Iterated Fractal Drums $\sim$ Some New Perspectives: Polyhedral Measures, Atomic Decompositions and Morse Theory

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Some New Perspectives: Polyhedral Measures, Atomic Decompositions and Morse Theory

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Abstract

In this research paper, we carry on our exploration of the connections between the Complex Fractal Dimensions of an iterated fractal drum (IFD) and the intrinsic properties of the fractal involved – in our present case, the Weierstrass Curve.
In order to gain a better understanding of the differential operators associated to this everywhere singular object, we use atomic decompositions, which enable us to characterize Besov spaces on the Weierstrass Curve $\Gamma_{\mathcal{W}}$ and then, by means of trace theorems which extend the results of Alf Jonsson and Hans Wallin obtained in the case of a d-set, to identify the trace of the classical Sobolev spaces on $\Gamma_{\mathcal{W}}$. For this purpose, we construct a specific polyhedral measure, which is done by means of a sequence of polygonal neighborhoods of the Curve. We then determine the order of the fractal Laplacian on $\Gamma_{\mathcal{W}}$, thanks to the connections between Sobolev spaces and the usual Laplacian.
Moreover, we lay out some of the foundations of an extension of Morse theory dedicated to fractals, where the Complex Fractal Dimensions appear to play a major role, by means of level sets connected to the successive prefractal approximations.
In the end, we envision the Weierstrass Curve as the projection of a 3-dimensional vertical comb, where each horizontal row is associated to the k^{th} cohomogical infinitesimal, the *fractal signature* of the k^{th} prefractal approximation, according to our previous results on fractal cohomology.

Keywords: Weierstrass Curve, iterated fractal drum (IFD), Complex Dimensions of an IFD, box-counting (or Minkowski) dimension, cohomology infinitesimal, polyhedral measure, atomic decomposition, trace theorems, order of the fractal Laplacian, fractal Morse theory, fractal Morse indexes.

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Contents

1 Introduction

In [DL24a], [DL24c], we introduced the concept of *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs), by analogy with the *relative fractal drums* (RFDs) involved, for instance, in the case of the Cantor Staircase, in [LRŽ17a], Section 5.5.4, as well as in [LRŽ17b] and in [LRŽ18]. Those iterated fractal drums simply consist in a sequence of appropriate tubular neighborhoods of prefractal polygonal approximations of the Curve.

By exploring the connections between the Complex Dimensions of those IFDs and the cohomological properties of a fractal object, we showed, in [DL24c], that the functions belonging to the cohomology groups associated to the Curve are obtained, by induction, as (finite or infinite) sums indexed by the underlying Complex Dimensions. In particular, to each prefractal approximation of the given iterated fractal drum, we associate the *minimal real part* of the Complex Dimensions involved. Contrary to fractal tube formulas, which are obtained for small values of a positive parameter ε, the aforementioned expansions are only valid for the value of the (multi-scales) *cohomology infinitesimal* ε associated to the scaling relationship obeyed by the Weierstrass Curve (or else, for a smaller positive infinitesimal). (See also [DL24a] for an exposition of these results, as well as of the computation of the Complex Dimensions via a fractal tube formula obtained in [DL22].)

Our differentials δ and $\bar{\delta}$ (see [DL24c]) enable us to define the associated Laplacian $\delta\,\bar{\delta} + \bar{\delta}\,\delta$. This naturally raises questions as to the possible connections with the usual Laplacian (i.e., the one of classical analysis). Based on the seminal works of Alf Jonsson and Hans Wallin in [Wal89], [JW84], a hint was that it could involve the Complex Dimensions, insofar as, in the case of a d-set $\mathcal{F} \subset \mathbb{R}^n$, for $n \in \mathbb{N}^\star$ and $d > 0$, characterizations of the restrictions of classical Sobolev spaces to $\mathcal{F}$ can be obtained by means of trace theorems. More precisely, the restrictions to $\mathcal{F}$ of those Sobolev spaces are obtained as Besov spaces $B^{p,q}_\beta(\mathcal{F})$ on $\mathcal{F}$, where the index β depends explicitly on d. (Recall that $\mathcal{F} \subset \mathbb{R}^n$ is a *d-set* if it has finite and positive d dimensional Hausdorff measure.)

In the present research paper, we hereafter extend the results of Alf Jonsson and Hans Wallin in the case of our Weierstrass IFDs. This requires, in particular, the construction of a specific measure (called the polyhedral measure) on the Weierstrass IFD, which is done by means of a sequence of polygons – a polygonal neighborhood of the Curve. This enables us to define the Besov spaces on the Weierstrass Curve, by means of an atomic decomposition.

We next recall a few important facts about atomic decompositions. They came into play in the 1970s, in order to obtain a better characterization of the smoothness, in a more general context of the "measure of smoothness", which is the expression used by Hans Triebel in the Preface of his book [Tri06], page xi. Atomic decompositions were studied by several authors. We may cite, for instance, Ronald R. Coifman in [Coi74], where the author provides an explicit representation theorem for functions belonging to the Hardy space $\mathcal{H}^p(\mathbb{R}) = \left\{ f \in L^p(\mathbb{R}) \, , \, \mathcal{R}_i(f) \in L^p(\mathbb{R}) \right\}$, for $0 < p \leqslant 1$, (where $\mathcal{R}_i(f)$ denotes the i^{th} Riesz transform of the function f) by means of p-atoms, i.e., functions a_p with compact support and vanishing k^{th} order moments $\displaystyle\int_{-\infty}^{\infty} x^k \, a_p(x) \, dx$, for all integers $k \in \left\{ 0, \ldots, \left[\frac{1}{p} \right] - 1 \right\}$, where $[.]$ denotes the integer part.

The n-dimensional case of distributions in $\mathcal{H}^p(\mathbb{R}^n)$, for $n \geqslant 1$, was then developed by Robert H. Latter in [Lat78]. The latter results were extended by Alberto P. Calderon to distributions in parabolic $\mathcal{H}^p$ spaces, for $0 < p \leqslant \infty$, in [Cal77]. Extensions to the case of homogeneous Besov spaces were given by Michael Frazier and Björn Jawerth in [FJ85].

However, one had to wait until the work of Hans Triebel for the extension to d-sets of $\mathbb{R}^n$, with $0 < d < n$; see [Tri06], Chapter 8, Section 8.1.3, on pages 350–356. Along these lines, further extensions were obtained by Maryia Kabanava in the case of nested fractals; see [Kab12].

In our present context, we follow and extend Maryia Kabanava's approach, which enables us to characterize Besov spaces on the Weierstrass Curve. Thanks to the connections between Sobolev spaces and the usual Laplacian, we are then able to connect explicitly the order of the restriction of the usual Laplacian on the Curve – or, rather, on the IFD – and the maximal real part of the Complex Dimensions, namely, the Minkowski Dimension of the Curve.

Note that whereas atomic decompositions may appear as *magical* tools, which, in a sense, convert the study of function spaces into the study of sequence spaces, they have drawbacks, insofar as all the information relative to the involved function space is not contained in the associated sequence space. Indeed, the coefficients of the optimal atomic decomposition of a function f generally only depends on f. One can think of *local*, vs *global* decompositions. In order to overcome the resulting loss of information, a slightly different approach was presented by Hans Triebel in [Tri97]; see, especially, in the case of general Besov spaces, Chapter 4, Section 20, building on the spectral theory for fractal pseudodifferential operators. More precisely, the function spaces on the involved fractal (viewed as a d-set) are directly adapted from the case of the boundary $\partial\Omega \subset \mathbb{R}^{n-1}$ of a C^∞-domain in $\mathbb{R}^n$, where $\partial\Omega$ is reduced "via an atlas of finitely many C^∞ charts to corresponding spaces on $\mathbb{R}^{n-1}$" (the quote is from Section 20.1, on page 151, in. *loc. cit.*).

Still to circumvent the aforementioned drawbacks of atomic decompositions, significant advances were then made by Ingrid Daubechies, who introduced localized, compactly supported wavelets; see the expositary paper [Dau92]. In our present context, we cannot, for the moment, apply such elegant methods. However, we retain the important idea that an essential – and unavoidable step – is to question the quality of the sequence involved in the atomic decomposition. In the specific case of a closed subset $\mathcal{F}$ of $\mathbb{R}^n$, the above disadvantages were also highlighted by Alf Jonsson in [Jon09], where the author explains that, contrary to the (more general) case studied by Michael Frazier and Björn Jawerth in [FJ85], "the definition of $B_\alpha^{p,q}(\mathcal{F})$ is not satisfactory from the point of view that it is not constructive" (the quote is from page 586, in. *loc. cit.*). It is for this reason that Alf Jonsson considered the (even) more specific case of sets preserving Markov's Inequality. In its original form, that goes back to 1889 (see [Mar48]), Markov's Inequality states that, for any nonnegative integer n, and any complex polynomial P of degree d_P:

$$\|P'\|_{[-1,1]} \leqslant d_P^2 \, \|P\|_{[-1,1]} \, .$$

By extension, a closed nonempty subset $\mathcal{F}$ of $\mathbb{R}^n$ *preserves Markov's Inequality* if, for all positive integers N, any real polynomial P with d_P variables, of degree at most equal to N, any point $X \in \mathcal{F}$, and any real number $r \in]0,1]$:

$$\max_{\mathcal{F} \cap \mathcal{B}(X,r)} |\nabla P| \leqslant \frac{c\,(P,n,N,\mathcal{F})}{r} \max_{\mathcal{F} \cap \mathcal{B}(X,r)} |P| \, ,$$

where $c\,(P,n,N,\mathcal{F})$ denotes a positive constant that depend on n, P, N and $\mathcal{F}$.

Such an assumption enabled Alf Jonsson to build polynomial approximations, by means of orthogonal polynomials. Such an approach therefore ensures "more satisfactory trace theorems" (the quote is from page 595, in *loc. cit.*). Indeed, all is finally about describing the smoothness properties of functions, where the degree of the polynomials involved in the approximation (defined on the intersection of $\mathcal{F}$ with squares or balls) provides the informations related to the level of smoothness. In this light, we clearly understand that Markov's Inequality ensures the *quality* of the approximation.

Going further, in the case when $p \geqslant 1$, Alf Jonsson introduces a space $\widetilde{B}_\alpha^{p,q}(\mathcal{F})$, obtained via local polynomial approximation, and proves that it is equal to the Besov space $B_\alpha^{p,q}(\mathcal{F})$ obtained thanks to atomic decomposition.

Now, a very interesting point is that fractal curves generally preserve Markov's Inequality. For instance, it is the case of the von Koch's snowflake, as is studied in the work by Hans Wallin in [Wal91], where the author provides the trace to the (fractal) boundary for functions in a Sobolev space.

in our present context, the Weierstrass Curve $\Gamma_{\mathcal{W}}$, also preserves Markov's Inequality, as in proved in Property 4.2, on page 39. This ensures that our characterization of Besov spaces on $\Gamma_{\mathcal{W}}$ is obtained in a constructive manner

Switching to more geometric topics, following our quest for the connections between the Complex Dimensions and the intrinsic properties of a fractal, we showed, in our previous work [DL24c], that, in the case of junction vertices, i.e., points belonging to consecutive prefractal graphs, the aforementioned maximal real part of the Complex Dimensions, associated to the cohomology group involved, changes. Hence, a change of shape – when one switches from a prefractal approximation, to the consecutive polygonal approximation, also corresponds to the occurrence of new polygons – is closely connected to a change of fractal dimensions. In this light, it was natural to explore further connections between the Complex Dimensions of a fractal object – the Weierstrass Curve, or the Weierstrass IFDs – and a suitable analog of Morse theory: given the Complex Dimensions and the fractal Morse indexes, can we, in some sense, reconstruct the fractal? Towards the end of the paper, we begin to lay the foundations for addressing this challenging and very interesting inverse topological and geometric problem.

Our main results in the present setting can be found in the following places:

i. In Section 3, Property 3.6, on page 29, where we introduce the polyhedral measure on the Weierstrass IFD. In particular, we prove that this polyhedral measure is well defined, as well as nontrivial, and is a bounded and singular Borel measure on the Weierstrass Curve, whose total mass is positive and given by a suitable renormalization procedure (see Theorem 3.7, on page 30.

ii. In Section 4, where our polyhedral measure enables us to extend the aforementioned results of Alf Jonsson and Hans Wallin to the case of the Weierstrass IFD. More precisely, we define *the atomic decomposition* of a function defined on the IFD; see Definition 4.6, on page 45. We also give a direct proof of the fact that the Weierstrass Curve satisfies Markov's Inequality; see Property 4.2, on page 39. We can then define and characterize the Besov spaces on the IFD; see Definition 4.8, on page 49, along with Property 4.8, on page 52. In the end, we establish a trace theorem in this context (Theorem 4.11, on page 57), and (in Corollary 4.12, on page 57) deduce from it the order of the fractal Laplacian on the IFD, which is in agreement with previous results of Robert S. Strichartz in the case of the Sierpiński Gasket $\mathcal{SG}$ in [Str03]; namely, the fractal Laplacian is not of order 2, unlike for the classical Laplacian. We also note that our trace theorem 4.11, on page 57, connects, in particular, the Sobolev space on a suitable open neighborhood of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ with the Besov space on $\Gamma_{\mathcal{W}}$, for the appropriate values of the exponents involved.

iii. In Section 5, on page 60, where we use the Complex Dimensions, along with the fractal cohomology, of the Weierstrass IFD, obtained and developed by the authors in [DL22], [DL24c] (see also [DL24a]), in order to propose an extension of Morse theory devoted to fractals. In particular, a *minimal real Complex Dimension* is associated to each prefractal approximation (in Definition 5.4, on page 63), along with *a cohomological vertex integer*, associated, this time, to each vertex of the prefractal approximation; see Definition 5.5, on page 63. We also define new Morse indexes – applicable to fractal curves such as the Weierstrass Curve; see Definition 5.10, on page 70. Furthermore, we introduce the notion of a *fractal Morse function* (in Definition 5.9, on page 70), and show, in Property 5.7, on page 70, that the Weierstrass function is a fractal Morse function.

iv. In Section 6, on page 73, we explain how the Weierstrass Curve can be viewed as the projection of a 3-dimensional vertical comb, the teeth of which are directly connected with the cohomological vertex integers. This provides us with new research directions for future work, in connection with our fractal Morse theory.

For a thorough discussion of the theory of Complex Dimensions, we refer the interested reader to [LvF13] and to [LRŽ17a], in the case of fractal strings and of (relative) higher-dimensional fractal drums, respectively; see also the survey article on the subject, [Lap19].

In closing this introduction, we mention that, for clarity and by necessity of concision, we work throughout this paper with the important example of the Weierstrass Curve and the associated Weierstrass IFD. We stress, however, that we expect that our main results will extend to a large class of fractal curves and their associated IFDs – as well as, eventually, to a large class of higher-dimensional fractal manifolds obtained by means of polyhedral prefractal approximations.

2 Geometry of the Weierstrass Curve

We begin by reviewing the main geometric properties of the Weierstrass Curve (and of the associated IFD), which will be key to our work in the rest of this paper.

Henceforth, we place ourselves in the Euclidean plane, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by (x, y). The horizontal and vertical axes will be respectively referred to as $(x'x)$ and $(y'y)$.

Notation 1 (Set of all Nonnegative Integers, and Intervals).

As in Bourbaki [Bou04] (Appendix E. 143), we denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of all nonnegative integers, and set $\mathbb{N}^\star = \mathbb{N} \setminus \{0\}$.

Given a, b with $-\infty \leqslant a \leqslant b \leqslant \infty$, $]a,b[= (a,b)$ denotes an open interval, while, for example, $]a,b] = (a,b]$ denotes a half-open, half-closed interval.

Notation 2 (Wave Inequality Symbol).

Given two positive-valued functions f and g, defined on a subset $\mathcal{I}$ of $\mathbb{R}$, we use the following notation, for all $x \in \mathcal{I}$: $f(x) \lesssim g(x)$ when there exists a strictly positive constant C such that, for all $x \in \mathcal{I}$, $f(x) \leqslant C\, g(x)$.

Notation 3 (Weierstrass Parameters).

In the sequel, λ and N_b are two real numbers such that

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^\star \quad \text{and} \quad \lambda\, N_b > 1 \,. \qquad (\clubsuit)$$

Note that this implies that $N_b > 1$; i.e., $N_b \geqslant 2$, if $N_b \in \mathbb{N}^\star$, as will be the case in this paper.

As is explained in [Dav19], we deliberately made the choice to introduce the notation N_b which replaces the initial number b, in so far as, in Hardy's paper [Har16] (in contrast to Weierstrass' original article [Wei75]), b is any positive real number satisfying $\lambda\, b > 1$, whereas we deal here with the specific case of a positive integer, which accounts for the natural notation N_b.

Definition 2.1 (Weierstrass Function, Weierstrass Curve).

We consider the *Weierstrass function* $\mathcal{W}$ (also called, in short, the *$\mathcal{W}$-function*) defined, for any real number x, by

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) \,.$$

We call the associated graph the *Weierstrass Curve*, and denote it by $\Gamma_{\mathcal{W}}$.

Due to the one-periodicity of the $\mathcal{W}$-function (since $N_b \in \mathbb{N}^\star$), from now on, and without loss of generality, we restrict our study to the interval $[0,1[= [0,1)$. Note that $\mathcal{W}$ is continuous, and hence, bounded on all of $\mathbb{R}$. In particular, $\Gamma_{\mathcal{W}}$ is a (nonempty) compact subset of $\mathbb{R}^2$.

Property 2.1 (Scaling Properties of the Weierstrass Function, and Consequences [DL22]).

Since, for any real number x, $\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right)$, one also has

$$\mathcal{W}(N_b\, x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^{n+1}\, x\right) = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) = \frac{1}{\lambda}\left(\mathcal{W}(x) - \cos\left(2\,\pi\, x\right)\right) \,,$$

which yields, for any strictly positive integer m and any j in $\{0, \dots, \#V_m\}$,

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \lambda\,\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}\right) + \cos\left(\frac{2\,\pi\,j}{(N_b-1)\,N_b^m}\right).$$

By induction, one then obtains that

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \lambda^m\,\mathcal{W}\left(\frac{j}{(N_b-1)}\right) + \sum_{k=0}^{m-1}\lambda^k\,\cos\left(\frac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m}\right).$$

Definition 2.2 (**Weierstrass Complexified Function**).

We introduce the *Weierstrass Complexified function* $\mathcal{W}_{comp}$, defined, for any real number x, by

$$\mathcal{W}_{comp}(x) = \sum_{n=0}^{\infty}\lambda^n\,e^{2\,i\,\pi\,N_b^n\,x}. \tag{1}$$

Clearly, $\mathcal{W}_{comp}$ is also a continuous and 1-periodic function on $\mathbb{R}$.

Notation 4 (**Logarithm**).

Given $y>0$, $\ln y$ denotes the natural logarithm of y, while, given $a>1$, $\ln_a y = \dfrac{\ln y}{\ln a}$ denotes the logarithm of y in base a; so that, in particular, $\ln = \ln_e$.

Notation 5 (**Minkowski Dimension and Hölder Exponent**).

For the parameters λ and N_b satisfying condition (♣) (see Notation 3, on page 6), we denote by

$$D_{\mathcal{W}} = 2 + \frac{\ln\lambda}{\ln N_b} = 2 - \ln_{N_b}\frac{1}{\lambda} \in \,]1,2[\tag{2}$$

the box–counting dimension (or Minkowski dimension) of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, which happens to be equal to its Hausdorff dimension [KMPY84], [BBR14],
[She18], [Kel17]. We point out that the results in our previous papers([DL23b] and [DL23a] respectively), also provide a direct geometric proof of the fact that $D_{\mathcal{W}}$, the Minkowski dimension (or box-counting dimension) of $\Gamma_{\mathcal{W}}$, exists and takes the above values, as well as of the fact that $\mathcal{W}$ is Hölder continuous with *optimal* Hölder exponent

$$2 - D_{\mathcal{W}} = -\frac{\ln\lambda}{\ln N_b} = \ln_{N_b}\frac{1}{\lambda}.$$

Convention (The Weierstrass Curve as a Cyclic Curve).

In the sequel, we identify the points $(0, \mathcal{W}(0))$ and $(1, \mathcal{W}(1)) = (1, \mathcal{W}(0))$. This is justified by the fact that the Weierstrass function $\mathcal{W}$ is 1-periodic, since N_b is an integer.

Remark 2.1. The above convention makes sense, because, in addition to the periodicity property of the $\mathcal{W}$–function, the points $(0, \mathcal{W}(0))$ and $(1, \mathcal{W}(1))$ have the same vertical coordinate.

Property 2.2 (Symmetry with Respect to the Vertical Line $x = \frac{1}{2}$).

Since, for any $x \in [0,1]$,

$$\mathcal{W}(1-x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\pi N_b^n - 2\pi N_b^n x\right) = \mathcal{W}(x)\,,$$

the Weierstrass Curve is symmetric with respect to the vertical straight line $x = \frac{1}{2}$.

Proposition 2.3 (Nonlinear and Noncontractive Iterated Function System (IFS)).

Following our previous work [Dav18], *we approximate the restriction $\Gamma_{\mathcal{W}}$ to $[0,1[\times\mathbb{R}$, of the Weierstrass Curve, by a sequence of graphs, built via an iterative process. For this purpose, we use the nonlinear iterated function system (IFS) consisting of a finite family of C^∞ maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ and denoted by*

$$\mathcal{T}_{\mathcal{W}} = \left\{T_0, \ldots, T_{N_b-1}\right\}\,,$$

where, for any integer i belonging to $\{0, \ldots, N_b - 1\}$ and any point (x,y) of $\mathbb{R}^2$,

$$T_i(x,y) = \left(\frac{x+i}{N_b}, \lambda\, y + \cos\left(2\pi\left(\frac{x+i}{N_b}\right)\right)\right)\,.$$

Note that unlike in the classical situation, these maps are not contractions. Nevertheless, $\Gamma_{\mathcal{W}}$ can be recovered from this IFS in the usual way, as we next explain.

Property 2.4 (**Attractor of the IFS [Dav18], [Dav19]**).

The Weierstrass Curve $\Gamma_{\mathcal{W}}$ *is the attractor of the IFS* $\mathcal{T}_{\mathcal{W}}$*, and hence, is the unique nonempty compact subset* $\mathcal{K}$ *of* $\mathbb{R}^2$ *satisfying* $\mathcal{K} = \bigcup_{i=0}^{N_b-1} T_i(\mathcal{K})$*; in particular, we have that* $\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$.

Notation 6 (**Fixed Points**).

For any integer i belonging to $\{0,\dots,N_b-1\}$, we denote by

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b-1}, \frac{1}{1-\lambda}\cos\left(\frac{2\pi i}{N_b-1}\right)\right)$$

the unique fixed point of the map T_i; see [Dav19].

Definition 2.3 (**Sets of Vertices, Prefractals**).

We denote by V_0 the ordered set (according to increasing abscissae) of the points

$$\{P_0, \dots, P_{N_b-1}\} .$$

The set of points V_0 – where, for any integer i in $\{0,\dots,N_b-2\}$, the point P_i is linked to the point P_{i+1} – constitutes an oriented finite graph, ordered according to increasing abscissae, which we will denote by $\Gamma_{\mathcal{W}_0}$. Then, V_0 is called *the set of vertices* of the graph $\Gamma_{\mathcal{W}_0}$.

For any positive integer m, i.e., for $m \in \mathbb{N}^\star$, we set $V_m = \bigcup_{i=0}^{N_b-1} T_i(V_{m-1})$.

The set of points V_m, where two consecutive points are linked, is an oriented finite graph, ordered according to increasing abscissae, called the $\boldsymbol{m^{th}}$ **order** $\boldsymbol{\mathcal{W}}$**-prefractal**. Then, V_m is called *the set of vertices* of the prefractal $\Gamma_{\mathcal{W}_m}$; see Figure 2, on page 14.

Property 2.5 (**Density of the Set** $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ **in the Weierstrass Curve [DL22]**).

The set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ *is dense in the Weierstrass Curve* $\Gamma_{\mathcal{W}}$.

Definition 2.4 (**Adjacent Vertices, Edge Relation**).

For any $m \in \mathbb{N}$, the prefractal graph $\Gamma_{\mathcal{W}_m}$ is equipped with an edge relation $\underset{m}{\sim}$, as follows: two vertices X and Y of $\Gamma_{\mathcal{W}_m}$ (i.e., two points belonging to V_m) will be said to be *adjacent* (i.e., *neighboring* or *junction points*) if and only if the line segment $[X, Y]$ is an edge of $\Gamma_{\mathcal{W}_m}$; we then write $X \underset{m}{\sim} Y$. Note that this edge relation depends on m, which means that points adjacent in V_m might not remain adjacent in V_{m+1}.

We refer to part *iv.* of Property 2.6, on page 10, along with Figure 1, on page 11, for the definition of the polygons $\mathcal{P}_{m,k}$ and $\mathcal{Q}_{m,k}$ associated with the Weierstrass Curve.

Property 2.6. [Dav18] *For any $m \in \mathbb{N}$, the following statements hold:*

i. $V_m \subset V_{m+1}$.

ii. $\#V_m = (N_b - 1)\, N_b^m + 1$, *where $\#V_m$ denotes the number of elements in the finite set V_m.*

iii. The prefractal graph $\Gamma_{\mathcal{W}_m}$ has exactly $(N_b - 1)\, N_b^m$ edges.

iv. The consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$ are the vertices of N_b^m simple nonregular polygons $\mathcal{P}_{m,k}$. For any strictly positive integer m, the junction point between two consecutive polygons $\mathcal{P}_{m,k}$ and $\mathcal{P}_{m,k+1}$ is the point

$$\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m} \right) \right) \quad , \quad 1 \leqslant k \leqslant N_b^m - 1 \,.$$

Hence, the total number of junction points is $N_b^m - 1$. For instance, in the case $N_b = 3$, the polygons are all triangles; see Figure 1, on page 11.

We call extreme vertices *of the polygon $\mathcal{P}_{m,k}$ the junction points*

$$\mathcal{V}_{initial}\left(\mathcal{P}_{m,k}\right) = \left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m} \right) \right),$$

for $0 \leqslant k \leqslant N_b^m - 1$, *and*

$$\mathcal{V}_{end}\left(\mathcal{P}_{m,k}\right) = \left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m} \right) \right),$$

for $0 \leqslant k \leqslant N_b^m - 2$.

In the sequel, we will denote by $\mathcal{P}_0$ ***the initial polygon****, whose vertices are the fixed points of the maps T_i, $0 \leqslant i \leqslant N_b - 1$, introduced in Notation 6 and Definition 2.3, on page 9, i.e., $\left\{P_0, \ldots, P_{N_b-1}\right\}$; see, again, Figure 1, on page 11.*

In the same way, the consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$, distinct from the fixed points P_0 and P_{N_b-1} (see Notation 6, on page 9), are also the vertices of $N_b^m - 1$ simple nonregular polygons $\mathcal{Q}_{m,j}$, for
$1 \leqslant j \leqslant N_b^m - 1$, this time with at most N_b sides. For any integer j such that $1 \leqslant j \leqslant N_b^m - 2$, one obtains each polygon $\mathcal{Q}_{m,j}$ by connecting the point number j to the point number $j+1$ if $j \equiv i \bmod N_b$, for $1 \leqslant i \leqslant N_b - 1$, and the point number j to the point number $j - N_b + 1$ if $j \equiv 0 \bmod N_b$.

As previously, we call extreme vertices *of the polygon $\mathcal{Q}_{m,k}$ the points*

$$\mathcal{V}_{initial}\left(\mathcal{Q}_{m,k}\right) = \left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m} \right) \right),$$

for $1 \leqslant k \leqslant N_b^m - 1$, *and*

$$\mathcal{V}_{end}\left(\mathcal{Q}_{m,k}\right) = \left(\frac{(N_b-1)(k+1)}{(N_b-1)N_b^m}, \mathcal{W}\left(\frac{(N_b-1)(k+1)}{(N_b-1)N_b^m}\right)\right),$$

for $1 \leqslant k \leqslant N_b^m - 2$.

Definition 2.5 (**Polygonal Sets**).

