On Abel-Jacobi Maps of Lagrangian Families

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Abstract

We study in this article the cohomological properties of Lagrangian families on projective hyper-Kähler manifolds. First, we give a criterion for the vanishing of Abel-Jacobi maps of Lagrangian families. Using this criterion, we show that under a natural condition, if the variation of Hodge structures on the degree 1 cohomology of the fibers of the Lagrangian family is maximal, its Abel-Jacobi map is trivial. We also construct Lagrangian families on generalized Kummer varieties whose Abel-Jacobi map is not trivial, showing that our criterion is optimal.

0 Introduction

Let X be a projective hyper-Kähler manifold [1], that is, a simply connected complex projective manifold whose space of holomorphic 2-forms is generated by a nowhere degenerate 2-form σ_X. The dimension of X is an even number $2n$. A Lagrangian subvariety L of X is a dimension n irreducible possibly singular subvariety of X such that, denoting $j : \tilde{L} \to L \hookrightarrow X$ a desingularisation of L, $j^*\sigma_X = 0$ in $H^0(\tilde{L}, \Omega^2_{\tilde{L}})$.

In this article, we will be considering Lagrangian families of a hyper-Kähler manifold X.

Definition 0.1. A Lagrangian family of a hyper-Kähler manifold X is a diagram

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{q} & X \\ \big\downarrow{\scriptstyle p} & & \\ B & & \end{array} \tag{1}$$

where p is flat and projective, $\mathscr{L}$ and B are connected quasi-projective manifolds and q maps birationally the general fiber $L_b := p^{-1}(b)$, $b \in B$, to a Lagrangian subvariety of X. In what follows, we will denote j_b the composition $L_b \hookrightarrow \mathscr{L} \to X$.

Lagrangian families were studied by Voisin in [14, 15] in different contexts, as generalizations of Lagrangian fibrations. While Lagrangian fibrations do not exist in general on projective hyper-Kähler manifolds (as this forces the Picard number to be at least 2), Lagrangian families (and even Lagrangian coverings for which q is dominant) are relatively common. See, for example, constructions in [8, 11] on Lagrangian families (coverings) on Fano varieties of lines of cubic fourfolds and on Hilbert schemes of K3 surfaces. Works of Voisin [14, 15] indicate that the existence of Lagrangian coverings implies important properties of the hyper-Kähler manifold in question. For example, it is shown in [15] that a very general projective hyper-Kähler fourfold admitting a Lagrangian covering satisfies Lefschetz standard conjecture for degree 2 cohomology. Studies of examples of Lagrangian families lead us to consider the following problem, which is motivated in the article [13].

Problem 0.2. Consider a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$ given by a diagram as in (1). What can be said of the map

$$\begin{array}{cccc} \psi_{\mathscr{L}}: & B & \to & CH^n(X) \\ & b & \mapsto & q_*(L_b) \end{array} \; ? \tag{2}$$

Let us first explain the relation between this problem and article [13]. In [13], the following conjecture is proposed and proved for some cases.

Conjecture 0.3. Let L and L' be two *constant cycle* Lagrangian subvarieties of a hyper-Kähler manifold X. If the cohomological classes $[L] = [L']$ in $H^{2n}(X, \mathbb{Q})$, then L is rationally equivalent to L' as algebraic cycles in X.

Here, a subvariety Z of a smooth projective variety X is called a *constant cycle* subvariety, if any two points z, z' in Z are rationally equivalent in X, or equivalently, if the image of the Gysin map $i_* : CH_0(Z) \to CH_0(X)$ is $\mathbb{Z}$ where $i : Z \hookrightarrow X$ is the inclusion map. The notion of constant cycle subvarieties is first proposed in [7] for curves in K3 surfaces. It is then studied in a more general setting in [13]. The condition constant cycle is a strong condition for Lagrangian subvarieties. In contrast with Lagrangian subvarieties, constant cycle Lagrangian subvarieties cannot deform into families.

Lemma 0.4. *Constant cycle Lagrangian subvarieties of X are rigid as constant cycle subvarieties.*

Proof Following the notations in [13], let

$$S_n X := \{x \in X : \text{ the rational equivalence orbit of } x \text{ has dimension} \geq n\}.$$

As is shown in [13, Theorem 1.3], $S_n X$ is a countable union of irreducible varieties of dimension $\leq n$ and constant cycle Lagrangian subvarieties of X are exactly irreducible components of $S_n X$ of dimension n. Hence, constant cycle Lagrangian subvarieties of X are rigid. □

One may now wonder if the condition *constant cycle* can be dropped in Conjecture 0.3. This motivates us to think about the Problem 0.2. Since the Lagrangian subvarieties in the Lagrangian family have the same cohomological class, if the map $\Psi_{\mathscr{L}}$ defined in Problem 0.2 is nonconstant, then we cannot drop the condition *constant cycle* in Conjecture 0.3, presenting the subtlety of Conjecture 0.3.

A weaker invariant of algebraic cycles in a projective manifold is the Abel-Jacobi invariant [16, Chapter 12]. Let us denote $\Phi_X^n : CH^n(X)_{hom} \to J^{2n-1}(X)$ the Abel-Jacobi map of X. We will recall the definitions and basic properties of the Abel–Jacobi invariant and the Abel–Jacobi maps in the beginning of Section 1. If two cycles homologeous to 0 are rationally equivalent, then they have the same Abel-Jacobi invariant in the intermediate Jacobians (see Proposition 1.2). Problem 0.2 thus motivates the following question.

Problem 0.5. Consider a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$ given by a diagram as in (1). Let $0 \in B$ be a point. Under which conditions is the Abel-Jacobi map

$$\begin{array}{rccc} \Psi_{\mathscr{L}}^{AJ}: & B & \to & J^{2n-1}(X) \\ & b & \mapsto & \Phi_X^n(q_*(L_b - L_0)) \end{array} \tag{3}$$

trivial?

Note that $\Psi_{\mathscr{L}}^{AJ}(b) = \Phi_X^n(\Psi_{\mathscr{L}}(b) - \Psi_{\mathscr{L}}(0))$. Problem 0.5 can be viewed as a first step to study Problem 0.2.

In this article, we first give a criterion for the vanishing of the Abel-Jacobi map (3) for Lagrangian families of a hyper-Kähler manifold (see also Proposition 1.5).

Proposition 0.6. *Consider a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$ as in (1), satisfying the following condition :*

♣ *For general $b \in B$, the contraction by $q^*\sigma_X$ gives an isomorphism $\lrcorner q^*\sigma_X : T_{B,b} \stackrel{\cong}{\to} H^0(L_b, \Omega_{L_b})$.*

Then the Abel-Jacobi map (3) is trivial if and only if for general $b \in B$, the restriction map

$$j_b^* : H^{2n-1}(X,\mathbb{Q}) \to H^{2n-1}(L_b,\mathbb{Q})$$

is zero.

