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ASYMPTOTICS FOR THE GREEN'S FUNCTIONS OF A TRANSIENT REFLECTED BROWNIAN MOTION IN A WEDGE

SANDRO FRANCESCHI, IRINA KOURKOVA, AND MAXENCE PETIT

Abstract. We consider a transient Brownian motion reflected obliquely in a two-dimensional wedge. A precise asymptotic expansion of Green's functions is found in all directions.

To this end, we first determine a kernel functional equation connecting the Laplace transforms of the Green's functions. We then extend the Laplace transforms analytically and study its singularities. We obtain the asymptotics applying the saddle point method to the inverse Laplace transform on the Riemann surface generated by the kernel.

1. Introduction

Context. Since its introduction in the 1980s, reflected Brownian motion in a cone has been much studied [29, 30, 44], particularly for its deep links with queuing systems as an approximate model in heavy traffic [27, 40]. Some seminal work has determined the recurrent or transient nature of this process in dimension two [45, 33] but also in higher dimension which is a much more complex issue [6, 4, 3, 8]. The literature is full of studies of its stationary distribution in the recurrent case, such as the study of its asymptotics, which has generated a great deal of work [28, 10, 11, 23, 39, 41], numerical methods developed to compute it [7, 9] or the determination of explicit expressions of its stationary density [19, 21, 1, 32, 12, 24, 2, 25]. The transient case, which is a little less studied, is also the subject of several articles which study its escape probability along the axes [20], its absorption probability at the vertex [26, 15] or its Green's functions also called occupation density [14, 22].

In this article we consider a transient obliquely reflected Brownian motion in a cone of angle $\beta \in (0,\pi)$ with two different reflection laws from two boundary rays of the cone. We denote by $\widetilde{g}(\rho\cos(\omega),\rho\sin(\omega))$ the Green's function of this process in polar coordinates. Green's function is the average time density that the process spends at a point on the cone. The article determines the asymptotics of $\widetilde{g}(\rho\cos(\omega),\rho\sin(\omega))$ as $\rho\to\infty$ and $\omega\to\omega_0$ for any given angle $\omega_0\in[0,\beta]$. See Theorem 1 when $\omega_0\in(0,\beta)$ and Theorem 2 when $\omega_0=0$ or β. It extends results of [14] in two aspects. First, asymptotics results are obtained in any convex two dimensional cone with two different reflection laws from its boundaries. While in the half plane of [14] the Laplace transform of Green's function is easily made explicit, this is not the case of a cone as above. Laplace transforms of Green functions in this case are expressed in [22] in terms of some integrals as solutions of Riemann boundary problems, which hardly suit for further analysis. Second, Theorem 1 provides Green function's asymptotic in any direction of the cone and not only along straight rays as in [14], namely when the angle ω above tends to a given angle ω_0 and not just equals it. The asymptotic depends on the rate of convergence of $\omega\to\omega_0$ and allows to determine the full Martin boundary of the process.

In [23] the asymptotic of the stationary distribution for recurrent Brownian motion in a cone is found along all regular directions $\omega_0\in(0,\beta)$, while some special directions ω_0 are left for future work. The asymptotics is obtained by studying the singularities and applying the saddle point method to the inverse Laplace transform of the stationary distribution. This article applies the approach of [23] to Green's functions and provides further developments: the new techniques allow to treat all special directions where asymptotic depends deeper of the rate of convergence of ω to ω_0. This is the case

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when $\omega_0 = 0$ or β, see Theorem 2, and also when the *saddle point meet a pole* of the boundary Laplace transform, see Theorem 3.

The tools used in this paper are inspired by methods introduced by Malyshev [38], which studies the asymptotic of the stationary distribution for random walks in the quarter plane. Articles studying asymptotics in line with Malshev's approach pursued in that direction, such as [35], which studies the Martin boundary of random walks in the quadrant; [36], which extends these methods to the join-the-shorter-queue issue; and [34], which studies the asymptotics of the occupation measure for random walks in the quarter plane with drift absorbed at the axes. Fayolle and Iasnogorodski [16] also developed a method to determine explicit expression for generating functions thanks to Riemann and Carlemann boundary value problem. Then Fayolle, Iansogorodski and Malyshev deepened and merged their analytic approach for random walks in the quadrant in the famous book [17]. Finally, the article [23] is the pioneer paper which began to extend their approach to continuous stochastic processes in the quadrant to compute asymptotics of stationary distributions. The paper [14] is the first paper studying the asymptotics of Green's functions using this analytic approach.

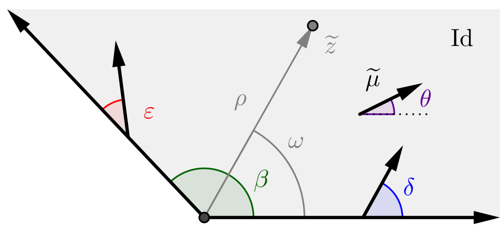


FIGURE 1. The cone of angle β, the reflection angles δ and ε and the drift $\widetilde{\mu}$ with its direction θ. In grey the point $\widetilde{z}$ of polar coordinates ρ and ω.

Main results. We consider an obliquely reflected standard Brownian motion in a cone of angle $\beta \in (0, \pi)$ starting from $\widetilde{z}_0$, of reflection angles $\delta \in (0, \pi)$ and $\varepsilon \in (0, \pi)$ and of drift $\widetilde{\mu}$ of angle $\theta \in (0, \beta)$ with the horizontal axis, see Figure 1. We assume that

$$\delta + \varepsilon < \beta + \pi.$$

This well known condition ensures that the process is a semi-martingale reflected Brownian motion [46, 47]. The reflected Brownian motion will be properly defined in the next section. The process is transient since we assumed that $\theta \in (0, \beta)$ which means that the drift belongs to the cone. If we assume that p_t is the transition probability of this process, the Green's function is defined for $\widetilde{z}$ inside the cone by

$$\widetilde{g}(\widetilde{z}) = \int_0^\infty \widetilde{p}_t(\widetilde{z}_0, \widetilde{z}) \mathrm{d}t.$$

For $\omega \in (0, \beta)$ and $\rho > 0$ we will denote $\widetilde{z} = (\rho \cos \omega, \rho \sin \omega)$ the polar coordinates in the cone. Note that the tilde symbol $\widetilde{\ }$ stands for quantities linked to the standard reflected Brownian motion in the β-cone. The same notations without the tilde symbol will stand for the corresponding process in the quadrant $\mathbb{R}_+^2$, see Remark 1.3 bellow.

Before presenting our results in more detail, we need to make the following remark.

Remark 1.1 (Notation)**.** *Along all this article, we will use the symbol $\sim$ to express an asymptotic expansion of a function. If for some functions f and g_k we state that $f(x) \sim \sum_{k=1}^n g_k(x)$ when $x \to x_0$, it means that $g_k(x) = o(g_{k-1}(x))$ and that $f(x) - \sum_{k=1}^n g_k(x) = o(g_n(x))$ when $x \to x_0$.*

We now state the main result of the article. We define the angles

$$\omega^* := \theta - 2\delta \quad \text{and} \quad \omega^{**} := \theta + 2\epsilon.$$

We can remark that $\omega^* < \theta < \omega^{**}$.

Theorem 1 (Asymptotics in the general case)**.** *We consider a standard reflected Brownian motion in a wedge of opening β, of reflection angles δ and ε and a drift $\widetilde{\mu}$ of angle θ, see Figure 1. Then, the Green's function $\widetilde{g}(\rho\cos\omega, \rho\sin\omega)$ of this process has the following asymptotics when $\omega \to \omega_0 \in (0, \beta)$ and $\rho \to \infty$, for all $n \in \mathbb{N}$:*

- *If $\omega^* < \omega_0 < \omega^{**}$ then*

$$\widetilde{g}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to\omega_0}}{\sim} e^{-2\rho|\widetilde{\mu}|\sin^2\left(\frac{\omega-\theta}{2}\right)} \frac{1}{\sqrt{\rho}} \sum_{k=0}^{n} \frac{\widetilde{c_k}(\omega)}{\rho^k} \tag{1.1}$$

- *If $\omega_0 < \omega^*$ then*

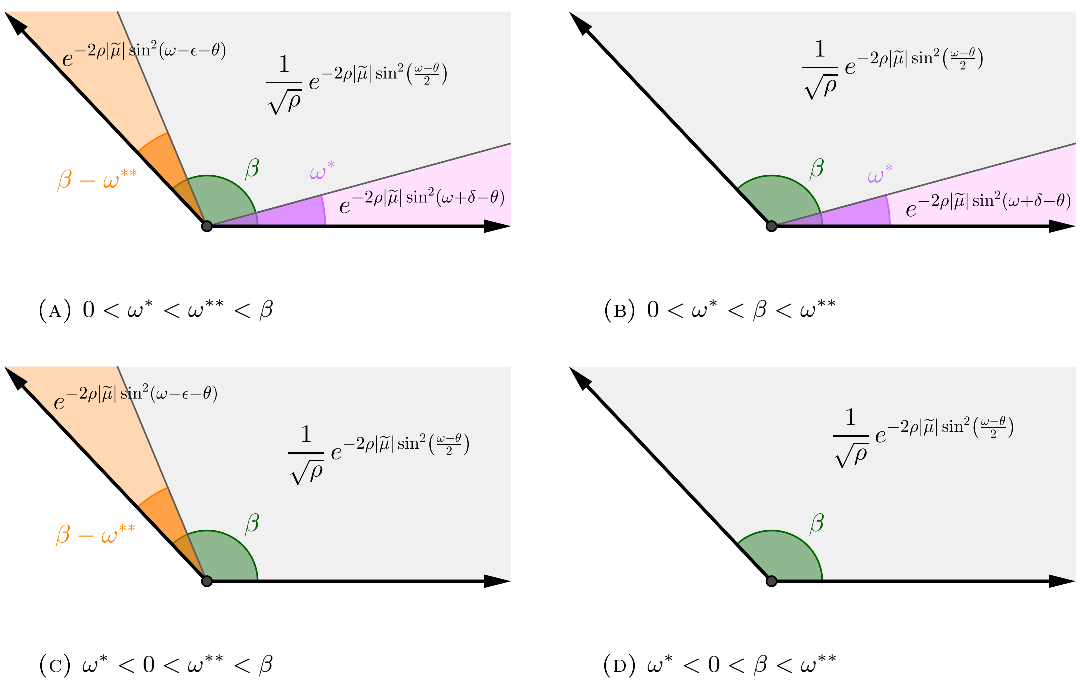
$$\widetilde{g}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to\omega_0}}{\sim} c^* e^{-2\rho|\widetilde{\mu}|\sin^2(\omega+\delta-\theta)} + e^{-2\rho|\widetilde{\mu}|\sin^2\left(\frac{\omega-\theta}{2}\right)} \frac{1}{\sqrt{\rho}} \sum_{k=0}^{n} \frac{\widetilde{c_k}(\omega)}{\rho^k} \tag{1.2}$$

- *If $\omega^{**} < \omega_0$ then*

$$\widetilde{g}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty \\ \omega\to\omega_0}}{\sim} c^{**} e^{-2\rho|\widetilde{\mu}|\sin^2(\omega-\epsilon-\theta)} + e^{-2\rho|\widetilde{\mu}|\sin^2\left(\frac{\omega-\theta}{2}\right)} \frac{1}{\sqrt{\rho}} \sum_{k=0}^{n} \frac{\widetilde{c_k}(\omega)}{\rho^k} \tag{1.3}$$

where c^* and c^{**} are positive constants and $c_k(\omega)$ are constants depending on ω such that $\widetilde{c_k}(\omega) \underset{\omega\to\omega_0}{\longrightarrow} \widetilde{c_k}(\omega_0)$.

There are four cases which are illustrated by Figure 2.



(A) $0 < \omega^* < \omega^{**} < \beta$

(B) $0 < \omega^* < \beta < \omega^{**}$

(C) $\omega^* < 0 < \omega^{**} < \beta$

(D) $\omega^* < 0 < \beta < \omega^{**}$

FIGURE 2. Asymptotics of the Green's function determined in Theorem 1 according to the direction ω_0: four different cases according to the value of angles $\omega^* = \theta - 2\delta$ and $\omega^{**} = \theta + 2\epsilon$. When ω_0 is in the grey region the asymptotics is given by (1.1), in the purple region by (1.2), in the orange region by (1.3).

Our second result states the asymptotics near the edges when $\omega \to 0$ or $\omega \to \beta$.

Theorem 2 (Asymptotics along the edges)**.** *We now assume that $\omega_0 = 0$ and let $\rho \to \infty$ and $\omega \to \omega_0 = 0$. In these case, we have $\widetilde{c_0}(\omega) \underset{\omega\to 0}{\sim} c'\omega$ and $\widetilde{c_1}(\omega) \underset{\omega\to 0}{\sim} c''$ for some non-negative constants c' and c'' which are non null at least when $\omega^* < 0$. Then, the Green's function $\widetilde{g}(\rho\cos\omega, \rho\sin\omega)$ has the following asymptotics:*

- *When $\omega^* < 0$ the asymptotics given by (1.1) remains valid. In particular, we have*
$$\widetilde{g}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty\\ \omega\to 0}}{\sim} e^{-2\rho|\widetilde{\mu}|\sin^2\left(\frac{\omega-\theta}{2}\right)} \frac{1}{\sqrt{\rho}} \left(c'\omega + \frac{c''}{\rho} \right)$$
- *When $\omega^* > 0$ the asymptotics given by (1.2) remains valid. In particular, we have*
$$\widetilde{g}(\rho\cos\omega, \rho\sin\omega) \underset{\substack{\rho\to\infty\\ \omega\to 0}}{\sim} c^* e^{-2\rho|\widetilde{\mu}|\sin^2(\omega+\delta-\theta)}.$$

Therefore, when $\omega^* < 0$, there is a competition between the two first terms of the sum $\sum_{k=0}^{n} \frac{\widetilde{c_k}(\omega)}{\rho^k}$ to know which one is dominant between $c'\omega$ and $\frac{c''}{\rho}$. More precisely:

- *If $\rho\sin\omega \underset{\substack{\rho\to\infty\\ \omega\to 0}}{\longrightarrow} \infty$ then the first term is dominant.*
- *If $\rho\sin\omega \underset{\substack{\rho\to\infty\\ \omega\to 0}}{\longrightarrow} c > 0$ then both terms contribute, they have the same order of magnitude.*
- *If $\rho\sin\omega \underset{\substack{\rho\to\infty\\ \omega\to 0}}{\longrightarrow} 0$ then the second term is dominant.*

A symmetric result holds when we take $\omega_0 = \beta$. The asymptotics given by (1.1) remains valid when $\beta < \omega^{**}$ and (1.3) remain valid when $\omega^{**} < \beta$ and there is a competition between the two first terms of the sum to know which one is dominant which depends of the limit of $\rho\sin(\beta-\omega)$.

We will explain later in Propositions 11.1 and 11.2 that ω^* and ω^{**} correspond in some sense to the poles of the Laplace transforms of the Green's functions and that ω correspond to the saddle point obtained when we will inverse the Laplace transform. Our third result states the asymptotics when the saddle point *meet* the poles which means when $\omega \to \omega^*$ or $\omega \to \omega^{**}$.

Theorem 3 (Asymptotics when the saddle point *meet* a pole)**.** *We now assume that $\omega_0 = \omega^* = \theta - 2\delta$ and let $\omega \to \omega^*$ and $\rho \to \infty$. Then, the Green's function $\widetilde{g}(\rho\cos\omega, \rho\sin\omega)$ has the following asymptotics:*

- *When $\rho(\omega-\omega^*)^2 \to 0$ then asymptotics is given by (1.2) but the constant c^* of the first term has to be replaced by $\frac{1}{2}c^*$.*
- *When $\rho(\omega-\omega^*)^2 \to c > 0$ for some constant c then:*
  - *If $\omega < \omega^*$ the asymptotics is still given by (1.2) but the constant c^* of the first term has to be replaced by $\frac{1}{2}c^*(1+\Phi(\sqrt{c}A))$ for some constant A.*
  - *If $\omega > \omega^*$ the asymptotics is still given by (1.2) but the constant c^* of the first term has to be replaced by $\frac{1}{2}c^*(1-\Phi(\sqrt{c}A))$ for some constant A.*

  In the previous items we denoted $\Phi(z) := \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)dt$.
- *When $\rho(\omega-\omega^*)^2 \to \infty$ then:*
  - *If $\omega < \omega^*$ the asymptotics is given by (1.2)*
  - *If $\omega > \omega^*$ the asymptotics is given by (1.1) and we have $\widetilde{c_0}(\omega) \underset{\omega\to\omega^*}{\sim} \frac{c}{\omega-\omega^*}$ for some constant c.*

A symmetric result holds when we assume that $\omega_0 = \omega^{**} = \theta + 2\epsilon$.

The following remark concerns the Martin boundary.

Remark 1.2 (Martin boundary)**.** *The Martin boundary associated to this process can be computed from the asymptotics of the Green's function obtained in the previous theorems. The corresponding harmonic functions can also be obtained thanks to the constants of the dominant terms of the asymptotics. See Section 6 of [14] which briefly reviews some elements of the theory in a similar context.*

We are now going to explain how to go from a standard Brownian motion reflected in a convex cone to a reflected Brownian motion reflected in a quadrant by adjusting the covariance matrix. This will be useful because our strategy of proof is to first establish our results in the quadrant for a general covariance matrix and then to extend the results to all convex cones.

Remark 1.3 (Equivalence between cones and quandrant)**.** *There is a bijective equivalence between the following two families of models:*

- Standard *reflected Brownian motions (i.e. identity covariance matrix) in* any convex cone *of angle* $\beta \in (0, \pi)$*,*
- *Reflected Brownian motions in a* quadrant *of* any covariance matrix *of the form*

$$\begin{pmatrix} 1 & -\cos\beta \\ -\cos\beta & 1 \end{pmatrix}.$$

In Section 11 this equivalence is established by means of a simple linear transformation defined in (11.2). Therefore, all the results established for one of these two families can be transposed directly to the other family.

Furthermore, any reflected Brownian motion in a general convex cone and with a general covariance matrix can always be reduced via a simple linear transformation to a Brownian motion of one of the two families of models mentioned above.

In the three previous theorems we consider a Brownian motion which is *standard*, i.e. of covariance matrix identity. But all the results stated above may easily be extended to all covariance matrices thanks to the simple linear transformation mentioned in the previous remark. The next remark explains how to proceed, in line with what is stated in Section 11.

Remark 1.4 (Generalisation to any covariance matrix in any convex cone)**.** *We consider* $\widehat{Z}_t$ *an obliquely reflected Brownian motion in a cone of angle* $\widehat{\beta}_0 \in (0, \pi)$ *starting from* $\widehat{z}_0$*, of reflection angles* $\widehat{\delta}$ *and* $\widehat{\varepsilon}$*, of drift* $\widehat{\mu}$ *of angle* $\widehat{\theta}$ *and of covariance matrix* $\widehat{\Sigma}$*. We introduce the angle* $\widehat{\beta}_1 := \arccos\left(-\frac{\widehat{\sigma}_{12}}{\sqrt{\widehat{\sigma}_{11}\widehat{\sigma}_{22}}}\right) \in (0, \pi)$ *and the linear transformation*

$$\widehat{T} := \begin{pmatrix} \frac{1}{\sin\widehat{\beta}_1} & \cot\widehat{\beta}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\widehat{\sigma}_{11}}} & 0 \\ 0 & \frac{1}{\sqrt{\widehat{\sigma}_{22}}} \end{pmatrix}$$

Then, the process $\widetilde{Z}_t := \widehat{T}\widehat{Z}_t$ *is an obliquely reflected* standard *Brownian motion in a cone of angle* $\beta \in (0, \pi)$ *starting from* $\widetilde{z}_0 := \widehat{T}\widehat{z}_0$*, of reflection angles* δ *and* ε *and of drift* $\widetilde{\mu} := \widehat{T}\widehat{\mu}$ *of angle* θ*. The angle parameters are in* $(0, \pi)$ *and are determined by*

$$\tan\beta = \frac{\sin\widehat{\beta}_1}{\frac{1}{\tan\widehat{\beta}_0}\sqrt{\frac{\widehat{\sigma}_{22}}{\widehat{\sigma}_{11}}} + \cos\widehat{\beta}_1}, \qquad \tan\theta = \frac{\sin\widehat{\beta}_1}{\frac{1}{\tan\widehat{\theta}}\sqrt{\frac{\widehat{\sigma}_{22}}{\widehat{\sigma}_{11}}} + \cos\widehat{\beta}_1},$$

$$\tan\delta = \frac{\sin\widehat{\beta}_1}{\frac{1}{\tan\widehat{\delta}}\sqrt{\frac{\widehat{\sigma}_{22}}{\widehat{\sigma}_{11}}} + \cos\widehat{\beta}_1}, \qquad \tan(\beta-\varepsilon) = \frac{\sin\widehat{\beta}_1}{\frac{1}{\tan(\widehat{\beta}_0-\widehat{\varepsilon})}\sqrt{\frac{\widehat{\sigma}_{22}}{\widehat{\sigma}_{11}}} + \cos\widehat{\beta}_1}.$$

Thanks to this linear transformation, we obtain the following relation between the Green's function of $\widehat{Z}_t$ *denoted by* $\widehat{g}(\widehat{z})$ *for* $\widehat{z}$ *inside the cone of angle* $\widehat{\beta}_0$ *and the Green's function of* $\widetilde{Z}_t$ *denoted by* $\widetilde{g}(\widetilde{z})$ *for* $\widetilde{z}$ *inside the cone of angle* β*:*

$$\widehat{g}(\widehat{z}) = \frac{1}{\sqrt{\det\widehat{\Sigma}}}\widetilde{g}(\widehat{T}\widehat{z}).$$

Therefore, the previous formula allows us to extend our results from $\widetilde{g}$ *to* $\widehat{g}$*.*

Plan and strategy of proof. In this article, the results will be first established in a quadrant for any covariance matrix and then transferred to any cone in the last section.

The first step in solving our problem is to determine a functional equation relating the Laplace transforms of Green's functions in the quadrant and on the edges, see Section 2. Next, we continue these Laplace transforms and study their singularities, see Section 3. Then, we use the inversion Laplace transform formula combined with the functional equation to express the Green's functions as a sum of simple integrals, see Section 4. Doetsch's book [13] is one of the leading references on Laplace transforms. To determine the asymptotics, we first use complex analysis to obtain Tauberian results which link the poles of the Laplace transforms to the asymptotics of the Green's functions. Then, we use a double refinement of the classical saddle-point method: the uniform method of the steepest descent. One of the reference books on this classical approach is those of Fedoryuk [18]. Appendix A, which gives a generalized version of the classical Morse Lemma by introducing a parameter dependency, will be useful to refine this saddle-point method. Section 5 studies the saddle point, Section 6 explains how we shift the integration contour and thus determines the contribution of the encountered poles to the asymptotics. Section 7 shows that some part of the new integration contour are negligible. Section 8 establishes the contribution of the saddle point to the asymptotics and states the main result. Section 9 studies the asymptotics along axes and Section 10 the asymptotics in the technical case where the saddle point meet a pole. Appendix B states a technical result useful to this section. Finally, Section 11 explains how to transfer to any convex cone the asymptotics results obtained in the quadrant in the previous sections and thus concludes the proof of Theorems 1, 2 and 3.

2. Convergence of Laplace transforms and functional equation

Transient reflected Brownian motion in a cone. Let $(Z_t)_{t\geq 0} = (z_0 + \mu t + B_t + RL_t)_{t\geq 0}$ be a (continuous) semimartingale reflected Brownian motion (SRBM) in $\mathbb{R}^2_+$ on a filtered probability space where $\mu = (\mu_1, \mu_2)^\top \in \mathbb{R}^2$ is the drift, Σ the covariance matrix associated to the Brownian motion B, $R = (r_{ij})_{1\leqslant i,j\leqslant 2} \in \mathbb{R}^{2\times 2}$ the reflection matrix, and $(L_t)_{t\geq 0} = ((L^1_t, L^2_t)^\top)_{t\geq 0}$ the local times on the edges associated to the process. We will assume that $\det(\Sigma) > 0$, i.e. that Σ is positive-definite. See Figure 3 to visualize the parameters of this process. We recall the following classical result concerning the existence of such a process, see for example [42, 46].

Proposition 2.1 (Existence and uniqueness of SRBM). *There exists an SRBM with parameters (μ, Σ, R) if and only if Σ is a covariance matrix and R is completely-$\mathcal{S}$, i.e.*

$$r_{11} > 0,\ r_{22} > 0,\ \text{and } [\,\det(R) > 0 \quad \text{or} \quad r_{21},\ r_{12} > 0\,]. \tag{2.1}$$

In this case, the SRBM is unique in law and defines a Feller continuous strong Markov process.

Condition (2.1) will therefore be required throughout the article. The recurrence and transience conditions of those processes are well known, see [33, 45]. In our case, the SRBM will be systematically transient because of the following assumption of positive drift, which will be held throughout the rest of the article.

Assumption 1 (Positivity of the drift). *We assume that $\mu_1 > 0$ and $\mu_2 > 0$.*

Note that this assumption is equivalent to that made in the introduction: $\theta \in (0, \beta)$.

Green's function. We are working in the transient case and we will focus on the Green's functions.

Definition 2.2 (Green's measures and densities). *The Green's measure G inside the quadrant is defined by*

$$G(z_0, A) := \mathbb{E}_{z_0}\left[\int_0^\infty \mathbb{1}_A(Z_t)dt\right] = \int_A g(z)dz$$

for $z_0 \in \mathbb{R}^2_+$ and $A \subset \mathbb{R}^2$ and admits a density g with respect to the Lebesgue measure. The density g is called the Green's function.

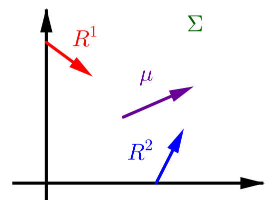


FIGURE 3. SRBM parameters in the quadrant: drift μ, reflection vectors R^1 and R^2 and covariance matrix Σ.

For $i \in \{1,2\}$, we define H_i the Green's measures on the edges of the quadrant which also has densities h_i with respect to the Lebesgue measure, namely

$$H_i(z_0, A) := \mathbb{E}_{z_0}\left[\int_0^\infty \mathbb{1}_A(Z_t) dL_t^i\right] = \int_A h_i(z)dz.$$

The measure H_1 has its support on the vertical axis and H_2 has its support on the horizontal axis.

