Success and failure of attempts to improve the accuracy of Raviart–Thomas mixed finite elements in curved domains

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Abstract

Several important problems in Mechanics can be efficiently solved using Raviart-Thomas mixed finite element methods. Whenever the domain of interest has a curved boundary the methods of this family for N-simplexes are the natural choice. But in this case the question arises on the best way to prescribe normal flux conditions across the boundary, if any. It is generally acknowledged that the normal component of the flux variable should preferably not take up corresponding prescribed values at nodes shifted to the boundary of the approximating polytope in the underlying normal direction. This is because an accuracy downgrade is to be expected, as shown in [1]. In that work an order-preserving technique was studied, based on a parametric version of these elements with curved simplexes. In this work an alternative with straight-edged triangles for two-dimensional problems is examined. The key feature of this approach is a Petrov-Galerkin formulation, in which the test-flux space is a little different from the shape-flux space. Based on previous author's experience with this technique, as applied to Lagrange finite elements, it would lead to an overall accuracy improvement here as well. The experimentation reported hereafter provides **examples** and **counterexamples** confirming or not such an expectation, depending on the unknown field of the mixed problem at hand.

Keywords: Accuracy improvement; Curved domains; Mixed finite elements; Neumann conditions; Raviart-Thomas; Straight-edged triangles.

1 Introduction

This work deals with a specific type of Petrov-Galerkin formulation, designed to preserve the order and/or to improve the accuracy of finite element methods to solve boundary value problems posed in domains having a smooth curved boundary, on which degrees of freedom (DOFs) are prescribed. In previous work the author applied it to conforming Lagrange and Hermite finite element methods to solve both two- and three-dimensional second and fourth order elliptic equations, in the aim of preserving their order higher than one in the inherent energy norm, with straight-edged triangles or tetrahedra. More recently it was shown that this technique has the very same effect, as applied to nonconforming finite element methods, even those having boundary-prescribed DOFs different from values at vertexes or function values at other points of an edge or a face of the elements. For all these contributions the author refers to [7] and references therein. Although this approach can be adopted for any type of element geometry, it appears most effective as combined to N-simplexes. The technique under consideration is simple to implement as it deals only with polynomial algebra. Moreover it definitively eliminates the need for curved elements, and consequently the use of non affine mappings.

Here we examine possible advantages of this formulation, in the framework of the solution of second-order boundary value problems in a smooth plane domain, with Neumann conditions prescribed on a boundary portion, by Raviart-Thomas mixed finite elements for N-simplexes. We recall that in

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this case the normal components of the flux variable in the underlying mixed formulation are prescribed. Our premise is that the approximate flux variable should preferably not take up corresponding prescribed values at nodes shifted to the boundary of the approximating polytope. Moreover, even when there is no node shift, in order to improve accuracy, presumably the prescribed flux on the boundary should be normal to the true boundary and not to the one of the approximating polytope. Actually a technique following these principles was studied in [1] for Raviart-Thomas mixed elements, based on a parametric version of theirs with curved simplexes. In that work the authors showed that globally their approach does bring about better accuracy. In contrast, in this paper the aforementioned Petrov-Galerkin formulation with straight-edged triangles is employed, to tackle this problem. Our experiments clearly advocate in favor of this approach in some cases, while in some others accuracy does not improve, even if in no case it really downgrades either.

2 The model problem

Our numerical study in connection with the Raviart-Thomas mixed finite elements, will be carried out for the Poisson first order system. Before recalling it, we observe that accuracy enhancement is strongly dependent on the solution regularity. Hence although our technique can be applied to mixed Dirichlet-Neumann boundary conditions in very general situations, in order to make sure that the solution of the problem at hand will has the required regularity for its theoretical order to prevail, we consider the following model equation.