For any $m \in \mathbb{N}$, we introduce the following polygonal sets

$$\mathcal{P}_m = \left\{\mathcal{P}_{m,k}\,,\, 0 \leqslant k \leqslant N_b^m - 1\right\} \quad \text{and} \quad \mathcal{Q}_m = \left\{\mathcal{Q}_{m,k}\,,\, 0 \leqslant k \leqslant N_b^m - 2\right\}.$$

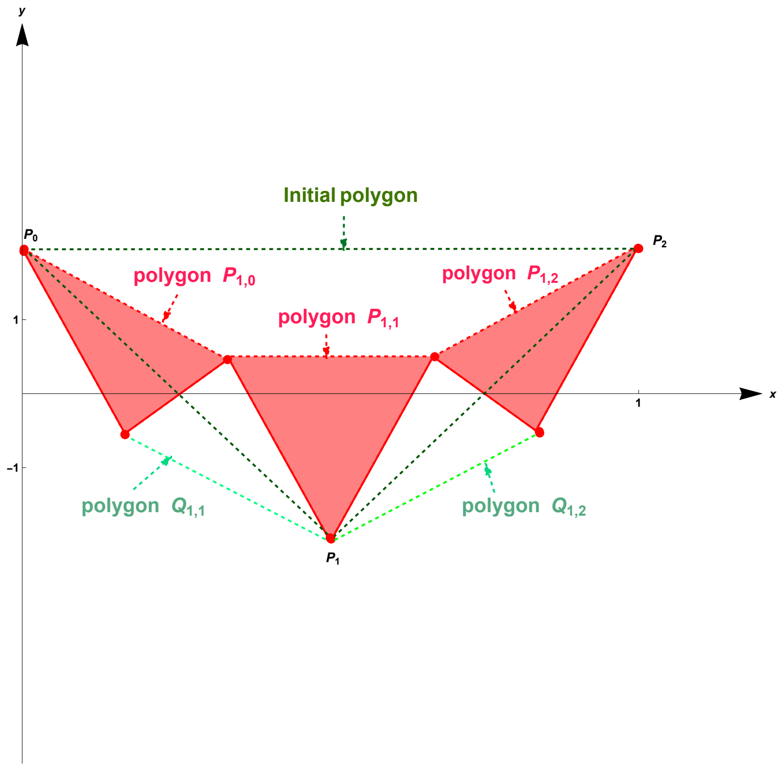


Figure 1: **The initial polygon $\mathcal{P}_0$, and the respective polygons $\mathcal{P}_{1,0}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, $\mathcal{Q}_{1,1}$, $\mathcal{Q}_{1,2}$, in the case when $\lambda = \dfrac{1}{2}$ and $N_b = 3$. (See also Figure 2, on page 14.)**

Notation 7. For any $m \in \mathbb{N}$, we denote by:

ii. $X \in \mathcal{P}_m$ (resp., $X \in \mathcal{Q}_m$) a vertex of a polygon $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$ (resp., a vertex of a polygon $\mathcal{Q}_{m,k}$, with $1 \leqslant k \leqslant N_b^m - 2$).

ii. $\mathcal{P}_m \bigcup \mathcal{Q}_m$ the reunion of the polygonal sets $\mathcal{P}_m$ and $\mathcal{Q}_m$, which consists in the set of all the vertices of the polygons $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$, along with the vertices of the polygons $\mathcal{Q}_{m,k}$, with $1 \leqslant k \leqslant N_b^m - 2$. In particular, $X \in \mathcal{P}_m \bigcup \mathcal{Q}_m$ simply denotes a vertex in $\mathcal{P}_m$ or $\mathcal{Q}_m$.

iii. $\mathcal{P}_m \bigcap \mathcal{Q}_m$ the intersection of the polygonal sets $\mathcal{P}_m$ and $\mathcal{Q}_m$, which consists in the set of all the vertices of both a polygon $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$, and a polygon $\mathcal{Q}_{m,k'}$, with $1 \leqslant' k \leqslant N_b^m - 2$.

Definition 2.6 (**Vertices of the Prefractals, Elementary Lengths, Heights and Angles**).

Given a strictly positive integer m, we denote by $\left(M_{j,m}\right)_{0\leqslant j\leqslant (N_b-1)\,N_b^m}$ *the set of vertices* of the prefractal graph $\Gamma_{\mathcal{W}_m}$. One thus has, for any integer j in $\left\{0,\ldots,(N_b-1)\,N_b^m\right\}$:

$$M_{j,m} = \left(\frac{j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right).$$

We also introduce, for any integer j in $\left\{0,\ldots,(N_b-1)\,N_b^m - 1\right\}$:

i. the elementary horizontal lengths:

$$L_m = \frac{1}{(N_b-1)\,N_b^m}\,;$$

ii. the elementary lengths:

$$l_{j,j+1,m} = d\left(M_{j,m}, M_{j+1,m}\right) = \sqrt{L_m^2 + h_{j,j+1,m}^2}\,,$$

where $h_{j,j+1,m}$ is defined in *iii.* just below.

iii. the elementary heights:

$$h_{j,j+1,m} = \left|\mathcal{W}\left(\frac{j+1}{(N_b-1)\,N_b^m}\right) - \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right|\,;$$

iv. the minimal height:

$$h_m^{inf} = \inf_{0\leqslant j\leqslant (N_b-1)\,N_b^m-1} h_{j,j+1,m}\,, \qquad (3)$$

along with the maximal height:

$$h_m = \sup_{0\leqslant j\leqslant (N_b-1)\,N_b^m-1} h_{j,j+1,m}\,, \qquad (4)$$

v. the geometric angles:

$$\theta_{j-1,j,m} = \left((y'y), \left(\widehat{M_{j-1,m}M_{j,m}}\right)\right) \quad , \quad \theta_{j,j+1,m} = \left((y'y), \left(\widehat{M_{j,m}M_{j+1,m}}\right)\right),$$

where $(y'y)$ denotes the vertical axis, which yield **the following value of the geometric angle between consecutive edges**, namely,
$\left[M_{j-1,m}\, M_{j,m}, M_{j,m}\, M_{j+1,m}\right]$, with arctan = $\tan^{-1}$:

$$\theta_{j-1,j,m} + \theta_{j,j+1,m} = \arctan \frac{L_m}{h_{j-1,j,m}} + \arctan \frac{L_m}{h_{j,j+1,m}} \,.$$

(Note that, of course, $\theta_{j-1,j,m} = \arctan \dfrac{L_m}{h_{j-1,j,m}}$ and $\theta_{j,j+1,m} = \arctan \dfrac{L_m}{h_{j,j+1,m}}$.)

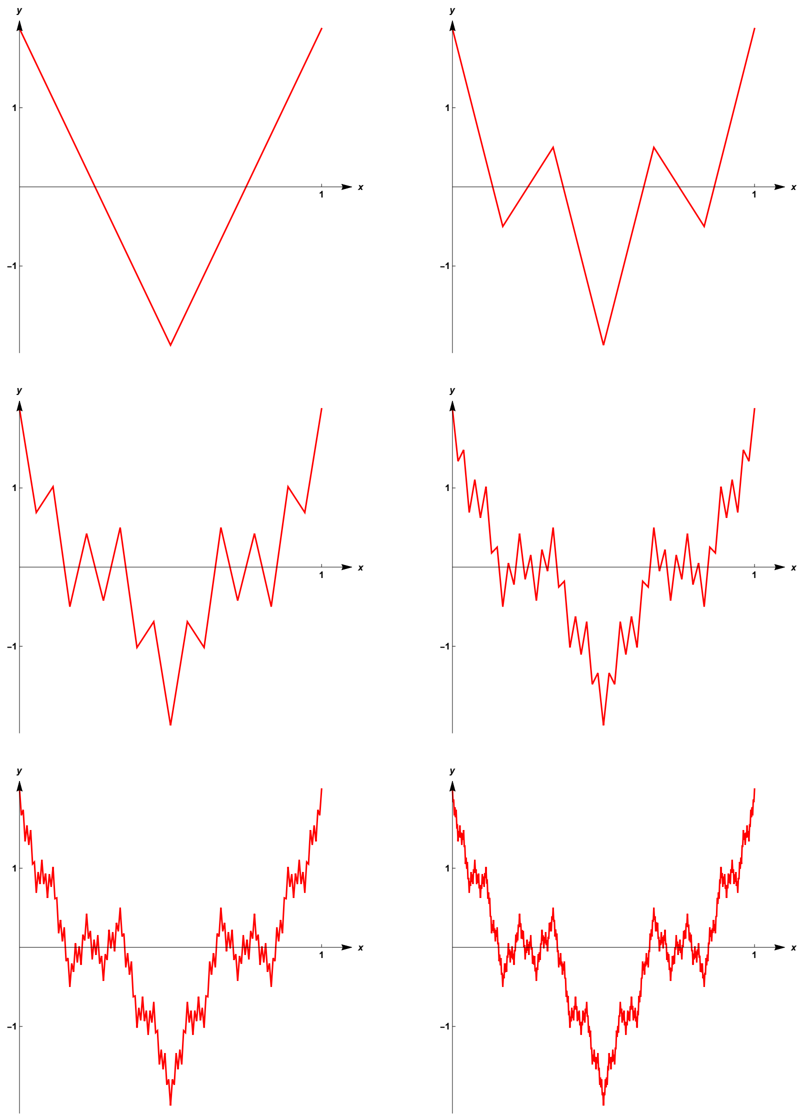


Figure 2: **The prefractal graphs $\Gamma_{\mathcal{W}_0}$, $\Gamma_{\mathcal{W}_1}$, $\Gamma_{\mathcal{W}_2}$, $\Gamma_{\mathcal{W}_3}$, $\Gamma_{\mathcal{W}_4}$, $\Gamma_{\mathcal{W}_5}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$. For example, $\Gamma_{\mathcal{W}_1}$ is on the right side of the top row, while $\Gamma_{\mathcal{W}_4}$ is on the left side of the bottom row.**

Property 2.7. *For the geometric angle $\theta_{j-1,j,m}$, $0 \leqslant j \leqslant (N_b-1)\,N_b^m$, $m \in \mathbb{N}$, we have the following relation:*

$$\tan\theta_{j-1,j,m} = \frac{h_{j-1,j,m}}{L_m}\,.$$

Property 2.8 (**A Consequence of the Symmetry with Respect to the Vertical Line $x = \frac{1}{2}$**).

For any strictly positive integer m and any j in $\{0,\ldots,\#V_m\}$, we have that

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \mathcal{W}\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}\right),$$

which means that the points

$$\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}\right)\right) \quad \text{and} \quad \left(\frac{j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right)$$

are symmetric with respect to the vertical line $x = \frac{1}{2}$; *see Figure 3, on page 15.*

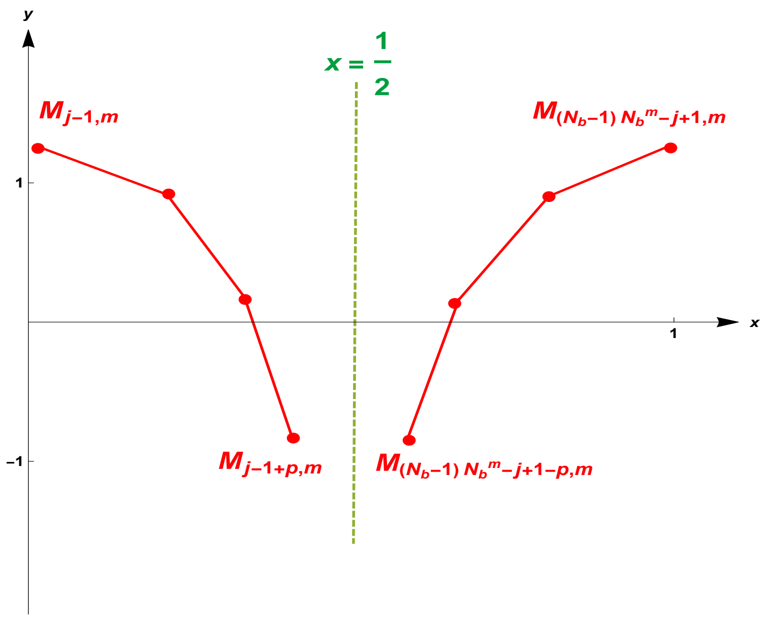


Figure 3: **Symmetric points with respect to the vertical line $x = \frac{1}{2}$.**

Definition 2.7 (**Left-Side and Right-Side Vertices**).

Given nonnegative integers m, k such that $0 \leqslant k \leqslant N_b^m - 1$, and a polygon $\mathcal{P}_{m,k}$, we define:

i. The set of its *left-side vertices* as the set of the first $\left[\dfrac{N_b-1}{2}\right]$ vertices, where $[y]$ denotes the integer part of the real number y.

ii. The set of its *right-side vertices* as the set of the last $\left[\dfrac{N_b-1}{2}\right]$ vertices.

When the integer N_b is odd, we define the bottom vertex as the $\left(\dfrac{N_b-1}{2}\right)^{th}$ one; see Figure 4, on page 16.

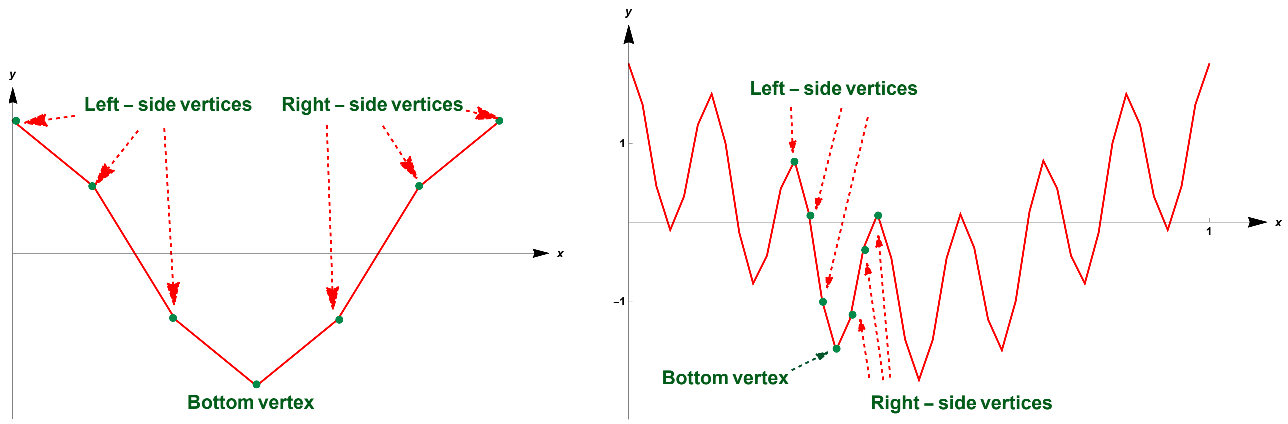


Figure 4: **The Left and Right-Side Vertices.**

Property 2.9 ([DL22]).

For any integer j in $\{0, \ldots, N_b - 1\}$:

$$\mathcal{W}\left(\frac{j}{N_b - 1}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\pi N_b^n \frac{j}{(N_b - 1)}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(\frac{2\pi j}{N_b - 1}\right) = \frac{1}{1-\lambda} \cos\left(\frac{2\pi j}{N_b - 1}\right).$$

Property 2.10 ([DL22]).

For $0 \leqslant j \leqslant \dfrac{(N_b - 1)}{2}$ (resp., for $\dfrac{(N_b - 1)}{2} \leqslant j \leqslant N_b - 1$), we have that

$$\mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \leqslant 0 \quad \left(\textit{resp.,}\ \mathcal{W}\left(\frac{j+1}{N_b - 1}\right) - \mathcal{W}\left(\frac{j}{N_b - 1}\right) \geqslant 0\right).$$

Notation 8 (**Signum Function**).

The *signum function* of a real number x is defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if} \quad x < 0, \\ 0, & \text{if} \quad x = 0, \\ +1, & \text{if} \quad x > 0. \end{cases}$$

Property 2.11 ([DL22]).

Given any strictly positive integer m, we have the following properties:

i. For any j in $\{0, \ldots, \#V_m\}$, the point

$$\left(\frac{j}{(N_b - 1) N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^m}\right)\right)$$

is the image of the point

$$\left(\frac{j}{(N_b - 1) N_b^{m-1}} - i, \mathcal{W}\left(\frac{j}{(N_b - 1) N_b^{m-1}} - i\right)\right),$$

which can also be written as

$$\left(\frac{j - i\,(N_b - 1) N_b^{m-1}}{(N_b - 1) N_b^{m-1}}, \mathcal{W}\left(\frac{j - i\,(N_b - 1) N_b^{m-1}}{(N_b - 1) N_b^{m-1}}\right)\right)$$

under the map T_i, where $i \in \{0, \ldots, N_b - 1\}$ is arbitrary.

Consequently, for $0 \leqslant j \leqslant N_b - 1$, ***the j^{th} vertex of the polygon*** $\mathcal{P}_{m,k}$*,*
$0 \leqslant k \leqslant N_b^m - 1$, i.e., the point

$$\left(\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m} \right) \right),$$

is the image of the point

$$\left(\frac{(N_b-1)\left(k-i\,(N_b-1)\,N_b^{m-1}\right)+j}{(N_b-1)\,N_b^{m-1}}, \mathcal{W}\left(\frac{(N_b-1)\left(k-i\,(N_b-1)\,N_b^{m-1}\right)+j}{(N_b-1)\,N_b^{m-1}} \right) \right);$$

under the map T_i, where $i \in \{0,\ldots,N_b-1\}$ is again arbitrary. This latter point is also ***the j^{th} vertex of the polygon*** *$\mathcal{P}_{m-1,k-i\,(N_b-1)\,N_b^{m-1}}$. Therefore, there is an exact correspondance between vertices of the polygons at consecutive steps $m-1$, m.*

ii. *Given j in $\{0,\ldots,N_b-2\}$ and k in $\{0,\ldots,N_b^m-1\}$, we have that*

$$sgn\left(\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m} \right) - \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m} \right) \right) = sgn\left(\mathcal{W}\left(\frac{j+1}{N_b-1} \right) - \mathcal{W}\left(\frac{j}{N_b-1} \right) \right).$$

Proof.

i. Given $m \in \mathbb{N}^\star$, let us consider $i \in \{0,\ldots,N_b-1\}$. The image of the point

$$\left(\frac{j}{(N_b-1)\,N_b^{m-1}} - i, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}} - i \right) \right)$$

under the map T_i is obtained by applying the analytic expression given in Property 2.3, on page 8, to the coordinates of this point, which, thanks to Property 2.1, on page 6 above, yields the expected result, namely,

$$\left(\frac{j}{(N_b-1)\,N_b^m}, \lambda\,\underbrace{\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}} - i \right)}_{\substack{\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}} \right) \\ \text{(by 1-periodicity)}}} + \cos \frac{2\,\pi\,j}{(N_b-1)\,N_b^m} \right),$$

i.e., also,

$$\left(\frac{j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m} \right) \right).$$

ii. See [DL22].

□

Property 2.12 (**Lower Bound and Upper Bound for the Elementary Heights [DL22]**).

For any strictly positive integer m and any j in $\{0,\ldots,(N_b-1)\,N_b^m\}$, we have the following estimates, where L_m is the elementary horizontal length introduced in part i. of Definition 2.6, on page 12:

$$C_{inf}\,L_m^{2-D_{\mathcal{W}}} \leqslant \underbrace{|\mathcal{W}\left((j+1)\,L_m\right)-\mathcal{W}\left(j\,L_m\right)|}_{h_{j,j+1,m}} \leqslant C_{sup}\,L_m^{2-D_{\mathcal{W}}}\,, \tag{5}$$

where the finite and positive constants C_{inf} and C_{sup} are given by

$$C_{inf} = (N_b-1)^{2-D_{\mathcal{W}}} \min_{0\leqslant j\leqslant N_b-1,\, \mathcal{W}\left(\frac{j+1}{N_b-1}\right)\neq\mathcal{W}\left(\frac{j}{N_b-1}\right)} \left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right|$$

and

$$C_{sup} = (N_b-1)^{2-D_{\mathcal{W}}} \left(\max_{0\leqslant j\leqslant N_b-1}\left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right| + \frac{2\,\pi}{(N_b-1)\,(\lambda\,N_b-1)}\right).$$

One should note, in addition, that these constants C_{inf} and C_{sup} depend on the initial polygon $\mathcal{P}_0$.

As a consequence, we also have that

$$C_{inf}\,L_m^{2-D_{\mathcal{W}}} \leqslant h_m^{inf} \leqslant C_{sup}\,L_m^{2-D_{\mathcal{W}}} \quad \text{and} \quad C_{inf}\,L_m^{2-D_{\mathcal{W}}} \leqslant h_m \leqslant C_{sup}\,L_m^{2-D_{\mathcal{W}}}\,,$$

where h_m^{inf} and h_m respectively denote the minimal and maximal heights introduced in part iv. of Definition 2.6, on page 12.

Theorem 2.13 (**Sharp Local Discrete Reverse-Hölder Properties of the Weierstrass Function [DL22]**).

For any $m\in\mathbb{N}$, let us consider a pair of real numbers (x,x') such that

$$x = \frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m} = \left((N_b-1)\,k+j\right)\,L_m\,,$$

and

$$x' = \frac{(N_b-1)\,k+j+\ell}{(N_b-1)\,N_b^m} = \left((N_b-1)\,k+j+\ell\right)\,L_m\,,$$

where $0\leqslant k\leqslant N_b^m-1$. We then have the following (discrete, local) reverse-Hölder inequality*, with sharp Hölder exponent $-\dfrac{\ln\lambda}{\ln N_b}=2-D_{\mathcal{W}}$:*

$$C_{inf}\,|x'-x|^{2-D_{\mathcal{W}}} \leqslant \left|\mathcal{W}(x')-\mathcal{W}(x)\right|\,,$$

where $(x,\mathcal{W}(x))$ and $\left(x',\mathcal{W}(x')\right)$ are adjacent vertices of the same m^{th} prefractal approximation, $\Gamma_{\mathcal{W}_m}$, with $m\in\mathbb{N}$ arbitrary. Here, C_{inf} is given as in Property 2.12 just above, on page 19 .

Corollary 2.14 (**Optimal Hölder Exponent for the Weierstrass Function (see [DL22])**).

The local reverse Hölder property of Theorem 2.13 just above – in conjunction with the Hölder condition satisfied by the Weierstrass function (see also [Zyg02], Chapter II, Theorem 4.9, on page 47) – shows that the Codimension

$$2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b} \in]0,1[$$

is the best (i.e., optimal) Hölder exponent for the Weierstrass function (as was originally shown, by a completely different method, by G. H. Hardy in [Har16]).

Note that, as a consequence, since the Hölder exponent is strictly smaller than one, it follows that the Weierstrass function $\mathcal{W}$ is nowhere differentiable.

Corollary 2.15 (**Coming from Property** 2.11, on page 17)**.**

Thanks to Property 2.12, on page 19, one may now write, for any strictly positive integer m and any integer j in $\left\{0, \ldots, (N_b - 1)\, N_b^m - 1\right\}$:

i. *for the elementary heights:*

$$h_{j-1,j,m} = L_m^{2-D_{\mathcal{W}}}\, \mathcal{O}(1) \; ; \tag{6}$$

ii. *for the elementary quotients:*

$$\frac{h_{j-1,j,m}}{L_m} = L_m^{1-D_{\mathcal{W}}}\, \mathcal{O}(1) \; , \tag{7}$$

and where

$$0 < C_{inf} \leqslant \mathcal{O}(1) \leqslant C_{sup} < \infty \, .$$

Recall from Property 2.12, on page 19, that C_{inf} and C_{sup} are independent of $m \in \mathbb{N}^\star$.

Corollary 2.16 (**Nonincreasing Sequence of Geometric Angles** (**Coming from Property** 2.11, on page 17; see [DL22])).

*For the **geometric angles*** $\theta_{j-1,j,m}$, $0 \leqslant j \leqslant (N_b-1)\,N_b^m$, $m \in \mathbb{N}$, *introduced in part v. of Definition 2.6, on page 12, we have the following result:*

$$\tan\theta_{j-1,j,m} = \frac{L_m}{h_{j-1,j,m}}(N_b-1) > \tan\theta_{j-1,j,m+1}\,,$$

which yields

$$\theta_{j-1,j,m} > \theta_{j-1,j,m+1} \quad \text{and} \quad \theta_{j-1,j,m+1} \lesssim L_m^{D_{\mathcal{W}}-1}\,.$$

Corollary 2.17 (**Local Extrema** (**Coming from Property 2.11, on page 17; see** [**DL22**])).

i. The set of local maxima of the Weierstrass function on the interval $[0,1]$ *is given by*

$$\left\{\left(\frac{(N_b-1)\,k}{N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{N_b^m}\right)\right) \,:\, 0 \leqslant k \leqslant N_b^m-1,\ m \in \mathbb{N}\right\},$$

and corresponds to the extreme vertices of the polygons $\mathcal{P}_{m,k}$ *and* $\mathcal{Q}_{m,k}$ *(see Property 2.6, on page 10) at a given step* m *(i.e., they are the vertices connecting consecutive polygons; see part iv. of Property 2.6, on page 10).*

ii. For odd values of N_b*, the set of local minima of the Weierstrass function on the interval* $[0,1]$ *is given by*

$$\left\{\left(\frac{(N_b-1)\,k+\frac{N_b-1}{2}}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k+\frac{N_b-1}{2}}{(N_b-1)\,N_b^m}\right)\right) \,:\, 0 \leqslant k \leqslant N_b^m-1,\ m \in \mathbb{N}\right\},$$

and corresponds to the bottom vertices of the polygons $\mathcal{P}_{m,k}$ *and* $\mathcal{Q}_{m,k}$ *at a given step* m*; see also part iv. of Property 2.6, on page 10.*

Property 2.18 (**Existence of Reentrant Angles** [**DL22**]).

i. The initial polygon $\mathcal{P}_0$*, admits **reentrant interior angles**, at a vertex* P_j*, with* $0 < j \leqslant N_b-1$*, in the sense that, with the right-hand rule (according to which angles are measured in a counterclockwise direction), we have that*

$$\left(\widehat{(P_jP_{j+1}),(P_jP_{j-1})}\right) > \pi\,,$$

for

$$0 < j \leqslant \frac{N_b-3}{4} \quad \text{or} \quad \frac{3\,N_b-1}{4} \leqslant j < N_b-1$$

(see Figure 5, on page 22), which does not occur for values of $N_b < 7$*.*

The number of reentrant angles is then equal to $2\left[\frac{N_b-3}{4}\right]$*.*

ii. At a given step $m \in \mathbb{N}^\star$*, with the above convention, a polygon* $\mathcal{P}_{m,k}$ *admits reentrant interior angles in the sole cases when* $N_b \geqslant 7$*, at vertices* M_{k+j}*,* $1 \leqslant k \leqslant N_b^m$*,* $0 < j \leqslant N_b - 1$*, as well as in the case when*

$$0 < j \leqslant \frac{N_b - 3}{4} \quad or \quad \frac{3\,N_b - 1}{4} \leqslant j < N_b - 1\,.$$

The number of reentrant angles is then equal to $2\,N_b^m \left[\dfrac{N_b - 3}{4}\right]$.

Remark 2.2. Note that due to the respective definitions of the polygons $\mathcal{P}_{m,k}$ and $\mathcal{Q}_{m,k}$, the existence of reentrant interior angles for $\mathcal{P}_{m,k}$ at a vertex M_{k+j},
for $1 \leqslant k \leqslant N_b^m$, $0 < j \leqslant N_b - 1$, also results in the existence of reentrant interior angles for $\mathcal{Q}_{m,k}$ at the vertices M_{k+j-1}, $1 \leqslant k \leqslant N_b^m$, $1 < j \leqslant N_b - 1$ and M_{k+j+1},
$1 \leqslant k \leqslant N_b^m$, $0 < j \leqslant N_b - 2$.

