:

$$h_{j-1,j,m} = L_m^{2-D_{\mathcal{W}}}\,\mathcal{O}(1) \;;$$

ii. for the elementary quotients:

$$\frac{h_{j-1,j,m}}{L_m} = L_m^{1-D_{\mathcal{W}}}\,\mathcal{O}(1) \;,$$

and where

$$0 < C_{inf} \leqslant \mathcal{O}(1) \leqslant C_{sup} < \infty \cdot$$

Corollary 2.16 (**Nonincreasing Sequence of Geometric Angles** (**Coming from Property** 2.11; see [DL22b])).

For the ***geometric angles*** $\theta_{j-1,j,m}$, $0 \leqslant j \leqslant (N_b-1)\,N_b^m$, $m \in \mathbb{N}$, *introduced in part v. of Definition 2.6, we have the following result:*

$$\tan\theta_{j-1,j,m} = \frac{L_m}{h_{j-1,j,m}}(N_b-1) > \tan\theta_{j-1,j,m+1}\,,$$

which yields

$$\theta_{j-1,j,m} > \theta_{j-1,j,m+1} \quad \textit{and} \quad \theta_{j-1,j,m+1} \lesssim L_m^{D_{\mathcal{W}}-1} \cdot$$

Corollary 2.17 (**Local Extrema** (**Coming from Property** 2.11; see [DL22b])).

i. The set of local maxima of the Weierstrass function on the interval $[0, 1]$ *is given by*

$$\left\{\left(\frac{(N_b-1)\,k}{N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{N_b^m}\right)\right) \,:\, 0 \leqslant k \leqslant N_b^m - 1,\, m \in \mathbb{N}\right\},$$

and corresponds to the extreme vertices of the polygons $\mathcal{P}_{m,k}$ *and* $\mathcal{Q}_{m,k}$ *(see Property 2.6) at a given step* m *(i.e., they are the vertices connecting consecutive polygons; see part iv. of Property 2.6).*

ii. For odd values of N_b*, the set of local minima of the Weierstrass function on the interval* $[0, 1]$ *is given by*

$$\left\{\left(\frac{(N_b-1)\,k + \frac{N_b-1}{2}}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k + \frac{N_b-1}{2}}{(N_b-1)\,N_b^m}\right)\right) \,:\, 0 \leqslant k \leqslant N_b^m - 1,\, m \in \mathbb{N}\right\},$$

and corresponds to the bottom vertices of the polygons $\mathcal{P}_{m,k}$ *and* $\mathcal{Q}_{m,k}$ *at a given step* m*; see also part iv. of Property 2.6.*

Property 2.18 (**Existence of Reentrant Angles [DL22b]**).

i. The initial polygon $\mathcal{P}_0$*, admits* ***reentrant interior angles****, at a vertex* P_j*, with* $0 < j \leqslant N_b - 1$*, in the sense that, with the right-hand rule, according to which angles are measured in a counterclockwise direction* $\left(\widehat{(P_jP_{j+1}),(P_jP_{j-1})}\right) > \pi$*, in the case when*

$$0 < j \leqslant \frac{N_b-3}{4} \quad or \quad \frac{3\,N_b-1}{4} \leqslant j < N_b - 1$$

(see Figure 5), which does not occur for values of $N_b < 7$.

The number of reentrant angles is then equal to $2\left[\frac{N_b-3}{4}\right]$.

ii. At a given step $m \in \mathbb{N}^\star$*, with the above convention, a polygon* $\mathcal{P}_{m,k}$ *admits reentrant interior angles in the sole cases when* $N_b \geqslant 7$*, at vertices* M_{k+j}*,* $1 \leqslant k \leqslant N_b^m$*,* $0 < j \leqslant N_b - 1$*, as well as in the case when*

$$0 < j \leqslant \frac{N_b-3}{4} \quad or \quad \frac{3\,N_b-1}{4} \leqslant j < N_b - 1 \cdot$$

The number of reentrant angles is then equal to $2\,N_b^m\left[\frac{N_b-3}{4}\right]$.

Remark 2.2. Note that due to the respective definitions of the polygons $\mathcal{P}_{m,k}$ and $\mathcal{Q}_{m,k}$, the existence of reentrant interior angles for $\mathcal{P}_{m,k}$ at a vertex M_{k+j}, $1 \leqslant k \leqslant N_b^m$, $0 < j \leqslant N_b - 1$, also results in the existence of reentrant interior angles for $\mathcal{Q}_{m,k}$ at the vertices M_{k+j-1}, $1 \leqslant k \leqslant N_b^m$, $1 < j \leqslant N_b - 1$ and M_{k+j+1}, $1 \leqslant k \leqslant N_b^m$, $0 < j \leqslant N_b - 2$.

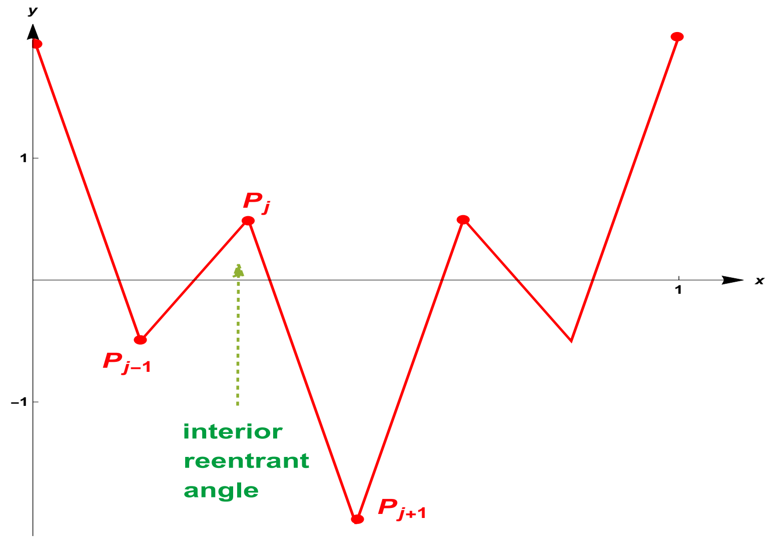


Figure 5: **An interior reentrant angle. Here, $N_b = 7$ and $\lambda = \frac{1}{2}$.**

3 Polyhedral Measure on the Weierstrass IFD

Our results on fractal cohomology obtained in [DL22c] (see also [DL22b], [DL22c]) have highlighted the role played by specific threshold values for the number $\varepsilon > 0$ at any step $m \in \mathbb{N}$ of the prefractal graph approximation; namely, the m^{th} *cohomology infinitesimal* introduced in Definition 3.1 just below.

Definition 3.1 (m^{th} **Cohomology Infinitesimal and Infinitesimal Sequence [DL22b], [DL22c]).**

From now on, given any $m \in \mathbb{N}$, we will call m^{th} *cohomology infinitesimal* the number $\varepsilon^m > 0$ which, modulo a multiplicative constant equal to $\frac{1}{N_b - 1}$, i.e., $\varepsilon^m = \frac{1}{N_b - 1}\,\frac{1}{N_b^m}$ (recall that $N_b > 1$), stands as the elementary horizontal length introduced in part *i.* of Definition 2.6, i.e.,

$$\frac{1}{N_b^m}\,.$$

Observe that, clearly, ε itself – and not just ε^m – depends on m; hence, we should really write $\varepsilon^m = (\varepsilon_m)^m$, for all $m \in \mathbb{N}$.

Also, since $N_b > 1$, ε^m satisfies the following asymptotic behavior,

$$\varepsilon^m \to 0\,, \text{ as } m \to \infty,$$

which, naturally, results in the fact that the larger m, the smaller ε^m. It is for this reason that we call ε^m – or rather, the *infinitesimal sequence* $\left(\varepsilon^m\right)_{m=0}^{\infty}$ of positive numbers tending to zero as $m \to \infty$, with $\varepsilon^m = (\varepsilon_m)^m$, for each $m \in \mathbb{N}$ – an *infinitesimal*. Note that this m^{th} cohomology infinitesimal is the one naturally associated to the scaling relation of Property 2.1 above from [DL22b].

In the sequel, it is also useful to keep in mind that – since, by definition, $\varepsilon_m = \frac{1}{N_b}\left(\frac{1}{N_b-1}\right)^{\frac{1}{m}}$, for every $m \in \mathbb{N}$ – the sequence of positive numbers $(\varepsilon_m)_{m=0}^{\infty}$ itself satisfies

$$\varepsilon_m \sim \frac{1}{N_b}\,, \text{ as } m \to \infty \ ;$$

i.e., $\varepsilon_m \to \frac{1}{N_b}$, as $m \to \infty$. In particular, $\varepsilon_m \not\to 0$, as $m \to \infty$, but, instead, ε_m decreases and tends to a strictly positive and finite limit.

In the sequel, an *infinitesimal* ϵ will refer to any sequence $(\epsilon_m)_{m\in\mathbb{N}}$ such that, for any $m \in \mathbb{N}$,

$$0 < \epsilon_m \leqslant \underbrace{\frac{1}{(N_b-1)\,N_b^m}}_{\varepsilon_m^m} \cdot$$

Note that this implies that

$$\lim_{m\to\infty} \epsilon_m = 0 \cdot$$

We refer to part *iv.* of Property 2.6 above, along with Figure 1, for the definition of the polygons $\mathcal{P}_{m,j}$ (resp., $\mathcal{Q}_{m,j}$) associated with the Weierstrass Curve in the next definition, as well as throughout the rest of this section. See also Definition 2.5 where the polygonal families are introduced.

Definition 3.2 (Power of a Vertex of the Prefractal Graph $\Gamma_{\mathcal{W}_m}$, $m \in \mathbb{N}^\star$, with Respect to the Polygonal Families $\mathcal{P}_m$ and $\mathcal{Q}_m$).

Given a strictly positive integer m, a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ will be said:

i. of power one relative to the polygonal family $\mathcal{P}_m$ if X belongs to (to be understood in the sense that X is a vertex of) one and only one N_b-gon $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$;

ii. of power $\frac{1}{2}$ *relative to the polygonal family* $\mathcal{P}_m$ if X is a common vertex to two consecutive N_b-gons $\mathcal{P}_{m,j}$ and $\mathcal{P}_{m,j+1}$, $0 \leqslant j \leqslant N_b^m - 2$;

iii. of power zero relative to the polygonal family $\mathcal{P}_m$ if X does not belong to (to be understood in the sense that X is not a vertex of) any N_b-gon $\mathcal{P}_{m,j}$,
$0 \leqslant j \leqslant N_b^m - 1$.

Similarly, given $m \in \mathbb{N}$, a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ is said:

i. of power one relative to the polygonal family $\mathcal{Q}_m$ if X belongs to (as above, to be understood in the sense that X is a vertex of) one and only one N_b-gon $\mathcal{Q}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$;

ii. of power $\frac{1}{2}$ *relative to the polygonal family* $\mathcal{Q}_m$ if X is a common vertex to two consecutive N_b-gons $\mathcal{Q}_{m,j}$ and $\mathcal{Q}_{m,j+1}$, $0 \leqslant j \leqslant N_b^m - 2$;

iii. of power zero relative to the polygonal family $\mathcal{Q}_m$ if X does not belong to (as previously, to be understood in the sense that X is not a vertex of) any N_b-gon $\mathcal{Q}_{m,j}$,
$0 \leqslant j \leqslant N_b^m - 2$.

Notation 9. In the sequel, given a strictly positive integer m, the *power of a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ relative to the polygonal families $\mathcal{P}_m$ and $\mathcal{Q}_m$* will be respectively denoted by:

$$p(X,\mathcal{P}_m) \quad \text{and} \quad p(X,\mathcal{Q}_m)\cdot$$

Notation 10 (Lebesgue Measure (on $\mathbb{R}^2$)).

In the sequel, we denote by $\mu_{\mathcal{L}}$ the Lebesgue measure on $\mathbb{R}^2$.

Notation 11. For any $m \in \mathbb{N}$, and any vertex X of V_m, we set:

$$\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m) = \begin{cases} \dfrac{1}{N_b}\, p(X,\mathcal{P}_m)\,\mu_{\mathcal{L}}(\mathcal{P}_{m,j}), & \text{if} \quad X \notin \mathcal{Q}_m \text{ vertex of } \mathcal{P}_{m,j},\ 0 \leqslant j \leqslant N_b^m - 1, \\ \dfrac{1}{N_b}\, p(X,\mathcal{Q}_m)\,\mu_{\mathcal{L}}(\mathcal{Q}_{m,j}), & \text{if} \quad X \notin \mathcal{P}_m \text{ vertex of } \mathcal{Q}_{m,j},\ 1 \leqslant j \leqslant N_b^m - 2, \\ \dfrac{1}{2\,N_b}\left\{p(X,\mathcal{P}_m)\,\mu_{\mathcal{L}}(\mathcal{P}_{m,j}) + p(X,\mathcal{Q}_m)\,\mu_{\mathcal{L}}(\mathcal{Q}_{m,j})\right\}, & \text{if } X \in \mathcal{P}_m \cap \mathcal{Q}_m \text{ vertex of } \mathcal{P}_{m,j} \text{ and } \mathcal{Q}_{m,j},\ 1 \leqslant j \leqslant N_b^m - 2\cdot \end{cases}$$

Property 3.1. *For any $m \in \mathbb{N}$, and any pair (X,Y) of adjacent vertices of V_m belonging to the same polygon $\mathcal{P}_{m,j}$, with $0 \leqslant j \leqslant N_b^m - 1$ (resp., $\mathcal{Q}_{m,j}$, with $0 \leqslant j \leqslant N_b^m - 2$), we have that*

$$d_{eucl}(X,Y) = \sqrt{h_{jm}^2 + L_m^2} > |h_{jm}|\,,$$

which, due to the inequality given in Property 2.12, ensures that

$$\frac{1}{d_{eucl}(X,Y)} < \frac{1}{|h_{jm}|} \lesssim L_m^{D_{\mathcal{W}}-2} \lesssim N_b^{(2-D_{\mathcal{W}})\,m}\cdot$$

At the same time, we also have that

$$d_{eucl}(X,Y) \lesssim h_m \lesssim L_m^{2-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-2)\,m}\cdot$$

Proof. This follows at once from Property 2.12.

□

Corollary 3.2. *For any $m \in \mathbb{N}$, any natural integer j of $\{0,\cdots, N_b^m - 1\}$, and any pair of points (X,Y) of $\mathcal{P}_{m,j}$ or of $\mathcal{Q}_{m,j}$, we have that*

$$\frac{1}{d_{eucl}(X,Y)} \lesssim L_m^{D_{\mathcal{W}}-2} \lesssim N_b^{(2-D_{\mathcal{W}})\,m}\,,$$

and

$$d_{eucl}(X,Y) \lesssim h_m \lesssim L_m^{2-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-2)\,m}\cdot$$

Property 3.3. *For any $m \in \mathbb{N}$, and any vertex X of V_m:*

$$\mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) \lesssim h_m\, L_m \lesssim L_m^{3-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-3)\,m},$$

and

$$\mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) \lesssim h_m\, L_m \lesssim L_m^{3-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-3)\,m}.$$

Proof. This also directly comes from Property 2.12.

□

Definition 3.3 (**Trace of a Polygon on the Weierstrass Curve**).

Given $m \in \mathbb{N}$, and $0 \leqslant j \leqslant N_b^m - 1$ (resp., $0 \leqslant j \leqslant N_b^m - 2$), of extreme vertices $\mathcal{V}_{initial}\left(\mathcal{P}_{m,j}\right) \in V_m$ and $\mathcal{V}_{end}\left(\mathcal{P}_{m,j}\right) \in V_m$ (resp., $\mathcal{V}_{initial}\left(\mathcal{Q}_{m,j}\right) \in V_m$ and $\mathcal{V}_{end}\left(\mathcal{Q}_{m,j}\right) \in V_m$; see Definition 2.6), we define *the trace* of the polygon $\mathcal{P}_{m,j}$ (resp., $\mathcal{Q}_{m,j}$) on the Weierstrass Curve as the set $tr_{\gamma_{\mathcal{W}}}\left(\mathcal{P}_{m,j}\right)$ (resp., $tr_{\gamma_{\mathcal{W}}}\left(\mathcal{Q}_{m,j}\right)$) of points $\left\{\mathcal{V}_{initial}\left(\mathcal{P}_{m,j}\right), M_\star, \mathcal{V}_{end}\left(\mathcal{P}_{m,j}\right)\right\}$ (resp., $\left\{\mathcal{V}_{initial}\left(\mathcal{Q}_{m,j}\right), M_\star, \mathcal{V}_{end}\left(\mathcal{Q}_{m,j}\right)\right\}$), where we denote by $M_\star$ any point of the Weierstrass Curve strictly located between $\mathcal{V}_{initial}\left(\mathcal{P}_{m,j}\right)$ and $\mathcal{V}_{end}\left(\mathcal{P}_{m,j}\right)$ (resp., $\mathcal{V}_{initial}\left(\mathcal{Q}_{m,j}\right)$ and $\mathcal{V}_{end}\left(\mathcal{Q}_{m,j}\right)$).