Throughout the article one should be kept in mind that in the notations g and h_i we omit the dependence on the starting point z_0.

Proof. In the recurrent case, Harrison and Williams proved in [31] that the invariant measure has a density according to the Lebesgue measure. The proof done in that article extends to the transient case and justify the existence of a density with respect to the Lebesgue measure for the Green's measures. Indeed, the proof of Lemma (9) of section 7 in [31] shows that for a Borel set A of Lebesgue measure 0, we have

$$\mathbb{E}\left[\int_0^{+\infty} \mathbb{1}_A(Z_t)dt\right] = 0.$$

This is even an equivalence, but we don't need it here. Since the proof does not requires the recurrence property, this gives the desired result by the Radon Nikodym theorem. The same argument applies to the densities of H_i for $i = 1, 2$, see theorem (1), section 8 in [31]. □

Remark 2.3 (Partial differential equation). *Let us denote $\mathcal{L} = \frac{1}{2}\nabla \cdot \Sigma\nabla + \mu \cdot \nabla$ the generator of the SRBM inside the quadrant and $\mathcal{L}^* = \frac{1}{2}\nabla \cdot \Sigma\nabla - \mu \cdot \nabla$ its dual operator. Then, the Green's function g satisfies*

$$\mathcal{L}^* g = -\delta_{z_0}$$

in the sense of distributions $\mathcal{D}'((\mathbb{R}_+^*)^2)$.

Let us define the matrix $R^* = 2\Sigma - R\ \mathrm{diag}(R)^{-1}\mathrm{diag}(\Sigma)$. We denote R_1^* and R_2^* the two columns of R^*. Then, the following boundary conditions holds

$$\begin{cases} \partial_{R_1^*} g(z) - 2\mu_1 g(z) = 0 \text{ for } z \in \{0\} \times \mathbb{R}_+ \\ \partial_{R_2^*} g(z) - 2\mu_2 g(z) = 0 \text{ for } z \in \mathbb{R}_+ \times \{0\} \end{cases}$$

where $\partial_{R_i^*} = R_i^* \cdot \nabla$.

Sketch of proof of the remark. The partial differential equation of the Green's function and its boundary conditions are derived from the forward equation of the transition kernel establish in [29], see Equation (8.3). However, we provides here a direct elementary proof of the fact that $\mathcal{L}^* g = -\delta_{z_0}$. Let $\varphi \in C_c^\infty((\mathbb{R}_+^*)^2)$. We apply Ito's formula and we take the expectation,

$$\mathbb{E}[\varphi(Z_t)] = \varphi(z_0) + \mathbb{E}\left[\int_0^t \mathcal{L}\varphi(Z_s)ds\right].$$

One may remark that there are no boundary terms since φ cancel on a neighborhood of the boundaries. Since we are in the transient case and since φ is bounded, the left term converges to 0 while t goes to infinity by the dominated convergence theorem. Since successive derivatives of φ are bounded, $\mathcal{L}\varphi(a,b)$ is bounded by an exponential function up to a multiplication constant. Thanks to convergence domain of the Laplace transform (see Proposition 2.6 bellow), we obtain by dominated convergence that $\varphi(z_0) = -\mathbb{E}\left[\int_0^{+\infty} \mathcal{L}\varphi(Z_s)ds\right] = -\int_{\mathbb{R}_+^2} \mathcal{L}\varphi(z)g(z)dz$ which implies that $\mathcal{L}^*g = -\delta_{z_0}$. □

Furthermore, it is preferable to have continuity of the Green's function to talk about their asymptotic behaviour. This is the subject of the following comment.

Remark 2.4 (Smoothness of Green's functions). *By the strictly elliptic regularity theorem, we may deduce from $\mathcal{L}^*g = -\delta_{z_0}$ that the density g has a $\mathcal{C}^\infty$ version on $(\mathbb{R}_+^*)^2\backslash\{z_0\}$. We won't go into more detail here about the proof of this result. In the remainder of this article, we will assume that this property is true and that g is continuous on $(\mathbb{R}_+^2)^* \setminus \{z_0\}$.*

Laplace transform and functional equation.

Definition 2.5 (Laplace transform of Green's functions). *For $(x,y) \in \mathbb{C}^2$ we define the Laplace transforms of the Green's measures by*

$$\varphi(x,y) := \mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,y)\cdot Z_t}dt\right] = \int_{\mathbb{R}_+^2} e^{(x,y)\cdot z}g(z)dz$$

and

$$\varphi_1(y) := \mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,y)\cdot Z_t}dL_t^1\right] = \int_{\mathbb{R}_+} e^{yb}h_1(b)db, \quad \varphi_2(x) := \mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,y)\cdot Z_t}dL_t^2\right] = \int_{\mathbb{R}_+} e^{xa}h_2(a)da.$$

Let us remark that φ_1 does not depend on x and φ_2 does not depend on y. One has to remember the dependence on the starting point z_0 even though we omit it in the notations.

Since Green's measures are not probability measures, the convergence of their Laplace transforms is not guaranteed. For example $\varphi(0)$ is not finite. Convergence domains had already been studied in [22] but we need stronger results. The following proposition establishes the convergence when the real part of x and y is negative.

Proposition 2.6 (Convergence of the Laplace transform). *Assuming that $\mu_1 > 0$ and $\mu_2 > 0$,*

- *$\varphi_1(y)$ converges (at least) on $y \in \{y \in \mathbb{C}, \Re(y) < 0\}$*
- *$\varphi_2(x)$ converges (at least) on $x \in \{x \in \mathbb{C}, \Re(x) < 0\}$*
- *$\varphi(x,y)$ converges (at least) on $(x,y) \in \{(x,y) \in \mathbb{C}^2, \Re(x) < 0 \text{ and } \Re(y) < 0\}$.*

Before proving this proposition, we state the functional equation that will be central in this article. First, we need to define for $(x,y) \in \mathbb{C}^2$ the following polynomials

$$\begin{cases} \gamma(x,y) = \frac{1}{2}(x,y)\cdot\Sigma(x,y) + (x,y)\cdot\mu = \frac{1}{2}(\sigma_{11}x + 2\sigma_{12}xy + \sigma_{22}y^2) + \mu_1 x + \mu_2 y \\ \gamma_1(x,y) = R^1\cdot(x,y) = r_{11}x + r_{21}y \\ \gamma_2(x,y) = R^2\cdot(x,y) = r_{12}x + r_{22}y \end{cases}$$

where R^1, R^2 are the two columns of the reflection matrix R. The polynomial γ is called the kernel.

Proposition 2.7 (Functional equation). *If $\Re(x) < 0$ and $\Re(y) < 0$, then*

$$-\gamma(x,y)\varphi(x,y) = \gamma_1(x,y)\varphi_1(y) + \gamma_2(x,y)\varphi_2(x) + e^{(x,y)\cdot z_0}. \tag{2.2}$$

The proofs of these two proposition are deeply linked. So we'll be gathering their proofs.

Proof of Propositions 2.6 and 2.7. The main idea of the proof is to take the expectation of Itô's formula applied to the SRBM and to use a sign argument to justify the limit when $t \to +\infty$. The beginning of the proof is inspired of the Proposition 5 of [22].

Let $(x,y)\in(\mathbb{R}_-^*)^2$, Itô's formula applied to $f(z):=e^{(x,y)\cdot z}$ gives

$$f(Z_t)-f(z_0)=\int_0^t \nabla f(Z_s).dB_s+\int_0^t \mathcal{L}f(Z_s)ds+\sum_{i=1}^2\int_0^t R_i\cdot\nabla f(Z_s)dL_s^i \tag{2.3}$$

$$=\int_0^t \nabla f(Z_s).dB_s+\gamma(x,y)\int_0^t e^{(x,y)\cdot Z_s}ds+\sum_{i=1}^2\gamma_i(x,y)\int_0^t e^{(x,y)\cdot Z_s}dL_s^i \tag{2.4}$$

where $\mathcal{L}=\frac{1}{2}\nabla\cdot\Sigma\nabla+\mu\cdot\nabla$ is the generator of the Brownian motion. Since $(x,y)\in(\mathbb{R}_-^*)^2$, the integral $\int_0^t\nabla f(Z_s).dB_s$ is a martingale (its quadratic variation is bounded by $C.t$ for a constant $C>0$) and its expectation cancels out. Therefore,

$$\mathbb{E}_{z_0}\left[e^{(x,y)\cdot Z_t}\right]-e^{(x,y)\cdot z_0}-\gamma(x,y)\mathbb{E}_{z_0}\left[\int_0^t e^{(x,y)\cdot Z_s}ds\right] \tag{2.5}$$
$$=\mathbb{E}_{z_0}\left[\gamma_1(x,y)\int_0^t e^{(x,y)\cdot Z_s}dL_s^1+\gamma_2(x,y)\int_0^t e^{(x,y)\cdot Z_s}dL_s^2\right].$$

The expectations in the left-hand side of the previous equation are finite because $(x,y)\in(\mathbb{R}_-^*)^2$, the first one is bounded by 1 and the second one by t. This implies that the expectation of the right-hand side is also finite.

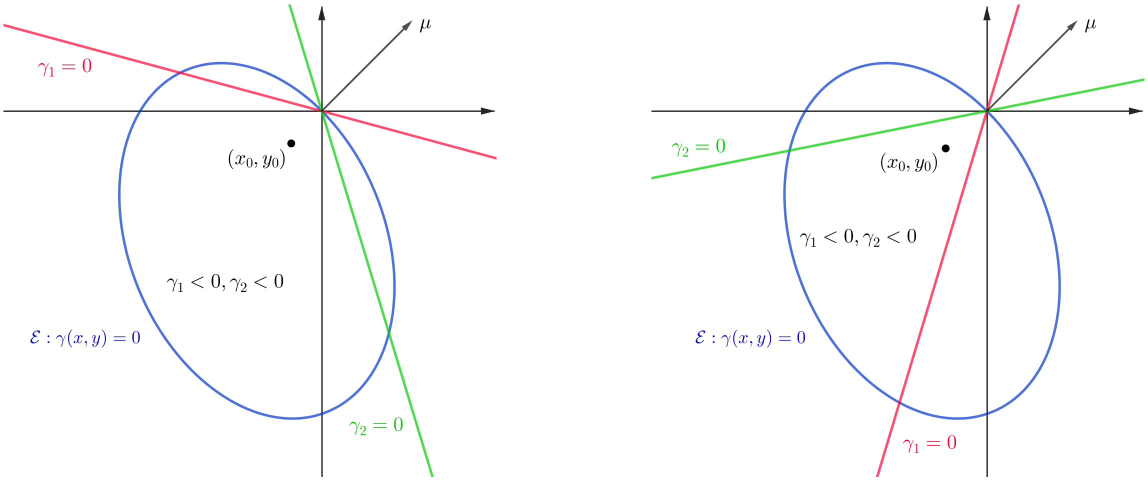
The aim now is to take the limit of (2.5) when t goes to infinity to show the finiteness of the Laplace transforms and the functional equation. First, since $(x,y)\in(\mathbb{R}_-^*)^2$ and $\|Z_t\|\underset{t\to\infty}{\longrightarrow}+\infty$ a.s., the expectation $\mathbb{E}_x\left[e^{(x,y)\cdot Z_t}\right]$ converges toward 0 when $t\to\infty$ by the dominated convergence theorem. Secondly, by the monotone convergence theorem the expectation $\mathbb{E}_{z_0}\left[\int_0^t e^{(x,y)\cdot Z_s}ds\right]$ converges in $[0,\infty]$ to $\varphi(x,y)=\mathbb{E}_{z_0}\left[\int_0^\infty e^{(x,y)\cdot Z_s}ds\right]$.

Let us assume for a moment that it is possible to choose $(x_0,y_0)\in(\mathbb{R}_-^*)^2$ such that $\gamma(x_0,y_0)<0$, $\gamma_1(x_0,y_0)<0$ and $\gamma_2(x_0,y_0)<0$. We use a proof by contradiction assuming that we have $\mathbb{E}_{z_0}\left[\int_0^\infty e^{(x_0,y_0)\cdot Z_s}ds\right]=+\infty$. Since $\gamma(x_0,y_0)<0$, it implies that the left-hand side of (2.5) will be positive for t large enough. But, since $\gamma_1(x_0,y_0)<0$ and $\gamma_2(x_0,y_0)<0$, the right-hand side of (2.5) is always negative. We obtain a contradiction and we deduce that $\varphi(x_0,y_0)=\mathbb{E}_{z_0}\left[\int_0^\infty e^{(x_0,y_0)\cdot Z_s}ds\right]$ is finite. Hence the limit of the right-hand side of (2.5) is also finite and converges by the monotone convergence theorem to $\gamma_1(x_0,y_0)\varphi_1(y_0)+\gamma_2(x_0,y_0)\varphi_2(x_0)$. We deduce that $\varphi_1(y_0)$ and $\varphi_2(x_0)$ are also finite and that the functional equation (2.2) is satisfied in (x_0,y_0). This implies that for all x and y in $\mathbb{C}$ such that $\Re x<x_0$ and $\Re y<y_0$ the Laplace transforms $\varphi(x,y)$, $\varphi_1(y)$ and $\varphi_2(x)<\infty$ are finite and the functional equation (2.2) is satisfied by taking the limit of (2.5) when $t\to\infty$.

All that remains is to show that we can always choose x_0 and y_0 *as close to* 0 *as we like*, such that $(x_0,y_0)\in(\mathbb{R}_-^*)^2$, $\gamma(x_0,y_0)<0$, $\gamma_1(x_0,y_0)<0$ and $\gamma_2(x_0,y_0)<0$ and the proof of Propositions 2.6 and 2.7 will be complete. Let us denote $\mathcal{E}$ the ellipse of equation $\gamma(x,y)=0$. One may observe that the interior of the ellipse $\mathcal{E}$ defined by $\gamma(x,y)<0$ contains a neighbourhood of 0 intersecting $(\mathbb{R}_-^*)^2$ by Assumption 1 on the positivity of the drift. Indeed, the drift is an external normal to the ellipse at $(0,0)$. We consider two cases coming from the existence condition of the process (2.1). First case, $r_{11}>0$, $r_{22}>0$, $r_{12}>0$ and $r_{21}>0$, see Figure 4a. In this case, one may see directly see that $\gamma_1(x,y)<0$ and $\gamma_2(x,y)<0$ on $(\mathbb{R}_-^*)^2$. It is therefore easy to pick (x_0,y_0) close enough to $(0,0)$ which satisfies the required conditions. Second case, $r_{11}>0$, $r_{22}>0$ and $\det(R)>0$, see Figure 4b. In this case, the cone defined by $\gamma_1<0$ and $\gamma_2<0$ has a non-empty intersection with $(\mathbb{R}_-^*)^2$. Hence, one can still choose (x_0,y_0) as close as we want to $(0,0)$ inside the desired cone and the ellipse $\mathcal{E}$.

□

The following lemma follows from the functional equation and states that the boundary green's densities h_1 and h_2 are equal, up to some constant, to the bivariate green's function g on the axes.



(A) Case $r_{11} > 0$, $r_{22} > 0$, $r_{12} > 0$ and $r_{21} > 0$.

(B) Case $r_{11} > 0$, $r_{22} > 0$ and $\det R > 0$.

FIGURE 4. For $(x, y) \in \mathbb{R}^2$, illustration of the domain where $\gamma_1 < 0$ and $\gamma_2 < 0$.

Proposition 2.8 (Green's densities on the boundaries)**.** *The Green's density g is related to the boundary Green's densities h_i by the formulas*

$$r_{11}h_1(b) = \frac{\sigma_{11}}{2}g(0, b) \quad \textit{and} \quad r_{22}h_2(a) = \frac{\sigma_{22}}{2}g(a, 0).$$

Proof. The initial value formula of a Laplace transform gives

$$x\varphi(x, y) \underset{x\to-\infty}{\longrightarrow} -\int_0^\infty e^{yb}g(0, b)db.$$

Therefore, by dividing the functional equation (2.2) by x and taking the limit when x tends to infinity, we obtain

$$\frac{1}{2}\sigma_{11}\int_0^\infty e^{yb}g(0, b)db = r_{11}\varphi_1(y) = r_{11}\int_0^\infty e^{yb}h_1(b)db$$

which implies the result. □

3. CONTINUATION AND PROPERTIES OF $\varphi_1(x)$ AND $\varphi_2(y)$

The first step of this analytic approach is to study the kernel.

Lemma 3.1 (Kernel study)**.** (i) *Equation $\gamma(x, y) = 0$ determines an algebraic function $Y(x)$ [resp. $X(y)$] with two branches*

$$Y^{\pm}(x) = \frac{1}{\sigma_{22}}\Big(- \sigma_{12}x - \mu_2 \pm \sqrt{(\sigma_{12}^2 - \sigma_{11}\sigma_{22})x^2 + 2(\mu_2\sigma_{12} - \mu_1\sigma_{22})x + \mu_2^2}\Big)$$

The function $Y(x)$ [resp. $X(y)$] has two branching points x_{min} and x_{max} [resp. y_{min} and y_{max}] given by

$$x_{min} = \frac{\mu_2\sigma_{12} - \mu_1\sigma_{22} - \sqrt{D_1}}{\det(\Sigma)}, \quad x_{max} = \frac{\mu_2\sigma_{12} - \mu_1\sigma_{22} + \sqrt{D_1}}{\det(\Sigma)},$$

$$y_{min} = \frac{\mu_1\sigma_{12} - \mu_2\sigma_{11} - \sqrt{D_2}}{\det(\Sigma)}, \quad y_{max} = \frac{\mu_1\sigma_{12} + \mu_2\sigma_{11} - \sqrt{D_2}}{\det(\Sigma)},$$

where $D_1 = (\mu_2\sigma_{12} - \mu_1\sigma_{22})^2 + \mu_2^2\det(\Sigma)$ and $D_2 = (\mu_1\sigma_{12} - \mu_2\sigma_{11})^2 + \mu_1^2\det(\Sigma)$. Both of them are real and $x_{min} < 0 < x_{max}$ [resp. $y_{min} < y_{max}$]. The branches of $Y(x)$ [resp.

$X(y)$] take real values if and only if $x \in [x^{min}, x^{max}]$ [resp. $y \in [y_{min}, y_{max}]$]. Furthermore $Y^-(0) = -\frac{2\mu_2}{\sigma_{22}} < 0$, $Y^-(x_{max}) < 0$, $Y^+(0) = 0$, $Y^+(x_{max}) < 0$. See Figure 5.

(ii) *For any $u \in \mathbb{R}$*

$$\mathrm{Re}Y^{\pm}(u+iv) = \frac{1}{\sigma_{22}}\Big(-\sigma_{12}u - \mu_2 \pm \frac{1}{\sqrt{2}}\sqrt{(u-x_{min})(x_{max}-u)+v^2+|(u+iv-x_{min})(x_{max}-u-iv)|}\Big).$$

(iii) *Let $\delta = \infty$ if $\sigma_{12} \geq 0$ and $\delta = -\mu_2/\sigma_{12} - x_{max} > 0$ if $\sigma_{12} < 0$. Then for some $\epsilon > 0$ small enough*

$$\mathrm{Re}Y^-(u+iv) < 0 \quad \text{for } u \in]-\epsilon,\ x_{max}+\delta[, \quad v \in \mathbb{R}.$$

Proof. Points (i) and (ii) follow from elementary considerations. The fact that $Y^+(x_{max}) < 0$ implies the inequality $-\sigma_{12}x_{max} - \mu_2 < 0$, so that $\delta > 0$. Furthermore by (ii) $\mathrm{Re}\,Y^-(u+iv) \leq \mathrm{Re}Y^-(u)$ which is strictly negative for $u \in]-\epsilon, x_{max}+\delta[$ by the analysis made in (i). □

Lemma 3.2 (Continuation of the Laplace transform). *Function $\varphi_2(x)$ can be meromorphically continued to the (cut) domain*

$$\{x = u+iv \mid u < x_{max}+\delta, v \in \mathbb{R}\} \setminus [x_{max}, x_{max}+\delta] \tag{3.1}$$

by the formula :

$$\varphi_2(x) = \frac{-\gamma_1(x, Y^-(x))\varphi_1(Y^-(x)) - \exp\left(a_0x + b_0Y^-(x)\right)}{\gamma_2(x, Y^-(x))}. \tag{3.2}$$

A symmetric continuation formula holds for φ_1.

Proof. By Lemma 3.1 (iii) for any $x = u+iv$ with $u \in]-\epsilon, 0[$ the following equation is valid.

$$\gamma(x, Y^-(x))\varphi(x, Y^-(x)) = \gamma_1(x, Y^-(x))\varphi_1(Y^-(x)) + \gamma_2(x, Y^-(x))\varphi_2(x) + \exp(a_0x + b_0Y^-(x)).$$

Since $\gamma(x, Y^-(x)) = 0$, the statement follows. □

Proposition 3.3 (Poles of the Laplace transform, necessary condition). (i) *$x = 0$ is not a pole of $\varphi_2(x)$, so that $\varphi_2(0) = \mathbb{E}[L^2_\infty] < +\infty$. The local time spent by the process on the horizontal axis is finite.*

(ii) *If x^* is a pole of $\varphi_2(x)$ in the domain (3.1), then $(x^*, Y^-(x^*))$ is a unique non-zero solution of the system of two equations*

$$\gamma(x,y) = 0, \qquad \gamma_2(x,y) = r_{12}x + r_{22}y = 0. \tag{3.3}$$

Moreover, x^* is real and belongs to $]0, x_{max}[$. Furthermore, this solution exist only if

$$x_{max}r_{12} + Y^{\pm}(x_{max})r_{22} > 0.$$

(iii) *If y^{**} is a pole of $\varphi_1(y)$, then $(X^-(y^{**}), y^{**})$ is a unique non-zero solution of the system of two equations*

$$\gamma(x,y) = 0, \qquad \gamma_1(x,y) = r_{11}x + r_{21}y = 0. \tag{3.4}$$

Moreover, y^{**} is real and belongs to $]0, y_{max}[$. Furthermore, this solution exist only if

$$y_{max}r_{21} + X^{\pm}(y_{max})r_{11} > 0.$$

When these solutions exist we have

$$x^* = 2\frac{\mu_2\frac{r_{12}}{r_{22}} - \mu_1}{\sigma_{11} - 2\sigma_{12}\frac{r_{12}}{r_{22}} + \sigma_{22}\left(\frac{r_{12}}{r_{22}}\right)^2} \quad \text{and} \quad y^{**} = 2\frac{\mu_1\frac{r_{21}}{r_{11}} - \mu_2}{\sigma_{11}\left(\frac{r_{21}}{r_{11}}\right)^2 - 2\sigma_{12}\frac{r_{21}}{r_{11}} + \sigma_{22}}. \tag{3.5}$$

Then, we define

$$y^* := Y^+(x^*) \quad \text{and} \quad x^{**} := X^+(y^*).$$

See Figure 5 to visualize all these points.

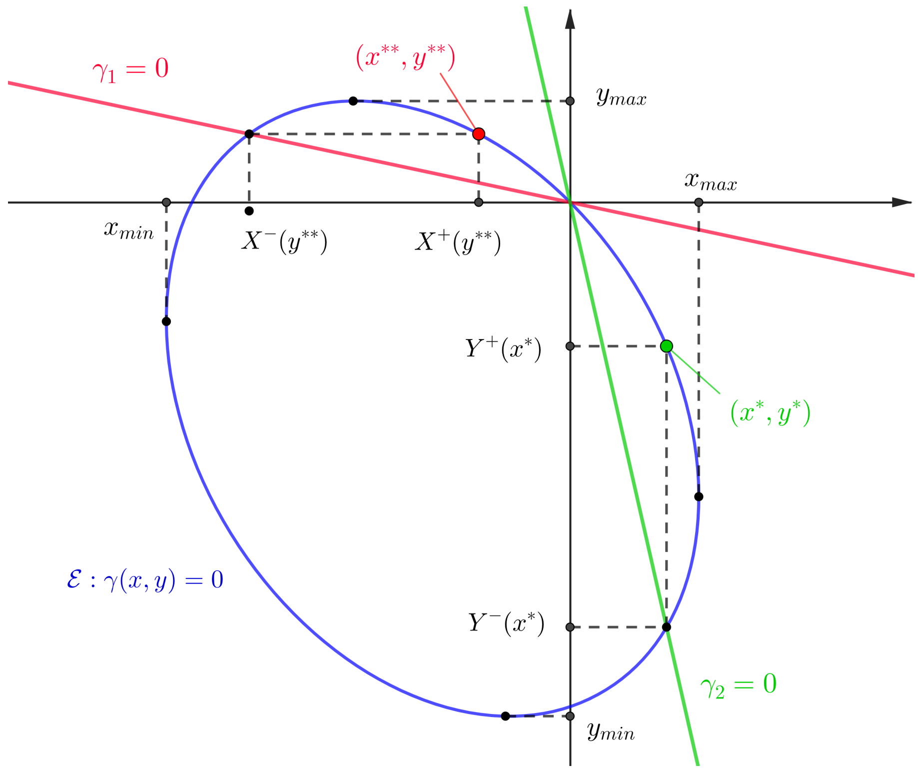


FIGURE 5. In the real plane (x, y), graphic representation of poles x^* and y^{**} when both exist.

Proof. (i) The observation that $\gamma_2(0, Y^-(0)) = r_{22} \times Y^-(0) \neq 0$ implies the first statement.