The condition ♣ is natural. According to [11, Proposition 2.4], the deformations of a *smooth* Lagrangian subvariety are non-obstructed, and a local deformation is still Lagrangian. Therefore, if we take (B,b) to be a germ of the Hilbert scheme of deformations of a *smooth* Lagrangian subvariety $L \subset X$, and $\mathscr{L} \to B$ the corresponding family, then condition ♣ holds since $T_{B,b} \cong H^0(L_b, N_{L_b/X})$ by unobstructedness and $\lrcorner\sigma_X : H^0(L_b, N_{L_b/X}) \to H^0(L_b, \Omega_{L_b})$ is an isomorphism for a smooth Lagrangian variety.

Using this criterion, we give a response to Problem 0.5.

Theorem 0.7. *Consider a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$ given by a diagram as in (1), satisfying condition ♣. Assume that the variation of Hodge structures on the degree 1 cohomology of the fibers of $p : \mathscr{L} \to B$ is maximal, i.e., the period map*

$$\begin{array}{rccc} \mathscr{P}: & B & \to & Gr(h^{1,0}(L), H^1(L,\mathbb{C})) \\ & b & \mapsto & H^{1,0}(L_b) \subset H^1(L_b,\mathbb{C}) \cong H^1(L,\mathbb{C}), \end{array} \tag{4}$$

where L is a general fiber of $p : \mathscr{L} \to B$, is generically a local immersion. Then the Abel-Jacobi map (3) is trivial.

This response to Problem 0.5 is conditional. However, it can be shown that the condition "maximal variation of Hodge structures" cannot be dropped. In fact, we construct in Section 4 Lagrangian families satisfying ♣ for which the Abel-Jacobi map is shown to be nontrivial using Proposition 0.6. The variation of weight 1 Hodge structures of the constructed Lagrangian families is not maximal.

In Section 3, we shall explore under which conditions the variation of weight 1 Hodge structures is maximal. Let $H^2(X,\mathbb{Q})_{tr}$ be the orthogonal complement of $NS(X)_{\mathbb{Q}}$ in $H^2(X,\mathbb{Q})$ with respect to the Beauville-Bogomolov-Fujiki form q of X (see [1]) and let $b_2(X)_{tr}$ be the dimension of $H^2(X,\mathbb{Q})_{tr}$ .We prove the following result (see also Proposition 3.2):

Proposition 0.8. *Consider a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$ given by a diagram as in (1), satisfying condition ♣. Assume that the Mumford-Tate group of the Hodge structure $H^2(X,\mathbb{Q})$ is maximal, i.e. it is the special orthogonal group of $(H^2(X,\mathbb{Q})_{tr}, q)$, and assume that $b_2(X)_{tr} \geq 5$. If $h^{1,0}(L_b)$ is smaller than $2^{\lfloor \frac{b_2(X)_{tr}-3}{2} \rfloor}$, then the variation of weight 1 Hodge structures of p is maximal.*

Corollary 0.9. *Under the same assumptions as in Proposition 0.8, the Abel-Jacobi map (3) is trivial.*

Let $p : \mathscr{L} \to B$, $q : \mathscr{L} \to X$ be a Lagrangian family. Up to shrinking B, we may assume that the map $p : \mathscr{L} \to B$ is smooth. Let

$$\pi : \mathscr{A} := Alb(\mathscr{L}/B) \to B$$

be the relative Albanese variety of $p : \mathscr{L} \to B$. In the proof of Theorem 0.7 and Proposition 0.8, we use a similar construction to those in [9, 14] to get a holomorphic 2-form $\sigma_{\mathscr{A}}$ on $\mathscr{A}$. If we assume the condition ♣, $\pi : \mathscr{A} \to B$ is a Lagrangian fibration with respect to $\sigma_{\mathscr{A}}$(see Section 2). It is interesting to notice that, by this construction, under condition ♣, we can translate the problem concerning Lagrangian families to a problem concerning Lagrangian fibrations. However, the total space of the Lagrangian fibration is no longer a hyper-Kähler manifold, but a completely integrable system over an open subset of the base, as studied for instance in [6, Chapter 7].

The organisation of the article is as follows. In Section 1, we prove Proposition 0.6. In Section 2, we construct a Lagrangian fibration structure on the relative Albanese variety and use it to prove Theorem 0.7. In Section 3, we discuss the condition on the maximality of the variation of Hodge structures. In Section 4, we construct Lagrangian families satisfying ♣ whose Abel-Jacobi map is nontrivial, showing that Theorem 0.7 is optimal.

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1 A Criterion

In this section, we establish a criterion for the vanishing of the Abel-Jacobi map (3). Let us first recall the Abel–Jacobi invariant and the Abel–Jacobi map defined by Griffiths. We follow the presentation in [16, Chapter 12]. Throughout this presentation only, X is a compact Kähler manifold.

Definition 1.1. The k-th intermediate Jacobian $J^{2k-1}(X)$ is the complex torus

$$J^{2k-1}(X) = H^{2k-1}(X,\mathbb{C})/(F^kH^{2k-1}(X)\oplus H^{2k-1}(X,\mathbb{Z})).$$

The Abel-Jacobi map

$$\begin{array}{rccc} \Phi_X^k: & CH^n(X)_{hom} & \to & J^{2n-1}(X) \\ & Z=\partial\Gamma & \mapsto & \int_\Gamma \end{array}$$

associates to any cycle $Z=\partial\Gamma$ homologous to zero the current of integration along Γ (See [16, 12.1.2]). For an algebraic cycle Z of codimension k in X, the associated an element $\Phi_X^k(Z)$ in $J^{2k-1}(X)$ [16, 12.1.2] is called the Abel–Jacobi invariant of the cycle Z. The triviality of Abel–Jacobi invariant is a weaker condition than the rational equivalence.

Proposition 1.2. *Let Z be an algebraic cycle that is rationally equivalent to 0 in X. Then the Abel–Jacobi invariant of Z is 0.*

Proof It is proven in [16, Lemme 21.19]. □

Let $\mathscr{X}\subset B\times X$ be a flat family of subvarieties of codimension k in X. More precisely, B is a connected complex manifold, and $\mathscr{X}$ is a subvariety of codimension k in $B\times X$, flat over B. Let $p:\mathscr{X}\to B$ and $q:\mathscr{X}\to X$ be the projection maps to the two components. Let $0\in B$ be a reference point.