Definition 3.4 (**Sequence of Domains Delimited by the Weierstrass IFD**).

We introduce *the sequence of domains delimited by the Weierstrass IFD* as the sequence $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ of open, connected polygonal sets $\left(\mathcal{P}_m \cup \mathcal{Q}_m\right)_{m\in\mathbb{N}}$, where, for each $m \in \mathbb{N}$, $\mathcal{P}_m$ and $\mathcal{Q}_m$ respectively denote the polygonal sets introduced in Definition 2.5.

Property 3.4 (**Domain Delimited by the Weierstrass IFD**).

We call domain, delimited by the Weierstrass IFD*, the set, which is equal to the following limit,*

$$\mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \lim_{m\to\infty} \mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right),$$

where the convergence is interpreted in the sense of the Hausdorff metric on $\mathbb{R}^2$; see Remark 3.1 below. In fact, we have that

$$\mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \Gamma_{\mathcal{W}}.$$

Proof. Note, first, that the sequence $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ can be replaced by its closure $\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$, with closed polygons $\overline{\mathcal{P}}_{m,j}$ and $\overline{\mathcal{Q}}_{m,j}$, instead of open polygons $\mathcal{P}_{m,j}$ and $\mathcal{Q}_{m,j}$, used in the counterpart of Definition 3.4 just above. We can then easily prove that the sequence $\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ converges to the Weierstrass Curve $\Gamma_{\mathcal{W}}$. This simply comes from the fact that, given any (positive) infinitesimal $\epsilon = \left(\epsilon_m\right)_{m\in\mathbb{N}}$ (in the sense of Definition 3.1), there exists an integer m_0 such that

$$\forall\, m \geqslant m_0 : \quad \overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right) \subset \mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \epsilon_m\right),$$

where $\mathcal{D}\left(\Gamma_{\mathcal{W}_m},\epsilon_m\right)$ denotes the (tubular) (m,ϵ_m)-neighborhood of the Weierstrass IFD introduced in [DL22b]; namely,

$$\mathcal{D}\left(\Gamma_{\mathcal{W}_m},\epsilon_m\right)=\left\{M\in\mathbb{R}^2\,,\,d\left(M,\Gamma_{\mathcal{W}_m}\right)\leqslant\epsilon_m\right\}\cdot$$

This also ensures that

$$\lim_{m\to\infty}\mu_{\mathcal{L}}\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)\right)=0\cdot$$

□

Remark 3.1. In our proof, we have considered the limit of the sequence $\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$, in the set-theoretic sense. In fact, we could have considered, instead, the Hausdorff limit of this sequence; i.e., by using the Hausdorff metric $d_{\mathcal{H}}$ on $\mathbb{R}^2$. This would not have changed our result, since

$$d_{\mathcal{H}}\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right),\Gamma_{\mathcal{W}}\right)=d_{\mathcal{H}}\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right),\Gamma_{\mathcal{W}}\right)\cdot$$

As we explained in our proof just above, given any positive infinitesimal $\epsilon_m>0$, there exists an integer m_0 such that,

$$\forall\,m\geqslant m_0:\quad\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)\subset\mathcal{D}\left(\Gamma_{\mathcal{W}_m},\epsilon_m\right)\cdot$$

This ensures, for all $m\geqslant m_0$, any j in $\left\{0,\cdots,N_b^m-1\right\}$, any point $P\in\mathcal{P}_{m,j}\subset\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)$ (resp., $Q\in\mathcal{Q}_{m,j}\subset\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)$), and any point $M\in tr_{\gamma_{\mathcal{W}}}\left(\mathcal{P}_{m,j}\right)$ (resp., $M\in tr_{\gamma_{\mathcal{W}}}\left(\mathcal{Q}_{m,j}\right)$), that the Hausdorff distance $d_{\mathcal{H}}(P,M)$ (resp., $d_{\mathcal{H}}(Q,M)$) between P (resp., Q) and M is such that

$$d_{\mathcal{H}}(P,M)=\max\left\{\underbrace{\sup_{M'\in tr_{\gamma_{\mathcal{W}}}(\mathcal{P}_{m,j})}d_{eucl}(P,M')}_{\lesssim\,\epsilon_m}\quad,\quad\underbrace{\sup_{P'\in\mathcal{P}_{m,j}}d_{eucl}(P',M)}_{\lesssim\,\epsilon_m}\right\}\lesssim\epsilon_m$$

$$\left(\text{resp., }d_{\mathcal{H}}(Q,M)=\max\left\{\underbrace{\sup_{M'\in tr_{\gamma_{\mathcal{W}}}(\mathcal{Q}_{m,j})}d_{eucl}(Q,M')}_{\lesssim\,\epsilon_m}\quad,\quad\underbrace{\sup_{Q'\in\mathcal{Q}_{m,j}}d_{eucl}(Q',M)}_{\lesssim\,\epsilon_m}\right\}\lesssim\epsilon_m\right),$$

as desired. It follows that

$$d_{\mathcal{H}}\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right),\Gamma_{\mathcal{W}}\right)\lesssim\epsilon_m\underset{m\to\infty}{\to}0\cdot$$

Hence, $\mathcal{D}\left(\Gamma_{\mathcal{W}}\right)$ is the Hausdorff limit of $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$, and so, by the uniqueness of such a limit, we deduce that $\mathcal{D}\left(\Gamma_{\mathcal{W}}\right)=\Gamma_{\mathcal{W}}$.

Notation 12 (Minimal and Maximal Values of the Weierstrass Funnction $\mathcal{W}$ on $[0,1]$).

We set

$$m_{\mathcal{W}}=\min_{t\in[0,1]}\mathcal{W}(t)=-\frac{1}{1-\lambda}\quad,\quad M_{\mathcal{W}}=\max_{t\in[0,1]}\mathcal{W}(t)=\frac{1}{1-\lambda}\cdot$$

Notation 13. Henceforth, for a given $m \in \mathbb{N}$, the notation $\sum\limits_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m}$ means that the associated finite sum runs through all of the vertices of the polygons belonging to the sets $\mathcal{P}_m$ and $\mathcal{Q}_m$ introduced in Definition 2.5; see also Notation 7 following that definition.

Property 3.5. *Given a continuous function u on $[0,1] \times [m_{\mathcal{W}}, M_{\mathcal{W}}]$, we have that, for any $m \in \mathbb{N}$, and any vertex X of V_m:*

$$\left|\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m)\, u(X)\right| \leqslant \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \left(\max_{[0,1]\times[m_{\mathcal{W}}, M_{\mathcal{W}}]} |u|\right) \lesssim N_b^{-(3-D_{\mathcal{W}})\,m} \cdot$$

Consequently, with the notation of Definition 3.1, we have that

$$\varepsilon^{m\,(D_{\mathcal{W}}-2)} \left|\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m)\, u(X)\right| \lesssim \varepsilon^{-m} \cdot$$

Sinc the sequence $\left(\sum\limits_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \varepsilon^{-m}\right)_{m \in \mathbb{N}}$ *is a positive and increasing sequence (the number of vertices involved increases as m increases), this ensures the existence of the finite limit*

$$\lim_{m \to \infty} \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m)\, u(X)\,,$$

where we have used Notation 13.

Proof. Thanks to Property 3.3, for any $m \in \mathbb{N}$, and any vertex X of V_m, we have that

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim N_b^{(D_{\mathcal{W}}-3)\,m} \quad \text{and} \quad \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim N_b^{(D_{\mathcal{W}}-3)\,m} \cdot$$

We then recall from Section 2 that, for any $m \in \mathbb{N}$, the total number of polygons $\mathcal{P}_m$ is N_b^m, while the total number of polygons $\mathcal{Q}_m$ is equal to $N_b^m - 1$; see Property 2.6. We then have that

$$\sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim N_b^m\, N_b^{(D_{\mathcal{W}}-3)\,m}\,;$$

i.e.,

$$\sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim N_b^{(D_{\mathcal{W}}-2)\,m}\,,$$

or, equivalently, due to the relation between the m^{th} *cohomology infinitesimal* ε^m introduced in Definition 3.1 and the Weierstrass parameter N_b ,

$$\sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim \varepsilon^{m\,(2-D_{\mathcal{W}})}\,,$$

which, as desired, ensures the existence of the finite limit

$$\left(\max_{[0,1]\times[m_{\mathcal{W}}, M_{\mathcal{W}}]} |u|\right) \lim_{m \to \infty} \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \cdot$$

□

Property 3.6 (Polyhedral Measure on the Weierstrass IFD).

We introduce the polyhedral measure on the Weierstrass IFD, *denoted by μ, such that for any continuous function u on the Weierstrass Curve, with the use of Notation 11 and 13,*

$$\int_{\Gamma_{\mathcal{W}}} u\, d\mu = \lim_{m\to\infty} \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{X\,\in\,\mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\, u\left(X\right)\,, \quad (\star)$$

which, thanks to Property 3.4, can also be understood in the following way,

$$\int_{\Gamma_{\mathcal{W}}} u\, d\mu = \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} u\, d\mu\,\cdot$$

Remark 3.2. In a sense, our polyhedral measure can be seen as a measure which is an extension of the Riemann integral, where the step functions are replaced by upper and lower affine functions which approximate the Weierstrass Curve.

Theorem 3.7.

The polyhedral measure μ is well defined, positive, as well as a bounded, nonzero, Borel measure on $\mathcal{D}\left(\Gamma_{\mathcal{W}}\right)$. The associated total mass is given by

$$\mu\left(\mathcal{D}\left(\Gamma_{\mathcal{W}}\right)\right) = \lim_{m\to\infty} \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{X\,\in\,\mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\,, \quad (\star\star)$$

and satisfies the following estimate:

$$\mu\left(\mathcal{D}\left(\Gamma_{\mathcal{W}}\right)\right) \leqslant \frac{2}{N_b}\left(N_b-1\right)^2 C_{sup}\,\cdot \quad (\star\star\star)$$

Furthermore, the support of μ coincides with the entire curve:

$$supp\, \mu = \mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \Gamma_{\mathcal{W}}\,\cdot$$

In addition, μ is the weak limit as $m\to\infty$ of the following discrete measures *(or Dirac Combs), given, for each $m\in\mathbb{N}$, by*

$$\mu_m = \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{X\,\in\,\mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\,\delta_X\,,$$

where ε denotes the cohomological infinitesimal introduced in Definition 3.1, δ_X is the Dirac measure concentrated at X, and we have used Notation 11 for $\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)$, along with Notation 13.

Proof.

i. μ is a well defined measure.

Indeed, according to Proposition 3.5, the map φ

$$u \mapsto \varphi(u) = \int_{\Gamma_{\mathcal{W}}} u\, d\mu$$

is a well defined linear functional on the space $C(\Gamma_{\mathcal{W}})$ of real-valued, continuous functions on $\Gamma_{\mathcal{W}}$. Hence, by a well-known argument, it is a continuous linear functional on $C(\Gamma_{\mathcal{W}})$, equipped with the *sup* norm. Since $\Gamma_{\mathcal{W}}$ is compact, and in light of $(\star)$ in Property 3.6, μ is a bounded, Radon measure, with total mass $\varphi(1) = \mu(\mathcal{D}(\Gamma_{\mathcal{W}}))$, also given by $(\star\star)$, and where 1 denotes the constant function equal to 1 on $\Gamma_{\mathcal{W}}$.

Then, according to the Riesz representation theorem, the associated positive Borel measure (still denoted by μ) is a bounded and positive Borel measure with the same total mass $\mu(\mathcal{D}(\Gamma_{\mathcal{W}})) = \mu(\Gamma_{\mathcal{W}})$.

ii. The nonzero measure – Estimates for the total mass of μ.

For $0 \leqslant j \leqslant N_b^m - 1$, each polygon $\mathcal{P}_{m,j}$ is contained in a rectangle of height at most equal to $(N_b - 1)\, h_m$ (where h_m is the maximal height introduced in part *iv.* of Definition 2.6), and of width at most equal to $(N_b - 1)\, L_m$. This ensures that the Lebesgue measure of each polygon $\mathcal{P}_{m,j}$ is at most equal to $(N_b - 1)^2\, h_m\, L_m$. We now recall that, thanks to Property 2.12, for any $m \in \mathbb{N}$, we have the following estimate

$$h_m \leqslant C_{sup}\, L_m^{2-D_{\mathcal{W}}} ,$$

where

$$C_{sup} = (N_b - 1)^{2-D_{\mathcal{W}}} \left(\max_{0 \leqslant j \leqslant N_b - 1} \left| \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \right| + \frac{2\pi}{(N_b - 1)\,(\lambda\, N_b - 1)} \right) \cdot$$

Consequently, the Lebesgue measure $\mu_{\mathcal{L}}(\mathcal{P}_{m,j})$ of each polygon $\mathcal{P}_{m,j}$ is such that

$$\mu_{\mathcal{L}}(\mathcal{P}_{m,j}) \leqslant (N_b - 1)^2\, C_{sup}\, L_m^{3-D_{\mathcal{W}}} \cdot \qquad (\approx\!\approx)$$

In the same way, for $0 \leqslant j \leqslant N_b^m - 2$, the Lebesgue measure $\mu_{\mathcal{L}}(\mathcal{P}_{m,j})$ of each polygon $\mathcal{Q}_{m,j}$ is such that

$$\mu_{\mathcal{L}}(\mathcal{Q}_{m,j}) \leqslant (N_b - 1)^2\, C_{sup}\, L_m^{3-D_{\mathcal{W}}} \cdot \qquad (\approx\!\approx\!\approx\!\approx)$$

We then deduce that, for any vertex X of V_m,

$$\mu(X, \mathcal{P}_m, \mathcal{Q}_m) \leqslant \frac{1}{N_b}\,(N_b - 1)^2\, C_{sup}\, L_m^{3-D_{\mathcal{W}}} \cdot$$

Hence, since the total number of polygons involved is at most equal to $2\, N_b^m - 1 \leqslant 2\, N_b^m$, we can deduce that

$$\sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \leqslant \frac{2\, N_b^m}{N_b}\,(N_b - 1)^2\, C_{sup}\, L_m^{3-D_{\mathcal{W}}} ,$$

or, equivalently,

$$\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\leqslant 2\,\frac{\varepsilon^{-m}}{N_b}\left(N_b-1\right)^2 C_{sup}\,\varepsilon^{m\,(3-D_{\mathcal{W}})}\,.$$

We then have that

$$\varepsilon^{m\,(D_{\mathcal{W}}-2)}\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\leqslant\frac{2}{N_b}\left(N_b-1\right)^2 C_{sup}<\infty\,,$$

from which we can deduce that the polyhedral measure is a bounded measure.

For $0\leqslant j\leqslant N_b^m-1$, each polygon $\mathcal{P}_{m,j}$ (which is convex) contains an inscribed circle, whose Lebesgue measure is greater than $\dfrac{h_m^{inf}\,L_m}{C_{N_b}}$, where h_m^{inf} is the minimal height introduced in part *iv.* of Definition 2.6, and where C_{N_b} is a strictly positive constant, which depends on the value of the integer N_b (depending on the number of sides of the polygon, i.e., depending on the value of this integer, we can express the radius of this circle in function of the side lengths of the polygon). We now recall that, thanks to Property 2.12, for any $m\in\mathbb{N}$, we have the following estimate,

$$C_{inf}\,L_m^{2-D_{\mathcal{W}}}\leqslant h_m^{inf}\,,$$

where

$$C_{inf}=\left(N_b-1\right)^{2-D_{\mathcal{W}}}\min_{0\leqslant j\leqslant N_b-1}\left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right|>0\cdot$$

Consequently, the Lebesgue measure $\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)$ of each polygon $\mathcal{P}_{m,j}$ is such that

$$\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)\geqslant\frac{h_m^{inf}\,L_m}{C_{N_b}}\geqslant\frac{C_{inf}\,L_m^{3-D_{\mathcal{W}}}}{C_{N_b}}\cdot$$

In the same way, for $0\leqslant j\leqslant N_b^m-2$, the Lebesgue measure $\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)$ of each polygon $\mathcal{Q}_{m,j}$ is such that

$$\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)\geqslant\frac{h_m^{inf}\,L_m}{C_{N_b}}\geqslant\frac{C_{inf}\,L_m^{3-D_{\mathcal{W}}}}{C_{N_b}}\cdot$$

We then deduce that, for any vertex X of V_m,

$$\mu\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\geqslant\frac{1}{N_b}\,\frac{C_{inf}\,L_m^{3-D_{\mathcal{W}}}}{C_{N_b}}\cdot$$

Hence, since the total number of polygons involved is greater than $N_b^m-1\geqslant\dfrac{N_b^m}{2}$, we can deduce that

$$\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\geqslant\frac{N_b^m}{2}\,\frac{C_{inf}\,L_m^{3-D_{\mathcal{W}}}}{N_b\,C_{N_b}}\,,$$

or, equivalently,

$$\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\geqslant\frac{\varepsilon^{-m}}{2\left(N_b-1\right)}\,\frac{C_{inf}\,\varepsilon^{m\,(3-D_{\mathcal{W}})}}{N_b\,C_{N_b}}\cdot$$

We then have that

$$\varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{X\,\in\,\mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\,(X,\mathcal{P}_m,\mathcal{Q}_m) \geqslant \frac{1}{2\,(N_b-1)}\,\frac{C_{inf}}{N_b\,C_{N_b}} > 0\,,$$

from which, upon passing to the limit when $m\to\infty$, we can deduce that the polyhedral measure is a nonzero measure, and that its total mass satisfies inequality $(\star\star\star)$.

iii. The support of μ coincides with the entire curve $\Gamma_{\mathcal{W}}$.