Let Ω be a two-dimensional smooth domain and Γ be its boundary, consisting of two non intersecting portions Γ_0 and Γ_1 with $meas(\Gamma_1) > 0$. We denote by $\mathbf{n}$ the outer normal vector to Γ and by V, either the space $L^2(\Omega)$ if $\Gamma_0 \neq \emptyset$, or its subspace $L^2_0(\Omega)$ consisting of those functions g such that $\int_\Omega g = 0$ otherwise. Now given $f \in V$ the problem to solve is,

$$\left\{ \begin{array}{ll} \text{Find } (\mathbf{p}; u) \text{ with } \int_\Omega u = 0 \text{ if } \Gamma_0 = \emptyset, & \text{such that} \\ -\nabla \cdot \mathbf{p} = f \text{ and } \mathbf{p} - \nabla u = \mathbf{0} & \text{in } \Omega; \\ u = 0 \text{ on } \Gamma_0 \text{ and } \mathbf{p} \cdot \mathbf{n} = 0 \text{ on } \Gamma_1 & . \end{array} \right. \tag{1}$$

Referring to [4], let $\mathbf{Q}$ be the subspace of $\mathbf{H}(div, \Omega)$ of those fields $\mathbf{q}$ such that $\mathbf{q} \cdot \mathbf{n} = 0$ on Γ_1. Then denoting by $(\cdot, \cdot)$ the standard inner product of $L^2(\Omega)$, problem (1) can be recast in the following equivalent variational form:

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{p}; u) \in \mathbf{Q} \times V \text{ such that,} \\ -(\nabla \cdot \mathbf{p}, v) = (f, v) \; \forall v \in V; \\ (\mathbf{p}, \mathbf{q}) + (u, \nabla \cdot \mathbf{q}) = 0 \; \forall \mathbf{q} \in \mathbf{Q}. \end{array} \right. \tag{2}$$

We shall deal with the Raviart-Thomas mixed finite element method for triangles [4] known as RT_k, separately for $k = 0$ and $k > 0$. This is because, in the former case, we will also study the effect of different ways to prescribe normal fluxes across Γ_1, in connection with the Hermite analog of RT_0 introduced in [5].

We recall that, provided u belongs to $H^{k+2}(\Omega)$ (see e.g. [2]), this method is of order $k + 1$ in the usual norm $||| \cdot |||$ of the space $\mathbf{H}(div, \Omega) \times L^2(\Omega)$. Denoting the standard norm of $L^2(\Omega)$ by $\| \cdot \|$, the norm $||| \cdot |||$ is defined by $|||(\mathbf{q}; v)||| = \left[\|\mathbf{q}\|^2 + \|\nabla \cdot \mathbf{q}\|^2 + \|v\|^2 \right]^{1/2}$. From the assumption that Γ_0 and Γ_1 have no common points, the above regularity of u holds if Γ is of the C^k-class and $f \in H^k(\Omega)$.

3 Method description

To begin with we give some notations related to the finite- element meshes.

3.1 Meshes and related sets

Let $\{\mathcal{T}_h\}_h$ be a regular family of meshes in the sense of [2], consisting of straight-edged triangles satisfying the usual compatibility conditions for the finite element method, h being the maximum edge length of all the triangles in the mesh $\mathcal{T}_h$. Every element of this mesh is considered to be a closed set and $\mathcal{T}_h$ is assumed to fit Ω in such a way that all the vertexes of the polygon Ω_h lie on Γ, where Ω_h is the interior of $\cup_{T \in \mathcal{T}_h} T$. The boundary of Ω_h is denoted by Γ_h and $\Gamma_{1,h}$ is the portion of Γ_h having a non empty intersection with Γ_1.We assume that any element in $\mathcal{T}_h$ has at most one edge contained in Γ_h.
Let $\mathcal{T}_{1,h}$ be the subset of $\mathcal{T}_h$ consisting of triangles T having one edge on $\Gamma_{1,h}$, say e_T. Referring to Figure 1, for every $T \in \mathcal{T}_{1,h}$ we denote by O_T the vertex of T not belonging to Γ_1, and denote by $\mathbf{n}_T$ the unit outer normal vector to Γ_h (i.e. to $\Gamma_{1,h}$); we also define Δ_T to be the closed set delimited by Γ_1 and the edge e_T of T whose end-points belong to Γ_1.

