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A CHARACTERIZATION OF STRICTLY UNIMODAL DISTRIBUTION FUNCTIONS BY THEIR CONCENTRATION FUNCTIONS[1]

Unimodal distribution functions

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The purpose of this article is to prove that a distribution function is strictly unimodal if and only if its concentration function is strictly unimodal.

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1. Introduction and main result

Let F be a (right continuous) distribution function on $\mathbb{R} = (-\infty, +\infty)$ and set $x' = \inf \{x : F(x) > 0\}$ and $x'' = \sup \{x : F(x) < 1\}$. F is said to be unimodal if and only if there exists at least one value $x = x^*$ such that F is convex on $(-\infty, x^*)$ and concave on $(x^*, +\infty)$; x^* is called a mode of F. In this case $F = \alpha H_{x^*} + (1-\alpha) A$, where $0 \leq \alpha \leq 1$ is the saltus of F at $x = x^*$, $H_{x^*}(x) = 0$ for $x < x^*$, $H_{x^*}(x) = 1$ for $x \geq x^*$, and A is an absolutely continuous unimodal distribution function with the same mode $x = x^*$; clearly A' has at most one relative strict maximum which is, if it exists, at $x = x^*$. A characterization of unimodal distribution functions by their characteristic functions was given by A. Ja. Hinčin (see, e.g., [2, p.92], Theorem 4.5.1): F is unimodal with mode $x^* = 0$ if and only if its characteristic function f has the form $f(t) = (1/t) \int_0^t g(u)\,du$, $t \in \mathbb{R}$, where g is a characteristic function.

Further let $Q_F(\ell) = \sup \{F(x+\ell) - F^-(x) : x \in \mathbb{R}\}$ for $\ell \geq 0$ and $Q_F(\ell) = 0$ otherwise, where F^- denotes the left limit of F, be the (Lévy) concentration function of F. Clearly Q_F is a distribution function. If there is no ambiguity we shall simply write Q instead of Q_F. It is known (see, e.g., [1, p.4-9] that Q is subadditive, $Q(0) = \sup \{F(x) - F^-(x) : x \in \mathbb{R}\}$, and that for every $\ell \geq 0$ there exists $x_\ell \in \mathbb{R}$ such that $Q(\ell) = F(x_\ell + \ell) - F^-(x_\ell)$. If $L = \sup \{\ell : Q(\ell) < 1\}$, then x_L is unique and $x' = x_L$, $x'' = x_L + L$.

Our purpose is to characterize unimodal distribution functions by their concentration functions. If F is unimodal, it is easily seen that Q is unimodal with unique mode $\ell^* = 0$. The converse of this assertion is unfortunately not true. Take, e.g.,

$$F(x) = \begin{cases} 0 & \text{for } x < 0 , \\ x/4 & \text{for } 0 \le x < 1 , \\ 1/4 & \text{for } 1 \le x < 2 , \\ (x-1)/4 & \text{for } 2 \le x < 4 , \\ (x+2)/8 & \text{for } 4 \le x < 6 , \\ 1 & \text{for } x \ge 1 . \end{cases}$$

Then

$$Q(\ell) = \begin{cases} 0 & \text{for } \ell < 0 , \\ \ell/4 & \text{for } 0 \le \ell < 2 , \\ (\ell+2)/8 & \text{for } 2 \le \ell < 6 , \\ 1 & \text{for } \ell \ge 6 . \end{cases}$$

This situation leads us to strenghten the definition of a unimodal distribution function. We shall say that F is _strictly unimodal_ if and only if F is unimodal with a mode at $x = x^*$, F is strictly convex on (x',x^*) , and F is strictly concave on (x^*,x'') . It follows immediately that $x = x^*$ is the unique mode of F . Let us remark that one or both of the sets (x',x^*) and (x^*,x'') may be empty. Since $F = \alpha H_{x^*} + (1-\alpha)A$, it follows that in this case A' has exactly one relative strict maximum at $x = x^*$. Moreover F is strictly unimodal with mode $x^* = 0$ if and only if its characteristic function f has the form $f(t) = (1/t) \int_0^t g(u)\,du$, $t \in \mathbb{R}$, where g is the characteristic function of a distribution function G which is strictly increasing on the interval $\{x{:}0<G(x)<1\}$.