This simply comes from the proof given in *ii.* just above that the measure μ is nonzero. In the case of a positive, continuous function u defined on the Weierstrass Curve, we have that

$$\varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{X\,\in\,\mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\,(X,\mathcal{P}_m,\mathcal{Q}_m)\;u(X) \geqslant \frac{1}{2\,(N_b-1)}\,\frac{C_{inf}}{N_b\,C_{N_b}}\left(\min_{\Gamma_{\mathcal{W}}}|u|\right) > 0\,\cdot$$

Hence, upon passing to the limit when $m\to\infty$, we deduce that $\varphi(u) = \displaystyle\int_{\Gamma_{\mathcal{W}}} u\,d\mu > 0$, and thus, $\varphi(u)\neq 0$, from which the claim follows easily.

Indeed, otherwise, if $\operatorname{supp}\mu\neq\Gamma_{\mathcal{W}}$, there exists $M\in\Gamma_{\mathcal{W}}\setminus\operatorname{supp}\mu$, and thus, by Urisohn's lemma (see, e.g., [Fol99], [Rud87] or [Rud91]), there exists $u\in C\,(\Gamma_{\mathcal{W}})$ and an open neighborhood $\mathcal{V}(M)$ of M in $\Gamma_{\mathcal{W}}$ disjoint from $\operatorname{supp}\mu$ and such that

$$u(M)=1 \quad,\quad 0\leqslant u\leqslant 1 \quad,\quad \text{and } u_{|\Gamma_{\mathcal{W}}\setminus\mathcal{V}(M)} = 0\,\cdot$$

Hence, by the above argument, $\varphi(u)\neq 0$, which contradicts the fact that $M\notin\operatorname{supp}\mu$ (see, e.g., loc. cit.).

iv. μ is a singular measure.

First, note that

$$\mu^{\mathcal{L}}\,(\Gamma_{\mathcal{W}}) = 0\,,$$

because $D_{\mathcal{W}}<2$, and, up to a multiplicative positive constant, $\mu^{\mathcal{L}}$ coincides with the 2-dimensional measure on $\mathbb{R}^2$. Now, since $\operatorname{supp}\mu\subset\Gamma_{\mathcal{W}}$, and $\mu^{\mathcal{L}}\,(\Gamma_{\mathcal{W}})=0$, it follows that μ is supported on a set of Lebesgue measure zero, which precisely implies that μ (viewed as a Boreal measure on the rectangle $[0,1]\times[m_{\mathcal{W}},M_{\mathcal{W}}]$ in the obvious way), is singular with respect to the restriction of $\mu^{\mathcal{L}}$ to this rectangle.

v. μ is the weak limit of the discrete measures μ_m.

Indeed, this follows at once from the latter part of Property 3.5, according to which, for every $u\in\mathcal{C}\,(\Gamma_{\mathcal{W}})$,

$$\int_{\Gamma_{\mathcal{W}}} u\,d\mu = \lim_{m\to\infty}\int_{\Gamma_{\mathcal{W}}} u\,d\mu_m\,,$$

as desired.

This completes the proof of Theorem 3.7.

□

Remark 3.3. This choice of measure is obtained in the same manner as the one we introduced while working on the Arrowhead Curve [Dav20]. In our present case, however we define the measure more precisely, as well as establish several new properties.

Considering a two-dimensional measure is both essential and natural in so far as we will consider geometric conditions and two-dimensional nets in Section 4.

4 Atomic Decompositions – Trace Theorems, and Consequences

Notation 14 (**Set of Polynomials in Two Real Variables**).

In the sequel, $\mathbb{R}[X,Y]$ denotes the set of all real polynomials in two real variables.
Given $k \in \mathbb{N}$, we will denote by $\mathcal{P}ol_k \subset \mathbb{R}[X,Y]$ the set of all real polynomials in two real variables of degree at most equal to k.

Definition 4.1 (**Two-Dimensional π_r-net ([Wal91], page 119)**).

Given a strictly positive real number r, we will call *two-dimensional π_r-net* a tessellation of $\mathbb{R}^2$ into half-open, non-overlapping squares of side lengths r, obtained by intersecting $\mathbb{R}^2$ with lines orthogonal to the coordinate axes.

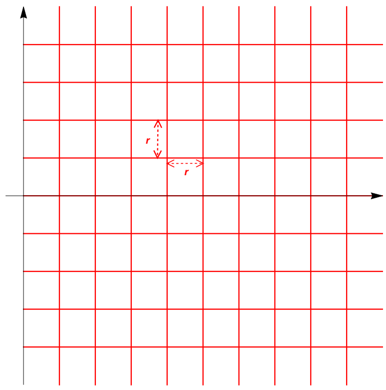


Figure 6: A two-dimensional π_r-net.

Definition 4.2 (**Two-Dimensional Polygonal $\pi_{\mathcal{W},m}$-Net, $m \in \mathbb{N}$**).

Given a strictly positive integer m, we call *two-dimensional polygonal $\pi_{\mathcal{W},m}$-net* a tessellation of $\mathbb{R}^2$ into half-open N_b-gons of side lengths at most equal to $\sqrt{2}\,h_m$ which contains the set of polygons

$$\left\{ \bigcup_{j=0}^{N_b^m-1} \mathcal{P}_{m,j} \right\} \bigcup \left\{ \bigcup_{k=1}^{N_b^m-2} \mathcal{Q}_{m,k} \right\}.$$

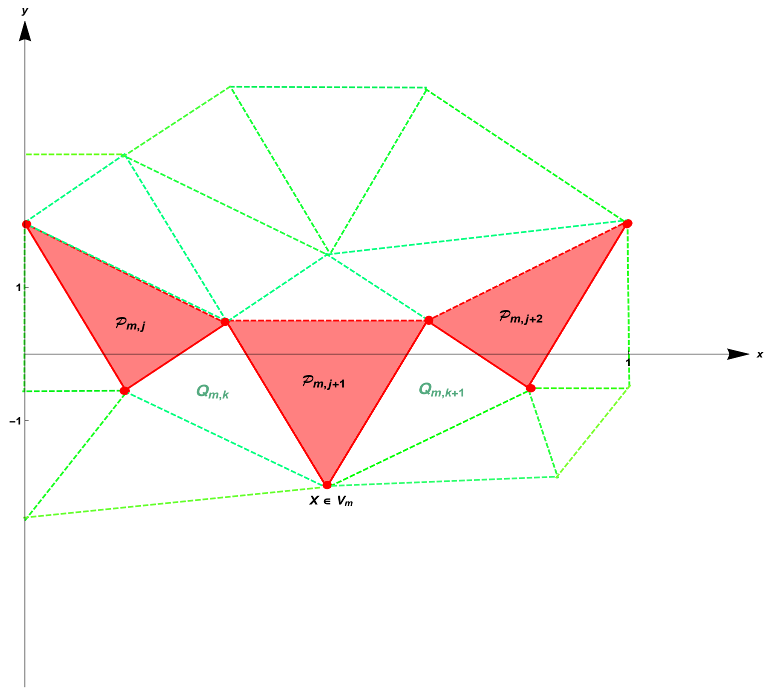


Figure 7: A two-dimensional polygonal $\pi_{\mathcal{W},m}$-net. Note that the polygons are not necessarily isometric.

Property 4.1. *Given a strictly positive integer m, the following properties hold:*

i. For any integer $j \in \{0, \cdots, N_b^m - 1\}$, and any pair of vertices $(X,Y) \in \left(V_m \cap \mathcal{P}_{m,j}\right)^2$:

$$d_{eucl}(X,Y) \lesssim N_b\, h_m \lesssim N_b^{-m\,(2-D_{\mathcal{W}})}\,.$$

ii. For any integer $j \in \{1, \cdots, N_b^m - 2\}$, and any pair of vertices $(X,Y) \in \left(V_m \cap \mathcal{Q}_{m,j}\right)^2$:

$$d_{eucl}(X,Y) \lesssim N_b\, h_m \lesssim N_b^{-m\,(2-D_{\mathcal{W}})}\,.$$

Proof. This simply comes from the fact that the polygons have N_b sides, and that two adjacent vertices are distant from at most an Euclidean distance equal to h_m.

□

Notation 15 (**Set of Piecewise Polynomial Functions on a Polygonal Net**).

Given a pair of natural integers (k, m), and a polygonal $\pi_{\mathcal{W},m}$-net, we denote by $\mathcal{P}ol_k\left(\pi_{\mathcal{W},m}\right)$ the set of non-smooth splines of degree k on $\pi_{\mathcal{W},m}$, i.e., piecewise polynomial functions on $\pi_{\mathcal{W},m}$:

$$\mathcal{P}ol_k\left(\pi_{\mathcal{W},m}\right) = \left\{ spline \text{ such that for any polygon } \mathcal{P} \in \pi_{\mathcal{W},m}, \text{ there exists } P \in \mathcal{P}ol_k : \quad spline_{|\mathcal{P}} = P_{|\mathcal{P}} \right\}$$

Definition 4.3 (**Atoms (Generalization of [Kab12])**).

Given a strictly positive real number $s<1$, a real number $p>1$, two natural integers m and $j\in\left\{0,\cdots,N_b^m-1\right\}$, a function $f_{m,j}$ defined on the prefractal graph $\Gamma_{\mathcal{W}_m}$ will be called a $(\mathcal{P}_{m,j},s,p)$-*atom* if the following three conditions are satisfied:

i. $\operatorname{Supp} f_{m,j}\subset\mathcal{P}_{m,j}$;

ii. $\forall\, X\in V_m\cap\mathcal{P}_{m,j}\,:\qquad \left|f_{m,j}(X)\right|\lesssim\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)^{\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}},$
or, equivalently,

$$\forall\, X\in V_m\cap\mathcal{P}_{m,j}\,:\qquad \left|f_{m,j}(X)\right|\lesssim\left(N_b^{(D_{\mathcal{W}}-3)\,m}\right)^{\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}}\;;$$

iii. $\forall\,(X,Y)\in\left(V_m\cap\mathcal{P}_{m,j}\right)^2\,:\quad \left|f_{m,j}(X)-f_{m,j}(Y)\right|\lesssim d_{eucl}(X,Y)\,\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}},$
or, equivalently,

$$\forall\,(X,Y)\in\left(V_m\cap\mathcal{P}_{m,j}\right)^2\,:\qquad \left|f_{m,j}(X)-f_{m,j}(Y)\right|\lesssim d_{eucl}(X,Y)\,\left(N_b^{(D_{\mathcal{W}}-3)\,m}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}\,.$$

Similarly, given a strictly positive real number $s<1$, a real number $p>1$, two natural integers m and $j\in\left\{1,\cdots,N_b^m-2\right\}$, a function $g_{m,j}$ on the prefractal graph $\Gamma_{\mathcal{W}_m}$ will be called a $(\mathcal{Q}_{m,j},s,p)$-*atom* if the following three conditions are satisfied:

i. $\operatorname{Supp} g_{m,j}\subset\mathcal{Q}_{m,j}$;

ii. $\forall\, X\in V_m\cap\mathcal{Q}_{m,j}\,:\qquad \left|g_{m,j}(X)\right|\lesssim\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)^{\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}},$
or, equivalently,

$$\forall\, X\in V_m\cap\mathcal{Q}_{m,j}\,:\qquad \left|g_{m,j}(X)\right|\lesssim\left(N_b^{(D_{\mathcal{W}}-3)\,m}\right)^{\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}}\;;$$

iii. $\forall\,(X,Y)\in\left(V_m\cap\mathcal{Q}_{m,j}\right)^2\,:\quad \left|g_{m,j}(X)-g_{m,j}(Y)\right|\lesssim d_{eucl}(X,Y)\,\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}},$
or, equivalently,

$$\forall\,(X,Y)\in\left(V_m\cap\mathcal{Q}_{m,j}\right)^2\,:\qquad \left|g_{m,j}(X)-g_{m,j}(Y)\right|\lesssim d_{eucl}(X,Y)\,\left(N_b^{(D_{\mathcal{W}}-3)\,m}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}\,.$$

Remark 4.1 (**Atoms as Hölder Functions**).

Insofar as,

$$\forall\,(X,Y)\in\left(V_m\cap\mathcal{P}_{m,j}\right)^2\,:\quad d_{eucl}(X,Y)\lesssim N_b^{-m\,(2-D_{\mathcal{W}})}\,,$$

the above condition *ii.* for a $(\mathcal{P}_{m,j},s,p)$-atom can be also written as

$$\forall\,(X,Y)\in\left(V_m\cap\mathcal{P}_{m,j}\right)^2\,:\qquad \left|f_{m,j}(X)-f_{m,j}(Y)\right|\lesssim N_b^{-m\,(2-D_{\mathcal{W}})+(D_{\mathcal{W}}-3)\,m\left(\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}\right)}\,,$$

which corresponds to a Hölder exponent of

$$1-\frac{(3-D_{\mathcal{W}})}{2-D_{\mathcal{W}}}\left(\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}\right).$$

Note that, due to their definition, $(\mathcal{P}_{m,j},s,p)$-atoms are necessarily continuous.

An entirely similar property holds if the polygons $\mathcal{P}_{m,j}$ are replaced by their counterpart $\mathcal{Q}_{m,j}$, and the $(\mathcal{P}_{m,j},s,p)$-atoms are replaced by the $(\mathcal{Q}_{m,j},s,p)$-atoms.

Remark 4.2 (**Atoms Associated with the Weierstrass Function**).

In the case of the Weierstrass function $\mathcal{W}$, for any $m\in\mathbb{N}$, and any pair (X,Y) of adjacent vertices in V_m, we have that

$$|\mathcal{W}(X)-\mathcal{W}(Y)|\lesssim d_{eucl}(X,Y)^{2-D_{\mathcal{W}}}.$$

By the triangle inequality, we immediately deduce that,

$$\forall\,(X,Y)\in\left(V_m\cap\mathcal{P}_{m,j}\right)^2\ \left(\text{resp., }\left(V_m\cap\mathcal{Q}_{m,j}\right)^2\right):\quad |\mathcal{W}(X)-\mathcal{W}(Y)|\lesssim d_{eucl}(X,Y)^{2-D_{\mathcal{W}}}.$$

At the same time, thanks to the estimates given in Property 3.3 and in the proof of part *ii.* of Theorem 3.7 (see ($\approx\approx$) and ($\approx\approx\approx\approx$)), for any polygon $\mathcal{P}_{m,j}$ (resp., $\mathcal{Q}_{m,j}$), we have that

$$N_b^{(D_{\mathcal{W}}-3)\,m}\lesssim\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)\lesssim N_b^{(D_{\mathcal{W}}-3)\,m}\quad\left(\text{resp., }N_b^{(D_{\mathcal{W}}-3)\,m}\lesssim\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)\lesssim N_b^{(D_{\mathcal{W}}-3)\,m}\right),$$

or, equivalently,

$$d_{eucl}(X,Y)^{\frac{3-D_{\mathcal{W}}}{2-D_{\mathcal{W}}}}\lesssim\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)\lesssim d_{eucl}(X,Y)^{\frac{3-D_{\mathcal{W}}}{2-D_{\mathcal{W}}}}\quad\left(\text{resp., }d_{eucl}(X,Y)^{\frac{3-D_{\mathcal{W}}}{2-D_{\mathcal{W}}}}\lesssim\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)\lesssim d_{eucl}(X,Y)^{\frac{3-D_{\mathcal{W}}}{2-D_{\mathcal{W}}}}\right).$$

We then deduce that

$$\forall\,(X,Y)\in\left(V_m\cap\mathcal{P}_{m,j}\right)^2\ \left(\text{resp., }\left(V_m\cap\mathcal{Q}_{m,j}\right)^2\right):$$

$$|\mathcal{W}(X)-\mathcal{W}(Y)|\lesssim d_{eucl}(X,Y)\,\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)^{(1-D_{\mathcal{W}})\,\frac{2-D_{\mathcal{W}}}{3-D_{\mathcal{W}}}}$$

$$\left(\text{resp., }|\mathcal{W}(X)-\mathcal{W}(Y)|\lesssim d_{eucl}(X,Y)\,\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)^{(1-D_{\mathcal{W}})\,\frac{2-D_{\mathcal{W}}}{3-D_{\mathcal{W}}}}\right).$$

It then follows that

$$\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}=(1-D_{\mathcal{W}})\,\frac{2-D_{\mathcal{W}}}{3-D_{\mathcal{W}}},\quad\text{i.e.,}\quad s=\frac{D_{\mathcal{W}}}{p}+D_{\mathcal{W}}\,\frac{(1-D_{\mathcal{W}})\,(2-D_{\mathcal{W}})}{3-D_{\mathcal{W}}},$$

and that the restriction of the Weierstrass function to each polygon $\mathcal{P}_{m,j}$, (resp., $\mathcal{Q}_{m,j}$) is a $(\mathcal{P}_{m,j},s,p)$-*atom* (resp., a $(\mathcal{Q}_{m,j},s,p)$-*atom*).

