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ON THE EXISTENCE OF MASS MINIMIZING RECTIFIABLE G CHAINS IN FINITE DIMENSIONAL NORMED SPACES

by Thierry DE PAUW & Ioann VASILYEV (*)

Abstract. — We introduce the notion of density contractor of dimension m in a finite dimensional normed space X. If $m+1 = \dim X$, this includes the area contracting projectors on hyperplanes whose existence was established by H. Busemann. If $m = 2$, density contractors are an ersatz for such projectors and their existence, established here, follows from works by D. Burago and S. Ivanov. Once density contractors are available, the corresponding Plateau problem admits a solution among rectifiable G chains, regardless of the group of coefficients G. This is obtained as a consequence of the lower semicontinuity of the m dimensional Hausdorff mass, of which we offer two proofs. One of these is based on a new type of integral geometric measure.

Résumé. — Nous introduisons la notion de contracteur de densité de dimension m, dans un espace normé X de dimension finie. Lorsque $m+1 = \dim X$, celle-ci inclut les projecteurs contractants sur les hyperplans, dont l'existence a été établie par H. Busemann. Lorsque $m = 2$, les contracteurs de densité constituent un ersatz à ces projecteurs, et leur existence, établie ci-dessous, découle de travaux de D. Burago et S. Ivanov. En présence de contracteurs de densité, le problème de Plateau correspondant admet une solution parmi les G chaînes rectifiables, indépendamment du groupe de coefficients G. Ceci est une conséquence de la semi-continuité inférieure de la masse de Hausdorff, dont nous proposons deux démonstrations. L'une d'elles repose sur un nouveau type de mesure intégrale géométrique.

1. Foreword

The classical tools of Geometric Measure Theory, developed by H. Federer and W.H. Fleming, provide the following setting for the Plateau problem. Here, the ambient space $X = \ell_2^n$ is the n dimensional Euclidean space

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and m is an integer comprised between 1 and $n-1$. Given an $m-1$ dimensional rectifiable cycle[(1)] with integer multiplicity and compact support $B \in \mathscr{R}_{m-1}(X, \mathbf{Z})$, we consider the variational problem

$$(\mathscr{P}) \begin{cases} \text{minimize } \mathscr{M}(T), \\ \text{among } T \in \mathscr{R}_m(X, \mathbf{Z}) \text{ with } \partial T = B\,. \end{cases}$$

As $\partial B = 0$, there are competitors indeed, one being provided by the cone construction. The mass of a competitor T is

$$\mathscr{M}(T) = \int_A |\overrightarrow{T}| \mathrm{d}\mathscr{H}^m \tag{1.1}$$

where A is the underlying countably $(\mathscr{H}^m, m)$ rectifiable set on which T concentrates, $|\overrightarrow{T}|$ is the absolute value of its algebraic multiplicity and $\mathscr{H}^m$ is the Hausdorff measure associated with the ambient Euclidean structure.

Problem $(\mathscr{P})$ admits a minimizing sequence $T_1, T_2, \ldots$, with each T_k supported in the convex hull of $\operatorname{supp} B$. This is because we may replace, if needed, T_k by $\pi_\# T_k$, where $\pi : X \to X$ is the nearest point projection on the convex hull of $\operatorname{supp} B$. Since $\operatorname{Lip} \pi \leqslant 1$, it follows that $\mathscr{M}(\pi_\# T_k) \leqslant \mathscr{M}(T_k)$. Such sequence is relatively compact with respect to H. Whitney's flat norm – a consequence of the deformation theorem and rectifiability theorem [15]. Its accumulation points T are minimizers of problem $(\mathscr{P})$ because the mass is lower semicontinuous with respect to convergence in the flat norm:

$$\mathscr{M}(T) \leqslant \liminf_k \mathscr{M}(T_k)\,. \tag{1.2}$$

The convergence in flat norm,

$$\mathscr{F}(T - T_k) = \inf \left\{ \mathscr{M}(R) + \mathscr{M}(S) : \begin{array}{l} R \in \mathscr{R}_m(X, \mathbf{Z}), S \in \mathscr{R}_{m+1}(X, \mathbf{Z}) \\ \text{and } T - T_k = R + \partial S \end{array} \right\} \tag{1.3}$$

is in this case equivalent to the weak* convergence as currents.

The same method applies to proving the existence of a mass minimizing chain when, in $(\mathscr{P})$ the group of coefficients $\mathbf{Z}$ is replaced by a cyclic group $\mathbf{Z}_q$ [13, 4.2.26], a finite group [16], or more generally a locally compact normed Abelian group G that does not contain any nontrivial curve of finite length, according to the work of B. White [24, 25]. For instance, $G = \mathbf{Z}_2$ allows for considering a nonorientable minimal surface bounded by the Möbius strip, whereas $G = \mathbf{Z}_3$ allows for considering a minimal surface bounded by a triple Möbius strip, singular along a spine where infinitesimally three half planes meet at equal angles.

(1) When $m = 1$, an $m-1$ dimensional cycle is understood with respect to reduced homology.

We now describe two ways of understanding why mass is lower semicontinuous with respect to convergence in the flat norm.

(1) *Orthogonal projectors are contractions,* in particular they reduce mass: If $W \subseteq X$ is an m dimensional affine subspace, $\pi_W : X \to X$ is the orthogonal projector onto W and σ is an m dimensional simplex in X, then $\mathscr{H}^m(\pi_W(\sigma)) \leqslant \mathscr{H}^m(\sigma)$. To see how this is related to the lower semicontinuity of mass, assume (as we may, according to the strong approximation theorem [13, 4.2.22]) that $T, T_1, T_2, \dots$ are all polyhedral chains. Thus $T - (T_k + R_k) = \partial S_k$, with $\mathscr{M}(R_k)$ small. If we are able to infer from this that $\mathscr{M}(T) \leqslant \mathscr{M}(T_k + R_k)$, then we will be done. Simplifying even further, we assume that $T = g[\![\sigma]\!]$ is associated with a single simplex contained in some m dimensional affine subspace $W \subseteq X$, and we write $T_k + R_k = \sum_j g_j [\![\sigma_j]\!]$, with the σ_j nonoverlapping. Since $\pi_{W\,\#}(T - (T_k + R_k))$ is an m dimensional cycle with compact support in W, the constancy theorem implies that $T = \pi_{W\,\#} T = \pi_{W\,\#}(T_k + R_k)$. Therefore,

$$\begin{aligned}\mathscr{M}(T) &= \mathscr{M}\left(\sum_j \pi_{W\,\#}(g_j[\![\sigma j]\!])\right)\\ &\leqslant \sum_j |g_j| \mathscr{H}^m\left(\pi_{W\,\#}(\sigma_j)\right) \leqslant \sum_j |g_j| \mathscr{H}^m(\sigma_j) = \mathscr{M}(T_k + R_k).\end{aligned}$$

(2) *Hausdorff measure coincides with integral geometric measure.* In other words, the mass of T can be recovered from the mass of its 0 dimensional slices $\langle T, \pi, y\rangle$, corresponding to all orthogonal maps $\pi \in \mathbf{O}^*(n,m)$ from $\mathbf{R}^n$ to $\mathbf{R}^m$, and $y \in \mathbf{R}^m$. Specifically, if $\boldsymbol{\theta}^*_{n,m}$ denotes an $\mathbf{O}(n)$ invariant probability measure on $\mathbf{O}^*(n,m)$, then

$$\mathscr{M}(T) = \boldsymbol{\beta}_1(n,m)^{-1} \int_{\mathbf{O}^*(n,m)} \mathrm{d}\boldsymbol{\theta}^*_{n,m}(\pi) \int_{\mathbf{R}^m} \mathscr{M}(\langle T, \pi, y\rangle) \mathrm{d}\mathscr{L}^m(y) \tag{1.4}$$

for some suitable constant $\boldsymbol{\beta}_1(n,m) > 0$, see [13, 2.10.15 and 3.2.26]. The mass $\mathscr{M}(Z)$ of a 0 dimensional chain $\mathscr{R}_0(X, \mathbf{Z}) \ni Z = \sum_j g_j \boldsymbol{\delta}_{x_j}$ (where the x_j's are distinct) is simply the finite sum $\mathscr{M}(Z) = \sum_j |g_j|$. Now $\mathscr{M} : \mathscr{R}_0(X, G) \to \mathbf{R}$ is lower semicontinuous with respect to convergence in the flat norm, and if $\mathscr{F}(T - T_k) \to 0$ rapidly, then for every π one infers that $\mathscr{F}(\langle T, \pi, y\rangle - \langle T_k, \pi, y\rangle) \to 0$, for almost every y. It therefore ensues from Fatou's Lemma that

$$\begin{aligned}\mathscr{M}(T) &\leqslant \boldsymbol{\beta}_1(n,m)^{-1}\int_{\mathbf{O}^*(n,m)} \mathrm{d}\boldsymbol{\theta}^*_{n,m}(\pi)\int_{\mathbf{R}^m} \liminf_k \mathscr{M}(\langle T_k,\pi,y\rangle)\mathrm{d}\mathscr{L}^m(y)\\ &\leqslant \liminf_k \boldsymbol{\beta}_1(n,m)^{-1}\int_{\mathbf{O}^*(n,m)} \mathrm{d}\boldsymbol{\theta}^*_{n,m}(\pi)\int_{\mathbf{R}^m} \mathscr{M}(\langle T_k,\pi,y\rangle)\mathrm{d}\mathscr{L}^m(y)\\ &= \liminf_k \mathscr{M}(T_k)\,.\end{aligned}$$

Of interest to us in this paper is the case when the Euclidean norm of the ambient space is replaced with another, arbitrary norm $\|\cdot\|$. Thus, we consider G chains in a finite dimensional normed space X. The notions of m dimensional rectifiable G chains and their convergence in flat norm are not affected, thus the compactness tool for applying the direct method of the calculus of variations is available without modification. Hausdorff measure $\mathscr{H}^m_{\|\cdot\|}$ however has changed in the process, to the extent that the corresponding new Hausdorff mass

$$\mathscr{M}_H(T) = \int_A |\overrightarrow{T}|\mathrm{d}\mathscr{H}^m_{\|\cdot\|}\,,$$

$T \in \mathscr{R}_m(X,\mathbf{Z})$, is not known in general to be lower semicontinuous. In light of the two methods evoked above, we note that:

(1) If $W \subseteq X$ is an m dimensional subspace, there may not exist a projector $\pi : X \to W$ with $\operatorname{Lip}\pi \leqslant 1$. Perhaps the simplest case is when $X = \ell^3_\infty$ and $W = X \cap \{(x_1,x_2,x_3) : x_1+x_2+x_3 = 0\}$.
(2) In a specific sense, there is no integral geometric formula available for the Hausdorff measure in a normed space, according to R. Schneider [20].

Notwithstanding the first observation, H. Busemann established the existence, in codimension 1, of projectors that reduce the area. More precisely, if $W \subseteq X$ is an affine subspace of dimension $m = \dim X - 1$, then there exists a projector $\pi : X \to X$ onto W such that $\mathscr{H}^m_{\|\cdot\|}(\pi(\sigma)) \leqslant \mathscr{H}^m_{\|\cdot\|}(\sigma)$, whenever σ is an m dimensional simplex in X. This is enough for implementing successfully method (1) and proving the existence of an $\mathscr{M}_H$ minimizing rectifiable G chain, when $m = \dim X - 1$. In fact, in Theorem 9.2 below we show that there exists at least one minimizer supported in the convex hull of its boundary, even though not all minimizers have this extra property.

If $1 < m < \dim X - 1$, the existence of projectors (onto m dimensional affine subspaces) that reduce area remains conjectural. D. Burago and S. Ivanov were able [5] to design a calibration method in case $m = 2$, proving that flat 2 dimensional disks minimize their Hausdorff measure $\mathscr{H}^2_{\|\cdot\|}$ among Lipschitz competing surfaces with coefficients in $G = \mathbf{R}$ or $G = \mathbf{Z}_2$.

Even though in that case there may not exist area reducing projectors, we show here how their method leads to the notion of density contractors, *i.e.* a probability measure on the space of linear mappings $\pi : X \to X$ with rank at most 2, playing the analogous role of a single projector as far as comparing areas is concerned. The fact that we need to consider arbitrary linear mappings π, possibly with large Lipschitz constant, is a source of some technical complications.

In fact, we develop in the present paper an axiomatic theory of density contractors in arbitrary dimension m and establish their existence when $m = 2$ or $m = \dim X - 1$, see Theorem 5.10. We prove how density contractors can be used to compare the Hausdorff mass of chains Theorem 5.11, and the Hausdorff measure of sets Theorem 5.12. We show that once density contractors exist in dimension m, the Plateau problem admits $\mathscr{M}_H$ minimizers with coefficients in arbitrary locally compact groups of coefficients that satisfy B. White's condition stated above, see Theorem 8.1. The minimizers obtained here, following [2] see Theorem 3.1, are merely locally rectifiable and we do not know in general whether they may be further required to be compactly supported, except in the important cases when $m = \dim X - 1$ or when we assume a uniform lower bound on the norm of nonzero coefficients in G.

The main concern is indeed to show that $\mathscr{M}_H : \mathscr{R}_m(X, G) \to \mathbf{R}$ is lower semicontinuous with respect to convergence in the flat norm. In the spirit of this introduction, we obtain two proofs of this fact, granted the existence of density contractors.

(1) In Section 4, we establish a necessary and sufficient condition for the lower semicontinuity of $\mathscr{M}_H$ in terms of a triangle inequality of polyhedral cycles, see Theorem 4.3 and Theorem 4.5. In Section 5, where the notion of density contractor is introduced, we give a simple proof that their existence implies this triangle inequality for cycles, see Theorem 5.3.

(2) In Section 6, we define a new type of integral geometric measure associated with density contractors, Section 6.3. Rather than integrating the mass of slices as in (1.4), which would correspond to a measure $\mathscr{I}^m_1$ in the notation of [13], we consider a measure obtained from Caratheodory's method II applied locally to the supremum of the averaged measure of projected sets (averaged and projected by means of a density contractor), corresponding to $\mathscr{I}^m_\infty$ in

the notation of [13]. Rather than essential suprema, we consider locally actual suprema, thus more in the spirit of the Gross measure, [13, 2.10.4(1)]. We show that the corresponding newly defined Gross measure coincides with the Hausdorff measure of rectifiable sets, see Theorem 6.4. Furthermore, in the spirit of [3] we show that the corresponding Gross mass defined in Section 7.1 is lower semicontinuous with respect to flat convergence, Theorem 7.3.

To close this introduction we mention that D. Burago and S. Ivanov introduced a notion of semi-ellipticity for a density, *i.e.* a function defined on a Grassmann cone $GC(n,m)$. This notion depends on the group of coefficients considered – they consider only subgroups of $\mathbf{R}$, whether $\mathbf{R}$ itself, $\mathbf{Z}$, or $\mathbf{Q}$. For those groups G, semi-ellipticity with respect to G is equivalent to the triangle inequality for cycles with coefficients in G, see Definition 4.2. In [4] they establish that if a density is semi-elliptic with respect to $\mathbf{R}$ then it is extendibly convex, *i.e.* it extends over the whole $\bigwedge_m \mathbf{R}^n$ to a convex functional. That of course applies to showing our own lower semicontinuity result in case $G = \mathbf{Z}$ or $G = \mathbf{Q}$. Their techniques do not seem to establish the semi-ellipticity (say in case $m = 2$ for the Busemann–Hausdorff density) with respect to all normed Abelian groups of coefficients – in particular these authors develop an ad hoc proof in their paper [5] for the special case when $G = \mathbf{Z}_2$. The semi-ellipticity for arbitrary G is one of the results following from the present paper, and the new notion of density contractor we introduce here.

Groups of coefficients that are not subgroups of $\mathbf{R}$ have arisen in many applications of which we now mention a few. The group $G = \mathbf{Z}_2$ corresponds to unorientable surfaces. The group $G = \mathbf{Z}_3$ of integers modulo 3 is also useful as mentioned in an example above. All other cyclic groups $\mathbf{Z}_q$ might be considered as well. In the context of least energy configurations of immiscible fluids, it is relevant to consider the group G which is a free $\mathbf{Z}_2$ module generated by a finite set, see [23]. The notion of size minimizing current was introduced by F.J. Almgren to model some soap films, see also H. Federer's paper [14]. The size norm of a group G is defined by $|g| = 1$, if $g \neq 0$, and $|g| = 0$, when $g = 0$. The group G is there to realize some boundary condition in the homological sense, whereas the size norm is considered in order to minimize the volume of the underlying rectifiable set without taking into account the weight of the coefficients. Existence of size minimizers is not known in general, but see for instance [18] or [8]. In case of the Euclidean norm, semi-ellipticity is classically obtained via the Cauchy–Crofton formula. In optimal transportation theory (and in many natural structures such as the human blood vessels system), it has been argued that paths minimize a cost that takes into account a certain weight to a power different than 1. This boils down to considering, for instance, the group of coefficients $G = \mathbf{R}$ endowed with the norm $G \to \mathbf{R} : g \mapsto |g|^\alpha$, for some $0 < \alpha < 1$ (the case $\alpha = 1$ corresponds to usual mass, whereas the case $\alpha = 0$ corresponds to size). See for instance [26], or [10] for the case of higher dimensional surfaces. One may also consider a totally disconnected group, such as p-adic numbers. All these fall into the context developed in Geometric Measure Theory in recent years, see for instance [25] or [11].

It is our pleasure to record helpful discussions with Ph. Bouafia and G. Godefroy.

2. Preliminaries

2.1. Hausdorff distance

If X is a metric space and $A, B \subseteq X$ are compact, we define their *Hausdorff distance* as

$$\operatorname{dist}_{\mathscr{H}}(A,B) = \inf \left\{ \delta > 0 : A \subseteq \mathbf{B}(B,\delta) \text{ and } B \subseteq \mathbf{B}(A,\delta) \right\}$$

where $\mathbf{B}(A,\delta) = X \cap \{x : \operatorname{dist}(x,A) \leqslant \delta\}$. The following are rather obvious.

(1) *Suppose $f, f_1, f_2, \dots$ are continuous mappings from X to Y such that $f_k \to f$ locally uniformly as $k \to \infty$, and $A \subseteq X$ is compact. It follows that $\operatorname{dist}_{\mathscr{H}}(f_k(A), f(A)) \to 0$ as $k \to \infty$.*

(2) *Suppose $A, A_1, A_2, \dots$ are compact subsets of X such that $\operatorname{dist}_{\mathscr{H}}(A_k, A) \to 0$ as $k \to \infty$, and $f : X \to Y$ is continuous. Then $\operatorname{dist}_{\mathscr{H}}(f(A_k), f(A)) \to 0$ as $k \to \infty$.*

Let X be a finite dimensional real linear space. Given a norm ν on X we let $B_\nu = X \cap \{x : \nu(x) \leqslant 1\}$ denote its unit (closed) ball. If ν_1 and ν_2 are two norms on X, we define $\delta(\nu_1,\nu_2) = \inf\{\lambda > 0 : B_{\nu_1} \subseteq \lambda B_{\nu_2} \text{ and } B_{\nu_2} \subseteq \lambda B_{\nu_1}\}$. One readily checks that $\log \delta$ is well defined and a distance on the set of norms on X. When we will consider a convergent sequence of norms on X it will be relative to this distance.