Definition 1.3. The Abel–Jacobi map of the family $\mathscr{X}\subset B\times X$ with respect to the reference point $0\in B$ is the map

$$\begin{array}{rccc} \Psi_{\mathscr{L}}^{AJ}: & B & \to & J^{2k-1}(X) \\ & b & \mapsto & \Phi_X^k(q_*p^*(b)-q_*p^*(0)). \end{array}$$

Theorem 1.4 (Griffiths)**.** *We have the following properties about the Abel–Jacobi map.*
(i) The Abel–Jacobi map $\Psi_{\mathscr{L}}^{AJ}:B\to J^{2k-1}(X)$ is holomorphic.
(ii) The image of the differential of $\Psi_{\mathscr{L}}^{AJ}$ at any point lies in $H^{k-1,k}(X)\subset H^{2k-1}(X,\mathbb{C})$.

Proof (i) is proven in [16, Théorème 12.4] and (ii) is proven in [16, Corollary 12.19 and Remarque 12.20]. □

This concludes the presentation of Abel–Jacobi invariants and Abel–Jacobi maps. From now on, let X be a hyper-Kähler manifold of dimension $2n$. With the notation $j_b:L_b\to X$ as in the introduction, we prove

Proposition 1.5. *Consider a Lagrangian family of hyper-Kähler manifold X of dimension $2n$ given by a diagram as in (1).*
(a) If for general $b \in B$, the restriction map

$$j_b^* : H^{2n-1}(X,\mathbb{Q}) \to H^{2n-1}(L_b,\mathbb{Q})$$

is zero, then the Abel-Jacobi map (3) is trivial.
(b) If condition ♣ holds (see Proposition 0.6), then the converse of (a) holds.

Remark 1.6. The cohomology group $H^{2n-1}(L_b,\mathbb{Q})$ has a Hodge structure of weight $2n-1$ and level 1. By Hodge symmetry and using the fact $j_b^* : H^{2n-1}(X,\mathbb{Q}) \to H^{2n-1}(L_b,\mathbb{Q})$ is a morphism of Hodge structures, we conclude that $j_b^* : H^{2n-1}(X,\mathbb{Q}) \to H^{2n-1}(L_b,\mathbb{Q})$ is zero, if and only if $j_b^* : H^{n-1,n}(X) \to H^{n-1,n}(L_b)$ is zero.

Proof Let $d\Psi^{AJ}_{\mathscr{L},b} : T_{B,b} \to H^n(X,\Omega_X^{n+1})$ denote the differential of the Abel-Jacobi map $\Psi^{AJ}_{\mathscr{L}}$ at point $b \in B$ (Theorem 1.4). Let $j_{b*} : H^0(L_b,\Omega_{L_b}) \to H^n(X,\Omega_X^{n+1})$ be the Gysin map, which is the Serre dual to the following composition

$$j_b^* : H^n(X,\Omega_X^{n-1}) \to H^n(L_b,\Omega^{n-1}_{\mathscr{L}|L_b}) \to H^n(L_b,\Omega^{n-1}_{L_b}). \tag{5}$$

By the above remark, the proposition follows from the following lemma and the fact that

$$\cup\sigma_X : H^n(X,\Omega_X^{n-1}) \to H^n(X,\Omega_X^{n+1})$$

is an isomorphism since $\wedge\sigma_X : \Omega_X^{n-1} \to \Omega_X^{n+1}$ is a vector bundle isomorphism. □

Lemma 1.7. *The following diagram is commutative:*

$$\begin{array}{ccc} T_{B,b} & \xrightarrow{d\Psi^{AJ}_{\mathscr{L},b}} & H^n(X,\Omega_X^{n-1}) \\ \downarrow{\lrcorner q^*\sigma_X} & & \downarrow{\cup\sigma_X} \\ H^0(L_b,\Omega_{L_b}) & \xrightarrow{j_{b*}} & H^n(X,\Omega_X^{n+1}). \end{array} \tag{6}$$

Proof We are going to show that the Serre dual of the diagram (6) is commutative.

Let $L^\bullet\Omega^i_{\mathscr{L}|L_b}$ be the Leray filtration [16, Chapter 16] induced on the vector bundle $\Omega^i_{\mathscr{L}|L_b}$ by the exact sequence

$$0 \to p^*\Omega_{B,b} \to \Omega_{\mathscr{L}|L_b} \to \Omega_{L_b} \to 0,$$

and defined by $L^j\Omega^i_{\mathscr{L}|L_b} = p^*\Omega^j_{B,b} \wedge \Omega^{i-j}_{\mathscr{L}|L_b}$. Since L_b is supposed to be Lagrangian, $q^*\sigma_X \in H^0(L_b, L^1\Omega^2_{\mathscr{L}|L_b})$ and thus the cup product

$$\cup q^*\sigma_X : \Omega^\bullet_{\mathscr{L}|L_b} \to \Omega^{\bullet+2}_{\mathscr{L}|L_b}$$

sends $L^k\Omega^\bullet_{\mathscr{L}|L_b}$ to $L^{k+1}\Omega^{\bullet+2}_{\mathscr{L}|L_b}$. Denoting $\overline{q^*\sigma_X}$ the image of $q^*\sigma_X$ in $H^0(L_b, Gr^1_L\Omega^2_{\mathscr{L}|L_b}) \cong H^0(L_b,\Omega_{L_b}) \otimes \Omega_{B,b}$, this implies the existence of the following commutative diagram

$$\begin{array}{ccc} L^1\Omega^{n+1}_{\mathscr{L}|L_b} = \Omega^{n+1}_{\mathscr{L}|L_b} & \longrightarrow & K_{L_b} \otimes p^*\Omega_{B,b} = Gr^1_L\Omega^{n+1}_{\mathscr{L}|L_b} \\ \uparrow{\cup q^*\sigma_X} & & \uparrow{\cup\overline{q^*\sigma_X}} \\ L^0\Omega^{n-1}_{\mathscr{L}|L_b} = \Omega^{n-1}_{\mathscr{L}|L_b} & \longrightarrow & \Omega^{n-1}_{L_b} = Gr^0_L\Omega^{n-1}_{\mathscr{L}|L_b}, \end{array} \tag{7}$$

where K_{L_b} is the canonical bundle of L_b. Taking the n-th cohomology of (7) and combine it with $q^* : H^n(X,\Omega_X^\bullet) \to H^n(L_b,\Omega^\bullet_{\mathscr{L}|L_b})$, we get the following commutative ladder

$$\begin{array}{ccccc} H^n(X,\Omega_X^{n+1}) & \xrightarrow{q^*} & H^n(L_b,\Omega^{n+1}_{\mathscr{L}|L_b}) & \longrightarrow & H^n(L_b,K_{L_b}\otimes p^*\Omega_{B,b}) \cong \Omega_{B,b} \\ \uparrow{\cup\sigma_X} & & \uparrow{\cup q^*\sigma_X} & & \uparrow{\cup\overline{q^*\sigma_X}} \\ H^n(X,\Omega_X^{n-1}) & \xrightarrow{q^*} & H^n(L_b,\Omega^{n-1}_{\mathscr{L}|L_b}) & \longrightarrow & H^n(L_b,\Omega^{n-1}_{L_b}). \end{array} \tag{8}$$