Definition 4.4 (Atomic Decomposition of a Function Defined on the Weierstrass Curve).

Given a continuous function f on the Weierstrass Curve, we will say that f admits *an atomic decomposition* in the following form:

$$\begin{aligned} f &= \lim_{m\to\infty} \left\{ \sum_{j=0}^{N_b^m-1} \sum_{X \text{ vertex of } \mathcal{P}_{m,j}} p(X,\mathcal{P}_m)\, \lambda_{f,m,j,X}\, f_{m,j,X} + \sum_{j=1}^{N_b^m-2} \sum_{X \text{ vertex of } \mathcal{Q}_{m,j}} p(X,\mathcal{Q}_m)\, \lambda_{g,m,j,X}\, g_{m,j} \right\} \\ &= \lim_{m\to\infty} \left\{ \sum_{j=0}^{N_b^m-1} \sum_{X \text{ vertex of } \mathcal{P}_{m,j},\, X\notin \mathcal{Q}_m} \lambda_{f,m,j,X}\, f_{m,j,X} \right. \\ &\quad + \sum_{j=1}^{N_b^m-2} \sum_{X \text{ vertex of } \mathcal{P}_{m,j},\, X\in \mathcal{P}_m\cap\mathcal{Q}_m} \left\{ \lambda_{f,m,j,X}\, f_{m,j,X} + \lambda_{g,m,j,X}\, g_{m,j} \right\} \\ &\quad \left. + \sum_{j=1}^{N_b^m-2} \sum_{X \text{ vertex of } \mathcal{Q}_{m,j},\, X\notin \mathcal{P}_m} \lambda_{g,m,j,X}\, g_{m,j} \right\}, \end{aligned}$$

where, for any $m \in \mathbb{N}$, the functions $f_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$ and $g_{m,j}$, $1 \leqslant j \leqslant N_b^m - 2$ are respectively $(\mathcal{P}_{m,j}, s, p)$ and $(\mathcal{Q}_{m,j}, s, p)$-atoms, $s < 1$, $p > 1$, while the coefficients $\lambda_{f,m,j}$, $0 \leqslant j \leqslant N_b^m - 1$ and $\lambda_{g,m,j}$, $1 \leqslant j \leqslant N_b^m - 2$, denote real numbers.

For the sake of simplicity, we will write the above decomposition in the following briefer form:

$$f = \lim_{m\to\infty} \sum_{X\in\mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X} = \lim_{m\to\infty} \sum_{X\in\mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{f,m}\, \tilde{f}_m \,,$$

where

$$\tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X} = \begin{cases} \lambda_{f,m,j,X}\, f_{m,j,X} \,, \text{if } X \notin \mathcal{Q}_m \,, X \text{ is a vertex of } \mathcal{P}_{m,j} \,, 0 \leqslant j \leqslant N_b^m - 1 \,, \\ \lambda_{g,m,j,X}\, g_{m,j,X} \,, \text{if } X \notin \mathcal{P}_m \,, X \text{ is a vertex of } \mathcal{Q}_{m,j} \,, 1 \leqslant j \leqslant N_b^m - 2 \,, \\ \lambda_{f,m,j,X}\, f_{m,j,X} + \lambda_{g,m,j,X}\, g_{m,j,X}\, f_{m,j,X}, \text{if } X \in \mathcal{P}_m \cap \mathcal{Q}_m \,, X \text{ is a vertex of } \mathcal{P}_{m,j} \text{ and } \mathcal{Q}_{m,j}, \\ \qquad 1 \leqslant j \leqslant N_b^m - 2 \,. \end{cases}$$

For any $m \in \mathbb{N}$, we say that $\tilde{\lambda}_{f,m}$ is the m^{th}-*atomic coefficient.*

From now on, the functions $\tilde{f}_{m,X}$ and $\tilde{f}_m$ will be called (m, s, p')-*atoms.* In the sequel, we will use the most suitable notation among these two possibilities.

Property 4.2 (Atomic Decomposition of Spline Functions in $\mathcal{P}ol_k\left(\pi_{N_b^{(D_\mathcal{W}-3)\,n}}\right)$, $n \in \mathbb{N}$).

Given a pair (n, k) of natural integers, a spline function (denoted by spline*) in $\mathcal{P}ol_k\left(\pi_{N_b^{(D_\mathcal{W}-3)\,n}}\right)$ admits* an atomic decomposition *of the form*

$$spline = \lim_{m\to\infty} \sum_{X\in\mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{s,m,X}\, \widetilde{spline}_{m,X} \,.$$

$$\mathcal{P}ol_k(\pi_r) = \left\{ spline \text{ such that for any square } \mathcal{C}_r \in \pi_r, \text{ there exists } P \in \mathcal{P}ol_k : \ spline_{|\mathcal{C}_r} = P_{|\mathcal{C}_r} \right\}$$

Proof. This directly comes from the definition of functions of $\mathcal{P}ol_k\left(\pi_{N_b^n}\right)$ as piecewise polynomial functions.

□

Property 4.3. *Given the polyhedral measure μ on the Weierstrass Curve $\Gamma_{\mathcal{W}}$, and a continuous function f on $\Gamma_{\mathcal{W}}$, of atomic decomposition*

$$f = \lim_{m\to\infty} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X}\,,$$

we have that

$$\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} f\,d\mu = \lim_{m\to\infty} \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X}\,\mu\,(X,\mathcal{P}_m,\mathcal{Q}_m)\,\cdot$$

Proof. This simply comes from the Definition 3.6 of the polyhedral measure on the Weierstrass IFD.

□

Remark 4.3. Such a decomposition makes sense since the set of vertices $(V_m)_{m\in\mathbb{N}}$ is dense in $\Gamma_{\mathcal{W}}$. Thus, because we deal with continuous functions, given any point X of the Weierstrass Curve, there exists a sequence $(X_m)_{m\in\mathbb{N}}$ such that

$$f(X) = \lim_{m\to\infty} f(X_m)\,,$$

where, for any $m\in\mathbb{N}$, X_m belongs to the prefractal graph $\Gamma_{\mathcal{W}_m}$.

We can naturally write $f(X_m)$ as

$$f(X_m) = \sum_{Y_m\in V_m} f(Y_m)\,\delta_{X_m Y_m}(X_m)\,,$$

where δ is the classical Kronecker symbol; i.e.,

$$\forall\, Y_m \in V_m\ :\quad \delta_{X_m Y_m}(Y_m) = \begin{cases} 1\,, & \text{if} \quad Y_m = X_m\,, \\ 0\,, & \text{else.} \end{cases}$$

This, of course, yields

$$f(X) = \lim_{m\to\infty} \sum_{Y_m\in V_m} f(Y_m)\,\delta_{X_m Y_m}(Y_m)\,\cdot$$

Now, we can go a little further and, as in [Str06], introduce spline functions $\psi^m_{X_m}$ such that

$$\forall\, Y \in \Gamma_{\mathcal{W}}\ :\quad \psi^m_{X_m}(Y) = \begin{cases} \delta_{X_m Y_m}\,, & \forall\, Y \in V_m \\ 0\,, & \forall\, Y \notin V_m\,, \end{cases}$$

and write

$$f(X) = \lim_{m\to\infty} \sum_{Y_m\in V_m} f(Y_m)\,\psi^m_{X_m}(Y_m)\,,$$

which is nothing but the application of the Weierstrass approximation theorem. In particular, spline functions are a natural choice for atoms.

Convention. In the sequel, all functions f considered on the Weierstrass Curve are implicitely supposed to be Lebesgue measurable.

Definition 4.5 (L^p**-Norm of a Function on the Weierstrass Curve).**

Given a function f on the Weierstrass Curve, we define its L^p-norm via

$$\begin{aligned}\|f\|_{L^p(\Gamma_{\mathcal{W}})} &= \left(\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} |f|^p \, d\mu\right)^{\frac{1}{p}} \\ &= \left(\lim_{m\to\infty} \varepsilon^{m\,(D_{\mathcal{W}}-2)} \, \mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) \left|f^p(X)\right|\right)^{\frac{1}{p}} .\end{aligned}$$

Property 4.4 (L^p**-Norm of a Function on the Weierstrass Curve Defined by Means of an Atomic Decomposition).**

Given a positive integer p, and a continuous function f on $\Gamma_{\mathcal{W}}$, whose absolute value $|f|$ is defined by means of an atomic decomposition as

$$|f| = \lim_{m\to\infty} \sum_{X\in\mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{|f|,m,X} \, \widetilde{|f|}_{m,X} \, ,$$

its L^p-norm for the measure μ is given by

$$\begin{aligned}\|f\|_{L^p(\Gamma_{\mathcal{W}})} &= \left(\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} |f|^p \, d\mu\right)^{\frac{1}{p}} \\ &= \left(\lim_{m\to\infty} \varepsilon^{m\,(D_{\mathcal{W}}-3)} \sum_{X\in\mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) \tilde{\lambda}^p_{|f|,m,j,X} \, \widetilde{|f|}^p_{m,j,X}\right)^{\frac{1}{p}}\end{aligned} .$$

In particular, we have that

$$\begin{aligned}\|f\|_{L^1(\Gamma_{\mathcal{W}})} &= \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} |f| \, d\mu \\ &= \lim_{m\to\infty} \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{X\in\mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) \tilde{\lambda}_{|f|,m,j,X} \, \widetilde{|f|}_{m,j,X}\end{aligned} .$$

Remark 4.4. In the above definition, two limits are a priori considered at the same time; the limit associated to the integral, with respect to the polyhedral measure μ, namely,

$$\lim_{m\to\infty} \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{X\in\mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) |f(X)|^p \, ,$$

and the limit associated to the atomic decomposition

$$\lim_{m\to\infty} \sum_{X\in\mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{|f|,m,j,X} \, \widetilde{|f|}^p_{m,j,X} \, .$$

In fact, these two limits coincide.

Definition 4.6 (**Besov Space on the Weierstrass Curve (Extension of the result given by Theorem 6, page 135, in [JW84])**).

Given $k \in \mathbb{N}$, a real number α such that

$$k < \alpha \leqslant k+1 \,,$$

and two real numbers p and q greater or equal to 1, the *Besov space* $B^{p,q}_\alpha\left(\Gamma_{\mathcal{W}}\right)$ is defined as the set of functions $f \in L^p(\mu)$ such that there exists a sequence $(c_m)_{m\in\mathbb{N}} \in \ell^q$ of nonnegative real numbers such that for every $\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}$-net, one can find a spline function $spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right) \in \mathcal{P}ol_{[\alpha]}\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)$ satisfying

$$\left\| f - spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)\right\|_{L^p(\mu)} \leqslant N_b^{(D_{\mathcal{W}}-3)\,m\,\alpha}\, c_m \,, \text{ for all } m \in \mathbb{N}, \qquad \left(\mathcal{C}ond_{Besov\,spline}\right)$$

where, if we write the respective atomic decompositions of f and $spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)$ as

$$f = \lim_{m\to\infty} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X} \,,$$

and

$$spline\left(\pi_{\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}}\right) = \lim_{m\to\infty} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \tilde{\lambda}_{spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right),m,X}\, \widetilde{spline}\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)_{m,X} \,,$$

we then have that

$$\left\| f - spline\left(\pi_{\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}}\right)\right\|^p_{L^p(\mu)} = \lim_{m\to\infty} \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right) \tilde{\lambda}^p_{\left|f-spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)\right|,m,X} \left|\widetilde{f - spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)}\right|^p_{m,X} \,;$$

here, for the sake of simplicity, we have introduced the atomic decomposition of $\left|f - spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)\right|$ as

$$\left|f - spline\left(\pi_{\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}}\right)\right| = \lim_{m\to\infty} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \tilde{\lambda}_{\left|f-spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)\right|,m,X} \left|\widetilde{f - spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)}\right|_{m,X} \,.$$

Remark 4.5. The atomic decomposition used in [Kab12] is obtained by introducing small neighborhoods of the curve under study (union of balls). Our polygonal domain introduced in Definition 4.2 appears to be a more natural choice. Indeed, unlike the aforementioned balls, the polygons involved do not overlap with each other, which works better for the required nets.

Definition 4.7 (**Besov Norm**).

Given $k \in \mathbb{N}$, a real number α such that

$$k < \alpha \leqslant k+1 \,,$$

and two real numbers p and q greater or equal to 1, one can define, as in [Wal91], the $B^{p,q}_\alpha\left(\Gamma_{\mathcal{W}}\right)$-*norm* of a function f defined on the Weierstrass Curve as

$$\|f\|_{B^{p,q}_\alpha(\Gamma_{\mathcal{W}})} = \|f\|_{L^p(\Gamma_{\mathcal{W}})} + \inf\left\{\sum_{n\in\mathbb{N}} c_n^q\right\}^{\frac{1}{q}} ,$$

where the infimum is taken over all the sequences $(c_m)_{m\in\mathbb{N}} \in \ell^q$ of nonnegative real numbers involved in condition $\left(\mathcal{C}ond_{Besov\,spline}\right)$ in Definition 4.6 just above.

Yet, in order to obtain a characterization of the Besov space $B^{p,q}_\alpha\left(\Gamma_{\mathcal{W}}\right)$ by means of its norm, it is more useful to deal with the equivalent norm given by

$$\|f\|_{B^{p,q}_\alpha(\Gamma_{\mathcal{W}})} = \|f\|_{L^p(\Gamma_{\mathcal{W}})} + \left\{\iint_{(T,Y)\in\Gamma^2_{\mathcal{W}}} \frac{|f(T)-f(Y)|^q}{d_{eucl}^{D_{\mathcal{W}}+\alpha q}(T,Y)}\, d\mu^2\right\}^{\frac{1}{q}} \cdot$$

Remark 4.6 (**Alternative Definition of a Besov Space on the Weierstrass Curve**).

One of the interesting properties of Besov spaces is that they can be defined in two different ways: as given previously in Definition 4.6 by means of a polynomial approximation, which provides information on the degree of regularity of the functions involved; also, as the set of functions of a finite specific norm as in Definition 4.7. This latter definition is all the more interesting, because it enables one to make the link with discrete and fractal Laplacians, by means of the fractional difference quotients involved. Our present case is, of course, distinct from the classical one on $\mathbb{R}^N$, with $N \in \mathbb{N}$. Yet, the case of $\mathbb{R}^N$ enables us to understand the underlying connection; namely,

$$B^{p,q}_\alpha\left(\mathbb{R}^N\right) = \left\{ f \in L^p(\mathbb{R}^N) \,,\, \sum_{|j|\leq k} \left\|D^j f\right\|_{L^p(\Gamma_{\mathcal{W}})} + \sum_{|j|=k} \left\{\int_{\mathbb{R}^N} \frac{\|\Delta_h f\|^q_{L^p(\Gamma_{\mathcal{W}})}}{|h|^{n+(\alpha-k)\,q}}\, dh\right\}^{\frac{1}{q}} < \infty \right\},$$

where Δ_h denotes the usual first difference, defined here by

$$\forall\, t \in \Gamma_{\mathcal{W}},\ \forall\, h \in \mathbb{R}^n \,, \quad \Delta_h f(t) = f(t+h) - f(t) \cdot$$

For any $m \in \mathbb{N}$, we have that

$$\begin{aligned}
\varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{T\in\mathcal{P}_m} \mu^{\mathcal{L}}\left(T,\mathcal{P}_m,\mathcal{Q}_m\right)\, \tilde{\lambda}_{f,m} \left|\Delta_m \tilde{f}_m(T)\right|^q &= \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2,\, Y\underset{m}{\sim}T} \mu^{\mathcal{L}}\left(T,\mathcal{P}_m,\mathcal{Q}_m\right)\, \tilde{\lambda}_{\star,m} \frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|^q}{d_{eucl}^{D_{\mathcal{W}}+(\alpha-k)\,q}(T,Y)} \\
&\leq \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2} \mu^{\mathcal{L}}\left(T,\mathcal{P}_m,\mathcal{Q}_m\right)\, \tilde{\lambda}_{\star,m} \frac{\left|\tilde{f}_m(T)-\tilde{f}_m(Y)\right|^q}{d_{eucl}^{D_{\mathcal{W}}+(\alpha-k)\,q}(T,Y)} ,
\end{aligned}$$

where, for the sake of simplicity, we have denoted by $\tilde{\lambda}_{\star,m}$ the atomic coefficients involved.