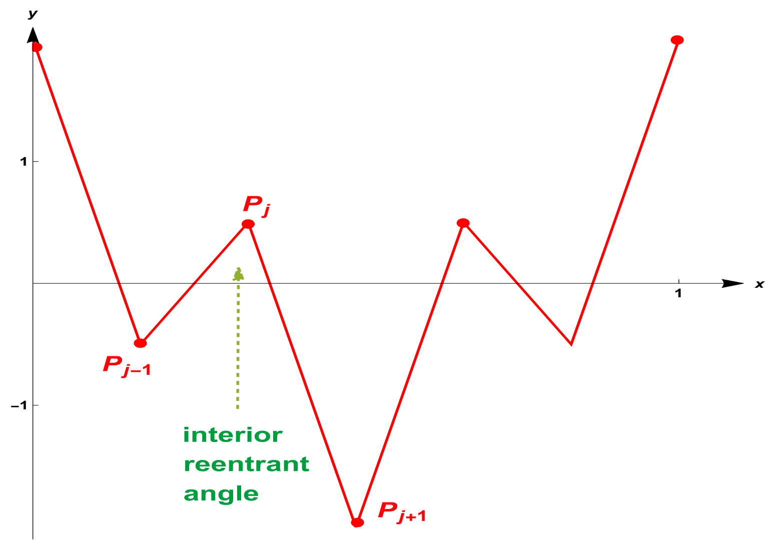


Figure 5: **An interior reentrant angle. Here,** $N_b = 7$ **and** $\lambda = \frac{1}{2}$**.**

3 Polyhedral Measure on the Weierstrass IFD

Our results on fractal cohomology obtained in [DL24c] (see also [DL22], [DL24c]) have highlighted the role played by specific threshold values for the number $\varepsilon_m^m > 0$ at any step $m \in \mathbb{N}$ of the prefractal graph approximation; namely, the m^{th} *cohomology infinitesimal* introduced in Definition 3.1, on page 23 just below.

Definition 3.1 (m^{th} **Cohomology Infinitesimal [DL22], [DL24c] and** m^{th} **Intrinsic Cohomology Infinitesimal [DL23b], [DL23a]).** From now on, given any $m \in \mathbb{N}$, we will call m^{th} *cohomology infinitesimal* the number $\varepsilon_m^m > 0$ which also corresponds to the elementary horizontal length introduced in part *i.* in Definition 2.6, on page 12; i.e., $\varepsilon_m^m = (\varepsilon_m)^m = \dfrac{1}{N_b - 1}\,\dfrac{1}{N_b^m}$.

Observe that, clearly, ε_m itself – and not just ε_m^m – depends on m.

In addition, since $N_b > 1$, ε_m^m satisfies the following asymptotic behavior,

$$\varepsilon_m^m \to 0\,, \text{ as } m \to \infty,$$

which, naturally, results in the fact that the larger m, the smaller ε_m^m. It is for this reason that we call ε_m^m – or rather, the *sequence* $\left(\varepsilon_m^m\right)_{m=0}^{\infty}$ of positive numbers tending to zero as $m \to \infty$, with $\varepsilon_m^m = (\varepsilon_m)^m$, for each $m \in \mathbb{N}$ – an *infinitesimal*, called *the cohomology infinitesimal.* Note that this cohomology infinitesimal is the one naturally associated to the scaling relation of Proposition 2.1, on page 6.

In the sequel, it is also useful to keep in mind that the sequence of positive numbers $(\varepsilon_m)_{m=0}^{\infty}$ itself satisfies

$$\varepsilon_m \sim \frac{1}{N_b}\,, \text{ as } m \to \infty \ ;$$

i.e., $\varepsilon_m \to \dfrac{1}{N_b}$, as $m \to \infty$. In particular, $\varepsilon_m \not\to 0$, as $m \to \infty$, but, instead, ε_m tends to a strictly positive and finite limit.

We also introduce, given any $m \in \mathbb{N}$, the m^{th} *intrinsic cohomology infinitesimal*, denoted by $\varepsilon^m > 0$, such that

$$\varepsilon^m = \frac{1}{N_b^m}\,,$$

where

$$\varepsilon = \frac{1}{N_b}\,.$$

We call ε *the intrinsic scale*, or *intrinsic subdivision scale.*

Note that

$$\varepsilon_m^m = \frac{\varepsilon^m}{N_b - 1}$$

and that the m^{th} intrinsic cohomology infinitesimal ε^m is asymptotic (when m tends to ∞) to the m^{th} cohomology infinitesimal ε_m^m.

Remark 3.1 (**Addressing Numerical Estimates**).

From a practical point of view, an important question is the value of the ratio

$$\frac{\text{Cohomology infinitesimal}}{\text{Maximal height}} = \frac{\varepsilon_m^m}{h_m}\ ;$$

see relation (4), on page 12.

Thanks to the estimates given in relation (7), on page 20, we have that

$$\frac{\varepsilon_m^m}{h_m} = L_m^{1-D_{\mathcal{W}}}\,\mathcal{O}(1) = \varepsilon_m^{m\,(1-D_{\mathcal{W}})}\,\mathcal{O}(1)\ ,$$

with

$$0 < C_{inf} \leqslant \mathcal{O}(1) \leqslant C_{sup} < \infty\,.$$

Given $q \in \mathbb{N}^\star$, we then have

$$\frac{1}{10^q}\,C_{inf} \leqslant \frac{\varepsilon_m^m}{h_m} \leqslant \frac{1}{10^q}\,C_{sup}$$

when

$$\frac{C_{inf}}{10^q} \leqslant e^{(1-D_{\mathcal{W}})\,\ln L_m} \leqslant \frac{C_{sup}}{10^q}\,,$$

or, equivalently, when

$$-\frac{1}{\ln N_b}\,\ln\left((N_b-1)\left(\frac{C_{sup}}{10^q}\right)^{\frac{1}{1-D_{\mathcal{W}}}}\right) \leqslant m \leqslant -\frac{1}{\ln N_b}\,\ln\left((N_b-1)\left(\frac{C_{inf}}{10^q}\right)^{\frac{1}{1-D_{\mathcal{W}}}}\right).$$

Numerical values for $N_b = 3$ and $\lambda = \frac{1}{2}$ yield:

i. For $q = 1$: $2 \leqslant m \leqslant 3$.

ii. For $q = 2$: $7 \leqslant m \leqslant 9$.

iii. For $q = 3$: $13 \leqslant m \leqslant 15$.

Hence, when m increases, the ratio decreases, and tends to 0. This numerical – but very practical and explicit argument – also accounts for our forthcoming neighborhoods, of width equal to the cohomology infinitesimal.

We refer to part *iv.* of Property 2.6, on page 10 above, along with Figure 1, on page 11, for the definition of the polygons $\mathcal{P}_{m,j}$ (resp., $\mathcal{Q}_{m,j}$) associated with the Weierstrass Curve in the next definition, as well as throughout the rest of this section. See also Definition 2.5, on page 11 where the polygonal families are introduced.

Definition 3.2 (**Power of a Vertex of the Prefractal Graph $\Gamma_{\mathcal{W}_m}$, $m \in \mathbb{N}^\star$, with Respect to the Polygonal Families $\mathcal{P}_m$ and $\mathcal{Q}_m$**).

Given a strictly positive integer m, a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ will be said to be:

i. *of power one relative to the polygonal family* $\mathcal{P}_m$ if X belongs to (to be understood in the sense that X is a vertex of) one and only one N_b-gon $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$;

ii. of power $\frac{1}{2}$ relative to the polygonal family $\mathcal{P}_m$ if X is a common vertex to two consecutive N_b-gons $\mathcal{P}_{m,j}$ and $\mathcal{P}_{m,j+1}$, $0 \leqslant j \leqslant N_b^m - 2$;

iii. of power zero relative to the polygonal family $\mathcal{P}_m$ if X does not belong to (to be understood in the sense that X is not a vertex of) any N_b-gon $\mathcal{P}_{m,j}$,
$0 \leqslant j \leqslant N_b^m - 1$.

Similarly, given $m \in \mathbb{N}$, a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ is said to be:

i. of power one relative to the polygonal family $\mathcal{Q}_m$ if X belongs to (as above, to be understood in the sense that X is a vertex of) one and only one N_b-gon $\mathcal{Q}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 2$;

ii. of power $\frac{1}{2}$ relative to the polygonal family $\mathcal{Q}_m$ if X is a common vertex to two consecutive N_b–gons $\mathcal{Q}_{m,j}$ and $\mathcal{Q}_{m,j+1}$, $0 \leqslant j \leqslant N_b^m - 3$;

iii. of power zero relative to the polygonal family $\mathcal{Q}_m$ if X does not belong to (as previously, to be understood in the sense that X is not a vertex of) any N_b-gon $\mathcal{Q}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 2$.

Notation 9. In the sequel, given a strictly positive integer m, the *power of a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ relative to the polygonal families $\mathcal{P}_m$ and $\mathcal{Q}_m$* will be respectively denoted by

$$p(X, \mathcal{P}_m) \quad \text{and} \quad p(X, \mathcal{Q}_m)\,.$$

Notation 10 (Lebesgue Measure (on $\mathbb{R}^2$)).

In the sequel, we denote by $\mu_{\mathcal{L}}$ the Lebesgue measure on $\mathbb{R}^2$.

Notation 11. For any $m \in \mathbb{N}$, and any vertex X of V_m, we set:

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) = \begin{cases} \dfrac{1}{N_b}\, p(X, \mathcal{P}_m) \displaystyle\sum_{0 \leqslant j \leqslant N_b^m - 1,\, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}(\mathcal{P}_{m,j}) \,, \text{ if } X \notin \mathcal{Q}_m \,, \\ \dfrac{1}{N_b}\, p(X, \mathcal{Q}_m) \displaystyle\sum_{1 \leqslant j \leqslant N_b^m - 2,\, X \text{ vertex of } \mathcal{Q}_{m,j}} \mu_{\mathcal{L}}(\mathcal{Q}_{m,j}) \,, \text{ if } X \notin \mathcal{P}_m \,, \\ \dfrac{1}{2\,N_b} \left\{ p(X, \mathcal{P}_m) \displaystyle\sum_{0 \leqslant j \leqslant N_b^m - 1,\, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}(\mathcal{P}_{m,j}) + p(X, \mathcal{Q}_m) \displaystyle\sum_{1 \leqslant j \leqslant N_b^m - 2,\, X \text{ vertex of } \mathcal{Q}_{m,j}} \mu_{\mathcal{L}}(\mathcal{Q}_{m,j}) \right\}, \\ \text{if } X \in \mathcal{P}_m \cap \mathcal{Q}_m \,. \end{cases}$$

Notation 12 (Euclidean Distance).

We hereafter denote by d_{Eucl} the Euclidean distance.

Property 3.1. *For any $m \in \mathbb{N}$, and any pair (X, Y) of adjacent vertices of V_m belonging to the same polygon $\mathcal{P}_{m,j}$, with $0 \leqslant j \leqslant N_b^m - 1$ (resp., $\mathcal{Q}_{m,j}$, with $0 \leqslant j \leqslant N_b^m - 2$), we have that*

$$d_{Eucl}(X,Y) = \sqrt{h_{jm}^2 + L_m^2} > |h_{jm}| ,$$

which, due to the inequality given in Property 2.12, on page 19, ensures that

$$\frac{1}{d_{Eucl}(X,Y)} < \frac{1}{|h_{jm}|} \lesssim L_m^{D_{\mathcal{W}}-2} \lesssim N_b^{(2-D_{\mathcal{W}})\,m} .$$

At the same time, we also have that

$$d_{Eucl}(X,Y) \lesssim h_m \lesssim L_m^{2-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-2)\,m} .$$

Proof. This follows at once from Property 2.12, on page 19.

□

Corollary 3.2. *For any $m \in \mathbb{N}$, any integer j of $\{0, \ldots, N_b^m - 1\}$, and any pair of points (X,Y) of $\mathcal{P}_{m,j}$ or of $\mathcal{Q}_{m,j}$, we have that*

$$\frac{1}{d_{Eucl}(X,Y)} \lesssim L_m^{D_{\mathcal{W}}-2} \lesssim N_b^{(2-D_{\mathcal{W}})\,m} ,$$

and

$$d_{Eucl}(X,Y) \lesssim h_m \lesssim L_m^{2-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-2)\,m} .$$

Property 3.3. *For any $m \in \mathbb{N}$, and any vertex X of V_m:*

$$\mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) \lesssim h_m\, L_m \lesssim L_m^{3-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-3)\,m} ,$$

and

$$\mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) \lesssim h_m\, L_m \lesssim L_m^{3-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-3)\,m} .$$

Proof. This also directly comes from Property 2.12, on page 19.

□

Definition 3.3 (**Trace of a Polygon on the Weierstrass Curve**).

Given $m \in \mathbb{N}$, and $0 \leqslant j \leqslant N_b^m - 1$ (resp., $0 \leqslant j \leqslant N_b^m - 2$), of extreme vertices $\mathcal{V}_{initial}\left(\mathcal{P}_{m,j}\right) \in V_m$ and $\mathcal{V}_{end}\left(\mathcal{P}_{m,j}\right) \in V_m$ (resp., $\mathcal{V}_{initial}\left(\mathcal{Q}_{m,j}\right) \in V_m$ and $\mathcal{V}_{end}\left(\mathcal{Q}_{m,j}\right) \in V_m$; see Definition 2.6, on page 10), we define *the trace* of the polygon $\mathcal{P}_{m,j}$ (resp., $\mathcal{Q}_{m,j}$) on the Weierstrass Curve as the set $tr_{\gamma_{\mathcal{W}}}\left(\mathcal{P}_{m,j}\right)$ (resp., $tr_{\gamma_{\mathcal{W}}}\left(\mathcal{Q}_{m,j}\right)$) of points $\left\{\mathcal{V}_{initial}\left(\mathcal{P}_{m,j}\right), M_\star, \mathcal{V}_{end}\left(\mathcal{P}_{m,j}\right)\right\}$ (resp., $\left\{\mathcal{V}_{initial}\left(\mathcal{Q}_{m,j}\right), M_\star, \mathcal{V}_{end}\left(\mathcal{Q}_{m,j}\right)\right\}$), where we denote by $M_\star$ any point of the Weierstrass Curve strictly located between $\mathcal{V}_{initial}\left(\mathcal{P}_{m,j}\right)$ and $\mathcal{V}_{end}\left(\mathcal{P}_{m,j}\right)$ (resp., $\mathcal{V}_{initial}\left(\mathcal{Q}_{m,j}\right)$ and $\mathcal{V}_{end}\left(\mathcal{Q}_{m,j}\right)$).

Definition 3.4 (Sequence of Domains Delimited by the Weierstrass IFD).

We introduce *the sequence of domains delimited by the Weierstrass IFD* as the sequence $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ of open, connected polygonal sets $(\mathcal{P}_m \cup \mathcal{Q}_m)_{m\in\mathbb{N}}$, where, for each $m \in \mathbb{N}$, $\mathcal{P}_m$ and Q_m respectively denote the polygonal sets introduced in Definition 2.5, on page 11.

Property 3.4 (Domain Delimited by the Weierstrass IFD).

We call domain, delimited by the Weierstrass IFD*, the set, which is equal to the following limit,*

$$\mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \lim_{m\to\infty} \mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right) ,$$

where the convergence is interpreted in the sense of the Hausdorff metric on $\mathbb{R}^2$; see Remark 3.2, on page 27 below. In fact, we have that

$$\mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \Gamma_{\mathcal{W}} .$$

Proof. Note, first, that the sequence $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ can be replaced by its closure $\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$, with closed polygons $\overline{\mathcal{P}}_{m,j}$ and $\overline{\mathcal{Q}}_{m,j}$, instead of open polygons $\mathcal{P}_{m,j}$ and $\mathcal{Q}_{m,j}$, used in the counterpart of Definition 3.4, on page 27 just above. We can then easily prove that the sequence $\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ converges to the Weierstrass Curve $\Gamma_{\mathcal{W}}$. This simply comes from the fact that, given any (positive) infinitesimal $\epsilon = \left(\varepsilon_m^m\right)_{m\in\mathbb{N}}$ (in the sense of Definition 3.1, on page 23), there exists an integer m_0 such that

$$\forall\, m \geqslant m_0 : \quad \overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right) \subset \mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m\right) ,$$

where $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m\right)$ denotes the (tubular) (m, ε_m^m)-neighborhood of the Weierstrass IFD introduced in [DL22]; namely,

$$\mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m\right) = \left\{M \in \mathbb{R}^2 ,\, d\left(M, \Gamma_{\mathcal{W}_m}\right) \leqslant \varepsilon_m^m\right\} .$$

This also ensures that

$$\lim_{m\to\infty} \mu_{\mathcal{L}}\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)\right) = 0 .$$

□

Remark 3.2. In our proof, we have considered the limit of the sequence

$$\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$$

in the set–theoretic sense. In fact, we could have considered, instead, the Hausdorff limit of this sequence; i.e., by using the Hausdorff metric $d_{\mathcal{H}}$ on $\mathbb{R}^2$. This would not have changed our result, since

$$d_{\mathcal{H}}\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right), \Gamma_{\mathcal{W}}\right) = d_{\mathcal{H}}\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right), \Gamma_{\mathcal{W}}\right) .$$

As we explained in our proof just above, there exists an integer m_0 such that,

$$\forall\, m \geqslant m_0 : \quad \overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right) \subset \mathcal{D}\left(\Gamma_{\mathcal{W}_m}, \varepsilon_m^m\right) .$$

This ensures, for all $m \geqslant m_0$, any j in $\left\{0,\ldots,N_b^m-1\right\}$, any point $P \in \mathcal{P}_{m,j} \subset \overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)$ (resp., $Q \in \mathcal{Q}_{m,j} \subset \overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right)$), and any point $M \in tr_{\gamma_{\mathcal{W}}}\left(\mathcal{P}_{m,j}\right)$ (resp., $M \in tr_{\gamma_{\mathcal{W}}}\left(\mathcal{Q}_{m,j}\right)$), that the Hausdorff distance $d_{\mathcal{H}}(P,M)$ (resp., $d_{\mathcal{H}}(Q,M)$) between P (resp., Q) and M is such that

$$d_{\mathcal{H}}(P,M) = \max\left\{\underbrace{\sup_{M' \in tr_{\gamma_{\mathcal{W}}}(\mathcal{P}_{m,j})} d_{Eucl}(P,M')}_{\lesssim \varepsilon_m^m} \quad , \quad \underbrace{\sup_{P' \in \mathcal{P}_{m,j}} d_{Eucl}(P',M)}_{\lesssim \varepsilon_m^m}\right\} \lesssim \varepsilon_m^m$$

$$\left(\text{resp., } d_{\mathcal{H}}(Q,M) = \max\left\{\underbrace{\sup_{M' \in tr_{\gamma_{\mathcal{W}}}(\mathcal{Q}_{m,j})} d_{Eucl}(Q,M')}_{\lesssim \varepsilon_m^m} \quad , \quad \underbrace{\sup_{Q' \in \mathcal{Q}_{m,j}} d_{Eucl}(Q',M)}_{\lesssim \varepsilon_m^m}\right\} \lesssim \varepsilon_m^m\right),$$

as desired. It follows that

$$d_{\mathcal{H}}\left(\overline{\mathcal{D}}\left(\Gamma_{\mathcal{W}_m}\right),\Gamma_{\mathcal{W}}\right) \lesssim \varepsilon_m^m \underset{m\to\infty}{\to} 0\,.$$

Hence, $\mathcal{D}\left(\Gamma_{\mathcal{W}}\right)$ is the Hausdorff limit of $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$, and so, by the uniqueness of such a limit, we deduce that $\mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \Gamma_{\mathcal{W}}$.

Notation 13 (**Minimal and Maximal Values of the Weierstrass Funnction $\mathcal{W}$ on $[0,1]$).**

We set

$$m_{\mathcal{W}} = \min_{t\in[0,1]} \mathcal{W}(t) = -\frac{1}{1-\lambda} \quad , \quad M_{\mathcal{W}} = \max_{t\in[0,1]} \mathcal{W}(t) = \frac{1}{1-\lambda}\,.$$

Notation 14. Henceforth, for a given $m \in \mathbb{N}$, the notation $\displaystyle\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}$ means that the associated finite sum runs through all of the vertices of the polygons belonging to the sets $\mathcal{P}_m$ and $\mathcal{Q}_m$ introduced in Definition 2.5, on page 11; see also Notation 7, on page 12 following that definition.

Property 3.5. *Given a continuous function u on $[0,1]\times[m_{\mathcal{W}},M_{\mathcal{W}}]$, we have that, for any $m \in \mathbb{N}$, and any vertex X of V_m:*

$$\left|\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\,u\left(X\right)\right| \leqslant \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\left(\max_{[0,1]\times[m_{\mathcal{W}},M_{\mathcal{W}}]}|u|\right) \lesssim N_b^{-(3-D_{\mathcal{W}})\,m}\,.$$

Consequently, with the notation of Definition 3.1, on page 23, we have that

$$\varepsilon_m^{m\,(D_{\mathcal{W}}-2)}\left|\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\,u\left(X\right)\right| \lesssim \varepsilon_m^{-m}\,.$$

Since the sequence $\left(\displaystyle\sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \varepsilon_m^{-m} \right)_{m \in \mathbb{N}}$ *is a positive and increasing sequence (note that the number of vertices involved increases as m increases), this ensures the existence of the finite limit*

$$\lim_{m \to \infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\,(X, \mathcal{P}_m, \mathcal{Q}_m)\; u\,(X)\;,$$

where we have used Notation 14, on page 28.

Proof. Thanks to Property 3.3, on page 26, for any $m \in \mathbb{N}$, and any vertex X of V_m, we have that

$$\mu^{\mathcal{L}}\,(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim N_b^{(D_{\mathcal{W}}-3)\,m} \quad \text{and} \quad \mu^{\mathcal{L}}\,(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim N_b^{(D_{\mathcal{W}}-3)\,m}\,.$$

We then recall from Section 2 that, for any $m \in \mathbb{N}$, the total number of polygons $\mathcal{P}_m$ is N_b^m, while the total number of polygons $\mathcal{Q}_m$ is equal to $N_b^m - 1$; see Property 2.6, on page 10. We then have that

$$\sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\,(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim N_b^m\, N_b^{(D_{\mathcal{W}}-3)\,m}\,;$$

i.e.,

$$\sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\,(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim N_b^{(D_{\mathcal{W}}-2)\,m}\,,$$

or, equivalently, due to the relation between the m^{th} *cohomology infinitesimal* ε_m^m introduced in Definition 3.1, on page 23, and the Weierstrass parameter N_b ,

$$\sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\,(X, \mathcal{P}_m, \mathcal{Q}_m) \lesssim \varepsilon_m^{m\,(2-D_{\mathcal{W}})}\,,$$

which, as desired, ensures the existence of the finite limit

$$\left(\max_{[0,1]\times[m_{\mathcal{W}}, M_{\mathcal{W}}]} |u| \right) \lim_{m \to \infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\,(X, \mathcal{P}_m, \mathcal{Q}_m)\,.$$

□

Property 3.6 (Polyhedral Measure on the Weierstrass IFD).

We introduce the polyhedral measure on the Weierstrass IFD, *denoted by* μ, *such that for any continuous function u on the Weierstrass Curve, with the use of Notation 11, on page 25, and Notation 14, on page 28,*

$$\int_{\Gamma_{\mathcal{W}}} u\, d\mu = \lim_{m \to \infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\,(X, \mathcal{P}_m, \mathcal{Q}_m)\; u\,(X)\;, \quad (\star) \tag{8}$$

which, thanks to Property 3.4, on page 27, can also be understood in the following way,

$$\int_{\Gamma_{\mathcal{W}}} u\, d\mu = \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} u\, d\mu\,.$$

Remark 3.3. In a sense, our polyhedral measure can be seen as a measure which is an extension of the Riemann integral, where the step functions are replaced by upper and lower affine functions which approximate the Weierstrass Curve.

Theorem 3.7.

The polyhedral measure μ is well defined, positive, as well as a bounded, nonzero, Borel measure on $\mathcal{D}(\Gamma_{\mathcal{W}})$. The associated total mass is given by

$$\mu(\mathcal{D}(\Gamma_{\mathcal{W}})) = \mu(\Gamma_{\mathcal{W}}) = \lim_{m\to\infty} \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) \,, \quad (\star\star) \tag{9}$$

and satisfies the following estimate:

$$\mu(\mathcal{D}(\Gamma_{\mathcal{W}})) \leqslant \frac{2}{N_b}(N_b-1)^2 C_{sup} \,. \quad (\star\star\star) \tag{10}$$

Furthermore, the support of μ coincides with the entire curve:

$$supp\, \mu = \mathcal{D}(\Gamma_{\mathcal{W}}) = \Gamma_{\mathcal{W}} \,.$$

In addition, μ is the weak limit as $m \to \infty$ of the following discrete measures *(or Dirac Combs), given, for each $m \in \mathbb{N}$, by*

$$\mu_m = \varepsilon_m^{m(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m)\, \delta_X \,,$$

where ε_m^m denotes the m^{th} cohomology infinitesimal introduced in Definition 3.1, on page 23, δ_X is the Dirac measure concentrated at X, and we have used Notation 11, on page 25, for $\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m)$, along with Notation 14, on page 28.

Proof.

For the sake of simplicity, we restrict ourselves to the case when $N_b < 7$. When $N_b \geqslant 7$, there are reentrant interior angles (see Property 2.18, on page 21), which result in the fact that for $m \in \mathbb{N}^\star$ and $0 \leqslant j \leqslant N_b^m - 1$, each polygon $\mathcal{P}_{m,j}$ is not necessarily convex. This significantly lengthens the corresponding proof which, however, consists of a mere, but cumbersome adaptation to this configuration of the present proof.

i. μ is a well defined measure.

Indeed, according to Proposition 3.5, on page 28, the map φ

$$u \mapsto \varphi(u) = \int_{\Gamma_{\mathcal{W}}} u\, d\mu$$

is a well defined linear functional on the space $C(\Gamma_{\mathcal{W}})$ of real-valued, continuous functions on $\Gamma_{\mathcal{W}}$. Hence, by a well-known argument, it is a continuous linear functional on $C(\Gamma_{\mathcal{W}})$, equipped with the *sup* norm. Since $\Gamma_{\mathcal{W}}$ is compact, and in light of relation (8) in Property 3.6, on page 29, μ is a bounded, Radon measure, with total mass $\varphi(1) = \mu(\mathcal{D}(\Gamma_{\mathcal{W}}))$, also given by inequality (9), and where 1 denotes the constant function equal to 1 on $\Gamma_{\mathcal{W}}$.

Then, according to the Riesz representation theorem, the associated positive Borel measure (still denoted by μ) is a bounded and positive Borel measure with the same total mass $\mu\left(\mathcal{D}\left(\Gamma_{\mathcal{W}}\right)\right)=\mu\left(\Gamma_{\mathcal{W}}\right)$.

ii. The nonzero measure – Estimates for the total mass of μ.

For $0\leqslant j\leqslant N_b^m-1$, each polygon $\mathcal{P}_{m,j}$ is contained in a rectangle of height at most equal to $(N_b-1)\,h_m$ (where h_m is the maximal height introduced in part *iv.* of Definition 2.6, on page 12), and of width at most equal to $(N_b-1)\,L_m$. This ensures that the Lebesgue measure of each polygon $\mathcal{P}_{m,j}$ is at most equal to $(N_b-1)^2\,h_m\,L_m$. We now recall that, thanks to Property 2.12, on page 19, for any $m\in\mathbb{N}$, we have the following estimate:

$$h_m\leqslant C_{sup}\,L_m^{2-D_{\mathcal{W}}}\,,$$

where

$$C_{sup}=(N_b-1)^{2-D_{\mathcal{W}}}\left(\max_{0\leqslant j\leqslant N_b-1}\left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right|+\frac{2\,\pi}{(N_b-1)\,(\lambda\,N_b-1)}\right).$$

Consequently, the Lebesgue measure $\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)$ of each polygon $\mathcal{P}_{m,j}$ is such that

$$\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)\leqslant(N_b-1)^2\,C_{sup}\,L_m^{3-D_{\mathcal{W}}}\,. \tag{11}$$

In the same way, for $0\leqslant j\leqslant N_b^m-2$, the Lebesgue measure $\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)$ of each polygon $\mathcal{Q}_{m,j}$ is such that

$$\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)\leqslant(N_b-1)^2\,C_{sup}\,L_m^{3-D_{\mathcal{W}}}\,. \tag{12}$$

We then deduce that, for any vertex X of V_m,

$$\mu\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\leqslant\frac{1}{N_b}\,(N_b-1)^2\,C_{sup}\,L_m^{3-D_{\mathcal{W}}}\,.$$

Hence, since the total number of polygons involved is at most equal to
$2\,N_b^m-1\leqslant 2\,N_b^m$, we can deduce that

$$\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\leqslant\frac{2\,N_b^m}{N_b}\,(N_b-1)^2\,C_{sup}\,L_m^{3-D_{\mathcal{W}}}\,,$$

or, equivalently,

$$\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\leqslant 2\,\frac{\varepsilon^{-m}}{N_b}\,(N_b-1)^2\,C_{sup}\,\varepsilon_m^{m\,(3-D_{\mathcal{W}})}\,.$$

We then have that

$$\varepsilon_m^{m\,(D_{\mathcal{W}}-2)}\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m}\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\leqslant\frac{2}{N_b}\,(N_b-1)^2\,C_{sup}<\infty\,,$$

from which we can deduce that the polyhedral measure is a bounded measure.