(1) *Given $\nu, \nu_1, \nu_2, \dots$ a sequence of norms on X, $\delta(\nu,\nu_j) \to 1$ as $j \to \infty$ if and only if $\nu_j(x) \to \nu(x)$ as $j \to \infty$ for every $x \in X$.*

This follows from the relation $\nu(x) = \inf\{t > 0 : x \in tB_\nu\}$ and the fact that the pointwise convergence of the ν_j, $j = 1, 2, \dots$, to ν implies their uniform convergence on bounded subsets of X, due to their convexity.

(2) *Given two norms ν_1 and ν_2 on X and $W \subseteq X$ a linear subspace, one has $\delta(\nu_1|_W, \nu_2|_W) \leqslant \delta(\nu_1,\nu_2)$.*

2.2. Hausdorff outer measures

Given a norm ν on X we associate with it the Hausdorff outer measures $\mathscr{H}_\nu^m$, $m \in \{1, \dots, \dim X\}$, see for instance [13, 2.10.2]. It is the case that (the restriction to $\mathscr{B}(X)$ of) $\mathscr{H}_\nu^{\dim X}$ is a Haar measure on X. It therefore follows from the uniqueness of Haar measure and the Borel regularity of Hausdorff measures that if ν_1 and ν_2 are norms on X then there exists $0 < \beta(\nu_1, \nu_2) < \infty$ such that $\mathscr{H}_{\nu_1}^{\dim X} = \beta(\nu_1, \nu_2)\mathscr{H}_{\nu_2}^{\dim X}$. In order to estimate $\beta(\nu_1, \nu_2)$ in terms of the distance between ν_1 and ν_2 we recall the following central result of H. Busemann [6] (see also [17, Lemma 6] or [22, 7.3.6]):

$$\mathscr{H}_\nu^{\dim X}(B_\nu) = \boldsymbol{\alpha}(\dim X) \tag{2.1}$$

is independent of ν. Here $\boldsymbol{\alpha}(m) = \mathscr{L}^m(\mathbf{B}(0,1))$ is the Lebesgue measure of the unit Euclidean ball in $\mathbf{R}^m$, and a constant used in the definition of $\mathscr{H}_\nu^m$. The identity $\boldsymbol{\alpha}(\dim X) = \mathscr{H}_{\nu_1}^{\dim X}(B_{\nu_1}) = \beta(\nu_1, \nu_2)\mathscr{H}_{\nu_2}^{\dim X}(B_{\nu_1})$, the definition of $\delta(\nu_1, \nu_2)$ and the homogeneity of Hausdorff measures imply that

$$\delta(\nu_1, \nu_2)^{-\dim X} \leqslant \beta(\nu_1, \nu_2) \leqslant \delta(\nu_1, \nu_2)^{\dim X}. \tag{2.2}$$

We will also use the following observation: $\mathscr{H}_{\nu,\delta}^{\dim X} = \mathscr{H}_\nu^{\dim X}$ for all $0 < \delta \leqslant \infty$, where the former are the δ size approximating outer measures.

2.3. Grassmannian

Given X and $m \in \{1, \dots, \dim X - 1\}$ we let $\mathbf{G}_m(X)$ denote the set of m dimensional linear subspaces of X. In order to give $\mathbf{G}_m(X)$ a topology we equip first X with an inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $|\cdot|$. Given $W \in \mathbf{G}_m(X)$ we let $\pi_W : X \to X$ denote the orthogonal projection onto W. We then define $d(W_1, W_2) = |||\pi_{W_1} - \pi_{W_2}|||$, $W_1, W_2 \in \mathbf{G}_m(X)$, where $|||\cdot|||$ is the operator norm. In the remaining part of this paper, an inner product structure will always be fixed on X.

Let $W_1, W_2, \dots$ be a sequence in $\mathbf{G}_m(X)$. Choose $e_1^k, \dots, e_n^k$ to be an orthonormal basis of X such that $\operatorname{span}\{e_1^k, \dots, e_m^k\} = W_k$. Possibly passing to a subsequence we may assume $e_j^k \to e_j$ as $k \to \infty$, $j = 1, \dots, n$, and clearly $e_1, \dots, e_n$ is an orthonormal basis of X. Define $W = \operatorname{span}\{e_1, \dots, e_m\} \in \mathbf{G}_m(X)$. Given $x \in X$ a simple calculation shows that $|(\pi_W - \pi_{W_k})(x)| \leqslant |x| \sum_{j=1}^n |(\pi_W - \pi_{W_k})(e_j)| \leqslant |x| \sum_{j=1}^n \left|e_j^k - e_j\right|$. Thus $d(W, W_k) \to 0$.

2.4. Busemann–Hausdorff density

Now given two norms ν_1 and ν_2 on X and $W \in \mathbf{G}_m(X)$ we claim that there exists $0 < \beta(\nu_1, \nu_2, W) < \infty$ such that

$$\mathscr{H}^m_{\nu_1} \llcorner W = \beta(\nu_1, \nu_2, W) \mathscr{H}^m_{\nu_2} \llcorner W.$$

This is because (the restriction to $\mathscr{B}(W)$ of) $\mathscr{H}^m_{\nu_i} \llcorner W$, $i = 1, 2$, are both Haar measures on W.

In the remaining part of this paper we will consider a given fixed norm $\|\cdot\|$ on X. Comparing it with the underlying reference inner product norm $|\cdot|$, we define the *Busemann–Hausdorff density* function $\psi : \mathbf{G}_m(X) \to \mathbf{R}$ as $\psi(W) = \beta(\|\cdot\|, |\cdot|, W)$, corresponding to each $m \in \{1, \dots, \dim X - 1\}$ (we omit both $\|\cdot\|$ and m in the notation for ψ). It follows that $\psi(W)$ is characterized by the identity $\mathscr{H}^m_{\|\cdot\|}(W \cap E) = \psi(W) \mathscr{H}^m_{|\cdot|}(W \cap E)$ for any Borel $E \subseteq X$ such that one of the (and therefore both) both measures appearing there are nonzero and finite. Letting respectively $E = B_{|\cdot|}$ and $E = B_{\|\cdot\|}$ and referring to (2.1) we find that

$$\psi(W) = \frac{\mathscr{H}^m_{\|\cdot\|}\left(W \cap B_{|\cdot|}\right)}{\boldsymbol{\alpha}(m)} = \frac{\boldsymbol{\alpha}(m)}{\mathscr{H}^m_{|\cdot|}\left(W \cap B_{\|\cdot\|}\right)}. \tag{2.3}$$

Proposition 2.1. — *The Busemann–Hausdorff density $\psi : \mathbf{G}_m(X) \to \mathbf{R}$ is continuous.*

Proof. — Let $W, W_1, W_2, \dots$ be members of $\mathbf{G}_m(X)$ such that $W_j \to W$ as $j \to \infty$. We consider linear isometries $f_j : \ell_2^m \to (X, |\cdot|)$, $j = 1, 2, \dots$, whose range is W_j. By compactness a subsequence of $f_1, f_2, \dots$, still denoted $f_1, f_2, \dots$, converges pointwise to some linear isometry $f : \ell_2^m \to (X, |\cdot|)$. Since $W_j \to W$ as $j \to \infty$ we infer that the range of f is W. Now we define norms $\nu, \nu_1, \nu_2, \dots$ on $\mathbf{R}^m$ by $\nu(\xi) = \|f(\xi)\|$ and $\nu_j(\xi) = \|f_j(\xi)\|$, $\xi \in \mathbf{R}^m$, so that $f : (\mathbf{R}^m, \nu) \to (W, \|\cdot\|)$ and $f_j : (\mathbf{R}^m, \nu_j) \to (W_j, \|\cdot\|)$ are also isometries. Therefore $f_* \mathscr{H}^m_{|\cdot|} = \mathscr{H}^m_{|\cdot|} \llcorner W$, $f_{j\,*} \mathscr{H}^m_{|\cdot|} = \mathscr{H}^m_{|\cdot|} \llcorner W_j$, $f_* \mathscr{H}^m_{\nu} = \mathscr{H}^m_{\|\cdot\|} \llcorner W$ and $f_{j\,*} \mathscr{H}^m_{\nu_j} = \mathscr{H}^m_{\|\cdot\|} \llcorner W_j$, $j = 1, 2, \dots$. Since $\nu_j \to \nu$ pointwise it follows from (1), p. 641 and (2.2) that $\beta_{\ell_2^m}(\nu_j, |\cdot|) \to \beta_{\ell_2^m}(\nu, |\cdot|)$ and hence also $\beta_X(\|\cdot\|, |\cdot|, W_j) \to \beta_X(\|\cdot\|, |\cdot|, W)$ as $j \to \infty$. As the argument can be repeated for any subsequence of the original sequence $W_1, W_2, \dots$ the proof is complete. $\square$

3. Plateau problem

In this section we describe the setting in which we will state the Plateau problem. Groups of polyhedral, rectifiable and flat chains have been studied in [11, 13, 16, 24, 25]. We now provide a very quick overview.

3.1. Polyhedral, rectifiable, and flat G chains

We let $(G, |\cdot|)$ be an Abelian group equipped with a norm $|\cdot|$ which turns it into a complete metric space. As before $(X, \|\cdot\|)$ is a finite dimensional normed linear space and $m \in \{1, \dots, \dim X - 1\}$. With an m dimensional oriented simplex σ in X and a group element $g \in G$ we associate an object $g[\![\sigma]\!]$. We consider equivalence classes of formal sums of these $\sum_{k=1}^{\kappa} g_k [\![\sigma_k]\!]$. The equivalence identifies $(-g)[\![-\sigma]\!] = g[\![\sigma]\!]$ (where $-\sigma$ has orientation opposite to that of σ) and $g[\![\sigma]\!] = \sum_{k=1}^{\kappa} g[\![\sigma_k]\!]$ if $\sigma_1, \dots, \sigma_\kappa$ is a simplicial partition of σ with the same orientation. We let $\mathscr{P}_m(X, G)$ denote the group whose elements are these *polyhedral G chains of dimension m*. A boundary operator $\partial : \mathscr{P}_m(X, G) \to \mathscr{P}_{m-1}(X, G)$ is defined as usual. Given $P \in \mathscr{P}_m(X, G)$ we define its *Hausdorff mass* by the formula

$$\mathscr{M}_H(P) = \sum_{k=1}^{\kappa} |g_k| \mathscr{H}^m_{\|\cdot\|}(\sigma_k),$$

where $P = \sum_{k=1}^{\kappa} g_k [\![\sigma_k]\!]$ and the $\sigma_1, \dots, \sigma_\kappa$ are chosen to be nonoverlapping. This definition does not depend upon the choice of such a decomposition of P. If we want to insist that the mass is defined with respect to the norm $\|\cdot\|$ we will write $\mathscr{M}_{H,\|\cdot\|}(P)$ instead of $\mathscr{M}_H(P)$ to avoid confusion, as another mass $\mathscr{M}_{H,|\cdot|}(P)$ is readily available as well. Both are equivalent. The *flat norm* of $P \in \mathscr{P}_m(X, G)$ is then defined as

$$\mathscr{F}(P) = \inf \left\{ \mathscr{M}_{H,|\cdot|}(Q) + \mathscr{M}_{H,|\cdot|}(R) : \begin{array}{l} Q \in \mathscr{P}_m(X, G), R \in \mathscr{P}_{m+1}(X, G) \\ \text{and } P = Q + \partial R \end{array} \right\}.$$

The completion of $\mathscr{P}_m(X, G)$ with respect to $\mathscr{F}$ is the group $\mathscr{F}_m(X, G)$ whose members are called *flat G chains of dimension m*. One important feature of this group is that with a Lipschitz map $f : X \to Y$ one can associate a push-forward morphism $f_\# : \mathscr{F}_m(X, G) \to \mathscr{F}_m(Y, G)$ that commutes with ∂. The m dimensional Lipschitz G chains of X are then defined to be the members of $\mathscr{F}_m(X, G)$ of the form $\sum_{k=1}^{\kappa} f_{k\#} P_k$ where $P_k \in \mathscr{P}_m(\ell_\infty^m, G)$ and $f_k : \ell_\infty^m \to X$ is Lipschitz, $k = 1, \dots, \kappa$. One further defines the subgroup $\mathscr{R}_m(X,G) \subseteq \mathscr{F}_m(X,G)$ whose members, called *rectifiable G chains of dimension m*, have compact support and are limits in the $\mathscr{M}_H$ norm of sequences of Lipschitz G chains of dimension m. With such $T \in \mathscr{R}_m(X,G)$ is associated a countably $(\mathscr{H}^m_{\|\cdot\|}, m)$ rectifiable Borel subset $\operatorname{set}_m \|T\| \subseteq X$ and for almost every $x \in \operatorname{set}_m \|T\|$ a so-called G orientation $\boldsymbol{g}(x)$ which consists of a nonzero group element and an orientation of the approximate tangent space of $\operatorname{set}_m \|T\|$ at x. The *Hausdorff mass* of T is defined as

$$\mathscr{M}_H(T) = \int_{\operatorname{set}_m \|T\|} \big| \boldsymbol{g}(x) \big| \, \mathrm{d}\mathscr{H}^m_{\|\cdot\|}(x)\,.$$

The definition is consistent with the previous one in case T is polyhedral.

3.2. Hausdorff mass and lower semicontinuity

The Hausdorff Euclidean mass $\mathscr{M}_{H,|\cdot|} : \mathscr{R}_m(X,G) \to \mathbf{R}$ is lower semicontinuous with respect to $\mathscr{F}$ convergence whereas the lower semicontinuity of $\mathscr{M}_{H,\|\cdot\|} : \mathscr{R}_m(X,G) \to \mathbf{R}$ is unknown in general. This is the main topic of the present paper.

3.3. Locally rectifiable G chains

If $T \in \mathscr{F}_m(X,G)$ and $u : X \to \mathbf{R}$ is Lipschitz then the restriction $T \llcorner \{u < r\}$ is defined for $\mathscr{L}^1$ almost every $r \in \mathbf{R}$. We here define $\mathscr{R}^{\mathrm{loc}}_m(X,G)$ to be the subgroup of $\mathscr{F}_m(X,G)$ of those T such that for every bounded open set $U \subseteq X$, letting $u(x) = \operatorname{dist}(x,U)$, there exists $R \in \mathscr{R}_m(X,G)$ such that $T \llcorner \{u < r\} = R \llcorner \{u < r\}$ for $\mathscr{L}^1$ almost every $0 < r \leqslant 1$. These do not necessarily have compact support. We call these *locally rectifiable G chains of dimension m* and we define

$$\mathscr{M}_H(T) = \sup \big\{ \mathscr{M}_H(R \llcorner U) : \ U \text{ and } R \text{ are as above} \big\}\,.$$

One checks this is consistent with the preceding number.

3.4. Compactness

The following is a consequence of the deformation theorem proved in this context by B. White [24]. If $K \subseteq X$ is compact, $\lambda > 0$ and $(G, |\cdot|)$ is locally compact then

$$\mathscr{F}_m(X,G) \cap \big\{ T : \operatorname{supp} T \subseteq K \text{ and } \mathscr{M}_{H,|\cdot|}(T) + \mathscr{M}_{H,|\cdot|}(\partial T) \leqslant \lambda \big\}$$

is $\mathscr{F}$ compact.

3.5. White groups

We say that $(G, \|\cdot\|)$ is a *White group* if G does not contain any non-trivial curve of finite length. Of course if $(G, \|\cdot\|)$ is totally disconnected then it is White, for instance $G = \mathbf{Z}$, $G = \mathbf{Z}_q$ for any $q = 2, 3, \ldots$, or $G = \mathbf{Z}_2^{\mathbf{N}}$ the Cantor group. The group $(\mathbf{R}, |\cdot|)$ is not White, but $(\mathbf{R}, |\cdot|^p)$ is whenever $0 < p < 1$. The reason for considering those groups is the following result, see [25] : If $(G, \|\cdot\|)$ is a White group, $T \in \mathscr{F}_m(X, G)$ and $\mathscr{M}_{H,|\cdot|}(T) < \infty$ then $T \in \mathscr{R}_m(X, G)$. The cases $G = \mathbf{Z}$ and $G = \mathbf{Z}_q$ go back to [13, 4.2.16(3) and 4.2.26]. Together with the preceding number we obtain the following compactness result: If $K \subseteq X$ is compact, $\lambda > 0$ and $(G, \|\cdot\|)$ is locally compact and White then

$$\mathscr{R}_m(X, G) \cap \left\{T : \operatorname{supp} T \subseteq K \text{ and } \mathscr{M}_{H,|\cdot|}(T) + \mathscr{M}_{|\cdot|}(\partial T) \leqslant \lambda\right\}$$

is $\mathscr{F}$ compact.

3.6. Isoperimetric inequality

Here we assume that the group of coefficients G verifies the following assumption:

$$0 < \inf\{\|g\| : g \in G \text{ and } g \neq 0_G\}\,. \tag{3.1}$$

If $m \geqslant 2$ and $R \in \mathscr{R}_{m-1}(X, G)$ is so that $\partial R = 0$, then there exists $S \in \mathscr{R}_m(X, G)$ with $\partial S = R$ and $\mathscr{M}_{H,|\cdot|}(S) \leqslant C(m, \dim X, \|\cdot\|)\mathscr{M}_{H,|\cdot|}(R)^{\frac{m}{m-1}}$.

The constant $C(m, \dim X, \|\cdot\|)$ depends on m, $\dim X$ and the infimum in (3.1). The proof of this isoperimetric inequality is the same as in [13, 4.2.10] with the deformation theorem [13, 4.2.9] replaced with [24].

3.7. Quasiminimizing chains

We say that $T \in \mathscr{R}_m^{\text{loc}}(X, G)$ is *quasiminimizing* if there exists $C > 0$ with the following property. For every closed Euclidean ball K and every $S \in \mathscr{R}_m(X, G)$ such that $\operatorname{supp} S \subseteq K$ and $\partial S = 0$ one has $\mathscr{M}_{H,|\cdot|}(T \llcorner K) \leqslant C\mathscr{M}_{H,|\cdot|}(T \llcorner K + S)$.