(ii) If x^* is a pole of $\varphi_2(x)$, then $(x^*, Y^-(x^*))$ should be a solution the system (3.3) above by by the continuation formula (3.2) and the continuity of φ_1 [resp. φ_2] on $\{\Re y \leq 0\}$ [resp. $\{\Re x \leq 0\}$]. This system has one solution $(0, 0)$ and the second one (x°, y°), which is necessarily real. Then $x^\circ \in [x_{min}, x_{max}]$ and y° is either $Y^-(x^\circ)$ or $Y^+(x^\circ)$. But x° can be a pole of $\varphi_2(x)$, if only it is within $]0, x_{max}]$ and $y^\circ = Y^-(x^\circ)$. This last condition implies $\frac{r_{12}}{r_{22}} > \frac{-Y^\pm(x_{\max})}{x_{max}}$. □

Proposition 3.4 (Poles of the Laplace transforms, sufficient condition)**.** *The pole x^* (resp. y^{**}) of φ_2 (resp. φ_1) exists if (and only if) $x_{max}r_{12} + Y^\pm(x_{max})r_{22} > 0$ (resp. $y_{max}r_{21} + X^\pm(y_{max})r_{11} > 0$).*

Proof. The conditions of the previous proposition are necessary. The next two lemmas prove the sufficiency. In those, we denote the dependence of Laplace transforms with the initial condition z_0 by $\varphi_1^{z_0}, \varphi_2^{z_0}$ instead of φ_1, φ_2. The proof is done for x^*, but is of course symmetrical for y^{**}. □

Lemma 3.5 (Existence of the pole for a starting point)**.** *If $x_{max}r_{12} + Y^\pm(x_{max})r_{22} > 0$, there exists $z_0 \in \mathbb{R}_+^2$ such that x^* is a pole of $\varphi_2^{z_0}$.*

Proof. The denominator of the continuation formula (3.2) vanishes since we assume that $x_{max}r_{12} + Y^\pm(x_{max})r_{22} > 0$. We are looking for a z_0 such that the numerator doesn't vanish at x^*, which will imply that z_0 is a pole of φ_2. If $\gamma_1(x^*, Y^-(x^*)) \geq 0$, this is obvious thanks to the exponential term and a sign argue. We suppose now that $-C := \gamma_1(x^*, Y^-(x^*)) < 0$. We make a proof by contradiction assuming that

$$\forall z_0 = (a_0, b_0) \in \mathbb{R}_+^2, \quad -C\varphi_1^{(a_0,b_0)}(Y^-(x^*)) + e^{a_0 x^* + b_0 Y^-(x^*)} = 0. \tag{3.6}$$

Let T be the stopping time defined by the first hitting time of the axis $\{x = 0\}$, i.e. $T = \inf\{t \geq 0, Z_t^1 = 0\}$ with $Z = (Z^1, Z^2)$. (It is possible that $T = +\infty$). Firstly, since the the Stieltjes measure

dL^1 is supported by $\{Z^1 = 0\}$ and since Z is a strong Markov process, for a starting point $z_0 = (a_0, b_0)$ we have:

$$\varphi_1^{(a_0,b_0)}(Y^-(x^*)) = \mathbb{E}_{(a_0,b_0)}\left[\int_T^{+\infty} e^{Z_t^2.Y^-(x^*)} dL_t^1 \mathbb{1}_{T<+\infty}\right] \tag{3.7}$$

$$= \mathbb{E}_{(a_0,b_0)}\left[\mathbb{E}_{Z_T}\left[\int_0^{+\infty} e^{Z_t^2.Y^-(x^*)} dL_t^1\right] \mathbb{1}_{T<+\infty}\right] \tag{3.8}$$

$$= \mathbb{E}_{(a_0,b_0)}\left[\varphi_1^{(0,Z_T^2)}(Y^-(x^*))\mathbb{1}_{T<+\infty}\right]. \tag{3.9}$$

Conditioning by the value of Z_T^2, using (3.6) and $Y^-(x^*) \le 0$, we get :

$$\varphi_1^{(a_0,b_0)}(Y^-(x^*)) = \int_0^{+\infty} \varphi_1^{(0,b)}(Y^-(x^*))\mathbb{P}_{(a_0,b_0)}(T<+\infty, Z_T^2 = db) \tag{3.10}$$

$$= \int_0^{+\infty} \frac{1}{C} e^{0.x^* + bY^-(x^*)}\mathbb{P}_{(a_0,b_0)}(T<+\infty, Z_T^2 = db) \le \frac{1}{C}\mathbb{P}_{(a_0,b_0)}(T<+\infty) \le \frac{1}{C}. \tag{3.11}$$

But, (a_0, b_0) can be chosen such that $e^{a_0 x^* + b_0 Y^-(x^*)}$ is as huge as wanted because $x^* > 0$. This is in contradiction with (3.6). □

Lemma 3.6 (Existence of a pole for all starting points)**.** *If x^* is a pole of $\varphi_2^{z_0}$ for some $z_0 \in \mathbb{R}_+^2$, then x^* is a pole of $\varphi_2^{z_0'}$ for every $z_0' \in \mathbb{R}_+^2$.*

The proof of this lemma is postponed bellow Proposition 3.8 since it needs this proposition to be established.

Lemma 3.7 (Nature of the branching point of φ_2)**.** *Let $x \to x_{max}$ with $x < x_{max}$, we have*

- *If $\gamma_2(x_{max}, Y^-(x_{max})) = 0$, i.e. $x^* = x_{max}$, then*
$$\varphi_2(x) = \frac{C}{\sqrt{x_{max} - x}} + O(1)$$
for a constant $C > 0$.
- *If $\gamma_2(x_{max}, Y^-(x_{max})) \neq 0$, then*
$$\varphi_2(x) = C_1 + C_2\sqrt{x_{max} - x} + O(x_{max} - x)$$
for constants $C_1 \in \mathbb{R}$ and $C_2 > 0$.

Proof. Thanks to Lemma 3.1, Y^- can be written as $Y^-(x) = Y^-(x_{max}) - c\sqrt{x_{max} - x} + O(x_{max} - x)$ where $c > 0$. Let's carry out an elementary asymptotic expansion of the quotient of the continuation formula (3.2). First of all,

$$\frac{1}{\gamma_2(x, Y^-(x))} = \frac{1}{\gamma_2(x_{max}, Y^-(x_{max})) - r_{22}c\sqrt{x_{max} - x} + O(x_{max} - x)}$$
$$= \begin{cases} \frac{-1}{r_{22}c\sqrt{x_{max}-x}} & \text{if } \gamma_2(x_{max}, Y^-(x_{max}) = 0, \\ \frac{1}{\gamma_2(x_{max},Y^-(x_{max}))}\left(1 + \frac{r_{22}c\sqrt{x_{max}-x}}{\gamma_2(x_{max},Y^-(x_{max}))} + O(x_{max}-x)\right) & \text{if } \gamma_2(x_{max}, Y^-(x_{max})) \neq 0. \end{cases}$$

Secondly, for the numerator,

$$\begin{aligned} &\gamma_1(x, Y^-(x))\varphi_1(Y^-(x)) + e^{a_0 x + b_0 Y^-(x)} = \\ &\quad \left(\gamma_1(x_{max}, Y^-(x_{max})) - r_{21}c\sqrt{x_{max} - x} + O(x_{max} - x)\right) \\ &\quad \times \left(\varphi_1(Y^-(x_{max})) - c\varphi_1'(Y^-(x_{max}))\sqrt{x_{max} - x} + O(x_{max} - x)\right) \\ &\quad + e^{a_0 x_{max} + b_0 Y^-(x_{max})}(1 - cb_0\sqrt{x_{max} - x} + O(x_{max} - x)) \end{aligned} \tag{3.12}$$

Combining the two asymptotic expansions, we obtain the desired formula with

$$C = \frac{\gamma_1(x_{max}, Y^-(x_{max}))\varphi_1(Y^-(x_{max})) + e^{a_0 x_{max} + b_0 Y^-(x_{max})}}{r_{22} c}$$

and

$$C_2 = \frac{1}{\gamma_2(x_{max}, Y^-(x_{max}))}\Big[r_{21} c \varphi_1(Y^-(x_{max})) + c\gamma_1(x_{max}, Y^-(x_{max}))\varphi_1'(Y^-(x_{max})) + c b_0 e^{a_0 x_{max} + b_0 Y^-(x_{max})} - \frac{r_{22} c}{\gamma_2(x_{max}, Y^-(x_{max}))}\Big(\gamma_1(x_{max}, Y^-(x_{max}))\varphi_1(Y^-(x_{max})) + e^{a_0 x_{max} + b_0 Y^-(x_{max})}\Big)\Big]$$

□

The following proposition states the asymptotics of the Green's functions h_1 and h_2 on the boundaries. We note that we obtain the same asymptotics as in Theorem 2 and 5 with $\alpha \to 0$, which is consistent with the link made between h_1, h_2 and g in Proposition 2.8.

Proposition 3.8 (Asymptotics of the Green's functions on the boundary h_1 and h_2)**.** *In this lemma we denote by c a constant which can be different from one line to another.*

(1) *Suppose that we have a pole $x^* \in]0, x_{max}[$ for φ_2. Then, the Green's function h_2 has the following asymptotics*
$$h_2(u) \underset{u\to\infty}{\sim} c e^{-x^* u}.$$
(2) *Suppose that $x^* = x_{max}$, then*
$$h_2(u) \underset{u\to\infty}{\sim} c u^{-1/2} e^{-x_{max} u}.$$
(3) *Suppose that there is no pole in $]0, x_{max}[$ and that $x^* \neq x_{max}$, then,*
$$h_2(u) \underset{u\to\infty}{\sim} c u^{-3/2} e^{-x_{max} u}.$$

A symmetric result holds for h_1.

Proof. The result directly follows from classical Tauberian inversion lemmas which link the asymptotic of a function at infinity to the first singularity of its Laplace transform (which is here given in Lemma 3.7). We refer here Theorem 37.1 of Doetsch's book [13] and more precisely we apply the special case stated in Lemma C.2 of [10]. To apply this lemma we have to verify the analyticity and the convergence to 0 at infinity of φ_2 in a domain $\mathcal{G}_\delta(x_{max}) := \{z \in \mathbb{C} : z \neq x_{max}, |\arg(z - x_{max})| > \delta\}$ for some $\delta \in (0, \pi/2)$. But this follows directly from the continuation procedure of Lemma 3.2 :the exponential part of the continuation formula (3.2) tends to 0 in a domain $\mathcal{G}_\delta(x_{max})$ for some $\delta \in (0, \pi/2)$ by using (ii) of lemma 3.1. Note that the convergence to 0 also follows from Lemma C.1. Then, Lemma 3.7 gives the nature at the branching point x_{max} which is the smallest singularity except in the case where there is a pole in $]0, x_{max}[$, then the pole x^* is the smallest singularity. □

Remark 3.9. *We can remark in the proof of Lemma 3.7 that $O(1)$ and $O(x_{max} - x)$ of this lemma are locally uniform according to z_0. Which means that $\sup_{z_0' \in V} \left| \varphi_2^{(z_0')}(x) - \frac{C^{(z_0')}}{\sqrt{x_{max} - x}} \right| = O(1)$ as $x \to x^*$ when $\gamma_2(x_{max}, Y^-(x_{max})) = 0$ for a sufficiently small neighborhood V of z_0 (and the same holds for $O(x_{max} - x)$ in the other case). This imply that the results of Proposition 3.8 hold locally uniformly in z_0. Indeed it is enough to adapt the Tauberian lemmas of* [13] *used in the proof of Proposition 3.8 in a slightly more technical but quite similar way. Note that the constants c of this proposition depend continuously on z_0.*

Proof of Lemma 3.6. Let $z_0 = (a_0, b_0)$ be a starting point such that x^* is a pole of $\varphi_2^{z_0}$. Then, the continuation formula (3.2) implies that $-\gamma_1(x^*, Y^-(x^*))\varphi_1^{z_0}(Y^-(x^*)) - \exp\big(a_0 x^* + b_0 Y^-(x^*)\big) \neq 0$. By continuity with respect to the starting point (which follows from the integral formula given in [22] or from [37]), there exists a neighbourhood V of z_0 such that $-\gamma_1(x^*, Y^-(x^*))\varphi_1^{z_0'}(Y^-(x^*)) - \exp\big(a_0' x^* +$

$b_0'Y^-(x^*)\big) \neq 0$ for all $z_0' = (a_0', b_0') \in V$. Therefore, by the continuation formula, x^* is a pole of $\varphi_2^{z_0'}$ for all $z_0' \in V$. From Proposition 3.8 and by continuity of the constant of this proposition according to z_0' we deduce the following. If x^* is a pole of $\varphi_2^{z_0'}$, there exists a constant c such that for all $z_0' \in V$ we have $h_2^{(z_0')}(u) = ce^{-x^*u}(1+o(1))$ (notice that $o(1)$ is uniform in z_0' in the sense of Remark 3.9 and that c is continuous in z_0'). For $z_0'' \in \mathbb{R}^2$ we introduce the stopping time

$$T_V := \inf\{t > 0 : Z_t^{z_0''} \in V\}$$

where $Z_t^{z_0''}$ denotes the process starting from z_0''. By the strong Markov property applied to T_V we have for some constant C and when $u \to \infty$,

$$h_2^{z_0''}(u) \geqslant \mathbb{P}_{z_0''}(T_V < \infty) \inf_{z_0' \in V} h_2^{z_0'}(u) = Ce^{-x^*u}(1+o(1)).$$

We deduce by Proposition 3.8 that z_0'' is necessarily a pole. $\square$

We conclude this section with the following lemma which will be needed in Section 6.

Lemma 3.10 (Boundedness of the Laplace transform)**.** *Let $\eta \in]0, \delta[$, we have*

$$\sup_{\substack{u \in [X^{\pm}(y_{\max})-\eta, x_{max}+\eta] \\ |v|>\epsilon}} |\varphi_2(u+iv)| < \infty.$$

Proof. Clearly, for any $x = u + iv$ with $u < 0$, $|\varphi_2(u+iv)| \leq \varphi_2(u)$. Then for any $\epsilon > 0$,

$$\sup_{u \in [X^{\pm}(y_{\max})-\eta, -\epsilon]} |\varphi_2(u+iv)| < \infty. \tag{3.13}$$

For any $x = u+iv$ with $u \in [-\epsilon, x_{max} + \eta]$ Lemma 3.2 applies and gives the representation (3.2). Let us consider all its terms. By Lemma 3.1 (ii), for any fixed $u \in \mathbb{R}$, function $\mathrm{Re}Y^-(u+iv)$ is strictly decreasing as $|v|$ goes from 0 to infinity. Moreover for any $u \in [-\epsilon, x_{max} + \delta]$

$$\mathrm{Re}Y^-(u+iv) \leq -\frac{1}{\sqrt{2}\sigma_{22}}|v|.$$

Then,

$$|\varphi_1(Y^-(u+iv))| \leq \varphi_1(\mathrm{Re}\, Y^-(u+iv)) \leq \varphi_1\left(\frac{-1}{\sqrt{2}\sigma_{22}}|v|\right) \leq \varphi_1(0). \tag{3.14}$$

By Lemma 3.3 (i) $\varphi_1(0) < \infty$. It follows that

$$\sup_{u \in [-\epsilon, x_{max}+\delta]} \varphi_1(Y^-(u+iv)) < \infty. \tag{3.15}$$

By Lemma 3.1 (i) there exists a constant $d_1 > 0$ such that

$$|\gamma_1(u+iv, Y^-(u+iv))| \leq d_1|v|, \quad \forall u \in [-\epsilon, x_{max}+\eta], \ |v| \geq \epsilon. \tag{3.16}$$

Note that $|\gamma_2(u+iv, Y^-(u+iv))| \geq |r_{12}u + r_{22}\mathrm{Re}Y^-(u+iv)|$. Then by Lemma 3.1 (ii) and also by Lemma 3.3 (ii) there exists a constant $d_2 > 0$ such that

$$|\gamma_2(u+iv, Y^-(u+iv))| \geq d_2|v|, \quad \forall u \in [-\epsilon, x_{max}+\eta], \ |v| \geq \epsilon. \tag{3.17}$$

Finally by Lemma 3.1 (ii)

$$|\exp(a_0(u+iv) + b_0Y^-(u+iv))| = \exp(a_0u + b_0\mathrm{Re}\, Y^-(u+iv)) \leq \exp\left(\left(a_0 - b_0\frac{\sigma_{12}}{\sigma_{22}}\right)u - \frac{b_0}{\sqrt{2}\sigma_{22}}|v|\right) \tag{3.18}$$

for any $u \in [-\epsilon, x_{max} + \eta]$ and v with $|v| > \epsilon$. Then the estimate (3.13), the representation (3.2) combined with the estimates (3.15), (3.16), (3.17) and (3.18) lead to the statement of the lemma. $\square$

4. Inverse Laplace transform: from a double integral to simple integrals

By the Laplace transform inversion formula ([13, Theorem 24.3 and 24.4] and [5]), for any $\epsilon > 0$ small enough,

$$g(a,b) = \frac{1}{(2\pi i)^2} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi(x,y) \exp(-ax-by) dxdy.$$

in the sense of principal value convergence.

Lemma 4.1 (Inverse Laplace transform as a sum of simple integrals)**.** *Let $z_0 = (a_0, b_0)$ be the starting point of the process. For any $(a,b) \in \mathbb{R}_+^2$ where either $a > a_0, b > 0$ or $b > b_0, a > 0$ the following representation holds :*

$$g(a,b) = I_1(a,b) + I_2(a,b) + I_3(a,b)$$

where

$$I_1(a,b) = \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi_2(x) \gamma_2(x, Y^+(x)) \exp(-ax - bY^+(x)) \frac{dx}{\gamma_y'(x, Y^+(x))},$$

$$I_2(a,b) = \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi_1(y) \gamma_1(X^+(y), y) \exp(-aX^+(y) - by) \frac{dy}{\gamma_x'(X^+(y), y)},$$

$$I_3(a,b) = \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \exp(a_0 x + b_0 Y^+(x)) \exp(-ax - bY^+(x)) \frac{dx}{\gamma_y'(x, Y^+(x))} \quad \text{if } b > b_0,$$

$$I_3(a,b) = \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \exp(a_0 X^+(y) + b_0 y) \exp(-aX^+(y) - by) \frac{dy}{\gamma_x'(X^+(y), y)} \quad \text{if } a > a_0.$$

The two different formulas for I_3 will be useful in Section 9 to study the asymptotics along the axes.

Proof. For any $\epsilon > 0$ small enough $\gamma(-\epsilon, -\epsilon) < 0$. Then

$$\text{Re}\gamma(-\epsilon + iv_1, -\epsilon + iv_2) < 0 \quad \forall v_1, v_2 \in \mathbb{R} \tag{4.1}$$

since Σ is a covariance matrix. Then by (2.2)

$$g(a,b) = \frac{-1}{(2\pi i)^2} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{\gamma_1(x,y)\varphi_1(y) + \gamma_2(x,y)\varphi_2(x) + \exp(a_0 x + b_0 y),}{\gamma(x,y)} \exp(-ax-by) dxdy$$

Now, let us consider for example the second term. It can be written as

$$\frac{-1}{(2\pi i)^2} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \varphi_2(x) \exp(-ax) \Big(\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{\gamma_2(x,y)}{\gamma(x,y)} \exp(-by) dy \Big) dx.$$

Note that the convergence in the sense of the principal value of this integral can be guaranteed by integrating by parts. Now, it just remains to show that

$$\frac{-1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{\gamma_2(x,y)}{\gamma(x,y)} \exp(-by) dy = \frac{\gamma_2(x, Y^+(x))}{\gamma_y'(x, Y^+(x))} \exp\big(-bY^+(x)\big) \tag{4.2}$$

Let $x = -\epsilon$. The equation $\gamma(-\epsilon, y) = 0$ has two solutions $Y^+(-\epsilon) > 0$ and $Y^-(-\epsilon) < 0$. (In fact, for $\epsilon > 0$ small enough $Y^+(-\epsilon)$ is close to $Y^+(0) = 0$ staying positive and $Y^-(-\epsilon)$ is close to $Y^-(0) = -2\mu_2/\sigma_{22} < 0$). Let $x = -\epsilon + iv$. Functions $Y^+(-\epsilon + iv)$ and $Y^-(-\epsilon + iv)$ are continuous in v. By (4.1) their real parts do not equal $-\epsilon$ for no $v \in \mathbb{R}$. Thus $\text{Re}Y^+(-\epsilon + iv) > -\epsilon$ and $\text{Re}Y^-(-\epsilon + iv) < -\epsilon$ for all $v \in \mathbb{R}$. Let us construct the contour $[-\epsilon - iR, -\epsilon + iR] \cup \{t + iR, \mid t \in [-\epsilon, 0]\} \cup \{Re^{it} \mid t \in]-\pi/2 + \pi/2[\} \cup \{t - iR, \mid t \in [-\epsilon, 0]\}$, see Figure 6.

For any fixed $x = -\epsilon + iv$, the integral over this contour taken in the counter-clockwise direction of the function $\frac{\gamma_2(x,y)}{\gamma(x,y)} \exp(-by)$ equals the residue of this function multiplied by $2\pi i$, which is exactly the result announced in (4.2). It suffices to show that the integral over $\{t + iR \mid t \in [-\epsilon, 0]\} \cup \{Re^{it} \mid$

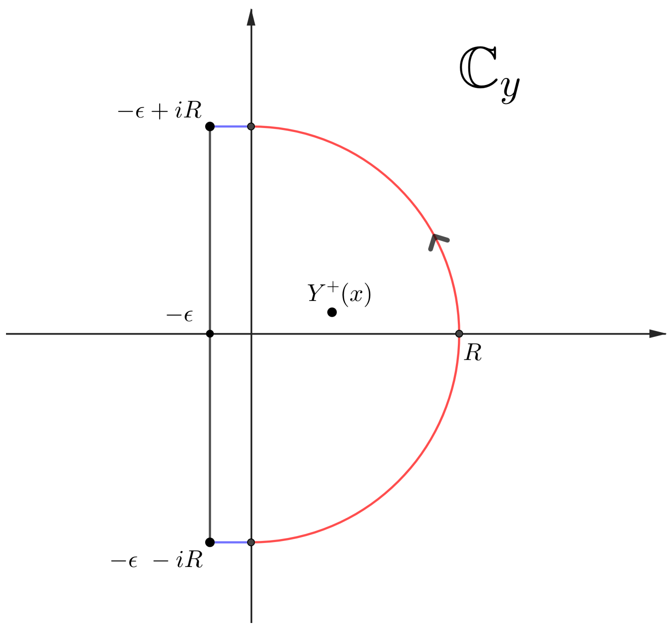


FIGURE 6. Integral contour in the complex plane $\mathbb{C}_y$, with the pole $Y^+(x)$.

$t \in]-\pi/2+\pi/2[\} \cup \{t - iR \mid t \in [-\epsilon, 0]\}$ converges to zero as $R \to \infty$. The integral over the half of the circle $\{Re^{it} \mid t \in]-\pi/2+\pi/2[\}$ equals

$$\int_{-\pi/2}^{\pi/2} \frac{\gamma_2(x, Re^{it})}{\gamma(x, Re^{it})} \exp(-bRe^{it}) iRe^{it} dt.$$

We have $\sup_{R>R_0} \sup_{t\in]-\pi/2,\pi/2[} \left| \frac{\gamma_2(x,Re^{it})}{\gamma(x,Re^{it})} iRe^{it} \right| < \infty$ for $R_0 = R_0(x) > 0$ large enough, while $|\exp(-bRe^{it})| = \exp(-bR\cos t) \to 0$ as $R \to \infty$ for any $t \in]-\pi/2, \pi/2[$ since $b > 0$. Hence, the integral over the half of the circle converges to zero as $R \to \infty$ by the dominated convergence theorem. Let us look at the integral over segment $\{t + iR \mid t \in [-\epsilon, 0]\}$. For any fixed $x = -\epsilon + iv$, there exists a constant $C(x) > 0$ such that for any R large enough

$$\sup_{u\in[-\epsilon,0]} \left| \frac{\gamma_2(x, u+iR)}{\gamma(x, u+iR)} \right| \leq \frac{C(x)}{R}.$$

Therefore

$$\left| \int_{-\epsilon}^{0} \frac{\gamma_2(x, u+iR)}{\gamma(x, u+iR)} \exp(-b(u+iR)) du \right| \leq \epsilon \exp(b\epsilon) \frac{C(x)}{R} \underset{R\to\infty}{\longrightarrow} 0.$$

The representation of $I_1(a, b)$ follows.

The reasoning is the same for the third term. The integral over the half of the circle equals

$$\int_{-\pi/2}^{\pi/2} \frac{\exp(-(b-b_0)Re^{it})}{\gamma(x, Re^{it})} iRe^{it} dt.$$

We have $\sup_{R>R_0} \sup_{t\in]-\pi/2,\pi/2[} \left| \frac{1}{\gamma(x,Re^{it})} iRe^{it} \right| < \infty$ while $|\exp(-(b-b_0)Re^{it})| = \exp(-(b-b_0)R\cos t) \to 0$ as $R \to \infty$ for any $t \in]-\pi/2, \pi/2[$ since $b - b_0 > 0$. The integral over the half of the circle converges to zero as $R \to \infty$ by the dominated convergence theorem once again. For any fixed $x = -\epsilon + iv$, there exists a constant $C(x) > 0$ such that for any R large enough

$$\sup_{u\in[-\epsilon,0]} \left| \frac{1}{\gamma(x, u+iR)} \right| \leq \frac{C(x)}{R^2}.$$

Therefore

$$\Big|\int_{-\epsilon}^{0}\frac{\exp(-(b-b_0)(u+iR))}{\gamma(x,u+iR)}du\Big|\leq \epsilon\exp((b-b_0)\epsilon)\frac{C(x)}{R^2}\to 0,\quad R\to\infty.$$

The representations for $I_2(a,b)$ and $I_3(a,b)$ with $a>a_0$ are obtained in the same way. □

Remark. Let us introduce some notations a, b, c, $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ by

$$\gamma(x,y)=a(x)y^2+b(x)y+c(x)=\widetilde{a}(y)x^2+\widetilde{b}(y)x+\widetilde{c}(y). \tag{4.3}$$

Then functions in the integrand can be represented as

$$\gamma'_y(x,Y^+(x))=a(x)(Y^+(x)-Y^-(x))=2a(x)Y^+(x)+b(x)=\sqrt{b^2(x)-4a(x)c(x)} \tag{4.4}$$

$$\gamma'_x(X^+(y),y)=\widetilde{a}(y)(X^+(y)-X^-(y))=2\widetilde{a}(y)X^+(y)+\widetilde{b}(y)=\sqrt{\widetilde{b}^2(y)-4\widetilde{a}(y)\widetilde{c}(y)} \tag{4.5}$$

All these forms will be used in the following.

5. Saddle point and contour of the steepest descent

Our aim is to study the integrals I_1, I_2 and I_3 of Lemma 4.1 using the saddle point method. One of the reference books about this approach is those of Fedoryuk [18].