Lemma 1.8. *The composite in the first row of the diagram (8) coincides with the dual of $d\Psi_{\mathscr{L},b}$.*

Proof Let $\bar{p} : \bar{\mathscr{L}} \to \bar{B}$, $\bar{q} : \bar{\mathscr{L}} \to X$ be a relative completion of $p : \mathscr{L} \to B$ with respect to the morphism $q : \mathscr{L} \to X$. More precisely, $\bar{B}$ is a smooth projective variety, $\bar{p}$ is a flat morphism extending p, and $\bar{q}$ is a morphism extending q. The extended family has a Abel–Jacobi map $\Psi^{AJ}_{\bar{\mathscr{L}}} : \bar{B} \to J^{2n-1}(X)$ that induces a morphism of complex tori $\psi : \mathrm{Alb}(\bar{B}) \to J^{2n-1}(X)$ whose differential is given by the morphism of Hodge structures ([16, Théorème 12.17])

$$\bar{q}_*\bar{p}^* : H^{2d-1}(\bar{B},\mathbb{Z}) \to H^{2n-1}(X,\mathbb{Z}),$$

where $d = \dim B$. It is well-known that the differential of the Albanese map $\mathrm{alb} : \bar{B} \to \mathrm{Alb}(\bar{B})$ at a point $b \in B$ is given by the dual of the evaluation map $ev_b : H^0(\bar{B},\Omega_{\bar{B}}) \to \Omega_{\bar{B},b}$. Hence, the dual of $d\Psi^{AJ}_{\bar{\mathscr{L}},b}$ is given by the correspondance $\bar{p}_*\bar{q}^* : H^n(X,\Omega^{n+1}_X) \to H^0(\bar{B},\Omega_{\bar{B}})$ composed with the evaluation map $ev_b : H^0(\bar{B},\Omega_{\bar{B}}) \to \Omega_{\bar{B},b} \cong \Omega_{B,b}$. The domain $\bar{B}$ can be restricted to B before the evaluation map $ev_b : H^0(\bar{B},\Omega_{\bar{B}}) \to \Omega_{B,b}$. Therefore, the dual of $d\Psi^{AJ}_{\mathscr{L},b}$ is given by the correspondance $p_*q^* : H^n(X,\Omega^{n+1}_X) \to H^0(B,\Omega_B)$ composed with the evaluation map $ev_b : H^0(B,\Omega_B) \to \Omega_{B,b}$. The Gysin map $p_* : H^n(\mathscr{L},\Omega^{n+1}_{\mathscr{L}}) \to H^0(B,\Omega_B)$ is given by the Leray filtration. Taken together, the dual of $d\Psi^{AJ}_{\mathscr{L},b}$ is given by

$$H^n(X,\Omega^{n+1}_X) \xrightarrow{q^*} H^n(\mathscr{L},\Omega^{n+1}_{\mathscr{L}}) \to H^n(\mathscr{L},K_{\mathscr{L}/B} \otimes p^*\Omega_B) \to H^0(B,R^np_*\Omega_{\mathscr{L}/B} \otimes \Omega_B) \xrightarrow{ev_b} \Omega_{B,b}.$$

Since the restriction to the fiber L_b and taking the Leray filtration are commutative processes, the above composite of maps is equal to the one that takes the restriction to the fiber L_b first and then takes the Leray filtration. The latter one is exactly the first row of the diagram (8). □

As in (5), the composite in the second row is j_b^*. Taking into account of Lemma 1.8, the diagram 8 is indeed the Serre dual of the diagram (6). This concludes the proof of Lemma 6. □

2 Lagrangian Fibrations

In this section, we associate to any Lagrangian family satisfying condition ♣ a Lagrangian fibration with the help of a construction appeared in [9, 14], and use this Lagrangian fibration to prove Theorem 0.7.

General Constructions

Let (1) be a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$. We fix a relative polarization of $\mathscr{L} \to B$ given by a hyperplane section of X. Let

$$\pi : \mathscr{A} := Alb(\mathscr{L}/B) \to B$$

be the relative Albanese variety of $p : \mathscr{L} \to B$.

Lemma 2.1. *Let l be the relative dimension of $p : \mathscr{L} \to B$. Then there exist an open dense subset $B_0 \subset B$ and a finite covering $B'_0 \to B_0$ such that, denoting $p'_0 : \mathscr{L}'_0 \to B'_0$ the base change of p under $B'_0 \to B_0 \hookrightarrow B$ and $\pi'_0 : \mathscr{A}'_0 \to B'_0$ the relative Albanese variety of p'_0, there is a cycle $Z_0 \in CH^l(\mathscr{A}'_0 \times_{B'_0} \mathscr{L}'_0)$ such that*

$$[Z_0]^* : p'_{0*}\Omega_{\mathscr{L}'_0/B'_0} \to \pi'_{0*}\Omega_{\mathscr{A}'_0/B'_0}$$

is an isomorphism.

Proof Let $B_0 \subset B$ be the subset of regular points of $p : \mathscr{L} \to B$. For $b \in B_0$, let $C_b \subset L_b$ be a complete intersection curve and J_{C_b} the Jacobian variety of C_b. By Lefschetz theorem on hyperplane sections, $j_* : J_{C_b} \to$

$A_b := Alb(L_b)$ is surjective. By the semi-simplicity of polarized Hodge structures, there exists a $\mathbb{Q}$-section $s : A_b \to J_{C_b}$ of j_*, i.e., there exists $N > 0$ such that $j_* \circ s = N \cdot id_{A_b}$. On $J_{C_b} \times C_b$, we have the Poincaré divisor $d_b \in CH^1(J_{C_b} \times C_b)$ such that $[d_b]^* : H^1(C_b, \mathbb{Q}) \to H^1(J_{C_b}, \mathbb{Q})$ is an isomorphism of Hodge structures, exhibiting the inverse of the pull-back of the Jacobi map $jac_b : C_b \to J_{C_b}$. Let us consider

$$\begin{array}{ccc} A_b \times C_b & \xrightarrow{(s, id_{C_b})} & J_{C_b} \times C_b \\ \Big\downarrow {\scriptstyle (id_{A_b}, j)} & & \\ A_b \times L_b & & \end{array}$$