There exists a constant $0 < \eta = \eta(m, \dim X, \|\cdot\|)$ with the following property. If $T \in \mathscr{R}_m^{\text{loc}}(X, G)$ is quasiminimizing, $x \in \operatorname{supp} T$, $r > 0$ and $\mathbf{B}(x, r) \cap \operatorname{supp} \partial T = \emptyset$ then $\mathscr{M}_{H,|\cdot|}(T \llcorner \mathbf{B}(x, r)) \geqslant \eta r^m$.

The proof in case $m \geqslant 2$ is as in [13, 5.1.6 page 523]. Fix $x \in \operatorname{supp} T$. For each $\mathbf{B}(x,r)$ as in the statement we define $T_r = T \llcorner \mathbf{B}(x,r)$. One has $\partial T_r = \langle T, |\cdot|, r\rangle \in \mathscr{R}_{m-1}(X,G)$ for $\mathscr{L}^1$ almost such r, and $\mathbf{M}(\langle T,u,r\rangle) \leqslant f'(r)$, where $f(r) = \mathbf{M}(T_r)$, according to [13, 4.2.1]. For such r it follows from the isoperimetric inequality Section 3.6, in case $m \geqslant 2$, that there exists $T'_r \in \mathscr{R}_m(X,G)$ such that $\partial T'_r = \langle T,u,r\rangle = \partial T_r$ and $\mathscr{M}_{H,|\cdot|}(T'_r)^{\frac{m-1}{m}} \leqslant \sigma \mathscr{M}_{H,|\cdot|}(\langle T,u,r\rangle) \leqslant \sigma f'(r)$ where $\sigma = C(m, \dim X, |\!\cdot\!|)^{\frac{m-1}{m}}$. Pushing forward the chain T'_r by the nearest point projection on the closed Euclidean ball $\mathbf{B}(x,r)$ one can readily achieve $\operatorname{spt} T'_r \subseteq \mathbf{B}(x,r)$. Therefore on letting $S_r = T'_r - T_r$ one has $\partial S_r = 0$, $\operatorname{spt} S_r \subseteq \mathbf{B}(x,r)$. Since T is quasiminimizing, $f(r) = \mathscr{M}_{H,|\cdot|}(T_r) \leqslant C\mathscr{M}_{H,|\cdot|}(T_r + S_r) = C\mathscr{M}_{H,|\cdot|}(T'_r) \leqslant C\sigma^{\frac{m}{m-1}} f'(r)^{\frac{m}{m-1}}$. Since $x \in \operatorname{spt} T$ we notice that $f(r) > 0$ and the above inequality yields $\left(f(r)^{\frac{1}{m}}\right)' \geqslant m^{-1}\sigma^{-1}C^{\frac{1-m}{m}}$. The conclusion then follows on integrating this inequality and applying [13, 2.9.19] to the nondecreasing function $f^{\frac{1}{m}}$.

In the easier case when $m = 1$ we start by noticing that $x \in \operatorname{supp} T$ and $\mathbf{B}(x,r) \cap \operatorname{supp} \partial T = \emptyset$ imply $\operatorname{Bdry} \mathbf{B}(x,\rho) \cap \operatorname{supp} T \neq \emptyset$ for every $0 < \rho < r$. Therefore

$$\begin{aligned}\mathscr{M}_{H,|\cdot|}(T \llcorner \mathbf{B}(x,r)) &= \int_{\mathbf{B}(x,r)} |\boldsymbol{g}(z)| \,\mathrm{d}(\mathscr{H}^1_{|\cdot|} \llcorner \operatorname{set}_1 \|T\|)(z)\\ &\geqslant \int_0^r \mathrm{d}\mathscr{L}^1(\rho) \sum_{z \in \operatorname{set}_1 \|T\| \cap \operatorname{Bdry} \mathbf{B}(x,\rho)} |\boldsymbol{g}(z)| \,\mathrm{d}\mathscr{H}^0(z) \geqslant \varepsilon r\end{aligned}$$

where ε is the infimum in (3.1).

We now state a particular Plateau problem: that of minimizing the *Hausdorff mass* in the context of rectifiable G chains in a finite dimensional normed space.

Theorem 3.1. — *Assume that*

(A) $(X, \|\cdot\|)$ *is a finite dimensional normed space and* $(G, |\!\cdot\!|)$ *is an Abelian normed locally compact White group;*
(B) $1 \leqslant m \leqslant \dim X - 1$*;*
(C) $\mathscr{M}_H : \mathscr{R}_m(X,G) \to \mathbf{R}$ *is lower semicontinuous with respect to* $\mathscr{F}$ *convergence;*
(D) $B \in \mathscr{R}_{m-1}(X,G)$ *and* $\partial B = 0$.

It follows that the Plateau problem

$$(\mathscr{P}) \begin{cases} \textit{minimize } \mathscr{M}_H(T), \\ \textit{among } T \in \mathscr{R}^{\mathrm{loc}}_m(X,G) \textit{ such that } \partial T = B, \end{cases}$$

admits a solution. If one further assumes that

$$0 < \inf \{ \,|\, g \,|\, : g \in G \text{ and } g \neq 0_G \}$$

then each solution of $(\mathscr{P})$ *has compact support.*

Two comments are in order.

(1) The nontrivial assumption is Item (C), that the Hausdorff mass be lower semicontinuous. The remaining sections of this paper are devoted to establishing hypothesis Item (C) in case $m = 2$ or $m = \dim X - 1$ and G is arbitrary.

(2) There is another technical problem with applying the direct method of calculus of variations. In order to apply the compactness theorem 3.5 one would need to exhibit a minimizing sequence supported in some given compact set. *In case X is Euclidean* this is done as follows. If T is such that $\partial T = B$ and $\operatorname{supp} B \subseteq \mathbf{B}(0,R)$, one considers the nearest point projection $f : X \to \mathbf{B}(0,R)$. Since $\operatorname{Lip} f \leqslant 1$ one has $\mathscr{M}_H(f_\# T) \leqslant \mathscr{M}_H(T)$, and also $\partial f_\# T = f_\# \partial T = B$. Pushing forward along f each member of a given minimizing sequence produces a new minimizing sequence of chains all supported in $\mathbf{B}(0,R)$. In case the norm is not Euclidean this process cannot be repeated as the map

$$f : X \to \mathbf{B}(0,1) : x \mapsto \begin{cases} x & \text{if } \|x\| \leqslant 1, \\ \frac{x}{\|x\|} & \text{if } \|x\| > 1, \end{cases}$$

has merely $\operatorname{Lip} f \leqslant 2$ and nothing better in general. This leads one to show that a minimizing sequence is necessarily *tight* and perform a diagonal argument as in [2]. Here we are not able to do better in general except under the extra assumption on the group G stated at the end of Theorem 3.1, or when $m = \dim X - 1$. In fact in the last section, see Theorem 9.2, we show that if $m = \dim X - 1$ then there exists a minimizing chain whose support is contained in the convex hull of $\operatorname{supp} B$.

Proof. — Let $T_1, T_2, \dots$ be a minimizing sequence. As in [2, Proof of Theorem 1.1] we start by showing that this sequence is tight:

$$\lim_{r \to \infty} \sup_{k=1,2,\dots} \mathscr{M}_H \left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(0,r)^c \right) = 0 \,. \tag{3.2}$$

Let $\gamma = \inf(\mathscr{P})$, $\varepsilon > 0$, $\alpha = \varepsilon^{-1} \sup_{k=1,2,\dots} \mathscr{M}_H(T_k - T_1) < \infty$, choose $r > 0$ such that $\mathscr{M}_H \left(T_1 \llcorner \mathbf{B}_{\|\cdot\|}(0,r)^c \right) \leqslant \varepsilon$ and choose $h > 0$ such that

$$\alpha < \int_r^{r+h} r^{-1} \mathrm{d}\mathscr{L}^1(r) \,. \tag{3.3}$$

Fix $k = 1, 2, \ldots$. Since

$$\int_r^{r+h} \mathscr{M}_H(\langle T_k - T_1, \|\cdot\|, \rho\rangle) \mathrm{d}\mathscr{L}^1(\rho) \leqslant 2C_{m,1}\mathscr{M}_H(T_k - T_1) \tag{3.4}$$

according to [11, 3.7.1(9)], there exists $r \leqslant r_k \leqslant r+h$ such that $\langle T_k - T_1, \|\cdot\|, r_k\rangle \in \mathscr{R}_{m-1}(X, G)$ and

$$\mathscr{M}_H(\langle T_k - T_1, \|\cdot\|, r_k\rangle) \leqslant r_k^{-1} 2C_{m,1}\mathscr{M}_H(T_k - T_1)\alpha^{-1}\,, \tag{3.5}$$

for if not the combination of (3.3) and (3.4) would lead to a contradiction. Letting $S_k = \boldsymbol{\delta}_0 \times \langle T_k - T_1, \|\cdot\|, r_k\rangle \in \mathscr{R}_m(X, G)$ we recall that $\mathscr{M}_H(S_k) \leqslant 2C'_m r_k \mathscr{M}_H(\langle T_k - T_1, \|\cdot\|, r_k\rangle) \leqslant 4C'_m C_{m,1}\varepsilon$ (see for instance [9, §2.5]) and $\partial S_k = \langle T_k - T_1, \|\cdot\|, r_k\rangle$ since $\partial\langle T_k - T_1, \|\cdot\|, r_k\rangle = -\langle \partial(T_k - T_1), \|\cdot\|, r_k\rangle = 0$. Since also $\langle T_k - T_1, \|\cdot\|, r_k\rangle = \partial(T_k - T_1) \llcorner \mathbf{B}_{\|\cdot\|}(0, r_k)$ we see that $\partial R_k = B$, where $R_k = (T_k - T_1) \llcorner \mathbf{B}_{\|\cdot\|}(0, r_k) + T_1 - S_k$. Therefore $\gamma \leqslant \mathscr{M}_H(R_k)$. Now $R_k = T_k \llcorner \mathbf{B}_{\|\cdot\|}(0, r_k) + T_1 \llcorner \mathbf{B}_{\|\cdot\|}(0, r_k)^c - S_k$ and accordingly

$$\begin{aligned}
\gamma &\leqslant \mathscr{M}_H(R_k) \\
&\leqslant \mathscr{M}_H\left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(0, r+h)\right) + \mathscr{M}_H\left(T_1 \llcorner \mathbf{B}_{\|\cdot\|}(0, r)^c\right) + \mathscr{M}_H(S_k) \\
&\leqslant \mathscr{M}_H(T_k) - \mathscr{M}_H\left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(0, r+h)^c\right) + (1 + 4C'_m C_{m,1})\varepsilon\,.
\end{aligned} \tag{3.6}$$

Choosing k_0 such that $\mathscr{M}_H(T_k) - \gamma \leqslant \varepsilon$ whenever $k \geqslant k_0$ it follows that

$$\sup_{k=k_0, k_0+1, \ldots} \mathscr{M}_H\left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(0, r+h)^c\right) \leqslant (2 + 4C'_m C_{m,1})\varepsilon\,. \tag{3.7}$$

As $\lim_{\rho\to\infty} \mathscr{M}_H\left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(0, \rho)^c\right) = 0$ for each $k = 1, \ldots, k_0 - 1$ it readily follows that

$$\limsup_{\rho\to\infty} \sup_{k=1,2,\ldots} \mathscr{M}_H\left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(0, \rho)^c\right) \leqslant (2 + 4C'_m C_{m,1})\varepsilon\,. \tag{3.8}$$

Since $\varepsilon > 0$ is arbitrary (3.2) is established.

We consider all $j = 1, 2, \ldots$ large enough for $\operatorname{supp} B \subseteq \mathbf{U}_{\|\cdot\|}(0, j)$. Applying inductively the compactness theorem to suitable subsequences of $\left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(0, j)\right)_k$ the diagonal argument yields one subsequence of $(T_k)_k$, still denoted as such, and one sequence $(\widehat{T}_j)_j$ of members of $\mathscr{R}_m(X, G)$ such that $\mathscr{F}\left(\widehat{T}_j - T_k \llcorner \mathbf{B}_{\|\cdot\|}(0, j)\right) \to 0$ as $k \to \infty$, for every j. We show that $(\widehat{T}_j)_j$ is $\mathscr{F}$ Cauchy. Given $\varepsilon > 0$ we choose $r > 0$ such that the supremum in (3.2) is bounded above by ε. Given j_1, j_2 larger than r, there exists k_i, $i = 1, 2$, such that $\mathscr{F}\left(\widehat{T}_{j_i} - T_k \llcorner \mathbf{B}_{\|\cdot\|}(0, j_i)\right) \leqslant \varepsilon$ whenever $k \geqslant k_i$. For

$k = \max\{k_1, k_2\}$ we thus have

$$\begin{aligned}\mathscr{F}\left(\widehat{T}_{j_1} - \widehat{T}_{j_2}\right) \leqslant & \mathscr{F}\left(\widehat{T}_{j_1} - T_k \llcorner \mathbf{B}_{\|\cdot\|}(0,j_1)\right) \\ & + \mathscr{F}\left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(0,j_1) - T_k \llcorner \mathbf{B}_{\|\cdot\|}(0,j_2)\right) \\ & + \mathscr{F}\left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(0,j_2) - \widehat{T}_{j_2}\right) \\ \leqslant & \, 3\varepsilon\,.\end{aligned}$$

Let $T \in \mathscr{F}_m(X,G)$ be the $\mathscr{F}$ limit of this sequence. Given a bounded open subset $U \subseteq X$, $u(x) = \operatorname{dist}_{\|\cdot\|}(U,x)$, and $r_0 > 0$ we select j_0 sufficiently large for $\mathbf{B}_{\|\cdot\|}(U,r_0) \subseteq \mathbf{U}_{\|\cdot\|}(0,j_0)$. We first recall that for $\mathscr{L}^1$ almost every $0 < r \leqslant r_0$ one has $\mathscr{F}\left(T \llcorner \{u<r\} - \widehat{T}_j \llcorner \{u<r\}\right) \to 0$ as $j \to \infty$, [11, 5.2.3(2)]. Furthermore if $j \geqslant j_0$ then $\widehat{T}_j \llcorner \{u<r\} = \widehat{T}_{j_0} \llcorner \{u<r\}$ by the definition of $\widehat{T}_j$. Therefore $T \llcorner \{u<r\} = \widehat{T}_{j_0} \llcorner \{u<r\}$, and thus $T \in \mathscr{R}^{\mathrm{loc}}_m(X,G)$. Now clearly $\partial \widehat{T}_j = B$ for each j and thus $\partial T = B$. Finally it remains to show that $\mathscr{M}_H(T) \leqslant \gamma$. Let $U \subseteq X$ be open and bounded and choose j_0 such that $\mathbf{B}_{\|\cdot\|}(U,1) \subseteq \mathbf{U}_{\|\cdot\|}(0,j_0)$. Now if $R \in \mathscr{R}_m(X,G)$ is so that $R \llcorner \{u<r\} = T \llcorner \{u<r\}$ for almost every $0 < r \leqslant 1$ then also $R \llcorner \{u<r\} = \widehat{T}_{j_0} \llcorner \{u<r\}$ for such r. Therefore

$$\mathscr{M}_H(R \llcorner U) \leqslant \mathscr{M}_H\left(\widehat{T}_{j_0}\right) \leqslant \liminf_k \mathscr{M}_H\left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(0,j_0)\right) \leqslant \gamma\,,$$

according to hypothesis (C). Since U is arbitrary we conclude that

$$\mathscr{M}_H(T) = \gamma.$$

We now turn to proving that each solution T of the optimization problem $(\mathscr{P})$ has compact support in case G verifies (3.1). Since the norms $|\cdot|$ and $\|\cdot\|$ are equivalent, we note that T is a quasiminimizing chain (with respect to the Euclidean Hausdorff mass, Section 3.7). Assume if possible that $\operatorname{supp} T \sim \mathbf{B}(\operatorname{supp}(\partial T),1)$ is not totally bounded. There would exist $0 < r < 1$ and a sequence $x_1, x_2, \dots$ in $\operatorname{supp} T$ such that $\mathbf{B}(x_k,r) \cap \operatorname{supp} \partial T = \emptyset$ for each $k = 1,2,\dots$, and $|x_k - x_{k'}| > 2r$ whenever $k \neq k'$. For every $\kappa = 1,2,\dots$ one has $\gamma = \mathscr{M}_{H,|\cdot|}(T) \geqslant \sum_{k=1}^{\kappa} \mathscr{M}_{H,|\cdot|}(T \llcorner \mathbf{B}(x_k,r)) \geqslant \kappa\eta r^m$ according to Section 3.7, thus $r = 0$, a contradiction. □

4. A general criterion for lower semicontinuity

We will need the following technical fact. Given $T, T_1, T_2, \dots$ in $\mathscr{F}_m(X,G)$ we say that $T_1, T_2, \dots$ *converges rapidly* to T if $\sum_j \mathscr{F}(T_j - T) < \infty$. Of course any convergent sequence admits a rapidly convergent subsequence. The following is a consequence for instance of [11, 5.2.1(3)].