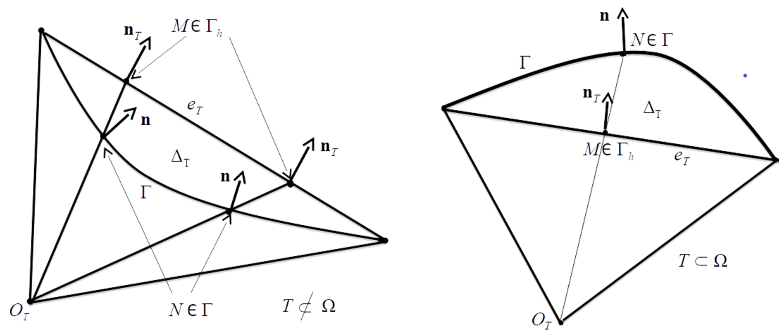

Figure 1: Normal flux degrees of freedom on Γ_h and Γ for RT_1 (left) and RT_0 (right)

3.2 Petrov-Galerkin formulation

Let us introduce three spaces $\mathbf{P}_h^k$, $\mathbf{Q}_h^k$ and V_h^k associated with $\mathcal{T}_h$.
• V_h^k is the space consisting of functions $v \in L^2(\Omega_h)$, whose restriction to every $T \in \mathcal{T}_h$ is a polynomial of degree less than or equal to k for $k \geq 0$, that vanish at a given point of $\Omega_h \cap \Omega$ in case $\Gamma_0 = \emptyset$.
• $\mathbf{Q}_h^k$ is the space of fields in $\mathbf{q} \in \mathbf{H}(div, \Omega_h)$, whose restriction to every $T \in \mathcal{T}_h$ belongs to the standard Raviart-Thomas space RT_k [4], fulfilling $\mathbf{q} \cdot \mathbf{n}_T = 0$ along e_T for every $T \in \mathcal{T}_{1,h}$.
• $\mathbf{P}_h^k$ is the space of fields $\mathbf{r}$ satisfying the following conditions:
1. $\mathbf{r} \in \mathbf{H}(div, \Omega_h)$;
2. The restriction of any $\mathbf{r}$ to every $T \in \mathcal{T}_h$ is a field belonging to the standard Raviart-Thomas space RT_k [4] of degree less than or equal to k;
3. $\mathbf{r}$ is also defined in $\bar{\Omega} \setminus \bar{\Omega}_h$ in such a way that, $\forall \Delta_T$ which is not a subset of $T \in \mathcal{T}_{1,h}$, the expression of $\mathbf{r}$ in T extends to points in $\Delta_T \setminus T$;
4. $\forall T \in \mathcal{T}_{1,h}$, $\mathbf{r}(N) \cdot \mathbf{n} = 0$ for all N which is the nearest intersections with Γ of the half-line with origin at the vertex O_T of T and $k+1$ points M of the edge e_T chosen to be the Gauss-Legendre points if k is odd or zero, or the Gauss-Lobatto points otherwise [3] (see Figure 1 for $k = 1$ and $k = 0$) [1].

$\mathbf{P}_h^k$ is a non empty finite-dimensional space, at least if h is sufficiently small. This result can be established following the main lines of [7] and references therein.