The remainder of this paper is devoted to the proof of the following:

THEOREM. F _is strictly unimodal if and only if_ Q _is strictly unimodal_.

2. Auxiliary results

Let ϕ be a real-valued function defined on $\mathbb{R}$ and consider its four derived numbers at $x \in \mathbb{R}$:

$$D^+\phi(x) = \limsup_{h \downarrow 0} [\phi(x+h)-\phi(x)]/h ,$$

$$D_+\phi(x) = \liminf_{h \downarrow 0} [\phi(x+h)-\phi(x)]/h ,$$

$$D^-\phi(x) = \limsup_{h \downarrow 0} [\phi(x)-\phi(x-h)]/h ,$$

$$D_-\phi(x) = \liminf_{h \downarrow 0} [\phi(x)-\phi(x-h)]/h .$$

If $D^+\phi(x) = D_+\phi(x)$ we shall write $\Delta^+\phi(x)$ for this common value which represents the right derivative of ϕ at $x \in \mathbb{R}$, and similarly $\Delta^-\phi(x)$ for its left derivative at $x \in \mathbb{R}$.

LEMMA 1. For all $\ell > 0$ we have

$$\max(D_-F^-(x_\ell) , D_+F(x_\ell+\ell)) \le D_+Q(\ell) ,$$

$$\min(D_+F^-(x_\ell) , D_-F(x_\ell+\ell)) \ge D_-Q(\ell) .$$

The above inequalities hold also if D_- and D_+ are replaced by D^- and D^+ respectively.

PROOF. For $0 < a < b$ we have for all $x,y \in \mathbb{R}$

$$\frac{F(x+b)-F^-(x)-F(x_a+a)+F^-(x_a)}{b-a} \le \frac{Q(b)-Q(a)}{b-a} \le \frac{F(x_b+b)-F^-(x_b)-F(y+a)+F^-(y)}{b-a} .$$

Take first $x = x_a$, $y = x_b$, and then $x = x_a + a-b$, $y = x_b + b-a$. Lemma 1 follows after passage to the limit.

Let us remark that if $F'(x_\ell)$, $F'(x_\ell+\ell)$, and $Q'(\ell)$ exist, then Lemma 1 states that $Q'(\ell) = F'(x_\ell) = F'(x_\ell+\ell)$. Moreover if $F'(x_\ell)$ exists and

Q is concave on $(0,+\infty)$, then, by Lemma 1, $Q'(\ell)$ exists.

LEMMA 2. Suppose that Q is strictly unimodal. Then

(1) $$F(x_\ell+\ell+h) - F^-(x_\ell+h) < Q(\ell)$$

for all h, $0 < |h| \le \ell$, and for all ℓ such that $Q(\ell) < 1$.

PROOF. Since Q is strictly unimodal, we have

$$\tfrac{1}{2}[F(x+\ell+h)-F^-(x)] + \tfrac{1}{2}[F(y+\ell-h)-F^-(y)] \le [Q(\ell-h)-Q(\ell+h)]/2 < Q(\ell)$$

for all $x,y \in \mathbb{R}$ and $0 < |h| \le \ell$. Take $x = x_\ell$ and $y = x_\ell+h$. We get

$$\tfrac{1}{2}[F(x_\ell+\ell+h) - F^-(x_\ell+h)] + \tfrac{1}{2}[F(x_\ell+\ell) - F^-(x_\ell)] < Q(\ell) ,$$

and consequently we obtain (1).

LEMMA 3. Suppose that Q is strictly unimodal and let $\ell_1 \ge 0$ be such that $\frac{1}{2} \le Q(\ell_1) < 1$. Then x_ℓ is unique for all $\ell \in (\ell_1,L)$.

PROOF. Suppose by contraposition that $Q(\ell) = F(x^1_\ell+\ell) - F^-(x^1_\ell) = F(x^2_\ell+\ell) - F^-(x^2_\ell)$ and $x^1_\ell < x^2_\ell$. Then by Lemma 2 it follows that $x^2_\ell > x^1_\ell + \ell$ and hence $F(x^2_\ell+\ell) \ge 2Q(\ell)$. Since $\ell > \ell_1$ and since Q is strictly unimodal we get $F(x^2_\ell+\ell) > 1$ and we are led to a contradiction.