and define $Z_b := (id_{A_b}, j)_*(s, id_{C_b})^*(d_b) \in CH^l(A_b \times L_b)$. Then $[Z_b]^* : H^1(L_b, \mathbb{Q}) \to H^1(A_b, \mathbb{Q})$ is given by the composition

$$H^1(L_b, \mathbb{Q}) \xrightarrow{j^*} H^1(C_b, \mathbb{Q}) \xrightarrow{d_b^*} H^1(J_{C_b}, \mathbb{Q}) \xrightarrow{s^*} H^1(A_b, \mathbb{Q}), \tag{9}$$

which is an isomorphism by the definition of s. In fact, let $(j_*)^* : H^1(A_b, \mathbb{Q}) \to H^1(J_{C_b}, \mathbb{Q})$ be the pull-back map of $j_* : J_{C_b} \to A_b = Alb(L_b)$. Precomposing the left-hand-side of $(j_*)^* : H^1(A_b, \mathbb{Q}) \to H^1(J_{C_b}, \mathbb{Q})$ with the natural identification $H^1(Alb(L_b), \mathbb{Q}) \cong H^1(L_b, \mathbb{Q})$, and post-composing the right hand side with the pull-back of the Jacobi map $jac_b^* : H^1(J_{C_b}, \mathbb{Q}) \to H^1(C_b, \mathbb{Q})$, we would get the pull-back map $j^* : H^1(L_b, \mathbb{Q}) \to H^1(C_b, \mathbb{Q})$. Since $jac_b^* : H^1(J_{C_b}, \mathbb{Q}) \to H^1(C_b, \mathbb{Q})$ is the inverse of $d_b^* : H^1(C_b, \mathbb{Q}) \to H^1(J_{C_b}, \mathbb{Q})$, the composite $d_b^* \circ j^* : H^1(L_b, \mathbb{Q}) \to H^1(J_{C_b}, \mathbb{Q})$ of the first two maps of the composition (9) is exactly the pull-back map of $j_* : J_{C_b} \to A_b = Alb(L_b)$ at the level of cohomology, after the natural identification $H^1(Alb(L_b), \mathbb{Q}) \cong H^1(L_b, \mathbb{Q})$. Since $j_* \circ s = N \cdot id_{A_b}$, we get the desired isomorphism.

The cycles Z_b are defined fiberwise, but standard arguments [17, Chapter 3] show that they can be constructed in family over a smooth generically finite cover $B_0' \to B_0$. Let us spell out the standard arguments. By the theory of Hilbert schemes, there are countably many connected projective B_0-schemes $H_1, \ldots, H_i, \ldots$ together with the universal families of cycles $\mathscr{Z}_1, \ldots, \mathscr{Z}_i, \ldots$ such that for each $i \in \mathbb{N}$, every fiber of $\mathscr{Z}_i \to H_i$ is a cycle $Z_b \in \mathscr{Z}^1(A_b \times L_b)$ such that $[Z_b]^* : H^1(L_b, \mathbb{Q}) \to H^1(A_b, \mathbb{Q})$ is an isomorphism. By the construction of the previous paragraph, the structure map $\bigcup_{i \in \mathbb{N}} H_i \to B_0$ is surjective, thus there is $i \in \mathbb{N}$, such that $H_i \to B_0$ is surjective. Since $H_i \to B_0$ is projective, we may take a multisection of the map $B_0' \to B_0$, and the universal family $\mathscr{Z}_i$ restrited to B_0 gives the desired 1-cycle. □

For the sake of simplicity, we shall note B_0, $\mathscr{L}_0$ and $\mathscr{A}_0$ instead of B_0', $\mathscr{L}_0'$ and $\mathscr{A}_0'$. We define a holomorphic 2-form $\sigma_{\mathscr{A}_0}$ on $\mathscr{A}_0$ by setting

$$\sigma_{\mathscr{A}_0} := [Z_0]^* q_0^* \sigma_X, \tag{10}$$

where $q_0 : \mathscr{L}_0 \to X$ is the natural map.

Proposition 2.2. *(a) The 2-form $\sigma_{\mathscr{A}_0}$ is closed.*
(b) $\sigma_{\mathscr{A}_0}$ vanishes on fibers of $\pi_0 : \mathscr{A}_0 \to B_0$.
(c) The composite morphism $\kappa : T_{B_0} \xrightarrow{\lrcorner q_0^* \sigma_X} p_{0*}\Omega_{\mathscr{L}_0/B_0} \xrightarrow{[Z_0]^*} \pi_{0*}\Omega_{\mathscr{A}_0/B_0}$ is given by the contraction $\lrcorner \sigma_{\mathscr{A}_0}$.

Proof (a) Let $Z_q := (id, q_0)_* Z_0 \in CH(\mathscr{A}_0 \times X)$. Then by the projection formula, $\sigma_{\mathscr{A}_0} = [Z_q]^* \sigma_X$. Let $\mathscr{A}'$ be a projective completion of $\mathscr{A}_0$. Then Z_q extends to a cycle $\bar{Z}_q$ of $\mathscr{A}' \times X$. $\sigma_{\mathscr{A}_0}$ extends to a 2-form $\sigma_{\mathscr{A}'} := [\bar{Z}_q]^* \sigma_X$ which is automatically closed since $\mathscr{A}'$ is projective. Thus, $\sigma_{\mathscr{A}_0} = \sigma_{\mathscr{A}'|\mathscr{A}_0}$ is also closed.

(b) Since Z_0 is a cycle in $\mathscr{A}_0 \times_{B_0} \mathscr{L}_0 \subset \mathscr{A}_0 \times \mathscr{L}_0$, the morphism $[Z_0]^* : H^*(\mathscr{L}_0) \to H^*(\mathscr{A}_0)$ preserves the Leray filtrations on both sides. Therefore, $\sigma_{\mathscr{A}_0} \in H^0(\mathscr{A}_0, \pi_0^* \Omega_{B_0} \wedge \Omega_{\mathscr{A}_0}) \subset H^0(\mathscr{A}_0, \Omega^2_{\mathscr{A}_0})$ since $q_0^* \sigma_X \in H^0(\mathscr{L}_0, p_0^* \Omega_{B_0} \wedge \Omega_{\mathscr{L}_0})$ by the definition of Lagrangian families. Therefore, $\sigma_{\mathscr{A}_0}$ vanishes on the fibers of $\pi_0 : \mathscr{A}_0 \to B_0$.