For $0\leqslant j\leqslant N_b^m-1$, each polygon $\mathcal{P}_{m,j}$ (which is convex) contains an inscribed circle, whose Lebesgue measure is greater than $\dfrac{h_m^{inf}\,L_m}{C_{N_b}}$, where h_m^{inf} is the minimal height introduced in part *iv.* of Definition 2.6,on page 12, and where C_{N_b} is a strictly positive constant, which depends on the value of the integer N_b (depending on the number of sides of the polygon, i.e., depending on the value of this integer, we can express the radius of this circle in function of the side lengths of the polygon). We now recall that, thanks to Property 2.12, on page 19, for any $m \in \mathbb{N}$, we have the following estimate:

$$C_{inf}\,L_m^{2-D_{\mathcal{W}}} \leqslant h_m^{inf}\,,$$

where

$$C_{inf} = (N_b-1)^{2-D_{\mathcal{W}}} \min_{0\leqslant j\leqslant N_b-1,\,\mathcal{W}\left(\frac{j+1}{N_b-1}\right)\neq\mathcal{W}\left(\frac{j}{N_b-1}\right)} \left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right|\,.$$

Consequently, the Lebesgue measure $\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)$ of each polygon $\mathcal{P}_{m,j}$ is such that

$$\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right) \geqslant \frac{h_m^{inf}\,L_m}{C_{N_b}} \geqslant \frac{C_{inf}\,L_m^{3-D_{\mathcal{W}}}}{C_{N_b}}\,.$$

In the same way, for $0 \leqslant j \leqslant N_b^m-2$, the Lebesgue measure $\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)$ of each polygon $\mathcal{Q}_{m,j}$ is such that

$$\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right) \geqslant \frac{h_m^{inf}\,L_m}{C_{N_b}} \geqslant \frac{C_{inf}\,L_m^{3-D_{\mathcal{W}}}}{C_{N_b}}\,.$$

We then deduce that, for any vertex X of V_m,

$$\mu\left(X,\mathcal{P}_m,\mathcal{Q}_m\right) \geqslant \frac{1}{N_b}\,\frac{C_{inf}\,L_m^{3-D_{\mathcal{W}}}}{C_{N_b}}\,.$$

Hence, since the total number of polygons involved is greater than $N_b^m-1 \geqslant \dfrac{N_b^m}{2}$, we can deduce that

$$\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right) \geqslant \frac{N_b^m}{2}\,\frac{C_{inf}\,L_m^{3-D_{\mathcal{W}}}}{N_b\,C_{N_b}}\,,$$

or, equivalently,

$$\sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right) \geqslant \frac{\varepsilon_m^{-m}}{2\,(N_b-1)}\,\frac{C_{inf}\,\varepsilon_m^{m\,(3-D_{\mathcal{W}})}}{N_b\,C_{N_b}}\,.$$

We then have that

$$\varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right) \geqslant \frac{1}{2\,(N_b-1)}\,\frac{C_{inf}}{N_b\,C_{N_b}} > 0\,,$$

from which, upon passing to the limit when $m \to \infty$, we can deduce that the polyhedral measure is a nonzero measure, and that its total mass satisfies inequality (10), on page 30.

iii. The support of μ coincides with the entire curve $\Gamma_{\mathcal{W}}$.

This simply comes from the proof given in *ii.* just above that the measure μ is nonzero. In the case of a positive, continuous function u defined on the Weierstrass Curve, we have that

$$\varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X\in\mathcal{P}_m\bigcup\mathcal{Q}_m} \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\,u(X) \geqslant \frac{1}{2\,(N_b-1)}\,\frac{C_{inf}}{N_b\,C_{N_b}}\left(\min_{\Gamma_{\mathcal{W}}}|u|\right) > 0\,.$$

Hence, upon passing to the limit when $m \to \infty$, we deduce that

$$\varphi(u) = \int_{\Gamma_{\mathcal{W}}} u \, d\mu > 0 \,,$$

and thus, $\varphi(u) \neq 0$, from which the claim follows easily.

Indeed, otherwise, if $\operatorname{supp} \mu \neq \Gamma_{\mathcal{W}}$, there exists $M \in \Gamma_{\mathcal{W}} \setminus \operatorname{supp} \mu$, and thus, by Urisohn's lemma (see, e.g., [Fol99], [Rud87] or [Rud91]), there exists $u \in C(\Gamma_{\mathcal{W}})$ and an open neighborhood $\mathcal{V}(M)$ of M in $\Gamma_{\mathcal{W}}$ disjoint from $\operatorname{supp} \mu$ and such that

$$u(M) = 1 \quad , \quad 0 \leqslant u \leqslant 1 \quad , \quad \text{and } u_{|\Gamma_{\mathcal{W}} \setminus \mathcal{V}(M)} = 0 \,.$$

Hence, by the above argument, $\varphi(u) \neq 0$, which contradicts the fact that $M \notin \operatorname{supp} \mu$ (see, e.g., *loc. cit.*).

iv. μ is a singular measure.

First, note that

$$\mu^{\mathcal{L}} (\Gamma_{\mathcal{W}}) = 0 \,,$$

because $D_{\mathcal{W}} < 2$, and, up to a multiplicative positive constant, $\mu^{\mathcal{L}}$ coincides with the 2-dimensional measure on $\mathbb{R}^2$. Now, since $\operatorname{supp} \mu \subset \Gamma_{\mathcal{W}}$, and $\mu^{\mathcal{L}} (\Gamma_{\mathcal{W}}) = 0$, it follows that μ is supported on a set of Lebesgue measure zero, which precisely implies that μ (viewed as a Borel measure on the rectangle $[0,1] \times [m_{\mathcal{W}}, M_{\mathcal{W}}]$ in the obvious way), is singular with respect to the restriction of $\mu^{\mathcal{L}}$ to this rectangle.

v. μ is the weak limit of the discrete measures μ_m.

Indeed, this follows at once from the latter part of Property 3.5, on page 28, according to which, for every $u \in \mathcal{C}(\Gamma_{\mathcal{W}})$,

$$\int_{\Gamma_{\mathcal{W}}} u \, d\mu = \lim_{m \to \infty} \int_{\Gamma_{\mathcal{W}}} u \, d\mu_m \,,$$

as desired.

This completes the proof of Theorem 3.7, given on page 30.

□

Remark 3.4. This choice of measure is obtained in the same manner as the one we introduced while working on the Arrowhead Curve [Dav20]. In our present case, however we define the measure more precisely, as well as establish several new properties.

Considering a two-dimensional measure is both essential and natural in so far as we will consider geometric conditions and two-dimensional nets in Section 4.

4 Atomic Decompositions – Trace Theorems, and Consequences

We begin this section by discussing Markov's Inequality and a closely related smoothness condition. Our goal, in this first part of Section 4, is to show that the Weierstrass Curve satifsfies Markov's Inequality (see Property 4.2, on page 39).

Definition 4.1 (**Markov's Inequality on a Subset $\mathcal{F}$ of $\mathbb{R}^n$ (see [JW84], Chapter II, Definition 2, on pages** 34-35)**).**

A closed, nonempty subset $\mathcal{F}$ of $\mathbb{R}^n$ preserves Markov's Inequality if, for all positive integers N, any real polynomial P with d_P variables, of degree at mots equal to N, any point $X \in \mathcal{F}$, and any real number $r \in]0,1]$:

$$\max_{\mathcal{F}\cap\mathcal{B}(X,r)} |\nabla P| \leqslant \frac{c\,(P,n,N,\mathcal{F})}{r} \max_{\mathcal{F}\cap\mathcal{B}(X,r)} |P|\,,$$

where $c\,(P,n,N,\mathcal{F})$ denotes a positive constant that depend on n, P, N and $\mathcal{F}$.

Definition 4.2 (**Smoothness Geometric Condition (see [JW84], Chapter II, Theorem 2, on page** 38)**).**

A closed, nonempty subset $\mathcal{F}$ of $\mathbb{R}^n$ satisfies the *smoothness geometric condition* $\mathcal{G}_{smooth}$ if, for any strictly positive number ϵ, there exists a point $X \in \mathcal{F}$, a real number $r \in]0,1]$, and a normed vector $b \in \mathbb{R}^n$ such that the intersection

$$\mathcal{F}\cap\mathcal{B}(X,r)$$

is contained in a band of the type

$$\left\{x \in \mathbb{R}^d\,,\, |b.(x-X)| < \varepsilon\, r\right\}\,.$$

Remark 4.1. A smooth curve satisfies the geometric condition $\mathcal{G}_{smooth}$.
In a sense, the Markov inequality means that locally, the curve is sufficiently flat.

Theorem 4.1 (**Necessary and Sufficient Condition for a Subset $\mathcal{F}$ of $\mathbb{R}^n$ to Satisfy Markov's Inequality (see [JW84], Chapter II, Theorem 2, on pages** 38)**).**

A closed, nonempty subset $\mathcal{F}$ of $\mathbb{R}^n$ preserves Markov's Inequality if and only if, for every strictly positive number ϵ, the geometric condition $\mathcal{G}_{smooth}$ does not hold.

Proof.

The following proof is inspired by the proof given in [JW84], Chapter II, on pages 38-39, but rewritten and containing much more detail and precision.

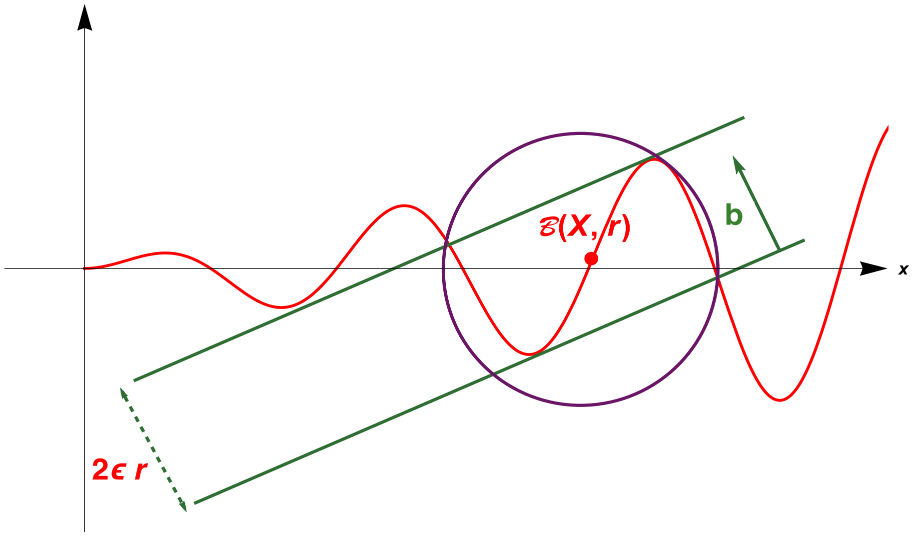


Figure 6: A smooth curve satisfying the geometric condition $\mathcal{G}_{smooth}$.

i. Let us assume that the geometric condition $\mathcal{G}_{smooth}$ holds. Then, for all $\epsilon > 0$, there exists $X = (x_1, \ldots, x_n) \in \mathcal{F}$, along with $r \in]0,1]$ and a unit vector
$b = (b_1 \ldots, b_n) \in \mathbb{R}^n$ such that

$$\mathcal{F} \cap \mathcal{B}(X,r) \subset \left\{x \in \mathbb{R}^n \,,\, |b \cdot (x - X)| < \epsilon\, r\right\} \,,$$

which of course, in the case when $\epsilon < 1$ implies that

$$\mathcal{F} \cap \mathcal{B}(X,r) \subset \left\{x \in \mathbb{R}^n \,,\, |b \cdot (x - X)| \leqslant \epsilon < 1\right\} \,.$$

By considering the polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$ such that

$$P(x) = b \cdot (x - X) = b_1\,(x_1 - X_1) + \ldots + b_n\,(x_n - X_n)\,,$$

we obtain that

$$0 < \max_{\mathcal{F} \cap \mathcal{B}(X,r)} |P| \leqslant \max_{x \in \mathbb{R}^n} |b \cdot (x - X)| < 1\,.$$

At the same time, we have that

$$\forall\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n \;: \quad \nabla P(x) = b\,,$$

which, since b is a unit vector (i.e., $|b| = 1$) ensures that

$$\max_{\mathcal{F} \cap \mathcal{B}(X,r)} |\nabla P| = |b| = 1 > \max_{\mathcal{F} \cap \mathcal{B}(X,r)} |P|\,.$$

Consequently, there cannot exist any positive constant $c\,(P, n, N, \mathcal{F})$ depending on n, P, N and $\mathcal{F}$ such that

$$\max_{\mathcal{F} \cap \mathcal{B}(X,r)} |\,\nabla\, P| \leqslant \frac{c\,(P, n, N, \mathcal{F})}{r} \max_{\mathcal{F} \cap \mathcal{B}(X,r)} |P|\,.$$

Indeed, note that, in such a case, we would have that

$$\max_{\mathcal{F} \cap \mathcal{B}(X,r)} |P| \geqslant \frac{r}{c\,(P, n, N, \mathcal{F})} \max_{\mathcal{F} \cap \mathcal{B}(X,r)} |\,\nabla\, P| \geqslant \frac{\max_{\mathcal{F} \cap \mathcal{B}(X,r)} |\,\nabla\, P|}{c\,(P, n, N, \mathcal{F})} > \frac{\max_{\mathcal{F} \cap \mathcal{B}(X,r)} |P|}{c\,(P, n, N, \mathcal{F})}\,,$$

where $\max\limits_{\mathcal{F}\cap\mathcal{B}(X,r)} |P| > 0$.

ii. Conversely, let us assume the existence of a strictly positive real number ϵ such that the geometric condition $\mathcal{G}_{smooth}$ does not hold.

By considering a first degree polynomial $P \in \mathbb{R}[X_1,\ldots,X_n]$ such that

$$\forall\, x=(x_1,\ldots,x_n) \in \mathbb{R}^n : \quad P(x) = b\cdot(x-X_0)+a = \sum_{j=1}^{n} b_j\,(x_j - X_{j,0}) + a\,,$$

where $b=(b_1\ldots,b_n)\in\mathbb{R}^n$ is a unit vector, $a\in\mathbb{R}$ and $X_0=\left(X_{0,1}\ldots,X_{0,n}\right)$ belongs to $\mathcal{F}$, we then obtain that

$$\forall\, x=(x_1,\ldots,x_n) \in \mathbb{R}^n : \quad \nabla P(x) = b$$

and, obvioulsy,

$$\forall\, x=(x_1,\ldots,x_n) \in \mathbb{R}^n : \quad |\nabla P(x)| = |b| = 1\,.$$

Since the geometric condition $\mathcal{G}_{smooth}$ does not hold, there exists $\epsilon>0$ such that, given $r\in]0,1]$, we can then find a point

$$X_1=\left(X_{1,1}\ldots,X_{1,n}\right)\in\mathcal{F}\cap\mathcal{B}(X_0,r)\,,$$

satisfying

$$|b\cdot(X_1-X_0)|\geqslant\epsilon\,r\,,$$

i.e., equivalently,

$$|P(X_1)-P(X_0)|\geqslant\epsilon\,r\,.$$

Consequently,

$$|P(X_1)|\geqslant\epsilon\,r+|P(X_0)|=\epsilon\,r+|a|\Rightarrow\frac{1}{r}\,|P(X_1)|\geqslant\epsilon+\frac{|a|}{r}\,.$$

We then have to consider the following two configurations:

$\rightsquigarrow$ $\underline{\dfrac{r}{k}\leqslant|a|\leqslant r\text{, where } k>1 :}$

In this case, we have that

$$\frac{1}{r}\,|P(X_1)|\geqslant\epsilon+\frac{|a|}{r}\geqslant\epsilon+\frac{1}{k}\,.$$

We then obtain that

$$\frac{1}{\left(\varepsilon+\frac{1}{k}\right)r}\max_{\mathcal{F}\cap\mathcal{B}(X,r)}|P|\geqslant\frac{1}{\left(\varepsilon+\frac{1}{k}\right)r}\,|P(X_1)|\geqslant 1\,,$$

or, equivalently,

$$\frac{1}{\left(\varepsilon+\frac{1}{k}\right)r}\max_{\mathcal{F}\cap\mathcal{B}(X,r)}|P|\geqslant\underbrace{\max_{\mathcal{F}\cap\mathcal{B}(X,r)}|\nabla P|}_{1}\,,$$

in which case the inequality holds.

$\rightsquigarrow$ $\underline{|a| > r}$:

In this case, we have that

$$\frac{1}{r}\,|P(X_1)| \geqslant \varepsilon + 1\,.$$

We then have that

$$\frac{1}{(1+\varepsilon)\,r}\,\max_{\mathcal{F}\cap\mathcal{B}(X,r)}|P| \geqslant \frac{1}{(1+\varepsilon)\,r}\,|P(X_1)| \geqslant 1\,,$$

or, equivalently,

$$\frac{1}{(1+\varepsilon)\,r}\,\max_{\mathcal{F}\cap\mathcal{B}(X,r)}|P| \geqslant \underbrace{\max_{\mathcal{F}\cap\mathcal{B}(X,r)}|\nabla P|}_{1}\,,$$

in which case the inequality also holds.

The remainder of the proof, i.e., when $P \in \mathbb{R}[X_1,\ldots,X_n]$ is not a first degree polynomial $P \in \mathbb{R}[X_1,\ldots,X_n]$, is obtained by induction. For the sake of concision, we only present the case when P is of degree $N \geqslant 2$ and is of the following form:

$$\forall\, x = (x_1,\ldots,x_n) \in \mathbb{R}^n : \quad P(x) = \left(b\cdot(x - X_0)\right)^N\,,$$

where $b = (b_1\,\ldots,b_n) \in \mathbb{R}^n$ is a unit vector and $X_0 = \left(X_{0,1}\,\ldots,X_{0,n}\right)$ belongs to $\mathcal{F}$. In this case, we have that

$$\forall\, x = (x_1,\ldots,x_n) \in \mathbb{R}^n : \quad \nabla P(x) = N\,\left(b\cdot(x - X_0)\right)^{N-1}\,b$$

and, consequently,

$$\forall\, x = (x_1,\ldots,x_n) \in \mathbb{R}^n : \quad |\nabla P(x)| = N\,|b\cdot(x - X_0)|^{N-1}\,.$$

Again, since the geometric condition $\mathcal{G}_{smooth}$ does not hold, there exists $\epsilon > 0$ such that, given $r \in]0,1]$, we can then find a point

$$X_1 = \left(X_{1,1}\,\ldots,X_{1,n}\right) \in \mathcal{F}\cap\mathcal{B}(X_0,r)$$

satisfying

$$|b\cdot(X_1 - X_0)| \geqslant \epsilon\, r\,.$$

We then have that, for all $x = (x_1,\ldots,x_n) \in \mathcal{F}\cap\mathcal{B}(X_0,r)$,

$$\max_{\mathcal{F}\cap\mathcal{B}(X_0,r)}|P| \geqslant |b\cdot(X_1 - X_0)|\;|b\cdot(x - X_0)|^{N-1}\,,$$

which ensures that

$$\max_{\mathcal{F}\cap\mathcal{B}(X_0,r)}|P| \geqslant |b\cdot(X_1 - X_0)|\,\max_{\mathcal{F}\cap\mathcal{B}(X_0,r)}|b\cdot(x - X_0)|^{N-1}\,,$$

i.e., equivalently,

$$\max_{\mathcal{F}\cap\mathcal{B}(X_0,r)}|P| \geqslant \frac{\epsilon\, r}{N}\,\max_{\mathcal{F}\cap\mathcal{B}(X_0,r)}|\nabla P|\,,$$

in which case the inequality holds.

When, more generally, P is of degree $N \geqslant 2$, of the following form:

$$\forall\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \quad P(x) = \sum_{k=0}^{N} \beta_k \, \left(b_k \cdot (x - X_0)\right)^k = \sum_{k=0}^{N} \beta_k \, \left(b_k \cdot (x - X_0)\right)^k ,$$

where, for $0 \leqslant k \leqslant N$, β_k denotes a real number, while $b_k = \left(b_{k,1} \ldots, b_{k,n}\right) \in \mathbb{R}^n$ is a unit vector and $X_0 = \left(X_{0,1} \ldots, X_{0,n}\right)$ belongs to $\mathcal{F}$, we apply the result given in [JW84], on page 37 (Proposition 4), which states that if Markov's Inequality is valid for all polynomials P of degree 1, then $\mathcal{F}$ preserves Markov's Inequality. It is based on the fact that, since the geometric condition $\mathcal{G}_{smooth}$ does not hold, for $0 \leqslant k \leqslant N$, there exists $\epsilon > 0$ such that, given $r \in]0,1]$, we can then find a point

$$X_1 = \left(X_{1,k} \ldots, X_{1,n}\right) \in \mathcal{F} \cap \mathcal{B}(X_0, r)$$

satisfying

$$|b_k \cdot (X_1 - X_0)| \geqslant \epsilon\, r$$

and hence, obviously,

$$|\beta_k| \, |b_k \cdot (X_1 - X_0)|^k \geqslant |\beta_k| \, \epsilon^k \, r^k .$$

This ensures that

$$\max_{\mathcal{F} \cap \mathcal{B}(X_0,r)} |P| \geqslant \sum_{k=0}^{N} |\beta_k| \, |b_k \cdot (X_1 - X_0)|^k \geqslant \sum_{k=0}^{N} |\beta_k| \, \epsilon^k \, r^k .$$

We also have that

$$\max_{\mathcal{F} \cap \mathcal{B}(X_0,r)} |P| \lesssim \sum_{k=0}^{N} |\beta_k| \, r^k \quad .$$

Since, at the same time,

$$\forall\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \quad \nabla P(x) = \sum_{k=0}^{N} k \, \left(b_k \cdot (x - X_{0,k})\right)^{k-1} \beta_k$$

and, consequently,

$$\forall\, x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \quad |\nabla P(x)| \leqslant \sum_{k=0}^{N} k \, |\beta_k| \, \left|b_k \cdot (x - X_{0,k})\right|^{k-1} ,$$

we have, in the same manner, that

$$\max_{\mathcal{F} \cap \mathcal{B}(X_0,r)} |\nabla P| \lesssim \sum_{k=0}^{N} k \, |\beta_k| \, r^{k-1} \lesssim \frac{1}{r} \max_{\mathcal{F} \cap \mathcal{B}(X_0,r)} |P| ,$$

which is the desired result.

□

Property 4.2 (The Weierstrass Curve Satisfies Markov's Inequality).

The Weierstrass Curve does not satisfy the geometric condition $\mathcal{G}_{smooth}$ and, therefore, satisfies Markov's Inequality.

Remark 4.2. A general result that could directly be applied to the Weierstrass Curve is given in the book by A. Jonsson and H. Wallin ([JW84], page 39):

Theorem 4.3. *If $\mathcal{F} \subset \mathbb{R}^n$ is a d-dimensional set such that $d > n-1$, then $\mathcal{F}$ preserves Markov's Inequality.*

Yet, the proof requires the use of d-measures. We propose a specific geometric proof that applies to fractal curves.

Proof. Building upon our results on the polygonal neighborhoods of the Weierstrass Curve (see [DL23b]), we can restrict ourseves to the m^{th} prefractal approximations $\Gamma_{\mathcal{W}_m}$, for $m \in \mathbb{N}^\star$ sufficiently large. Indeed, thanks to Property 2.5, on page 9, the set $V^\star = \bigcup\limits_{n\in\mathbb{N}} V_n$ is dense in the Weierstrass Curve $\Gamma_{\mathcal{W}_m}$.

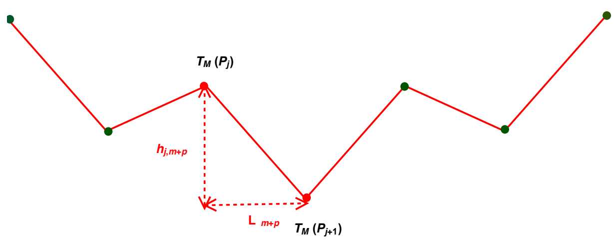
Let us consider a point $X = X_m$ belonging to the m^{th} prefractal approximation $\Gamma_{\mathcal{W}_m}$, for $m \in \mathbb{N}^\star$ sufficiently large. We presently aim at determining a strictly positive number ϵ such that the intersection $\mathcal{B}(X_m, L_m) \cap \Gamma_{\mathcal{W}_m}$ (where L_m denotes the elementary horizontal length, introduced in part i. of Property 2.6, on page 12) cannot be contained in a band of the type

$$D = \left\{x \in \mathbb{R}^2,\, |b\cdot(x-X)| < \epsilon\, L_m\right\} = \left\{x \in \mathbb{R}^2,\, |b\cdot(x-X_m)| < \epsilon\, L_m\right\},$$

where b denotes a unit vector of $\mathbb{R}^2$.

Intuitively, one understands that the geometric condition cannot hold, because of the specific properties of the maps T_i, $0 \leqslant i \leqslant N_b - 1$, which extend distances vertically (see [Dav19]).

Thus, the simplest way to show that the condition cannot be satisfied could be achieved by finding a nonnegative integer p and an adjacent vertex $Y \in \Gamma_{\mathcal{W}_{m+p}}$ of X such that Y does not belong to the band D.



The Euclidean distance $d_{Eucl}(X,Y)$ is given by

$$d_{Eucl}(X,Y) = \sqrt{h_{j,m+p}^2 + L_{m+p}^2} < \sqrt{h_{m+p}^2 + L_{m+p}^2} < \sqrt{2}\, h_{m+p}\,,$$

where h_{m+p} is the $(m+p)^{th}$ maximal height introduced in relation (4), on page 12.

Thanks to inequality 5, on page 19, there exists a nonnegative integer p such that

$$\sqrt{2}\, h_{m+p} < L_m \,,$$

since

$$\frac{h_{m+p}}{L_m} \leqslant \frac{C_{sup}\, L_{m+p}^{2-D_{\mathcal{W}}}}{L_m} \,,$$

where the constant C_{sup} is the constant involved in Property 2.12, on page 19, and where

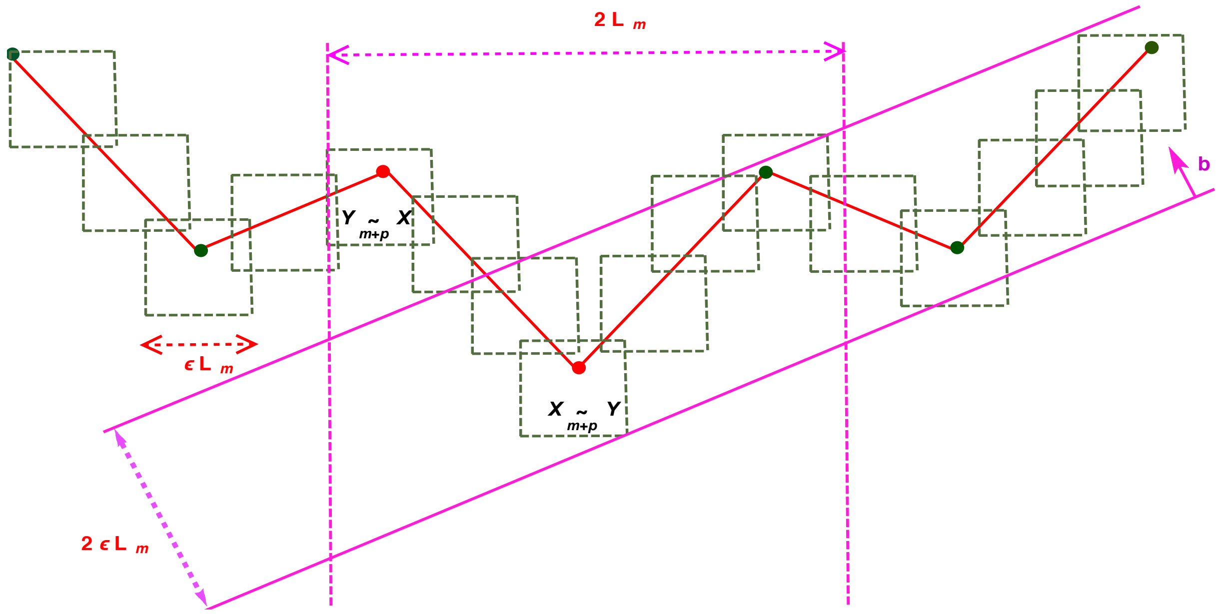
$$\frac{L_{m+p}^{2-D_{\mathcal{W}}}}{L_m} = \frac{L_m^{1-D_{\mathcal{W}}}}{N_b^{(3-D_{\mathcal{W}})\,p}} = \frac{1}{(N_b-1)^{1-D_{\mathcal{W}}}\, N_b^{(3-D_{\mathcal{W}})\,p+(1-D_{\mathcal{W}})\,m}} \,.$$

In the case when $N_b - 1 > 1$, the positive integer p has thus to be choosen such that:

$$(3-D_{\mathcal{W}})\,p > (D_{\mathcal{W}}-1)\,m \,.$$

As is done in the previous work of the first author in [Dav22], we recall that a convenient cover of $\Gamma_{\mathcal{W}_m}$ between X and an adjacent vertex $Y \in \Gamma_{\mathcal{W}_{m+p}}$ of X requires at least:

$$\frac{h_{m+p}\times L_{m+p}}{\epsilon^2\, L_m^2} \leqslant \frac{C_{inf}\, L_{m+p}^{3-D_{\mathcal{W}}}}{\epsilon^2\, L_m^2} \quad \text{squares of side length } \epsilon\, L_m \,,$$



where

$$\frac{L_{m+p}^{3-D_{\mathcal{W}}}}{L_m^2} = \frac{L_m^{3-D_{\mathcal{W}}}}{L_m^2\, N_b^{(3-D_{\mathcal{W}})\,p}} = \frac{L_m^{1-D_{\mathcal{W}}}}{N_b^{(3-D_{\mathcal{W}})\,p}} \,.$$

We thus deal with the restriction of the band D to a vertical band of width $2\,L_m$, which can be thus covered by at most

$$C \times \frac{2\,\epsilon\, L_m \times 2\, L_m}{(\epsilon\, L_m)^2} = \frac{4\,C}{\epsilon} \,, \quad \text{with } C > 0 \,,$$

squares of side length equal to $\epsilon\, L_m$.