Proposition 4.1. — *Assume that $T, T_1, T_2, \dots$ belong to $\mathscr{F}_m(X;G)$ and that $(T_j)_j$ converges rapidly to T. If $u : X \to \mathbf{R}$ is Lipschitzian then $\mathscr{F}(T \llcorner \{u > r\} - T_j \llcorner \{u > r\}) \to 0$ as $j \to \infty$, for almost every $r \in \mathbf{R}$.*

Definition 4.2. — *We say that the triple $\big((X, \|\cdot\|), (G, |\cdot|), m\big)$ satisfies the* triangle inequality for cycles *whenever the following holds. For every integer $\kappa \geqslant 2$, every oriented simplexes $\sigma_1, \dots, \sigma_\kappa$ of dimension m in X and every $g_1, \dots, g_\kappa \in G$, if $\partial \sum_{k=1}^{\kappa} g_k [\![\sigma_k]\!] = 0$ then*

$$\mathscr{M}_H(g_1 [\![\sigma_1]\!]) \leqslant \sum_{k=2}^{\kappa} \mathscr{M}_H(g_k [\![\sigma_k]\!]) .$$

Theorem 4.3. — *The following are equivalent.*

(1) *The triple $\big((X, \|\cdot\|), (G, |\cdot|), m\big)$ satisfies the triangle inequality for cycles.*
(2) *The Hausdorff mass $\mathscr{M}_H : \mathscr{P}_m(X;G) \to \mathbf{R}_+$ is lower semicontinuous with respect to $\mathscr{F}$ convergence.*

Proof. — In order to establish that (1) $\Rightarrow$ (2) we let $P, P_1, P_2, \dots$ belong to $\mathscr{P}_m(X;G)$ and be such that $\mathscr{F}(P - P_j) \to 0$ as $j \to \infty$. Thus $P - P_j = Q_j + \partial R_j$, $Q_j \in \mathscr{P}_m(X;G)$, $R_j \in \mathscr{P}_{m+1}(X;G)$ and $\mathscr{M}_H(Q_j) + \mathscr{M}_H(R_j) \to 0$ as $j \to \infty$. Considering first the case when $P = g[\![\sigma]\!]$, we notice that $\partial(P - P_j - Q_j) = \partial(\partial R_j) = 0$ whence $\mathscr{M}_H(P) \leqslant \mathscr{M}_H(P_j + Q_j)$ follows from hypothesis (1) for all $j = 1, 2, \dots$. Accordingly, $\mathscr{M}_H(P) \leqslant \liminf_j \mathscr{M}_H(P_j + Q_j) = \liminf_j \mathscr{M}_H(P_j)$. Turning to the general case we write $P = \sum_{k=1}^{\kappa} g_k [\![\sigma_k]\!]$ where the σ_k are nonoverlapping. Letting $W_k \in \mathbf{G}_m(X)$ have a translate containing σ_k, we define $U_k = \operatorname{Conv}(\mathring{\sigma}_k \cup (x_k + V_k))$ where $\mathring{\sigma}_k$ is the relative interior of σ_k, x_k is the barycenter of σ_k and V_k is a convex polyhedral neighborhood of zero in some complementary subspace of W_k. Choosing those V_k small enough we can assume that $U_1, \dots, U_\kappa$ are pairwise disjoint. We next define $u_k = \operatorname{dist}(\cdot, U_k^c)$. We observe that (A) the sets $\sigma_k \cap \{u_k > r\}$ are simplicial (*i.e.* of the type $\sigma_{k,r}$) and (B) the sets $\{u_k > r\}$ are (convex open) polyhedra. Given $\varepsilon > 0$ we choose $\rho > 0$ such that $\mathscr{H}^m(\sigma_k \cap \{u_k \leqslant r\}) \leqslant \varepsilon \kappa^{-1} |g_k|^{-1}$ whenever $0 < r \leqslant \rho$ and $k = 1, \dots, \kappa$, whence also $\mathscr{M}_H(P) \leqslant \varepsilon + \sum_{k=1}^{\kappa} \mathscr{M}_H(P \llcorner \{u_k > r\})$. Replacing the original sequence $(P_j)_j$ by a subsequence if necessary we may assume that $\liminf_j \mathscr{M}_H(P_j) = \lim_j \mathscr{M}_H(P_j)$ and that $(P_j)_j$ converges rapidly to P. Thus there exists $0 < r \leqslant \rho$ such that $\mathscr{F}(P \llcorner \{u_k > r\} - P_j \llcorner \{u_k > r\}) \to 0$ as $j \to \infty$ for each

$k = 1, \dots, \kappa$ according to Proposition 4.1. It then follows from observations Item (A) and Item (B) above, and from the first case of this proof, that

$$\mathscr{M}_H(P \llcorner \{u_k > r\}) \leqslant \liminf_j \mathscr{M}_H(P_j \llcorner \{u_k > r\}) .$$

In turn,

$$\begin{aligned}\mathscr{M}_H(P) - \varepsilon \leqslant \sum_{k=1}^{\kappa} \mathscr{M}_H(P \llcorner \{u_k > r\}) \leqslant \sum_{k=1}^{\kappa} \liminf_j \mathscr{M}_H(P_j \llcorner \{u_k > r\}) \\ \leqslant \liminf_j \sum_{k=1}^{\kappa} \mathscr{M}_H(P_j \llcorner \{u_k > r\}) \leqslant \lim_j \mathscr{M}_H(P_j) .\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary the proof is complete.

We prove (2) $\Rightarrow$ (1) by contraposition. Let $\sigma_1, \dots, \sigma_\kappa$ be simplexes in X and $g_1, \dots, g_\kappa$ be elements of G such that

$$\eta := \mathscr{M}_H(g_1[\![\sigma_1]\!]) - \sum_{k=2}^{\kappa} \mathscr{M}_H(g_k[\![\sigma_k]\!]) > 0 \tag{4.1}$$

and $\partial(P - Q) = 0$ where we have abbreviated $P = g_1[\![\sigma_1]\!]$ and $Q = \sum_{k=2}^{\kappa} g_k[\![\sigma_k]\!]$. There is no restriction to assume that σ_1 does not overlap any of the σ_k, $k = 2, \dots, \kappa$. If indeed σ_1 and some σ_k overlap then we replace the summand $g_k[\![\sigma_k]\!]$ in Q by a sum $S_{k,\delta} = \sum_{i=0}^{m} g_k \boldsymbol{\delta}_{p_{k,\delta}} \times [\![\tau_i]\!]$ where τ_i runs over the facets of σ_k (properly oriented so that the cycle condition is preserved) and where $p_{k,\delta}$ belongs to X, but not to the affine plane containing σ_k, and is a distance δ apart from (say) the barycenter of $\mathring{\sigma}_k$. It follows from Proposition 2.1 that $\mathscr{M}_H(S_{k,\delta}) \to \mathscr{M}_H(g_k[\![\sigma_k]\!])$ as $\delta \to 0$. Thus (4.1) is preserved upon choosing $\delta > 0$ small enough, and clearly the new simplexes replacing σ_k do not overlap σ_1.

Let $x_0, x_1, \dots, x_m \in X$ be such that $[\![\sigma_1]\!] = [\![x_0, x_1, \dots, x_m]\!]$. For an integer j we define $I_j = \mathbb{N}^m \cap \{(\alpha_1, \dots, \alpha_m) : \alpha_1 + \ldots + \alpha_m \leqslant j - 1\}$ and corresponding to each $\alpha \in I_j$ we define an affine bijection $f_{j,\alpha} : X \to X$ by the formula

$$f_{j,\alpha}(x) = \left(x_0 + \sum_{i=1}^{m} \alpha_i \left(\frac{x_i - x_0}{j} \right) \right) + \frac{x - x_0}{j} ,$$

and we notice that $\operatorname{Lip} f_{j,\alpha} = j^{-1}$ and that

$$\mathscr{M}_H(f_{j,\alpha\#} g_1[\![\sigma_1]\!]) = j^{-m} \mathscr{M}_H(g_1[\![\sigma_1]\!]).$$

We introduce

$$P_j = P + \sum_{\alpha \in I_j} f_{j,\alpha\,\#}(Q - P) \in \mathscr{P}_m(X;G)\,.$$

Since $\partial(Q-P) = 0$ there exists $R \in \mathscr{P}_{m+1}(X;G)$ such that $Q - P = \partial R$. Therefore

$$\begin{aligned}\mathscr{F}(P_j - P) &= \mathscr{F}\left(\partial \sum_{\alpha \in I_j} f_{j,\alpha\,\#}R\right) \leqslant \mathscr{M}_H\left(\sum_{\alpha \in I_j} f_{j,\alpha\,\#}R\right) \\ &\leqslant (\operatorname{card} I_j) \max_{\alpha \in I_j} (\operatorname{Lip} f_{j,\alpha})^{m+1} \mathscr{M}_H(R) \leqslant j^m \left(\frac{1}{j}\right)^{m+1} \mathscr{M}_H(R)\,,\end{aligned}$$

whence $\mathscr{F}(P_j - P) \to 0$ as $j \to \infty$. In order to estimate the mass of P_j we note that

$$P_j = P - \sum_{\alpha \in I_j} f_{j,\alpha\,\#}P + \sum_{\alpha \in I_j} f_{j,\alpha\,\#}Q$$

and that the $f_{j,\alpha\,\#}P$ are nonoverlapping subchains of P. Therefore

$$\begin{aligned}\mathscr{M}_H(P_j) &\leqslant \mathscr{M}_H(P) - \sum_{\alpha \in I_j} \mathscr{M}_H\left(f_{j,\alpha\,\#}P\right) + \sum_{\alpha \in I_j} \mathscr{M}_H\left(f_{j,\alpha\,\#}Q\right) \\ &\leqslant \mathscr{M}_H(P)\left(1 - \left(\frac{1}{j}\right)^m (\operatorname{card} I_j)\right) + \mathscr{M}_H(Q)\left(\frac{1}{j}\right)^m (\operatorname{card} I_j) \\ &= \mathscr{M}_H(P) - \eta \left(\frac{1}{j}\right)^m (\operatorname{card} I_j)\,.\end{aligned}$$

Since $\lim_j (\operatorname{card} I_j) j^{-m} = 1/2$ we infer that

$$\liminf_j \mathscr{M}_H(P_j) \leqslant \mathscr{M}_H(P) - \frac{\eta}{2} < \mathscr{M}_H(P)\,. \qquad \square$$

We will need two versions of approximation of rectifiable G chains. One is established in [9, Theorem 4.2] and the other one is below.

THEOREM 4.4 (Approximation Theorem). — *Let $T \in \mathscr{R}_m(X,G)$ and $\varepsilon > 0$. There exist $P \in \mathscr{P}_m(X,G)$ and $f : X \to X$ a diffeomorphism of class C^1 with the following properties:*

(1) $\max\{\operatorname{Lip}_{\|\cdot\|}(f), \operatorname{Lip}_{\|\cdot\|}(f^{-1})\} \leqslant 1 + \varepsilon$;
(2) $\|f(x) - x\| \leqslant \varepsilon$ *for every* $x \in X$;
(3) $f(x) = x$ *whenever* $\operatorname{dist}_{\|\cdot\|}(x, \operatorname{supp} T) \geqslant \varepsilon$;
(4) $\mathscr{M}_H(P - f_\# T) \leqslant \varepsilon$.

There are three minor differences with [13, 4.2.20] which require some comments: the coefficients group G is not necessarily $\mathbf{Z}$; we merely hypothesize that T be rectifiable (thus ∂T could have infinite mass and conclusion (4) does not involve the normal mass); and our conclusions involve the $\|\cdot\|$ metric in the ambient space X (of course that would not seriously affect conclusions (2), (3) and (4) but there is something to say about conclusion (1)).

Sketch of proof. — The proof uses [13, 3.1.23] whose only metric aspects regard the Lipschitz constants of f and f^{-1}, and the balls $\mathbf{U}(b,r)$ and $\mathbf{U}(b,tr)$. We fix a Euclidean structure $|\cdot|$ on X. A careful inspection of the proof of [13, 3.1.23] reveals that it holds with Euclidean balls $\mathbf{U}(b,r)$ and $\mathbf{U}(b,tr)$ unchanged and Lipschitz constants $\operatorname{Lip}_{\|\cdot\|}(f)$ and $\operatorname{Lip}_{\|\cdot\|}(f^{-1})$ with respect to the ambient norm $\|\cdot\|$. Indeed bounding from above these Lipschitz constants involves only estimating the operator norm of $Df(x) - \mathrm{id}_X$; this requires an extra factor accounting for the fact $\|\cdot\|$ and the Euclidean norm are equivalent, which can be absorbed in the definition of ε.

The remaining part of the proof mimics that of [13, 4.2.19]. We let $\|T\| = |\boldsymbol{g}|\, \mathscr{H}^m_{\|\cdot\|} \llcorner A$, $A = \operatorname{set}_m \|T\|$. As A is countably $(\mathscr{H}^m_{|\cdot|}, m)$ rectifiable there exist (at most) countably many m dimensional C^1 submanifolds M_j of X, $j \in J$, such that $\|T\|(\mathbf{R}^n \sim \cup_{j\in J} M_j) = 0$. We classically check that for $\|T\|$ almost every $x \in \mathbf{R}^n$ there exists $j(x) \in J$ such that $x \in M_{j(x)}$ and $\Theta^m(\|T\| \llcorner (\mathbf{R}^n \sim M_{j(x)}), x) = 0$. Given $\widehat{\varepsilon} > 0$, for all such x there exists $r(x) > 0$ such that for each $0 < r < r(x)$

(A) the above version of [13, 3.1.23] applies with $t = (1+\widehat{\varepsilon})^{-1}$ at scale r and point x to $M_{j(x)}$;
(B) $\mathscr{M}_H\left(T \llcorner \mathbf{B}(x,tr) - T \llcorner M_{j(x)} \cap \mathbf{B}(x,tr)\right) \leqslant \widehat{\varepsilon} \mathscr{M}_H\left(T \llcorner \mathbf{B}(x,tr)\right)$;
(C) $\mathscr{M}_H\left(T \llcorner \mathbf{B}(x,tr) - T \llcorner \mathbf{B}(x,r)\right) \leqslant 2C(1-t^m)\mathscr{M}_H\left(T \llcorner \mathbf{B}(x,r)\right)$ (where $C > 0$ is such that $\mathscr{H}^m_{\|\cdot\|} \leqslant C \mathscr{H}^m_{|\cdot|}$).

According to the Besicovitch–Vitali covering Theorem there exists a disjointed family of balls $\mathbf{B}(x_k, r_k)$, $k = 1, 2, \ldots$, whose centers are as before and $0 < r_k < r(x_k)$, and $\|T\|(\mathbf{R}^n \sim \cup_{k=1}^{\infty} \mathbf{B}(x_k, r_k)) = 0$. Thus $\|T\|(\mathbf{R}^n \sim \cup_{k=1}^{\kappa} \mathbf{B}(x_k, r_k)) \leqslant \widehat{\varepsilon}$ for some κ. For each $k = 1, \ldots, \kappa$ we associate with $t = 1 - \widehat{\varepsilon}$, $M_{j(x_k)}$, x_k and r_k the C^1 diffeomorphism f_k of X according to (the above version of) Proposition [13, 3.1.23], and we infer from its last conclusion that $f_{k\#}(T \llcorner M_{j(x_k)} \cap \mathbf{B}(x_k, tr_k))$ is an m dimensional G chain supported in an affine m dimensional subspace of X and thus corresponds to a G valued L_1 function, therefore $\mathscr{M}_H(f_{k\#}(T \llcorner M_{j(x_k)} \cap \mathbf{B}(x_k, tr_k)) - P_k) \leqslant \widehat{\varepsilon}\kappa^{-1}$ for some $P_k \in \mathscr{P}_m(X, G)$ as in [12, Lemma 3.2]. Letting $f : X \to X$

coincide with f_k in the ball $\mathbf{B}(x_k, r_k)$, $k = 1, \dots, \kappa$, and with id_X otherwise, one readily checks that conclusions (1), (2) and (3) hold. Finally,

$$\begin{aligned}\mathscr{M}_H(P - f_\# T) &\leqslant \sum_{k=1}^{\kappa} \mathscr{M}_H \left(P_k - f_{k\,\#}(T \llcorner M_{j(x_k)} \cap \mathbf{B}(x_k, r_k))\right) \\ &\qquad + \sum_{k=1}^{\kappa} \mathscr{M}_H \left(T \llcorner \mathbf{B}(x_k, tr_k) - T \llcorner M_{j(x_k)} \cap \mathbf{B}(x_k, tr_k)\right) \\ &\qquad + \mathscr{M}_H \left(f_\# T \llcorner \left(\mathbf{R}^n \sim \cup_{k=1}^{\kappa} \mathbf{B}(x_k, r_k)\right)\right) \\ &\leqslant \widehat{\varepsilon} + \widehat{\varepsilon}(1+\widehat{\varepsilon})^m \mathscr{M}_H(T) + (1+\widehat{\varepsilon})^m \widehat{\varepsilon}. \end{aligned}$$ □

The following is inspired by the proof of [13, 5.1.5].

Theorem 4.5. — *The following are equivalent.*

(1) *The triple* $\left((X, \|\cdot\|), (G, |\cdot|), m\right)$ *satisfies the triangle inequality for cycles.*
(2) *The Hausdorff mass* $\mathscr{M}_H : \mathscr{R}_m(X; G) \to \mathbf{R}_+$ *is lower semicontinuous with respect to* $\mathscr{F}$ *convergence.*

Proof. — That (2) $\Rightarrow$ (1) follows from Theorem 4.3. We now prove the reciprocal proposition holds. Let $T, T_1, T_2, \dots$ be members of $\mathscr{R}_m(X, G)$ such that $\mathscr{F}(T - T_j) \to 0$ as $j \to \infty$. We first establish the conclusion in the particular case when $T \in \mathscr{P}_m(X, G)$ is polyhedral. According to [9, Theorem 4.2] there are $P_j \in \mathscr{P}_m(X, G)$ such that $\mathscr{F}(T_j - P_j) \leqslant j^{-1}$ and $\mathscr{M}_H(P_j) \leqslant j^{-1} + \mathscr{M}_H(T_j)$, $j = 1, 2, \dots$. Therefore $\mathscr{F}(T - P_j) \to 0$ as $j \to \infty$ and it follows from Theorem 4.3 that

$$\mathscr{M}_H(T) \leqslant \liminf_j \mathscr{M}_H(P_j) \leqslant \liminf_j \mathscr{M}_H(T_j)\,.$$

Turning to the general case we associate with T and $\varepsilon > 0$ a polyhedral chain P and a C^1 diffeomorphism f as in Theorem 4.4. Letting $E \in \mathscr{R}_m(X, G)$ be such that $P = f_\# T + E$ we see that $\mathscr{M}_H(E) \leqslant \varepsilon$. Observing that $\mathscr{F}(f_\# T - f_\# T_j) \to 0$ as $j \to \infty$) we infer that also $\mathscr{F}(P - (E + f_\# T_j)) \to 0$ as $j \to \infty$ and thus

$$\mathscr{M}_H(P) \leqslant \liminf_j \mathscr{M}_H(E + f_\# T_j) \leqslant \varepsilon + (1+\varepsilon)^m \liminf_j \mathscr{M}_H(T_j)$$

according to the particular case treated first. Now since $T = (f^{-1})_\#(P - E)$ we conclude that

$$\begin{aligned}\mathscr{M}_H(T) &\leqslant (1+\varepsilon)^m \mathscr{M}_H(P - E) \\ &\leqslant (1+\varepsilon)^m \left(2\varepsilon + (1+\varepsilon)^m \liminf_j \mathscr{M}_H(T_j)\right)\,.\end{aligned}$$

As $\varepsilon > 0$ is arbitrary the proof is complete. □

5. Density contractors

5.1. Spaces of linear homomorphisms

We let $\operatorname{Hom}(X,X)$ denote the linear space of linear homomorphisms $X \to X$. It is equipped with its usual norm $|||\pi||| = \max\{|\pi(x)| : x \in X \text{ and } |x| \leqslant 1\}$ corresponding to the Euclidean norm $|\cdot|$ of X.