(c) By Lemma 2.1, $[Z_0]^*$ induces an isomorphism $H^0(B_0, \Omega_{B_0} \otimes p_{0*}\Omega_{\mathscr{L}_0/B_0}) \to H^0(B_0, \Omega_{B_0} \otimes \pi_{0*}\Omega_{\mathscr{A}_0/B_0})$ which sends $\lrcorner q_0^* \sigma_X$ to $\lrcorner \sigma_{\mathscr{A}_0}$. □

By (b) and (c) of the above proposition, we get the following diagram that is commutative up to a sign:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{\mathscr{A}_0/B_0} & \longrightarrow & T_{\mathscr{A}_0} & \longrightarrow & \pi_0^* T_{B_0} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle (\pi_0^*\kappa)^*} & & \downarrow{\scriptstyle \lrcorner\sigma_{\mathscr{A}_0}} & & \downarrow{\scriptstyle \pi_0^*\kappa} & & \\ 0 & \longrightarrow & \pi_0^*\Omega_{B_0} & \longrightarrow & \Omega_{\mathscr{A}_0} & \longrightarrow & \Omega_{\mathscr{A}_0/B_0} & \longrightarrow & 0. \end{array} \tag{11}$$

Here, the commutativity of the second square is dual to Proposition 2.2 (c). Since the dual of $\lrcorner\sigma_{\mathscr{A}_0} : T_{\mathscr{A}_0} \to \Omega_{\mathscr{A}_0}$ is given by $-\lrcorner\sigma_{\mathscr{A}_0} : T_{\mathscr{A}_0} \to \Omega_{\mathscr{A}_0}$, the first square is anti-commutative.

Lemma 2.3. *If condition ♣ (see Proposition 0.6) holds for all $b \in B_0$, then $\sigma_{\mathscr{A}_0}$ is nowhere degenerate on $\mathscr{A}_0$.*

Proof If condition ♣ holds, then $\pi_0^*\kappa : \pi_0^* T_{B_0} \to \Omega_{\mathscr{A}_0/B_0}$ is an isomorphism. By the commutativity of (11) and the five lemma, $\lrcorner\sigma_{\mathscr{A}_0} : T_{\mathscr{A}_0} \to \Omega_{\mathscr{A}_0}$ is an isomorphism, which means that $\sigma_{\mathscr{A}_0}$ is nowhere degenerate. □

Symmetry

Let (1) be a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$. We fix a relative polarization of $\mathscr{L} \to B$ given by a hyperplane section of X. Let $b \in B$ be a general point. The infinitesimal variation of Hodge structures on degree 1 cohomology of the fibers of $p : \mathscr{L} \to B$ at b is given by (see [16, Lemme 10.19])

$$\bar{\nabla} : T_{B,b} \to Hom(H^0(L_b, \Omega_{L_b}), H^1(L_b, \mathscr{O}_{L_b})).$$

Precomposed with the map $\lrcorner q^*\sigma_X : T_{B,b} \to H^0(L_b, \Omega_{L_b})$, the map $\bar{\nabla}$ induces a bilinear map

$$\begin{array}{rccc} S : & T_{B,b} \times T_{B,b} & \to & H^1(L_b, \mathscr{O}_{L_b}) \\ & (u,v) & \mapsto & \bar{\nabla}_u(v \lrcorner q^*\sigma_X). \end{array} \tag{12}$$

Proposition 2.4. *The bilinear map S is symmetric in the sense that $S(u,v) = S(v,u)$ for any $u, v \in T_{b,B}$.*

Proof By Griffiths' transversality [16, Chapter 17], $S(u,v) = \rho(u) \lrcorner (v \lrcorner q^*\sigma_X)$, where $\rho : T_{B,b} \to H^1(L_b, T_{L_b})$ is the Kodaira-Spencer map. Therefore, we need to show that the following diagram is commutative

$$\begin{array}{ccc} T_{B,b} \otimes T_{B,b} & \xrightarrow{\rho \otimes id} & H^1(L_b, T_{L_b}) \otimes T_{B,b} \\ \downarrow{\scriptstyle \lrcorner q^*\sigma_X \otimes id} & & \downarrow{\scriptstyle \beta} \\ H^0(L_b, \Omega_{L_b}) \otimes T_{B,b} & \xrightarrow{\alpha} & H^1(L_b, \mathscr{O}_{L_b}), \end{array} \tag{13}$$

where $\alpha(\omega, v) = \rho(v) \lrcorner \omega$ and $\beta(u, \chi) = u \lrcorner (\chi \lrcorner q^*\sigma_X)$.

To see the commutativity of (13), restrict the commutative ladder (11) to L_b and apply the cohomology on L_b, then the commutativity of (11) implies the commutativity of (13). Indeed, (13) is the connecting homomorphism of the cohomology of (11) tensored by $T_{B,b}$. □

Remark 2.5. When condition ♣ is satisfied, the symmetry of S comes from the completely integrable system structure on $(\mathscr{A}_0, \sigma_{\mathscr{A}_0})$. What we proved is in fact the symmetry of

$$\begin{array}{rccc} S' : & T_{B,b} \times T_{B,b} & \to & H^1(A_b, \mathscr{O}_{A_b}) \\ & (u,v) & \mapsto & \bar{\nabla}_u(v \lrcorner \sigma_{\mathscr{A}_0}). \end{array} \tag{14}$$

Fixing a relative polarisation on $\mathscr{A}_0 \to B_0$, we have natural isomorphisms (always under ♣): $H^1(A_b, \mathscr{O}_{A_b}) \cong H^0(A_b, \Omega_{A_b})^* \cong T^*_{B,b}$ and we can thus view S' as an element in $T^*_{B,b} \otimes T^*_{B,b} \otimes T^*_{B,b}$. If this relative polarisation is principal, Donagi and Markman proved in [6, Lemma 7.2] that S' lies in $Sym^3 T^*_{B,b}$. This result is called "weak cubic condition" in [6, Lemma 7.2]. See also [12, Theorem 4.4]

Now we are ready to prove the main theorem.

Proof of Theorem 0.7. Under the assumptions of Theorem 0.7, assume by contradiction that the Abel–Jacobi map (3) is not constant. In what follows, we fix a relative polarization on $p_0 : \mathscr{L}_0 \to B_0$ induced from a hyperplane section of X, so that $R^{2n-1}p_{0*}\mathbb{Q} \cong R^1 p_{0*}\mathbb{Q}$. By Proposition 0.6, the morphism

$$j^* : H^{2n-1}(X,\mathbb{Q}) \to R^{2n-1}p_{0*}\mathbb{Q} \cong R^1 p_{0*}\mathbb{Q}$$

of variations of Hodge structures on an open subset $B_0 \subset B$ containing b is not zero. Hence, there is a non-zero locally constant sub-variation of Hodge strutures $I := \mathrm{Im} j^* \subset R^1 p_{0*}\mathbb{Q}$. Since I is locally constant, for any $\omega \in I_b^{1,0}$ and $u \in T_{B,b}$, $\nabla_u(\omega) = 0$. Recall that ♣ means that $\lrcorner q^*\sigma_X : T_{B,b} \to H^0(L_b, \Omega_{L_b})$ is bijective. Let $F := (\lrcorner q^*\sigma_X)^{-1}(I^{1,0}) \subset T_{B,b}$. Then by the symmetry of S given by Proposition 2.4, F lies in the kernel of $\bar{\nabla}$, which contradicts our assuption that the variation of Hodge structures is maximal.