Note that more points are involved on the right side of the inequality; indeed, on the left side, only adjacent points are considered.

Remark 4.7.

i. This enables us to obtain the first side of the comparison relation required in order to establish the equivalence of norms. It is the side that plays the most significant part in the definition of Besov spaces on the Weierstrass Curve. Thus, characterizing Besov spaces on $\Gamma_{\mathcal{W}}$ by means of the norm defined in Definition 4.7 is directly associated to the definition of a sequence of (suitably renormalized) discrete graph Laplacians $(\Delta_m)_{m \in \mathbb{N}}$ on the sequence of prefractal approximations $\left(\Gamma_{\mathcal{W}_m}\right)_{m \in \mathbb{N}}$. In a sense, it is also connected to the existence of the limit

$$\lim_{m \to \infty} \Delta_m$$

by means of an equivalent pointwise formula expressed in terms of integrals, somehow the counterpart, in a way, of the one which is well known in the case of the fractal Laplacian on the Sierpiński Gasket [Kig01], [Str06].

ii. The difficulty, in our context, is to obtain an equivalent formulation of the definition of Besov spaces with the sequence of discrete Laplacians alluded to in part *i*. Clearly, a discrete Laplacian corresponds to the usual first difference Δ_h. Working with discrete Laplacians, along with atomic decompositions of functions, leads to expressions of the following form:

$$\lim_{m \to \infty} \varepsilon^{2\,m\,(D_{\mathcal{W}}-2)} \sum_{(T,Y) \in (\mathcal{P}_m \bigcup \mathcal{Q}_m)^2,\, Y \underset{m}{\sim} T} \mu^{\mathcal{L}}\,(T, \mathcal{P}_m, \mathcal{Q}_m)\; \mu^{\mathcal{L}}\,(Y, \mathcal{P}_m, \mathcal{Q}_m)\; \tilde{\lambda}_{f,m}\, \frac{\left|\tilde{f}_m(T) - \tilde{f}_m f(Y)\right|^q}{d_{eucl}^{D_{\mathcal{W}}+(\alpha-k)\,q}(T,Y)}\,,$$

Property 4.5 (**Characterization of Besov Spaces (Sufficient and Necessary Condition)**).

Given $k \in \mathbb{N}$, a real number α such that

$$k < \alpha \leqslant k+1\,,$$

two real numbers p and q greater or equal to 1, and a continuous function f given by means of an atomic decomposition of the form

$$f = \lim_{m \to \infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X}$$

belongs to the Besov space $B_\alpha^{p,q}\,(\Gamma_{\mathcal{W}})$ if and only if the following two conditions are satisfied,

$$(3 - D_{\mathcal{W}})\left\{q\left(\frac{1}{p} - \frac{s-1}{D_{\mathcal{W}}}\right)\right\} + (2 - D_{\mathcal{W}})\,(D_{\mathcal{W}} + (\alpha-1)\,q) < 2\,, \quad (\mathcal{C}ond_{Besov})$$

and

$$\frac{D_{\mathcal{W}}}{3 - D_{\mathcal{W}}} + \frac{D_{\mathcal{W}}}{p} \leqslant s\,, \quad (\mathcal{C}ond_{L^p})$$

where the real number $s \in \,]0,1[$ has been introduced in Definition 4.3.

Proof.

i. We have that

$$\|f\|^q_{B^{p,q}_\alpha(\Gamma_{\mathcal{W}})} = \|f\|^p_{L^p(\Gamma_{\mathcal{W}})} + \int_{T\in\Gamma_{\mathcal{W}}}\int_{Y\in\Gamma_{\mathcal{W}}} \frac{|f(T)-f(Y)|^q}{d_{eucl}^{D_{\mathcal{W}}+\alpha q}(T,Y)}\,d\mu^2\,,$$

where

$$\begin{aligned}\int_{T\in\Gamma_{\mathcal{W}}}\int_{Y\in\Gamma_{\mathcal{W}}} \frac{|f(T)-f(Y)|^q}{d_{eucl}^{D_{\mathcal{W}}+\alpha q}(T,Y)}\,d\mu^2 &= \lim_{m\to\infty}\varepsilon^{2m(D_{\mathcal{W}}-2)}\sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2}\mu\left(T,\mathcal{P}_m,\mathcal{Q}_m\right)\,\mu\left(Y,\mathcal{P}_m,\mathcal{Q}_m\right)\,\frac{|f(T)-f(Y)|^q}{d_{eucl}^{D_{\mathcal{W}}+\alpha q}(T,Y)}\\ &= \lim_{m\to\infty}\varepsilon^{2m(D_{\mathcal{W}}-2)}\sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2}\mu^{\mathcal{L}}\left(T,\mathcal{P}_m,\mathcal{Q}_m\right)\,\mu^{\mathcal{L}}\left(Y,\mathcal{P}_m,\mathcal{Q}_m\right)\,\tilde\lambda_{f,m}\,\frac{\left|\tilde f_m(T)-\tilde f_m(Y)\right|^q}{d_{eucl}^{D_{\mathcal{W}}+\alpha q}(T,Y)}\,.\end{aligned}$$

Note that since the function f is continuous, the atomic coefficients $\lambda_{\star,m}$ are necessarily bounded (since $\Gamma_{\mathcal{W}}$ is compact, and hence, f is bounded).

Since we deal with the atomic decomposition of f, due to Definition 4.3, part *iii.*, we have that

$$\mu^{\mathcal{L}}\left(T,\mathcal{P}_m,\mathcal{Q}_m\right)\,\mu^{\mathcal{L}}\left(Y,\mathcal{P}_m,\mathcal{Q}_m\right)\,\tilde\lambda_{f,m}\,\frac{\left|\tilde f_n(T)-\tilde f_m(Y)\right|^q}{d_{eucl}^{D_{\mathcal{W}}+\alpha q}(T,Y)} \lesssim \mu^{\mathcal{L}}\left(T,\mathcal{P}_m,\mathcal{Q}_m\right)\,\mu^{\mathcal{L}}\left(Y,\mathcal{P}_m,\mathcal{Q}_m\right)\,\tilde\lambda_{f,m}\,\frac{\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{d}-\frac{1}{p}}}{d_{eucl}^{D_{\mathcal{W}}+\alpha q-q}(T,Y)}\,.$$

We also have that

$$\frac{1}{d_{eucl}(T,Y)} < \frac{1}{|h_{jm}|} \lesssim L_m^{D_{\mathcal{W}}-2} \lesssim N_b^{(2-D_{\mathcal{W}})\,m}\,,$$

and

$$d_{eucl}(T,Y) \lesssim h_m \lesssim L_m^{2-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-2)\,m}\,.$$

Moreover,

$$\mu^{\mathcal{L}}\left(T,\mathcal{P}_m,\mathcal{Q}_m\right) \lesssim h_m\,L_m \lesssim L_m^{3-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-3)\,m}$$

and, in the same way,

$$\mu^{\mathcal{L}}\left(Y,\mathcal{P}_m,\mathcal{Q}_m\right) \lesssim N_b^{(D_{\mathcal{W}}-3)\,m}\,.$$

We therefore deduce that

$$\mu^{\mathcal{L}}\left(T,\mathcal{P}_m,\mathcal{Q}_m\right)\,\mu^{\mathcal{L}}\left(Y,\mathcal{P}_m,\mathcal{Q}_m\right)\,\tilde\lambda_{f,m}\,\frac{\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{d}-\frac{1}{p}}}{d_{eucl}^{D_{\mathcal{W}}+\alpha q-q}(T,Y)} \lesssim N_b^{2\,m\,(D_{\mathcal{W}}-3)}\,\tilde\lambda_{f,m}\,\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}\,N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)}\,,$$

and

$$\begin{aligned}&\varepsilon^{2\,m\,(D_{\mathcal{W}}-2)}\,\mu^{\mathcal{L}}\left(T,\mathcal{P}_m,\mathcal{Q}_m\right)\,\mu^{\mathcal{L}}\left(Y,\mathcal{P}_m,\mathcal{Q}_m\right)\,\tilde\lambda_{f,m}\,\frac{\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{d}-\frac{1}{p}}}{d_{eucl}^{D_{\mathcal{W}}+\alpha q-q}(T,Y)}\\ &\lesssim \varepsilon^{2\,m\,(D_{\mathcal{W}}-2)}\,N_b^{2\,m\,(D_{\mathcal{W}}-3)}\,\tilde\lambda_{f,m}\,\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}\,N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)}\,,\end{aligned}$$

$$\int_{T\in\Gamma_{\mathcal{W}}}\int_{Y\in\Gamma_{\mathcal{W}}}\frac{\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{d}-\frac{1}{p}}}{d_{eucl}^{D_{\mathcal{W}}+\alpha\,q-q}(T,Y)}\,d\mu^2 =$$

$$= \lim_{m\to\infty}\varepsilon^{2\,m\,(D_{\mathcal{W}}-2)}\sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2}\mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\,\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m)\,\frac{\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}}{d_{eucl}^{D_{\mathcal{W}}+\alpha\,q-a}(T,Y)}$$

$$\lesssim \lim_{m\to\infty}\varepsilon^{2\,m\,(D_{\mathcal{W}}-2)}\sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2}\mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\,\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m)\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)}$$

$$\lesssim \lim_{m\to\infty}\varepsilon^{2\,m\,(D_{\mathcal{W}}-2)}\sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2}N_b^{2\,m\,(D_{\mathcal{W}}-3)}\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)}\cdot$$

The function f will thus belong to the Besov space $B_\alpha^{p,q}(\Gamma_{\mathcal{W}})$ provided that

$$\sum_{m\in\mathbb{N}}\varepsilon^{2\,m\,(D_{\mathcal{W}}-2)}\,N_b^{2\,m\,(D_{\mathcal{W}}-3)}\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)}<\infty\,,$$

or, equivalently,

$$\sum_{m\in\mathbb{N}}N_b^{-2\,m\,(D_{\mathcal{W}}-2)}\,N_b^{2\,m\,(D_{\mathcal{W}}-3)}\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)}<\infty\,,$$

converges, which requires that

$$-2\,(D_{\mathcal{W}}-2)+2\,(D_{\mathcal{W}}-3)-q\,(3-D_{\mathcal{W}})\left(\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}\right)+(2-D_{\mathcal{W}})\,(D_{\mathcal{W}}+\alpha\,q-q)<0\,,$$

i.e.,

$$(3-D_{\mathcal{W}})\left\{q\left(\frac{1}{p}-\frac{s-1}{D_{\mathcal{W}}}\right)\right\}+(2-D_{\mathcal{W}})\,(D_{\mathcal{W}}+(\alpha-1)\,q)<2\cdot\quad(\mathcal{C}ond_{Besov})$$

At the same time, thanks to Property 4.4, and in light of Definition 4.3, we have that

$$\begin{aligned}
\|f\|^p_{L^p(\Gamma_{\mathcal{W}})} &= \lim_{m\to\infty}\varepsilon^{m\,(D_{\mathcal{W}}-3)}\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)\,\tilde{\lambda}^p_{|f|,m,j,X}\,\widetilde{|f|}^p_{m,j,X}\\
&\lesssim \lim_{m\to\infty}\varepsilon^{D_{\mathcal{W}}-3}\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)\,\tilde{\lambda}^p_{|f|,m,j,X}\,\underbrace{\mu_{\mathcal{L}}(\mathcal{P}_{m,j})}_{\lesssim\,\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)}{}^{\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}}\\
&\lesssim \lim_{m\to\infty}\varepsilon^{m\,(D_{\mathcal{W}}-3)}\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\underbrace{\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)}_{\lesssim\,N_b^{(D_{\mathcal{W}}-3)\,m}}{}^{1+\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}}\\
&\lesssim \lim_{m\to\infty}N_b^{m\,(3-D_{\mathcal{W}})}\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}N_b^{(D_{\mathcal{W}}-3)\,m\left(1+\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}\right)}\\
&\lesssim \lim_{m\to\infty}N_b^m\,N_b^{m\,(3-D_{\mathcal{W}})}\,N_b^{(D_{\mathcal{W}}-3)\,m\left(1+\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}\right)}\,.
\end{aligned}$$

The convergence of this latter expression requires that

$$1+3-D_{\mathcal{W}}+(D_{\mathcal{W}}-3)\left(1+\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}\right)=1+(D_{\mathcal{W}}-3)\left(\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}\right)\leqslant 0\,,$$

i.e.,

$$1\leqslant(3-D_{\mathcal{W}})\left(\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}\right),$$

or, equivalently,

$$\frac{D_{\mathcal{W}}}{3-D_{\mathcal{W}}}+\frac{D_{\mathcal{W}}}{p}\leqslant s\,\cdot\quad(\mathcal{C}ond_{L^p})$$

ii. Conversely, we can check that if the above conditions $(\mathcal{C}ond_{Besov})$ and $(\mathcal{C}ond_{L^p})$ are satisfied,

$$\|f\|_{L^p(\Gamma_{\mathcal{W}})}+\left\{\iint_{(T,Y)\in\Gamma^2_{\mathcal{W}}}\frac{|f(T)-f(Y)|^q}{d_{eucl}^{D_{\mathcal{W}}+\alpha\,q}(T,Y)}\,d\mu^2\right\}^{\frac{1}{q}}<\infty\,,$$

as desired. □

Corollary 4.6 (The Specific Case of $B^{p,p}_\beta(\Gamma_{\mathcal{W}})$).

Given $k\in\mathbb{N}$, a real number p greater or equal to 1, we set

$$\beta=k-\frac{2-D_{\mathcal{W}}}{p}\,\cdot$$

A function f, given by means of an atomic decomposition of the form:

$$f=\lim_{m\to\infty}\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\tilde{\lambda}_{f,m,X}\,\tilde{f}_{m,X}\,,$$

belongs to the Besov space $B^{p,p}_\beta(\Gamma_{\mathcal{W}})$ if and only if

$$(3-D_{\mathcal{W}})\left\{p\left(\frac{1}{p}-\frac{s-1}{D_{\mathcal{W}}}\right)\right\}+(2-D_{\mathcal{W}})\,(D_{\mathcal{W}}+(\beta-1)\,p)<2\,,$$

and

$$\frac{D_{\mathcal{W}}}{3-D_{\mathcal{W}}}+\frac{D_{\mathcal{W}}}{p}\leqslant s\,\cdot$$

Definition 4.8 (Trace of an $L^1_{loc}(\mathbb{R}^2)$ Function on the Weierstrass Curve).

Along the lines of [**?**], page 15, or [Wal91], we will say that an $L^1_{loc}(\mathbb{R}^2)$ function f is *strictly defined* at a vertex X of the Weierstrass Curve if the following limit exists and is given by

$$\bar{f}(X)=\lim_{m\to\infty}\frac{1}{\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)}\sum_{Y\sim X}f(Y)<\infty\,\cdot$$

This enables us to define *the trace* $f_{|\Gamma_{\mathcal{W}}}$ of the function f on the Weierstrass Curve, via

$$\forall\,X\in\Gamma_{\mathcal{W}}\,:\,f_{|\Gamma_{\mathcal{W}}}(X)=\bar{f}(X)\,\cdot$$

Remark 4.8. The trace $\bar{f}$ of an $L^1_{loc}(\mathbb{R}^2)$ function thus naturally admits an atomic decomposition of the form

$$\bar{f} = \lim_{m\to\infty} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \tilde{\lambda}_{\bar{f},m,X}\,\tilde{\bar{f}}_{m,X}\,\cdot$$

The following simple property was already used implicitly when introducing the polyhedral measure, earlier in Section 3.

Property 4.7 (**The Compact Set of $\mathbb{R}^2$ which Contains the Weierstrass Curve**).

The Weierstrass Curve $\Gamma_{\mathcal{W}}$ is contained in the following compact set of $\mathbb{R}^2$,

$$\Omega_{\mathcal{W}} = [0,1]\times[m_{\mathcal{W}},M_{\mathcal{W}}]\,,$$

where $m_{\mathcal{W}}$ and $M_{\mathcal{W}}$ respectively denote the minimal and maximal values of the Weierstrass function $\mathcal{W}$ on $[0,1]$, introduced in Notation 12.

Notation 16 (**Interior of the Compact Set $\Omega_{\mathcal{W}}$**).

We hereafter denote by $\mathring{\Omega}_{\mathcal{W}}$ the interior of the compact set $\Omega_{\mathcal{W}}$.

Definition 4.9 (**Sobolev Space on the Open Set $\mathring{\Omega}_{\mathcal{W}}$**).