Therefore, provided that

$$\frac{4\,C}{\epsilon} < \frac{C_1\, L_{m+p}^{3-D_{\mathcal{W}}}}{\epsilon^2\, L_m^2}\,,$$

i.e.,

$$\epsilon < \frac{C_{inf}}{4\,C}\, \frac{L_{m+p}^{3-D_{\mathcal{W}}}}{L_m^2}$$

or, equivalently

$$\epsilon < \frac{C_{inf}}{4\,C}\, \frac{L_m^{1-D_{\mathcal{W}}}}{N_b^{(3-D_{\mathcal{W}})\,p}}\,,$$

the vertex Y does not belong to the band D, which is the expected result.

□

Remark 4.3. We can check that our upper bound

$$\frac{C_{inf}}{4\,C}\, \frac{L_m^{1-D_{\mathcal{W}}}}{N_b^{(3-D_{\mathcal{W}})\,p}}$$

is explicitly a very small one, in fact as small as one chooses, whereas the one given in the book [JW84], on page 40, is of the following form:

$$\text{Constant}^{\frac{1}{d-(n-1)}} \quad , \quad 1 < d < n\,,$$

where the real positive constant is not explicitly known, in particular, it is not specified wether the constant is greater or less than 1.

Notation 15 (Set of Polynomials in Two Real Variables).

In the sequel, $\mathbb{R}[X,Y]$ denotes the set of all real polynomials in two real variables.
Given $k \in \mathbb{N}$, we will denote by $\mathcal{P}ol_k \subset \mathbb{R}[X,Y]$ the set of all real polynomials in two real variables of degree at most equal to k.

Definition 4.3 (Two-Dimensional π_r-Net ([Wal91], on page 119)).

Given a strictly positive real number r, we will call *two-dimensional π_r-net* a tessellation of $\mathbb{R}^2$ into half-open, non-overlapping squares of side lengths r, obtained by intersecting $\mathbb{R}^2$ with lines orthogonal to the coordinate axes.

Definition 4.4 (Two-Dimensional Polygonal $\pi_{\mathcal{W},m}$-Net, $m \in \mathbb{N}$).

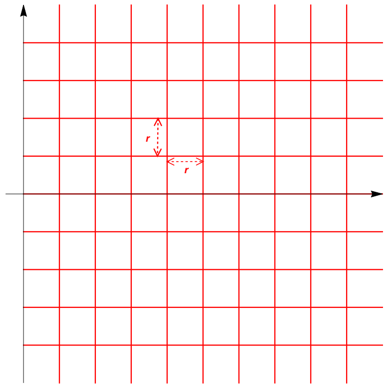


Figure 7: A two-dimensional π_r-net.

Given a strictly positive integer m, we call *two-dimensional polygonal $\pi_{\mathcal{W},m}$-net* a tessellation of $\mathbb{R}^2$ into half-open N_b-gons of side lengths at most equal to $\sqrt{2}\,h_m$ which contains the set of polygons

$$\left\{\bigcup_{j=0}^{N_b^m-1}\mathcal{P}_{m,j}\right\}\bigcup\left\{\bigcup_{k=1}^{N_b^m-2}\mathcal{Q}_{m,k}\right\}.$$

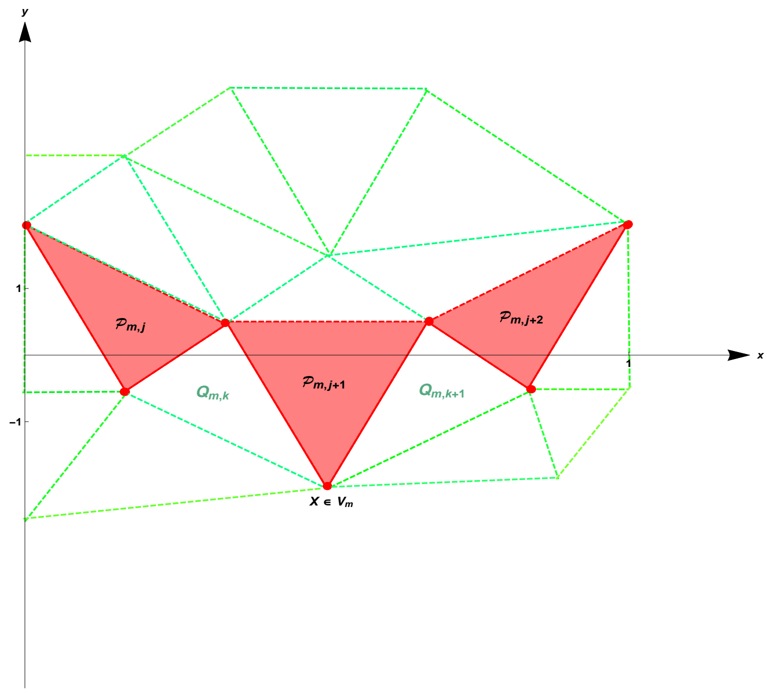


Figure 8: A two-dimensional polygonal $\pi_{\mathcal{W},m}$-net. Note that the polygons are not necessarily isometric.

Property 4.4. *Given a strictly positive integer m, the following properties hold:*

i. For any integer $j \in \{0,\ldots,N_b^m-1\}$, and any pair of vertices $(X,Y) \in \left(V_m \cap \mathcal{P}_{m,j}\right)^2$:

$$d_{Eucl}(X,Y) \lesssim N_b\,h_m \lesssim N_b^{-m\,(2-D_{\mathcal{W}})}.$$

ii. For any integer $j \in \{1,\ldots,N_b^m-2\}$, and any pair of vertices $(X,Y) \in \left(V_m \cap \mathcal{Q}_{m,j}\right)^2$:

$$d_{Eucl}(X,Y) \lesssim N_b\, h_m \lesssim N_b^{-m\,(2-D_{\mathcal{W}})}\,.$$

Proof. This simply follows from the fact that the polygons have N_b sides, and that two adjacent vertices are distant from at most an Euclidean distance equal to h_m.

□

Notation 16 (**Set of Piecewise Polynomial Functions on a Polygonal Net**).

Given a pair of nonnegative integers (k,m), and a polygonal $\pi_{\mathcal{W},m}$-net, we denote by $\mathcal{P}ol_k\left(\pi_{\mathcal{W},m}\right)$ the set of non-smooth splines of degree k on $\pi_{\mathcal{W},m}$, i.e., piecewise polynomial functions on $\pi_{\mathcal{W},m}$:

$$\mathcal{P}ol_k\left(\pi_{\mathcal{W},m}\right) = \Big\{ spline \text{ such that for any polygon } \mathcal{P} \in \pi_{\mathcal{W},m}, \\ \text{there exists } P \in \mathcal{P}ol_k : \quad spline_{|\mathcal{P}} = P_{|\mathcal{P}} \Big\}\,.$$

Definition 4.5 (**Atoms (Generalization of [Kab12])**).

Given a strictly positive real number $s<1$, a real number $p>1$, two nonnegative integers m and $j \in \left\{0,\ldots,N_b^m-1\right\}$, a function $f_{m,j}$ defined on the prefractal graph $\Gamma_{\mathcal{W}_m}$ will be called a $(\mathcal{P}_{m,j},s,p)$–*atom* if the following three conditions are satisfied:

i. Supp $f_{m,j} \subset \mathcal{P}_{m,j}$;

ii. $\forall\, X \in V_m \cap \mathcal{P}_{m,j}$:

$$\left|f_{m,j}(X)\right| \lesssim \mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)^{\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}}$$

or, equivalently,

$$\left|f_{m,j}(X)\right| \lesssim \left(N_b^{(D_{\mathcal{W}}-3)\,m}\right)^{\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}} \;;$$

iii. $\forall\,(X,Y) \in \left(V_m \cap \mathcal{P}_{m,j}\right)^2$:

$$\left|f_{m,j}(X)-f_{m,j}(Y)\right| \lesssim d_{Eucl}(X,Y)\,\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}\,,$$

or, equivalently,

$$\left|f_{m,j}(X)-f_{m,j}(Y)\right| \lesssim d_{Eucl}(X,Y)\,\left(N_b^{(D_{\mathcal{W}}-3)\,m}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}\,.$$

Similarly, given a strictly positive real number $s<1$, a real number $p>1$, two nonnegative integers m and $j \in \left\{1,\ldots,N_b^m-2\right\}$, a function $g_{m,j}$ on the prefractal graph $\Gamma_{\mathcal{W}_m}$ will be called a $(\mathcal{Q}_{m,j},s,p)$–*atom* if the following three conditions are satisfied:

i. $\operatorname{Supp} g_{m,j} \subset \mathcal{Q}_{m,j}$;

ii. $\forall\, X \in V_m \cap \mathcal{Q}_{m,j}$:

$$\left| g_{m,j}(X) \right| \lesssim \mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j} \right)^{\frac{s}{D_{\mathcal{W}}} - \frac{1}{p}} ,$$

or, equivalently,

$$\left| g_{m,j}(X) \right| \lesssim \left(N_b^{(D_{\mathcal{W}}-3)\, m} \right)^{\frac{s}{D_{\mathcal{W}}} - \frac{1}{p}} ;$$

iii. $\forall\, (X,Y) \in \left(V_m \cap \mathcal{Q}_{m,j} \right)^2$:

$$\left| g_{m,j}(X) - g_{m,j}(Y) \right| \lesssim d_{Eucl}(X,Y)\, \mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j} \right)^{\frac{s-1}{D_{\mathcal{W}}} - \frac{1}{p}} ,$$

or, equivalently,

$$\left| g_{m,j}(X) - g_{m,j}(Y) \right| \lesssim d_{Eucl}(X,Y) \left(N_b^{(D_{\mathcal{W}}-3)\, m} \right)^{\frac{s-1}{D_{\mathcal{W}}} - \frac{1}{p}} .$$

Remark 4.4 (**Atoms as Hölder Functions**).

Insofar as,

$$\forall\, (X,Y) \in \left(V_m \cap \mathcal{P}_{m,j} \right)^2 : \quad d_{Eucl}(X,Y) \lesssim N_b^{-m\,(2-D_{\mathcal{W}})} ,$$

the above condition *ii.* for a $(\mathcal{P}_{m,j}, s, p)$-atom can be also written as

$$\forall\, (X,Y) \in \left(V_m \cap \mathcal{P}_{m,j} \right)^2 : \qquad \left| f_{m,j}(X) - f_{m,j}(Y) \right| \lesssim N_b^{-m\,(2-D_{\mathcal{W}}) + (D_{\mathcal{W}}-3)\, m \left(\frac{s-1}{D_{\mathcal{W}}} - \frac{1}{p} \right)} ,$$

which corresponds to a Hölder exponent of

$$1 - \frac{(3 - D_{\mathcal{W}})}{2 - D_{\mathcal{W}}} \left(\frac{s-1}{D_{\mathcal{W}}} - \frac{1}{p} \right) .$$

Note that, due to their definition, $(\mathcal{P}_{m,j}, s, p)$-atoms are necessarily continuous.

An entirely similar property holds if the polygons $\mathcal{P}_{m,j}$ are replaced by their counterpart $\mathcal{Q}_{m,j}$, and the $(\mathcal{P}_{m,j}, s, p)$-atoms are replaced by the $(\mathcal{Q}_{m,j}, s, p)$-atoms.

Remark 4.5 (**Atoms Associated with the Weierstrass Function**).

In the case of the Weierstrass function $\mathcal{W}$, for any $m \in \mathbb{N}$, and any pair (X,Y) of adjacent vertices in V_m, we have that

$$|\mathcal{W}(X) - \mathcal{W}(Y)| \leqslant d_{Eucl}(X,Y)^{2-D_{\mathcal{W}}} .$$

By the triangle inequality, we immediately deduce that,

$\forall\,(X,Y)\,\in\,\left(V_m\cap\mathcal{P}_{m,j}\right)^2\,\left(\text{resp., }\left(V_m\cap\mathcal{Q}_{m,j}\right)^2\right):$

$$|\mathcal{W}(X)-\mathcal{W}(Y)|\leqslant d_{Eucl}(X,Y)^{2-D_{\mathcal{W}}}\,.$$

At the same time, thanks to the estimates given in Property 2.12, on page 19, and in part *ii.* of the proof of Theorem 3.7, on page 30 (see relation (11) and relation (12)), for any polygon $\mathcal{P}_{m,j}$ (resp., $\mathcal{Q}_{m,j}$), we have that

$$N_b^{(D_{\mathcal{W}}-3)\,m}\lesssim\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)\lesssim N_b^{(D_{\mathcal{W}}-3)\,m}\quad\left(\text{resp., }N_b^{(D_{\mathcal{W}}-3)\,m}\lesssim\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)\lesssim N_b^{(D_{\mathcal{W}}-3)\,m}\right),$$

or, equivalently,

$$d_{Eucl}(X,Y)^{\frac{3-D_{\mathcal{W}}}{2-D_{\mathcal{W}}}}\lesssim\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)\lesssim d_{Eucl}(X,Y)^{\frac{3-D_{\mathcal{W}}}{2-D_{\mathcal{W}}}}$$
$$\left(\text{resp., }d_{Eucl}(X,Y)^{\frac{3-D_{\mathcal{W}}}{2-D_{\mathcal{W}}}}\lesssim\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)\lesssim d_{Eucl}(X,Y)^{\frac{3-D_{\mathcal{W}}}{2-D_{\mathcal{W}}}}\right).$$

We then deduce that

$\forall\,(X,Y)\,\in\,\left(V_m\cap\mathcal{P}_{m,j}\right)^2\,\left(\text{resp., }\left(V_m\cap\mathcal{Q}_{m,j}\right)^2\right):$

$$|\mathcal{W}(X)-\mathcal{W}(Y)|\lesssim d_{Eucl}(X,Y)\,\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)^{(1-D_{\mathcal{W}})\,\frac{2-D_{\mathcal{W}}}{3-D_{\mathcal{W}}}}$$
$$\left(\text{resp., }|\mathcal{W}(X)-\mathcal{W}(Y)|\lesssim d_{Eucl}(X,Y)\,\mu_{\mathcal{L}}\left(\mathcal{Q}_{m,j}\right)^{(1-D_{\mathcal{W}})\,\frac{2-D_{\mathcal{W}}}{3-D_{\mathcal{W}}}}\right).$$

It then follows that

$$\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}=(1-D_{\mathcal{W}})\,\frac{2-D_{\mathcal{W}}}{3-D_{\mathcal{W}}},\quad\text{i.e.,}\quad s=\frac{D_{\mathcal{W}}}{p}+D_{\mathcal{W}}\,\frac{(1-D_{\mathcal{W}})\,(2-D_{\mathcal{W}})}{3-D_{\mathcal{W}}},$$

and that the restriction of the Weierstrass function to each polygon $\mathcal{P}_{m,j}$, (resp., $\mathcal{Q}_{m,j}$) is a $(\mathcal{P}_{m,j},s,p)$-*atom* (resp., a $(\mathcal{Q}_{m,j},s,p)$-*atom*).

Definition 4.6 (**Atomic Decomposition of a Function Defined on the Weierstrass Curve**).

Given a continuous function f on the Weierstrass Curve, we will say that f admits *an atomic decomposition* in the following form:

$$\begin{aligned}
f &= \lim_{m\to\infty}\left\{\sum_{j=0}^{N_b^m-1}\ \sum_{X\text{ vertex of }\mathcal{P}_{m,j}} p\,(X,\mathcal{P}_m)\ \lambda_{f,m,j,X}\,f_{m,j,X}\right.\\
&\qquad\left.+\sum_{j=1}^{N_b^m-2}\ \sum_{X\text{ vertex of }\mathcal{Q}_{m,j}} p\,(X,\mathcal{Q}_m)\ \lambda_{g,m,j,X}\,g_{m,j}\right\}\\
&= \lim_{m\to\infty}\left\{\sum_{j=0}^{N_b^m-1}\ \sum_{X\text{ vertex of }\mathcal{P}_{m,j},\,X\notin\mathcal{Q}_m}\lambda_{f,m,j,X}\,f_{m,j,X}\right.\\
&\qquad+\sum_{j=1}^{N_b^m-2}\ \sum_{X\text{ vertex of }\mathcal{P}_{m,j},\,X\in\mathcal{P}_m\cap\mathcal{Q}_m}\left\{\lambda_{f,m,j,X}\,f_{m,j,X}+\lambda_{g,m,j,X}\,g_{m,j}\right\}\\
&\qquad\left.+\sum_{j=1}^{N_b^m-2}\ \sum_{X\text{ vertex of }\mathcal{Q}_{m,j},\,X\notin\mathcal{P}_m}\lambda_{g,m,j,X}\,g_{m,j}\right\},
\end{aligned}$$

where, for any $m \in \mathbb{N}$, the functions $f_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$ and $g_{m,j}$, $1 \leqslant j \leqslant N_b^m - 2$ are respectively $(\mathcal{P}_{m,j}, s, p)$ and $(\mathcal{Q}_{m,j}, s, p)$-atoms, $s < 1$, $p > 1$, while the coefficients $\lambda_{f,m,j}$, $0 \leqslant j \leqslant N_b^m - 1$ and $\lambda_{g,m,j}$, $1 \leqslant j \leqslant N_b^m - 2$, denote real numbers.

For the sake of simplicity, we will write the above decomposition in the following briefer form, where the two summation signs $\displaystyle\sum_{j=0}^{N_b^m-1}$ and $\displaystyle\sum_{X \text{ vertex of } \mathcal{P}_{m,j}}$ have been condensed in the equivalent summation sign $\displaystyle\sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m}$ as follows:

$$f = \lim_{m\to\infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X} = \lim_{m\to\infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{f,m}\, \tilde{f}_m \,,$$

where

$$\tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X} = \begin{cases} \lambda_{f,m,j,X}\, f_{m,j,X} \text{, if } X \notin \mathcal{Q}_m \,,\; X \text{ is a vertex of } \mathcal{P}_{m,j} \,,\; 0 \leqslant j \leqslant N_b^m - 1 \,, \\ \lambda_{g,m,j,X}\, g_{m,j,X} \text{, if } X \notin \mathcal{P}_m \,,\; X \text{ is a vertex of } \mathcal{Q}_{m,j} \,,\; 1 \leqslant j \leqslant N_b^m - 2 \,, \\ \lambda_{f,m,j,X}\, f_{m,j,X} + \lambda_{g,m,j,X}\, g_{m,j,X}\, f_{m,j,X} ,\; \text{if } X \in \mathcal{P}_m \cap \mathcal{Q}_m \,, \\ \qquad X \text{ is a vertex of } \mathcal{P}_{m,j} \text{ and } \mathcal{Q}_{m,j},\; 1 \leqslant j \leqslant N_b^m - 2 \,. \end{cases}$$

For any $m \in \mathbb{N}$, we say that $\tilde{\lambda}_{f,m}$ is the m^{th}-*atomic coefficient.*

From now on, the functions $\tilde{f}_{m,X}$ and $\tilde{f}_m$ will be called (m, s, p')-*atoms.* In the sequel, we will use the most suitable notation among these two possibilities.

Property 4.5 (Atomic Decomposition of Spline Functions in $\mathcal{P}ol_k\left(\pi_{N_b^{(D_\mathcal{W}-3)\,n}}\right)$, $n \in \mathbb{N}$).

Given a pair (n, k) of nonnegative integers, a spline function (denoted by spline*) in $\mathcal{P}ol_k\left(\pi_{N_b^{(D_\mathcal{W}-3)\,n}}\right)$ admits* an atomic decomposition *of the form*

$$spline = \lim_{m\to\infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{s,m,X}\, \widetilde{spline}_{m,X} \,.$$

Proof. This directly comes from the definition of functions of $\mathcal{P}ol_k\left(\pi_{N_b^n}\right)$ as piecewise polynomial functions.

□

Property 4.6. *Given the polyhedral measure μ on the Weierstrass Curve $\Gamma_\mathcal{W}$, and a continuous function f on $\Gamma_\mathcal{W}$, of atomic decomposition*

$$f = \lim_{m\to\infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X} \,,$$

we have that

$$\int_{\mathcal{D}(\Gamma_\mathcal{W})} f\, d\mu = \lim_{m\to\infty} \varepsilon_m^{m\,(D_\mathcal{W}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X}\, \mu\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) .$$

Note that since the Weierstrass Curve $\Gamma_{\mathcal{W}}$ satisfies Markov's Inequality (see Property 4.2 above, on page 39), this ensures a good quality *of our atomic decomposition, in the sense that the corresponding and forthcoming characterization of Besov spaces on $\Gamma_{\mathcal{W}}$ will be obtained in a constructive way, along the lines of the work by Alf Jonsson in [Jon09].*

Proof. This simply follows from the definition of the polyhedral measure on the Weierstrass Curve provided (and justified) by Property 3.6, on page 29.

□

Remark 4.6. Such a decomposition makes sense since the set of vertices $(V_m)_{m\in\mathbb{N}}$ is dense in $\Gamma_{\mathcal{W}}$. Thus, because we deal with continuous functions, given any point X of the Weierstrass Curve, there exists a sequence $(X_m)_{m\in\mathbb{N}}$ such that

$$f(X) = \lim_{m\to\infty} f(X_m)\,,$$

where, for any $m \in \mathbb{N}$, X_m belongs to the prefractal graph $\Gamma_{\mathcal{W}_m}$.

We can naturally write $f(X_m)$ as

$$f(X_m) = \sum_{Y_m \in V_m} f(Y_m)\,\delta_{X_m Y_m}(X_m)\,, \tag{13}$$

where δ is the classical Kronecker symbol; i.e.,

$$\forall\, Y_m \in V_m \;:\quad \delta_{X_m Y_m}(Y_m) = \begin{cases} 1\,, & \text{if} \quad Y_m = X_m\,, \\ 0\,, & \text{else.} \end{cases} \tag{14}$$

This, of course, yields

$$f(X) = \lim_{m\to\infty} \sum_{Y_m \in V_m} f(Y_m)\,\delta_{X_m Y_m}(Y_m)\,.$$

Now, we can go a little further and, as in [Str06], introduce spline functions $\psi^m_{X_m}$ such that

$$\forall\, Y \in \Gamma_{\mathcal{W}} \;:\quad \psi^m_{X_m}(Y) = \begin{cases} \delta_{X_m Y_m}\,, & \forall\, Y \in V_m \\ 0\,, & \forall\, Y \notin V_m\,, \end{cases}$$

and write

$$f(X) = \lim_{m\to\infty} \sum_{Y_m \in V_m} f(Y_m)\,\psi^m_{X_m}(Y_m)\,,$$

which is nothing but an application in this context of the Weierstrass approximation theorem. In particular, spline functions are a natural choice for atoms.

Convention. In the sequel, all functions f considered on the Weierstrass Curve are implicitly supposed to be Lebesgue measurable.

Definition 4.7 (L^p**-Norm of a Function on the Weierstrass Curve).**

Given a function f on the Weierstrass Curve, we define its L^p-norm as follows:

$$\begin{aligned}\|f\|_{L^p(\Gamma_{\mathcal{W}})} &= \left(\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} |f|^p \, d\mu\right)^{\frac{1}{p}} \\ &= \left(\lim_{m\to\infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) \left|f^p(X)\right|\right)^{\frac{1}{p}}.\end{aligned}$$

Property 4.7 (L^p**-Norm of a Function on the Weierstrass Curve Defined by Means of an Atomic Decomposition).**

Given a positive integer p*, and a continuous function* f *on* $\Gamma_{\mathcal{W}}$*, whose absolute value* $|f|$ *is defined by means of an atomic decomposition as*

$$|f| = \lim_{m\to\infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{|f|,m,X} \, \widetilde{|f|}_{m,X} \,,$$

the L^p*-norm of* f *with respect to the measure* μ *is given by*

$$\begin{aligned}\|f\|_{L^p(\Gamma_{\mathcal{W}})} &= \left(\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} |f|^p \, d\mu\right)^{\frac{1}{p}} \\ &= \left(\lim_{m\to\infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-3)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) \tilde{\lambda}^p_{|f|,m,j,X} \, \widetilde{|f|}^p_{m,j,X}\right)^{\frac{1}{p}}.\end{aligned}$$

In particular, we have that

$$\begin{aligned}\|f\|_{L^1(\Gamma_{\mathcal{W}})} &= \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} |f| \, d\mu \\ &= \lim_{m\to\infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) \tilde{\lambda}_{|f|,m,j,X} \, \widetilde{|f|}_{m,j,X} \,.\end{aligned}$$

Remark 4.7. In the above definition, two limits are a priori considered at the same time; the limit associated to the integral, with respect to the polyhedral measure μ, namely,

$$\lim_{m\to\infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) |f(X)|^p \,,$$

and the limit associated to the atomic decomposition

$$\lim_{m\to\infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{|f|,m,j,X} \, \widetilde{|f|}^p_{m,j,X} \,.$$

In fact, these two limits coincide.

Definition 4.8 (**Besov Space on the Weierstrass Curve (Extension to the Weierstrass Curve of the result given by Theorem 6, on page 135, in [JW84])**).

Given $k \in \mathbb{N}$, a real number α such that

$$k < \alpha \leqslant k + 1\,,$$

and two real numbers p and q greater or equal than 1, the *Besov space* $B^{p,q}_{\alpha}\left(\Gamma_{\mathcal{W}}\right)$ is defined as the set of functions $f \in L^p(\mu)$ such that there exists a sequence $(c_m)_{m\in\mathbb{N}} \in \ell^q$ of nonnegative real numbers such that for every $\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}$-net, one can find a spline function $spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right) \in \mathcal{P}ol_{[\alpha]}\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)$ satisfying

$$\left\| f - spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)\right\|_{L^p(\mu)} \leqslant N_b^{(D_{\mathcal{W}}-3)\,m\,\alpha}\, c_m\,,\ \text{for all } m \in \mathbb{N}, \qquad \left(\mathcal{C}ond_{Besov\,spline}\right)$$

where, if we write the respective atomic decompositions of f and $spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)$ as

$$f = \lim_{m\to\infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{f,m,X}\, \tilde{f}_{m,X}\,,$$

and

$$spline\left(\pi_{\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}}\right) = \lim_{m\to\infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right),m,X}\, \widetilde{spline}\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)_{m,X}\,,$$

we then have that

$$\begin{aligned}&\left\| f - spline\left(\pi_{\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}}\right)\right\|^p_{L^p(\mu)} = \\ &= \lim_{m\to\infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\left(X, \mathcal{P}_m, \mathcal{Q}_m\right) \tilde{\lambda}^p_{\left|f-spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)\right|,m,X} \left|f - \widetilde{spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)}\right|^p_{m,X};\end{aligned}$$

here, for the sake of simplicity, we have introduced the atomic decomposition of $\left|f - spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)\right|$ as

$$\begin{aligned}&\left|f - spline\left(\pi_{\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}}\right)\right| = \\ &= \lim_{m\to\infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{\left|f-spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)\right|,m,X} \left|f - \widetilde{spline\left(\pi_{N_b^{(D_{\mathcal{W}}-3)\,m}}\right)}\right|_{m,X}.\end{aligned}$$

Remark 4.8. The atomic decomposition used in [Kab12] is obtained by introducing small neighborhoods of the curve under study (union of balls). Our polygonal domain introduced in Definition 4.4, on page 41 appears to be a more natural choice. Indeed, unlike the aforementioned balls, the polygons involved do not overlap with each other, which works better for the required nets.