Given $m \in \{1, \dots, \dim X - 1\}$ we define the metric subspace

$$\operatorname{Hom}_m(X,X) = \operatorname{Hom}(X,X) \cap \{\pi : \operatorname{rank} \pi \leqslant m\}$$

which is closed, as well as its own subspace

$$\operatorname{Hom}_{m,\mathrm{inv}}(X,X) = \operatorname{Hom}_m(X,X) \cap \{\pi : \operatorname{rank} \pi = m\}$$

which is relatively open. We further define the map

$$\operatorname{Hom}_{m,\mathrm{inv}}(X,X) \to \mathbf{G}_m(X) : \pi \mapsto W_\pi$$

so that $\operatorname{im} \pi = W_\pi$ and we claim it is continuous. Indeed if $\pi, \pi_1, \pi_2, \dots$ belong to $\operatorname{Hom}_{m,\mathrm{inv}}(X,X)$, $\pi_k \to \pi$ and $\operatorname{im} \pi_k = W_k$ then we choose orthonormal bases $e_1^k, \dots, e_n^k$ of X such that $\operatorname{span}\{e_1^k, \dots, e_m^k\} = W_k$. Each subsequence of these bases admits a subsequence (still denoted the same way) such that $e_j^k \to e_j$ as $k \to \infty$, $j = 1, \dots, n$, where $e_1, \dots, e_n$ is some orthonormal basis of X. Given $x \in X$ write $\pi_k(x) = \sum_{j=1}^m t_j^k e_j^k$ with $\sum_{j=1}^m (t_j^k)^2 \leqslant |x|^2 \Gamma^2$ where $\Gamma = \sup_k |||\pi_k||| < \infty$. If $i = m+1, \dots, n$ then $\langle e_i, \pi(x) \rangle = \lim_k \langle e_i, \pi_k(x) \rangle = \lim_k \sum_{j=1}^m t_j^k \langle e_i, e_j^k \rangle = 0$. Therefore $\operatorname{im} \pi \subseteq \operatorname{span}\{e_{m+1}, \dots, e_n\}^\perp = W$ where $W = \operatorname{span}\{e_1, \dots, e_m\}$. Since $\operatorname{rank} \pi = m$ we infer $\operatorname{im} \pi = W$. As $d(W, W_k) \to 0$ according to Section 2.3, the asserted continuity follows.

Finally, corresponding to $W \in \mathbf{G}_m(X)$ we define

$$\operatorname{Hom}(X,W) = \operatorname{Hom}(X,X) \cap \{\pi : \operatorname{im} \pi \subseteq W\}.$$

Proposition 5.1. — *Let $A \subseteq X$ be such that $\mathscr{H}^m_{\|\cdot\|}(A) < \infty$ and define $f_A : \operatorname{Hom}_m(X,X) \to \mathbf{R}$ by the formula $f_A(\pi) = \mathscr{H}^m_{\|\cdot\|}(\pi(A))$. It follows that*

1. *If A is compact then f_A is upper semicontinuous;*
2. *If A is compact and convex then f_A is continuous;*
3. *If A is Borel then f_A is Borel.*

Proof.

(1). — We start with the following remark. Given $\pi \in \mathrm{Hom}_m(X,X)$ let $W \in \mathbf{G}_m(X)$ be such that $\operatorname{im}\pi \subseteq W$. Observe that in the definition of $\mathscr{H}^m_{\|\cdot\|,\delta}(\pi(A))$, $0 < \delta \leqslant \infty$, one can restrict to covers of $\pi(A)$ by subsets of W. By means of a linear homomorphism $W \to \mathbf{R}^m$ one transforms this number to the $\mathscr{H}^m_{\nu,\delta}$ measure of a subset of $\mathbf{R}^m$ with respect to some norm ν in $\mathbf{R}^m$, therefore $\mathscr{H}^m_{\|\cdot\|,\delta}(\pi(A)) = \mathscr{H}^m_{\|\cdot\|}(\pi(A))$ according to Section 2.2. We let $\delta = \infty$. Consider a sequence $\pi_1, \pi_2, \dots$ in $\mathrm{Hom}_m(X,X)$ converging to π. Letting $\varepsilon > 0$ we choose a cover $(E_i)_{i\in I}$ of $\pi(A)$ such that $\sum_{i\in I} \boldsymbol{\alpha}(m)2^{-m}(\operatorname{diam}_{\|\cdot\|} E_i)^m < \varepsilon + \mathscr{H}^m_{\|\cdot\|,\infty}(\pi(A))$. Notice there is no restriction to assume the E_i are open in X. Define $E = \cup_{i\in I} E_i$ and observe the compactness of $\pi(A)$ implies there exists $r > 0$ such that $\mathbf{U}(\pi(A), r) \subseteq E$. Now if k is sufficiently large then $\pi_k(A) \subseteq \mathbf{U}(\pi(A), r) \subseteq \cup_{i\in I} E_i$ – because $\pi_k(A) \to \pi(A)$ in Hausdorff distance according to Section 2.1(1) – therefore $\mathscr{H}^m_{\|\cdot\|,\infty}(\pi_k(A)) \leqslant \sum_{i\in I} \boldsymbol{\alpha}(m)2^{-m}(\operatorname{diam}_{\|\cdot\|} E_i)^m < \varepsilon + \mathscr{H}^m_{\|\cdot\|,\infty}(\pi(A))$. By our initial remark it follows that

$$\limsup_k \mathscr{H}^m_{\|\cdot\|}(\pi_k(A)) \leqslant \varepsilon + \mathscr{H}^m_{\|\cdot\|}(\pi(A)).$$

Since $\varepsilon > 0$ is arbitrary the proof of (1) is complete.

(3). — Choose a nondecreasing sequence of compact subsets of A, say $A_1, A_2, \dots$ such that $\mathscr{H}^m_{\|\cdot\|}(A \sim A_j) \to 0$. Given $\pi \in \mathrm{Hom}_m(X,X)$ we notice that

$$\begin{aligned} 0 &\leqslant \mathscr{H}^m_{\|\cdot\|}(\pi(A)) - \mathscr{H}^m_{\|\cdot\|}(\pi(A_j)) = \mathscr{H}^m_{\|\cdot\|}(\pi(A) \sim \pi(A_j)) \\ &\leqslant \mathscr{H}^m_{\|\cdot\|}(\pi(A \sim A_j)) \\ &\leqslant (\operatorname{Lip}_{\|\cdot\|} \pi)^m \mathscr{H}^m_{\|\cdot\|}(A \sim A_j) \to 0\,. \end{aligned}$$

In other words $f_{A_j} \to f_A$ pointwise. Since each f_{A_j} is upper semicontinuous according to (1) the conclusion follows.

(2). — Since f_A is upper semicontinuous according to (1) it remains to establish it is lower semicontinuous as well. We start with the particular case when the norm $\|\cdot\| = |\cdot|$ is Euclidean. Let $\pi, \pi_1, \pi_2, \dots$ be members of $\mathrm{Hom}_m(X,X)$ such that $\pi_k \to \pi$. Choose $W \in \mathbf{G}_m(X)$ with $\operatorname{im}\pi \subseteq W$ and let π_W denote the orthogonal projection onto W. Since $\pi_k(A) \to \pi(A)$ in Hausdorff distance, Section 2.1(1), $\pi_W(\pi_k(A)) \to \pi_W(\pi(A))$ as well, Section 2.1(2). Furthermore $\pi_W(\pi(A)) = \pi(A)$. Recall that any Haar measure on W, restricted to the collection of compact convex subsets of W, is continuous with respect to Hausdorff distance, [13, 3.2.36]. Therefore

$\lim_k \mathscr{H}^m_{|\cdot|}(\pi_W(\pi_k(A)) = \mathscr{H}^m_{|\cdot|}(\pi(A))$. Since we are in the Euclidean setting, $\mathscr{H}^m_{|\cdot|}(\pi_W(\pi_k(A)) \leqslant |||\pi_W|||^m \mathscr{H}^m_{|\cdot|}(\pi_k(A)) = \mathscr{H}^m_{|\cdot|}(\pi_k(A))$, and finally $\mathscr{H}^m_{|\cdot|}(\pi(A)) \leqslant \liminf_k \mathscr{H}^m_{|\cdot|}(\pi_k(A))$ which completes the proof in case $\|\cdot\| = |\cdot|$.

We now turn to the general case. If $\operatorname{rank}\pi < m$ then clearly $\mathscr{H}^m_{\|\cdot\|}(\pi(A)) = 0 \leqslant \liminf_k \mathscr{H}^m_{\|\cdot\|}(\pi_k(A))$. We henceforth assume that $\operatorname{rank}\pi = m$ and thus also $\operatorname{rank}\pi_k = m$ if k is sufficiently large. Recalling that $\mathscr{H}^m_{\|\cdot\|}(\pi(A)) = \beta(\|\cdot\|, |\cdot|, W_\pi)\mathscr{H}^m_{|\cdot|}(\pi(A)) = \psi(W_\pi)\mathscr{H}^m_{|\cdot|}(\pi(A))$ the lower semicontinuity in the variable π follows from the particular case, from the continuity of $\pi \mapsto W_\pi$, Section 5.1 and from the continuity of ψ, Proposition 2.1. $\square$

Definition 5.2. — *A density contractor on $W \in \mathbf{G}_m(X)$ is a Borel probability measure μ on* $\operatorname{Hom}_m(X,X)$ *such that*

(1) *μ is supported in* $\operatorname{Hom}(X,W)$, i.e.

$$\mu(\operatorname{Hom}_m(X,X) \sim \operatorname{Hom}(X,W)) = 0;$$

(2) *If $V \in \mathbf{G}_m(X)$ and $A \subseteq V$ is Borel then*

$$\int_{\operatorname{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}(\pi(A))\mathrm{d}\mu(\pi) \leqslant \mathscr{H}^m_{\|\cdot\|}(A)\,,$$

with equality when $V = W$.

When there will be several norms under consideration on X, in order to avoid confusion we will insist that μ is a density contractor on W with respect to $\mathscr{H}^m_{\|\cdot\|}$.

In view of the preceding section, density contractors are useful for the following reason.

Theorem 5.3. — *Assume that $(X, \|\cdot\|)$ and $m \in \{1, \dots, \dim X - 1\}$ have the following property: Every $W \in \mathbf{G}_m(X)$ admits a density contractor with respect to $\mathscr{H}^m_{\|\cdot\|}$. It follows that for every complete normed Abelian group $(G, |\!|\cdot|\!|)$ the triple $\big((X, \|\cdot\|), (G, |\!|\cdot|\!|), m\big)$ satisfies the triangle inequality for cycles.*

Proof. — Let $P = \sum_{k=1}^{\kappa} g_k[\![\sigma_k]\!] \in \mathscr{P}_m(X,G)$, with the $\sigma_1, \dots, \sigma_\kappa$ nonoverlapping. Since the statement to check is invariant under translation of P we may assume 0 belongs to the support of $g_1[\![\sigma_1]\!]$. Let $\pi \in \operatorname{Hom}(X,W)$ where $W \in \mathbf{G}_m(X)$ is the m dimensional subspace of X containing σ_1. Now $\pi_\# P \in \mathscr{P}_m(W,G)$, $\partial\pi_\# P = 0$ and $\pi_\# P$ has compact support. Applying the Constancy Theorem [12, Theorem 6.2] with a large m cube $Q \subseteq W$

such that $\operatorname{spt}(\pi_\# P) \subseteq \operatorname{int} Q$ we find that $\pi_\# P = 0$. Therefore

$$g_1[\![\pi(\sigma_1)]\!] = \pi_\#(g_1[\![\sigma_1]\!]) = -\pi_\# \sum_{k=2}^{\kappa} g_k[\![\sigma_k]\!] = -\sum_{k=2}^{\kappa} g_k[\![\pi(\sigma_k)]\!] .$$

The triangular inequality for $\mathscr{M}_H$ thus implies

$$\begin{aligned} \| g_1 \| \mathscr{H}^m_{\|\cdot\|}(\pi(\sigma_1)) &= \mathscr{M}_H(g_1[\![\pi(\sigma_1)]\!]) \\ &\leqslant \sum_{k=2}^{\kappa} \mathscr{M}_H(g_k[\![\pi(\sigma_k)]\!]) = \sum_{k=2}^{\kappa} \| g_k \| \mathscr{H}^m_{\|\cdot\|}(\pi(\sigma_k)) . \end{aligned}$$

Now let μ be a density contractor for W. Integrating the above inequality with respect to μ yields the sought for inequality:

$$\begin{aligned} \mathscr{M}_H(g_1[\![\sigma_1]\!]) = \| g_1 \| \mathscr{H}^m_{\|\cdot\|}(\sigma_1) &= \| g_1 \| \int_{\operatorname{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}(\pi(\sigma_1)) \mathrm{d}\mu(\pi) \\ &\leqslant \sum_{k=2}^{\kappa} \| g_k \| \int_{\operatorname{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}(\pi(\sigma_k)) \mathrm{d}\mu(\pi) \\ &\leqslant \sum_{k=2}^{\kappa} \| g_k \| \mathscr{H}^m_{\|\cdot\|}(\sigma_k) = \sum_{k=2}^{\kappa} \mathscr{M}_h(g_k[\![\sigma_k]\!]) . \end{aligned}$$ □

Remark 5.4. — The following are two trivial cases of existence of density contractors. We recall that if $\pi : X \to W$ is Lipschitzian then $\mathscr{H}^m_{\|\cdot\|}(\pi(A)) \leqslant (\operatorname{Lip}\pi)^m \mathscr{H}^m_{\|\cdot\|}(A)$, for every $A \subseteq X$, where the Lipschitz constant $\operatorname{Lip}\pi$ is with respect to the norm $\|\cdot\|$ of X and W. Thus in case π is a projector onto W (*i.e.* $\pi|_W = \mathrm{id}_W$) and $\operatorname{Lip}\pi = 1$ then $\boldsymbol{\delta}_\pi$ is readily a density contractor on W.

(1) If the norm $\|\cdot\| = |\cdot|$ is Euclidean and m is arbitrary then the orthogonal projector $\pi : X \to W$ verifies the above condition.
(2) If the norm $\|\cdot\|$ is arbitrary and $m = 1$ then there exists a projector $\pi : X \to W$ with $\operatorname{Lip}\pi = 1$. Indeed letting w be a unit vector spanning W, we choose $\alpha \in X^*$ such that $\operatorname{Lip}\alpha = 1$ and $\alpha(w) = 1$, according to Hahn's theorem, and we define $\pi(x) = \alpha(x)w$.

There does not always exist a projector $\pi : X \to W$ with $\operatorname{Lip} = 1$, even when $m + 1 = \dim X = 3$. For instance when $X = \ell^3_\infty$ and $W = X \cap \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$ any projector $\pi : X \to W$ has $\operatorname{Lip}\pi \geqslant 1 + 1/7$. This does not rule out the possibility that there be a projector onto W that decreases the area $\mathscr{H}^2_{\|\cdot\|_\infty}$; such projector actually exists according to the following classical result of H. Busemann.

Theorem 5.5 (Busemann, 1949). — *Let $m = \dim X - 1$. For every $W \in \mathbf{G}_m(X)$ there exists a projector π onto W with the following property.*

For every $V \in \mathbf{G}_m(X)$ and every Borel set $A \subseteq V$ one has $\mathscr{H}^m_{\|\cdot\|}(\pi(A)) \leqslant \mathscr{H}^m_{\|\cdot\|}(A)$.

In fact H. Busemann shows that the function

$$X \to \mathbf{R} : u \mapsto \frac{|u|\boldsymbol{\alpha}(m)}{\mathscr{H}^m_{|\cdot|}\left(B_{\|\cdot\|} \cap \operatorname{span}\{u\}^\perp\right)}$$

is convex (hence a norm on X), see [7] or [22, §7.1]. The existence of the projector then follows for instance as in [1, Theorem 4.13].

Theorem 5.6 (Burago–Ivanov [5]). — *Let $m = 2$ and assume the norm $\|\cdot\|$ is crystalline*[2]*. Given $W \in \mathbf{G}_2(X)$ we let $u_1, \dots, u_{2p}$ denote a collection of distinct unit vectors in W, numbered in consecutive order, such that $-u_i = u_{p+i}$ for each $i = 1, \dots, p$, and containing all the vertices of the polygon $W \cap B_{\|\cdot\|}$. Let $\alpha_i \in X^*$ be a supporting functional of $B_{\|\cdot\|}$ such that $\alpha_i|_{\operatorname{Conv}\{u_i, u_{i+1}\}} = 1$ and $\lambda_i = 2\mathscr{H}^2_{|\cdot|}(W \cap B_{\|\cdot\|})^{-1}\mathscr{H}^2_{|\cdot|}(\operatorname{Conv}\{0, u_i, u_{i+1}\})$, $i = 1, \dots, p$. Define $\omega \in \bigwedge^2 X$ by the formula*

$$\omega = \boldsymbol{\alpha}(2) \sum_{1 \leqslant i < j \leqslant p} \lambda_i \lambda_j . \alpha_i \wedge \alpha_j \,.$$

It follows that for every $V \in \mathbf{G}_2(X)$ one has

$$|\omega(v_1 \wedge v_2)| \leqslant \boldsymbol{\alpha}(2) \sum_{1 \leqslant i < j \leqslant p} \lambda_i \lambda_j \left|\langle \alpha_i \wedge \alpha_j, v_1 \wedge v_2 \rangle\right| \leqslant \psi(V)$$

where v_1, v_2 is an orthonormal basis of V, with both inequalities becoming equalities when $V = W$.

This is the main result of [5] (see the discussion at the beginning of Section 2 and Proposition 2.2 therein). We should point out that our formulation differs from that in [5] in two respects: First, we stress the middle inequality in the conclusion; Second, there may be more unit vectors $u_1, \dots, u_{2p}$ in our statement than there are vertices of $W \cap B_{\|\cdot\|}$ but same argument as in [5] applies in this slightly more general situation (which is needed in the proof of Theorem 5.9). We now proceed to showing how it leads to the existence of density contractors in case $m = 2$. We start with two easy and useful observations.

Proposition 5.7. — *Let μ be a Borel probability measure on $\operatorname{Hom}_m(X, X)$, $W \in \mathbf{G}_m(X)$, and assume μ is supported in $\operatorname{Hom}(X, W)$. The following are equivalent.*

(1) *μ is a density contractor on W;*

[2] *i.e.* its unit ball $B_{\|\cdot\|}$ is a polytope

(2) *For every $V \in \mathbf{G}_m(X)$ there exists some Borel subset $A \subseteq V$ with $0 < \mathscr{H}^m_{\|\cdot\|}(A) < \infty$ and*

$$\int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}(\pi(A))\mathrm{d}\mu(\pi) \leqslant \mathscr{H}^m_{\|\cdot\|}(A)\,,$$

with equality when $V = W$;

(3) *For every $V \in \mathbf{G}_m(X)$ there exists some Borel subset $A \subseteq V$ with $0 < \mathscr{H}^m_{|\cdot|}(A) < \infty$ and*

$$\int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_{|\cdot|}(\pi(A))\mathrm{d}\mu(\pi) \leqslant \frac{\psi(V)}{\psi(W)} \mathscr{H}^m_{|\cdot|}(A)\,,$$

with equality when $V = W$.