Let us remark that the proof of Lemma 3 also shows that x_{ℓ_1} is unique. Moreover if $Q(0) \ge \frac{1}{2}$, then x_ℓ is unique for all $0 < \ell < L$.

LEMMA 4. Suppose that Q is strictly unimodal and let $\ell_1 \ge 0$ be such that $\frac{1}{2} \le Q(\ell_1) < 1$. Then x_ℓ is continuous on (ℓ_1,L).

PROOF. Let $\ell \in (\ell_1,L)$ and take $\lambda_n \in (\ell_1,L)$ for all $n \ge 1$ such that $\lambda_n \to \ell$. Then $Q(\lambda_n) = F(x_{\lambda_n} + \lambda_n) - F^-(x_{\lambda_n}) \to Q(\ell)$. Since $\lambda_n \to \ell$, there is $0 < M_1 < +\infty$ such that $\lambda_n \le M_1$ for all $n \ge 1$. Let us now take $M_2 > 0$

such that $1-F(M_2) + F^-(-M_2) < \frac{1}{2}$. Then $x_{\lambda_n} \in K = [-M_1-M_2-1,\ M_1+M_2+1]$ for all $n \geq 1$.

Further, take a subsequence $x_{\lambda_{n_k}} \to \xi \in K$ such that either $F^-(x_{\lambda_{n_k}}) \to F^-(\xi)$ or $F^-(x_{\lambda_{n_k}}) \to F(\xi)$, and either $F(x_{\lambda_{n_k}} + \lambda_{n_k}) \to F^-(\xi+\ell)$ or $F(x_{\lambda_{n_k}} + \lambda_{n_k}) \to F(\xi+\ell)$. Therefore $F(x_{\lambda_{n_k}} + \lambda_{n_k}) - F^-(x_{\lambda_{n_k}}) \to Q(\ell)$, and we get $\frac{1}{2} \leq Q(\ell) = F(\xi+\ell) - F^-(\xi)$. By Lemma 3, we conclude that $\xi = x_\ell$.

Let us remark that the proof of Lemma 4 also shows that x_ℓ is right continuous at $\ell = \ell_1$. Moreover it is possible that $x_L = x_\ell = x_{\ell_1}$ for all $\ell \in (\ell_1, L)$. As an example take $F(x) = 0$ for $x < 0$, $F(x) = \sqrt{x}$ for $0 \leq x < 1$, and $F(x) = 1$ for $x \geq 1$.

LEMMA 5. Suppose that Q is strictly unimodal and let $\ell_1 \geq 0$ be such that $\frac{1}{2} \leq Q(\ell_1) < 1$. Then F is convex on $(-\infty, x_{\ell_1})$, strictly convex on (x_L, x_{ℓ_1}), concave on $(x_{\ell_1}+\ell_1, +\infty)$, and strictly concave on $(x_{\ell_1}+\ell_1, x_L+L)$.

PROOF. By Lemma 4 there is for $x \in (x_L, x_{\ell_1})$ an $\ell \in (\ell_1, L)$ such that $x = x_\ell$. Now since Q is strictly unimodal we have for $0 \leq a < \ell < c$

$$\frac{\ell-a}{c-a}\left[F(u+c) - F^-(u)\right] + \frac{c-\ell}{c-a}\left[F(v+a) - F^-(v)\right] < Q(\ell) \tag{2}$$