Definition 4.9 (**Besov Norm**).

Given $k \in \mathbb{N}$, a real number α such that

$$k < \alpha \leqslant k + 1\,,$$

and two real numbers p and q greater or equal than 1, one can define, as in [Wal91], the $B_\alpha^{p,q}(\Gamma_{\mathcal{W}})$-*norm* of a function f defined on the Weierstrass Curve as

$$\|f\|_{B_\alpha^{p,q}(\Gamma_{\mathcal{W}})} = \|f\|_{L^p(\Gamma_{\mathcal{W}})} + \inf \left\{ \sum_{n\in\mathbb{N}} c_n^q \right\}^{\frac{1}{q}} ,$$

where the infimum is taken over all the sequences $(c_m)_{m\in\mathbb{N}} \in \ell^q$ of nonnegative real numbers involved in condition $(\mathcal{C}ond_{Besov\, spline})$ in Definition 4.8, on page 49 just above.

Remark 4.9 (**Alternative Definition of a Besov Space on the Weierstrass Curve**).

One of the interesting properties of Besov spaces is that they can be defined in two different ways: first, as given previously in Definition 4.8, on page 49, by means of a polynomial approximation, which provides information on the degree of regularity of the functions involved; also, as the set of functions of a finite specific norm as in Definition 4.9, on page 49. This latter definition is all the more interesting, because it enables one to establish connections with discrete and fractal Laplacians, by means of the fractional difference quotients involved. Our present case is, of course, distinct from the classical one on $\mathbb{R}^N$, with $N \in \mathbb{N}$. Yet, the case of $\mathbb{R}^N$, as it can be found in [JW84], Chapter 1, Section 1.5, on page 7, enables us to understand the underlying connection; namely, for $\alpha > 0$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant q < \infty$,

$$B_\alpha^{p,q}\left(\mathbb{R}^N\right) =$$

$$= \left\{ f \in L^p(\mathbb{R}^N) \,,\, \sum_{|j|\leqslant[\alpha]} \left\|D^j f\right\|_{L^p(\Gamma_{\mathcal{W}})} + \sum_{|j|=[\alpha]} \left\{ \int_{\mathbb{R}^N} \frac{\left\|\Delta_h^{first\, diff.}\, D^j f\right\|_{L^p(\Gamma_{\mathcal{W}})}^q}{|h|^{N+(\alpha-[\alpha])\,q}}\, dh \right\}^{\frac{1}{q}} < \infty \right\} ,$$

where $[\alpha]$ denotes the integer part of the positive number α, while $\Delta_h^{first\, diff.}$ is the usual first difference, defined here by

$$\forall\, t \in \Gamma_{\mathcal{W}},\ \forall\, h \in \mathbb{R}^n \ , \quad \Delta_h^{first\, diff.} f(t) = f(t+h) - f(t)\,.$$

The difficulty, in our case, is to obtain an equivalent formulation. For our forthcoming purpose of determining the order of the fractal Laplacian on the (fractal) Weierstrass Curve $\Gamma_{\mathcal{W}}$, in Corollary 4.12 below, on page 57, we will restrict ourselves to the case where $\alpha < 2$. We thus have to deal with expressions of the following form,

$$\left\{ \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} \frac{|f(T) + f(Y) - 2\,f(X)|^q}{|h|^{N+(\alpha-[\alpha])\,q}}\, dh \right\}^{\frac{1}{q}} .$$

Thanks to the Minkowski inequality, we of course have that, for all points (X,Y,T) in $(\mathcal{D}(\Gamma_{\mathcal{W}}))^3$,

$$\left\{ \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} \frac{|f(T) + f(Y) - 2\,f(X)|^q}{|h|^{N+(\alpha-[\alpha])\,q}}\, dh \right\}^{\frac{1}{q}} \leqslant$$

$$\leqslant \left\{ \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} \frac{|f(T) - f(X)|^q}{|h|^{N+(\alpha-[\alpha])\,q}}\, dh \right\}^{\frac{1}{q}} + \left\{ \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} \frac{|f(Y) - f(X)|^q}{|h|^{N+(\alpha-[\alpha])\,q}}\, dh \right\}^{\frac{1}{q}} .$$

In our present context, the term

$$\left\{ \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} \frac{|f(T)+f(Y)-2\,f(X)|^q}{|h|^{N+(\alpha-[\alpha])\,q}}\,dh \right\}^{\frac{1}{q}}, \tag{15}$$

where N has to be replaced by $D_{\mathcal{W}}$, is obtained thanks to the atomic decompositions by means of the sum of the following positive series,

$$\begin{aligned} \varepsilon_m^{3\,m\,(D_{\mathcal{W}}-2)} \sum_{(X,Y,T)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^3,\,Y\underset{m}{\sim}X,\,T\underset{m}{\sim}X} &\Bigg(\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m)\,\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m) \\ &\mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\;\tilde{\lambda}_{\star,m}\,\frac{\left|\tilde{f}_m(T)+\tilde{f}_m f(Y)-2\,\tilde{f}_m(Y)\right|^q}{d_{Eucl}^{D_{\mathcal{W}}+(\alpha-k)\,q}(X,Y)}\Bigg), \end{aligned} \tag{16}$$

since, for vertices $Y \underset{m}{\sim} X$ or $T \underset{m}{\sim} X$, we have, thanks to Property 2.12, on page 19, that

$$d_{Eucl}(X,Y) \lesssim d(X,T))$$

and where, for the sake of simplicity, we have denoted by $\tilde{\lambda}_{\star,m}$ the atomic coefficients involved.

As for the terms

$$\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} \frac{|f(Y)-f(X)|^q}{|h|^{N+(\alpha-[\alpha])\,q}}\,dh\,,$$

or, equivalently,

$$\int_{\mathcal{D}(\Gamma_{\mathcal{W}})} \frac{|f(T)-f(X)|^q}{|h|^{N+(\alpha-[\alpha])\,q}}\,dh\,,$$

they are obtained by means of the sum of the following positive series,

$$\begin{aligned} \varepsilon_m^{2\,m\,(D_{\mathcal{W}}-2)} \sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2,\,Y\underset{m}{\sim}T} &\Bigg(\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m) \\ &\mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\;\tilde{\lambda}_{\star,m}\,\frac{\left|\tilde{f}_m(T)-\tilde{f}_m(Y)\right|^q}{d_{Eucl}^{D_{\mathcal{W}}+(\alpha-k)\,q}(T,Y)}\Bigg). \end{aligned} \tag{17}$$

This ensures that, if the (positive) series in expression (17) just above converges, the (positive) series in expression (16) also converges, which ensures the existence of the integral given in expression (15) .

Going further, we may note that by applying the same reasoning, we could extend the validity to higher values of α.

Remark 4.10.

This enables us to obtain the first side of the comparison relation required in order to establish the equivalence of norms. It is the side that plays the most significant part in the definition of Besov spaces on the Weierstrass Curve. Thus, characterizing Besov spaces on $\Gamma_{\mathcal{W}}$ by means of the norm introduced in Definition 4.9, on page 49, is directly associated to the definition of a sequence of (suitably renormalized) discrete graph Laplacians $(\Delta_m)_{m\in\mathbb{N}}$ on the sequence of prefractal approximations $\left(\Gamma_{\mathcal{W}_m}\right)_{m\in\mathbb{N}}$. In a sense, it is also connected to the existence of the limit

$$\lim_{m\to\infty} \Delta_m$$

by means of an equivalent pointwise formula expressed in terms of integrals, somehow the counterpart, in a way, of the one which is well known in the case of the fractal Laplacian on the Sierpiński Gasket [Kig01], [Str06].

Property 4.8 (Characterization of Besov Spaces: Sufficient Condition).

Given a real number α *such that*

$$1 < \alpha \leqslant 2 \,,$$

two real numbers p *and* q *greater or equal than 1, and a continuous function* f *given by means of an atomic decomposition of the form*

$$f = \lim_{m\to\infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{f,m,X} \, \tilde{f}_{m,X}$$

belongs to the Besov space $B^{p,q}_\alpha(\Gamma_{\mathcal{W}})$ *if the following two conditions are satisfied,*

$$(3 - D_{\mathcal{W}}) \left\{ q \left(\frac{1}{p} - \frac{s-1}{D_{\mathcal{W}}} \right) \right\} + (2 - D_{\mathcal{W}}) \, (D_{\mathcal{W}} + (\alpha - 1)\, q) < 2 \,, \quad (\mathcal{C}ond_{Besov})$$

and

$$\frac{D_{\mathcal{W}}}{3 - D_{\mathcal{W}}} + \frac{D_{\mathcal{W}}}{p} \leqslant s \,, \quad (\mathcal{C}ond_{L^p})$$

where the real number $s \in \,]0,1[$ *has been introduced in Definition 4.5, on page 43.*

Remark 4.11. As was explained in the introduction, since the Weierstrass Curve preserves Markov's Inequality, this ensures the *quality* of the approximation, when it comes to characterize Besov spaces by means of atomic decompositions (see [Jon09]), since Besov spaces are then obtained in a constructive manner.

Proof. (Of Property 4.8)

Since $1 < \alpha \leqslant 2$, we have that

$$\|f\|^q_{B^{p,q}_\alpha(\Gamma_{\mathcal{W}})} \quad = \quad \|f\|^p_{L^p(\Gamma_{\mathcal{W}})} + \left(\iiint_{(X,Y,T) \in (\mathcal{D}(\Gamma_{\mathcal{W}}))^3} \frac{|f(T) + f(Y) - 2\, f(X)|^q}{d_{Eucl}^{D_{\mathcal{W}} + \alpha\, q}(T,Y)} \, d\mu^3 \right)^{\frac{1}{q}} .$$

By applying the results at the end of Remark 4.9, on page 50, we can restrict ourselves to expressions of the following form:

$$\left(\iint_{(T,Y) \in (\mathcal{D}(\Gamma_{\mathcal{W}}))^2} \frac{|f(T) - f(Y)|^q}{d_{Eucl}^{D_{\mathcal{W}} + \alpha\, q}(T,Y)} \, d\mu^2 \right)^{\frac{1}{q}} ,$$

where, via the atomic decomposition of f,

$$\iint_{(T,Y)\in\Gamma_{\mathcal{W}}^2} \frac{|f(T)-f(Y)|^q}{d_{Eucl}^{D_{\mathcal{W}}+\alpha\,q}(T,Y)}\,d\mu^2 =$$

$$= \lim_{m\to\infty} \varepsilon_m^{2\,m\,(D_{\mathcal{W}}-2)} \sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2} \mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\,\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m)\,\frac{|f(T)-f(Y)|^q}{d_{Eucl}^{D_{\mathcal{W}}+\alpha\,q}(T,Y)}$$

$$= \lim_{m\to\infty} \varepsilon_m^{2\,m\,(D_{\mathcal{W}}-2)} \sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2} \mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\,\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m)\,\tilde{\lambda}_{f,m}\,\frac{\left|\tilde{f}_m(T)-\tilde{f}_m(Y)\right|^q}{d_{Eucl}^{D_{\mathcal{W}}+\alpha\,q}(T,Y)}$$

and where, for the sake of simplicity, we have denoted by $\tilde{\lambda}_{\star,m}$ the atomic coefficients involved.

Note that since the function f is continuous, the atomic coefficients $\lambda_{\star,m}$ are necessarily bounded (since $\Gamma_{\mathcal{W}}$ is compact, and hence, f is bounded).

Since we deal with the atomic decomposition of f, due to part *iii.* of Definition 4.5, on page 43, we have that

$$\mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\,\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m)\,\tilde{\lambda}_{f,m}\,\frac{\left|\tilde{f}_n(T)-\tilde{f}_m(Y)\right|^q}{d_{Eucl}^{D_{\mathcal{W}}+\alpha\,q}(T,Y)} \lesssim$$

$$\lesssim \mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\,\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m)\,\tilde{\lambda}_{f,m}\,\frac{\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{d}-\frac{1}{p}}}{d_{Eucl}^{D_{\mathcal{W}}+\alpha\,q-q}(T,Y)}\,.$$

We also have that

$$\frac{1}{d_{Eucl}(T,Y)} < \frac{1}{|h_{jm}|} \lesssim L_m^{D_{\mathcal{W}}-2} \lesssim N_b^{(2-D_{\mathcal{W}})\,m}\,,$$

and

$$d_{Eucl}(T,Y) \lesssim h_m \lesssim L_m^{2-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-2)\,m}\,.$$

Moreover,

$$\mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m) \lesssim h_m\,L_m \lesssim L_m^{3-D_{\mathcal{W}}} \lesssim N_b^{(D_{\mathcal{W}}-3)\,m}$$

and, in the same way,

$$\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m) \lesssim N_b^{(D_{\mathcal{W}}-3)\,m}\,.$$

We therefore deduce that

$$\mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\,\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m)\,\tilde{\lambda}_{f,m}\,\frac{\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{d}-\frac{1}{p}}}{d_{Eucl}^{D_{\mathcal{W}}+\alpha\,q-q}(T,Y)} \lesssim$$

$$\lesssim N_b^{2\,m\,(D_{\mathcal{W}}-3)}\,\tilde{\lambda}_{f,m}\,\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}\,N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)}\,,$$

and

$$\varepsilon_m^{2\,m\,(D_{\mathcal{W}}-2)}\,\mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\,\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m)\,\tilde{\lambda}_{f,m}\,\frac{\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{d}-\frac{1}{p}}}{d_{Eucl}^{D_{\mathcal{W}}+\alpha\,q-q}(T,Y)} \lesssim$$

$$\lesssim \varepsilon_m^{2\,m\,(D_{\mathcal{W}}-2)}\,N_b^{2\,m\,(D_{\mathcal{W}}-3)}\,\tilde{\lambda}_{f,m}\,\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}\,N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)}\,,$$

along with

$$\int_{T\in\Gamma_{\mathcal{W}}}\int_{Y\in\Gamma_{\mathcal{W}}} \frac{\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{d}-\frac{1}{p}}}{d_{Eucl}^{D_{\mathcal{W}}+\alpha\,q-q}(T,Y)}\,d\mu^2 =$$

$$= \lim_{m\to\infty} \varepsilon_m^{2\,m\,(D_{\mathcal{W}}-2)} \sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2} \mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\,\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m)\times \frac{\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}}{d_{Eucl}^{D_{\mathcal{W}}+\alpha\,q-a}(T,Y)}$$

$$\lesssim \lim_{m\to\infty} \varepsilon_m^{2\,m\,(D_{\mathcal{W}}-2)} \sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2} \mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\,\mu^{\mathcal{L}}(Y,\mathcal{P}_m,\mathcal{Q}_m)\times \times\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}} N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)}$$

$$\lesssim \lim_{m\to\infty} \varepsilon_m^{2\,m\,(D_{\mathcal{W}}-2)} \sum_{(T,Y)\in(\mathcal{P}_m\bigcup\mathcal{Q}_m)^2} N_b^{2\,m\,(D_{\mathcal{W}}-3)}\times \times\left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}\, N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)}\,.$$

The function f will thus belong to the Besov space $B_\alpha^{p,q}(\Gamma_{\mathcal{W}})$ provided that

$$\sum_{m\in\mathbb{N}} \varepsilon_m^{2\,m\,(D_{\mathcal{W}}-2)}\, N_b^{2\,m\,(D_{\mathcal{W}}-3)} \left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}\, N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)} < \infty\,,$$

or, equivalently,

$$\sum_{m\in\mathbb{N}} N_b^{-2\,m\,(D_{\mathcal{W}}-2)}\, N_b^{2\,m\,(D_{\mathcal{W}}-3)} \left(N_b^{-m\,q\,(3-D_{\mathcal{W}})}\right)^{\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}}\, N_b^{(2-D_{\mathcal{W}})\,m\,(D_{\mathcal{W}}+\alpha\,q-q)} < \infty\,,$$

converges, which requires that

$$-2\,(D_{\mathcal{W}}-2)+2\,(D_{\mathcal{W}}-3)-q\,(3-D_{\mathcal{W}})\left(\frac{s-1}{D_{\mathcal{W}}}-\frac{1}{p}\right)+(2-D_{\mathcal{W}})\,(D_{\mathcal{W}}+\alpha\,q-q)<0\,,$$

i.e.,

$$(3-D_{\mathcal{W}})\left\{q\left(\frac{1}{p}-\frac{s-1}{D_{\mathcal{W}}}\right)\right\}+(2-D_{\mathcal{W}})\,(D_{\mathcal{W}}+(\alpha-1)\,q)<2\,. \quad (\mathcal{C}ond_{Besov})$$

At the same time, thanks to Property 4.7, on page 48, and in light of Definition 4.5, on page 43, we have that

$$\begin{aligned}
\|f\|^p_{L^p(\Gamma_{\mathcal{W}})} = & \lim_{m\to\infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-3)} \sum_{X\,\in\,\mathcal{P}_m\bigcup\mathcal{Q}_m} \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\,\tilde{\lambda}^p_{|f|,m,j,X}\,\widetilde{|f|}^p_{m,j,X} \\
\lesssim & \lim_{m\to\infty} \varepsilon^{D_{\mathcal{W}}-3} \sum_{X\,\in\,\mathcal{P}_m\bigcup\mathcal{Q}_m} \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)\,\tilde{\lambda}^p_{|f|,m,j,X}\,\underbrace{\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j}\right)}_{\lesssim\,\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)}{}^{\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}} \\
\lesssim & \lim_{m\to\infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-3)} \sum_{X\,\in\,\mathcal{P}_m\bigcup\mathcal{Q}_m} \underbrace{\mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right)}_{\lesssim\,N_b^{(D_{\mathcal{W}}-3)\,m}}{}^{1+\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}} \\
\lesssim & \lim_{m\to\infty} N_b^{m\,(3-D_{\mathcal{W}})} \sum_{X\,\in\,\mathcal{P}_m\bigcup\mathcal{Q}_m} N_b^{(D_{\mathcal{W}}-3)\,m\left(1+\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}\right)} \\
\lesssim & \lim_{m\to\infty} N_b^m\,N_b^{m\,(3-D_{\mathcal{W}})}\,N_b^{(D_{\mathcal{W}}-3)\,m\left(1+\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}\right)}\,.
\end{aligned}$$

The convergence of this latter expression requires that

$$1+3-D_{\mathcal{W}}+(D_{\mathcal{W}}-3)\left(1+\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}\right)=1+(D_{\mathcal{W}}-3)\left(\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}\right)\leqslant 0\,;$$

i.e.,

$$1\leqslant(3-D_{\mathcal{W}})\left(\frac{s}{D_{\mathcal{W}}}-\frac{1}{p}\right),$$

or, equivalently,

$$\frac{D_{\mathcal{W}}}{3-D_{\mathcal{W}}}+\frac{D_{\mathcal{W}}}{p}\leqslant s\,.\quad (\mathcal{C}ond_{L^p})$$

We can check that if the above conditions $(\mathcal{C}ond_{Besov})$ and $(\mathcal{C}ond_{L^p})$ are satisfied,

$$\|f\|_{L^p(\Gamma_{\mathcal{W}})}+\left\{\iiint_{(X,T,Y)\,\in\,(\mathcal{D}(\Gamma_{\mathcal{W}}))^3}\frac{|f(T)+f(Y)-2\,f(X)|^q}{d_{Eucl}^{D_{\mathcal{W}}+\alpha\,q}(T,Y)}\,d\mu^3\right\}^{\frac{1}{q}}<\infty\,,$$

as desired. □

Corollary 4.9 (**The Specific Case of** $B^{p,p}_\beta\left(\Gamma_{\mathcal{W}}\right)$)**.**

Given $k\in\mathbb{N}$*, a real number* p *greater or equal than 1, we set*

$$\beta=k-\frac{2-D_{\mathcal{W}}}{p}\,.$$

A function f*, given by means of an atomic decomposition of the form*

$$f=\lim_{m\to\infty}\sum_{X\,\in\,\mathcal{P}_m\bigcup\mathcal{Q}_m}\tilde{\lambda}_{f,m,X}\,\tilde{f}_{m,X}$$

belongs to the Besov space $B^{p,p}_\beta\left(\Gamma_{\mathcal{W}}\right)$ *if*

$$(3 - D_{\mathcal{W}}) \left\{ p \left(\frac{1}{p} - \frac{s-1}{D_{\mathcal{W}}} \right) \right\} + (2 - D_{\mathcal{W}}) \, (D_{\mathcal{W}} + (\beta - 1) \, p) < 2 \, ,$$

and

$$\frac{D_{\mathcal{W}}}{3 - D_{\mathcal{W}}} + \frac{D_{\mathcal{W}}}{p} \leqslant s \, .$$

Definition 4.10 (**Trace of an $L^1_{loc}(\mathbb{R}^2)$ Function on the Weierstrass Curve**).

Along the lines of [JW84], on page 15, or [Wal91], we will say that an $L^1_{loc}(\mathbb{R}^2)$ function f is *strictly defined* at a vertex X of the Weierstrass Curve if the following limit exists and is given by

$$\bar{f}(X) = \lim_{m \to \infty} \frac{1}{\mu^{\mathcal{L}} \left(X, \mathcal{P}_m, \mathcal{Q}_m \right)} \sum_{Y \sim X} \mu^{\mathcal{L}} \left(Y, \mathcal{P}_m, \mathcal{Q}_m \right) \, f(Y) < \infty \, .$$

This enables us to define *the trace* $f_{|\Gamma_{\mathcal{W}}}$ of the function f on the Weierstrass Curve, via

$$\forall \, X \, \in \, \Gamma_{\mathcal{W}} \, : f_{|\Gamma_{\mathcal{W}}}(X) = \bar{f}(X) \, .$$

Remark 4.12. The trace $\bar{f}$ of an $L^1_{loc}(\mathbb{R}^2)$ function thus naturally admits an atomic decomposition of the form

$$\bar{f} = \lim_{m \to \infty} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \tilde{\lambda}_{\bar{f},m,X} \, \tilde{\bar{f}}_{m,X} \, .$$

Also,

$$C_{inf} \, N_b^{(D_{\mathcal{W}}-2)\,m} \leqslant f(Y) \leqslant C_{sup} \, N_b^{(D_{\mathcal{W}}-2)\,m} \, .$$

$$C_{inf} \, N_b^{(D_{\mathcal{W}}-3)\,m} \, N_b^{(D_{\mathcal{W}}-2)\,m} \leqslant N_b^{(D_{\mathcal{W}}-3)\,m} \, f(Y) \leqslant C_{sup} \, N_b^{(D_{\mathcal{W}}-3)\,m} \, N_b^{(D_{\mathcal{W}}-2)\,m} \, .$$

The following simple property was already used implicitly when introducing the polyhedral measure, earlier in Section 3.

Property 4.10 (**The Compact Set of $\mathbb{R}^2$ which Contains the Weierstrass Curve**).

The Weierstrass Curve $\Gamma_{\mathcal{W}}$ is contained in the following compact set of $\mathbb{R}^2$,

$$\Omega_{\mathcal{W}} = [0,1] \times [m_{\mathcal{W}}, M_{\mathcal{W}}] \, ,$$

where $m_{\mathcal{W}}$ and $M_{\mathcal{W}}$ respectively denote the minimal and maximal values of the Weierstrass function $\mathcal{W}$ on $[0,1]$, introduced in Notation 13, on page 28.

Notation 17 (**Interior of the Compact Set $\Omega_{\mathcal{W}}$**).

We hereafter denote by $\mathring{\Omega}_{\mathcal{W}}$ the interior of the compact set $\Omega_{\mathcal{W}}$.

Definition 4.11 (Sobolev Space on the Open Set $\mathring{\Omega}_{\mathcal{W}}$).

Given $k \in \mathbb{N}$, and $p \geqslant 1$, we recall that the Sobolev space on the open set $\mathring{\Omega}_{\mathcal{W}} \subset \mathbb{R}^2$ (which boundary is a rectangle), denoted by $W_k^p\left(\mathring{\Omega}_{\mathcal{W}}\right)$, is given by

$$W_k^p\left(\mathring{\Omega}_{\mathcal{W}}\right) = \left\{f \in L^p\left(\mathring{\Omega}_{\mathcal{W}}\right) \ , \forall\, \alpha \text{ such that } |\alpha| \leqslant k \, , \, D^\alpha f \in L^p\left(\mathring{\Omega}_{\mathcal{W}}\right)\right\} ,$$

where $L^p\left(\mathring{\Omega}_{\mathcal{W}}\right)$ denotes the Lebesgue space of order p on $\mathring{\Omega}_{\mathcal{W}}$, while, for the multi-index α, $D^\alpha f$ is the classical partial derivative of order α, interpreted in the weak sense.

The following result is the counterpart, in our context, of the corresponding one obtained in [JW84] (Theorem 1 of Chapter VI, on page 141), or, in a more powerful way, when $p > 1$, in the case of the von Koch Snowflake (which preserves Markov's Inequality), as shown in [Wal91], Proposition 4, on page 120.

Theorem 4.11 (The Trace of Sobolev Spaces Viewed as Besov Spaces).

Given a positive integer k, and a real number $p \geqslant 1$, we set

$$\beta_{k,p} = k - \frac{2 - D_{\mathcal{W}}}{p} \, .$$

We then have that

$$W_k^p\left(\,\mathring{\Omega}_{\mathcal{W}}\right)_{|\Gamma_{\mathcal{W}}} = B^{p,p}_{\beta_{k,p}}\left(\Gamma_{\mathcal{W}}\right) \, .$$

Remark 4.13. Note that, as was previoulsy explained in the introduction, the fact that the Weierstrass Curve preserves Markov's Inequality plays an important part in this theorem. Indeed, it ensures that the correspondance between the Sobolev space and its trace on the fractal boundary, is obtained in a constructive way.

Corollary 4.12 (The Specific Case $k = 2$, and its Consequences – Order of the Fractal Laplacian).

Given a real number $p \geqslant 1$, we then have that

$$W_2^p\left(\,\mathring{\Omega}_{\mathcal{W}}\right)_{|\Gamma_{\mathcal{W}}} = B^{p,p}_{\beta_{2,p}}\left(\Gamma_{\mathcal{W}}\right) ,$$

where

$$\beta_{2,p} = 2 - \frac{2 - D_{\mathcal{W}}}{p} = 2 + \frac{1}{p}\,\frac{\ln \lambda}{\ln N_b} < 2 \, .$$

In particular, in the case when $p = 2$, which corresponds to

$$\beta_{2,2} = 2 - \frac{2 - D_{\mathcal{W}}}{2} \, ,$$

and provided that

$$(3 - D_{\mathcal{W}})\left\{2\left(\frac{1}{2} - \frac{s-1}{D_{\mathcal{W}}}\right)\right\} + (2 - D_{\mathcal{W}})\left(D_{\mathcal{W}} - 2\left(1 + \frac{2 - D_{\mathcal{W}}}{2}\right)\right) < 2 \, ,$$

i.e.,

$$1 - D_{\mathcal{W}} - 2\,(3 - D_{\mathcal{W}})\,\frac{s-1}{D_{\mathcal{W}}} + (2 - D_{\mathcal{W}})\,\left(2\,D_{\mathcal{W}} - 4\right) < 0\,,$$

or, equivalently,

$$s > 1 + D_{\mathcal{W}}\left(\frac{1 - D_{\mathcal{W}} + (2 - D_{\mathcal{W}})\,\left(2\,D_{\mathcal{W}} - 4\right)}{2\,(3 - D_{\mathcal{W}})}\right),$$

we then have that

$$W_2^2\,(\Omega_{\mathcal{W}})_{|\Gamma_{\mathcal{W}}} = B^{2,2}_{\beta_{2,2}}\,(\Gamma_{\mathcal{W}})\,,$$

where

$$\beta_{2,2} = 2 - \frac{1}{2}\,\frac{\ln\lambda}{\ln N_b} < 2\,.$$

Consequently, by analogy with the classical theories, the Laplacian on the Weierstrass Curve (see Remark 4.10, on page 51 above) arises as a differential operator of order $\beta_{2,2} \in \,]1,2[$.