Now let us set the problem associated with spaces $\mathbf{P}_h^h$, $\mathbf{Q}_h^k$ and V_h^k, whose solution is an approximation of the solution of (2). Extending f in $\Omega_h \setminus \Omega$ in such a way that such an extension belongs to $H^{k-1}(\Omega \cup \Omega_h)$. Still denoting the resulting function by f, and defining $(\cdot, \cdot)_h$ to be the standard inner

[1] The construction of the nodes associated with $\mathbf{P}_h^k$ located on Γ_1 advocated in item 4. is not mandatory. It is just a balanced choice, but there are many other possibilities to choose such $k+1$ nodes for our Petrov-Galerkin formulation to work fine.

product of $L^2(\Omega_h)$, the problem to solve is,

$$\begin{cases} \text{Find } (\mathbf{p}_h; u_h) \in \mathbf{P}_h^k \times V_h^k \text{ such that} \\ -(\nabla \cdot \mathbf{p}_h, v)_h = (f, v)_h \ \forall v \in V_h^k; \\ (\mathbf{p}_h, \mathbf{q})_h + (u_h, \nabla \cdot \mathbf{q})_h = 0 \ \forall \mathbf{q} \in \mathbf{Q}_h^k. \end{cases} \quad (3)$$

Using arguments similar to those exploited by the author in [6], provided h is sufficiently small we can prove that problem (3) has a unique solution, and moreover that it is uniformly stable. This opens wide doors towards proving that this method converges with order $k+1$ in the norm of $\mathbf{H}(div, \Omega_h) \times L^2(\Omega_h)$, as h goes to zero.

3.3 Petrov-Galerkin formulation for the Hermite analog of RT_0

Referring to [5] for a Hermite version of RT_0, we briefly recall it below as applied to problem (2). Denoting by $P_2(D)$ the space of polynomials of degree less than or equal to 2 in a bounded set D of $\Re^2$, let U_h^H and V_h^H be the finite-dimensional spaces defined as follows:

- $V_h^H := \{w |\ w \in L^2(\Omega_h),\ w_{|T} \in P_2(T)\ \forall T \in \mathcal{T}_h \text{ and } \nabla w \in \mathbf{Q}_h^0\}$;
- $U_h^H := \{w |\ u \in L^2(\Omega_h),\ w_{|T} \in P_2(T)\ \forall T \in \mathcal{T}_h \text{ and } \nabla w \in \mathbf{P}_h^0\}$.

Then we approximate u by $u_h^H \in U_h^H$ and $\mathbf{p}$ by ∇u_h^H, where u_h^H is the unique function fulfilling:

$$\begin{cases} a_h(u_h^H, w) = (f, w)_h \ \forall w \in V_h^H, \text{ with} \\ a_h(w, v) := (\nabla w, \nabla v)_h - (\Delta w, v)_h + (w, \Delta v)_h \ \forall w \in U_h^H \text{ and } \forall v \in V_h^H. \end{cases} \quad (4)$$

The main difference between (4) and (3) for $k=0$ is the fact that u is approximated in a space of incomplete quadratic functions in each triangle, though containing all linear functions. As pointed out in [5], this actually leads to second order approximations of u in $L^2(\Omega)$, whenever Ω is a convex polygon, while maintaining first order approximations of $\mathbf{p}$ by ∇u_h^H in $\mathbf{H}(div, \Omega)$. A similar behavior is expected in the case of smooth domains, which will be confirmed hereafter.

4 Numerical experiments

Let us now go into the main purpose of this paper, that is, numerical experimentation of the methods defined by (3) and (4).

4.1 Solution of a test-problem with RT_1

We checked the performance of (3) for $k=1$, by solving a test-problem in an annulus with inner radius $r_i = 1/2$ and outer radius $r_e = 1$, for an exact solution given by $u(x,y) = (r^2 - 2rr_i + 2r_ir_e - r_e^2)/2$, where $r = \sqrt{x^2+y^2}$. This function satisfies $u=0$ on Γ_0, namely, the circle given by $r = r_e$ and $\partial u/\partial r = 0$ on Γ_1, namely, the circle given by $r = r_i$. We computed with a quasi-uniform family of meshes for a quarter annulus, constructed for a quarter unit disk with $2L^2$ triangles for $L = 2^m$, by removing the $L^2/2$ triangles fully contained in the disk with radius $1/2$. For simplicity we set $h = 1/L$. In the upper part of Table 1 we supply the approximation errors of u, $\mathbf{p}$ and $\nabla \cdot \mathbf{p}$ measured in the norm of $L^2(\Omega_h)$, for formulation (3) with $k=1$, taking $m = 2,3,4,5,6$. In order to highlight eventual advantages of the Petrov-Galerkin approximation (3) over the standard Galerkin approach, consisting of replacing $\mathbf{P}_h^k$ by $\mathbf{Q}_h^k$ in (3), we present in the lower part of Table 1, the same kind of errors for the solution $(\bar{\mathbf{p}}_h; \bar{u}_h)$ obtained by the latter formulation.