Proof. — Recalling that $\mathscr{H}^m_{\|\cdot\|}(E) = \psi(V)\mathscr{H}^m_{|\cdot|}(E)$ whenever $E \subseteq V \in \mathbf{G}_m(X)$ and E is Borel, we infer at once that μ is a density contractor on W if and only if the condition in (3) holds for every Borel $A \subseteq V$. Thus clearly (3) is a consequence of (1), and is equivalent to (2). In order to establish that (2) implies (1) we fix $V \in \mathbf{G}_m(X)$ and we define

$$\phi(A) = \int_{\mathrm{Hom}(X,W)} \mathscr{H}^m_{\|\cdot\|}(\pi(A))\mathrm{d}\mu(\pi)\,,$$

$A \in \mathscr{B}(V)$, where $\mathscr{B}(V)$ is the σ-algebra consisting of Borel subsets of V. We must show that if $\phi(A) \leqslant \mathscr{H}^m_{\|\cdot\|}(A)$ (resp. $\phi(A) = \mathscr{H}^m_{\|\cdot\|}(A)$ when $V = W$) for some $A \in \mathscr{B}(V)$ such that $0 < \mathscr{H}^m_{|\cdot|}(A) < \infty$ then it holds for all $A \in \mathscr{B}(V)$. We define

$$G_{W,V} = \mathrm{Hom}(X, W) \cap \{\pi : \pi \mid V \text{ is injective}\}\,.$$

Clearly $G_{W,V}$ is an open subset of $\mathrm{Hom}(X, W)$ and $\mathscr{H}^m_{\|\cdot\|}(\pi(A)) = 0$ whenever $\pi \notin G_{W,V}$. When $\pi \in G_{W,V}$ we abbreviate $\phi_\pi(A) = \mathscr{H}^m_{\|\cdot\|}(\pi(A))$. Since $\pi|V$ is a homeomorphism from V to W it follows that ϕ_π is a measure on $\mathscr{B}(V)$. Since $\phi(A) = \int_{G_{W,V}} \phi_\pi(A)\mathrm{d}\mu(\pi)$, it ensues from the monotone convergence theorem that ϕ is a measure as well on $\mathscr{B}(V)$. If $h \in V$ and $\pi \in \mathrm{Hom}(X, W)$ then $\phi_\pi(A + h) = \phi_\pi(A)$ because π is linear and $\mathscr{H}^m_{\|\cdot\|}$ is translation invariant. Therefore ϕ is also translation invariant. Now either $\phi = 0$ and there is nothing to prove or ϕ is one of Haar measures on V and the conclusion follows from their uniqueness up to a multiplicative factor, because (the restriction to $\mathscr{B}(V)$ of) $\mathscr{H}^m_{\|\cdot\|}$ is also a Haar measure on V. $\square$

In our next observation we consider density contractors with respect to a sequence of norms on X. Rather than using the ambiguous notation $\|\cdot\|_j$, $j = 1, 2, \ldots$ for a sequence of norms, we prefer using ν_j, $j = 1, 2, \ldots$.

Proposition 5.8. — *Let $\nu, \nu_1, \nu_2, \dots$ be a sequence of norms on X, let $W \in \mathbf{G}_m(X)$ and let $\mu_1, \mu_2, \dots$, be a sequence of probability measures on* $\mathrm{Hom}_m(X, X)$ *all supported in* $\mathrm{Hom}(X, W)$. *We assume that*

(1) *Each μ_k is a density contractor on W with respect to $\mathscr{H}^m_{\nu_k}$, $k = 1, 2, \dots$;*
(2) *$\nu_k \to \nu$ as $k \to \infty$;*
(3) *The sequence $\mu_1, \mu_2, \dots$ is uniformly tight,* i.e.
$$\lim_{n\to\infty} \sup_{k=1,2,\dots} \mu_k \left(\mathrm{Hom}(X, W) \cap \{\pi : |||\pi||| \geqslant n\}\right) = 0\,;$$
(4) *There exists a compact convex set $A \subseteq W$ such that $0 < \mathscr{H}^m_{|\cdot|}(A)$ and*
$$\lim_{n\to\infty} \sup_{k=1,2,\dots} \int_{\mathrm{Hom}(X,W)\cap\{\pi:|||\pi|||\geqslant n\}} \mathscr{H}^m_{\nu_k}(\pi(A)) \mathrm{d}\mu_k(\pi) = 0\,.$$

It follows that there exists a density contractor on W with respect to $\mathscr{H}^m_\nu$.

Proof. — We first notice that $\mu_1, \mu_2, \dots$ admits a subsequence (still denoted the same way) converging tightly to some Borel probability measure μ on $\mathrm{Hom}_m(X, X)$, also supported in $\mathrm{Hom}(X, W)$, according to assumption (3) and Prokhorov's Theorem [19, Chapter II Theorem 6.7]. In other words

$$\int_{\mathrm{Hom}_m(X,X)} f(\pi)\mathrm{d}\mu_k(\pi) \to \int_{\mathrm{Hom}_m(X,X)} f(\pi)\mathrm{d}\mu(\pi) \text{ as } k \to \infty \tag{5.1}$$

whenever $f : \mathrm{Hom}_m(X, X) \to \mathbf{R}$ is continuous and bounded. Assumption (1) says that

$$\int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_{\nu_k}(\pi(A))\mathrm{d}\mu_k(\pi) \leqslant \mathscr{H}^m_{\nu_k}(A) \tag{5.2}$$

for every $k = 1, 2, \dots$, every $V \in \mathbf{G}_m(X)$ and every Borel $A \subseteq V$, with equality when $V = W$. Assumption (2) is that $\delta(\nu, \nu_k) \to 1$ when $k \to \infty$. According to Proposition 5.7 it suffices to establish that (5.2) holds with ν_k replaced by ν, μ_k replaced by μ, for some Borel $A \subseteq V$ such that $0 < \mathscr{H}^m_\nu(A) < \infty$, with equality when $V = W$. We will start by proving the inequality case.

Fix $V \in \mathbf{G}_m(X)$ and choose a compact convex set $A \subseteq V$ such that $0 < \mathscr{H}^m_{|\cdot|}(A)$, for instance $A = V \cap B_{|\cdot|}$. We define on $\mathrm{Hom}_m(X, X)$ the real valued functions $f, f_1, f_2, \dots$ by the formula $f(\pi) = \mathscr{H}^m_\nu(\pi(A))$ and $f_k(\pi) = \mathscr{H}^m_{\nu_k}(\pi(A))$, $k = 1, 2, \dots$. Letting $\mathscr{K}_{\mathrm{Conv}}(W)$ denote the space of nonempty compact convex subsets of W endowed with its Hausdorff metric, we notice that $\mathrm{Hom}(X, W) \to \mathscr{K}_{\mathrm{Conv}}(W) : \pi \mapsto \pi(A)$ is continuous since $\mathrm{dist}_{\mathscr{H}}(\pi(A), \widetilde{\pi}(A)) \leqslant |||\pi - \widetilde{\pi}|||\, \mathrm{diam}(A \cup \{0\})$. If $\mathscr{H}$ is a Haar measure on

W then its restriction $\mathscr{K}_{\mathrm{Conv}}(W) \to \mathbf{R} : C \mapsto \mathscr{H}(C)$ is continuous, see for instance [13, 3.2.36]. It therefore follows that the $f, f_1, f_2, \dots$ are all continuous. As

$$f_k(\pi) = \mathscr{H}^m_{\nu_k}(\pi(A)) \leqslant (\mathrm{Lip}_{\nu_k} \pi)^m \mathscr{H}^m_{\nu_k}(A) \tag{5.3}$$

we note that they do not need to be bounded. Letting $\Gamma = \sup_{k=1,2,\dots} \delta(\nu_k, |\cdot|)$ we note that $\Gamma < \infty$ and that $\mathrm{Lip}_{\nu_k}(\pi) \leqslant \Gamma^2 |||\pi|||$, $k = 1, 2, \dots$, as well as $\mathrm{Lip}_\nu(\pi) \leqslant \Gamma^2 |||\pi|||$ whenever $\pi \in \mathrm{Hom}_m(X, X)$. Given $n = 1, 2, \dots$ we choose a continuous (cut-off function) $\chi_n : \mathrm{Hom}_m(X, X) \to \mathbf{R}$ such that

$$\mathbb{1}_{\mathrm{Hom}_m(X,X) \cap \{\pi : |||\pi||| \leqslant n\}} \leqslant \chi_n \leqslant \mathbb{1}_{\mathrm{Hom}_m(X,X) \cap \{\pi : |||\pi||| \leqslant n+1\}} . \tag{5.4}$$

Since are $\chi_n f_k$ compactly supported, they are bounded. In fact

$$\|\chi_n f_k\|_\infty \leqslant \left(\Gamma^2 (n+1)\right)^m \Gamma^m \mathscr{H}^m_{|\cdot|}(A)$$

according to (5.3), (2.2) and (2), p. 641. We now show that $\chi_n f_1, \chi_n f_2, \dots$ converge uniformly to $\chi_n f$. Observe that

$$\begin{aligned} |f_k(\pi) - f(\pi)| &= (\beta(\nu_k, \nu, W) - 1)\, \mathscr{H}^m_\nu(\pi(A)) \\ &\leqslant (\beta(\nu_k, \nu, W) - 1)\, \Gamma^{2m} |||\pi|||^m \mathscr{H}^m_\nu(A), \end{aligned}$$

for every $\pi \in \mathrm{Hom}_m(X, X)$, whence

$$\|\chi_n f_k - \chi_n f\|_\infty \leqslant (\beta(\nu_k, \nu, W) - 1)\, \Gamma^{2m} (n+1)^m \mathscr{H}^m_\nu(A) .$$

Now $\delta(\nu_k|_W, \nu|_W)^{-m} \leqslant \beta(\nu_k, \nu, W) \leqslant \delta(\nu_k|_W, \nu|_W)^m$ according to (2.2) and $\delta(\nu_k|_W, \nu|_W) \to 1$ as $k \to \infty$ according to (2), p. 641 and assumption (2), thus $\|\chi_n f_k - \chi_n f\|_\infty \to 0$ as $k \to \infty$. This together with (5.1) yields classically that

$$\int_{\mathrm{Hom}_m(X,X)} \chi_n f_k \mathrm{d}\mu_k \to \int_{\mathrm{Hom}_m(X,X)} \chi_n f \mathrm{d}\mu \text{ as } k \to \infty , \tag{5.5}$$

simply because

$$\begin{aligned} &\left| \int_{\mathrm{Hom}_m(X,X)} \chi_n f_k \mathrm{d}\mu_k - \int_{\mathrm{Hom}_m(X,X)} \chi_n f \mathrm{d}\mu \right| \\ &\qquad \leqslant \|\chi_n f_k - \chi_n f\|_\infty + \left| \int_{\mathrm{Hom}_m(X,X)} \chi_n f \mathrm{d}\mu_k - \int_{\mathrm{Hom}_m(X,X)} \chi_n f \mathrm{d}\mu \right| . \end{aligned}$$

Now (5.5) and (5.2) imply that

$$\begin{aligned}\int_{\mathrm{Hom}_m(X,X)} & \chi_n(\pi)\mathscr{H}^m_\nu(\pi(A))\mathrm{d}\mu(\pi) \\ &= \lim_k \int_{\mathrm{Hom}_m(X,X)} \chi_n(\pi)\mathscr{H}^m_{\nu_k}(\pi(A))\mathrm{d}\mu_k(\pi) \\ &\leqslant \liminf_k \int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_{\nu_k}(\pi(A))\mathrm{d}\mu_k(\pi) \\ &\leqslant \liminf_k \mathscr{H}^m_{\nu_k}(A) \\ &= \left(\liminf_k \beta(\nu_k,\nu,V)\right)\mathscr{H}^m_\nu(A) \\ &= \mathscr{H}^m_\nu(A)\,.\end{aligned}$$

Letting $n \to \infty$ and referring to the monotone convergence theorem we obtain

$$\int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_\nu(\pi(A))\mathrm{d}\mu(\pi) \leqslant \mathscr{H}^m_\nu(A)\,. \tag{5.6}$$

The inequality case in Proposition 5.7(2) is now established and it remains only to show that the above becomes an equality when $V = W$. This is where assumption (4) turns up. We keep the same notations as above but we reason in the particular case when $V = W$ and A is the set given in assumption (4). Our extra information is that for each $n = 1, 2, \dots$ there exists $\varepsilon_n > 0$ with

$$\sup_{k=1,2,\dots} \int_{\mathrm{Hom}_m(X,X)} (1-\chi_n)\, f_k \mathrm{d}\mu_k \leqslant \varepsilon_n$$

and $\varepsilon_n \to 0$ as $n \to \infty$. Thus for every $n, k = 1, 2, \dots,$

$$\begin{aligned}&\left|\int_{\mathrm{Hom}_m(X,X)} f_k \mathrm{d}\mu_k - \int_{\mathrm{Hom}_m(X,X)} \chi_n f \mathrm{d}\mu\right| \\ &\qquad \leqslant \int_{\mathrm{Hom}_m(X,X)} (1-\chi_n)\, f_k \mathrm{d}\mu_k \\ &\qquad\qquad + \left|\int_{\mathrm{Hom}_m(X,X)} \chi_n f_k \mathrm{d}\mu_k - \int_{\mathrm{Hom}_m(X,X)} \chi_n f \mathrm{d}\mu\right| \\ &\qquad \leqslant \varepsilon_n + \left|\int_{\mathrm{Hom}_m(X,X)} \chi_n f_k \mathrm{d}\mu_k - \int_{\mathrm{Hom}_m(X,X)} \chi_n f \mathrm{d}\mu\right|\,.\end{aligned} \tag{5.7}$$

Recalling our assumption that μ_k is a density contractor on W we infer that

$$\lim_k \int_{\mathrm{Hom}_m(X,X)} f_k \mathrm{d}\mu_k = \lim_k \int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_{\nu_k}(\pi(A))\mathrm{d}\mu_k(\pi) \\ = \lim_k \mathscr{H}^m_{\nu_k}(A) = \mathscr{H}^m_{\nu}(A)\,.$$

Using this together with (5.5) and letting $k \to \infty$ in (5.7) we now find that

$$\left| \mathscr{H}^m_\nu(A) - \int_{\mathrm{Hom}_m(X,X)} \chi_n(\pi) \mathscr{H}^m_\nu(\pi(A))\mathrm{d}\mu(\pi) \right| \leqslant \varepsilon_n\,.$$

Letting $n \to \infty$ our conclusion becomes a consequence of the monotone convergence theorem. □

THEOREM 5.9. — *Let $(X, \|\cdot\|)$ be a finite dimensional normed space. Every $W \in \mathbf{G}_2(X)$ admits a density contractor with respect to $\mathscr{H}^2_{\|\cdot\|}$.*

We start with the case when $\|\cdot\|$ is crystalline. For use in the proof we introduce the notation $\mathrm{square}(v_1, v_2) = X \cap \{t_1 v_1 + t_2 v_2 : 0 \leqslant t_i \leqslant 1, i = 1, 2\}$ to refer to the square with two edges v_1, v_2, when these constitute an orthonormal family in X.

Proof of Theorem 5.9 in case $\|\cdot\|$ is crystalline. — In this case we shall prove that the density contractor μ can be chosen to verify the following two additional requirements. Here $\Gamma = \delta(\|\cdot\|, |\cdot|)$ and

$$\tau = \frac{\max_{i=1,\dots,p} |u_i \wedge u_{i+1}|_2}{\min_{i=1,\dots,p} |u_i \wedge u_{i+1}|_2}\,.$$

(1) *For every $n = 1, 2, \dots$ one has*

$$\mu\left(\mathrm{Hom}(X, W) \cap \{\pi : |||\pi||| \geqslant n\}\right) \leqslant \frac{4(4+\tau)\Gamma^8}{n^2}\,;$$

(2) *For every $n = 1, 2, \dots$ and every orthonormal basis v_1, v_2 of W one has*

$$\int_{\mathrm{Hom}(X,W)\cap\{\pi:|||\pi|||\geqslant n\}} \mathscr{H}^2_{\|\cdot\|}(\pi(\mathrm{square}(v_1, v_2)))\mathrm{d}\mu(\pi) \leqslant \frac{4(4+\tau)\boldsymbol{\alpha}(2)\Gamma^{10}}{n^2}\,.$$

We let $u_1, \dots, u_{2p}$, $\lambda_1, \dots, \lambda_p$ and $\alpha_1, \dots, \alpha_p$ be defined as in Theorem 5.6. For use in the definition of μ we define certain $\widetilde{\pi}_{i,j} \in \mathrm{Hom}(X, W)$, $i, j = 1, \dots, p$ with $i \neq j$, as follows:

$$\widetilde{\pi}_{i,j}(x) = \alpha_i(x) u_i + \alpha_j(x) u_j\,,$$

$x \in X$. We notice that

$$|||\widetilde{\pi}_{i,j}||| \leqslant 2\Gamma^2 . \tag{5.8}$$

Furthermore if v_1, v_2 is an orthonormal family in X then $\widetilde{\pi}_{i,j}(\mathrm{square}(v_1, v_2))$ is a parallelogram in W with sides $\widetilde{\pi}_{i,j}(v_1)$ and $\widetilde{\pi}_{i,j}(v_2)$, thus

$$\begin{aligned} \mathscr{H}^2_{|\cdot|}\left(\widetilde{\pi}_{i,j}(\mathrm{square}(v_1, v_2))\right) &= |\widetilde{\pi}_{i,j}(v_1) \wedge \widetilde{\pi}_{i,j}(v_2)|_2 \\ &= \left|\det\begin{pmatrix} \alpha_i(v_1) & \alpha_j(v_1) \\ \alpha_i(v_2) & \alpha_j(v_2) \end{pmatrix}\right| . |u_i \wedge u_j|_2 . \end{aligned} \tag{5.9}$$

We now normalize the $\widetilde{\pi}_{i,j}$. We define

$$\rho = \sqrt{\frac{\boldsymbol{\alpha}(2)}{2\psi(W)}} ,$$

and

$$\pi_{i,j} = \left(\frac{\rho}{\sqrt{|u_i \wedge u_j|_2}}\right) \widetilde{\pi}_{i,j}$$

in case $i \neq j$, and $\pi_{i,i} = 0$, $i = 1, \dots, p$. It clearly follows from (5.8) and (5.9) that

$$|||\pi_{i,j}||| \leqslant \frac{2\Gamma^2 \rho}{\sqrt{|u_i \wedge u_j|_2}} , \tag{5.10}$$

and

$$\mathscr{H}^2_{|\cdot|}\left(\pi_{i,j}(\mathrm{square}(v_1, v_2))\right) = \rho^2 \left|\det\begin{pmatrix} \alpha_i(v_1) & \alpha_j(v_1) \\ \alpha_i(v_2) & \alpha_j(v_2) \end{pmatrix}\right| . \tag{5.11}$$

We are now ready to define the Borel measure μ on $\mathrm{Hom}(X, W)$:

$$\mu = \sum_{i,j=1}^{p} \lambda_i \lambda_j \boldsymbol{\delta}_{\pi_{i,j}} . \tag{5.12}$$

From the definition of the λ_i we infer that $\sum_{i=1}^p \lambda_i = 1$ and therefore μ is readily a probability measure.