Saddle point. For $\alpha\in[0,2\pi[$ we define

$$(x(\alpha),y(\alpha)):=\text{argmax}_{(x,y):\gamma(x,y)=0}(x\cos\alpha+y\sin\alpha). \tag{5.1}$$

We are going to see that this point turns out to be the saddle point of the functions inside the exponentials of the integrals I_1, I_2 and I_3. See Figure 7 for a geometric interpretation of this point.

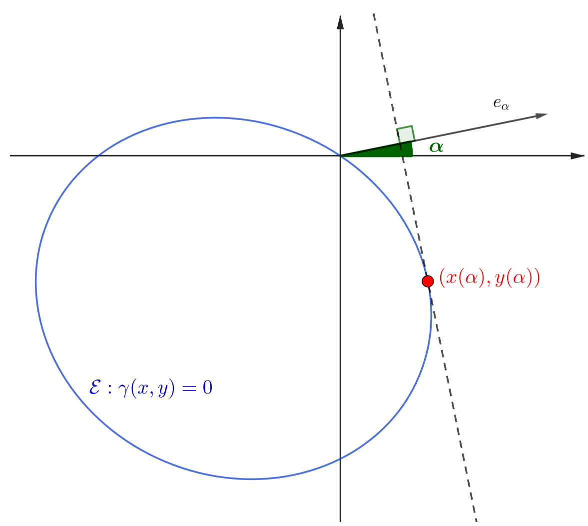


Figure 7. Graphic representation of the saddle point. We denote $e_\alpha=(\cos(\alpha),\sin(\alpha))$.

The map $\alpha:[0,2\pi[\to\{(x,y):\gamma(x,y)=0\}$ is a diffeomorphism. Functions $x(\alpha),y(\alpha)$ are in the class $C^\infty([0,2\pi])$. For any $\alpha\in[0,\pi/2]$ function $\cos(\alpha)x+\sin(\alpha)Y^+(x)$ reaches its maximum at the unique point on $[X^\pm(y_{max}),x_{max}]$ called $x(\alpha)$. This function is strictly increasing on $[X^\pm(y_{max}),x(\alpha)]$ and strictly decreasing on $[x(\alpha),x_{max}]$. Function $\cos(\alpha)X^+(y)+\sin(\alpha)y$ reaches its maximum on $[Y^\pm(x_{max}),y_{max}]$ at the unique point $y(\alpha)$. It is strictly increasing on $[Y^\pm(x_{max}),y(\alpha)]$ and strictly decreasing on $[y(\alpha),y_{max}]$.

Thus $x(0) = x_{max}$, $y(0) = Y^{\pm}(x_{max})$, $x(\pi/2) = X^{\pm}(y_{max})$, $y(\pi/2) = y_{max}$. Finally, $x(\alpha) = 0$ and $y(\alpha) = 0$ if $(\cos(\alpha), \sin(\alpha)) = \left(\frac{\mu_1}{\sqrt{\mu_1^2+\mu_2^2}}, \frac{\mu_2}{\sqrt{\mu_1^2+\mu_2^2}}\right)$. We denote this direction corresponding to the drift by α_μ.

Let's define the functions

$$F(x,\alpha) = -\cos(\alpha)x - \sin(\alpha)Y^+(x) + \cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha), \tag{5.2}$$
$$G(y,\alpha) = -\cos(\alpha)X^+(y) - \sin(\alpha)y + \cos(\alpha)x(\alpha) + \sin(\alpha)y(\alpha).$$

The function F appears to be (up to a constant) the function inside exponential of the integral I_1, and the function G appears to be (up to a constant) the function inside the exponential of the integral I_2, see Lemma 4.1. We have

$$F(x(\alpha),\alpha) = 0 \quad \forall \alpha \in [0,\pi/2]$$

and

$$F'_x(x(\alpha),\alpha) = 0 \quad \forall \alpha \in]0,\pi/2], \text{ but not at } \alpha = 0.$$

In the same way $G(y(\alpha),\alpha) = 0$ for any $\alpha \in [0,\pi/2]$ and $G'_y(y(\alpha),\alpha) = 0$ for any $\alpha \in [0,\pi/2[$ but not at $\alpha = \pi/2$. Then $(Y^+(x(\alpha)))' = -\text{ctan}(\alpha)$ and $(X^+(y(\alpha)))' = -\tan(\alpha)$.

Using the identities $\gamma(x, Y^+(x)) \equiv 0$ and $\gamma(X^+(y), y) \equiv 0$, we get :

$$(Y^+(x))'\Big|_{x=x(\alpha)} = -\frac{\gamma'_x(x(\alpha),y(\alpha))}{\gamma'_y(x(\alpha),y(\alpha))} = -\frac{\cos(\alpha)}{\sin(\alpha)}, \quad \alpha \in]0,\pi/2] \tag{5.3}$$
$$(X^+(y))'\Big|_{y=y(\alpha)} = -\frac{\gamma'_y(x(\alpha),y(\alpha))}{\gamma'_x(x(\alpha),y(\alpha))} = -\frac{\sin(\alpha)}{\cos(\alpha)}, \quad \alpha \in [0,\pi/2[$$
$$(Y^+(x))''\Big|_{x=x(\alpha)} = -\frac{\sigma_{11} + 2\sigma_{12}(-\text{ctan}\,(\alpha)) + \sigma_{22}(-\text{ctan}\,(\alpha))^2}{\gamma'_y(x(\alpha),y(\alpha))}$$
$$(X^+(y))''\Big|_{x=x(\alpha)} = -\frac{\sigma_{11}(-\tan(\alpha))^2 + 2\sigma_{12}(-\tan\,(\alpha)) + \sigma_{22}}{\gamma'_x(x(\alpha),y(\alpha))}$$

$$F''_x(x(\alpha),\alpha) = \frac{\sigma_{11}\sin^2(\alpha) + 2\sigma_{12}\sin(\alpha)\cos(\alpha) + \sigma_{22}\cos^2(\alpha)}{\gamma'_y(x(\alpha),y(\alpha))\sin\alpha} > 0 \quad \alpha \in]0,\pi/2], \tag{5.4}$$
$$G''_y(y(\alpha),\alpha) = \frac{\sigma_{11}\sin^2(\alpha) + 2\sigma_{12}\sin(\alpha)\cos(\alpha) + \sigma_{22}\cos^2(\alpha)}{\gamma'_x(x(\alpha),y(\alpha))\cos(\alpha)} > 0 \quad \alpha \in [0,\pi/2[,$$

the strict inequality coming from (4.4), (4.5) and the positive-definite form of Σ.

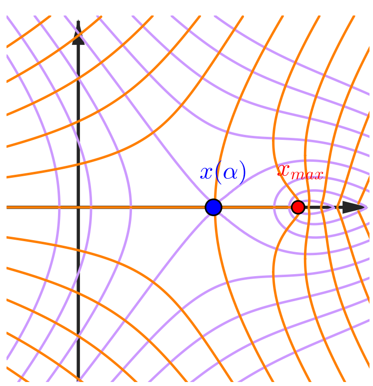


FIGURE 8. Level sets of $\Re(F)$ in purple and of $\Im(F)$ in orange. The saddle point $x(\alpha)$ is represented in blue and the branch point x_{max} is in red.

The values of $x(\alpha)$ and $y(\alpha)$ are given by the following formulas.

$$x(\alpha)=\frac{(\mu_2\sigma_{12}-\mu_1\sigma_{22})}{\det(\Sigma)}+\frac{1}{\det(\Sigma)}(\sigma_{22}-\tan(\alpha)\sigma_{12})\sqrt{\frac{\mu_2^2\sigma_{11}-2\mu_1\mu_2\sigma_{12}+\mu_1^2\sigma_{22}}{\sigma_{11}\tan^2(\alpha)-2\sigma_{12}\tan(\alpha)+\sigma_{22}}} \tag{5.5}$$

$$y(\alpha)=\frac{(\mu_1\sigma_{12}-\mu_2\sigma_{11})}{\det(\Sigma)}+\frac{1}{\det(\Sigma)}\left(\sigma_{11}-\frac{1}{\tan(\alpha)}\sigma_{12}\right)\sqrt{\frac{\mu_1^2\sigma_{22}-2\mu_1\mu_2\sigma_{12}+\mu_2^2\sigma_{11}}{\frac{\sigma_{22}}{\tan^2(\alpha)}-2\frac{\sigma_{12}}{\tan(\alpha)}+\sigma_{11}}} \tag{5.6}$$

Indeed, using the same calculations as in section 4.2 of [23], the equation $0=\frac{d}{dx}\left[\gamma(x,Y^+(x))\right]|_{x=x(\alpha)}$ combined with the first equation of (5.3) gives a linear relationship between $x(\alpha)$ and $y(\alpha)$. Injecting this condition in the polynomial equation $\gamma(x(\alpha),y(\alpha))=0$, we get two possible values for $x(\alpha)$ and $y(\alpha)$. The choice of the sign depends then of α and we get this expression.

Contour of the steepest descent. Before continuing, the reader should read Appendix A which states a parameter dependent Morse lemma. Let $\alpha_0\in]0,\pi/2]$. We apply Lemma A.1 to F defined in (5.2). Let us fix any $\epsilon\in]0,K[$ and consider any α such that $|\alpha-\alpha_0|<\eta$, where constants K and η are taken from the definition of $\Omega(0,\alpha_0)$ in Lemma A.1. Then, for any α we can construct the contour of the steepest descent

$$\Gamma_{x,\alpha}=\{x(it,\alpha)\mid t\in[-\epsilon,\epsilon]\}.$$

Clearly

$$F(x(it,\alpha),\alpha)=-t^2.$$

We denote by $x_\alpha^+=x(i\epsilon,\alpha)$ and $x_\alpha^-=x(-i\epsilon,\alpha)$ its ends. Then

$$F(x_\alpha^+,\alpha)=-\epsilon^2,\quad F(x_\alpha^-,\alpha)=-\epsilon^2. \tag{5.7}$$

Since $F''_x(x(\alpha),\alpha)\neq 0$, the contours in a neighborhood of $x(\alpha)$ where the function F is real are orthogonal, see Figure 8. One of them is the real axis. The other is the contour of the steepest descent, which is the orthogonal to the real axis. It follows that $\mathrm{Im} x_{\alpha_0}^+>0$ and $\mathrm{Im} x_{\alpha_0}^-<0$. By continuity of $x(i\epsilon,\alpha)$ on α for any $\eta>0$ small enough, there exists $\nu>0$ such that

$$\mathrm{Im}\, x_\alpha^+>\nu,\quad \mathrm{Im}\, x_\alpha^-<-\nu\quad \forall\alpha:|\alpha-\alpha_0|<\eta. \tag{5.8}$$

In the same way, for any $\alpha\in[0,\pi/2[$, we may define by the generalized Morse lemma the function $y(\omega,\alpha)$ w.r.t. $G(y,\alpha)$. Let $\alpha_0\in[0,\pi/2[$. We can construct the contour of the steepest descent

$$\Gamma_{y,\alpha}=\{y(it,\alpha)\mid t\in[-\epsilon,\epsilon]\}$$

with end points $y_\alpha^+=y(i\epsilon,\alpha)$ and $y_\alpha^-=y(-i\epsilon,\alpha)$ and the property analogous to (5.8).

We note that for any $\alpha=]0,\pi/2[$

$$\Gamma_{x,\alpha}=\underrightarrow{\overleftarrow{X^+(\Gamma_{y,\alpha})}},\quad \Gamma_{y,\alpha}=\underrightarrow{\overleftarrow{Y^+(\Gamma_{x,\alpha})}}. \tag{5.9}$$

The arrows mean that the direction has to be changed because of the facts that $(X^+(y))'\Big|_{y=y(\alpha)}<0$ and $(Y^+(x))'\Big|_{x=x(\alpha)}<0$.

Case where $\alpha_0=0$. In this case $\Gamma_{y,0}$ is now well defined, but not $\Gamma_{x,0}$ (since $F''(x(0),0)=\infty$), see Figure 9. We may define then

$$\Gamma_{x,0}=\underrightarrow{\overleftarrow{X^+(\Gamma_{y,0})}}$$

with end points $x_0^+=X^+(y_0^+)=x_{max}+\epsilon^2$ and $x_0^-=X^+(y_0^-)=x_{max}+\epsilon^2$. In fact, for $\alpha=0$, we have $G(y,0)=-X^+(y)+x_{max}$ and $G(y(i\epsilon,0),0)=-\epsilon^2$. Thus $\Gamma_{x,0}$ runs the real segment from $x_{max}+\epsilon^2$ to x_{max} and back $x_{\max}+\epsilon^2$. Figure 9 illustrates why this phenomenon happens when $\alpha=0$. Again by continuity on α we may find $\eta>0$ and $\nu>0$ small enough, such that

$$Re x_\alpha^+-x_{max}>\nu,\quad Re x_\alpha^--x_{max}>\nu,\quad \forall\alpha\in[0,\eta]. \tag{5.10}$$

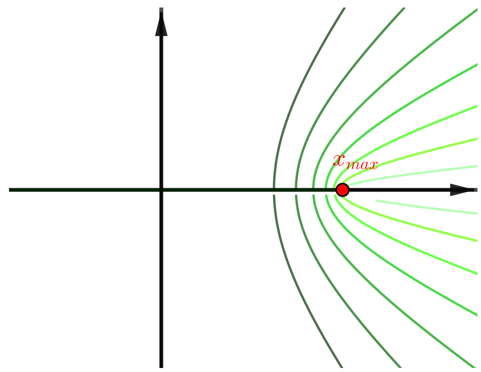


FIGURE 9. Steepest descent contour for $\Re(F)$ according to α, the closer alpha is to zero, the lighter green the corresponding contour becomes. When $\alpha \to 0$ this contour tends the half line $[x_{max}, \infty)$.

If $\alpha_0 = \pi/2$, $\Gamma_{x,\pi/2}$ is well defined, but not $\Gamma_{y,\pi/2}$. We put then

$$\Gamma_{y,\pi/2} = \overrightarrow{\overleftarrow{Y^+(\Gamma_{x,\pi/2})}}$$

with end points $y_\alpha^+ = Y^+(x_\alpha^+)$ and $y_\alpha^- = Y^+(x_\alpha^-)$.

6. Shift of the integration contours and contribution of the poles

We are going to define the integration contours of I_1, I_2 and I_3 thanks to the contours of the steepest descent studied in the previous section. First, let us define

$$S_{x,\alpha}^+ = \{x_\alpha^+ + it \mid t \geq 0\}, \quad S_{x,\alpha}^- = \{x_\alpha^- - it \mid t \geq 0\},$$

$$S_{y,\alpha}^+ = \{y_\alpha^+ + it \mid t \geq 0\}, \quad S_{y,\alpha}^- = \{y_\alpha^- - it \mid t \geq 0\}.$$

Now, let us construct the integration contours $T_{x,\alpha} = S_{x,\alpha}^- + \Gamma_{x,\alpha} + S_{x,\alpha}^+$ and $T_{y,\alpha} = S_{y,\alpha}^- + \Gamma_{y,\alpha} + S_{y,\alpha}^+$ for any $\alpha \in [0, \pi/2]$. See Figure 10 which illustrates these integration contours.

Case where the saddle point meet the pole. The only exception to define these contours will be for $\alpha \in [0, \pi/2]$ such that $x(\alpha) = x^* \in]0, x_{max}[$ is a pole of $\varphi_2(x)$ and $y(\alpha) = y^{**} \in]0, y^{max}[$ is a pole of $\varphi_1(y)$. We call these directions α^* and α^{**}, so that $x(\alpha^*) = x^*$, $y(\alpha^*) = Y^+(x^*) = y^*$, $y(\alpha^{**}) = y^{**}$, $x(\alpha^{**}) = X^+(y^{**}) = x^{**}$. When the poles x^* and y^{**} exists, we recall that by the Lemma 3.3 :

$$x^* = 2\frac{\mu_2 \frac{r_{12}}{r_{22}} - \mu_1}{\sigma_{11} - 2\sigma_{12}\frac{r_{12}}{r_{22}} + \sigma_{22}\left(\frac{r_{12}}{r_{22}}\right)^2} \quad \text{and} \quad y^{**} = 2\frac{\mu_1 \frac{r_{21}}{r_{11}} - \mu_2}{\sigma_{11}\left(\frac{r_{21}}{r_{11}}\right)^2 - 2\sigma_{12}\frac{r_{21}}{r_{11}} + \sigma_{22}}. \tag{6.1}$$

We also recall that by definition

$$y^* := Y^+(x^*) \quad \text{and} \quad x^{**} := X^+(y^{**}). \tag{6.2}$$

We remark that we have $y^* = -\frac{r_{12}}{r_{22}}x^*$ (resp. $x^{**} = -\frac{r_{21}}{r_{11}}y^{**}$) if and only if x^* (resp y^{**}) is not a pole of φ_2 (resp. φ_1) because of the condition on x^* and y^{**} to be poles.

If the pole x^* exists, then $\alpha^* \in]0, \alpha_\mu[$, and if y^* exists, then $\alpha^{**} \in]\alpha_\mu, \pi/2[$. We denote for convenience $\alpha^* = -\infty$ if x^* does not exist and $\alpha^{**} = +\infty$ if y^* does not exist.

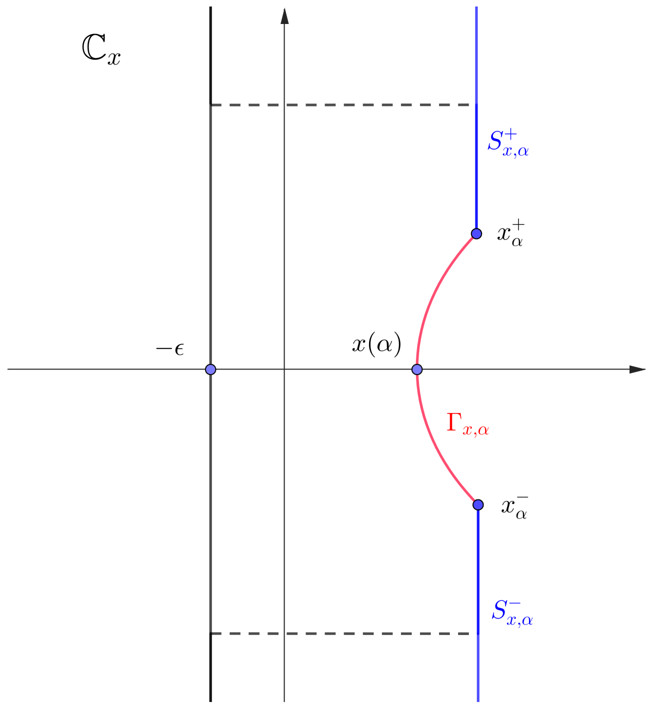


FIGURE 10. In the complex plane $\mathbb{C}_x$, shift of the integration contour passing through the saddle point along the steepest line.

If $\alpha = \alpha^* \in]0, \alpha_\mu[$, we modify in the definition of $T_{x,\alpha}$ the contour $\Gamma_{x,\alpha}$ by $\widetilde{\Gamma}_{x,\alpha}$, which is the half of the circle centred at $x(\alpha^*)$ going from $x^+_{\alpha^*}$ to $x^-_{\alpha^*}$ in the counter-clockwise direction. The same modification is made for $\alpha = \alpha^{**} \in]\alpha_\mu, \pi/2[$.

The next lemma perform the shift of the integration contour and take into account the contribution of the crossed poles. Recall that I_1, I_2 and I_3 are defined in Lemma 4.1.

Lemma 6.1 (Contribution of the poles to the asymptotics). *Let $\alpha \in [0, \pi/2]$. Then for any $a, b > 0$*

$$I_1(a,b) = \frac{\big(-\mathrm{res}_{x=x^*}\varphi_2(x)\big)\gamma_2(x^*,y^*)}{\gamma_y'(x^*,y^*)}\exp(-ax^*-by^*)\times \mathbf{1}_{\alpha<\alpha^*}$$
$$+\frac{1}{2\pi i}\int_{T_{x,\alpha}}\frac{\varphi_2(x)\gamma_2(x,Y^+(x))}{\gamma_y'(x,Y^+(x))}\exp(-ax-bY^+(x))dx,$$

$$I_2(a,b) = \frac{\big(-\mathrm{res}_{y=y^{**}}\varphi_1(y)\big)\gamma_1(x^{**},y^{**})}{\gamma_x'(x^{**},y^{**})}\exp(-ax^{**}-by^{**})\times \mathbf{1}_{\alpha>\alpha^{**}}$$
$$+\frac{1}{2\pi i}\int_{T_{y,\alpha}}\frac{\varphi_1(y)\gamma_1(X^+(y),y)}{\gamma_x'(X^+(y),y)}\exp(-aX^+(y)-by)dy,$$

$$I_3(a,b) = \frac{1}{2\pi i}\int_{T_{x,\alpha}}\exp((a_0-a)x+(b_0-b)Y^+(x))\frac{dx}{\gamma_y'(x,Y^+(x))} \quad \text{if } b>b_0$$

$$I_3(a,b) = \frac{1}{2\pi i}\int_{T_{x,\alpha}}\exp((a_0-a)X^+(y)+(b_0-b)y)\frac{dy}{\gamma_x'(X^+(y),y)} \quad \text{if } a>a_0.$$

One may remark that we have $\gamma_2(x^*,y^*)\mathrm{res}_{x^*}\varphi_2 < 0$ and $\gamma_1(x^{**},y^{**})\mathrm{res}_{y^{**}}\varphi_1 < 0$.

Proof. We start from the result of Lemma 4.1 and we use Cauchy theorem to shift the integration contour. We take into account the poles by the residue theorem noting that $x^* < x(\alpha)$ if and only if $\alpha < \alpha^*$ and that $y^{**} < y(\alpha)$ if and only if $\alpha^{**} < \alpha$. In order to get the representation of I_1 by shifting the contour, we want to show that the integrals on the dotted lines of Figure 10 tends to 0 when these lines goes to infinity. To do so, it suffices to show that for any $\eta > 0$ small enough ,

$$\sup_{u\in[X^+(y_{max})-\eta,x^{max}+\eta]}\Big|\frac{\varphi_2(u+iv)\gamma_2(u+iv,Y^+(u+iv))}{\gamma_y'(u+iv,Y^+(u+iv))}\exp(-a(u+iv)-bY^+(u+iv))\Big|\to 0,\quad\text{as } v\to\infty.$$

In reality, it would be sufficient to study the supremum on $[-\epsilon, x^{max}+\eta]$. By Lemma 3.10 for any $\epsilon > 0$,

$$\sup_{u\in[X^+(y_{max})-\eta,x^{max}+\eta],|v|\geq\epsilon}|\varphi_2(u+iv)|<\infty.$$

Let us observe that by (4.4)

$$\gamma_y'(x,Y^+(x))=\sqrt{b^2(x)-4a(x)c(x)}=\sqrt{(\sigma_{12}^2-\sigma_{11}\sigma_{22})x^2+2(\mu_2\sigma_{12}-\mu_1\sigma_{22})x+\mu_2^2}. \tag{6.3}$$

This function equals zero only at real points x_{min} and x_{max} and grows linearly in the absolute value as $|\Im x|\to\infty$. By Lemma 3.1 (i) function $|\gamma_2(x,Y^+(x))|$ grows linearly as well as $|\Im x|\to\infty$. Then for any $\epsilon > 0$

$$\sup_{\substack{u\in[X^+(y_{max})-\eta,x^{max}+\eta],\\|v|\geq\epsilon}}\Big|\frac{\gamma_2(u+iv,Y^+(u+iv))}{\gamma_y'(u+iv,Y^+(u+iv))}\exp(-a(u+iv))\Big|<\infty,$$

Finally,

$$\sup_{u\in[X^+(y_{max})-\eta,x^{max}+\eta]}|\exp(-bY^+(u+iv))|=\sup_{u\in[X^+(y_{max})-\eta,x^{max}+\eta]}\exp(-b\mathrm{Re}Y^+(u+iv))\to 0,$$

as $|v|\to\infty$ due to Lemma 3.1 (ii) and the fact that $b>0$. The other representations are obtained in the same way. In the representations of $I_3(a,b)$ we use the facts that $a-a_0>0$ and $b-b_0>0$. □

7. Exponentially negligible part of the asymptotic

Let's recall the integration contours $T_{x,\alpha}=S^-_{x,\alpha}+\Gamma_{x,\alpha}+S^+_{x,\alpha}$ and $T_{y,\alpha}=S^-_{y,\alpha}+\Gamma_{y,\alpha}+S^+_{y,\alpha}$ for any $\alpha\in[0,\pi/2]$. This section establishes a domination of the integrals on the contours $S^{\pm}_{x,\alpha}$ and $S^{\pm}_{y,\alpha}$. This domination is useful in the following sections to show that these integrals are negligible. We will see that the asymptotics of integrals I_1, I_2 and I_3 of contour $T_{x,\alpha}$ and $T_{y,\alpha}$ are given by the integrals on the lines of steepest descent $\Gamma_{x,\alpha}$ and $\Gamma_{y,\alpha}$.