for all $u, v \in \mathbb{R}$. Let us note that $0 < (\ell-a)/(c-a) = \alpha < 1$, $(c-\ell)/(c-a) = 1-\alpha$, $\ell = \alpha c + (1-\alpha)a$, $x_\ell = \alpha(x_\ell+\ell-c) + (1-\alpha)(x_\ell+\ell-a)$, and $x_\ell + \ell = \alpha(x_\ell+c) + (1-\alpha)(x_\ell+a)$. Let us now take in (2) $u = x_\ell + \ell - c$ and $v = x_\ell + \ell - a$. Then we get $\alpha F^-(x_\ell+\ell-c) + (1-\alpha)F^-(x_\ell+\ell-a) > F^-(x_\ell)$. Since $x_{\ell_1} \leq x_\ell + \ell$, F^- and therefore F is strictly convex on (x_L, x_{ℓ_1}). By a similar argument we get the convexity of F on a neighborhood of x_L provided that x_L is finite. Therefore F is convex on $(-\infty, x_{\ell_1})$ and strictly convex on (x_L, x_{ℓ_1}). Next let us take in (2) $u = v = x_\ell$; then analogously we get that F is concave on $(x_{\ell_1} + \ell_1, +\infty)$ and strictly concave on $(x_{\ell_1} + \ell, x_L + L)$.

As a straightforward consequence of Lemma 5 we get

LEMMA 6. *Let* $Q(0) \geq \frac{1}{2}$. *If* Q *is strictly unimodal, then* F *is strictly unimodal*.

LEMMA 7. *Suppose that* Q *is strictly unimodal and let* $\ell_1 \geq 0$ *such that* $\frac{1}{2} \leq Q(\ell_1) < 1$. *Then* x_ℓ *is nonincreasing and* $x_\ell + \ell$ *is nondecreasing on* (ℓ_1, L).

PROOF. Let $\ell_1 < \lambda_1 < \lambda_2 < L$. Since Q is strictly unimodal we have by Lemma 1 $\Delta^- F^-(x_{\lambda_2}) \leq \Delta^+ Q(\lambda_2) < \Delta^- Q(\lambda_1) \leq \Delta^+ F^-(x_{\lambda_1})$ and therefore by Lemma 5 $F(x_{\lambda_2}) \leq F(x_{\lambda_1})$ which implies $x_{\lambda_2} \leq x_{\lambda_1}$. Analogously from $\Delta^+ F(x_{\lambda_2} + \lambda_2) \leq \Delta^+ Q(\lambda_2) < \Delta^- Q(\lambda_1) \leq \Delta^- F(x_{\lambda_1} + \lambda_1)$ we get $x_{\lambda_1} + \lambda_1 \leq x_{\lambda_2} + \lambda_2$.

3. Proof of the main result

We can now prove the Theorem of Section 1 by making use of the auxiliary results given in Section 2.

Suppose that F is strictly unimodal with unique mode $x = x^*$. Then for any $\ell > 0$ we have $x_\ell \leq x^*$ and $x_\ell + \ell \geq x^*$. Let us take now $0 \leq \ell_1 < \ell_2 \leq L$ for $L < +\infty$ or $0 \leq \ell_1 < \ell_2 < +\infty$ for $L = +\infty$; then we can write for $0 \leq \alpha \leq 1$

$$\begin{aligned}
\alpha Q(\ell_1) + (1-\alpha) Q(\ell_2) &= \alpha\left[F(x_{\ell_1} + \ell_1) - F^-(x_{\ell_1})\right] + \\
&+ (1-\alpha)\left[F(x_{\ell_2} + \ell_2) - F^-(x_{\ell_2})\right] < \\
&< F(\alpha(x_{\ell_1} + \ell_1) + (1-\alpha)(x_{\ell_2} + \ell_2)) - F^-(\alpha x_{\ell_1} + (1-\alpha) x_{\ell_2}) \leq \\
&\leq Q(\alpha \ell_1 + (1-\alpha)\ell_2).
\end{aligned}$$

In order to prove the converse assertion, we begin by extending Lemma 5 using induction on r. Take $r = 1$ and let us call P_1 the property of F which was proven in Lemma 5. We go to $r = 2$; by Lemma 6 we can assume that $Q(0) < \frac{1}{2}$. In order to prove property P_2 of F we have to extend first Lemmas 3 and 4. Hence we begin by showing that if Q is strictly unimodal, $Q(0) < \frac{1}{2}$, and if $\ell_2 \geq 0$ is such that $1/3 \leq Q(\ell_2) < 2/3$, then x_ℓ is unique for all $\ell \in (\ell_2, L)$ and continuous on (ℓ_2, L).