In order to show that μ is a density contractor on W with respect to $\mathscr{H}^2_{\|\cdot\|}$ we will apply Proposition 5.7(3). Given $V \in \mathbf{G}_2(X)$ we choose v_1, v_2

an orthonormal basis of V and we let $A = \operatorname{square}(v_1, v_2)$. We observe that

$$\begin{aligned}
\int_{\operatorname{Hom}(X,W)} \mathscr{H}^2_{|\cdot|}(\pi(A))\mathrm{d}\mu(\pi) &= \sum_{i,j=1}^{p} \lambda_i\lambda_j \mathscr{H}^2_{|\cdot|}(\pi_{i,j}(A)) \\
&= 2\sum_{1\leqslant i<j\leqslant p} \lambda_i\lambda_j\rho^2 \left|\det\begin{pmatrix}\alpha_i(v_1) & \alpha_j(v_1)\\ \alpha_i(v_2) & \alpha_j(v_2)\end{pmatrix}\right| \\
&= 2\rho^2 \sum_{1\leqslant i<j\leqslant p} \lambda_i\lambda_j \left|\langle \alpha_i\wedge\alpha_j, v_1\wedge v_2\rangle\right| \\
&\leqslant 2\rho^2\boldsymbol{\alpha}(2)^{-1}\psi(V) \\
&= \frac{\psi(V)}{\psi(W)} \\
&= \frac{\psi(V)}{\psi(W)}\mathscr{H}^2_{|\cdot|}(A)
\end{aligned}$$

where the inequality follows from Theorem 5.6 and specializes to an equality in case $V = W$. This completes the proof that μ is a density contractor on W.

We now turn to establishing the extra properties (1) and (2) stated at the beginning of the proof. Given $n = 1, 2, \dots$ we let

$$\Pi_n = \operatorname{Hom}(X, W) \cap \{\pi : |||\pi||| \geqslant n\}\,.$$

If $i, j = 1, \dots, p$ and $\pi_{i,j} \in \Pi_n$ then

$$|u_i \wedge u_j|_2 \leqslant \frac{4\Gamma^4\rho^2}{n^2}$$

according to (5.10).

With $i = 1, \dots, p$ we associate $J^+_{n,i} = \{1, \dots, p\} \cap \{j : i < j \text{ and } \pi_{i,j} \in \Pi_n\}$ as well as $J^{+,a}_{n,i} = J^+_{n,i} \cap \{j : \langle u_i, u_j\rangle \geqslant 0\}$ and $J^{+,b}_{n,i} = J^+_{n,i} \cap \{j : \langle u_i, u_j\rangle < 0\}$. Assuming that $J^{+,a}_{n,i} \neq \emptyset$ we define $j^*_i = \max J^{+,a}_{n,i}$. If $j \in J^{+,a}_{n,i}$ and $i < j < j^*_i$ then

$$\operatorname{Conv}\{0, u_j, u_{j+1}\} \subseteq X \cap \{su_1 + tu_{j^*_i} : 0 \leqslant s \leqslant \Gamma^2 \text{ and } 0 \leqslant t \leqslant \Gamma^2\} =: S_i,$$

since the inner product of either u_j or u_{j+1} with either u_1 or $u_{j^*_i}$ is nonnegative and $\max\{|u_1|, |u_j|, |u_{j+1}|, |u_{j^*_i}|\} \leqslant \Gamma$. Furthermore, by definition of τ and the reasoning above we see that

$$\mathscr{H}^2_{|\cdot|}\left(\operatorname{Conv}\{0, u_{j^*_i}, u_{j^*_i+1}\}\right) \leqslant \tau\mathscr{H}^2_{|\cdot|}\left(\operatorname{Conv}\{0, u_{j^*_i-1}, u_{j^*_i}\}\right) \leqslant \mathscr{H}^2_{|\cdot|}(S_i)\,.$$

Therefore,

$$\begin{aligned}\sum_{j\in J^{+,a}_{n,i}}\lambda_j &= \frac{2}{\mathscr{H}^2_{|\cdot|}\left(B_{\|\cdot\|}\cap W\right)}\sum_{j\in J^{+,a}_{n,i}}\mathscr{H}^2_{|\cdot|}(\operatorname{Conv}(\{0,u_j,u_{j+1}\})\\ &=\frac{1}{\rho^2}\left(\mathscr{H}^2_{|\cdot|}\left(\bigcup_{j\in J^{+,a}_{n,i}\sim\{j^*_i\}}\operatorname{Conv}\{0,u_j,u_{j+1}\}\right)+\mathscr{H}^2_{|\cdot|}\left(\operatorname{Conv}\{0,u_{j^*_i},u_{j^*_i+1}\}\right)\right)\\ &\leqslant\frac{1+\tau}{\rho^2}\mathscr{H}^2_{|\cdot|}(S_i)\leqslant\frac{(1+\tau)\Gamma^4}{\rho^2}\left|u_i\wedge u_{j^*_i}\right|_2\leqslant\frac{4(1+\tau)\Gamma^8}{n^2}\,.\end{aligned}$$

In order to deal with indexes from $J^{+,b}_{n,i}$ we let $u_i^\perp\in W$ be such that $|u_i^\perp|=1$, $\langle u_i,u_i^\perp\rangle=0$, and $\langle u_i^\perp,u_j\rangle>0$, whenever $j\in J^{+,b}_{n,i}$. Given such a j, we let α_j be the acute angle between u_j and $-u_i$. Thus, $|u_i\wedge u_j|_2=|u_i||u_j|\sin(\alpha_j)\geqslant\Gamma^{-2}\sin(\alpha_j)$ and, in turn, $\alpha_j\leqslant 2\sin(\alpha_j)\leqslant 8\Gamma^6\rho^2n^{-2}$. Assuming that $J^{+,b}_{n,i}\neq\emptyset$ we define $j^i_*=\min J^{+,b}_{n,i}$ and we let C_i be the circular sector in W of radius Γ and comprised between two half-lines of direction u_p and $u_{j^i_*}$, respectively. Thus, $\operatorname{Conv}\{0,u_j,u_{j+1}\}\subseteq C_i$, for each $j\in J^{+,b}_{n,i}$. It follows that

$$\begin{aligned}\sum_{j\in J^{+,b}_{n,i}}\lambda_j &= \frac{2}{\mathscr{H}^2_{|\cdot|}\left(B_{\|\cdot\|}\cap W\right)}\sum_{j\in J^{+,b}_{n,i}}\mathscr{H}^2_{|\cdot|}(\operatorname{Conv}(\{0,u_j,u_{j+1}\})\\ &\leqslant\frac{1}{\rho^2}\mathscr{H}^2_{|\cdot|}(C_i)=\frac{\alpha_{j^i_*}\Gamma^2}{2\rho^2}\leqslant\frac{4\Gamma^8}{n^2}\,.\end{aligned}$$

We conclude that

$$\sum_{j\in J^{+}_{n,i}}\lambda_j\leqslant\frac{4(2+\tau)\Gamma^8}{n^2}\,.$$

Similarly we let $J^-_{n,i}=\{1,\dots,p\}\cap\{j:j<i\text{ and }\pi_{i,j}\in\Pi_n\}$. Reasoning analogously we obtain the slightly better

$$\sum_{j\in J^{-}_{n,i}}\lambda_j\leqslant\frac{8\Gamma^8}{n^2}\,.$$

Consequently,

$$\mu\left(\Pi_n\right)=\sum_{\substack{i,j=1\\ \pi_{i,j}\in\Pi_n}}^{p}\lambda_i\lambda_j=\sum_{i=1}^{p}\lambda_i\sum_{j\in J^-_{n,i}\cup J^+_{n,i}}\lambda_j\leqslant\frac{4(4+\tau)\Gamma^8}{n^2}\tag{5.13}$$

and the proof of (1) is complete.

Finally we observe that for all $i=1,\dots,p$ and $l=1,2$ one has $|\alpha_i(v_l)|\leqslant \|\alpha_i\|\|v_l\|\leqslant\Gamma$ and thus $\mathscr{H}^2_{|\cdot|}(\pi_{i,j}(\mathrm{square}(v_1,v_2)))\leqslant 2\Gamma^2\rho^2$, according to (5.11). It therefore follows from (5.13) that

$$\begin{aligned}\int_{\Pi_n}&\mathscr{H}^2_{\|\cdot\|}(\pi(\mathrm{square}(v_1,v_2)))\mathrm{d}\mu(\pi)\\ &=\psi(W)\sum_{\substack{i,j=1\\ \pi_{i,j}\in\Pi_n}}^{p}\lambda_i\lambda_j\mathscr{H}^2_{|\cdot|}(\pi_{i,j}(\mathrm{square}(v_1,v_2)))\\ &\leqslant\psi(W)2\Gamma^2\rho^2\mu(\Pi_n)=\frac{4(4+\tau)\boldsymbol{\alpha}(2)\Gamma^{10}}{n^2}\end{aligned}$$

and the proof of (2) is complete. □

Proof of Theorem 5.9 in the general case. — Fix $W\in\mathbf{G}_2(X)$. We start by choosing a sequence of crystalline norms $\nu_1,\nu_2,\dots$ such that $\nu_k\to\|\cdot\|$ as $k\to\infty$. This is done classically by choosing a finite k^{-1}-net $F_k\subseteq X\cap\{x:\|x\|=1\}$ and letting $B_{\nu_k}=\mathrm{Conv}(F_k\cup(-F_k))$. Now with each ν_k we will associate a density contractor μ_k on W with respect to $\mathscr{H}^2_{\nu_k}$ as in the first part of the proof, choosing the unit vectors $u_1^k,\dots,u_{2p_k}^k$ (recall the statement of Theorem 5.6) in order that all the $|u_i^k\wedge u_{i+1}^k|_2$ are nearly the same value. This is where we may have to add unit vectors to the list of vertices $v_1^k,\dots,v_{2q_k}^k$ of the polygon $W\cap B_{\nu_k}$. It can easily be done for the following reason: The formula $d_k(u,v)=|u\wedge v|_2$ defines a distance of unit vectors u, v lying in « between » v_1^k and $v_{q_k}^k$ such that each « segment » $[u,v]$ on the corresponding « half unit circle » can be partitioned into two segments $[u,w]$ and $[w,v]$ of same « length »; iterating this process we can readily achieve $\tau_k\leqslant 2$.

Now since $\nu_k\to\|\cdot\|$ as $k\to\infty$ we infer that $\sup_{k=1,2,\dots}\Gamma_k<\infty$. It therefore follows from the estimates (1) and (2) proved about μ_k in the first part of this proof that the sequence $\mu_1,\mu_2,\dots$ verifies the hypotheses of Proposition 5.8. The conclusion follows. □

THEOREM 5.10. — *Let $(X,\|\cdot\|)$ be a finite dimensional normed space and $m\in\{1,\dots,\dim X-1\}$. It follows that every $W\in\mathbf{G}(X)$ admits a density contractor with respect to $\mathscr{H}^m_{\|\cdot\|}$ when either $m=1$ or $m=2$ or $m=\dim X-1$.*

Proof. — The case $m=1$ follows from Hahn's theorem as in Remark 5.4(2) and the case $m=\dim X-1$ follows from Busemann's theorem 5.5. In both cases a density contractor on W is given by $\mu=\boldsymbol{\delta}_\pi$ where $\pi:X\to W$ is a projector that contracts $\mathscr{H}^m_{\|\cdot\|}$. The case $m=2$ is

Theorem 5.9 above, a consequence of Burago–Ivanov's Theorem 5.6 and the density contractor exhibited is not of the simple type $\boldsymbol{\delta}_\pi$. $\square$

Theorem 5.11. — *Let $(X, \|\cdot\|)$ be a finite dimensional normed space, let $(G, |\cdot|)$ be a complete normed Abelian group and let $m \in \{1, \dots, \dim X - 1\}$. Assume that $W \in \mathbf{G}_m(X)$, μ is a density contractor on W and $T \in \mathscr{R}_m(X, G)$. It follows that*

$$\int_{\mathrm{Hom}(X,W)\times W} \mathscr{M}(\langle T, \pi, y\rangle)\mathrm{d}\left(\mu \otimes \mathscr{H}^m_{\|\cdot\|}\right)(\pi, y) \leqslant \mathscr{M}_H(T)$$

and the above integrand is $\mu \otimes \mathscr{H}^m_{\|\cdot\|}$ measurable. Furthermore

$$\int_{\mathrm{Hom}(X,W)} \mathscr{M}_H(\pi_\# T)\mathrm{d}\mu(\pi) \leqslant \mathscr{M}_H(T)$$

with equality when $\operatorname{spt} T \subseteq W$, *and the above integrand is Borel measurable.*

In the above statement we wrote $\mathscr{M}(\langle T, \pi, y\rangle)$ without a subscript H. Indeed no reference to the norm $\|\cdot\|$ is needed at all, as both $\mathscr{R}_0(X, G)$ and the mass $\mathscr{M}$ defined on it are independent of $\|\cdot\|$. Members of $\mathscr{R}_0(X, G)$ are G valued atomic Borel measures on X, *i.e.* of the form $\sum_{j\in J} g_j \boldsymbol{\delta}_{x_j}$ with the x_j all distinct, and

$$\mathscr{M}\left(\sum_{j\in J} g_j \boldsymbol{\delta}_{x_j}\right) = \sum_{j\in J} |g_j| .$$

Proof. — We give both X and W a fixed orientation. We start with the first inequality in the case when $T = P \in \mathscr{P}_m(X, G)$ is polyhedral. We choose a decomposition $P = \sum_{k=1}^{\kappa} g_k [\![\sigma_k]\!]$ where the $\sigma_1, \dots, \sigma_\kappa$ are nonoverlapping. Let $\pi \in \mathrm{Hom}(X, W)$ be of maximal rank. For almost every $y \in W$ the following holds: For every $k = 1, \dots, \kappa$ either $\sigma_k \cap \pi^{-1}\{y\}$ is a singleton or it is empty. Furthermore, letting $F_y = (\cup_{k=1}^{\kappa} \sigma_k) \cap \pi^{-1}\{y\}$ one has

$$\langle P, \pi, y\rangle = \sum_{x\in F_y} (-1)^{\varepsilon(\pi,x)} \boldsymbol{\delta}_x$$

where $\varepsilon(\pi, x) = \pm 1$ according to whether $\pi|_{W_k} : W_k \to W$ preserves or not the orientation, where W_k is the affine m plane containing σ_k and is given the orientation of $[\![\sigma_k]\!]$. In particular, for those y,

$$\mathscr{M}(\langle P, \pi, y\rangle) = \sum_{k=1}^{\kappa} |g_k| \mathbb{1}_{\pi(\sigma_k)}(y) .$$

The same formula holds if π is not of maximal rank; in fact in that case both members of the identity above vanish for almost every y. That this function of (π, y) be Borel measurable can be established along the lines of [3, §5.4]. It now follows from Fubini's theorem and the property of density contractor that

$$\begin{aligned}\int_{\mathrm{Hom}(X,W)\times W} &\mathscr{M}(\langle P,\pi,y\rangle)\mathrm{d}\left(\mu\otimes\mathscr{H}^m_{\|\cdot\|}\right)(\pi,y)\\ &= \sum_{k=1}^{\kappa} \mid g_k \mid \int_{\mathrm{Hom}(X,W)} \mathscr{H}^m_{\|\cdot\|}(\pi(\sigma_k))\mathrm{d}\mu(W)\\ &\leqslant \sum_{k=1}^{\kappa} \mid g_k \mid \mathscr{H}^m_{\|\cdot\|}(\sigma_k) = \mathscr{M}_H(P)\,.\end{aligned}$$

We next assume that $T \in \mathscr{R}_m(X,G)$ and we choose a sequence $P_1, P_2, \dots$ in $\mathscr{P}_m(X,G)$ such that $\lim_k \mathscr{F}(P_k - T) = 0$ and $\limsup_k \mathscr{M}_H(P_k) \leqslant \mathscr{M}_H(T)$, see [9, Theorem 4.2]. Taking a subsequence if necessary we may assume that $P_1, P_2, \dots$ converges rapidly to T. Accordingly it follows from [11, 5.2.1(4)] that for every $\pi \in \mathrm{Hom}(X,W)$ one has, for $\mathscr{H}^m_{\|\cdot\|}$ almost every $y \in W$, $\mathscr{F}(\langle T,\pi,y\rangle - \langle P_k,\pi,y\rangle) \to 0$ as $k \to \infty$. Since $\mathscr{M} : \mathscr{R}_0(X,G) \to \mathbf{R}$ is $\mathscr{F}$ lower semicontinuous, [11, 4.4.1], we infer at once that $(\pi,y) \mapsto \mathscr{M}(\langle T,\pi,y\rangle)$ is $\mu\otimes\mathscr{H}^m_{\|\cdot\|}$ measurable, and that

$$\mathscr{M}(\langle T,\pi,y\rangle) \leqslant \liminf_k \mathscr{M}\left(\langle P_k,\pi,y\rangle\right)\,.$$

It then follows from Fatou's lemma that

$$\begin{aligned}\int_{\mathrm{Hom}(X,W)\times W} &\mathscr{M}(\langle T,\pi,y\rangle)\,\mathrm{d}\Big(\mu\otimes\mathscr{H}^m_{\|\cdot\|}\Big)(\pi,y)\\ &\leqslant \liminf_k \int_{\mathrm{Hom}(X,W)\times W} \mathscr{M}(\langle P_k,\pi,y\rangle)\,\mathrm{d}\Big(\mu\otimes\mathscr{H}^m_{\|\cdot\|}\Big)(\pi,y)\\ &\leqslant \liminf_k \mathscr{M}_H(P_k) \leqslant \mathscr{M}_H(T)\,.\end{aligned}$$