Given $k\in\mathbb{N}$, and $p\geqslant 1$, we recall that the Sobolev space on the open set $\mathring{\Omega}_{\mathcal{W}}\subset\mathbb{R}^2$, denoted by $W^p_k\left(\mathring{\Omega}_{\mathcal{W}}\right)$, is given by

$$W^p_k\left(\mathring{\Omega}_{\mathcal{W}}\right) = \left\{f\,\in\,L^p\left(\mathring{\Omega}_{\mathcal{W}}\right)\,,\,\forall\,\alpha\leqslant k\,,\,D^\alpha f\,\in\,L^p\left(\mathring{\Omega}_{\mathcal{W}}\right)\right\}\,,$$

where $L^p\left(\mathring{\Omega}_{\mathcal{W}}\right)$ denotes the Lebesgue space of order p on $\mathring{\Omega}_{\mathcal{W}}$, while, for the multi-index $\alpha\leqslant k$, $D^\alpha f$ is the classical partial derivative of order α, interpreted in the weak sense.

The following result is the counterpart, in our context, of the corresponding one obtained in [JW84], Chapter VI.

Theorem 4.8 (**The Trace of Sobolev Spaces as Besov Spaces**).

Given a positive integer k, and a real number $p\geqslant 1$, we set

$$\beta = k - \frac{2-D_{\mathcal{W}}}{p}\,\cdot$$

We then have that

$$W^p_k\left(\,\mathring{\Omega}_{\mathcal{W}}\right)_{|\Gamma_{\mathcal{W}}} = B^{p,p}_\beta\left(\Gamma_{\mathcal{W}}\right)\,\cdot$$

Corollary 4.9 (The Specific Case $k = 2$, and its Consequences – Order of the Fractal Laplacian).

Given a real number $p \geqslant 1$, we set

$$\beta_{k,p} = 2 - \frac{2 - D_{\mathcal{W}}}{p} = 2 - \frac{1}{p}\,\frac{\ln\lambda}{\ln N_b}\cdot$$

We then have that

$$W_2^p\left(\mathring{\Omega}_{\mathcal{W}}\right)_{|\Gamma_{\mathcal{W}}} = B^{p,p}_{\beta_{2,p}}\left(\Gamma_{\mathcal{W}}\right),$$

where

$$\beta_{2,p} = 2 - \frac{1}{p}\,\frac{\ln\lambda}{\ln N_b} = 2 + \frac{2 - D_{\mathcal{W}}}{p}\cdot$$

In particular, in the case where $p = 2$, which corresponds to

$$\beta_{2,2} = 2 + \frac{2 - D_{\mathcal{W}}}{2},$$

and provided that

$$(3 - D_{\mathcal{W}})\left\{2\left(\frac{1}{2} - \frac{s-1}{D_{\mathcal{W}}}\right)\right\} + (2 - D_{\mathcal{W}})\left(D_{\mathcal{W}} - 2\left(1 + \frac{2 - D_{\mathcal{W}}}{2}\right)\right) < 2,$$

i.e.,

$$1 - D_{\mathcal{W}} - 2\,(3 - D_{\mathcal{W}})\,\frac{s-1}{D_{\mathcal{W}}} + (2 - D_{\mathcal{W}})\,(2\,D_{\mathcal{W}} - 3) < 0,$$

or, equivalently,

$$s > 1 + D_{\mathcal{W}}\,\frac{1 - D_{\mathcal{W}} + (2 - D_{\mathcal{W}})\,(2\,D_{\mathcal{W}} - 3)}{2\,(3 - D_{\mathcal{W}})},$$

we then have that

$$W_2^2\left(\Omega_{\mathcal{W}}\right)_{|\Gamma_{\mathcal{W}}} = B^{2,2}_{\beta_{2,2}}\left(\Gamma_{\mathcal{W}}\right),$$

where

$$\beta_{2,2} = 2 - \frac{1}{2}\,\frac{\ln\lambda}{\ln N_b} > 2\cdot$$

Consequently, by analogy with the classical theories, the Laplacian on the Weierstrass Curve (see Remark 4.7 above) arises as a differential operator of order $\beta_{2,2} \in]2,3[$.

Remark 4.9 (**Connection with the Optimal Exponent of Hölder Continuity**).

We note that

$$\beta_{2,2} = 2 + \frac{\alpha_{\mathcal{W}}}{2},$$

where the Codimension $\alpha_{\mathcal{W}} = 2 - D_{\mathcal{W}} = -\frac{\ln\lambda}{\ln N_b} \in]0,1[$ is the best (i.e., optimal) Hölder exponent for the Weierstrass function, as was initially obtained by G. H. Hardy in [Har16]), and then, by a completely different method – geometrically – in [DL22b] (this latter result is recalled in Theorem 2.13 and in Corollary 2.14).

Property 4.10 (Connection with Fractional Derivatives).

In Definition 4.7, the Besov norm of a function f defined on the Weierstrass Curve involves the integral

$$\iint_{(T,Y)\in\Gamma_{\mathcal{W}}^2} \frac{|f(T)-f(Y)|^q}{d_{eucl}^{D_{\mathcal{W}}+\alpha q}(T,Y)}\, d\mu^2 \cdot \qquad (\maltese)$$

For $p=2$, $q=1$ and $\alpha = 3-D_{\mathcal{W}}$, i.e., if $f \in B^{2,1}_{3-D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$, the integral in $(\maltese)$ just above can be connected to the associated fractional derivative of order $\gamma = 2-D_{\mathcal{W}} \in]0,1[$, defined, for any vertex $Y \in V_{m_0} \subset \Gamma_{\mathcal{W}}$, with $m_0 \in \mathbb{N}$, by the following expression:

$$\begin{aligned} D^{2-D_{\mathcal{W}}} f(Y) &= \frac{2-D_{\mathcal{W}}}{\Gamma(D_{\mathcal{W}}-1)} \lim_{m\to\infty} \int_{T\in\mathcal{D}(\Gamma_{\mathcal{W}}),\, T \underset{m}{\sim} Y} \frac{|f(T)-f(Y)|}{d_{eucl}^{3-D_{\mathcal{W}}}(T,Y)}\, d\mu \\ &= \frac{2-D_{\mathcal{W}}}{\Gamma(D_{\mathcal{W}}-1)} \lim_{m\to\infty} \varepsilon^{m\,(D_{\mathcal{W}}-2)} \sum_{T \underset{m}{\sim} Y} \mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\, \lambda_{\star,m} \frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{eucl}^{3-D_{\mathcal{W}}}(T,Y)}\,, \end{aligned}$$

where, for any $m \in \mathbb{N}$, $\lambda_{\star,m}$ is the m^{th} scalar coefficient involved, and where Γ denotes the usual Gamma function.

Remark 4.10 (**On the Existence of Fractional Derivatives**).

Note that due to the first condition given in Definition 5.1 of Section 5 below, there exist strictly positive constants $\tilde{C}_{inf}$ and $\tilde{C}_{sup}$ such that, for all vertices $(T,Y) \in V_m \times V_m \subset \Gamma_{\mathcal{W}} \times \Gamma_{\mathcal{W}}$,

$$\tilde{C}_{inf}\, d_{eucl}^{2-D_{\mathcal{W}}}(T,Y) \leqslant \left|\tilde{f}_m(T)-\tilde{f}_m(Y)\right| \leqslant \tilde{C}_{sup}\, d_{eucl}^{2-D_{\mathcal{W}}}(T,Y) \cdot$$

Hence, we have that, for all vertices $(T,Y) \in V_m \times V_m \subset \Gamma_{\mathcal{W}} \times \Gamma_{\mathcal{W}}$,

$$\frac{\tilde{C}_{inf}}{d_{eucl}(T,Y)} \leqslant \frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{eucl}^{3-D_{\mathcal{W}}}(T,Y)} \leqslant \frac{\tilde{C}_{sup}}{d_{eucl}(T,Y)}\,,$$

or, equivalently, expressed in terms of the cohomology infinitesimal ε^m (see Definition 3.1),

$$\frac{\tilde{C}_{inf}}{\varepsilon^m} \leqslant \frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{eucl}^{3-D_{\mathcal{W}}}(T,Y)} \leqslant \frac{\tilde{C}_{sup}}{\varepsilon^m}\,,$$

which, in conjunction with the estimates given in Property 3.3, namely,

$$\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m) \lesssim \varepsilon^{m\,(3-D_{\mathcal{W}})}\,,$$

yields

$$\varepsilon^{m\,(D_{\mathcal{W}}-2)}\, \varepsilon^{m\,(3-D_{\mathcal{W}})} \frac{\tilde{C}_{inf}}{\varepsilon^m} \lesssim \varepsilon^{m\,(D_{\mathcal{W}}-2)}\, \mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m) \frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{eucl}^{3-D_{\mathcal{W}}}(T,Y)} \lesssim \varepsilon^{m\,(D_{\mathcal{W}}-2)}\, \varepsilon^{m\,(3-D_{\mathcal{W}})} \frac{\tilde{C}_{sup}}{\varepsilon^m}\,;$$

i.e.,

$$\tilde{C}_{inf} \lesssim \varepsilon^{m\,(D_{\mathcal{W}}-2)}\,\mu^{\mathcal{L}}\,(T,\mathcal{P}_m,\mathcal{Q}_m)\,\frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{eucl}^{3-D_{\mathcal{W}}}(T,Y)} \lesssim \tilde{C}_{sup}\,.$$

Since the vertex $Y \in V_m$ admits at most two adjacent vertices in V_m, we then deduce that

$$\tilde{C}_{inf}\,\left|\lambda_{\star,m}\right| \lesssim \varepsilon^{m\,(D_{\mathcal{W}}-2)}\,\mu^{\mathcal{L}}\,(T,\mathcal{P}_m,\mathcal{Q}_m)\,\left|\lambda_{\star,m}\right|\,\frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{eucl}^{3-D_{\mathcal{W}}}(T,Y)} \lesssim \tilde{C}_{sup}\,\left|\lambda_{\star,m}\right|\,.$$

Note that in the case of a continuous function f, the atomic coefficients $\lambda_{\star,m}$ are necessarily bounded (since $\Gamma_{\mathcal{W}}$ is compact, and hence, f is bounded).

5 Towards an Extension of Morse Theory

Classical Morse theory (se, e.g., [Bot82], [Bot88] and [Mil63]) enables us to explore the shape (i.e., the topology) of a smooth manifold by means of the study of the critical points of suitable smooth functions defined on the manifold. Such functions are required to be nondegenerate – in the sense that their Hessian determinant is nonzero at critical points – and are then called Morse functions.

In the classical Morse theory, i.e., for smooth manifolds, the height function plays a major role. More precisely, along with its critical points and its level sets, it encodes the information that enables us to reconstruct the manifold.

For fractal curves or IFDs such as, for instance, the Weierstrass Curve, this does not make sense anymore. Since fractals are involved, a change of shape occurs at each vertex of each prefractal approximation.

In particular, we will show that the Weierstrass function $\mathcal{W}$ is a fractal Morse function. Note (as in [DL22b], [DL22c], [DL22a]) that $\mathcal{W}$ can be viewed as a function on $\Gamma_{\mathcal{W}}$, namely, the identity function; see Remark 5.4 below.

By using some of the results of [DL22b], [DL22c], [DL22a], we hereafter begin to lay the foundations of a fractal Morse theory that should eventually enable us to explore the shape of *fractal manifolds* (viewed as higher-dimensional IFDs). The important example of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ – or, rather, of the associated Weierstrass IFD – sheds a useful light, especially when it comes to studying not only the (appropriately defined) critical points, but also other suitable (inflection) points of *fractal Morse functions*, and the remainaing points, which are themselves subject to a change of shape, in connection with the associated angle between the corresponding adjacent edges; i.e., a typical change of curvature.

In this light, the simple knowledge of the equivalent of the usual Morse indexes, along with a suitable analogue of the height function, does not appear as being sufficient when it comes to reconstruct the fractal IFD. The missing data can be obtained by means of the sequence of cohomological integers associated, at each step $m \in \mathbb{N}$ of the prefractal approximation, with the set of vertices V_m; see Definition 5.5.

Also, in our context, the sequence of sets of critical point introduced in Definition 5.6 is an increasing sequence. For this reason, instead of the height function, we choose to consider *the increasing reordering* of *the absolute heights*, as introduced in Definition 5.11 below. We believe that this increasing reordering still bears the fractality of the Weierstrass Curve.

As was previously encountered in Corollary 2.17, at any given level $m \in \mathbb{N}$ of the prefractal graph approximation of the Curve, the extreme and bottom vertices of the polygons $\mathcal{P}_{m,k}$, $0 \leqslant k \leqslant N_b^m - 1$, are respectively local maxima and minima of the Weierstrass function $\mathcal{W}$. Those points are isolated ones, a result that directly comes from the construction of the Weierstrass Curve. Then, there remains to identify points that could play the role of degenerate ones, and to define *fractal suited Morse functions*. Also, as was evoked in the introduction, the respective notions of *maximal Complex Dimension* and *cohomogical vertex integers* naturally arise from our results obtained in [DL22c] on the fractal cohomology of the Weierstrass Curve. For the sake of a better understanding, we next briefly recall those results.

Definition 5.1 (**Set of Functions of the Same Nature as the Weierstrass Function** $\mathcal{W}$ [**DL22c**]).

i. We say that a continuous, complex-valued function f, defined on $\Gamma_{\mathcal{W}} \supset V^\star$, is *of the same nature as the Weierstrass function* $\mathcal{W}$, if it satisfies local Hölder and reverse-Hölder properties analogous to those satisfied by the Weierstrass function $\mathcal{W}$; i.e., for any $m \in \mathbb{N}$ and for any pair of adjacent vertices (M, M') with respective affixes $(z, z') \in \mathbb{C}^2$, of the prefractal graph $\Gamma_{\mathcal{W}_m}$, we have that

$$\tilde{C}_{inf}\, |z' - z|^{2-D_{\mathcal{W}}} \leqslant \left| f(z') - f(z) \right| \leqslant \tilde{C}_{sup}\, |z' - z|^{2-D_{\mathcal{W}}} ,$$

where $\tilde{C}_{inf}$ and $\tilde{C}_{sup}$ denote suitable positive and finite constants possibly depending on f. This can be written, equivalently, as

$$\left| z - z' \right|^{2-D_{\mathcal{W}}} \lesssim \left| f(z) - f(z') \right| \lesssim \left| z - z' \right|^{2-D_{\mathcal{W}}} \cdot \qquad (\blacklozenge)$$

(Compare with Theorem 2.13 and Corollary 2.14 above.)

Hereafter, we will denote by $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ the set consisting of the continuous, complex-valued functions f, defined on $\Gamma_{\mathcal{W}} \supset V^\star$ and satisfying ($\blacklozenge$); see part *i.* just above.

ii. Moreover, we will denote by $\mathcal{H}\ddot{o}ld_{geom}\,(\Gamma_{\mathcal{W}}) \subset \mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ the subset of $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ consisting of the functions f of $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ which satisfy the following *additional geometric condition* ($\spadesuit$), again, for any pair of adjacent vertices (M, M') of the prefractal graph V_m with respective affixes $(z, z') \in \mathbb{C}^2$, and for $m \in \mathbb{N}$ arbitrary; namely,

$$\left| \arg\,(f(z)) - \arg\left(f(z')\right) \right| \lesssim |z - z'| \cdot \qquad (\spadesuit)$$

Remark 5.1. Note that, according to the results of [DL22b] and [DL22c], the Weierstrass function $\mathcal{W}$ belongs to $\mathcal{H}\ddot{o}ld_{geom}\,(\Gamma_{\mathcal{W}})$ – and hence, also, to $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$.

Definition 5.2 (**(m, p)-Fermion [DL22c]**).

By analogy with particle physics, given a pair of integers (m, p), with $m \in \mathbb{N}$ and $p \in \mathbb{N}^\star$, we will call (m, p)-*fermion* on V_m, with values in $\mathbb{C}$, any antisymmetric map f from V_m^{p+1} to $\mathbb{C}$. Note that these maps are not assumed to be multilinear (which would be meaningless here, anyway, since V_m is not a vector space).

For any $m \in \mathbb{N}$, an $(m,0)$-*fermion* on V_m (or a 0-*fermion*, in short) is simply a map f from V_m to $\mathbb{C}$. We adopt the convention according to which a 0-fermion on V_m is a 0-*antisymmetric map* on V_m.

In the sequel, for any $(m,p) \in \mathbb{N}^2$, we will denote by $\mathcal{F}^p\left(V_m,\mathbb{C}\right)$ the complex vector space of (m,p)-fermions on V_m, which makes it an abelian group with respect to the addition, with an external law from $\mathbb{C}\times\mathcal{F}^p\left(V_m,\mathbb{C}\right)$ to $\mathcal{F}^p\left(V_m,\mathbb{C}\right)$.

Definition 5.3 (**$(m-1,m)$-Differentials [DL22c]**).