Remark 4.14. Indeed, a fractal Laplacian arising in this way is non-local. Our aim is to sketch a connection between such a fractal Laplacian and the fractal Laplacian in the sense of Kigami, Strichartz and others, which is defined for certain classes of fractals by means of discrete approximations. The point is that those operators each correspond to a specific order of differentiation, which is not the same. As it can be found in [Str03], when the fractal Laplacian in the sense of Kigami, Strichartz and others, is involved, the associated function spaces are distinct from the function spaces obtained via traces, as is the case in the work of A. Jonsson and H. Wallin, which can be found, for instance, in [JW84].

Remark 4.15 (**Connection with the Optimal Exponent of Hölder Continuity**).

We note that

$$\beta_{2,2} = 2 - \frac{\alpha_{\mathcal{W}}}{2}\,,$$

where the Codimension $\alpha_{\mathcal{W}} = 2 - D_{\mathcal{W}} = -\dfrac{\ln\lambda}{\ln N_b} \in \,]0,1[$ is the best (i.e., optimal) Hölder exponent for the Weierstrass function, as was initially obtained by G. H. Hardy in [Har16]), and then, by a completely different method – geometrically – in [DL22] (this latter result is recalled in Theorem 2.13, on page 19, and in Corollary 2.14, on page 19).

Property 4.13 (**Connection with Fractional Derivatives**).

In Definition 4.9, on page 49, the Besov norm of a function f defined on the Weierstrass Curve involves the integral

$$\iint_{(T,Y)\in\Gamma^2_{\mathcal{W}}} \frac{|f(T) - f(Y)|^q}{d_{Eucl}^{D_{\mathcal{W}}+\alpha\,q}(T,Y)}\,d\mu^2\,. \tag{18}$$

For $p=2$, $q=1$ and $\alpha = 3-2\,D_{\mathcal{W}}$, i.e., if $f \in B^{2,1}_{3-2\,D_{\mathcal{W}}}(\Gamma_{\mathcal{W}})$, the integral in expression (18) just above can be connected to the associated fractional derivative of order

$$\gamma = 2 - D_{\mathcal{W}} \in \,]0,1[\,,$$

defined, subject to its existence, for any vertex $Y \in V_{m_0} \subset \Gamma_{\mathcal{W}}$, with $m_0 \in \mathbb{N}$, by the following expression:

$$D^{2-D_{\mathcal{W}}} f(Y) \quad = \quad \frac{2-D_{\mathcal{W}}}{\Gamma(D_{\mathcal{W}}-1)} \lim_{m\to\infty} \int_{T\in\mathcal{D}(\Gamma_{\mathcal{W}}),\, T \underset{m}{\sim} Y} \frac{|f(T)-f(Y)|}{d_{Eucl}^{3-D_{\mathcal{W}}}(T,Y)}\, d\mu\,,$$

i.e., equivalently,

$$D^{2-D_{\mathcal{W}}} f(Y) = $$

$$= \frac{2-D_{\mathcal{W}}}{\Gamma(D_{\mathcal{W}}-1)} \lim_{m\to\infty} \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{T \underset{m}{\sim} Y} \mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\; \lambda_{\star,m}\, \frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{Eucl}^{3-D_{\mathcal{W}}}(T,Y)}\,,$$

where, for any $m \in \mathbb{N}$, $\lambda_{\star,m}$ is the m^{th} scalar coefficient involved, and where Γ denotes the usual Gamma function.

Remark 4.16 (**Bounds for Fractional Derivatives**).

Note that due in the forthcoming case of functions f satisfying the first condition given in Definition 5.1 below, on page 61, there exist strictly positive constants $\tilde{C}_{inf}$ and $\tilde{C}_{sup}$ such that, for all vertices $(T,Y) \in V_m \times V_m \subset \Gamma_{\mathcal{W}} \times \Gamma_{\mathcal{W}}$,

$$\tilde{C}_{inf}\, d_{Eucl}^{2-D_{\mathcal{W}}}(T,Y) \leqslant \left|\tilde{f}_m(T)-\tilde{f}_m(Y)\right| \leqslant \tilde{C}_{sup}\, d_{Eucl}^{2-D_{\mathcal{W}}}(T,Y)\,.$$

Hence, we have that, for all vertices $(T,Y) \in V_m \times V_m \subset \Gamma_{\mathcal{W}} \times \Gamma_{\mathcal{W}}$,

$$\frac{\tilde{C}_{inf}}{d_{Eucl}(T,Y)} \leqslant \frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{Eucl}^{3-D_{\mathcal{W}}}(T,Y)} \leqslant \frac{\tilde{C}_{sup}}{d_{Eucl}(T,Y)}\,,$$

or, equivalently, expressed in terms of the m^{th} cohomology infinitesimal ε_m^m (see Definition 3.1, on page 23),

$$\frac{\tilde{C}_{inf}}{\varepsilon_m^m} \leqslant \frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{Eucl}^{3-D_{\mathcal{W}}}(T,Y)} \leqslant \frac{\tilde{C}_{sup}}{\varepsilon_m^m}\,,$$

which, in conjunction with the estimates given in Property 3.3, on page 26, namely,

$$\mu^{\mathcal{L}}(X,\mathcal{P}_m,\mathcal{Q}_m) \lesssim \varepsilon_m^{m\,(3-D_{\mathcal{W}})}\,,$$

yields

$$\varepsilon_m^{m\,(D_{\mathcal{W}}-2)}\, \varepsilon_m^{m\,(3-D_{\mathcal{W}})}\, \frac{\tilde{C}_{inf}}{\varepsilon_m^m} \lesssim \varepsilon_m^{m\,(D_{\mathcal{W}}-2)}\, \mu^{\mathcal{L}}(T,\mathcal{P}_m,\mathcal{Q}_m)\, \frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{Eucl}^{3-D_{\mathcal{W}}}(T,Y)}$$

and

$$\varepsilon_m^{m\,(D_{\mathcal{W}}-2)}\, \varepsilon_m^{m\,(3-D_{\mathcal{W}})}\, \frac{\tilde{C}_{inf}}{\varepsilon^m} \lesssim \varepsilon_m^{m\,(D_{\mathcal{W}}-2)}\, \varepsilon_m^{m\,(3-D_{\mathcal{W}})}\, \frac{\tilde{C}_{sup}}{\varepsilon_m^m}\,;$$

i.e.,

$$\tilde{C}_{inf} \lesssim \varepsilon_m^{m\,(D_{\mathcal{W}}-2)}\,\mu^{\mathcal{L}}\,(T,\mathcal{P}_m,\mathcal{Q}_m)\;\frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{Eucl}^{3-D_{\mathcal{W}}}(T,Y)} \lesssim \tilde{C}_{sup}\,.$$

Since the vertex $Y \in V_m$ admits at most two adjacent vertices in V_m, we then deduce that

$$\tilde{C}_{inf}\,\left|\lambda_{\star,m}\right| \lesssim \varepsilon_m^{m\,(D_{\mathcal{W}}-2)}\,\mu^{\mathcal{L}}\,(T,\mathcal{P}_m,\mathcal{Q}_m)\,\left|\lambda_{\star,m}\right|\;\frac{\left|\tilde{f}_m(T)-\tilde{f}_m f(Y)\right|}{d_{Eucl}^{3-D_{\mathcal{W}}}(T,Y)} \lesssim \tilde{C}_{sup}\,\left|\lambda_{\star,m}\right|\,.$$

Note that in the case of a continuous function f, the atomic coefficients $\lambda_{\star,m}$ are necessarily bounded (since $\Gamma_{\mathcal{W}}$ is compact, and hence, f is bounded).

5 Towards an Extension of Morse Theory

Classical Morse theory (see, e.g., [Bot82], [Bot88] and [Mil63]) enables us to explore the shape (i.e., the topology) of a smooth manifold by means of the study of the critical points of suitable smooth functions defined on the manifold. Such functions are required to be nondegenerate – in the sense that their Hessian determinant is nonzero at critical points – and are then called Morse functions.

In the classical Morse theory, i.e., for smooth manifolds, the height function plays a major role. More precisely, along with its critical points and its level sets, it encodes the information that enables us to reconstruct the manifold.

For fractal curves or IFDs such as, for instance, the Weierstrass Curve, this does not make sense anymore. Since fractals are involved, a change of shape occurs at each vertex of each prefractal approximation.

In particular, we will show that the Weierstrass function $\mathcal{W}$ is a fractal Morse function. Note (as in [DL22], [DL24c], [DL24a]) that $\mathcal{W}$ can be viewed as a function on $\Gamma_{\mathcal{W}}$, namely, the identity function; see Remark 5.4, on page 70 below.

By using some of the results of [DL22], [DL24c], [DL24a], we hereafter begin to lay the foundations of a fractal Morse theory that should eventually enable us to explore the shape of *fractal manifolds* (viewed as higher-dimensional IFDs). The important example of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ – or, rather, of the associated Weierstrass IFD – sheds a useful light, especially when it comes to studying not only the (appropriately defined) critical points, but also other suitable (inflection) points of *fractal Morse functions*, and the remainaing points, which are themselves subject to a change of shape, in connection with the associated angle between the corresponding adjacent edges; i.e., a typical change of curvature.

In this light, the simple knowledge of the equivalent of the usual Morse indexes, along with a suitable analogue of the height function, does not appear as being sufficient when it comes to reconstruct the fractal IFD. The missing data can be obtained by means of the sequence of cohomological integers associated, at each step $m \in \mathbb{N}$ of the prefractal approximation, with the set of vertices V_m; see Definition 5.5, on page 63.

Also, in our context, the sequence of sets of critical point introduced in Definition 5.6, on page 66 is an increasing sequence. For this reason, instead of the height function, we choose to consider *the increasing reordering* of *the absolute heights*, as introduced in Definition 5.11, on page 71 below. We believe that this increasing reordering still bears the fractality of the Weierstrass Curve.

As was previously encountered in Corollary 2.17, on page 21, at any given level $m \in \mathbb{N}$ of the prefractal graph approximation of the Curve, the extreme and bottom vertices of the polygons $\mathcal{P}_{m,k}$, $0 \leqslant k \leqslant N_b^m - 1$, are respectively local maxima and minima of the Weierstrass function $\mathcal{W}$. Those points are isolated ones, a result that directly comes from the construction of the Weierstrass Curve. Then, there remains to identify points that could play the role of degenerate ones, and to define *fractal suited Morse functions*. Also, as was alluded to in the introduction, the respective notions of *minimal real Complex Dimension* and *cohomogical vertex integer* naturally arise from our results obtained in [DL24c] on the fractal cohomology of the Weierstrass Curve. For the sake of a better understanding, we next briefly recall those results.

Definition 5.1 (**Set of Functions of the Same Nature as the Weierstrass Function $\mathcal{W}$ [DL24c]**).

i. We say that a continuous, complex-valued function f, defined on $\Gamma_{\mathcal{W}} \supset V^\star$, is *of the same nature as the Weierstrass function* $\mathcal{W}$, if it satisfies local Hölder and reverse-Hölder properties analogous to those satisfied by the Weierstrass function $\mathcal{W}$; i.e., for any $m \in \mathbb{N}$ and for any pair of adjacent vertices (M, M') with respective affixes $(z, z') \in \mathbb{C}^2$, of the prefractal graph $\Gamma_{\mathcal{W}_m}$, we have that

$$\tilde{C}_{inf} \, |z' - z|^{2-D_{\mathcal{W}}} \leqslant \left| f(z') - f(z) \right| \leqslant \tilde{C}_{sup} \, |z' - z|^{2-D_{\mathcal{W}}} \, ,$$

where $\tilde{C}_{inf}$ and $\tilde{C}_{sup}$ denote suitable positive and finite constants possibly depending on f. This can be written, equivalently, as

$$\left| z - z' \right|^{2-D_{\mathcal{W}}} \lesssim \left| f(z) - f(z') \right| \lesssim \left| z - z' \right|^{2-D_{\mathcal{W}}} . \quad (\blacklozenge) \tag{19}$$

(Compare with Theorem 2.13, on page 19, and Corollary 2.14, on page 19 above.)

Hereafter, we will denote by $\mathcal{H}\ddot{o}ld\left(\Gamma_{\mathcal{W}}\right)$ the set consisting of the continuous, complex-valued functions f, defined on $\Gamma_{\mathcal{W}} \supset V^\star$ and satisfying relation (19).

ii. Moreover, we will denote by $\mathcal{H}\ddot{o}ld_{geom}\left(\Gamma_{\mathcal{W}}\right) \subset \mathcal{H}\ddot{o}ld\left(\Gamma_{\mathcal{W}}\right)$ the subset of $\mathcal{H}\ddot{o}ld\left(\Gamma_{\mathcal{W}}\right)$ consisting of the functions f of $\mathcal{H}\ddot{o}ld\left(\Gamma_{\mathcal{W}}\right)$ which satisfy the following *additional geometric condition* (20), again, for any pair of adjacent vertices (M, M') of the prefractal graph V_m with respective affixes $(z, z') \in \mathbb{C}^2$, and for $m \in \mathbb{N}$ arbitrary; namely,

$$\left| \arg\left(f(z)\right) - \arg\left(f(z')\right) \right| \lesssim |z - z'|^{D_{\mathcal{W}}-1} . \quad (\spadesuit) \tag{20}$$

Remark 5.1. Note that, according to the results of [DL22] and [DL24c], the Weierstrass function $\mathcal{W}$ belongs to $\mathcal{H}\ddot{o}ld_{geom}\left(\Gamma_{\mathcal{W}}\right)$ – and hence, also, to $\mathcal{H}\ddot{o}ld\left(\Gamma_{\mathcal{W}}\right)$.

Definition 5.2 (**(m, p)-Fermion [DL24c]**).

By analogy with particle physics, given a pair of integers (m, p), with $m \in \mathbb{N}$ and $p \in \mathbb{N}^\star$, we will call (m, p)-*fermion* on V_m, with values in $\mathbb{C}$, any antisymmetric map f from V_m^{p+1} to $\mathbb{C}$. Note that these maps are not assumed to be multilinear (which would be meaningless here, anyway, since V_m is not a vector space).

For any $m \in \mathbb{N}$, an $(m,0)$-*fermion* on V_m (or a 0-*fermion*, in short) is simply a map f from V_m to $\mathbb{C}$. We adopt the convention according to which a 0-fermion on V_m is a 0-*antisymmetric map* on V_m.

In the sequel, for any $(m,p) \in \mathbb{N}^2$, we will denote by $\mathcal{F}^p(V_m,\mathbb{C})$ the complex vector space of (m,p)-fermions on V_m, which makes it an abelian group with respect to the addition, with an external law from $\mathbb{C} \times \mathcal{F}^p(V_m,\mathbb{C})$ to $\mathcal{F}^p(V_m,\mathbb{C})$.

Definition 5.3 (**$(m-1,m)$-Differentials [DL24c]**).

Here and thereafter, for $(m,p) \in \mathbb{N} \times \mathbb{N}^\star$, V_m^p denotes the p-fold product of V_m by itself.

Given a strictly positive integer m, we define the $(m-1,m)$-*differential* $\delta_{m-1,m}$ from $\mathcal{F}^0(V_m,\mathbb{C})$ to $\mathcal{F}^{N_b+1}(V_m,\mathbb{C})$, for any f in $\mathcal{F}^0(\Gamma_{\mathcal{W}},\mathbb{C})$ and any $\left(M_{i,m-1}, M_{i+1,m-1}, M_{j+1,m}, \ldots, M_{j+N_b-2,m}\right) \in V_m^{N_b+1}$ such that

$$M_{i,m-1} = M_{j,m} \quad \text{and} \quad M_{i+1,m-1} = M_{j+N_b,m}\,,$$

by

$$\delta_{m-1,m}(f)\left(M_{i,m-1}, M_{i+1,m-1}, M_{j+1,m}, \ldots, M_{j+N_b-1,m}\right) = c_{m-1,m}\left(\sum_{q=0}^{N_b}(-1)^q f\left(M_{j+q,m}\right)\right),$$

where $c_{m-1,m}$ denotes a suitable positive constant. The value of this constant is unimportant for the remainder of this discussion.

Theorem 5.1 (**Fractal Cohomology of the Weierstrass Curve [DL24c]**).

Within the set $\mathcal{H}\ddot{o}ld_{geom}(\Gamma_{\mathcal{W}})$ of continuous, complex-valued functions f, defined on the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset V^\star = \bigcup_{n\in\mathbb{N}} V_n$ (see part ii. of Definition 5.1, on page 61 above), let us consider the Complex (which can be called the Total Fractal Complex *of $\Gamma_{\mathcal{W}}$),*

$$H^\star = H^\bullet\left(\mathcal{F}^\bullet(\Gamma_{\mathcal{W}},\mathbb{C}),\delta^\bullet\right) = \bigoplus_{m=0}^{\infty} H^m\,,$$

where, for any positive integer m, with the additional convention $\delta_{-2,-1} = 0$ and $\delta_{-1,0} = 0$ (which ensures that $H^0 = \{0\}$), H^m is the cohomology group

$$H^m = \ker\delta_{m-1,m}/Im\,\delta_{m-2,m-1}\,,$$

which consist in maps the expression of which is obtained as the difference of an antisymmetric map with respect to the set of vertices $\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \ldots, M_{j+N_b^m-1,m}\right)$ and of an antisymmetric map with respect to the set of vertices $\left(M_{i,0}, M_{i+1,0}, M_{j+1,m-1}, \ldots, M_{j+N_b^{m-1}-1,m-1}\right)$.

Then, $H^\star$ is the set consisting of functions f on $\Gamma_{\mathcal{W}}$, viewed as 0-fermions (in the sense of Definition 5.2, on page 61), and, for any integer $m \geqslant 1$, of the restrictions to V_m of (m, N_b^m+1)-fermions, i.e., the restrictions to (the Cartesian product space) $V_m^{N_b^m+1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with N_b^m+1 variables (corresponding to the vertices of V_m), involving the restrictions to V_m of the continuous, complex-valued functions f on $\Gamma_{\mathcal{W}}$ – as, naturally, the aforementioned 0-fermions – satisfying the following convergent (and even, absolutely convergent)Taylor-like expansions (with $V^\star = \bigcup_{n\in\mathbb{N}} V_n$),

$$\forall\, M_{\star,\star} \in V^\star : \\ f\left(M_{\star,\star}\right) = \sum_{k=0}^{\infty} c_k\left(f, M_{\star,\star}\right)\, \varepsilon_k^{k\,(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_k\,\mathbf{p}} = \sum_{k=0}^{\infty} c_k\left(f, M_{\star,\star}\right)\, \varepsilon_k^{k\,(2-D_{\mathcal{W}})+i\,\ell_k\,\mathbf{p}}\,, \tag{21}$$

where, for each integer $k \geqslant 0$, $c_k\left(\star,\star\right) = c_k\left(f,\star\right) \in \mathbb{C}$, the number $\varepsilon_k^k > 0$ is the k^{th} cohomology infinitesimal introduced in Definition 3.1, on page 23, and where ℓ_k denotes an integer (in $\mathbb{Z}$) such that

$$0 \leqslant \left\{\ell_k\, \frac{\ln \varepsilon_k^k}{\ln N_b}\right\} \lesssim \frac{\varepsilon_k^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi}\,,$$

with $\{.\}$ being the fractional part.

Note that since the functions f involved are uniformly continuous on the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset V^\star$, and since the set $V^\star$ is dense in $\Gamma_{\mathcal{W}}$, they are uniquely determined by their restriction to $V^\star$, as given by relation (21), on page 63. We caution the reader, however, that at this stage of our investigations, we do not know wether $f(M)$ is given by an expansion analogous to the one in relation (21), for every $M \in \Gamma_{\mathcal{W}}$, rather than just for all $M \in V^\star$.

For each $M_\star = M_{\star,m} \in V^\star$, as is shown in [DL24c], the coefficients $c_k\left(\star,\star\right)$ (for any $k \in \mathbb{N}$) are the residues at the possible cohomological Complex Dimensions $-\left(k\,(2-D_{\mathcal{W}}) + i\,k\,\ell_k\,\mathbf{p}\right)$*) of a suitable global scaling zeta function evaluated at $M_\star$.*

The group $H^\star = \bigoplus_{m=0}^{\infty} H^m$ is called the total fractal cohomology group *of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (or else, of the Weierstrass function $\mathcal{W}$).*

Definition 5.4 (**Minimal (Real) Complex Dimension of a Prefractal Approximation**).

Given $m \in \mathbb{N}$, we define the *minimal real Complex Dimension* of the prefractal approximation $\Gamma_{\mathcal{W}_m}$ as

$$\omega_m = D_{\mathcal{W}} - m\,(2 - D_{\mathcal{W}})\,.$$

Remark 5.2. Clearly, the successive prefractal approximations play the role of level sets, in our present context.

We now recall the following results, obtained in [DL24c], [DL23b].

Definition 5.5 (**Cohomological Vertex Integer**).

Given $m \in \mathbb{N}$, and a vertex $M_{j,m} = M_{(N_b-1)\,k'+k",m} \in V_m$, of abscissa $\left((N_b-1)\,k' + k"\right)\,\varepsilon_m^m$, where $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k" \leqslant N_b - 1$, we introduce the *cohomological vertex integer* $\ell_{j,m}$ associated to the vertex $M_{j,m}$ (which is also the $(k")^{th}$ vertex of the polygon $\mathcal{P}_{m,k'}$; see part *iv.* of Property 2.6, on page 10), as

$$\ell_{j,m} = \ell_{k',k",m} = (N_b - 1)\, k' + k" \,. \tag{22}$$

$$\left((N_b - 1)\, k' + k"\right) \varepsilon_m^m = \left((N_b - 1)\, k' bis + k" bis\right) \varepsilon_{m+1}^{m+1} \,.$$

Depending on the context; that is,

i. when the cohomological vertex integer enables one to locate the vertex $M_{j,m}$.

ii. When it is used in a more general framework, i.e., in order to describe the generators of cohomology groups (see [DL24b]);

we will use the best suited notation between $\ell_{j,m}$, in case *i.*, or $\ell_{k',k",m}$, in case *ii.*

Proposition 5.2 (**Cross-Scales Paths, and Associated Sequence of Vertex Integers**).

Given $m \in \mathbb{N}$, $0 \leqslant j \leqslant \#V_m - 1$ and a vertex $M_{j,m} = M_{(N_b-1)\,k'+k",m}$ in V_m, with $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k" \leqslant N_b - 1$, we introduce the cross-scales path *$\mathcal{P}ath\left(P_{k"}, M_{j,m}\right)$, where $P_{k'}$ is the $(k')^{th}$ fixed point of the map $T_{k'}$ (see Proposition 2.3, on page 8, along with Notation 6, on page 9), as the ordered set $\left(M_{j_{k,m},k}\right)_{0\leqslant k\leqslant m}$ such that:*

i. For $0 \leqslant k \leqslant m$, each vertex $M_{j_{k,m},k}$ is in $V_k \setminus V_k \cap V_m$ (which means that $M_{j_{k,m},k}$ strictly belongs to V_k, i.e., it is in the k^{th} prefractal approximation $\Gamma_{\mathcal{W}_k}$, and not in $\Gamma_{\mathcal{W}_{k+1}}$).

ii. For $1 \leqslant k \leqslant m$, each vertex $M_{j_{k,m},k} = M_{(N_b-1)\,k'_{k,m}+k",k}$, with $0 \leqslant k'_{k,m} \leqslant N_b^k - 1$, is the image of the point $M_{j_{k-1,m},k-1}$ under the map T_i (see again Proposition 2.3, on page 8), where $i \in \{0, \dots, N_b - 1\}$ is the smallest admissible value. We thus also have that

$$M_{j_{k-1,m},k-1} = \left(\frac{(N_b-1)\left(k'_{k,m} - i\,(N_b-1)\,N_b^{k-1}\right) + k"}{(N_b-1)\,N_b^{k-1}}, \mathcal{W}\left(\frac{(N_b-1)\left(k'_{k,m} - i\,(N_b-1)\,N_b^{k-1}\right) + k"}{(N_b-1)\,N_b^{k-1}}\right)\right).$$

This latter point is also ***the $(k")^{th}$ vertex of the polygon*** *$k'_{k,m} - i\,(N_b-1)\,N_b^{k-1}$ (see part iv. of Property 2.6, on page 10).*

The sequence of vertex integers associated with the cross-scales path $\mathcal{P}ath\left(P_{k"}, M_{j,m}\right)$ (or, in short, and equivalently, also refered to as the sequence of vertex integers associated with $M_{j,m}$) is the sequence $\left(\ell_{j_{k,m},k}\right)_{0\leqslant k\leqslant m}$, where, for $0 \leqslant k \leqslant m$, $\ell_{j_{k,m},k}$ is the cohomological vertex integer associated with the vertex $M_{j_{k,m},k}$ *(see Definition 5.5, on page 63).*

Theorem 5.3 (**Complex Dimensions Series Expansion of the Complexified Weierstrass function $\mathcal{W}_{comp}$ and of the Weierstrass function $\mathcal{W}$**).

For any sufficiently large positive integer m (i.e., for all $m \geqslant m_0$, with $m_0 \in \mathbb{N}^\star$ optimal) and for any j in $\{0, \dots, \#V_m - 1\}$, we have the following fractal power series exact *expansion, indexed by the Complex Codimensions $k\,(D_{\mathcal{W}} - 2) + i\,k\,\ell_{j_{k,m},k}\,\mathbf{p}$, with $0 \leqslant k \leqslant m$,*

$$
\begin{aligned}
\mathcal{W}_{comp}\left(j\,\varepsilon_m^m\right) &= \mathcal{W}_{comp}\left(\frac{j\,\varepsilon^m}{N_b-1}\right) \\
&= \varepsilon^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}_{comp}\left(\frac{j}{N_b-1}\right)+\sum_{k=0}^{m-1} c_{k,j,m}\,\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_{k,j,m}\,\mathbf{p}} \qquad (23)\\
&= \sum_{k=0}^{m} c_{k,j,m}\,\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_{j_{k,m},k}\,\mathbf{p}}\,,
\end{aligned}
$$

where, for $0 \leqslant k \leqslant m$, ε^k *is the* k^{th} *intrinsic cohomology infinitesimal, introduced in Definition 3.1, on page 23, with* $\mathbf{p}=\dfrac{2\,\pi}{\ln N_b}$ *denoting the oscillatory period of the Weierstrass Curve, as introduced in [DL22] and where:*

i. $\ell_{j_{k,m},k} \in \mathbb{Z}$ *is the cohomological vertex integer associated with the vertex* $M_{j_{k,m},k}$ *(see Definition 5.5, on page 63, along with Proposition 5.2, on page 64).*

ii. $c_{m,j,m} = \mathcal{W}_{comp}\left(\dfrac{j}{N_b-1}\right)$ *and, for* $0 \leqslant k \leqslant m-1$, $c_{k,j,m} \in \mathbb{C}$ *is given by*

$$
c_{k,j,m} = exp\left(\frac{2\,i\,\pi}{N_b-1}\,j\,\varepsilon^{m-k}\right); \qquad (24)
$$

for $0 \leqslant k \leqslant m$, *the coefficient* $c_{k,j,m}$ *will also be referred to as* the k^{th} Weierstrass coefficient associated with the vertex $M_{j_{k,m},k} \in V_k$.