Lemma 7.1 (Negligibility of the integrals on $S^{\pm}_{x,\alpha}$ and $S^{\pm}_{y,\alpha}$). *For any couple $(a,b)\in\mathbb{R}^2_+$ we may define $\alpha(a,b)$ the angle in $[0,\pi/2]$ such that $\cos(\alpha)=\frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha)=\frac{b}{\sqrt{a^2+b^2}}$.*

- *Let $\alpha_0\in]0,\pi/2]$. Then for any η small enough and any $r_0>0$ there exists a constant $D>0$ such that for any couple (a,b) such that $\sqrt{a^2+b^2}>r_0$ and $|\alpha(a,b)-\alpha_0|<\eta$ we have*

$$\Big|\int_{S^+_{x,\alpha}}\frac{\varphi_2(x)\gamma_2(x,Y^+(x))}{\gamma_y'(x,Y^+(x))}\exp\big(-ax-bY^+(x)\big)dx\Big|\leq\frac{D}{b}\exp\Big(-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+b^2}\Big) \tag{7.1}$$

  and if furthermore $b>b_0$ we have

$$\Big|\int_{S^+_{x,\alpha}}\exp((a_0-a)x+(b_0-b)Y^+(x))\frac{dx}{\gamma_y'(x,Y^+(x))}\Big|\leq\frac{D}{b-b_0}\exp\Big(-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+(b-b_0)^2}\Big). \tag{7.2}$$

- *Let $\alpha_0\in[0,\pi/2[$. Then for any η small enough and any $r_0>0$ there exists a constant $D>0$ such that for any couple (a,b) such that $\sqrt{a^2+b^2}>r_0$, $|\alpha(a,b)-\alpha_0|\leq\eta$ we have*

$$\Big|\int_{S^+_{y,\alpha}}\frac{\varphi_1(y)\gamma_1(X^+(y),y)}{\gamma_x'(X^+(y),y)}\exp\big(-aX^+(y)-by\big)dy\Big|\leq\frac{D}{a}\exp\Big(-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{a^2+b^2}\Big). \tag{7.3}$$

and if furthermore $a > a_0$ we have

(7.4)
$$\Big|\int_{S^+_{y,\alpha}} \exp((a_0-a)X^+(y)+(b_0-b)y)\frac{dy}{\gamma'_x(X^+(y),y)}\Big| \leq \frac{D}{a-a_0}\exp\Big(-ax(\alpha)-by(\alpha)-\epsilon^2\sqrt{(a-a_0)^2+b^2}\Big).$$

The same estimations hold true for $S^-_{x,\alpha}$ and $S^-_{y,\alpha}$.

Proof. With definition (5.2) and notation (5.7), the estimate (7.1) can be written as

$$\Big|\int_{v>0} \frac{\varphi_2(x^+_\alpha+iv)\gamma_2(x^+_\alpha+iv,Y^+(x^+_\alpha+iv))}{\gamma'_y(x^+_\alpha+iv,Y^+(x^+_\alpha+iv))}\exp\big(-\sqrt{a^2+b^2}\big(F(x^+_\alpha+iv,\alpha)-F(x^+_\alpha,\alpha)\big)dx\Big| \leq \frac{D}{b} \tag{7.5}$$

with $\alpha=\alpha(a,b)$. Let us prove it.

Let first $\alpha_0 \in]0,\pi/2]$. If $\alpha_0\neq \pi/2$, let us fix $\eta>0$ so small that $\alpha_0-\eta>0$, and $\alpha_0+\eta\leq\pi/2$. If $\alpha_0=\pi/2$, let us fix any small $\eta>0$ and consider only $\alpha\in[\pi/2-\eta,\pi/2]$.

By Lemma 3.10 and remark (5.8)

$$\sup_{v\geq 0,|\alpha-\alpha_0|\leq\eta}|\varphi_2(x^+_\alpha+iv)|<\infty. \tag{7.6}$$

By the observation (4.4) $\gamma'_y(x,Y^+(x))=0$ only if $x=x_{min},x_{max}$. Then by (5.8) we have

$$\inf_{v\geq 0,|\alpha-\alpha_0|\leq\eta}|\gamma'_y(x^+_\alpha+iv,Y^+(x^+_\alpha+iv))|>0. \tag{7.7}$$

Furthermore again by (6.3) and Lemma 3.1 (ii) we have

$$\sup_{v\geq 0,|\alpha-\alpha_0|\leq\eta}\Big|\frac{\gamma_2(x^+_\alpha+iv,Y^+(x^+_\alpha+iv))}{\gamma'_y(x^+_\alpha+iv,Y^+(x^+_\alpha+iv))}\Big|<\infty. \tag{7.8}$$

Finally

$$|\exp\big(-\sqrt{a^2+b^2}(F(x^+_\alpha+iv,\alpha)-F(x^+_\alpha,\alpha))\big)|=\exp\big(-b(\mathrm{Re}Y^+(x^+_\alpha+iv)-\mathrm{Re}Y^+(x^+_\alpha))\big). \tag{7.9}$$

By Lemma 3.1 (ii) function $\mathrm{Re}Y^+(x^+_\alpha+iv)-\mathrm{Re}Y^+(x^+_\alpha)$ equals 0 at $v=0$ and strictly increasing as v goes from zero to infinity. Moreover, it grows linearly as $v\to\infty$: there exists a constant $c>0$ such that for any α such that $|\alpha-\alpha_0|\leq\eta$ and any v large enough

$$\mathrm{Re}Y^+(x^+_\alpha+iv)-\mathrm{Re}Y^+(x^+_\alpha)\geq cv. \tag{7.10}$$

It follows from (7.6), (7.8), (7.9) and (7.10) that the left hand side of (7.5) is bounded by

$$C\int_0^\infty \exp(-bcv)dv = C\times(cb)^{-1}$$

with some constant $C>0$ and all couples (a,b) with $|\alpha(a,b)-\alpha_0|\leq\eta$.

As for the integral (7.2), let us make the change of variables $B=b-b_0>0$. Next, we proceed exactly as for (7.1). The only different detail is the elementary estimation $\sup_{|\alpha-\alpha_0|\leq\eta,v>0}|\exp(a_0(x^+_\alpha+iv))|<\infty$. We obtain then the bound $\frac{D'}{B}\exp(-ax(\alpha)-By(\alpha)-\epsilon\sqrt{a^2+B^2})$ with some $D'>0$. Then with $D=D'\exp(b_0y(\alpha))$ the estimation (7.2) follows.

The proofs for (7.3) and (7.4) are symmetric. □

The previous lemma will be useful in Section 8 to establish the asymptotic when $\alpha_0\in]0,\pi/2[$. In the next lemma we are going to show the negligibility of the integrals in the two missing cases when $\alpha_0=0$ or $\pi/2$ which will be useful in Section 9.

Remark 7.2 (Pole and branching point)**.** *In the next lemma and in Section 9 and 10, we exclude the case $\gamma_2(x_{max},Y^\pm(x_{max}))=0$ [resp. $\gamma_1(X^\pm(y_{max}),y_{max})=0$] such that the branching point and the pole of $\varphi_2(x)$ coincides. This case correspond to $x^*=x_{max}$ [resp. $y^*=y_{max}$], i.e. $\alpha^*=0$ [resp. $\alpha^{**}=\pi/2$]. Note that we already obtained the asymptotics of h_1 and h_2 in these specific cases in Proposition 3.8.*

Lemma 7.3 (Negligibility of the integrals on $S^{\pm}_{x,\alpha}$ and $S^{\pm}_{y,\alpha}$, case where $\alpha_0 = 0$ or $\pi/2$)**.** *For any $\eta > 0$ small enough and any $r_0 > 0$ there exists a constant $D > 0$ such that for any couple (a, b) such that $\sqrt{a^2 + b^2} > r_0$ and $0 < \alpha(a, b) < \eta$ we have*

$$(7.11)\quad \Big|\int_{S^+_{x,\alpha}} \frac{\varphi_2(x)\gamma_2(x, Y^+(x))}{\gamma'_y(x, Y^+(x))} \exp\big(-ax - bY^+(x)\big)dx\Big| \leq D\exp\Big(-ax(\alpha) - by(\alpha) - \epsilon^2\sqrt{a^2+b^2}\Big)$$

and if furthermore $b > b_0$ we have

$$(7.12)\quad \Big|\int_{S^+_{x,\alpha}} \exp((a_0 - a)x + (b_0 - b)Y^+(x))\frac{dx}{\gamma'_y(x, Y^+(x))}\Big| \leq D\exp\Big(-ax(\alpha) - by(\alpha) - \epsilon^2\sqrt{a^2+(b-b_0)^2}\Big).$$

The same estimations hold true for $S^-_{x,\alpha}$. For any couple (a, b) such that $\sqrt{a^2 + b^2} > r_0$ and $0 < \pi/2 - \alpha(a, b) < \eta$, a symmetric result holds for the integrals on $S^+_{y,\alpha}$ and $S^-_{y,\alpha}$.

Proof. Let now $\alpha_0 = 0$ so that $x(\alpha_0) = x_{max}$. Our aim is to prove (7.11) which is reduced to the estimate

$$(7.13)\quad \Big|\int_{v>0} \frac{\varphi_2(x^+_\alpha + iv)\gamma_2(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))}{\gamma'_y(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))} \exp\big(-aiv - b(Y^+(x^+_\alpha + iv) - Y^+(x^+_\alpha))\big)dv\Big| \leq D$$

Let us fix any $\eta > 0$ small enough and consider $\alpha \in]0, \eta]$. By (4.4) the denominator $\gamma'_y(x, Y^+(x))$ has zero at $x = x_{max}$ but not at other points in a neighborhood of x_{max}. Then by (5.10) we have

$$(7.14)\quad \inf_{0\leq\alpha\leq\eta} |\gamma'_y(x^+_\alpha, Y^+(x^+_\alpha))| > 0.$$

The function $\varphi_2(x)$ has a branching point at $x_{\max}$. But it follows from the representation (3.2) that it is bounded in a neighborhood of x_{max} cut along the real segment due to Remark 7.2. Hence, this integral has no singularity at $v = 0$ for none $\alpha \in]0, \eta]$ so that

$$(7.15)\quad \sup_{0\leq\alpha\leq\eta} \frac{\varphi_2(x^+_\alpha)\gamma_2(x^+_\alpha, Y^+(x^+_\alpha))}{\gamma'_y(x^+_\alpha, Y^+(x^+_\alpha))} < \infty$$

Let us consider the asymptotic of the integrand in (7.13) as $v \to \infty$. It is clear that $Y^+(x^+_\alpha + iv)$ grows linearly as $v \to \infty$ and so do functions γ_2 and γ'_y of this argument. Function $\varphi_2(x_\alpha + iv)$ is defined by the formula of the analytic continuation

$$(7.16)\quad \varphi_2(x_\alpha + iv) = -\frac{\gamma_1(x^+_\alpha + iv, Y^-(x^+_\alpha + iv))\varphi_1(Y^-(x^+_\alpha + iv)) + \exp\big(a_0(x^+_\alpha + iv) + b_0Y^-(x^+_\alpha + iv)\big)}{\gamma_2(x^+_\alpha + iv, Y^-(x^+_\alpha + iv))}$$

We know that $Y^-(x^+_\alpha + iv)$ varies linearly as well as $v \to \infty$, and moreover $ReY^-(x_\alpha + iv) \leq -c_1 - c_2v$ for all $v \geq 0$ and $\alpha \in]0, \eta]$ with some $c_1, c_2 > 0$. Then by Lemma C.1 in Appendix

$$(7.17)\quad |\varphi_2(x^+_\alpha + iv)| \leq Cv^{\lambda-1}$$

for any $\alpha \in]0, \eta]$ and $v > V_0$ with some $C > 0$, $V_0 > 0$ and $\lambda < 1$. Hence the integrand

$$\frac{\varphi_2(x^+_\alpha + iv)\gamma_2(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))}{\gamma'_y(x^+_\alpha + iv, Y^+(x^+_\alpha + iv))}$$

is about $O(v^{\lambda-1})$ as $v \to \infty$. The positivity of $\mathrm{Re}Y^+(x^+_\alpha + iv) - \mathrm{Re}Y^+(x^+_\alpha)$ for any $v \geq 0$ and the inequality (7.10) in the exponent stay valid for any $\alpha \in]0, \eta]$, so that the exponential term is bounded in the absolute value by $\exp(-cbv)$ with some $c > 0$. But for η small enough, the assumption $\alpha(a, b) \in]0, \eta]$ implies the arbitrary smallness of b. In the limiting case $b = 0$ the integral in the l.h.s of (7.13) is not absolutely convergent. In order to prove the required estimate (7.13), we proceed by integration by parts, this integral equals

$$\frac{\varphi_2(x_\alpha^+ + iv)\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))}{\gamma_y'(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))(-ai - b(Y^+(x_\alpha^+ + iv))'_v)} \exp\Big(- aiv - b(Y^+(x_\alpha^+ + iv) - Y^+(x_\alpha))\Big) \Big|_{v=0}^{v=\infty} \tag{7.18}$$

$$- \int_0^\infty \Big(\frac{\varphi_2(x_\alpha^+ + iv)\gamma_2(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))}{\gamma_y'(x_\alpha^+ + iv, Y^+(x_\alpha^+ + iv))(-ai - b(Y^+(x_\alpha^+ + iv))'_v)}\Big)'_v \exp\big(-aiv - b(Y^+(x_\alpha^+ + iv) - Y^+(x_\alpha^+))\big)dv. \tag{7.19}$$

Note that although in this case $x_{\alpha_0} = x_{max}$ which is a branching point for $Y^+(x)$, we have the first and second derivative bounded

$$\sup_{\alpha\in[0,\eta]} \Big|Y(x_\alpha^+ + iv)'\Big|_{v=0}\Big| < \infty, \quad \sup_{\alpha\in[0,\eta]} \Big|Y(x_\alpha^+ + iv)''\Big|_{v=0}\Big| < \infty \tag{7.20}$$

by remark (5.10). Furthermore $Y^\pm(x_\alpha^+ + iv)'$ is of the constant order and $Y^\pm(x_\alpha^+ + iv)''$ is not greater than $O(1/v)$ as $v \to \infty$.

The term (7.18) at $v = 0$ is bounded in the absolute value by some constant due to (7.15) and (7.20). It converges to zero as $v \to \infty$ by all said above for any $\alpha \in [0, \infty]$, $a, b \geq 0$. To evaluate (7.19), we compute the derivative in its integrand and show that it is of the order $O(v^{\lambda-2})$ as $v \to \infty$. We skip technical details of this computation but outline fact that $\varphi_2(x_\alpha^+ + iv)'_v$ is computed via the representation (7.16) and $|\varphi_1(Y^-(x_\alpha^+ + iv))'_v|$ is evaluated again by Lemma C.1, namely it is of the order not greater than $O(v^{\lambda-2})$ as $v \to \infty$. Thus the integral (7.19) is absolutely convergent for any $a, b \geq 0$ and can be bounded by some constant as well. This finishes the proof of (7.11). The proof of (7.12) is symetric. □

Note that the proof of Lemma 7.1 uses essentially the result of Lemma 3.10 which bounds the Laplace transforms. The proof of Lemma 7.3, for its part, uses a stronger result stated in Appendix C which gives a more precise estimate of the Laplace transform near infinity.

Following the lines of the proof we could establish a better estimate, namely the one that the integral is bounded by some universal constant divided by a, but we do not need it for our purposes.

Remark 7.4 (Negligibility). *When $\alpha(a, b) \to \alpha_0 \in]0, \pi/2[$, Equations (7.1), (7.2), (7.3), (7.4) of Lemma 7.1 give quite satisfactory estimates to prove the negligibility of the integrals on the contours $S_{x,\alpha}^\pm$ and $S_{y,\alpha}^\pm$ with respect to integrals on contours $\Gamma_{x,\alpha}$ and $\Gamma_{y,\alpha}$, see Lemma 8.1 below. In fact*

$$\frac{\exp(-ax(\alpha) - by(\alpha) - \epsilon^2\sqrt{a^2+b^2})}{b} = o\Big(\frac{\exp(-ax(\alpha) - by(\alpha))}{\sqrt[4]{a^2+b^2}}\Big),$$

$$\frac{\exp(-ax(\alpha) - by(\alpha) - \epsilon^2\sqrt{a^2+b^2})}{a} = o\Big(\frac{\exp(-ax(\alpha) - by(\alpha))}{\sqrt[4]{a^2+b^2}}\Big).$$

When $\alpha(a, b) \to 0$ or $\pi/2$, Equations (7.11) and (7.12) of Lemma 7.3 give satisfactory estimates to prove the negligibility which will be useful in Section 9 to compute the asymptotics along the axes.

8. Essential part of the asymptotic and main theorem

This section is dedicated to the asymptotics of $g(a, b) = I_1 + I_2 + I_3$ when $\alpha(a, b) \to \alpha_0 \in]0, \pi/2[$. The next lemma determines the asymptotics of the integrals on the lines of steepest descent $\Gamma_{x,\alpha}$ and $\Gamma_{y,\alpha}$ of the shifted contours.

For any couple $(a, b) \in \mathbb{R}_+^2$ we define $\alpha(a, b)$ the angle in $[0, \pi/2]$ such that $\cos(\alpha) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha) = \frac{b}{\sqrt{a^2+b^2}}$ and we define $r \in \mathbb{R}_+$ such that $r = \sqrt{a^2 + b^2}$.

Lemma 8.1 (Contribution of the saddle point to the asymptotics). *Let $\alpha_0 \in]0,\pi/2[$. Let $\alpha(a,b) \to \alpha_0 =$ and $r = \sqrt{a^2+b^2} \to \infty$. Then for any $n \geq 0$ we have*

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{x,\alpha}} \frac{\varphi_2(x)\gamma_2(x,Y^+(x))}{\gamma_y'(x,Y^+(x))}\exp(-ax-bY^+(x))dx + \frac{1}{2\pi i}\int\limits_{\Gamma_{y,\alpha}} \frac{\varphi_1(y)\gamma_1(X^+(y),y)}{\gamma_x'(X^+(y),y)}\exp(-aX^+(y)-by)dy$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma_{x,\alpha}} \exp((a_0-a)X^+(y)+(b_0-b)y)\frac{dy}{\gamma_x'(X^+(y),y)}$$

$$(8.1) \qquad \sim \exp(-ax(\alpha(a,b))-by(\alpha(a,b)))\sum_{k=0}^{n}\frac{c_k(\alpha(a,b))}{\sqrt[4]{a^2+b^2}(a^2+b^2)^{k/2}}$$

with some constants $c_0(\alpha), c_1(\alpha), \ldots, c_n(\alpha)$ continuous at α_0. Namely

$$(8.2) \qquad c_0(\alpha) = \frac{\gamma_1(x(\alpha),y(\alpha))\varphi_1(y(\alpha)) + \gamma_2(x(\alpha),y(\alpha))\varphi_2(x(\alpha)) + \exp(a_0x(\alpha)+b_0y(\alpha))}{\sqrt{2\pi(\sigma_{11}\sin^2(\alpha)+2\sigma_{12}\sin(\alpha)\cos(\alpha)+\sigma_{22}\cos^2(\alpha))}} \times C(\alpha),$$

where

$$C(\alpha) = \sqrt{\frac{\sin(\alpha)}{\gamma_y'(x(\alpha),y(\alpha))}} = \sqrt{\frac{\cos(\alpha)}{\gamma_x'(x(\alpha),y(\alpha))}}.$$

Proof. Consider the first integral. Let us make the change of variables $x = x(it,\alpha)$, see Section 5 and Appendix A. Then it becomes

$$\frac{\exp(-ax(\alpha)-by(\alpha))}{2\pi}\int_{-\epsilon}^{\epsilon} f(it,\alpha)\exp(-\sqrt{a^2+b^2}t^2)dt$$

where

$$f(it,\alpha) = \frac{\varphi_2(x(it,\alpha))\gamma_2(x(it,\alpha),Y^+(x(it,\alpha)))}{\gamma_y'(x(it,\alpha),Y^+(x(it,\alpha)))}x_\omega'(it,\alpha).$$

Let us take $\Omega(\alpha_0)$ from Lemma A.1 with K and η defined in this lemma. For any $\alpha \in [\alpha_0-\eta,\alpha_0+\eta]$ and $t \in [-\epsilon,\epsilon]$ we have

$$\Big|f(it,\alpha) - \sum_{l=0}^{2n} f^{(l)}(0,\alpha)\frac{(it)^l}{l!}\Big| \leq C|t|^{2n+1}$$

with the constant

$$C = \sup_{\substack{|\omega|=K,\\ |\alpha-\alpha_0|\leq\eta}} \Big|\frac{f(\omega,\alpha)-\sum_{l=0}^{2n} f^{(l)}(0,\alpha)\frac{\omega^l}{l!}}{\omega^{2n+1}}\Big|$$

by the maximum modulus principle and the fact that $f(\omega,\alpha)$ is in class C^∞ in $\Omega(\alpha_0)$. The integral

$$\int_{-\epsilon}^{\epsilon} t^l\exp(-\sqrt{a^2+b^2}t^2)dt$$

equals 0 if l is odd. By the change of variables $s = \sqrt[4]{a^2+b^2}t$ it equals

$$\frac{(l-1)(l-3)...(1)}{2^{l/2}}\frac{\sqrt{\pi}}{(\sqrt[4]{a^2+b^2})^{l+1}} + O\Big(\frac{\exp(-\sqrt{a^2+b^2}\epsilon)}{(\sqrt[4]{a^2+b^2})^{l+1}}\Big), \quad \sqrt{a^2+b^2}\to\infty$$

if l is even. The constant comes from the fact that $\int_{-\infty}^{+\infty} t^l e^{-s^2}ds = \frac{(l-1)(l-3)...(1)}{2^{l/2}}\sqrt{\pi}$. By the same reason

$$\int_{-\epsilon}^{\epsilon}|t|^{2n+1}\exp(-\sqrt{a^2+b^2}t^2)dt = O\Big(\frac{1}{(\sqrt[4]{a^2+b^2})^{2n+2}}\Big), \quad \sqrt{a^2+b^2}\to\infty.$$

The representation (8.1) for the first integral follows with the constants

$$c_l^1(\alpha) = \frac{(l-1)(l-3)...(1)}{2^{l/2}}\frac{\sqrt{\pi}}{2\pi}\frac{(-1)^l f^{(2l)}(0,\alpha)}{(2l)!}.$$

In particular

$$c_0^1(\alpha) = \frac{1}{2\sqrt{\pi}} \times \frac{\gamma_2(x(\alpha), y(\alpha))\varphi_2(x(\alpha))}{\gamma_y'(x(\alpha), y(\alpha))} \times x_\omega'(0, \alpha).$$

Using the expressions (A.1) and (5.4), we get

$$c_0^1(\alpha) = \frac{\gamma_2(x(\alpha), y(\alpha))\varphi_2(x(\alpha))}{\sqrt{2\pi(\sigma_{11}\sin^2(\alpha) + 2\sigma_{12}\sin(\alpha)\cos(\alpha) + \sigma_{22}\cos^2(\alpha))}} \times \sqrt{\frac{\sin(\alpha)}{\gamma_y'(x(\alpha), y(\alpha))}}.$$

In the same way, using the variable y instead of x, we get the asymptotic expansions of the second and the third integral with constants $c_0^2(\alpha), \dots, c_n^2(\alpha), c_0^3(\alpha), \dots, c_n^3(\alpha)$. Namely,

$$c_0^2(\alpha) + c_0^3(\alpha) = \frac{\gamma_1(x(\alpha), y(\alpha))\varphi_1(y(\alpha)) + \exp(a_0 x(\alpha) + b_0 y(\alpha))}{\sqrt{2\pi(\sigma_{11}\sin^2(\alpha) + 2\sigma_{12}\sin(\alpha)\cos(\alpha) + \sigma_{22}\cos^2(\alpha))}} \times \sqrt{\frac{\cos(\alpha)}{\gamma_x'(x(\alpha), y(\alpha))}}.$$

By (5.3) $\sin(\alpha)\gamma_x'(x(\alpha), y(\alpha)) = \cos(\alpha)\gamma_y'(x(\alpha), y(\alpha))$. This implies the representation (8.1) and concludes the proof with $c_k(\alpha) = \sum_{i=1}^3 c_k^i(\alpha)$. □

Now, we summarise all our reasoning to show the main result. We will justify later that the constants $c_0(\alpha)$ are not zero.

Theorem 4 (Asymptotics in the quadrant, general case)**.** *We consider a reflected Brownian motion in the quadrant of parameters (Σ, μ, R) satisfying conditions of Proposition 2.1 and Assumption 1. Then, the Green's density function $g(r\cos(\alpha), r\sin(\alpha))$ of this process has the following asymptotics when $\alpha \to \alpha_0 \in (0, \pi/2)$ and $r \to \infty$, for all $n \in \mathbb{N}$ we have:*

- *If $\alpha^* < \alpha_0 < \alpha^{**}$ then*

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))} \frac{1}{\sqrt{r}} \sum_{k=0}^{n} \frac{c_k(\alpha)}{r^k} \tag{8.3}$$

- *If $\alpha_0 < \alpha^*$ then*

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} c^* e^{-r(\cos(\alpha)x^*+\sin(\alpha)y^*)} + e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))} \frac{1}{\sqrt{r}} \sum_{k=0}^{n} \frac{c_k(\alpha)}{r^k} \tag{8.4}$$

- *If $\alpha^{**} < \alpha_0$ then*

$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to\alpha_0}}{\sim} c^{**} e^{-r(\cos(\alpha)x^{**}+\sin(\alpha)y^{**})} + e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))} \frac{1}{\sqrt{r}} \sum_{k=0}^{n} \frac{c_k(\alpha)}{r^k} \tag{8.5}$$

where explicit expression of the saddle point coordinate $x(\alpha)$ and $y(\alpha)$ are given by (5.5) and (5.6), the coordinates of the poles x^*, y^*, y^{**}, x^{**} are given by (6.1) and (6.2), the constants are given by

$$c^* = \frac{(-res_{x=x^*}\varphi_2(x))\gamma_2(x^*, y^*)}{\gamma_y'(x^*, y^*)} > 0 \quad \textit{and} \quad c^{**} = \frac{(-res_{y=y^{**}}\varphi_1(y))\gamma_1(x^{**}, y^{**})}{\gamma_y'(x^{**}, y^{**})} > 0$$

and c_k are constants depending on α and such that $c_k(\alpha) \underset{\alpha\to\alpha_0}{\longrightarrow} c_k(\alpha_0)$ where $c_0(\alpha)$ is given by (8.2). We have $c_0(\alpha) > 0$ at least when $\alpha^* < \alpha_0 < \alpha^{**}$ where it gives the dominant term of the asymptotics (8.3).