Given a strictly positive integer m, we define the $(m-1,m)$-*differential* $\delta_{m-1,m}$ from $\mathcal{F}^0\left(V_m,\mathbb{C}\right)$ to $\mathcal{F}^{N_b+1}\left(V_m,\mathbb{C}\right)$, for any f in $\mathcal{F}^0\left(\Gamma_{\mathcal{W}},\mathbb{C}\right)$ and any $\left(M_{i,m-1},M_{i+1,m-1},M_{j+1,m},\cdots,M_{j+N_b-2,m}\right) \in V_m^{N_b+1}$ such that

$$M_{i,m-1} = M_{j,m} \quad \text{and} \quad M_{i+1,m-1} = M_{j+N_b,m} \,,$$

by

$$\delta_{m-1,m}(f)\left(M_{i,m-1},M_{i+1,m-1},M_{j+1,m},\cdots,M_{j+N_b-1,m}\right) = c_{m-1,m}\left(\sum_{q=0}^{N_b}(-1)^q\, f\left(M_{j+q,m}\right)\right),$$

where $c_{m-1,m}$ denotes a suitable positive constant.

Theorem 5.1 (**Fractal Cohomology of the Weierstrass Curve [DL22c]**).

Within the set $\mathcal{H}\ddot{o}ld_{geom}\left(\Gamma_{\mathcal{W}}\right)$ of continuous, complex-valued functions f, defined on the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset V^\star = \bigcup_{n\in\mathbb{N}} V_n$ (see part ii. of Definition 5.1 above), let us consider the Complex (which can be called the Total Fractal Complex *of $\Gamma_{\mathcal{W}}$),*

$$H^\star = H^\bullet\left(\mathcal{F}^\bullet\left(\Gamma_{\mathcal{W}},\mathbb{C}\right),\delta^\bullet\right) = \bigoplus_{m=0}^{\infty} H_m \,,$$

where, for any integer $m \geqslant 1$, and with the convention $H_0 = Im\,\delta_{-1,0} = \{0\}$, H_m is the cohomology group

$$H_m = \ker\delta_{m-1,m}/Im\,\delta_{m-2,m-1}\,.$$

Then, $H^\star$ is the set consisting of functions f on $\Gamma_{\mathcal{W}}$, viewed as 0-fermions (in the sense of Definition 5.2), and, for any integer $m \geqslant 1$, of the restrictions to V_m of (m, N_b^m+1)-fermions, i.e., the restrictions to (the Cartesian product space) $V_m^{N_b^m+1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with N_b^m+1 variables (corresponding to the vertices of V_m), involving the restrictions to V_m of the continuous, complex-valued functions f on $\Gamma_{\mathcal{W}}$ – as, naturally, the aforementioned 0-fermions – satisfying the following convergent (and even, absolutely convergent)Taylor-like expansions (with $V^\star = \bigcup_{n\in\mathbb{N}} V_n$),

$$\forall\, M_{\star,\star} \in V^\star : \quad f\left(M_{\star,\star}\right) = \sum_{k=0}^{\infty} c_k\left(f,M_{\star,\star}\right)\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_k\,\mathbf{p}} = \sum_{k=0}^{\infty} c_k\left(f,M_{\star,\star}\right)\varepsilon^{k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}\,, \quad (\spadesuit\spadesuit\spadesuit)$$

where, for each integer $k \geqslant 0$*,* $c_k(\star,\star) = c_k(f,\star) \in \mathbb{C}$*, the number* $\varepsilon^k > 0$ *is the* k^{th} *component of the* k^{th} *cohomology infinitesimal introduced in Definition 3.1, and where* ℓ_k *denotes an integer (in* $\mathbb{Z}$*) such that*

$$\left|\left\{\ell_k \frac{\ln\varepsilon}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon^{k(D_{\mathcal{W}}-1)}}{2\pi}\cdot$$

Note that since the functions f *involved are uniformly continuous on the Weierstrass Curve* $\Gamma_{\mathcal{W}} \supset V^\star$*, and since the set* $V^\star$ *is dense in* $\Gamma_{\mathcal{W}}$*, they are uniquely determined by their restriction to* $V^\star$*, as given by* (♠♠♠)*. We caution the reader, however, that at this stage of our investigations, we do not know wether* $f(M)$ *is given by an expansion analogous to the one in* (♠♠♠)*, for every* $M \in \Gamma_{\mathcal{W}}$*, rather than just for all* $M \in V^\star$*.*

For each $M_\star = M_{\star,m} \in V^\star$*, as is shown in [DL22c], the coefficients* $c_k(\star,\star)$ *(for any* $k \in \mathbb{N}$*) are the residues at the possible* cohomological Complex Dimensions $-(k(2-D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p})$*) of a suitable global scaling zeta function evaluated at* $M_\star$*.*

The group $H^\star = \bigoplus_{m=0}^{\infty} H_m$ *is called* the total fractal cohomology group *of the Weierstrass Curve* $\Gamma_{\mathcal{W}}$ *(or else, of the Weierstrass function* $\mathcal{W}$*).*

Definition 5.4 (**Maximal (Real) Complex Dimension of a Prefractal Approximation**).

Given $m \in \mathbb{N}$, we define the *maximal real Complex Dimension* of the prefractal approximation $\Gamma_{\mathcal{W}_m}$ as

$$\omega_m = -m\,(2 - D_{\mathcal{W}})\cdot$$

Remark 5.2. Clearly, the successive prefractal approximations play the role of level sets, in our present context.

We now to recall the following result, obtained in [DL22c].

Property 5.2 (**Complex Dimensions Series Expansion of the Weierstrass Complexified Function** $\boldsymbol{\mathcal{W}_{comp}}$ **[DL22c]**).

For any strictly positive integer m *and any* j *in* $\{0,\cdots,\#V_m\}$*, we have the following* exact *expansion, indexed by the Complex Codimensions* $k(D_{\mathcal{W}}-2) + i\,\ell_k\,\mathbf{p}$*, with* $0 \leqslant k \leqslant m$*,*

$$\mathcal{W}_{comp}\left(j\,\varepsilon^m\right) = (N_b-1)^{2-D_{\mathcal{W}}}\,c_m\,\varepsilon^{m(2-D_{\mathcal{W}})+i\,\ell_m\,\mathbf{p}} + (N_b-1)^{2-D_{\mathcal{W}}}\sum_{k=0}^{m-1} c_k\,\varepsilon^{k(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}},$$

where $c_m \in \mathbb{C}^\star$ *and* $\ell_m \in \mathbb{Z}$ *are such that*

$$c_m\,\varepsilon^{i\,\ell_m\,\mathbf{p}} = \mathcal{W}_{comp}\left(\frac{j}{N_b-1}\right), \quad (\diamond)$$

ε^m is the $(m-k)^{th}$ cohomology infinitesimal introduced in Definition 3.1, and where, for any integer k in $\{0, \cdots, m-1\}$, $c_k \in \mathbb{C}^\star$ and $\ell_k \in \mathbb{Z}$ are such that

$$(N_b - 1)^{2-D_{\mathcal{W}}}\, e^{i\,\frac{2\pi}{N_b-1}\,j\,\varepsilon^{m-k}} = c_k\,\varepsilon^{i\,\ell_k\,\mathbf{p}}\,, \quad (\diamond\diamond)$$

with $\mathbf{p} = \dfrac{2\pi}{\ln N_b}$ denoting the oscillatory period of the Weierstrass Curve, as introduced in [DL22b], and where, as above, ε^{m-k} is the $(m-k)^{th}$ cohomology infinitesimal introduced in Definition 3.1. Note that we write ε^{m-k} instead of ε_{m-k}^{m-k} in order to simplify the notation.

Definition 5.5 (**Cohomological Vertex Integer**).

Given $m \in \mathbb{N}$, and a vertex $M_{j,m} \in V_m$, of abscissa $j\,\varepsilon^m$, with $0 \leqslant j \leqslant \#V_m - 1$, we introduce the *cohomological vertex integer* associated to $M_{j,m}$ as the unique integer $\ell_{j,m} \in \mathbb{Z}$ such that

$$\varepsilon^{i\,\ell_{j,m}\,\mathbf{p}} = \frac{|c_m|\;\mathcal{W}_{comp}\left(\dfrac{j}{N_b-1}\right)}{c_m\,\left|\mathcal{W}_{comp}\left(\dfrac{j}{N_b-1}\right)\right|} \quad , \quad \left[\frac{\ell_{j,m}\,\ln\varepsilon}{\ln N_b}\right] = 0\,,$$

where $\mathbf{p} = \dfrac{2\pi}{\ln N_b}$ denotes the oscillatory period of the Weierstrass Curve, as introduced in [DL22b], $\mathcal{W}_{comp}$ is the complexified Weierstrass function (see Definition 2.2), $\varepsilon = \varepsilon_m$ stands for the cohomology infinitesimal introduced in Definition 3.1, and where the nonzero complex coefficient c_m has been introduced in Property 5.2 just above. Note that since ε_m depends on m, the integer $\ell_{j,m}$ itself also depends on the choice of $m \in \mathbb{N}$.

Remark 5.3. Note that each integer $\ell_{j,m}$ (choosen as in Definition 5.5 just above) is associated to the sequence of geometric angles introduced in part *v.* of Property 2.6; see also Definition 5.1. In this light, the cohomological vertex integer carries the information associated to an angle change between adjacent edges (and, hence, to a change of shape) when switching from $M_{j,m}$ to its consecutive neighbor $M_{j+1,m}$.

For the condition on the integer part $\left[\dfrac{\ell_{j,m}\,\ln\varepsilon}{\ln N_b}\right]$, we refer to [DL22c].

Definition 5.6 (**Sequence of Sets of Critical Points of the Weierstrass Curve**).

We define *the sequence of sets of critical points* of the Weierstrass function $\mathcal{W}$ – or, equivalently, of the Weierstrass IFD – as the sequence $\left(Crit\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ such that, for any $m \in \mathbb{N}$, the set $Crit\left(\Gamma_{\mathcal{W}_m}\right)$ is obtained as the union of the set of local extrema given in Corollary 2.17, and of the set of vertices with a reentrant angle of the prefractal approximation $\Gamma_{\mathcal{W}_m}$, as given in Property 2.18.

Definition 5.7 (**Topological Laplacian of Level** $m \in \mathbb{N}^\star$).

For any $m \in \mathbb{N}^\star$, and any real-valued function f, defined on the set V_m of the vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$, we introduce *the topological Laplacian of level* m, $\Delta_m^\tau(f)$, as applied to f, as follows:

$$\forall\, X \in V_m \setminus \partial V_m : \quad \Delta_m^\tau f(X) = \sum_{Y \in V_m,\, Y \underset{m}{\sim} X} \left(f(Y) - f(X)\right) \cdot$$

As a consequence, in the case of the Weierstrass function $\mathcal{W}$, we also have that

$$\forall\, X \in V_m \setminus \partial V_m : \quad \Delta_m^\tau \mathcal{W}(X) = \sum_{Y \in V_m,\, Y \underset{m}{\sim} X} \left(\mathcal{W}(Y) - \mathcal{W}(X)\right) \cdot$$

Note that we are excluding the case when $m = 0$ here, because $V_0 = \partial V_0$.

Proposition 5.3 (**Topological Laplacian of the Weierstrass function at Vertices of a Prefractal Graph Approximation**).

For any $m \in \mathbb{N}^\star$, any integer k in $\left\{0, \cdots, N_b^m - 1\right\}$, and any j in $\{1, \cdots, N_b - 2\}$, we have that

$$\Delta_m^\tau \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) = \mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) + \mathcal{W}\left(\frac{k\,(N_b-1)+j-1}{(N_b-1)\,N_b^m}\right) - 2\,\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) \cdot$$

Property 5.4 (**Sign of the Topological Laplacian of Level** $m \in \mathbb{N}^\star$).

i. For any positive integer m, and any k in $\left\{0, \cdots, N_b^m - 1\right\}$, we have that, for the initial vertex of a polygon $\mathcal{P}_{m,k}$,

$$\mathcal{W}\left(\frac{k\,(N_b-1)-1}{(N_b-1)\,N_b^m}\right) < \mathcal{W}\left(\frac{k\,(N_b-1)}{(N_b-1)\,N_b^m}\right),$$

along with

$$\mathcal{W}\left(\frac{k\,(N_b-1)+1}{(N_b-1)\,N_b^m}\right) < \mathcal{W}\left(\frac{k\,(N_b-1)}{(N_b-1)\,N_b^m}\right),$$

which enables us to deduce that

$$\Delta_m^\tau \mathcal{W}\left(\frac{k\,(N_b-1)}{(N_b-1)\,N_b^m}\right) < 0 \cdot$$

ii. When $N_b < 7$*, for any positive integer* m*, any* k *in* $\left\{0, \cdots, N_b^m - 1\right\}$*, and any* j *in* $\{1, \cdots, N_b - 2\}$*, one has:*

⇝ *For the left-side vertices in Definition 2.7, distinct from the initial one, of a polygon* $\mathcal{P}_{m,k}$*, we have that*

$$\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) < \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right),$$

along with

$$\left|\mathcal{W}\left(\frac{k\,(N_b-1)+j-1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right| > \left|\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right|,$$

which enables us to obtain that

$$\Delta_m^{\tau}\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) > 0\cdot$$

⇝ *For the right-side vertices in Definition 2.7, distinct from the last one, of a polygon* $\mathcal{P}_{m,k}$*, we have that*

$$\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) > \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right),$$

along with

$$\left|\mathcal{W}\left(\frac{k\,(N_b-1)+j-1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right| < \left|\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right|,$$

which enables us to obtain that

$$\Delta_m^{\tau}\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) > 0\cdot$$

iii. When $N_b \geqslant 7$*, for any* $m \in \mathbb{N}^\star$*, any integer* k *in* $\left\{0, \cdots, N_b^m - 1\right\}$*, and any* j *in* $\{1, \cdots, N_b - 2\}$ *such that*

$$0 < j \leqslant \frac{N_b-3}{4} \quad or \quad \frac{3\,N_b-1}{4} \leqslant j < N_b - 1,$$

we respectively have that:

⇝ *For the left-side vertices in Definition 2.7, distinct from the initial one, of a polygon* $\mathcal{P}_{m,k}$*,*

$$\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) < \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right),$$

along with

$$\left|\mathcal{W}\left(\frac{k\,(N_b-1)+j-1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right| < \left|\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right|,$$

which yields

$$\Delta_m^{\tau} \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) < 0 \cdot$$

$\rightsquigarrow$ <u>*For the right-side vertices in Definition 2.7, distinct from the last one, of a polygon $\mathcal{P}_{m,k}$,*</u>

$$\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) > \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right),$$

along with

$$\left|\mathcal{W}\left(\frac{k\,(N_b-1)+j-1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right| > \left|\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right|,$$

which enables us to obtain that

$$\Delta_m^{\tau} \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) > 0 \cdot$$

Proof. This is an immediate consequence of the proof of Property 2.18 above given in [DL22b].

□

Definition 5.8 (m^{th}**-Level Discrete Hessian**).

Given $m \in \mathbb{N}^\star$, any k in $\left\{0, \cdots, N_b^m - 1\right\}$, and any j in $\{1, \cdots, N_b - 2\}$, we define the m^{th}*-level discrete Hessian* $\mathcal{H}_m$ as follows:

$$\mathcal{H}_m\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) = \Delta_m^{\tau} \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) \cdot$$

The vertex

$$\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right)$$

is said *nondegenerate*, with respect to $\mathcal{H}_m$, if

$$\mathcal{H}_m\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) \neq 0 \cdot$$

Property 5.5 (**Absence of Degenerate Points for the Sequence of Discrete Hessians**).

Given $m \in \mathbb{N}^\star$, any integer k in $\left\{0, \cdots, N_b^m - 1\right\}$, and any j in $\{1, \cdots, N_b - 2\}$, the m^{th}-level discrete Hessian $\mathcal{H}_m$ *introduced in Definition 5.8 just above never vanishes.*

Proof. This is a direct consequence of Property 5.4.

□

Definition 5.9 (**Fractal Morse Function**).

A function f defined on the Weierstrass Curve $\Gamma_{\mathcal{W}}$ will be said to be *a fractal Morse function* it its critical points are nondegenerate; i.e., if, for any $m \in \mathbb{N}^\star$, its discrete Hessian $\mathcal{H}_m$ (see Definition 5.8) never vanishes.

Remark 5.4 (**The Weierstrass Function Viewed as the Identity Function on** $\Gamma_{\mathcal{W}}$).

As in [DL22c], we set, for any real number t in $[0,1]$,

$$\gamma_{\mathcal{W}}(t) = (t, \mathcal{W}(t)) \cdot$$

We then obtain the identity function on the Weierstrass Curve $\Gamma_{\mathcal{W}}$ as $\mathbb{1}_{\Gamma_{\mathcal{W}}}$. In this manner, the Weierstrass function can be viewed as the identity map on $\Gamma_{\mathcal{W}}$.

Property 5.6 (**The Weierstrass Function Viewed as a Fractal Morse Function**).

Since the discrete Hessian introduced in Definition 5.8 never vanishes, the Weierstrass function $\mathcal{W}$ is a fractal Morse function, in the sense of Definition 5.9 just above.

Proof. This is a direct consequence of Property 5.5, according to which the Weierstrass function $\mathcal{W}$ does not have any degenerate point.

□

Definition 5.10 (m^{th}**-Level Fractal Morse Index**).