For any $m \in \mathbb{N}$, *the complex numbers* $\left\{c_{0,j,m+1},\ldots,c_{m+1,j,m+1}\right\}$ *satisfy the following recurrence relations:*

$$
c_{m+1,j,m+1} = \mathcal{W}\left(\frac{j}{N_b-1}\right) = c_{m,j,m} \qquad (25)
$$

and

$$
\forall\, k \in \{1,\ldots,m\}: \quad c_{k,j,m+1} = c_{k-1,j,m}\,. \qquad (26)
$$

In addition, since relation (23) is valid for any $m \in \mathbb{N}^\star$ *(and since, clearly, relation (24) implies that the coefficients* $c_{k,j,m}$ *are nonzero for* $0 \leqslant k \leqslant m$*), we deduce that the associated Complex Dimensions (i.e., in fact, the Complex Dimensions associated with the complexified Weierstrass function) are*

$$
D_{\mathcal{W}} - k\,(2-D_{\mathcal{W}}) + i\,\ell_{j_{k,m},k}\,\mathbf{p} \qquad (27)
$$

where $0 \leqslant j \leqslant \#V_m - 1$, $0 \leqslant k \leqslant m$ *and* $\ell_{j_{k,m},k} \in \mathbb{Z}$ *is the cohomological vertex integer associated with the vertex* $M_{j_{k,m},k}$ *(see Definition 5.5, on page 63, along with Proposition 5.2, on page 64).*

This immediately ensures, for the Weierstrass function (i.e., the real part of the Complexified Weierstrass function $\mathcal{W}_{comp}$*), that, for any strictly positive integer* m *and for any* j *in* $\{0,\ldots,\#V_m-1\}$,

$$
\begin{aligned}
\mathcal{W}\left(j\,\varepsilon_m^m\right) &= \varepsilon^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}_{comp}\left(\frac{j}{N_b-1}\right)+\sum_{k=0}^{m-1}\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\mathcal{R}e\left(c_{k,j,m}\,\varepsilon_k^{i\,\ell_{j_{k,m},k}\,\mathbf{p}}\right)\\
&= \varepsilon^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}_{comp}\left(\frac{j}{N_b-1}\right)+\frac{1}{2}\sum_{k=0}^{m-1}\varepsilon^{k\,(2-D_{\mathcal{W}})}\left(c_{k,j,m}\,\varepsilon^{i\,\ell_{j_{k,m},k}\,\mathbf{p}}+\overline{c_{k,j,m}}\,\varepsilon^{-i\,\ell_{j_{k,m},k}\,\mathbf{p}}\right)\\
&= \frac{1}{2}\sum_{k=0}^{m}\varepsilon^{k\,(2-D_{\mathcal{W}})}\left(c_{k,j,m}\,\varepsilon^{i\,\ell_{j_{k,m},k}\,\mathbf{p}}+\overline{c_{k,j,m}}\,\varepsilon^{-i\,\ell_{j_{k,m},k}\,\mathbf{p}}\right).
\end{aligned}
\tag{28}
$$

More generally, for any strictly positive integer m and for any integer j,

$$
\mathcal{W}_{comp}\left(j\,\varepsilon^m\right)=\sum_{k=0}^{\infty}\varepsilon^{k\,(2-D_{\mathcal{W}})}\,c_{k,j,m}\,\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_{j_{k,m},k}\,\mathbf{p}}\,, \tag{29}
$$

where, for all $k\in\mathbb{N}$,

$$
c_{k,j,m}=\varepsilon^{2\,i\,\pi\,N_b^k\,j\,\varepsilon^m}\,. \tag{30}
$$

In addition, the Complex Dimensions associated with the (actual) Weierstrass function $\mathcal{W}$, are all exact and simple. Furthermore, they are given as in (27) above, on page 65, except for the fact that $i\,\ell_{j_{k,m},k}\,\mathbf{p}$ should be replaced by $\pm i\,\ell_{j_{k,m},k}\,\mathbf{p}$. Accordingly, as should be the case, the nonreal Complex Dimensions of the Weierstrass function come in complex conjugate pairs.

In particular, for any $k\in\mathbb{N}$, the only real Complex Dimension of $\mathcal{W}$ with abscissa $d_k=D_{\mathcal{W}}-k\,(2-D_{\mathcal{W}})$ is d_k itself, while the other Complex Dimensions of $\mathcal{W}$ with abscissa d_k consist of countably many nonreal complex conjugate pairs.

Remark 5.3. Note that each integer $\ell_{k,j,m}$ (choosen as in Definition 5.5, on page 63 just above) carries the information associated to the vertex point $M_{((N_b-1)\,k+j)\,,m}\in V_m$ (and, hence, to a change of shape) when switching from $M_{((N_b-1)\,k+j)\,,m}\in V_m$ to its consecutive neighbor $M_{((N_b-1)\,k+j+1)\,,m}\in V_m$.

Definition 5.6 (**Sequence of Sets of Critical Points of the Weierstrass Curve**).

We define *the sequence of sets of critical points* of the Weierstrass function $\mathcal{W}$ – or, equivalently, of the Weierstrass IFD – as the sequence $\left(Crit\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ such that, for any $m\in\mathbb{N}$, the set $Crit\left(\Gamma_{\mathcal{W}_m}\right)$ is obtained as the union of the set of local extrema of $\mathcal{W}$ given in Corollary 2.17, on page 21, and of the set of vertices with a reentrant angle of the prefractal approximation $\Gamma_{\mathcal{W}_m}$, as given in Property 2.18, on page 21.

Definition 5.7 (**Topological Laplacian of Level $m\in\mathbb{N}^\star$**).

For any $m\in\mathbb{N}^\star$, and any real-valued function f, defined on the set V_m of the vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$, we introduce *the topological Laplacian of level m*, $\Delta_m^{\mathcal{T}}(f)$, as applied to f, as follows:

$$\forall\, X \in V_m \setminus \partial V_m : \quad \Delta_m^{\tau} f(X) = \sum_{Y \in V_m,\, Y \underset{m}{\sim} X} \left(f(Y) - f(X) \right) .$$

As a consequence, in the case of the Weierstrass function $\mathcal{W}$, we also have that

$$\forall\, X \in V_m \setminus \partial V_m : \quad \Delta_m^{\tau} \mathcal{W}(X) = \sum_{Y \in V_m,\, Y \underset{m}{\sim} X} \left(\mathcal{W}(Y) - \mathcal{W}(X) \right) .$$

Here, ∂V_m consists of the two (identified) endpoints of V_m .

Note that we are excluding the case when $m = 0$ here, because $V_0 = \partial V_0$.

Proposition 5.4 (Topological Laplacian of the Weierstrass Function at Vertices of a Prefractal Graph Approximation).

For any $m \in \mathbb{N}^\star$, any integer k in $\left\{0, \ldots, N_b^m - 1\right\}$, and any j in $\{1, \ldots, N_b - 2\}$, we have that

$$\Delta_m^{\tau} \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m} \right) =$$
$$= \mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m} \right) + \mathcal{W}\left(\frac{k\,(N_b-1)+j-1}{(N_b-1)\,N_b^m} \right) - 2\,\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m} \right) .$$

Property 5.5 (Sign of the Topological Laplacian of Level $m \in \mathbb{N}^\star$).

i. For any positive integer m, and any k in $\left\{0, \ldots, N_b^m - 1\right\}$, we have that, for the initial vertex of a polygon $\mathcal{P}_m$

$$\mathcal{W}\left(\frac{k\,(N_b-1)-1}{(N_b-1)\,N_b^m} \right) < \mathcal{W}\left(\frac{k\,(N_b-1)}{(N_b-1)\,N_b^m} \right) ,$$

along with

$$\mathcal{W}\left(\frac{k\,(N_b-1)+1}{(N_b-1)\,N_b^m} \right) < \mathcal{W}\left(\frac{k\,(N_b-1)}{(N_b-1)\,N_b^m} \right) ,$$

which enables us to deduce that

$$\Delta_m^{\tau} \mathcal{W}\left(\frac{k\,(N_b-1)}{(N_b-1)\,N_b^m} \right) < 0 .$$

ii. When $N_b < 7$, for any positive integer m, any k in $\left\{0, \ldots, N_b^m - 1\right\}$, and any j in $\{1, \ldots, N_b - 2\}$, one has:

⇝ *For the left-side vertices in Definition 2.7, on page 16, distinct from the initial one, of a polygon $\mathcal{P}_{m,k}$, we have that*

$$\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m} \right) < \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m} \right) ,$$

along with

$$\left|\mathcal{W}\left(\frac{k\,(N_b-1)+j-1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right| > $$
$$> \left|\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right| \,,$$

which enables us to obtain that

$$\Delta_m^{\tau}\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) > 0\,.$$

⇝ *For the right-side vertices in Definition 2.7, on page 16, distinct from the last one, of a polygon $\mathcal{P}_{m,k}$, we have that*

$$\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) > \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\,,$$

along with

$$\left|\mathcal{W}\left(\frac{k\,(N_b-1)+j-1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right| < $$
$$< \left|\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right| \,,$$

which enables us to obtain that

$$\Delta_m^{\tau}\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) > 0\,.$$

iii. *When $N_b \geqslant 7$, for any $m \in \mathbb{N}^\star$, any integer k in $\{0,\ldots,N_b^m-1\}$, and any j in $\{1,\ldots,N_b-2\}$ such that*

$$0 < j \leqslant \frac{N_b-3}{4} \quad or \quad \frac{3\,N_b-1}{4} \leqslant j < N_b-1\,,$$

we respectively have that:

⇝ *For the left-side vertices in Definition 2.7, on page 16, distinct from the initial one, of a polygon $\mathcal{P}_{m,k}$,*

$$\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) < \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\,,$$

along with

$$\left|\mathcal{W}\left(\frac{k\,(N_b-1)+j-1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right| < $$
$$< \left|\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right| \,,$$

which yields

$$\Delta_m^{\tau}\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) < 0\,.$$

⇝ *For the right-side vertices in Definition 2.7, on page 16, distinct from the last one, of a polygon $\mathcal{P}_{m,k}$,*

$$\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right) > \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right),$$

with

$$\left|\mathcal{W}\left(\frac{k\,(N_b-1)+j-1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right| > $$
$$> \left|\mathcal{W}\left(\frac{k\,(N_b-1)+j+1}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right|,$$

which enables us to obtain that

$$\Delta_m^\tau \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) > 0\,.$$

Proof. This is an immediate consequence of the proof of Property 2.18 above, on page 21, given in [DL22].

□

Definition 5.8 (m^{th}**-Level Discrete Hessian**).

Given $m \in \mathbb{N}^\star$, any k in $\left\{0,\ldots,N_b^m-1\right\}$, and any j in $\{1,\ldots,N_b-2\}$, we define the m^{th}*-level discrete Hessian* $\mathcal{H}_m$ as follows:

$$\mathcal{H}_m\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) = \Delta_m^\tau \mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right).$$

The vertex

$$\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right)\right)$$

is said to be *nondegenerate*, with respect to $\mathcal{H}_m$, if

$$\mathcal{H}_m\left(\frac{k\,(N_b-1)+j}{(N_b-1)\,N_b^m}\right) \neq 0\,.$$

Property 5.6 (**Absence of Degenerate Points for the Sequence of Discrete Hessians**).

Given $m \in \mathbb{N}^\star$, any integer k in $\left\{0,\ldots,N_b^m-1\right\}$, and any j in $\{1,\ldots,N_b-2\}$, the m^{th}-level discrete Hessian $\mathcal{H}_m$ *introduced in Definition 5.8, on page 69 just above, never vanishes at the critical points of f.*

Proof. This is a direct consequence of Proposition 5.4, on page 67.

□

Definition 5.9 (Fractal Morse Function).

A function f defined on the Weierstrass Curve $\Gamma_{\mathcal{W}}$ will be said to be *a fractal Morse function* it its critical points are nondegenerate; i.e., if, for any $m \in \mathbb{N}^\star$, its discrete Hessian $\mathcal{H}_m$ (see Definition 5.8, on page 69), never vanishes at the critical points of f.

Remark 5.4 (**The Weierstrass Function Viewed as the Identity Function on** $\Gamma_{\mathcal{W}}$).

As in [DL24c], we set, for any real number t in $[0,1]$,

$$\gamma_{\mathcal{W}}(t) = (t, \mathcal{W}(t)) \ .$$

We then obtain the identity function on the Weierstrass Curve $\Gamma_{\mathcal{W}}$ as $\mathbb{1}_{\Gamma_{\mathcal{W}}}$. In this manner, the Weierstrass function can be viewed as the identity map on $\Gamma_{\mathcal{W}}$.

Property 5.7 (The Weierstrass Function Viewed as a Fractal Morse Function).

Since the discrete Hessian introduced in Definition 5.8, on page 69, never vanishes, the Weierstrass function $\mathcal{W}$ is a fractal Morse function, in the sense of Definition 5.9, on page 70 just above.

Proof. This is a direct consequence of Property 5.6, on page 69, according to which the Weierstrass function $\mathcal{W}$ does not have any degenerate point.

□

Definition 5.10 (m^{th}-Level Fractal Morse Index).

Given $m \in \mathbb{N}^\star$, and any j in $\{1, \ldots, \#V_m - 1\}$, the m^{th}*–level fractal Morse index* $\iota_{j,m}$ of the vertex $M_{j,m}$ is defined as follows:

$$\iota_{j,m} = \begin{cases} 1\,, & \text{if} \quad \mathcal{H}_m\left(\dfrac{j}{(N_b-1)\,N_b^m}\right) < 0\ , \\ 0\,, & \text{if} \quad \mathcal{H}_m\left(\dfrac{j}{(N_b-1)\,N_b^m}\right) > 0\,. \end{cases}$$

Remark 5.5. An overview of the values of the indexes for the different types of vertices involved is given in Table 1, on page 71.

Remark 5.6. One should note that, as in the classical Morse theory, the index of a nondegenerate critical point is equal to the dimension of the largest subspace of what plays the role of a *tangent space*

Vertex	**Junction point** (between consecutive polygons)	**Bottom point**	**Plain interior point (obtuse angle)**
Index	1	0	0

Vertex	**Left-side acute corner**	**Right-side acute corner**
Index	1	0

Table 1: An overview of the values of the indexes for the different types of vertices involved.

between two points, i.e., in our context, the edge that connects them (a line segment), where the Hessian is negative definite. It thus takes the value 1 at local maxima, and zero at local minima. It also takes the value zero at the right-side vertices with reentrant interior angles provided in Property 2.18, on page 21. This specific configuration corresponds, in a sense, to a sign change in the *curvature*.

Definition 5.11 (**Absolute Height Sequence of the Weierstrass Curve**).

We define *the absolute heights sequence* of the Weierstrass function $\mathcal{W}$ – or, equivalently, of the Weierstrass IFD – as the sequence of positive numbers

$$\left(Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m \in \mathbb{N}} ,$$

where, for any $m \in \mathbb{N}$,

$$Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right) = \left\{\mathcal{W}\left(\frac{j}{N_b^m}\right) + \frac{1}{1-\lambda} \, , \, 0 \leqslant j \leqslant \#V_m - 1\right\} .$$

Remark 5.7. The fact that, for any $m \in \mathbb{N}$ and any j in $\{0, \ldots, \#V_m - 1\}$, the value $Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)$ is positive simply comes from the fact that the minimum value of the Weierstrass function on $[0,1]$ is equal to $m_{\mathcal{W}} = -\dfrac{1}{1-\lambda}$; see Notation 13, on page 28 in Section 3.

Property 5.8 (**Fractal Morse Height Increasing Reordered Sequence**).

Given $m \in \mathbb{N}$, the set $Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)$ introduced in Definition 5.11, on page 71, admits an associated increasing reordered set.

Proof. This simply comes from the fact that, for any $m \in \mathbb{N}$, the set $Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)$ is a finite set of positive numbers, therefore allowing us to reorder the points of $Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)$ in increasing order. □

Definition 5.12 (Fractal Morse Height Increasing Reordered Sequence).

We define *the fractal Morse height reordered sequence* of the Weierstrass function $\mathcal{W}$ – or, equivalently, of the Weierstrass IFD – as the sequence of positive increasing reordered numbers $\left(Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\,\in\,\mathbb{N}}$ where, for any $m \in \mathbb{N}$,

$$Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_m}\right)$$

is the increasing reordered set associated to the finite set $Abs_{\mathcal{H}}\left(\Gamma_{\mathcal{W}_m}\right)$.

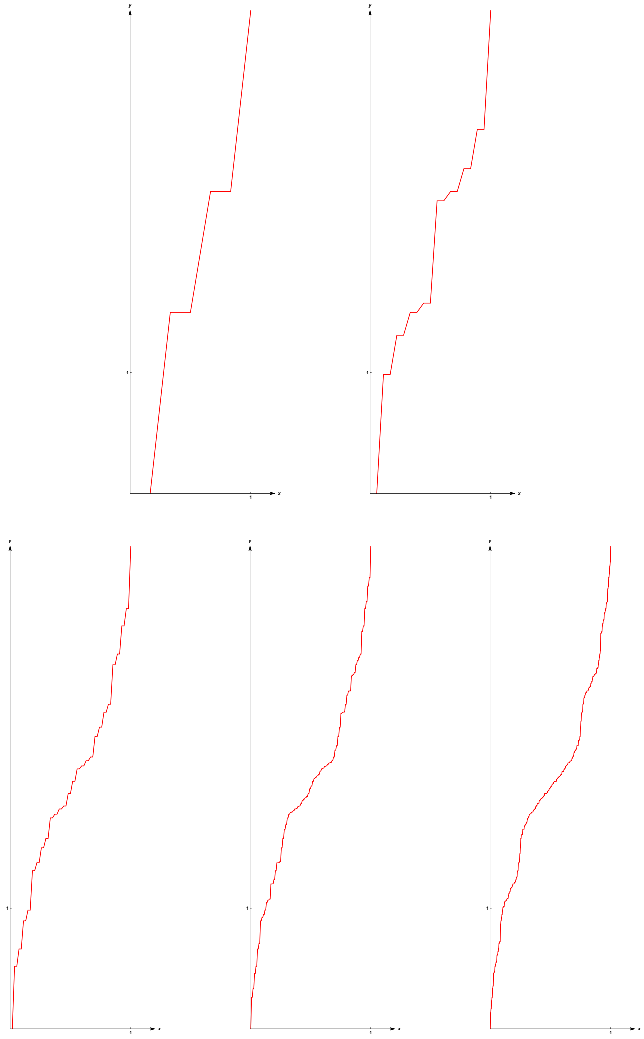


Figure 9: Plot of the fractal Morse heights $Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_1}\right)$, $Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_2}\right)$, $Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_3}\right)$, $Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_4}\right)$ and $Abs_{\mathcal{H}}^{reord}\left(\Gamma_{\mathcal{W}_5}\right)$, presented from top to bottom and from left to right.

6 Further Perspectives: The Weierstrass Curve as the Projection of a Vertical Comb

In this section, we place ourselves in the Euclidean plane of dimension 3, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by (x, y). The usual axes will be respectively referred to as $(x'x)$, $(y'y)$ and $(z'z)$.

Thanks to Property 2.1, on page 6, for any strictly positive integer m and any j in $\{0, \ldots, \#V_m\}$, we have that

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \lambda^m\,\mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} \lambda^k \cos\left(\frac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m}\right), \tag{31}$$

or, equivalently, expressed in terms of the k^{th} cohomology infinitesimal ε_k^k (see Definition 3.1, on page 23),

$$\frac{\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)}{(N_b-1)^{2-D_{\mathcal{W}}}} = \varepsilon_m^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} \varepsilon_k^{k\,(2-D_{\mathcal{W}})} \cos\left(\frac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m}\right). \tag{32}$$

If we consider the three-dimensional *vertical comb*, respectively comprised of the set of points

$$\left(\frac{j}{(N_b-1)\,N_b^m}, \frac{k}{m}, \varepsilon_k^{k\,(2-D_{\mathcal{W}})} \cos\left(\frac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m}\right)\right),$$

for $0 \leqslant j \leqslant \#V_m - 1$ and $0 \leqslant k \leqslant m-1$, along with the periodic set of points

$$\left(\frac{j}{(N_b-1)\,N_b^m}, 0, \varepsilon_m^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}\left(\frac{j}{N_b-1}\right)\right), \text{ for } 0 \leqslant j \leqslant \#V_m - 1\,,$$

we note that the expression in relation (32) just above corresponds to the superposition of the teeth of the comb; see Figures 10, on page 74 and Figure 11, on page 74.

We can thus envision the following three-dimensional *vertical comb*, respectively comprised of the set of *horizontal (rear) rows*,

$$\left(\frac{j}{(N_b-1)\,N_b^m}, \frac{k}{m}, \varepsilon_k^{k\,(2-D_{\mathcal{W}})}\right), \text{ for } 0 \leqslant k \leqslant m-1\,,$$

along with *the front row*

$$\left(\frac{j}{(N_b-1)\,N_b^m}, 0, \varepsilon_m^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}\left(\frac{j}{(N_b-1)}\right)\right), \text{ for } 0 \leqslant k \leqslant m-1\,,$$

and a moving observer who moves on this latter set of points (the front teeth of the comb): when it comes to a specific value of the integer k in $0, \ldots, m-1$, the observer looks at the associated comb under an angle of value

$$\vartheta_{j,k,m} = \frac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m} = \frac{2\,\pi\,j\,\varepsilon^{(m-k)\,(2-D_{\mathcal{W}})}}{(N_b-1)},$$

so that the expression in relation (32) just above corresponds to the successive superposition of the projections of the teeth of the combs; see Figure 12, on page 75, and Figure 13, on page 75. Namely, each horizontal row of the comb corresponds to a prefractal level set; see Figure 14, on page 76.

We note that

$$\vartheta_{j,k+1,m} = \frac{\vartheta_{j,k,m}}{\varepsilon} \, ,$$

which means that the projection angle increases as one gets closer to the front row.

Note also that in connection with the results of Section 5, the angle $\dfrac{2\,\pi\,N_b^k\,j}{(N_b-1)\,N_b^m}$ can be represented by a cohomological vertex integer $\ell_{j,k}$, as given in Definition 5.5, on page 63; see Figure 14, on page 76.

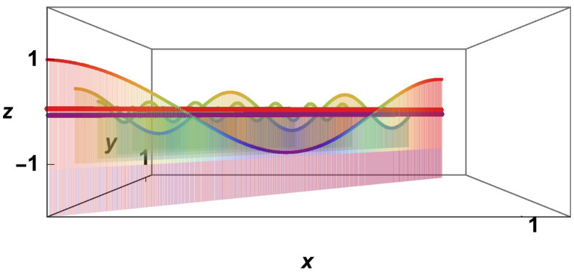


Figure 10: **The vertical comb, for $m = 5$ – front view.**

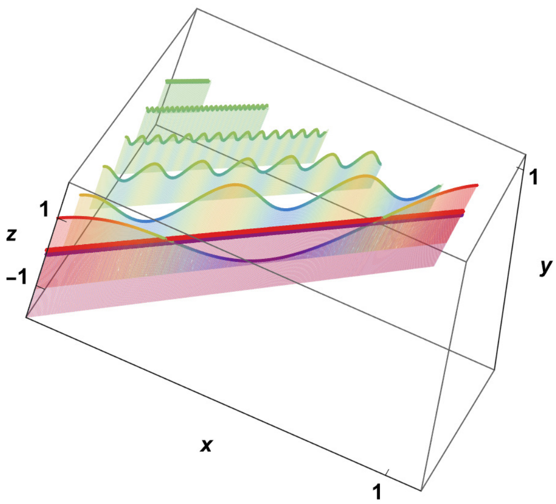


Figure 11: **The vertical comb, for $m = 5$ – side view.**

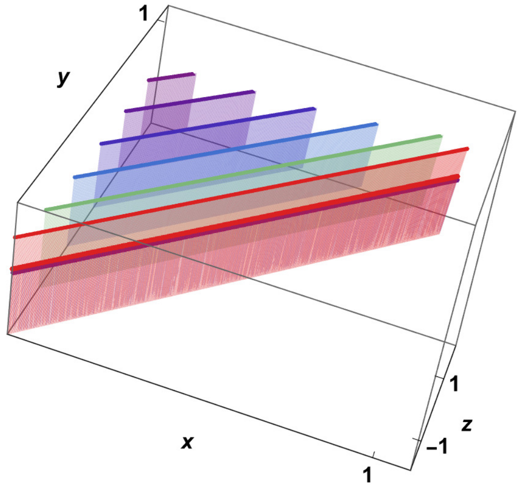


Figure 12: **The vertical comb, before projection, for $m = 5$ – side view.**

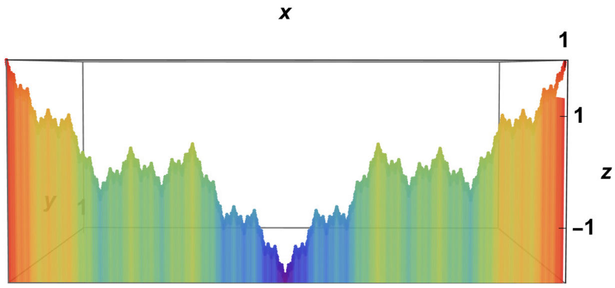


Figure 13: **The resulting vertical comb, after projection – face view.**

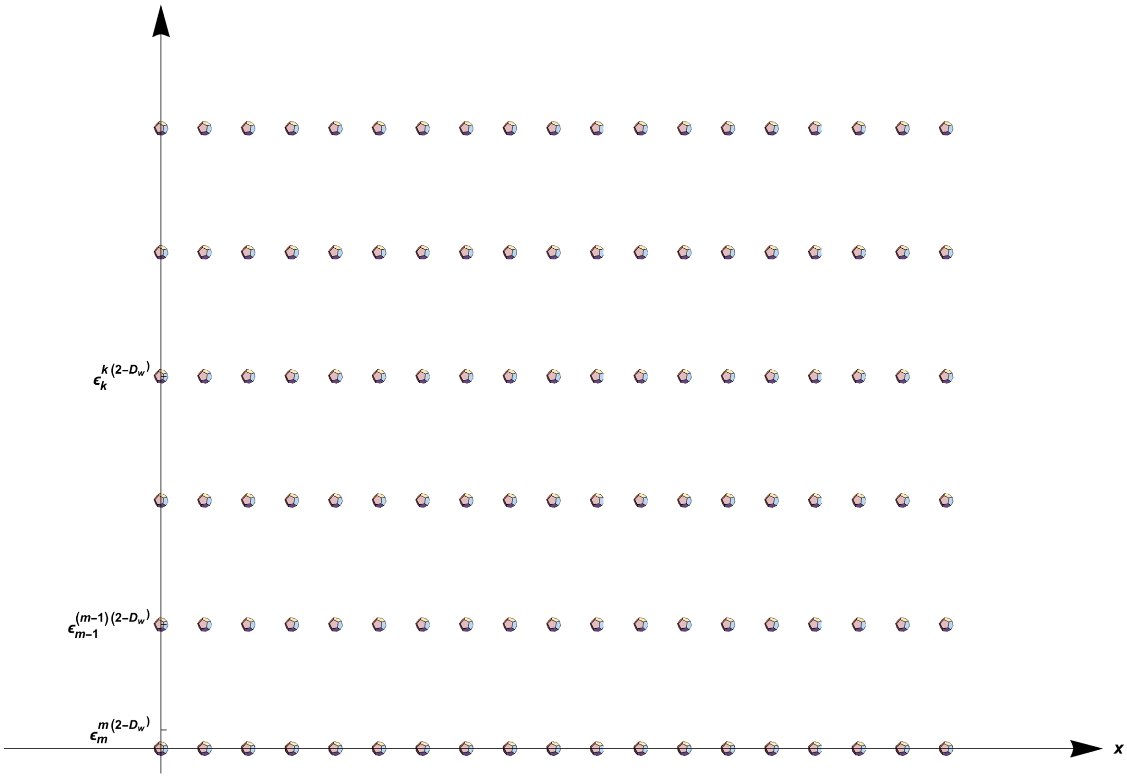


Figure 14: **The top view of the vertical comb. Each horizontal row coincides with a prefractal level set $\varepsilon_k^{k\,(2-D_{\mathcal{W}})}$, $0 \leqslant k \leqslant m$.**

In later work on the subject, it would be interesting to investigate the following open problem.

Problem

To what extent does the knowledge of the fractal Morse height reordered sequence given in Definition 5.12, on page 72, along with the fractal Morse indexes introduced in Definition 5.10, on page 70, the maximal real Complex Dimension introduced in Definition 5.4, on page 63, and the cohomological vertex integers given in Definition 5.5, on page 63, enable us to reconstruct the fractal (i.e., in our present setting, the Weierstrass Curve)?

7 Concluding Comments

All of the results obtained in this paper for the Weierstrass Curve (and for the associated Weierstrass IFD) are expected to extend naturally to a large class of fractal curves and their higher-dimensional analogues. Our methods and results should apply, in particular, to the Koch Curve (and the associated Koch IFD), for example, whose exact Complex Dimensions have been precisely determined by the authors in [DL23c], thereby extending the earlier results in [LP06].

We expect to consider these extensions to a more general setting of the notion of polyhedral measure, Sobolev and Besov spaces, along with the associated trace theorem, and of fractal Morse theory, in later work.

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