Proof. The theorem follows directly from several lemmas put together. By Lemma 4.1 which inverse the Laplace transform, $g(a, b)$ can be expressed as of the sum of three simple integrals $I_1 + I_2 + I_3$. Those integrals has been rewritten in Lemma 6.1, thanks to the residue theorem, as the sum of residues and integrals whose contour locally follows the steepest descent line through the saddle point. This has been done in Section 6 using Morse's Lemma, see Appendix A. Residues are present if $0 < x^* < x(\alpha)$ or $0 < y^{**} < y(\alpha)$. Besides, we proved in Lemma 7.1 the negligibility of the integrals of the lines $S_{x,\alpha}^{\pm}$ and $S_{y,\alpha}^{\pm}$ compared to the integrals on the steepest descent lines. The main asymptotics is then given by the poles plus the asymptotics of the steepest descent integrals. A disjunction of cases concerning the pole's contributions gives the three cases of the theorem (we recall that $\alpha^* < \alpha^{**}$). In the second case, when $\alpha_0 < \alpha^*$, φ_2 has a pole and then $c^* \neq 0$ because we have $\frac{r_{12}}{r_{22}} > \frac{-Y^{\pm}(x_{max})}{x_{max}}$ which imply $\gamma_2(x^*, y^*) \neq 0$. The same holds for c^{**}. Finally, Lemma 8.1 gives the desired asymptotic expansion of the integrals on the lines of the steepest descent. The fact that $c_0(\alpha_0) \neq 0$ when $\alpha^* < \alpha_0 < \alpha^{**}$ is postponed to Lemma 8.2 and Lemma 8.3. □

For the relevance of the asymptotics, constants $c_0(\alpha)$ shall not be zero at least when $\alpha^* < \alpha_0 < \alpha^{**}$, that is when the poles are not involved in the asymptotic. We divide the proof in two lemmas.

Most of the quantities studied so far depend on the starting point of the process, even if this dependence is not explicit in the notations. In the following, we add a power z_0 (or (a_0, b_0)) in the notation of the objects which correspond to a process whose starting point is $z_0 = (a_0, b_0)$. For example, we will note $h_1^{z_0}$ or $\varphi_1^{z_0}$ when we want to emphasise the dependency on the starting point.

Lemma 8.2 (Non nullity of the constant $c(\alpha)$ for at least a starting point)**.** *If $\alpha \in \left(0, \frac{\pi}{2}\right) \setminus \{\alpha^*, \alpha^{**}\}$, there exist some starting point $z_0 \in \mathbb{R}_+^2$ such that $c_0^{z_0}(\alpha) \neq 0$.*

Proof. Let $z_0 = (a_0, b_0)$ the starting point of the process. We proceed by contradiction assuming that $c_0^{(a_0,b_0)}(\alpha) = 0$ for all $a_0, b_0 \geq 0$. Since $x(\alpha) \leq 0$ or $y(\alpha) \leq 0$, we suppose without loss of generality that $y(\alpha) \leq 0$. We have then, by (8.2) and the continuation formula:

(8.6)
$$c_1\varphi_1^{(a_0,b_0)}(y(\alpha)) - c_2\varphi_1^{(a_0,b_0)}(Y^-(x(\alpha))) = \gamma_2(x(\alpha), y(\alpha))e^{a_0x(\alpha)+b_0Y^-(x(\alpha))} - \gamma_2(x(\alpha), Y^-(x(\alpha)))e^{a_0x(\alpha)+b_0y(\alpha)}$$

with $c_1 = \gamma_1(x(\alpha), Y^-(x(\alpha)))\gamma_2(x(\alpha), y(\alpha))$ and $c_2 = \gamma_1(x(\alpha), Y^-(x(\alpha)))\gamma_2(x(\alpha), Y^-(x(\alpha)))$. Remark that $\gamma_2(x(\alpha), Y^-(x(\alpha))) \neq 0$ since we assume $\alpha \neq \alpha^*$. The right term of (8.6) is unbounded on the set of all (a_0, b_0) belonging to $\mathbb{R}_+^2$ since $Y^-(x(\alpha)) < y(\alpha) = Y^+(x(\alpha))$. Then, it is sufficient to show that the supremum of the left term is bounded according to (a_0, b_0). We denote $h_1^{(a_0,b_0)}$ the density of H_1 according to the Lebesgue measure corresponding to the starting point (a_0, b_0). We have

$$c_1\varphi_1^{(a_0,b_0)}(y(\alpha)) - c_2\varphi_1^{(a_0,b_0)}(Y^-(x(\alpha))) = \int_0^\infty \left(c_1e^{y(\alpha)z} - c_2e^{Y^-(x(\alpha))z}\right) h_1^{(a_0,b_0)}(z)dz =: I. \tag{8.7}$$

Similarly to the proof of Lemma 3.5, we introduce T the first hitting time of the axis $\{x = 0\}$. By the strong Markov property, we obtain on the same way:

$$I = \mathbb{E}_{(a_0,b_0)}\left[\mathbb{1}_{T<+\infty}\mathbb{E}_{(0,Z_T^2)}\left[\int_0^{+\infty} \mathbb{1}_{\{0\}\times\mathbb{R}_+}(Z_t)\left(c_1e^{y(\alpha)Z_t^2} - c_2e^{Y^-(x(\alpha))Z_t^2}\right)dL_t^1\right]\right] \tag{8.8}$$

$$= \int_0^{+\infty}\int_0^{+\infty}\left(c_1e^{y(\alpha)z} - c_2e^{Y^-(x(\alpha))z}\right)h_1^{(0,y)}(z)dz\mathbb{P}(T < +\infty, Z_T^2 = dy) \tag{8.9}$$

$$= \int_0^{+\infty}\left(c_1\varphi_1^{(0,y)}(y_\alpha) - c_2\varphi_1^{(0,y)}(Y^-(x_\alpha))\right)\mathbb{P}(T < +\infty, Z_T^2 = dy). \tag{8.10}$$

Using the identity (8.6) in (8.10) (where we see the relevance of going to the y-axis), we get the bound

$$|I| \leq |\gamma_2(x(\alpha), y(\alpha))| + |\gamma_2(x(\alpha), Y^-(x(\alpha)))| \tag{8.11}$$

since $y(\alpha) \leq 0$. The right term of (8.6) is therefore bounded in (a_0, b_0), which is absurd according to what we said earlier in the proof. □

Lemma 8.3 (Non nullity of the constant $c(\alpha)$ for all starting points)**.** *For all $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that $\alpha^* < \alpha < \alpha^{**}$ and $z_0 \in \mathbb{R}_+^2$, we have $c_0^{z_0}(\alpha) \neq 0$.*

Proof. Let's denote $z_0 = (a_0, b_0)$ the point obtained in Lemma 8.2 such that $c_0^{z_0}(\alpha) \neq 0$. We use again the continuity of the Laplace transform in z_0 (see the proof of Lemma 3.6) to remark that $c^{z_0'}(\alpha) \neq 0$ for all z_0' in an open neighborhood V of z_0. Let $z_0'' \in \mathbb{R}_+^2$ be the starting point of the process $Z^{(z_0'')}$

and let $T=\inf\{t\geq 0, Z_t^{(z_0'')}\in V\}$ be the hitting time of V. We have $\mathbb{P}_{z_0''}(T<+\infty)=p>0$. By the strong Markov property,

$$g^{z_0''}(r\cos(\alpha),r\sin(\alpha))\geq p\inf_{z_0'\in V} g^{z_0'}(r\cos(\alpha),r\sin(\alpha)) \tag{8.12}$$

$$\geq p\inf_{z_0'\in V}\left[c_0^{z_0'}(\alpha)(1+o_{r\to\infty}(1))\right]e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{1}{\sqrt{r}}. \tag{8.13}$$

Furthermore, V can be chosen bounded and such that $\inf_{z_0'\in V}c_0^{z_0'}(\alpha)>0$. The issue is that the term $o_{r\to\infty}(1)$ may depend on z_0'. We then have to refer to the proof of Lemma 7.1. We remark that the only thing depending on the initial condition is the constant D of Lemma 7.1, which is based on Lemma 3.10. If the supremum on $z_0'\in V$ of the quantity of Lemma 3.10 is finite, then the result is true. This fact is verified easily from the proof of this lemma, because V is bounded and $\varphi_1^{z_0'}(0)$ is continuous in z_0'. □

9. Asymptotics along the axes : $\alpha\to 0$ or $\alpha\to\frac{\pi}{2}$

In this section we study the asymptotics of the Green's function g along the axes. We recall the assumption $\alpha^*\neq 0$ and $\alpha^{**}\neq\pi/2$ made in Remark 7.2.

Let us recall that for any couple $(a,b)\in\mathbb{R}_+^2$ we define $r=\sqrt{a^2+b^2}$ and $\alpha(a,b)$ the angle in $[0,\pi/2]$ such that $\cos(\alpha)=\frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha)=\frac{b}{\sqrt{a^2+b^2}}$.

Lemma 9.1 (Contribution of the saddle point to the asymptotics when $\alpha\to 0$ or $\pi/2$)**.**

(i) *Let $a\to\infty$, $b>0$ and $\alpha(a,b)\to 0$. Then the asymptotics of (8.1) remains valid with $c_0(\alpha)\to 0$ as $\alpha\to 0$. Moreover, we have $c_0(\alpha)\sim c'\alpha$ and $c_1(\alpha)\sim c''$ as $\alpha\to 0$ where c' and c'' are non null constants at least when $\alpha^*<0$.*

(ii) *When $b\to\infty$, $a>0$ and $\alpha(a,b)\to\pi/2$ the same result holds.*

Remark 9.2 (Competition between the two first term of the asymptotics)**.** *The previous lemma states that when $\alpha\to 0$ and $r\to\infty$, there is a competition between the first two terms of the sum of the asymptotic development given in (8.1), namely the first term $\frac{c_0(\alpha)}{\sqrt{r}}\sim\frac{c'\alpha}{\sqrt{r}}\sim\frac{c'b}{r\sqrt{r}}$ and the second term $\frac{c_1(\alpha)}{r\sqrt{r}}\sim\frac{c''}{r\sqrt{r}}$ may have the same order of magnitude. If $b\to 0$, the second term is dominant. If $b\to c$ where c is a positive constant they both contribute to the asymptotic. If $b\to\infty$ (and also $b=o(a)$ since $\alpha\to 0$), the first term is dominant.*

Proof. We first prove (i). For any α close to 0, $\Gamma_{x,\alpha}$ lies in a neighborhood of $x(\alpha)$. Using the continuation formula of $\varphi_2(x)$ (3.2), the definition of F (5.2), and the fact that $\Gamma_{x,\alpha}=\overleftrightarrow{X^+(\Gamma_{y,\alpha})}$ (5.9), the first integral of (8.1) becomes

$$\frac{e^{-ax(\alpha)-by(\alpha)}}{2i\pi}\int_{\overleftrightarrow{X^+(\Gamma_{y,\alpha})}}\frac{\gamma_2(x,Y^+(x))\Big(-\gamma_1(x,Y^-(x))\varphi_1(Y^-(x))-e^{a_0x+b_0Y^-(x)}\Big)}{\gamma_2(x,Y^-(x))\gamma_y'(x,Y^+(x))}$$
$$\times\exp\left(\sqrt{a^2+b^2}F(x,\alpha)\right)dx.$$

Let us make the change of variables $x=X^+(y)$. Taking into account the fact that $Y^+(X^+(y))=y$, the relation $\gamma_x'(X^+(y),y)(X^+(y))'+\gamma_y'(X^+(y),y)\equiv 0$ and the direction of $\overleftrightarrow{X^+(\Gamma_{y,\alpha})}$, the first integral becomes

$$\frac{e^{-ax(\alpha)-by(\alpha)}}{2i\pi}\int_{\Gamma_{y,\alpha}}\frac{\gamma_2(X^+(y),y)\Big(-\gamma_1(X^+(y),Y^-(X^+(y)))\varphi_1(Y^-(X^+(y)))-e^{a_0X^+(y)+b_0Y^-(X^+(y))}\Big)}{\gamma_2(X^+(y),Y^-(X^+(y)))\gamma_x'(X^+(y),y)}$$
$$\times\exp\left(\sqrt{a^2+b^2}G(y,\alpha)\right)dy. \tag{9.1}$$

Let us sum it up with the second and the third integral, for which we use the representation valid for $a > a_0$. Then we have to find the asymptotic of the integral

$$\frac{e^{-ax(\alpha)-by(\alpha)}}{2i\pi}\int_{\Gamma_{y,\alpha}}\frac{\gamma_2(X^+(y),Y^-(X^+(y)))H(X^+(y),y)-\gamma_2(X^+(y),y)H(X^+(y),Y^-(X^+(y)))}{\gamma_2(X^+(y),Y^-(X^+(y)))\gamma'_x(X^+(y),y)}$$
$$\times\exp\left(\sqrt{a^2+b^2}G(y,\alpha)\right)dy$$

where

$$H(X^+(y),y)=\gamma_1(X^+(y),y)\varphi_1(y)+\exp(a_0X^+(y)+b_0y).$$

Finally, let us note, that with notations (4.3)

$$Y^-(X^+(y))=\frac{c(X^+(y))}{a(X^+(y))\times Y^+(X^+(y))}=\frac{\sigma_{11}(X^+(y))^2+2\mu_1X^+(y)}{\sigma_{22}y}.$$

Function $X^+(y)$ is holomorphic in a neighborhood of $Y^\pm(x_{max})$. By (4.5) we have $\gamma'_x(X^+(y),y)=\sqrt{\widetilde{b}^2(y)-4\widetilde{a}(y)\widetilde{c}(y)}$ which is holomorphic in a neighborhood of $Y^\pm(x_{max})$ and different from zero. Finally $\gamma_2(x_{\max},Y^\pm(x_{max}))\neq 0$ by our assumption of Remark 7.2. It follows that the integrand in (9.1) is a holomorphic function in a neighborhood of $Y^\pm(x_{max})$. Then the whole saddle point procedure of Lemma 8.1 applied to $G(y,\alpha)$ with $\alpha=0$ and replacing the function $f(it,\alpha)$ by

$$f(it,\alpha)=[\gamma_2(X^+(y(it,\alpha)),Y^-(X^+(y(it,\alpha))))H(X^+(y(it,\alpha)),y(it,\alpha))-$$
$$\gamma_2(X^+(y(it,\alpha)),y(it,\alpha))H(X^+(y(it,\alpha)),Y^-(X^+(y(it,\alpha))))]$$
$$\times\frac{y'_\omega(it,\alpha)}{\gamma_2(X^+(y(it,\alpha)),Y^-(X^+(y(it,\alpha))))\gamma'_x(X^+(y(it,\alpha)),y(it,\alpha))}$$

where $y(it,\alpha)$ is the path given by the parameter Morse Lemma (Lemma A.1). We get the asymptotic development (8.1) as $\alpha\to 0$. We then have a competition $c_0(\alpha)+\frac{c_1(\alpha)}{r}+O\left(\frac{1}{r^2}\right)$ with $c_0(\alpha)=\frac{1}{2\sqrt{\pi}}f(0,\alpha)$ and $c_1(\alpha)=-\frac{1}{4\sqrt{\pi}}\frac{f''_\omega(0,\alpha)}{4!}$. When $\alpha\to 0$ we have $c_0(\alpha)\sim c'\alpha$ and $c_1(\alpha)\sim c''$ for a non null constants c' and c'', see Lemma 9.3 and Remark 9.4 bellow.

The proof of (ii) is exactly the same except that we use the other representation of $I_3(a,b)$. □

Lemma 9.3 (Non nullity of c')**.** *When $\alpha\to 0$ we have $c_0(\alpha)\sim c'\alpha$ and the constant c' is non null at least when $\alpha^*<0$.*

Proof. It is clear that $c_0(0)=0$ because $c_0(\alpha)$ coincides with (8.2) by uniqueness of asymptotic development and this expression tends to 0 as α goes to 0 due to $C(\alpha)$. Let us now deal with the behaviour of $c_0(\alpha)$ when $\alpha\to 0$ remembering that $c_0(\alpha)=\frac{1}{2\sqrt{\pi}}f(0,\alpha)$ where we use the notations of the proof of Lemma 9.1. First, we have thanks to Lemma 3.1

$$y(\alpha)-Y^-(X^+(y(\alpha)))=Y^+(X^+(y(\alpha))-Y^-(X^+(y(\alpha)))$$
$$=\frac{2}{\sigma_{22}}\sqrt{(\sigma_{11}\sigma_{22}-\sigma_{12}^2)(x_{max}-X^+(y(\alpha))(X^+(y(\alpha))-x_{min})}.$$

We also have $(X^+(y))'\Big|_{y=y(0)}=0$ and $(X^+(y))''\Big|_{y=y(0)}=-\frac{\sigma_{22}}{\gamma'_x(x_{max},Y^\pm(x_{max}))}$, so that

$$x_{max}-X^+(y(\alpha))=\frac{\sigma_{22}}{2\gamma'_x(x_{max},Y^\pm(x_{max}))}\alpha^2(1+o(1)),\ \text{as }\alpha\to 0.$$

Finally

$$y(\alpha)-Y^-(X^+(y(\alpha)))\sim\sqrt{\frac{2(\sigma_{11}\sigma_{22}-\sigma_{12}^2)(x_{max}-x_{min})}{2\sigma_{22}\gamma'_x(x_{max},Y^\pm(x_{max}))}}\times\alpha\sim\Pi\times\alpha,$$

where Π is defined as the constant in front of α.

Since $\gamma_2(x,y) = r_{12}x + r_{22}y$ and $\gamma_2(x_{max}, Y^-(x_{max}))\gamma_x'(x_{max}, Y^-(x_{max})) \neq 0$, we obtain that when $\alpha \to 0$

$$\begin{aligned} c_0(\alpha) &= \frac{-r_{22}H(x_{max}, Y^\pm(x^{max})) \times (\Pi\alpha) + \gamma_2(x_{max}, Y^\pm(x_{max}))H_y'(x_{max}, Y^\pm(x_{max})) \times (\Pi\alpha) + o(\alpha)}{\gamma_2(x_{max}, Y^-(x_{max}))\gamma_x'(x_{max}, Y^-(x_{max})) + o(1)} \\ &= \alpha(c' + o(1)) \end{aligned}$$

where c' is the corresponding constant.

Let us prove that $c' \neq 0$. We have to show that

$$-r_{22}H(x_{max}, Y^\pm(x^{max})) + \gamma_2(x_{max}, Y^\pm(x_{max}))H_y'(x_{max}, Y^\pm(x_{max})) \neq 0$$

i.e. that

$$-r_{22}\Big(\gamma_1(x_{max}, Y^\pm(x^{max}))\varphi_1(Y^\pm(x_{max})) + e^{a_0 x_{max} + b_0 Y^\pm(x_{max})}\Big) + \gamma_2(x_{max}, Y^\pm(x_{max}))\times$$
$$\Big(r_{21}\varphi_1(Y^\pm(x_{max})) + \gamma_1(x_{max}, Y^\pm(x_{max}))\varphi_1'(Y^\pm(x_{max})) + b_0 e^{a_0 x_{max} + b_0 Y^\pm(x_{max})}\Big) \neq 0$$

The equation can be rewritten as

$$c_1\varphi_1(Y^\pm(x_{max})) + c_2\varphi_1'(Y^\pm(x_{max})) \neq (c_3 + c_4 b_0)e^{a_0 x_{max} + b_0 Y^\pm(x_{max})} \tag{9.2}$$

with c_1, c_2, c_3, c_4 constants not depending on the initial conditions. Note that $c_3 = -r_{22} \neq 0$ by (2.1) and $c_4 = \gamma_2(x_{max}, Y^\pm(x_{max})) \neq 0$ by the assumption done in Remark 7.2. Furthermore, with the same method employed in the proof of Lemmas 3.5 and 8.2, the left term of (9.2) is bounded in (a_0, b_0). Since $x_{max} > 0$ and $Y^\pm(x_{max}) < 0$, the right term of (9.2) is not bounded in (a_0, b_0). Hence, (9.2) holds for at least one (a_0, b_0). By a similar argument developed in the proof Lemmas 3.6 and 8.3 (using the fact that $\alpha^* < 0$), we show that $c' \neq 0$ only for one starting point (a_0, b_0) imply that $c' \neq 0$ for all starting points. Finally, (9.2) holds for every initial condition which concludes the proof of the fact that $c_0(\alpha) \sim c'\alpha$ for a non null constant c'. □

Remark 9.4 (Non nullity of c'')**.** *We admit here that $c'' \neq 0$. A proof inspired by what has been done in the previous lemma to show that $c' \neq 0$ would work. The same techniques has also been employed in Lemmas 3.5 and 3.6 to characterize the poles by showing the non nullity of a constant and in Lemmas 8.2 and 8.3 to show the non nullity of $c_0(\alpha)$. Here, it would be too technical to be detailed and would not provide any additional elements of understanding compared with the previous applications of this method.*

We now have everything we need to prove our second main result which states the full asymptotic expansion of the Green's function g along the edges.

Theorem 5 (Asymptotics along the edges for the quadrant)**.** *We now assume that $\alpha_0 = 0$ and let $r \to \infty$ and $\alpha \to \alpha_0 = 0$. In this case, we have $c_0(\alpha) \underset{\alpha\to 0}{\sim} c'\alpha$ and $c_1(\alpha) \underset{\alpha\to 0}{\sim} c''$ for some constants c' and c'' which are non null at least when $\alpha^* < 0$. Then, the Green's function $g(r\cos(\alpha), r\sin(\alpha))$ has the following asymptotics:*

- *When $\alpha^* < 0$ the asymptotics given by (8.3) remains valid. In particular, we have*
$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to 0}}{\sim} e^{-r(\cos(\alpha)x(\alpha)+\sin(\alpha)y(\alpha))}\frac{1}{\sqrt{r}}\left(c'\alpha + \frac{c''}{r}\right).$$
- *When $\alpha^* > 0$ the asymptotics given by (8.4) remains valid. In particular, we have*
$$g(r\cos(\alpha), r\sin(\alpha)) \underset{\substack{r\to\infty \\ \alpha\to 0}}{\sim} c^* e^{-r(\cos(\alpha)x^*+\sin(\alpha)y^*)}.$$

Therefore, when $\alpha^* < 0$, there is a competition between the two first terms of the sum $\sum_{k=0}^n \frac{c_k(\alpha)}{r^k}$ to know which one is dominant between $c'\alpha$ and $\frac{c''}{r}$. More precisely:

- *If $r\sin\alpha \underset{\substack{r\to\infty \\ \alpha\to 0}}{\longrightarrow} \infty$ then the first term is dominant.*
- *If $r\sin\alpha \underset{\substack{r\to\infty \\ \alpha\to 0}}{\longrightarrow} c > 0$ then both terms contribute, they have the same order of magnitude.*

- *If* $r\sin\alpha \underset{\substack{r\to\infty\\ \alpha\to 0}}{\longrightarrow} 0$ *then the second term is dominant.*

A symmetric result holds when we take $\alpha_0 = \frac{\pi}{2}$. *The asymptotics given by* (8.3) *remains valid when* $\frac{\pi}{2} < \alpha^{**}$ *and* (8.5) *remain valid when* $\alpha^{**} < \frac{\pi}{2}$ *and there is a competition between the two first terms of the sum to know which one is dominant which depends of the limit of* $r\cos(\alpha)$.

Proof. The theorem follows directly from several lemmas put together. First, in Lemma 4.1 we inverse the Laplace transform and we express the Green's function g as the sum of three integrals. Then, in Lemma 6.1 we shift the integration contour of the integrals to reveal the contribution of the poles to the asymptotics applying the residue theorem. In Lemma 7.3 we show the negligibility of some integrals which imply that the asymptotic expansion is given by the integrals on the steepest descent contour. Finally, in Lemma 9.1 states the asymptotics expansion of these integrals given by the saddle point method. □

10. Asymptotics when a pole meet the saddle point : $\alpha \to \alpha^*$ or $\alpha \to \alpha^{**}$

In this section we study the asymptotics of the Green's function $g(r\cos\alpha, r\sin\alpha)$ when $\alpha \to \alpha_0$ in the special case where $\alpha_0 = \alpha^*$ or $\alpha_0 = \alpha^{**}$, that is when *the pole meet the saddle point*.

Let us introduce the following notation for shortness :

$$R(\alpha) = x'_\omega(0,\alpha) = \sqrt{\frac{2}{F''_{xx}(x(\alpha),\alpha)}} = \sqrt{\frac{2}{-\sin(\alpha)(Y^+(x))''\mid_{x(\alpha)}}} \tag{10.1}$$

$$= \sqrt{\frac{2\sin(\alpha)\gamma'_y(x(\alpha),y(\alpha))}{\sigma_{11}\sin^2(\alpha) + 2\sigma_{12}\sin(\alpha)\cos(\alpha) + \sigma_{22}\cos^2(\alpha)}}.$$

We recall that for $(a,b) \in \mathbb{R}^2_+$ we define $r = \sqrt{a^2+b^2}$ and $\alpha(a,b)$ the angle in $[0,\pi/2]$ such that $\cos(\alpha) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\alpha) = \frac{b}{\sqrt{a^2+b^2}}$.

Lemma 10.1 (Asymptotics of the integral on steepest descent line when $\alpha \to \alpha^*$)**.** *Let* $\alpha(a,b) \to \alpha^*$ *as* $r = \sqrt{a^2+b^2} \to \infty$. *Then*

$$I := \frac{1}{2\pi i} \int\limits_{\Gamma_{x,\alpha}} \frac{\gamma_2(x,Y^+(x))\varphi_2(x)}{\gamma'_y(x,Y^+(x))} \exp\left(\sqrt{a^2+b^2}F(x,\alpha(a,b))\right) dx$$

has the following asymptotics.