Given $m \in \mathbb{N}^\star$, and any j in $\{1, \cdots, \#V_m - 1\}$, the m^{th}*-level fractal Morse index* $\iota_{j,m}$ of the vertex $M_{j,m}$ is defined as follows:

$$\iota_{j,m} = \begin{cases} 1\,, & \text{if} \quad \mathcal{H}_m\left(\dfrac{j}{(N_b - 1)\, N_b^m}\right) < 0\,, \\ 0\,, & \text{if} \quad \mathcal{H}_m\left(\dfrac{j}{(N_b - 1)\, N_b^m}\right) > 0 \cdot \end{cases}$$

Remark 5.5. An overview of the values of the indexes for the different types of vertices involved is given in Table 1.

Vertex	**Junction point** (between consecutive polygons)	**Bottom point**	**Plain interior point (obtuse angle)**	**Left-side acute corner**	**Right-side acute corner**
Index	1	0	0	1	0

Table 1: An overview of the values of the indexes for the different types of vertices involved.

Remark 5.6. One should note that, as in the classical Morse theory, the index of a nondegenerate critical point is equal to the dimension of the largest subspace of what plays the role of a *tangent space* between two points, i.e., in our context, the edge that connects them (a line segment), where the Hessian is negative definite. It thus takes the value 1 at local maxima, and zero at local minima. It also takes the value zero at the right-side vertices with reentrant interior angles provided in Property 2.18. This specific configuration corresponds, in a sense, to a sign change in the *curvature*.

Definition 5.11 (**Absolute Height Sequence of the Weierstrass Curve**).

We define *the absolute heights sequence* of the Weierstrass function $\mathcal{W}$ – or, equivalently, of the Weierstrass IFD – as the sequence of positive numbers $\left(Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ where, for any $m \in \mathbb{N}$,

$$Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right) = \left\{\mathcal{W}\left(\frac{j}{N_b^m}\right) + \frac{1}{1-\lambda}\,,\ 0 \leqslant j \leqslant \#V_m - 1\right\}.$$

Remark 5.7. The fact that, for any $m \in \mathbb{N}$ and any j in $\{0, \cdots, \#V_m - 1\}$, the value $Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)$ is positive simply comes from the fact that the minimum value of the Weierstrass function on $[0,1]$ is equal to $m_{\mathcal{W}} = -\dfrac{1}{1-\lambda}$; see Notation 12 in Section 3.

Property 5.7 (**Fractal Morse Height Increasing Reordered Sequence**).

Given $m \in \mathbb{N}$, the set $Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)$ introduced in Definition 5.11 admits an associated increasing reordered set.

Proof. This simply comes from the fact that, for any $m \in \mathbb{N}$, the set $Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)$ is a finite set of positive numbers, therefore allowing us to reorder the points of $Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)$ in increasing order.

□

Definition 5.12 (**Fractal Morse Height Increasing Reordered Sequence**).

We define *the fractal Morse height reordered sequence* of the Weierstrass function $\mathcal{W}$ – or, equivalently, of the Weierstrass IFD – as the sequence of positive increasing reordered numbers $\left(Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ where, for any $m \in \mathbb{N}$, $Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_m}\right)$ is the increasing reordered set associated to the finite set $Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)$.

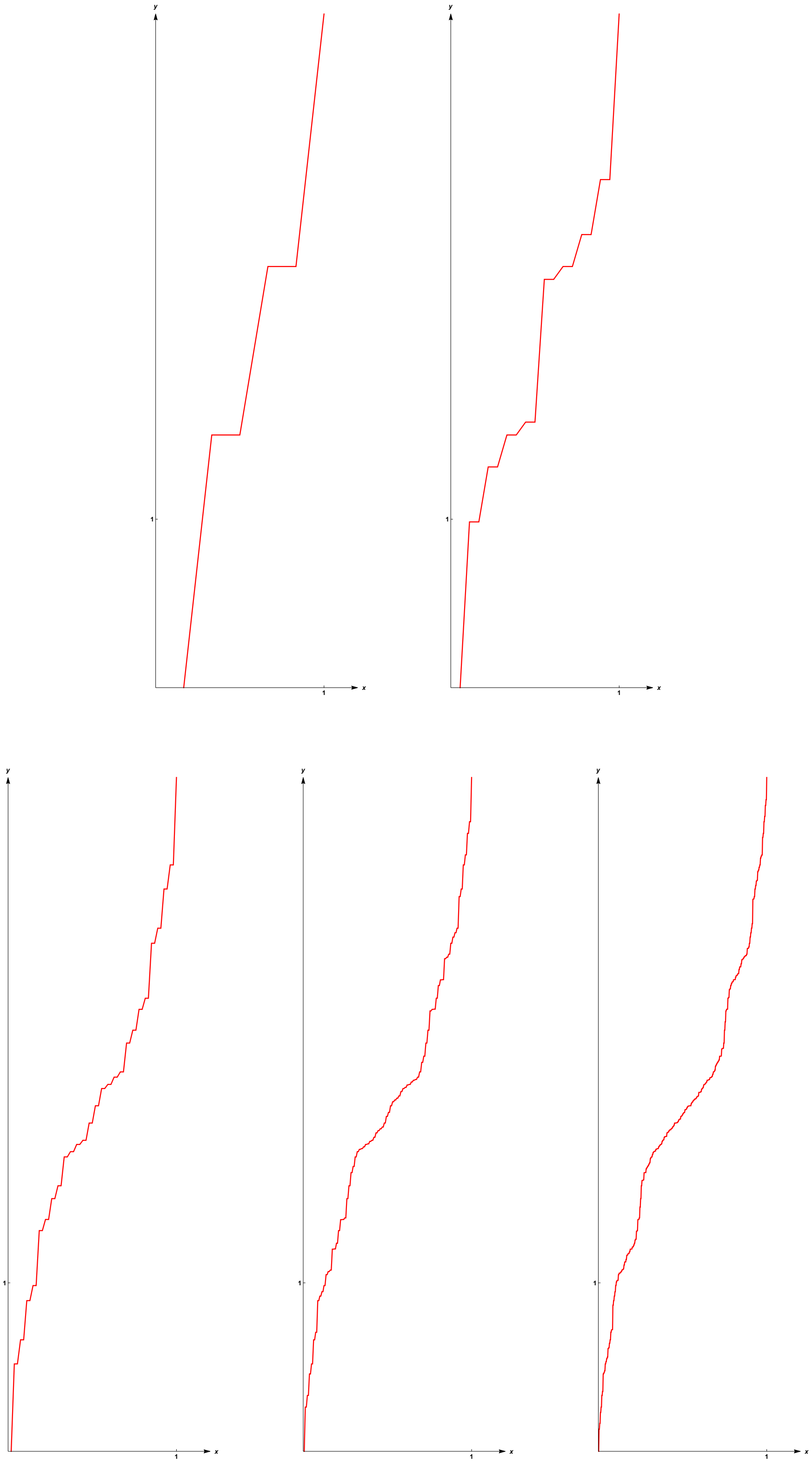


Figure 8: Plot of the fractal Morse heights $Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_1}\right)$, $Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_2}\right)$, $Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_3}\right)$, $Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_4}\right)$ and $Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_5}\right)$, presented from top to bottom and from left to right.

6 Further Perspectives: The Weierstrass Curve as the Projection of a Vertical Comb

In this section, we place ourselves in the Euclidean plane of dimension 3, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by (x, y). The usual axes will be respectively referred to as $(x'x)$, $(y'y)$ and $(z'z)$.

Thanks to Property 2.1, for any strictly positive integer m and any j in $\{0, \cdots, \#V_m\}$, we have that

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \lambda^m\,\mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} \lambda^k \cos\left(\frac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m}\right), \quad (❧)$$

or, equivalently, expressed in terms of the cohomology infinitesimal ε (see Definition 3.1),

$$\frac{\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)}{(N_b-1)^{2-D_{\mathcal{W}}}} = \varepsilon^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} \varepsilon^{k\,(2-D_{\mathcal{W}})} \cos\left(\frac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m}\right). \quad (❧\,❧)$$

If we consider the three-dimensional *vertical comb*, respectively comprised of the set of points

$$\left(\frac{j}{(N_b-1)\,N_b^m}, \frac{k}{m}, \varepsilon^{k\,(2-D_{\mathcal{W}})} \cos\left(\frac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m}\right)\right), \text{ for } 0 \leqslant j \leqslant \#V_m - 1 \text{ and } 0 \leqslant k \leqslant m-1,$$

along with the periodic set of points

$$\left(\frac{j}{(N_b-1)\,N_b^m}, 0, \varepsilon^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}\left(\frac{j}{N_b-1}\right)\right), \text{ for } 0 \leqslant j \leqslant \#V_m - 1,$$

we note that the expression in (❧ ❧) just above corresponds to the superposition of the teeth of the comb; see Figures 9 and 10.

We can thus envision the following three-dimensional *vertical comb*, respectively comprised of the set of *horizontal (rear) rows*,

$$\left(\frac{j}{(N_b-1)\,N_b^m}, \frac{k}{m}, \varepsilon^{k\,(2-D_{\mathcal{W}})}\right), \text{ for } 0 \leqslant k \leqslant m-1,$$

along with *the front row*

$$\left(\frac{j}{(N_b-1)\,N_b^m}, 0, \varepsilon^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}\left(\frac{j}{(N_b-1)}\right)\right), \text{ for } 0 \leqslant k \leqslant m-1,$$

and a moving observer who moves on this latter set of points (the front teeth of the comb): when it comes to a specific value of the integer k in $0, \cdots, m-1$, the observer looks at the associated comb under an angle of value

$$\vartheta_{j,k,m} = \frac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m} = \frac{2\,\pi\,j\,\varepsilon^{(m-k)\,(2-D_{\mathcal{W}})}}{(N_b-1)},$$

so that the expression in (❧❧) just above corresponds to the successive superposition of the projections of the teeth of the combs; see Figures 11 and 12. Namely, each horizontal row of the comb corresponds to a prefractal level set; see Figure 13.

We note that

$$\vartheta_{j,k+1,m} = \frac{\vartheta_{j,k,m}}{\varepsilon},$$

which means that the projection angle increases as one gets closer to the front row.

Note also that in connection with the results of Section 5, the angle $\dfrac{2\pi N_b^k j}{(N_b-1) N_b^m}$ can be represented by a cohomological vertex integer $\ell_{j,k}$, as given in Definition 5.5; see Figure 13.

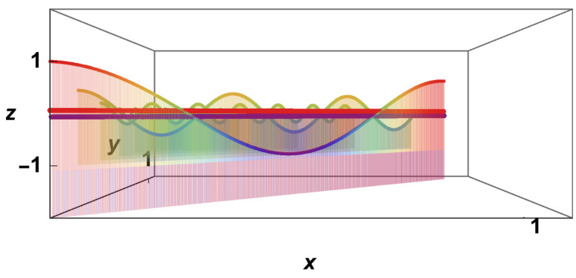


Figure 9: **The vertical comb, for $m = 5$ – front view.**

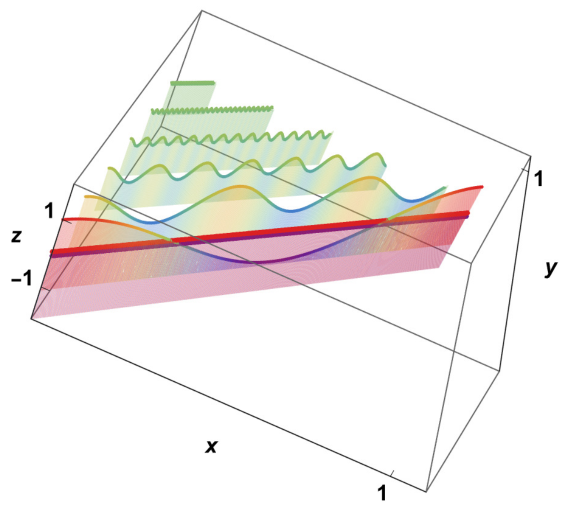


Figure 10: **The vertical comb, for $m = 5$ – side view.**

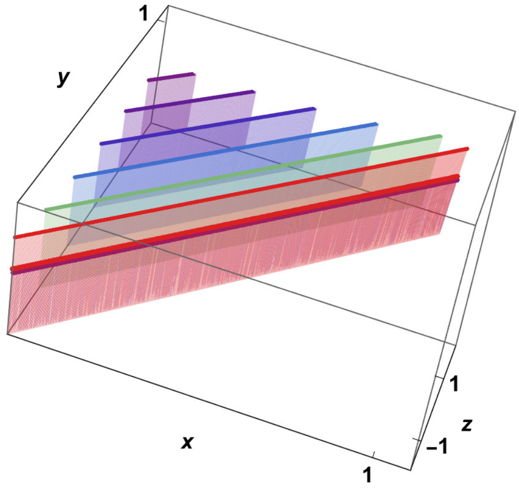


Figure 11: **The vertical comb, before projection, for $m = 5$ – side view.**

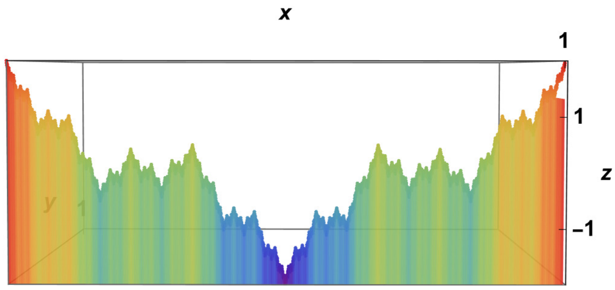


Figure 12: **The resulting vertical comb, after projection – face view.**

In later work on the subject, it would be interesting to investigate the following open problem.

Problem 1. To what extent does the knowledge of the fractal Morse height reordered sequence given in Definition 5.12, along with the fractal Morse indexes introduced in Definition 5.10, the maximal real Complex Dimension introduced in Definition 5.4, and the cohomological vertex integers given in Definition 5.5, enable us to reconstruct the fractal (i.e., in our present setting, the Weierstrass Curve)?

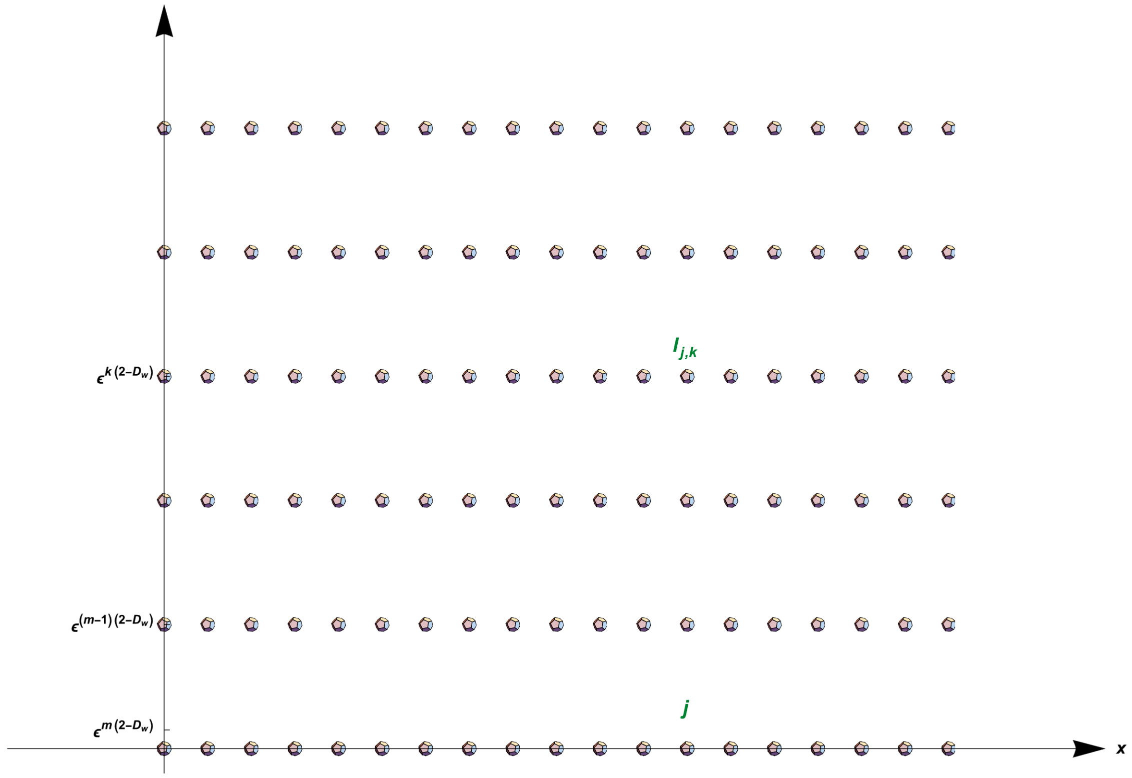


Figure 13: **The top view of the vertical comb. Each horizontal row coincides with a prefractal level set $\varepsilon^{k(2-D_{\mathcal{W}})}$, $0 \leqslant k \leqslant m$.**

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