h	$\longrightarrow$	$1/4$	$1/8$	$1/16$	$1/32$	$1/64$
$\| u_h - u \|_h$	$\longrightarrow$	0.19094E-2	0.47687E-3	0.11917E-3	0.29791E-4	0.74483E-5
$\| \mathbf{p}_h - \mathbf{p} \|_h$	$\longrightarrow$	0.19268E-2	0.48957E-3	0.12299E-3	0.30790E-4	0.77001E-5
$\| \nabla \cdot (\mathbf{p}_h - \mathbf{p}) \|_h$	$\longrightarrow$	0.41343E-2	0.10625E-2	0.26788E-3	0.67119E-4	0.16789E-4
$\| \bar{u}_h - u \|_h$	$\longrightarrow$	0.16376E-2	0.40823E-3	0.10198E-3	0.25491E-4	0.63733E-5
$\| \bar{\mathbf{p}}_h - \mathbf{p} \|_h$	$\longrightarrow$	0.38339E-2	0.96853E-3	0.24283E-3	0.60752E-4	0.15191E-4
$\| \nabla \cdot (\bar{\mathbf{p}}_h - \mathbf{p}) \|_h$	$\longrightarrow$	0.41343E-2	0.10625E-2	0.26788E-3	0.67119E-4	0.16789E-4

Table 1: L^2-errors for the Petrov-Galerkin and the Galerkin formulation of the RT_1 method

4.2 Solution of a test-problem with RT_0 and its Hermite analog

In this subsection we present the results obtained for a test-problem in the ellipse centered at the origin with half-axes equal to one and $e = 1/2$. We take an exact solution u satisfying only homogeneous Neumann boundary conditions, given by $u(x, y) = (e^2x^2 + e^4y^2 - x^4/2 - e^4y^4/2 - e^2x^2y^2)/2$. The same kind of meshes as above for the quarter unit disk are employed for a quarter ellipse with $2L^2$ triangles, for $L = 2^m$, this time with $m = 3, 4, 5, 6, 7$. Tables 2,3,4 display respectively the errors of u, $\mathbf{p}$ and $\nabla \cdot \mathbf{p}$, when $(\mathbf{p}; u)$ is approximated by:

- The pair $(\mathbf{p}_h; u_h)$ that solves (3) with $k = 0$;
- The pair $(\bar{\mathbf{p}}_h; \bar{u}_h)$ that solves the Galerkin analog of (3) with $k = 0$, by replacing $\mathbf{P}_h^0$ with $\mathbf{Q}_h^0$;
- The pair $(\nabla u_h^H; u_h^H)$, where u_h^H solves (4);
- The pair $(\nabla \bar{u}_h^H; \bar{u}_h^H)$, where $\bar{u}_h^H$ solves the Galerkin analog of (4), by replacing U_h^H with V_h^H.

h	$\longrightarrow$	$1/8$	$1/16$	$1/32$	$1/64$	$1/128$
$\| u_h - u \|_h$	$\longrightarrow$	0.42528E-3	0.17986E-3	0.85079E-4	0.41881E-4	0.20856E-4
$\| \bar{u}_h - u \|_h$	$\longrightarrow$	0.53435E-3	0.20712E-3	0.90368E-4	0.42781E-4	0.21005E-4
$\| u_h^H - u \|_h$	$\longrightarrow$	0.18500E-3	0.48191E-4	0.12493E-4	0.32565E-5	0.92059E-6
$\| \bar{u}_h^H - u \|_h$	$\longrightarrow$	0.32559E-3	0.99666E-4	0.29191E-4	0.83411E-5	0.24152E-5