(i) *Let* $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2 \to 0$, *then*

$$I \sim \frac{-1}{2} \frac{\gamma_2(x^*,y^*)\mathrm{res}_{x=x^*}\varphi_2}{\gamma'_y(x^*,y^*)} \qquad \text{if } \alpha(a,b) > \alpha^*,$$

$$I \sim \frac{1}{2} \frac{\gamma_2(x^*,y^*)\mathrm{res}_{x=x^*}\varphi_2}{\gamma'_y(x^*,y^*)} \qquad \text{if } \alpha(a,b) < \alpha^*.$$

(ii) *Let* $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2 \to c > 0$. *Let*

$$A(\alpha^*) = \frac{-x'_\alpha(\alpha^*)}{R(\alpha^*)}. \tag{10.2}$$

Then

$$I \sim \frac{-1}{2} \exp(cA^2(\alpha^*))\left(1 - \Phi\left(\sqrt{c}A(\alpha^*)\right)\right) \times \frac{\gamma_2(x^*,y^*)\mathrm{res}_{x=x^*}\varphi_2}{\gamma'_y(x^*,y^*)} \qquad \text{if } \alpha(a,b) > \alpha^*,$$

$$I \sim \frac{1}{2} \exp(cA^2(\alpha^*))\left(1 - \Phi\left(\sqrt{c}A(\alpha^*)\right)\right) \times \frac{\gamma_2(x^*,y^*)\mathrm{res}_{x=x^*}\varphi_2}{\gamma'_y(x^*,y^*)} \qquad \text{if } \alpha(a,b) < \alpha^*$$

where

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2)dt. \tag{10.3}$$

(iii) *Let* $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2 \to \infty$. *Then*

$$I \sim \frac{\gamma_2(x^*,y^*)R(\alpha^*)}{2\sqrt{\pi}\gamma_y'(x^*,y^*)} \times \frac{\mathrm{res}_{x=x^*}\varphi_2}{(x(\alpha(a,b))-x(\alpha^*))} \times \frac{1}{\sqrt[4]{a^2+b^2}}.$$

Proof. Starting the proof as in Lemma 8.1, we obtain again that

$$I \sim \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} f(it,\alpha(a,b))\exp(-\sqrt{a^2+b^2}t^2)dt \tag{10.4}$$

where

$$f(it,\alpha(a,b)) = \frac{\gamma_2(x(it,\alpha),Y^+(x(it,\alpha)))\varphi_2(x(it,\alpha))}{\gamma_y'(x(it,\alpha),Y^+(x(it,\alpha)))} \times x_\omega'(it,\alpha).$$

The function φ_2 is a sum of a holomorphic function and of the term $\frac{\mathrm{res}_{x^*}\varphi_2}{x-x^*}$ which after the change of variables takes the form $\frac{\mathrm{res}_{x^*}\varphi_2}{x(it,\omega)-x^*}$. This term should be worked out.

We have $x(0,\alpha^*)=x(\alpha^*)$. By the theorem of implicit function there exists a function $\omega(\alpha)$ in the class $\mathcal{C}^\infty$ such that

$$x(\omega(\alpha),\alpha) \equiv x^* \quad \forall \alpha : |\alpha-\alpha^*| \leq \tilde{\eta} \qquad \omega(\alpha^*)=0$$

with some $\tilde{\eta}$ small enough. Furthermore, differentiating this equality we get

$$\omega'(\alpha) = \frac{-x_\alpha'(\omega(\alpha),\alpha)}{x_\omega'(\omega(\alpha),\alpha)},$$

so that

$$\omega'(\alpha^*) = -\frac{x_\alpha'(\alpha^*)}{x_\omega'(0,\alpha^*)}.$$

The formula

$$\omega(\alpha) = \frac{(x(\alpha^*)-x(\alpha))}{x_\omega'(0,\alpha^*)}(1+o(1)) = \frac{(x(\alpha^*)-x(\alpha))}{R(\alpha^*)}(1+o(1)) \quad \text{as } \alpha\to\alpha^*. \tag{10.5}$$

provides the asymptotic of $\omega(\alpha)$ as $\alpha\to\alpha^*$. Note also that the main part of $\omega(\alpha)$ is real.

Let us introduce function

$$\Psi(\omega,\alpha) = \begin{cases} \frac{\omega-\omega(\alpha)}{x(\omega,\alpha)-x(\omega(\alpha),\alpha)} & \text{if } \omega\neq\omega(\alpha) \\ \frac{1}{x_\omega'(\omega(\alpha),\alpha)} & \text{if } \omega=\omega(\alpha) \end{cases}$$

This function is holomorphic in ω for any fixed α and continuous as a function of three real variables. We can write the integral (10.4) as

$$\frac{1}{2\pi i}\int_{-\epsilon}^{\epsilon} f(it,\alpha)(it-\omega(\alpha))\frac{\exp(-\sqrt{a^2+b^2}t^2)}{t+i\omega(\alpha)}dt.$$

There exists a constant $C>0$ such that

(10.6)

$$\Big|f(it,\alpha)(it-\omega(\alpha))-f(0,\alpha)(0-\omega(\alpha))\Big| \leq C|t|.\ \forall (it,\alpha)\in\widetilde{\Omega}(0,\alpha^*)=\{(\omega,\alpha):|\omega|\leq K, |\alpha-\alpha_0|\leq\min(\eta,\tilde{\eta})\}.$$

It suffices to take C the maximum of the modulus of $(f(\omega,\alpha)(\omega-\omega(\alpha))-f(0,\alpha)(0-\omega(\alpha)))\omega^{-1}$ on $\{(\omega,\alpha):|\omega|=K, |\alpha-\alpha_0|\leq\min(\eta_0,\tilde{\eta})\}$.

Moreover since $\Im\omega(\alpha)=o(\Re\omega(\alpha))$ as $\alpha\to\alpha^*$, then by Lemma (B.1) (i) for any α close to α^* the inequality

$$\frac{|t|}{|t+i\omega(\gamma)|} \leq 2$$

is valid for all $t\in\mathbb{R}$. The integral

$$\int_{\mathbb{R}} 2\exp(-\sqrt{a^2+b^2}t^2)dt = O(\frac{1}{\sqrt[4]{a^2+b^2}})$$

is of smaller order than any asymptotic announced in the statement of the lemma. Hence, it suffices to show that the integral

$$\frac{1}{2\pi i}\int_{-\epsilon}^{\epsilon} f(0,\alpha)(0-\omega(\alpha))\frac{\exp(-\sqrt{a^2+b^2}t^2)}{t+i\omega(\alpha)}dt$$

has the expected asymptotic. Note that by (10.5)

$$\varphi(x(\alpha))(-\omega(\alpha))x'_\omega(0,\alpha)\to \mathrm{res}_{x^*}\varphi \text{ as } \alpha\to\alpha^*,$$

so that

$$f(0,\alpha)(-\omega(\alpha))\to\frac{\gamma_2(x^*,y^*)}{\gamma'_y(x^*,y^*)}\times \mathrm{res}_{x^*}\varphi.$$

It remains to study

$$\frac{1}{2\pi i}\int_{-\epsilon}^{\epsilon}\frac{\exp(-\sqrt{a^2+b^2}t^2)}{t+i\omega(\alpha)}dt.$$

For any $t\in\mathbb{R}\setminus[-\epsilon,\epsilon]$ the denominator in the integral is bounded from below

$$|t+i\omega(\alpha)|\geq ||t|-\omega(\alpha)|\geq\epsilon-\omega(\alpha)\geq\epsilon/2$$

for any α close enough to α^* while

$$\int_{\mathbb{R}}\exp(-\sqrt{a^2+b^2}t^2)dt=O(\frac{1}{\sqrt[4]{a^2+b^2}})$$

is of smaller order than the one expected in the lemma. Finally it suffices to prove that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\exp(-\sqrt{a^2+b^2}t^2)}{t+i\omega(\alpha(a,b))}dt. \tag{10.7}$$

has the right asymptotic. By the change of variables, equation (10.7) is equals to

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\exp(-s^2)}{s+i\omega(\alpha)\sqrt[4]{a^2+b^2}}ds. \tag{10.8}$$

Let now $\alpha>\alpha^*$ [resp. $\alpha<\alpha^*$]. Then $x(\alpha)<x(\alpha^*)$ [resp. $x(\alpha)>x(\alpha^*)$] and by (10.5) $\Re\omega(\alpha)>0$ [resp. $\Re\omega(\alpha)<0$]. By Lemma B.1 (iii) this integral equals

$$\begin{aligned}&\frac{-1}{2}\exp(\sqrt{a^2+b^2}\omega^2(\alpha))\Big(1-\Phi(\sqrt[4]{a^2+b^2}\omega(\alpha))\Big)\quad \text{if } \alpha>\alpha^*\\ &\frac{1}{2}\exp(\sqrt{a^2+b^2}\omega^2(\alpha))\Big(1-\Phi(-\sqrt[4]{a^2+b^2}\omega(\alpha))\Big)\quad \text{if } \alpha<\alpha^*.\end{aligned}$$

If $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2\to c\geq 0$ then by (10.5) $\sqrt{a^2+b^2}\omega(\alpha(a,b))^2\to cA^2(\alpha^*)$, the results of (i) and (ii) are immediate. Let now $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2\to\infty$. Then by Lemma B.1 (ii) the asymptotic of this integral is

$$\frac{\sqrt{\pi}}{2\pi i\times(i\omega(\alpha(a,b)))\sqrt[4]{a^2+b^2}}$$

where the asymptotic of $\omega(\alpha(a,b))$ is found in (10.5). The result follows. □

For further use one may remark that

$$\begin{aligned}-\cos(\alpha)x^*-\sin(\alpha)y^* &= -\cos(\alpha)x(\alpha)-\sin(\alpha)Y^+(x(\alpha))+R^{-2}(\alpha^*)(x(\alpha)-x^*)^2(1+o(1))\\ &= -\cos(\alpha)x(\alpha)-\sin(\alpha)Y^+(x(\alpha))+A^2(\alpha^*)(\alpha-\alpha^*)^2(1+o(1)),\quad \alpha\to\alpha^*\end{aligned} \tag{10.9}$$

with notations $R(\alpha)$ and $A(\alpha)$ above (10.1) et (10.2).

In fact, by Taylor expansion at $x(\alpha)$ and definition of the saddle point (the first derivative is zero):

$$-\cos(\alpha)x^*-\sin(\alpha)y^*=-\cos(\alpha)x(\alpha)-\sin(\alpha)Y^+(x(\alpha))-\frac{1}{2}\sin(\alpha)(Y^+(x))''\mid_{x=x(\alpha)}(x(\alpha)-x^*)^2(1+o(1)),\ \ \alpha\to\alpha^*.$$

Let us remind that

$$-\frac{1}{2}\sin(\alpha)(Y^+(x))''\mid_{x=x(\alpha)}=(R(\alpha))^{-2}=R(\alpha^*)^{-2}(1+o(1)),\quad \alpha\to\alpha^*.$$

The following lemma is useful to determine the asymptotics of the value of I_1 found in Lemma 6.1.

Lemma 10.2 (Combined contribution of the pole and saddle point to the asymptotics when $\alpha\to\alpha^*$)**.** *Let $r=\sqrt{a^2+b^2}\to\infty$ and $\alpha(a,b)\to\alpha^*$. The sum*

(10.10)

$$I:=\frac{\big(-\text{res}_{x=x^*}\varphi_2(x)\big)\gamma_2(x^*,y^*)}{\gamma'_y(x^*,y^*)}\exp(-ax^*-by^*)\times\mathbf{1}_{\alpha<\alpha^*}+\int_{\Gamma_{x,\alpha}}\frac{\gamma_2(x,Y^+(x))\varphi_2(x)}{\gamma'_y(x,Y^+(x))}\exp(-ax-bY^+(x))dx$$

has the following asymptotic.

(i) *If $\alpha>\alpha^*$ and $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2\to\infty$. Then*

$$I\sim\frac{\exp(-ax(\alpha(a,b))-by(\alpha(a,b)))}{\sqrt[4]{a^2+b^2}}\frac{\gamma_2(x^*,y^*)}{\sqrt{2\pi}\sqrt{\sigma_{11}\sin^2(\alpha^*)+2\sigma_{12}\sin(\alpha^*)\cos(\alpha^*)+\sigma_{22}\cos^2(\alpha^*)}}\times\frac{\text{res}_{x^*}\varphi_2}{x(\alpha(a,b))-x^*}\times C(\alpha^*),$$

where

$$C(\alpha^*)=\sqrt{\frac{\sin(\alpha^*)}{\gamma'_y(x^*,y^*)}}=\sqrt{\frac{\cos(\alpha^*)}{\gamma'_x(x^*,y^*)}}.$$

(ii) *If $\alpha>\alpha^*$ and $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2\to c>0$, then*

$$I\sim\frac{-1}{2}\exp(-ax^*-by^*)\Big(1-\Phi\Big(\sqrt{c}A(\alpha^*)\Big)\Big)\times\frac{\gamma_2(x^*,y^*)\text{res}_{x=x^*}\varphi}{\gamma'_y(x^*,y^*)}.$$

where $A(\alpha^*)$ and Φ are defined in (10.2), (10.1) and (10.3).

(iii) *If $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2\to 0$, then*

$$I\sim\frac{-1}{2}\exp(-ax^*-by^*)\times\frac{\gamma_2(x^*,y^*)\text{res}_{x=x^*}\varphi}{\gamma'_y(x^*,y^*)}.$$

(iv) *If $\alpha<\alpha^*$ and $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2\to c>0$, then*

$$I\sim\frac{-1}{2}\exp(-ax^*-by^*)\Big(1+\Phi\Big(\sqrt{c}A(\alpha^*)\Big)\Big)\times\frac{\gamma_2(x^*,y^*)\text{res}_{x=x^*}\varphi}{\gamma'_y(x^*,y^*)}.$$

(v) *If $\alpha<\alpha^*$ and $\sqrt{a^2+b^2}(\alpha(a,b)-\alpha^*)^2\to\infty$, then*

$$I\sim\exp(-ax^*-by^*)\times\frac{-\gamma_2(x^*,y^*)\text{res}_{x=x^*}\varphi}{\gamma'_y(x^*,y^*)}.$$

Proof. Let us note that

$$\int_{\Gamma_{x,\alpha}}\frac{\gamma_2(x,Y^+(x))\varphi_2(x)}{\gamma'_y(x,Y^+(x))}\exp(-ax-bY^+(x))dx \tag{10.11}$$

$$=\exp(-ax(\alpha)-by(\alpha))\int_{\Gamma_{x,\alpha}}\frac{\gamma_2(x,Y^+(x))\varphi_2(x)}{\gamma'_y(x,Y^+(x))}\exp\big(-\sqrt{a^2+b^2}F(x,\alpha(a,b))\big)dx$$

(i) The result just follows from the representation (10.11) and Lemma 10.1 (iii) with $R(\alpha^*)$ defined in (10.1).

(ii) By (10.9) the representation (10.11) can be also written as

(10.12)

$$\exp(-ax^*-by^*-\sqrt{a^2+b^2}A^2(\alpha^*)(\alpha(a,b)-\alpha^*)^2(1+o(1)))\int_{\Gamma_{x,\alpha}}\frac{\gamma_2(x,Y^+(x))\varphi_2(x)}{\gamma'_y(x,Y^+(x))}\exp(-\sqrt{a^2+b^2}F(x,\alpha(a,b)))dx$$

The the result follows from Lemma 10.1 (ii).

(iii) We have to consider three subcases.

Let first $\alpha(a,b) > \alpha^*$. Then the announced result follows from Lemma 10.1 (i) and the representation (10.12) where $\sqrt{a^2+b^2}A^2(\alpha^*)(\alpha(a,b)-\alpha^*) \to 0$.

If $\alpha = \alpha^*$, then the contour $\Gamma_{x,\alpha}$ is special and a direct computation leads to the result. We refer to Lemma 19 of [14] which deals with a similar case.

If $\alpha(a,b) < \alpha^*$, then by Lemma 10.1 (i) the asymptotic of the second term of (10.10) is the same as in the case $\alpha(a,b) > \alpha^*$ but with the opposite sign. It should be summed with the first term. The sum of their constants $1/2 - 1$ provides the result.

(iv) By the representation (10.12) and Lemma 10.1 (ii) the asymptotic of the second term of (10.10) is the same as in the case (ii) but with opposite sign. It should be summed with the first term. The sum of their constants $\frac{1}{2}(1-\Phi(\sqrt{c}A(\alpha^*)))-1$ leads to the result.

(v) By Lemma (10.1) (iii) and the representation (10.12) the second term of (10.10) has the asymptotic

$$\exp\Big(-ax^*-by^*-\sqrt{a^2+b^2}A^2(\alpha^*)(\alpha(a,b)-\alpha^*)^2(1+o(1))\Big)$$
$$\times\frac{\gamma_2(x^*,y^*)R(\alpha^*)}{2\sqrt{\pi}\gamma_y'(x^*,y^*)}\times\frac{\mathrm{res}_{x=x^*}\varphi}{(x(\alpha(a,b))-x(\alpha^*))}\times\frac{1}{\sqrt[4]{a^2+b^2}}.$$

Since $\frac{\exp(-\sqrt{a^2+b^2}A^2(\alpha^*)(\alpha(a,b)-\alpha^*)^2)}{(\alpha(a,b)-\alpha^*)\sqrt[4]{a^2+b^2}}$ converges to 0 in this case, the order of the second term in (10.10) is clearly smaller than the one of the first term which dominates the asymptotics.

□

Remark 10.3 (Consistency of the results)**.** *The results of (i) and (v) are perfectly "continuous" with asymptotics along directions upper and lower than α^*. Namely, if in (i) we substitute $\varphi(x(\alpha))$ instead of $\frac{res_{x^*}\varphi}{x(\alpha(a,b))-x^*}$, we obtain the asymptotic for angles higher than α^*. The result (v) stays valid for angles lower than α^*.*

We now summarize the previous results to obtain our last main result.

Theorem 6 (Asymptotics in the quadrant when the saddle point *meet* a pole)**.** *We now assume that $\alpha_0 = \alpha^*$ and let $\alpha \to \alpha^*$ and $r \to \infty$. Then, the Green's density function $g(r\cos\alpha, r\sin\alpha)$ has the following asymptotics:*

- *When $r(\alpha-\alpha^*)^2 \to 0$ then the principal term of the asymptotics is given by (8.4) but the constant c^* of the first term has to be replaced by $\frac{1}{2}c^*$.*
- *When $r(\alpha-\alpha^*)^2 \to c > 0$ for some constant c then:*
  - *If $\alpha < \alpha^*$ the principal term of the asymptotics is still given by (8.4) but the constant c^* of the first term has to be replaced by $\frac{1}{2}c^*(1+\Phi(\sqrt{c}A))$ for some constant A.*
  - *If $\alpha > \alpha^*$ the principal term of the asymptotics is still given by (8.4) but the constant c^* of the first term has to be replaced by $\frac{1}{2}c^*(1-\Phi(\sqrt{c}A))$ for some constant A.*

  In the previous items we denoted $\Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)dt$.
- *When $r(\alpha-\alpha^*)^2 \to \infty$ then:*
  - *If $\alpha < \alpha^*$ the principal term of the asymptotics is given by (8.4).*
  - *If $\alpha > \alpha^*$ the principal term of the asymptotics is given by (8.3) and we have $c_0(\alpha) \underset{\alpha\to\alpha^*}{\sim} \frac{c}{\alpha-\alpha^*}$ for some constant c.*

A symmetric result holds when we assume that $\alpha_0 = \alpha^{**}$.

Proof. The theorem follows directly from several lemmas put together. The Green's function g is still given by the sum $I_1+I_2+I_3$, see Lemma 4.1. We apply again Lemma 6.1 to take into account the contribution of the poles and Lemma 7.1 which show the negligibility of some integrals in the final asymptotics. Furthermore, by the proof of Lemma 8.1, $I_2+I_3 = O\left(\frac{e^{-r\cos(\alpha^*)x(\alpha^*)-r\sin(\alpha^*)y(\alpha^*))}}{\sqrt{r}}\right)$ when $r\to\infty$ and $\alpha\to\alpha^*$ (recall that $\alpha^* < \alpha^{**}$). With Lemma 10.2, we see in each cases that I_2+I_3 is negligible compared to I_1 when $r\to\infty$ and $\alpha\to\alpha^*$. Indeed, in the case $\alpha>\alpha^*$ and $r(\alpha-\alpha^*)^2\to\infty$, the domination of I_1 is due to the term $\frac{1}{x(\alpha)-x^*}$. For the other cases, the domination of I_1 is due to the factor $\frac{1}{\sqrt{r}}$ in the asymptotics of I_2+I_3.

The proof is similar for $\alpha_0=\alpha^{**}$. □

11. Asymptotics in a cone

From the quadrant to the cone. Let us describe the linear transformation which maps the reflected Brownian motion in the quarter plane (of covariance matrix Σ and reflecting vectors R^1 and R^2) to a reflected Brownian motion in a wedge with identity covariance matrix. We take

$$\beta=\arccos\left(-\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}\right)\in(0,\pi) \tag{11.1}$$

and we define

$$T=\begin{pmatrix}\frac{1}{\sin\beta} & \cot\beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{\sigma_{11}}} & 0\\ 0 & \frac{1}{\sqrt{\sigma_{22}}}\end{pmatrix} \tag{11.2}$$

which satisfy $T\Sigma T^\top=\mathrm{Id}$. Then, if Z_t is a reflected Brownian motion in the quadrant of parameters (Σ,μ,R), the process $\widetilde{Z_t}=TZ_t$ is a reflected Brownian motion in the cone of angle β and of parameters $(T\Sigma T^\top,T\mu,TR)=(\mathrm{Id},\widetilde{\mu},TR)$. The new reflection matrix TR correspond to reflections of angles δ and ϵ defined in $(0,\pi)$ by

$$\tan\delta=\frac{\sin\beta}{\frac{r_{12}}{r_{22}}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}}+\cos\beta}\qquad\text{and}\qquad\tan\varepsilon=\frac{\sin\beta}{\frac{r_{21}}{r_{11}}\sqrt{\frac{\sigma_{11}}{\sigma_{22}}}+\cos\beta}. \tag{11.3}$$

The new drift has an angle $\theta=\arg\widetilde{\mu}$ with the horizontal axis, it satisfies

$$\tan\theta=\frac{\sin\beta}{\frac{\mu_1}{\mu_2}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}}+\cos\beta}. \tag{11.4}$$

The assumption $\mu_1>0$ and $\mu_2>0$ is equivalent to $\theta\in(0,\beta)$.

Green's functions in the cone. Let us denote g^{z_0} the density of $G(z_0,\cdot)$. For $z\in\mathbb{R}_+^2$ we have

$$g^{z_0}(z)=\int_0^\infty p_t(z_0,z)\mathrm{d}t.$$

Let us recall that we denote $\widetilde{G}(\widetilde{z_0},\widetilde{A})$ the Green measure of $\widetilde{Z_t}$ and $\widetilde{g}^{\widetilde{z_0}}(\widetilde{z})$ its density. It is straightforward to see that for $A\in\mathbb{R}_+^2$ we have $G(z_0,A)=\widetilde{G}(Tz_0,TA)$ and then

$$g^{z_0}(z)=|\det T|\widetilde{g}^{Tz_0}(Tz)=\frac{1}{\sqrt{\det\Sigma}}\widetilde{g}^{\widetilde{z_0}}(\widetilde{z}) \tag{11.5}$$

where $\widetilde{z_0}=Tz_0$ and $\widetilde{z}=Tz$.

Polar coordinates. For any $z=(a,b)\in\mathbb{R}_+^2$ we may define the polar coordinate in the quadrant $(r,\alpha)\in\mathbb{R}_+\times[0,\frac{\pi}{2}]$ by

$$z=(a,b)=(r\cos\alpha,r\sin\alpha). \tag{11.6}$$

We now define the polar coordinates in the β-cone (ρ,ω) by

$$\widetilde{z}=(\rho\cos\omega,\rho\sin\omega). \tag{11.7}$$

For $\widetilde{z}=Tz$ we obtain by a direct computation that

$$(r\cos\alpha,r\sin\alpha)=(\rho\sqrt{\sigma_{11}}\cos(\beta-\omega),\rho\sqrt{\sigma_{22}}\sin\omega). \tag{11.8}$$

and that

$$\tan\omega = \frac{\sin\beta}{\frac{1}{\tan\alpha}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}}+\cos\beta}. \tag{11.9}$$

We deduce that

$$\widetilde{g}^{\widetilde{z}_0}(\rho\cos\omega,\rho\sin\omega) = \sqrt{\det\Sigma}\ g^{z_0}(\rho\sqrt{\sigma_{11}}\cos(\beta-\omega),\rho\sqrt{\sigma_{22}}\sin\omega). \tag{11.10}$$

Saddle point. The ellipse $\mathcal{E} = \{(x,y)\in\mathbb{R}^2 : \gamma(x,y)=0\}$ can be easily parametrized by the following

$$\mathcal{E} = \{(\widetilde{x}(t),\widetilde{y}(t)) : t\in[0,2\pi]\},$$

where

$$\begin{cases}\widetilde{x}(t) = \frac{x_{max}+x_{min}}{2}+\frac{x_{max}-x_{min}}{2}\cos(t),\\ \widetilde{y}(t) = \frac{y_{max}+y_{min}}{2}+\frac{y_{max}-y_{min}}{2}\cos(t-\beta).\end{cases} \tag{11.11}$$

see Proposition 5 of [23]. Noticing that

$$-\cos\theta = \frac{x_{max}+x_{min}}{x_{max}-x_{min}}, \quad \text{and} \quad -\cos(\beta-\theta) = \frac{y_{max}+y_{min}}{y_{max}-y_{min}}$$

and that

$$2|\widetilde{\mu}| = \sqrt{\sigma_{11}}(x_{max}-x_{min})\sin\beta = \sqrt{\sigma_{22}}(y_{max}-y_{min})\sin\beta$$

we obtain

$$\begin{cases}\widetilde{x}(t) = \frac{|\widetilde{\mu}|}{\sqrt{\sigma_{11}}\sin\beta}(\cos t-\cos\theta) = \frac{2|\widetilde{\mu}|}{\sqrt{\sigma_{11}}\sin\beta}\sin(\frac{\theta-t}{2})\sin(\frac{t+\theta}{2})\\ \widetilde{y}(t) = \frac{|\widetilde{\mu}|}{\sqrt{\sigma_{22}}\sin\beta}(\cos(t-\beta)-\cos(\theta-\beta)) = \frac{2|\widetilde{\mu}|}{\sqrt{\sigma_{22}}\sin\beta}\sin(\frac{\theta-t}{2})\sin(\frac{t+\theta-2\beta}{2}).\end{cases} \tag{11.12}$$

The following result gives an expression of the saddle point in term of the polar coordinate in the cone.

Proposition 11.1 (Saddle point in polar coordinate)**.** *For $\alpha\in(0,\frac{\pi}{2})$ and $\omega\in(0,\beta)$ previously defined and linked by (11.9) we have*

$$(x(\alpha),y(\alpha)) = (\widetilde{x}(\omega),\widetilde{y}(\omega)) \tag{11.13}$$

where $(x(\alpha),y(\alpha))$ is the saddle point defined in (5.1).