Indeed, take $m_2 > \ell_2$ such that $Q(m_2) = 2/3$. (Note that in Lemmas 3 and 4 $m_1 = L$.) By Lemma 3, x_{m_2} is unique and consider the distribution function $G(x) = 0$ for $x < x_{m_2}$, $G(x) = 3F(x)/2$ for $x_{m_2} \leq x < x_{m_2} + m_2$, and $G(x) = 1$ for $x \geq x_{m_2} + m_2$. We show first that $Q_G(\ell) = 3Q(\ell)/2$ for all $\ell \leq m_2$, i.e., x_ℓ and $x_\ell + \ell$ lie in $(x_{m_2}, x_{m_2} + m_2)$ and are therefore for both F and G the same. Suppose by contraposition that $x_\ell < x_{m_2}$. Since Q is strictly unimodal we have $\Delta^- Q(\ell) \geq \Delta^+ Q(\ell) > \Delta^+ Q(m_2)$. On the other hand we have, by Lemmas 1 and 5, $\Delta^- Q(\ell) \leq \Delta^+ F^-(x_\ell) \leq \Delta^- F^-(x_{m_2}) \leq \Delta^+ Q(m_2)$ and we are led to a contradiction. By an analogous argument we have $x_\ell + \ell \leq$ $\leq x_{m_2} + m_2$. Next, by applying Lemmas 3 and 4 to G, we get our assertion.

Before continuing let us remark that $\ell_2 \leq \ell_1$, $x_L \leq x_{\ell_1} \leq x_{\ell_2}$, and $x_{\ell_2} + \ell_2 \leq x_{\ell_1} + \ell_1 \leq x_L + L$. Moreover $m_2 \leq m_1$.

Further, to get P_2 we have to show that F is convex on $(-\infty, x_{\ell_2})$, strictly convex on (x_L, x_{ℓ_2}), concave on $(x_{\ell_2} + \ell_2, +\infty)$, and strictly concave on $(x_{\ell_2} + \ell_2, x_L + L)$. Indeed, we get this assertion by proceeding in the same way as in the proof of Lemma 5 for G. As a straightforward consequence of P_2 we get that if $Q(0) \geq 1/3$ and if Q is strictly unimodal, then F is strictly unimodal. This is the extension of Lemma 6. Moreover Lemma 7 extends to the interval (ℓ_2,L).

Property P_{r+1} of F can be obtained from property P_r in the same way as we get P_2 from P_1. This means that, by induction on r, we have shown that if Q is strictly unimodal and if $\ell_r \geq 0$ is such that $1/(r+1) \leq Q(\ell_r) < 2/(r+1)$, then F is convex on $(-\infty, x_{\ell_r})$, strictly convex on (x_L, x_{ℓ_r}), concave on $(x_{\ell_r}+\ell_r, +\infty)$, and strictly concave on $(x_{\ell_r}+\ell_r, x_L+L)$. This is the extension of Lemma 5

From P_r we conclude that if $Q(0) \geq 1/r$ and if Q is strictly unimodal, then F is strictly unimodal. This is the extension of Lemma 6. Therefore our theorem is proven for $Q(0) > 0$. Moreover Lemma 7 extends to the interval (ℓ_r,L).

It remains to show that our theorem holds for $Q(0) = 0$, i.e., if and only if F is continuous. Take the nonincreasing sequence $\{m_r : r \geq 1\}$ such that $m_r \downarrow 0$ and $Q(m_r) = 2/(r+1) \downarrow 0$. Clearly $\ell_r \downarrow 0$ and since by the extention of Lemma 7 $x_{\ell_s} \leq x_{\ell_t} < x_{\ell_v} + \ell_v \leq x_{\ell_u} + \ell_u$ for arbitrary natural numbers s,t,u,v, such that $s < t$, $u < v$, we conclude that there is a value $x = x^*$ such that $x_{\ell_r} \uparrow x^*$ and $(x_{\ell_r} + \ell_r) \downarrow x^*$. By P_r, for any $r \geq 1$, we conclude that F is strictly unimodal with unique mode $x = x^*$.

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