Table 2: L^2-errors of u for four versions of the RT_0 method

h	$\longrightarrow$	$1/8$	$1/16$	$1/32$	$1/64$	$1/128$
$\| \mathbf{p}_h - \mathbf{p} \|_h$	$\longrightarrow$	0.28254E-2	0.14254E-2	0.71504E-3	0.35800E-3	0.18127E-3
$\| \bar{\mathbf{p}}_h - \mathbf{p} \|_h$	$\longrightarrow$	0.28596E-2	0.14308E-2	0.71581E-3	0.35811E-3	0.18127E-3
$\| \nabla(u_h^H - u) \|_h$	$\longrightarrow$	0.28256E-2	0.14255E-2	0.71505E-3	0.35800E-3	0.18127E-3
$\| \nabla(\bar{u}_h^H - u) \|_h$	$\longrightarrow$	0.28598E-2	0.14308E-2	0.71581E-3	0.35811E-3	0.18127E-3

Table 3: L^2-errors of $\mathbf{p} = \nabla u$ for four versions of the RT_0-method

h	$\longrightarrow$	$1/8$	$1/16$	$1/32$	$1/64$	$1/128$
$\parallel \nabla \cdot (\mathbf{p}_h - \mathbf{p}) \parallel_h$	$\longrightarrow$	0.17060E-1	0.85475E-2	0.42759E-2	0.21382E-2	0.10691E-2
$\parallel \nabla \cdot (\bar{\mathbf{p}}_h - \mathbf{p}) \parallel_h$	$\longrightarrow$	0.17060E-1	0.85475E-2	0.42759E-2	0.21382E-2	0.10691E-2
$\parallel \Delta(u_h^H - u) \parallel_h$	$\longrightarrow$	0.17060E-1	0.85475E-2	0.42759E-2	0.21382E-2	0.10691E-2
$\parallel \Delta(\bar{u}_h^H - u) \parallel_h$	$\longrightarrow$	0.17060E-1	0.85475E-2	0.42759E-2	0.21382E-2	0.10691E-2

Table 4: L^2-errors of $\nabla \cdot \mathbf{p} = \Delta u$ for four versions of the RT_0-method

5 Conclusions

From the results of the previous section, we can draw the following conclusions, preceded by **EX** for success or by **CO** for failure of the improvement attempt.

EX - The order in the norm $|||\cdot|||$ of the RT_1 method in the Petrov-Galerkin formulation (3) is two [2].
EX - In terms of the field $\mathbf{p}$, the RT_1 method works better in the Petrov-Galerkin formulation than in the Galerkin formulation.
CO - The approximations of $\nabla \cdot \mathbf{p}$ are the same for both methods above.
CO - As for the function u, the Petrov-Galerkin formulation and the Galerkin formulation for the RT_1 method are fairly equivalent, but the latter is a little more accurate.
EX - The Petrov-Galerkin formulation for the original RT_0 method is slightly more accurate in terms of the function u, as compared to the Galerkin formulation.
EX - The Hermite analog of the RT_0 method in both formulations is much more accurate than the original RT_0 method, as far as the function u is concerned.
CO - The four versions of the RT_0 method produce practically the same results for the field $\mathbf{p}$.
EX - The convergence rate of the solution u_h^H of (4) in the L^2-norm is almost equal to 2, while it downgrades to a value close to $7/4$ for the corresponding Galerkin formulation.

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[2] It is interesting to note that this is also the case of the Galerkin formulation without parametric elements, since the node shifts from Γ to Γ_h are not enough to spoil the exact interpolation of linear fields with the remaining DOFs of RT_1.