□

3 Maximal Variations

In this section, we study under what conditions could the variation of Hodge structures of a Lagrangian family be maximal. Consider a Lagrangian family of a hyper-Kähler manifold X of dimension $2n$ satisfying the condition ♣ given by the diagram as in (1). Let $U \subset B$ be a simply connected open subset of $B_0 \subset B$ and let

$$\begin{array}{rccc} \mathscr{P}: & U & \to & Gr(h^{1,0}(L), H^2(L,\mathbb{C})) \\ & b & \mapsto & H^{1,0}(L_b) \subset H^1(L_b,\mathbb{C}) \cong H^1(L,\mathbb{C}), \end{array} \tag{15}$$

be the local period map of the Lagrangian family.

In what follows, we are going to use a universal property of the Kuga-Satake construction proved in [10, Proposition 6].

Theorem 3.1 ([10])**.** *Let (H^2,q) be a polarized Hodge structure of hyper-Kähler type of dimension ≥ 5. Assume that the Mumford–Tate group of the Hodge structure on H^2 is maximal, namely the special orthogonal group of (H^2,q). Let H be a simple effective weight-1 Hodge structure, such that there exists an injective morphism of Hodge structures of bidegree $(-1,-1)$*

$$H^2 \hookrightarrow Hom(H,A)$$

for some weight-1 Hodge structure A. Then H is a direct summand of the Kuga–Satake Hodge structure $H^1_{KS}(H^2,q)$. In particular, $\dim H \geq 2^{\lfloor \frac{\dim H^2 - 1}{2} \rfloor}$.

With the same notations as in the introduction, we prove

Proposition 3.2. *Assume that the Mumford-Tate group of the Hodge structure $H^2(X,\mathbb{Q})$ is maximal, i.e. it is the special orthogonal group of $(H^2(X,\mathbb{Q})_{tr}, q)$ and assume $b_2(X)_{tr} \geq 5$. If the dimension of $H^{0,1}(L_b)$ is smaller than $2^{\lfloor \frac{b_2(X)_{tr}-3}{2} \rfloor}$ for a general fiber L_b of $p : \mathscr{L} \to B$, then the variation of weight 1 Hodge structure of p is maximal.*

Proof We use the same argument as in [10] where similar results were proved for Lagrangian fibrations. Assuming that the period map (15) is not generically an immersion, we are going to prove that $\dim H^{0,1}(L_b) \geq 2^{\lfloor \frac{b_2(X)_{tr}-3}{2} \rfloor}$. By assumption, the nonempty general fibers of $\mathscr{P}$ are of dimension ≥ 1. Let $b \in U$ be a general point and let B_b the fiber of $\mathscr{P}$ passing through b. Let $U_b = B_b \cap U$. Then the fibers of $\pi|_{U_b} : \mathscr{A}_{U_b} \to U_b$ are isomorphic with each other. Thus, up to a base change by a finite covering of U_b, we may assume $\pi|_{U_b} : \mathscr{A}_{U_b} \to U_b$ is trivial, i.e., $\mathscr{A}_{U_b} = U_b \times A_b$. Let $\pi_{F_b} : \mathscr{A}_{F_b} \to F_b$ be a smooth completion of $\pi|_{U_b}$, then $\mathscr{A}_{F_b}$ is birational to $F_b \times A_b$, which gives a morphism $H^2(\mathscr{A}_{F_b}) \to H^2(F_b \times A_b)$. Recall by Lemma 2.1, we get a morphism $[Z]^* : H^2(X) \to H^2(\mathscr{A})$ that sends σ_X to a holomorphic 2-form which is non-degenerate on $\mathscr{A}_U$. Finally, the rational map $\mathscr{A}_{F_b} \dashrightarrow \mathscr{A}$ induces $H^2(\mathscr{A}) \to H^2(\mathscr{A}_{F_b})$. Compositing all these maps, we get a morphism

$$\alpha : H^2(X)_{tr} \hookrightarrow H^2(X) \to H^2(\mathscr{A}) \to H^2(\mathscr{A}_{F_b}) \to H^2(F_b \times A_b) \to H^1(F_b) \otimes H^1(A_b), \tag{16}$$

where the last map is given by the projection in the Künneth decomposition.

Lemma 3.3. *$\alpha : H^2(X)_{tr} \to H^1(F_b) \otimes H^1(A_b)$ is injective.*

Proof Since $h^{2,0}(X) = 1$ and $H^{2,0}(X)$ is orthogonal to $NS(X)$ with respect to the Beauville-Bogomolov-Fujiki form, $H^2(X)_{tr}$ is a simple Hodge structure. Therefore, to show the injectivity of α it suffices to show that α is not zero. We claim that $\alpha(\sigma_X) \neq 0$. Indeed, Since A_b is Lagrangian with respect to $\sigma_{\mathscr{A}}$ (Proposition 2.2 (b)), in the Künneth's decomposition of $H^2(A_b \times F_b)$, the image of σ_X in $H^0(F_b) \otimes H^2(A_b)$ is zero. If furthermore $\alpha(\sigma_X) = 0$ in $H^1(F_b) \otimes H^1(A_b)$, then the image of σ_X on $F_b \times A_b$ comes from a 2-form on F_b, which has rank $\leq \dim F_b$. Therefore, the rank of $\sigma_{\mathscr{A}}$ has rank $\leq \dim F_b$ on $\mathscr{A}_{U_b}$. On the other hand, the codimension of $\mathscr{A}_{U_b}$ in $\mathscr{A}_U$ is $\dim B - \dim F_b$, and thus the non-degeneration of $\sigma_{\mathscr{A}_U}$ implies that $\sigma_{\mathscr{A}}$ has rank $\geq 2 \dim F_b$ on $\mathscr{A}_{U_b}$. This is a contradiction since we are assuming $\dim F_b \geq 1$. □

We are now in the position to use the universal property of the Kuga-Satake construction (see Theorem 3.1 above). Since $\alpha : H^2(X)_{tr} \to H^1(F_b) \otimes H^1(A_b)$ is nonzero, there is at least one simple direct factor A of A_b such that $H^2(X)_{tr} \to H^1(F_b) \otimes H^1(A)$ is nonzero thus injective. Taking H^2 as $H^2(X)_{tr}$, we conclude by Theorem 3.1 that

$$\dim H^{0,1}(L_b) = \dim A_b \geq \dim A \geq \frac{1}{2} \times 2^{\lfloor \frac{\dim H^2(X)_{tr} - 1}{2} \rfloor} = 2^{\lfloor \frac{b_2(X)_{tr} - 3}{2} \rfloor},$$

as desired. □

4 Example of a Lagrangian Family with Nontrivial Abel-Jacobi Map

Recall the construction of generalized Kummer varieties introduced in [1]. Let A be an abelian surface and $A^{[n+1]}$ the Hilbert scheme of length $n+1$ subschemes of A. Let $\mathrm{alb} : A^{[n+1]} \to A$ be the composition of the Hilbert-Chow morphism and the summation map

$$A^{[n+1]} \to A^{(n+1)} \to A.$$

Note that alb is an isotrivial fibration. The generalized Kummer variety $K_n(A)$ is defined to be the fiber of alb over $0 \in A$. As is shown in [1], $K_n(A)$ is a hyper-Kähler manifold of dimension $2n$.