Proof. Let $\alpha\in(0,\frac{\pi}{2})$, we are looking for the point $(x(\alpha),y(\alpha))$ which maximizes the quantity $x\cos\alpha+y\sin\alpha$ for (x,y) in the ellipse $\mathcal{E}=\{(x,y)\in\mathbb{R}^2:\gamma(x,y)=0\}$. We are looking for a $t\in(0,\beta)$ cancelling the derivative of $\widetilde{x}(t)\cos\alpha+\widetilde{y}(t)\sin\alpha$ w.r.t t. By (11.12) we obtain that $\widetilde{x}'(t)\cos\alpha+\widetilde{y}'(t)\sin\alpha=0$ if and only if

$$-\frac{1}{\sqrt{\sigma_{11}}}\sin t\cos\alpha-\frac{1}{\sqrt{\sigma_{22}}}\sin(t-\beta)\sin\alpha = 0.$$

Writing $\sin(t-\beta)=\sin t\cos\beta-\cos t\sin\beta$ it directly leads to $\tan t = \frac{\sin\beta}{\frac{1}{\tan\alpha}\sqrt{\frac{\sigma_{22}}{\sigma_{11}}}+\cos\beta}$. Then by (11.9) we obtain $\tan t=\tan\omega$ and we deduce that $t=\omega$ maximizes $\widetilde{x}(t)\cos\alpha+\widetilde{y}(t)\sin\alpha$ and therefore $(x(\alpha),y(\alpha))=(\widetilde{x}(\omega),\widetilde{y}(\omega))$. □

Poles. Let us recall that x^* is the pole of $\varphi_2(x)$ when it has one, and y^{**} is the pole of $\varphi_1(y)$ when it has one, see Proposition 3.4. We defined α^* and α^{**} such that $x(\alpha^*)=x^*$ and $y(\alpha^{**})=y^{**}$. Now, we may define the corresponding ω^* and ω^{**} linked by linked by formula (11.9) and such that

$$x^* = \widetilde{x}(\omega^*) = \widetilde{x}(-\omega^*) \quad \text{and} \quad y^{**} = \widetilde{y}(\omega^{**}) = \widetilde{y}(2\beta-\omega^{**}). \tag{11.14}$$

Proposition 11.2 (Poles in polar coordinate)**.** *We have*

$$\omega^* = \theta-2\delta \quad \textit{and} \quad \omega^{**} = \theta+2\epsilon. \tag{11.15}$$

We have, $\alpha<\alpha^*$ if and only if $\omega<\omega^*$, and $\alpha>\alpha^{**}$ if and only if $\omega>\omega^{**}$. Then, x^* is the pole of $\varphi_2(x)$ if and only if $\theta-2\delta>0$, and y^{**} is a pole of $\varphi_1(y)$ if and only if $\theta+2\epsilon<\beta$.

Proof. When the pole of φ_2 exists we have $\gamma_2(x^*, Y^-(x^*)) = 0$. Let us recall that in (6.2) we defined $y^* := Y^+(x^*) = \widetilde{y}(\omega^*)$. Therefore, we have $Y^-(x^*) = \widetilde{y}(-\omega^*)$. We are looking for the solutions of equation

$$\gamma_2(\widetilde{x}(t), \widetilde{y}(t)) = 0, \tag{11.16}$$

which is the intersection of the ellipse $\mathcal{E}$ and the line $\gamma_2 = 0$. There are two solutions, the first one is elementary and is given by $t = \theta$, that is $(\widetilde{x}(t), \widetilde{y}(t)) = (0, 0)$. The second one is by definition $(\widetilde{x}(-\omega^*), \widetilde{y}(-\omega^*)) = (x^*, Y^-(x^*))$. By (11.12), the equation (11.16) gives

$$r_{12} \frac{1}{\sqrt{\sigma_{11}}} \sin\left(\frac{-\omega^* + \theta}{2}\right) + r_{22} \frac{1}{\sqrt{\sigma_{22}}} \sin\left(\frac{-\omega^* + \theta}{2} - \beta\right) = 0$$

With some basic trigonometry, we obtain that

$$\tan \frac{-\omega^* + \theta}{2} = \frac{\sin \beta}{\frac{r_{12}}{r_{22}} \sqrt{\frac{\sigma_{22}}{\sigma_{11}}} + \cos \beta} = \tan(\delta).$$

We deduce that $\omega^* = \theta - 2\delta$. A symmetric computation leads to $\omega^{**} = \theta + 2\epsilon$. The necessary and sufficient condition for the existence of the poles comes from Proposition 3.4 and some trigonometry. The inequalities on α transfer on ω by equation (11.9). □

Asymptotics in the cone. We now compute the exponential decay rate in term of the polar coordinate in the cone.

Proposition 11.3 (Exponential decay rate)**.** *For α and ω previously defined and linked by (11.9) we have*

$$r \cos(\alpha) x(\alpha) + r \sin(\alpha) y(\alpha) = 2\rho |\widetilde{\mu}| \sin^2\left(\frac{\omega - \theta}{2}\right) \tag{11.17}$$

$$r \cos(\alpha) x(\alpha^*) + r \sin(\alpha) y(\alpha^*) = 2\rho |\widetilde{\mu}| \sin^2\left(\frac{2\omega - \omega^* - \theta}{2}\right) \tag{11.18}$$

Proof. By Equations (11.8) and (11.13) and some basic trigonometry properties we obtain the desired result. □

Proofs of Theorems 1, 2 and 3. Equation (11.5) and Propositions 11.1, 11.2, 11.3 combined to Theorem 4 (resp. Theorems 5 and 6), leads to Theorem 1 (resp. Theorems 2 and 3). □

Appendix A. Parameter-dependent Morse Lemma

The following lemma is a parameter-dependent Morse lemma. It is an intuitive result but we could not find it in the known literature.

Lemma A.1. *Assume that $\alpha_0 \in \mathbb{R}$ is a constant, $\alpha \mapsto x(\alpha)$ is a function which is $\mathcal{C}^\infty$ near α_0, and $(x, \alpha) \mapsto F(x, \alpha)$ is a function which is analytic as a function of the first variable x and $\mathcal{C}^\infty$ as a function of the second variable α near $(x(\alpha_0), \alpha_0)$. Furthermore we assume that for all α near α_0 we have*

$$F(x(\alpha), \alpha) = 0, \quad F'_x(x(\alpha), \alpha) = 0, \quad F''(x(\alpha), \alpha) > 0.$$

There exists a neighborhood of $(0, \alpha_0)$ in $\mathbb{C} \times \mathbb{R}$

$$\Omega(0, \alpha_0) = \{(\omega, \alpha) \in \mathbb{C} \times \mathbb{R} : |\omega| \leq K, |\alpha - \alpha_0| \leq \eta\}$$

with some $K, \eta > 0$ and a function $x(\omega, \alpha)$ defined in $\Omega(0, \alpha_0)$ such that

$$F(x(\omega, \alpha), \alpha) = \omega^2, \quad \forall \omega : |\omega| \leq K$$

$$x(0, \alpha) = x(\alpha) \quad \forall \alpha : |\alpha - \alpha_0| \leq \eta.$$

Furthermore $x(\omega,\alpha)$ is in the class $\mathcal{C}^\infty$ as function of three real variables $\Re\omega, \Im\omega, \alpha$ and holomorphic of ω for any fixed α. Finally

$$x'_\omega(0,\alpha) = \sqrt{\frac{2}{F''_x(x(\alpha),\alpha))}}. \tag{A.1}$$

Proof. This is an adaptation of Morse lemma to the dependence of the parameter a α. Consider $T(z,\alpha) = F(z + x(\alpha), \alpha)$. Then $T(0,\alpha) = 0$, $T'_z(0,\alpha) = 0$ and $T''_z(0,\alpha) = F''_x(x(\alpha),\alpha) > 0$ for any α close to α_0. Then the following representation holds

$$T(z,\alpha) = z^2 F''_x(x(\alpha),\alpha)/2 + z^3 h(z,\alpha) \tag{A.2}$$

which allows to define

$$S(z,\alpha) = z\sqrt{F''_x(x(\alpha),\alpha)/2 + zh(z,\gamma)}$$

with one of two branches of the square root. Let us choose the one that takes the value $+F''_x(x(\alpha),\alpha)/2$ at $z = 0$. Due to elementary properties of the function F and the fact that $x(\alpha)$ is in class $\mathcal{C}^\infty$, function $h(z,\alpha)$ in the representation of T above is in class $\mathcal{C}^\infty$ in a neighborhood of $\mathcal{O}(0,\alpha_0) \subset \mathbb{C} \times \mathbb{R}$ as a function of three real variables and also holomorphic in z for any fixed α. Furthermore

$$S'_z(0,\alpha_0) = F''_x(x(\alpha_0),\alpha_0)/2 \neq 0. \tag{A.3}$$

Then by the theorem of implicit function (the real one to establish the announced properties in $\mathbb{R}^3$ and the complex one to show the holomorphicity), there exists a function $z(\omega,\alpha)$ in a neighborhood of $(0,\alpha_0)$ which is in the class $\mathcal{C}^\infty$ in three variables and holomorphic in ω such that

$$S(z(\omega,\alpha),\alpha) \equiv \omega, \quad z(0,\alpha_0) = 0. \tag{A.4}$$

This means that $T(z(\omega,\alpha),\alpha) \equiv \omega^2$ for any couple (ω,α) in this neighborhood. In particular, function $z(0,\alpha)$ solves the equation $S(z,\alpha) \equiv 0$ in the variable z. Since $S'_z(0,\alpha_0) \neq 0$, a function in the class $\mathcal{C}^\infty$ of real variable α satisfying this equation and vanishing at α_0 is unique by the theorem of implicit function. But we know already that $S(0,\alpha) = 0$ for any α close to α_0. Hence, $z(0,\alpha) \equiv 0$ for any α close to α_0.

Let now

$$x(\omega,\alpha) = z(\omega,\alpha) + x(\alpha).$$

where $x(\alpha)$ is in the class $\mathcal{C}^\infty$. It satisfies all expected properties. Furthermore $F(x(\omega,\alpha),\alpha) \equiv \omega^2$. Differentiating this identity twice, we obtain (A.1). □

Appendix B. Technical Results

This following lemma is useful in Section 10 for finding out how the asymptotics behaves as the saddle point approaches the pole.

Lemma B.1. (i) *If $C > 0$ is such that $C^2 \geq 1 + \frac{B^2}{A^2}$, then*

$$\frac{|s|}{|s + i(A+iB)|} \leq C \quad \forall s \in \mathbb{R}.$$

(ii) *Let $|A| \to \infty$ and $B = o(A)$ as $|A| \to \infty$. Then*

$$\int_{-\infty}^{\infty} \frac{\exp(-s^2)}{s + i(A+iB)} ds \sim \frac{\sqrt{\pi}}{i(A+iB)}.$$

(iii) *Let*

$$\Pi(w) = \int_{-\infty}^{\infty} \frac{\exp(-s^2)}{s + iw} ds$$

with $\Re w \neq 0$. This function is holomorphic in each half plane $\{w : \Re w > 0\}$ and $\{w : \Re w < 0\}$ and can be made explicit:

$$\Pi(w) = \pi i \exp(w^2)(1 - \Phi(-w)) \quad \forall w : \ \Re w < 0$$
$$\Pi(w) = -\pi i \exp(w^2)(1 - \Phi(w)) \quad \forall z : \ \Re w > 0$$

where $\Phi(w) = \frac{2}{\sqrt{\pi}} \int_0^w \exp(-s^2)ds$.

Proof. (i) Elementary computation.

(ii) We have $\int_{-\infty}^{\infty} \frac{\exp(-s^2)}{i(A+iB)} ds = \frac{\sqrt{\pi}}{i(A+iB)}$. It suffices to show that

$$\int_{\mathbb{R}} \frac{|s|}{|s+i(A+iB)|} \exp(-s^2)ds$$

converges to 0 for any A with absolute value large enough to have $\frac{|A|}{|B|} \geq 1$. Then by (i) $\frac{|s|}{|s+i(A+iB)|} \leq 2$ for any $s \in \mathbb{R}$. Since the integral $\int_{\mathbb{R}} 2\exp(-s^2)ds$ converges, the dominated convergence theorem applies and we get the asymptotic.

(iii) Let us define for any $z > 0$ and $w > 0$

$$\Pi(z,w) = \int_{-\infty}^{\infty} \frac{\exp(-zs^2)}{s+iw} ds$$

Then

$$\begin{aligned}\Pi'_z(z,w) &= \int_{-\infty}^{\infty} \frac{-s^2\exp(-zs^2)}{s+iw} ds = \int_{-\infty}^{\infty} \frac{((iw)^2 - s^2 - (iw)^2)\exp(-zs^2)}{s+iw} ds \\ &= \int_{-\infty}^{\infty} (iw - s)\exp(-zs^2)ds + w^2 \int_{-\infty}^{\infty} \frac{\exp(-zs^2)}{s+iw} ds \\ &= iw\sqrt{\frac{\pi}{z}} + w^2\Pi(w,z).\end{aligned}$$

Solving this differential equation we get that $\Pi(w,z) = c(w,z)\exp(w^2 z)$ where $c'_z(w,z) = iw\sqrt{\frac{\pi}{z}}\exp(-w^2 z)$. Taking into account the fact that $\Pi(+\infty, w) = 0$, we obtain

$$\begin{aligned}\Pi(z,w) = -iw\sqrt{\pi}\exp(w^2 z)\int_z^{\infty} t^{-1/2}\exp(-w^2 t)dt &= -iw\sqrt{\pi}\exp(w^2 z)\int_{w\sqrt{z}}^{\infty} \exp(-s^2)ds \\ &= -iw\pi\exp(w^2 z)\Big(1 - \Phi(w\sqrt{z})\Big).\end{aligned}$$

Let now $z = 1$. Then

$$\Pi(1,w) = -\pi i \exp(w^2)(1 - \Phi(w))$$

for any real positive w. Using the holomorphicity of $\Phi(w)$ in $\{w \in \mathbb{C} : \Re w > 0\}$ we prove the statement (iii). Finally we note that for any w with $\Re w < 0$: $\Pi(-w) = -\Pi(w)$.

□

Appendix C. Green's functions near zero and Laplace transforms near infinity

We introduce the parameter

$$\lambda = \frac{\delta + \epsilon - \pi}{\beta}$$

where β is the angle of the cone, and ϵ and δ the angles of reflection which can be expressed in term of the covariance matrix Σ and the reflection matrix R, see Section 11. This parameter λ is well known in the SRBM literature and is usually denoted α but to avoid any confusion of notation we have called it lambda in this article. It is well known that existence conditions of the SRBM stated in (2.1) are equivalent to

$$\lambda < 1.$$

Lemma C.1 (Laplace transforms behaviour near infinity and Green's functions near zero)**.** *For some constants C_1 and C_2, the Laplace transforms φ_1 and φ_2 satisfy*

$$\varphi_1(y) \sim C_1 y^{\lambda-1} \text{ when } |y| \to \infty \quad \text{and} \quad \varphi_2(x) \sim C_2 x^{\lambda-1} \text{ when } |x| \to \infty \tag{C.1}$$

and their derivatives satisfy

$$\varphi'_1(y) \sim C_1(\lambda-1) y^{\lambda-2} \text{ when } |y| \to \infty \quad \text{and} \quad \varphi'_2(x) \sim C_2(\lambda-1) x^{\lambda-2} \text{ when } |x| \to \infty. \tag{C.2}$$

Furthermore, the Green's functions on the boundaries h_1 *and* h_2 *satisfy*

$$h_1(v) \sim C_1\Gamma(-\lambda+1)v^{-\lambda} \text{ when } |v| \to 0 \quad \text{and} \quad h_2(u) \sim C_2\Gamma(-\lambda+1)u^{-\lambda} \text{ when } |u| \to 0. \tag{C.3}$$

We have noted Γ *the gamma function.*

We give the sketch of the proof of the previous Lemma which rely on the resolution of Boundary Value Problem studied in [22] which cannot be fully detailed here due to technical aspects. This lemma is not crucial for establishing the results of this article. It is only used to simplify the proof of Lemma 7.3 which is useful only in the special case where we are looking for the asymptotics along the axes.

Sketch of proof. The article [22] states in Theorem 11 an explicit expression for the Laplace transform φ_1. This result is obtained by solving a Carleman Boundary Value Problem coming from the functional equation (2.2). The solution is the product of the solution of the corresponding homogeneous problem and an integral, namely

$$\varphi_1(y) = X(W(y)) \left(\frac{1}{2\pi} \int_{\mathcal{R}^-} \frac{g(t)}{X^+(t)} \frac{dt}{W(y) - W(t)} + C \right),$$

where we took the notations of Theorem 11 in [22] and its proof. Since $\frac{g(t)}{X^+(s)}$ converges to 0 when t tends to infinity, the integral $\frac{1}{2\pi}\int_0^1 \frac{g(t)}{X^+(t)} \frac{dt}{W(y)-W(t)}$ converges to a constant when $y \to \infty$ by classical complex analysis results, see (5.2.17) of [17]. The function $X(W(y))$ is the solution to the corresponding homogeneous BVP which is studied in detail in the recurrent case in [25]. Proposition 19 of [25] shows that $X(W(y))) \sim y^{\lambda-1}$ when y tends to infinity which concludes the proof of (C.1).

Integral Hardy–Littlewood Tauberian theorems (see for example Karamata's theorem and Ikehara's theorem [43, §7.4 & 7.5] and [13, Thm 33.3 & 33.7]) state that, with some hypothesis, for a function f and its Laplace transform $\mathcal{L}(f)$, for $\lambda \geqslant -1$, $f(t) \sim Ct^{-\lambda}$ when $t \to 0$ is equivalent to $\mathcal{L}(f)(x) \sim C\Gamma(-\lambda+1)x^{\lambda-1}$ when $x \to \infty$. Equation (C.3) follows from Tauberian theorem and from (C.1).

The proof of (C.2) follows from (C.3), from Tauberian theorem and from the properties of the derivative Laplace transform, namely $\mathcal{L}(tf(t)) = \frac{d}{dx}\mathcal{L}(f)(x)$. □

References

[1] F. Baccelli and G. Fayolle. Analysis of models reducible to a class of diffusion processes in the positive quarter plane. *SIAM J. Appl. Math.*, 47(6):1367–1385, 1987.

[2] M. Bousquet-Mélou, A. Elvey Price, S. Franceschi, C. Hardouin, and K. Raschel. On the stationary distribution of reflected Brownian motion in a wedge: differential properties. 2022. arXiv:2101.01562.

[3] M. Bramson. Positive recurrence for reflecting Brownian motion in higher dimensions. *Queueing Syst.*, 69(3-4):203–215, 2011.

[4] M. Bramson, J. Dai, and J. Harrison. Positive recurrence of reflecting Brownian motion in three dimensions. *Ann. Appl. Probab.*, 20(2):753–783, 2010.

[5] Y. Brychkov, H.-J. Glaeske, A. Prudnikov, and V. K. Tuan. *Multidimensional Integral Transformations.* CRC Press, Jan. 1992.

[6] H. Chen. A sufficient condition for the positive recurrence of a semimartingale reflecting Brownian motion in an orthant. *Ann. Appl. Probab.*, 6(3):758–765, 1996.

[7] J. Dai. *Steady-state analysis of reflected Brownian motions: Characterization, numerical methods and queueing applications.* ProQuest LLC, Ann Arbor, MI, 1990. Thesis (Ph.D.)–Stanford University.

[8] J. Dai and J. Harrison. Reflecting Brownian motion in three dimensions: a new proof of sufficient conditions for positive recurrence. *Math. Methods Oper. Res.*, 75(2):135–147, 2012.

[9] J. G. Dai and J. M. Harrison. Reflected Brownian motion in an orthant: numerical methods for steady-state analysis. *The Annals of Applied Probability*, 2(1):65–86, 1992.

[10] J. G. Dai and M. Miyazawa. Reflecting Brownian motion in two dimensions: exact asymptotics for the stationary distribution. *Stoch. Syst.*, 1(1):146–208, 2011.

[11] J. G. Dai and M. Miyazawa. Stationary distribution of a two-dimensional SRBM: geometric views and boundary measures. *Queueing Syst.*, 74(2-3):181–217, 2013.

[12] A. B. Dieker and J. Moriarty. Reflected Brownian motion in a wedge: sum-of-exponential stationary densities. *Electronic Communications in Probability*, 14:1–16, 2009.

[13] G. Doetsch. *Introduction to the Theory and Application of the Laplace Transformation.* Springer Berlin Heidelberg, Berlin, Heidelberg, 1974.

[14] P. A. Ernst and S. Franceschi. Asymptotic behavior of the occupancy density for obliquely reflected Brownian motion in a half-plane and Martin boundary. *The Annals of Applied Probability*, 31(6):2991 – 3016, 2021.

[15] P. A. Ernst, S. Franceschi, and D. Huang. Escape and absorption probabilities for obliquely reflected brownian motion in a quadrant. *Stochastic Processes and their Applications*, 142:634–670, 2021.

[16] G. Fayolle and R. Iasnogorodski. Two coupled processors: The reduction to a Riemann-Hilbert problem. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 47(3):325–351, Jan. 1979.

[17] G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random walks in the quarter plane*, volume 40 of *Probability Theory and Stochastic Modelling*. Springer, Cham, second edition, 2017. Algebraic methods, boundary value problems, applications to queueing systems and analytic combinatorics.

[18] M. V. Fedoryuk. Asymptotic methods in analysis. In *Analysis I*, pages 83–191. Springer, 1989.

[19] M. E. Foddy. *Analysis of Brownian motion with drift, confined to a quadrant by oblique reflection (diffusions, Riemann-Hilbert problem)*. ProQuest LLC, Ann Arbor, MI, 1984. Thesis (Ph.D.)–Stanford University.

[20] V. Fomichov, S. Franceschi, and J. Ivanovs. Probability of total domination for transient reflecting processes in a quadrant. *Advances in Applied Probability*, 54(4):1094–1138, 2022.

[21] G. J. Foschini. Equilibria for diffusion models of pairs of communicating computers—symmetric case. *IEEE Trans. Inform. Theory*, 28(2):273–284, 1982.

[22] S. Franceschi. Green's functions with oblique neumann boundary conditions in the quadrant. *Journal of Theoretical Probability*, 34(4):1775–1810, Dec 2021.

[23] S. Franceschi and I. Kourkova. Asymptotic expansion of stationary distribution for reflected Brownian motion in the quarter plane via analytic approach. *Stoch. Syst.*, 7(1):32–94, 2017.

[24] S. Franceschi and K. Raschel. Tutte's invariant approach for Brownian motion reflected in the quadrant. *ESAIM Probab. Stat.*, 21:220–234, 2017.

[25] S. Franceschi and K. Raschel. Integral expression for the stationary distribution of reflected Brownian motion in a wedge. *Bernoulli*, 25(4B):3673–3713, 2019.

[26] S. Franceschi and K. Raschel. A dual skew symmetry for transient reflected brownian motion in an orthant. *Queueing Systems*, 102(1):123–141, Oct 2022.

[27] J. M. Harrison. The diffusion approximation for tandem queues in heavy traffic. *Adv. in Appl. Probab.*, 10(4):886–905, 1978.

[28] J. M. Harrison and J. J. Hasenbein. Reflected Brownian motion in the quadrant: tail behavior of the stationary distribution. *Queueing Systems*, 61(2-3):113–138, Mar. 2009.

[29] J. M. Harrison and M. I. Reiman. On the distribution of multidimensional reflected Brownian motion. *SIAM J. Appl. Math.*, 41(2):345–361, 1981.

[30] J. M. Harrison and M. I. Reiman. Reflected Brownian motion on an orthant. *Ann. Probab.*, 9(2):302–308, 1981.

[31] J. M. Harrison and R. J. Williams. Brownian models of open queueing networks with homogeneous customer populations. *Stochastics*, 22(2):77–115, 1987.

[32] J. M. Harrison and R. J. Williams. Multidimensional reflected Brownian motions having exponential stationary distributions. *Ann. Probab.*, 15(1):115–137, 1987.

[33] D. G. Hobson and L. C. G. Rogers. Recurrence and transience of reflecting Brownian motion in the quadrant. *Math. Proc. Cambridge Philos. Soc.*, 113(2):387–399, 1993.

[34] I. Kourkova and K. Raschel. Random walks in $(\mathbb{Z}_+)^2$ with non-zero drift absorbed at the axes. *Bulletin de la Société Mathématique de France*, 139:341–387, 2011.

[35] I. A. Kourkova and V. A. Malyshev. Martin boundary and elliptic curves. *Markov Processes and Related Fields*, 4(2):203–272, 1998.

[36] I. A. Kourkova and Y. M. Suhov. Malyshev's Theory and JS-Queues. Asymptotics of Stationary Probabilities. *The Annals of Applied Probability*, 13(4):1313–1354, Nov. 2003.

[37] D. Lipshutz and K. Ramanan. Pathwise differentiability of reflected diffusions in convex polyhedral domains. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 55(3):1439 – 1476, 2019.

[38] V. A. Malyshev. Asymptotic behavior of the stationary probabilities for two-dimensional positive random walks. *Siberian Mathematical Journal*, 14(1):109–118, Jan. 1973.

[39] M. Miyazawa and M. Kobayashi. Conjectures on tail asymptotics of the marginal stationary distribution for a multidimensional SRBM. *Queueing Systems*, 68(3-4):251–260, Aug. 2011.

[40] M. I. Reiman. Open queueing networks in heavy traffic. *Math. Oper. Res.*, 9:441–458, 1984.

[41] A. Sarantsev. Reflected Brownian motion in a convex polyhedral cone: tail estimates for the stationary distribution. *J. Theoret. Probab.*, 30(3):1200–1223, 2017.

[42] L. M. Taylor and R. J. Williams. Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant. *Probab. Theory Related Fields*, 96(3):283–317, 1993.

[43] G. Tenenbaum. *Introduction to Analytic and Probabilistic Number Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

[44] S. R. S. Varadhan and R. J. Williams. Brownian motion in a wedge with oblique reflection. *Comm. Pure Appl. Math.*, 38(4):405–443, 1985.

[45] R. J. Williams. Recurrence classification and invariant measure for reflected Brownian motion in a wedge. *Ann. Probab.*, 13(3):758–778, 1985.

[46] R. J. Williams. Reflected Brownian motion in a wedge: semimartingale property. *Z. Wahrsch. Verw. Gebiete*, 69(2):161–176, 1985.
[47] R. J. Williams. Semimartingale reflecting Brownian motions in the orthant. *Stochastic Networks*, 13, 1995.

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