In this section, we are going to construct Lagrangian families of $X := K_n(A)$ for $n \geq 2$, satisfying condition ♣ and whose Abel-Jacobi map is *not* trivial.

For any $x \in A$, one defines a subvariety Z_x of $K_n(A)$ consisting of Artinian subschemes of A of length $n+1$ supported on x and $-nx$, with multiplicities n and 1, respectively. By [3, Proposition VI.1.1], Z_x is a *rational* variety of dimension $n-1$ if x is *not* an $(n+1)$-torsion point. Let $Z = \bigcup_{x \in A} Z_x$ and let $\pi : Z \to A$ send elements in Z_x to x. For any curve $C \subset A$, define $Z_C = \bigcup_{x \in C} Z_x$.

Now let B be a connected open subset of the Hilbert scheme of deformations of a smooth curve $C \subset A$ and $\mathscr{C} \to B$ the corresponding family.

Lemma 4.1. *$\{Z_C\}_{C \in B}$ is a Lagrangian family of $K_n(A)$ satisfying condition ♣.*

Proof Since for general C, Z_C is a fibration over a curve C whose general fibers are rational, any holomorphic 2-form on Z_C is 0. Furthermore, $\dim Z_C = n = \dim K_n(A)/2$. These imply that $\{Z_C\}_{C \in B}$ is a Lagrangian family.

We now show that this family satisfies condition ♣. Denoting $\mathscr{L}$ the total space of the family $\{Z_C\}_{C\in B}$ and L a general fiber, and using as before the following notation

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ B & & \end{array},$$

we need to show that $\lrcorner q^*\sigma_{K_n(A)} : H^0(L, N_{L/Z}) = H^0(L, N_{L/\mathscr{L}}) \to H^0(L, \Omega_L)$ is an isomorphism. Since the general fibers of π are rational, $q^*\sigma_{K_n(A)} = \pi^*\sigma_A$, where σ_A is the unique (up to coefficients) holomorphic 2-form on A. Therefore, we can conclude by the commutativity of the following diagram

$$\begin{array}{ccc} H^0(C, N_{C/A}) & \xrightarrow{\lrcorner\sigma_A} & H^0(C, \Omega_C) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ H^0(L, N_{L/Z}) & \xrightarrow{\lrcorner\pi^*\sigma_A} & H^0(L, \Omega_L) \end{array}$$

noting that the two vertical arrows are isomorphims since the fibers of π are rational, and that $\lrcorner\sigma_A : H^0(C, N_{C/A}) \to H^0(C, \Omega_C)$ is an isomorphism since σ_A is nondegenerate. □

Proposition 4.2. *The Abel-Jacobi map of the Lagrangian family $\{Z_C\}_{C\in B}$ is* not *trivial.*

Proof Let $i : C \hookrightarrow A$ be a general curve in the family $\mathscr{C} \to B$. By Proposition 1.5(a), it suffices to show that the restriction map $H^{2n-1}(X, \mathbb{Q}) \to H^{2n-1}(Z_C, \mathbb{Q})$ is nonzero.

Define an injective morphism

$$\begin{array}{cccc} \beta : & A \times A & \hookrightarrow & A^{(n+1)} \\ & (x, y) & \mapsto & n\{x\} + \{y\}, \end{array}$$

where we use the notation $\{x\} \in \mathscr{Z}_0(A)$ the 0-cycle of the point $x \in A$. Consider the following pull-back diagram defining a subvariety $Z' \subset A^{[n+1]}$

$$\begin{array}{ccc} Z' & \xhookrightarrow{\alpha} & A^{[n+1]} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle c} \\ A \times A & \xhookrightarrow{\beta} & A^{(n+1)} \end{array},$$

where $c : A^{[n+1]} \to A^{(n+1)}$ is the Hilbert-Chow morphism. Then $Z = Z' \cap K_n(A) \subset A^{[n+1]}$. We have the following commutative diagram where all three squares are pull-back diagrams

$$\begin{array}{ccccccc} Z & \hookrightarrow & & & X = K_n(A) & & \\ \downarrow{\scriptstyle \pi} & \searrow & & & & \searrow & \\ & & Z' & \xhookrightarrow{\alpha} & & & A^{[n+1]} \\ A & & \downarrow{\scriptstyle \pi'} & & & & \downarrow{\scriptstyle c} \quad \searrow{\scriptstyle \mathrm{alb}} \\ & \overset{f}{\searrow} & A \times A & & \xhookrightarrow{\beta} & & A^{(n+1)} \xrightarrow{\Sigma} A \end{array},$$

Here $f : A \to A \times A$ defined by $x \mapsto (x, -nx)$ is the fiber over $0 \in A$ of the trivial fibration $\sum \circ \beta : A \times A \to A$.

By [5, Corollary 5.1.5], $[Z']^* : H^{2n-1}(A^{[n+1]}, \mathbb{Q}) \to H^1(A \times A, \mathbb{Q})$ is surjective. Furthermore, the restriction map $f^* : H^1(A \times A, \mathbb{Q}) \to H^1(A, \mathbb{Q})$ is surjective since f is the fiber of a trivial fibration. These imply that $[Z]^* : H^{2n-1}(X, \mathbb{Q}) \to H^1(A, \mathbb{Q})$ is surjective. Finally, since the restriction map $i^* : H^1(A, \mathbb{Q}) \to H^1(C, \mathbb{Q})$ is injective by Lefschetz hyperplane theorem, the composition map $i^* \circ [Z]^* : H^{2n-1}(X, \mathbb{Q}) \to H^1(C, \mathbb{Q})$ is nonzero. This implies that the restriction map $H^{2n-1}(X, \mathbb{Q}) \to H^{2n-1}(Z_C, \mathbb{Q})$ is nonzero, as desired. □

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