We now turn to proving the second conclusion. As before we start with the case when $T = P \in \mathscr{P}_m(X,G)$ is polyhedral. The function $\mathrm{Hom}(X,W) \to \mathbf{R} : \pi \mapsto \mathscr{M}_H(\pi_\# P)$ is continuous. It is indeed the composition of $\mathscr{R}_m(W,G) \to \mathbf{R} : S \mapsto \mathscr{M}_H(S)$ and $\mathrm{Hom}(X,W) \to \mathscr{R}_m(X,G) : \pi \mapsto \pi_\# P$, the former being continuous since in fact $\mathscr{M}_H(S) = \psi(W)\mathscr{M}_{H,|\cdot|}(S) = \psi(W)\mathscr{F}(S)$ as $m = \dim W$. The latter is continuous as well according to the homotopy formula $\mathscr{F}(\pi_\# P - \widetilde{\pi}_\# P) \leqslant \max\{1+|||\pi|||, 1+|||\widetilde{\pi}|||\}^{m+1}\,\mathrm{diam}(\{0\}\cup \operatorname{spt} P)\mathscr{N}(P)$, $\pi,\widetilde{\pi} \in \mathrm{Hom}(X,W)$, see for instance [9, §2.6]. We next choose a decomposition $P = \sum_{k=1}^{\kappa} [\![\sigma_k]\!]$ where the $\sigma_1, \dots, \sigma_k$ are nonoverlapping, thus $\mathscr{M}_H(P) = \sum_{k=1}^{\kappa} \mid g_k \mid \mathscr{H}^m_{\|\cdot\|}(\sigma_k)$. Given $\pi \in \operatorname{Hom}(X,W)$ we notice that $\pi_\# P = \sum_{k=1}^{\kappa} g_k [\![\pi(\sigma_k)]\!]$, thus $\mathscr{M}_H(\pi_\# P) \leqslant \sum_{k=1}^{\kappa} \mid g_k \mid \mathscr{H}^m_{\|\cdot\|}(\pi(\sigma_k))$. Integrating over π with respect to μ we obtain

$$\begin{aligned}\int_{\operatorname{Hom}(X,W)} \mathscr{M}_H(\pi_\# P)\mathrm{d}\mu(\pi) &\leqslant \sum_{k=1}^{\kappa} \mid g_k \mid \int_{\operatorname{Hom}(X,W)} \mathscr{H}^m_{\|\cdot\|}(\pi(\sigma_k))\mathrm{d}\mu(\pi)\\ &\leqslant \sum_{k=1}^{\kappa} \mid g_k \mid \mathscr{H}^m_{\|\cdot\|}(\sigma_k) = \mathscr{M}_H(P)\,.\end{aligned}$$

In case $\operatorname{spt} P \subseteq W$ we reason as follows. Given $\pi \in \operatorname{Hom}(X,W)$ we note that $\mathscr{M}_H(\pi_\# P) = \sum_{k=1}^{\kappa} \mid g_k \mid \mathscr{H}^m_{\|\cdot\|}(\pi(\sigma_k))$ because either π has rank less than or equal to $m-1$ and both sides are clearly zero, or π has rank m and then the simplexes $\pi(\sigma_1),\dots,\pi(\sigma_\kappa)$ are nonoverlapping. Thus the first inequality above becomes an inequality. The second one as well because each $\sigma_k \subseteq W$. We now see that $T \in \mathscr{R}_m(X,G)$ and as we did before, we choose a sequence $P_1, P_2, \dots$ in $\mathscr{P}_m(X,G)$ such that $\lim_k \mathscr{F}(P_k - T) = 0$ and $\limsup_k \mathscr{M}_H(P_k) \leqslant \mathscr{M}_H(T)$, see [9, Theorem 4.2]. Since $\mathscr{F}(\pi_\# P_k - \pi_\# T) \leqslant \max\left\{|||\pi|||^m, |||\pi|||^{m+1}\right\}\mathscr{F}(P_k - T) \to 0$ as $k \to \infty$ we infer that $\mathscr{M}_H(\pi_\# P_k - \pi_\# T) \to 0$ as $k \to \infty$ because $\mathscr{M}_H = \psi(W)\mathscr{F}$. In particular $\pi \mapsto \mathscr{M}_H(\pi_\# T)$ is Borel measurable. It then follows from the dominated convergence theorem that

$$\begin{aligned}\int_{\operatorname{Hom}(X,W)} \mathscr{M}_H(\pi_\# T)\mathrm{d}\mu(\pi) &= \lim_k \int_{\operatorname{Hom}(X,W)} \mathscr{M}_H(\pi_\# P_k)\mathrm{d}\mu(\pi)\\ &\leqslant \limsup_k \mathscr{M}_H(P_k) \leqslant \mathscr{M}_H(T)\,.\end{aligned}$$

If $\operatorname{spt} T \subseteq W$ then applying [9, Theorem 4.2] in W instead of X, or applying [12, Lemma 3.2] we can guarantee that each $\operatorname{spt} P_k \subseteq W$. In that case also $\lim_k \mathscr{M}_H(P_k - T) = 0$ thus both inequalities above become equalities. □

Theorem 5.12. — *Let $(X, \|\cdot\|)$ be a finite dimensional normed space and let $m \in \{1,\dots,\dim X - 1\}$. Assume that $W \in \mathbf{G}_m(X)$, μ is a density contractor on W and $A \subseteq X$ is Borel measurable and countably $(\mathscr{H}^m_{\|\cdot\|}, m)$ rectifiable. It follows that*

$$\int_{\operatorname{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}(\pi(A))\mathrm{d}\mu(\pi) \leqslant \mathscr{H}^m_{\|\cdot\|}(A)$$

with equality when $A \subseteq W$, and the above integrand is Borel measurable.

Proof. — The measurability claim follows from Proposition 5.1(3). In order to prove the inequality there is of course no restriction to assume

$\mathscr{H}^m_{\|\cdot\|}(A) < \infty$. We will apply Theorem 5.11 with $G = \mathbf{Z}$. We start by choosing a Borel measurable orientation of the approximate tangent spaces of A, say $\xi : A \to \bigwedge_m X$, see [13, 3.2.25]. We let $T \in \mathscr{R}_m(X, G)$ be associated with the set A and the $\mathbf{Z}$ orientation $[1, \xi(x)]$ at almost every $x \in A$. Thus $\mathscr{M}_H(T) = \mathscr{H}^m_{\|\cdot\|}(A)$. Next we recall that for $\mathscr{H}^m_{\|\cdot\|}$ almost every $y \in W$

$$\langle T, \pi, y\rangle = \sum_{x \in A \cap \pi^{-1}\{y\}} (-1)^{\varepsilon(\pi,x)} \boldsymbol{\delta}_x \,,$$

where $\varepsilon(\pi, x) = \pm 1$ according to whether the restriction $\pi : \operatorname{ap}\operatorname{Tan}(\mathscr{H}^m \llcorner A, x) \to W$ preserves the orientation or not. In particular for $\mathscr{H}^m_{\|\cdot\|}$ almost every $y \in \pi(A)$ one has $\mathscr{M}(\langle T, \pi, y\rangle) \geqslant 1$. Our present conclusion then immediately follows from the first conclusion of Theorem 5.11. □

6. New type of Gross measures

6.1. Choice of density contractors

Given a finite dimensional normed space $(X, \|\cdot\|)$ and an integer $1 \leqslant m \leqslant \dim X - 1$ we say that $(X, \|\cdot\|)$ *admits density contractors of dimension* m if for every every $W \in \mathbf{G}_m(X)$ there exists a density contractor μ on W. Under this assumption the Axiom of Choice guarantees the existence of a choice map

$$\boldsymbol{\mu} : \mathbf{G}_m(X) \to \mathscr{M}_1(\operatorname{Hom}_m(X, X))\,,$$

i.e. such that $\boldsymbol{\mu}(W)$ is a density contractor on W, for each $W \in \mathbf{G}_m(X)$. Here $\mathscr{M}_1(\operatorname{Hom}_m(X, X))$ denotes the set of Borel probability measures on $\operatorname{Hom}_m(X, X)$. We call such $\boldsymbol{\mu}$ a *choice of density contractors in dimension* m and we write $\boldsymbol{\mu}_W$ instead of $\boldsymbol{\mu}(W)$. We shall show at the end of this section, Theorem 6.7 that such $\boldsymbol{\mu}$ can be chosen to be universally measurable, even though this extra property of a choice of density contractors will play no role in the other results presented here.

6.2. Convention

In the remaining part of this section we assume that $(X, \|\cdot\|)$ *admits density contractors of dimension* m *and we let* $W \mapsto \boldsymbol{\mu}_W$ *be a choice of density contractors in dimension* m.

6.3. Gross measure

With this data (m and $\boldsymbol{\mu}$) we shall now associate an outer measure on X denoted as $\mathscr{G}^m_{\boldsymbol{\mu}}$. To start with, for each Borel subset $A \subseteq X$ we define

$$\zeta_{\boldsymbol{\mu}}(A) = \sup_{W \in \mathbf{G}_m(X)} \int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}(\pi(A)) \mathrm{d}\boldsymbol{\mu}_W(\pi)\,.$$

Now with $A \subseteq X$ and $0 < \delta \leqslant \infty$ we associate

$$\mathscr{G}^m_{\boldsymbol{\mu},\delta}(A) \quad = \quad \inf \left\{ \sum_{j \in J} \zeta_{\boldsymbol{\mu}}(B_j) : \begin{array}{c} A \subseteq \cup_{j \in J} B_j \text{ and } \{B_j : j \in J\} \text{ is a} \\ \text{countable family of Borel subsets} \\ \text{of } X \text{ with } \operatorname{diam} B_j < \delta\,, j \in J \end{array} \right\},$$

as well as

$$\mathscr{G}^m_{\boldsymbol{\mu}}(A) = \sup_{\delta > 0} \mathscr{G}^m_{\boldsymbol{\mu},\delta}(A)\,.$$

All these are outer measures on X. Furthermore Borel sets are $\mathscr{G}^m_{\boldsymbol{\mu}}$ measurable and $\mathscr{G}^m_{\boldsymbol{\mu}}$ is Borel regular.

Our goal is to establish that $\mathscr{G}^m_{\boldsymbol{\mu}}(A) = \mathscr{H}^m_{\|\cdot\|}(A)$ in case A is countably $(\mathscr{H}^m_{\|\cdot\|}, m)$ rectifiable. We start with a trivial observation about rectangular matrices. Recall that $\Lambda(n,m)$ denotes the set of (strictly) increasing maps $\{1,\dots,m\} \to \{1,\dots,n\}$ which we may identify with their image.

Lemma 6.1. — *Let $A \in M_{m \times n}(\mathbf{R})$ and $B, B' \in M_{n \times m}(\mathbf{R})$ be such that*

$$B = \begin{pmatrix} I_m \\ 0 \end{pmatrix} \text{ and } B' = \begin{pmatrix} I_m \\ E \end{pmatrix}$$

for some $E \in M_{(n-m)\times m}(\mathbf{R})$ with $\|E\|_\infty \leqslant 1$. It follows that

$$|\det(AB) - \det(AB')| \leqslant \boldsymbol{c}_{\textit{Lemma 6.1}}(n,m)\|E\|_\infty \sum_{\substack{\lambda \in \Lambda(n,m) \\ \lambda \neq \{1,\dots,m\}}} |\det(A_\lambda)|$$

were A_λ is the square matrix whose k^{th} column coincides with the $\lambda(k)^{th}$ column of A.

Proof. — If $F \subseteq \{1,\dots,n\}$ is a nonempty set whose elements are numbered $j_1 < \ldots < j_p$ we let A_F denote the matrix whose k^{th} column is the j_k^{th} column of A. We also abbreviate $\lambda_0 = \{1,\dots,m\}$. It is immediate that $AB = A_{\lambda_0}$. Furthermore

$$\begin{aligned} AB' &= \begin{pmatrix} A_{\lambda_0} & \begin{pmatrix} A_{\{m+1\}} & \cdots & A_{\{n\}} \end{pmatrix} \end{pmatrix} \begin{pmatrix} I_m \\ E \end{pmatrix} \\ &= A_{\lambda_0} + \begin{pmatrix} A_{\{m+1\}} & \cdots & A_{\{n\}} \end{pmatrix} E\,. \end{aligned}$$

It follows that the k^{th} column of AB' is

$$A_{\{k\}} + \sum_{i=1}^{n-m} e_{i,k} A_{\{m+i\}}$$

where $E = (e_{i,j})_{\substack{i=1,\dots,n-m \\ j=1,\dots,m}}$. The conclusion now ensues from the multilinearity of the determinant. □

Proposition 6.2. — *Assume Section 6.2. Let $M \subseteq X$ be an m dimensional submanifold of X of class C^1, $a \in M$ and $0 < \varepsilon < 1$. There then exists $r_0 > 0$ and $W \in \mathbf{G}_m(X)$ with the following property. For every $0 < r \leqslant r_0$ one has*

$$\int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|} \left(\pi(M \cap \mathbf{B}_{\|\cdot\|}(a,r))\right) \mathrm{d}\boldsymbol{\mu}_W(\pi) \geqslant (1-\varepsilon)\mathscr{H}^m_{\|\cdot\|} \left(M \cap \mathbf{B}_{\|\cdot\|}(a,r)\right) .$$

Proof. — Abbreviate $W = \mathrm{Tan}(M,a)$. Owing to the translation invariance of the Hausdorff measure and to the linearity of π in the statement, replacing M by $M - a$ we may assume that $a = 0$. As usual we consider a Euclidean structure on X, and we let $C \geqslant 1$ be such that $C^{-1}\|x\| \leqslant |x| \leqslant C\|x\|$ for every $x \in X$ and $C^{-1} \leqslant \psi(W) \leqslant C$ for every $W \in \mathbf{G}_m(X)$ (recall Proposition 2.1). We let $0 < \widehat{\varepsilon} < 1$ be undefined for now.

We define

$$\mathrm{Hom}^{\sim}(X,W) = \mathrm{Hom}(X,W) \cap \{\pi : \mathrm{rank}\, \pi|_W \leqslant m-1\}$$

as well as

$$\mathrm{Hom}_{\mathrm{inv}}(X,W) = \mathrm{Hom}(X,W) \cap \{\pi : \mathrm{rank}\, \pi|_W = m\} ,$$

and we notice the first set is relatively closed and second one relatively open. For each $\kappa = 1, 2, \dots$ we define

$$\mathrm{Hom}_{\mathrm{inv},\kappa}(X,W) = \mathrm{Hom}_{\mathrm{inv}}(X,W) \cap \left\{\pi : \mathrm{Lip}_{|\cdot|}\, \pi \leqslant \kappa \text{ and } \mathrm{Lip}_{|\cdot|}\left(\pi|_W^{-1}\right) \leqslant \kappa\right\}.$$

Since $\mathrm{Lip}_{|\cdot|}\, \pi = |||\pi|||$, and since $\mathrm{Hom}_{\mathrm{inv}}(X,W) \to \mathrm{Hom}(W,W) : \pi \mapsto \pi|_W^{-1}$ is continuous, it follows that $\mathrm{Hom}_{\mathrm{inv},\kappa}(X,W)$ is relatively closed. Also,

$$\mathrm{Hom}_{\mathrm{inv}}(X,W) = \bigcup_{\kappa=1}^{\infty} \mathrm{Hom}_{\mathrm{inv},\kappa}(X,W) .$$

We abbreviate $H_\kappa = \mathrm{Hom}^{\sim}(X,W) \cup \mathrm{Hom}_{\mathrm{inv},\kappa}(X,W)$ and we infer from the monotone convergence theorem that there exists κ such that

$$\begin{aligned}(6.1)\quad \int_{H_\kappa} \mathscr{H}^m_{\|\cdot\|}\left(\pi(W\cap \mathbf{B}_{\|\cdot\|}(0,1))\right) \mathrm{d}\boldsymbol{\mu}_W(\pi) \\ \geqslant (1-\widehat{\varepsilon}) \int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}\left(\pi(W\cap \mathbf{B}_{\|\cdot\|}(0,1))\right) \mathrm{d}\boldsymbol{\mu}_W(\pi) \\ = (1-\widehat{\varepsilon})\, \mathscr{H}^m_{\|\cdot\|}\left(W\cap \mathbf{B}_{\|\cdot\|}(0,1)\right) = (1-\widehat{\varepsilon})\,\boldsymbol{\alpha}(m)\end{aligned}$$

where the last equality is H. Busemann's identity (2.1). For a reason that will become clear momentarily we then define

$$\check{\varepsilon} = \min\left\{\widehat{\varepsilon}, \frac{1}{2\kappa^2}\right\}.$$

There exists $r_1 > 0$ and $f : W \cap \mathbf{B}_{|\cdot|}(0,2r_1) \to W^\perp$ of class C^1 such that

(a) $\mathrm{Lip}_{|\cdot|} f < \check{\varepsilon}$;
(b) $M \cap \mathbf{B}_{|\cdot|}(0,r_1) \subseteq F(W\cap \mathbf{B}_{|\cdot|}(0,2r_1)) \subseteq M$,

where $F = \iota_W + \iota_{W^\perp}\circ f$, and ι_W and $\iota_{W^\perp}$ are the obvious inclusion maps. We let π_W denote the orthogonal projection onto W. It clearly follows that if $0<r\leqslant r_1/C$ then

(c) $F\left(\pi_W(M\cap \mathbf{B}_{\|\cdot\|}(0,r))\right) = M\cap \mathbf{B}_{\|\cdot\|}(0,r)$;

and furthermore,

(d) $$\begin{aligned} W\cap \mathbf{B}_{\|\cdot\|}\left(0,(1-C^2\widehat{\varepsilon})r\right) &\subseteq \pi_W(M\cap \mathbf{B}_{\|\cdot\|}(0,r)) \\ &\subseteq W\cap \mathbf{B}_{\|\cdot\|}\left(0,(1-C^2\widehat{\varepsilon})^{-1}r\right), \end{aligned}$$

because for each $x\in W\cap \mathbf{B}_{|\cdot|}(0,r_1)$ one has $\big|\|F(x)\|-\|x\|\big| \leqslant \|f(x)\| \leqslant C^2\widehat{\varepsilon}\|x\|$.

For the remaining part of this proof we fix $0<r\leqslant r_1/C$ and we abbreviate $D_r = \pi_W(M\cap \mathbf{B}_{\|\cdot\|}(0,r))$. For a while we turn to computing Hausdorff measures $\mathscr{H}^m_{|\cdot|}$ with respect to the Euclidean norm $|\cdot|$.

We claim that if $\pi\in \mathrm{Hom}^{\sim}(X,W)$ or $\pi\in\mathrm{Hom}_{\mathrm{inv},\kappa}(X,W)$, applying the (Euclidean) area formula to the mappings $\pi\circ F : W\to W$ and $\pi|_W : W \to W$, [13, 3.2.22] we obtain

$$(6.2)\quad \begin{aligned}\mathscr{H}^m_{|\cdot|}\left(\pi(M\cap \mathbf{B}_{\|\cdot\|}(0,r))\right) &= \mathscr{H}^m_{|\cdot|}\left(\pi(F(D_r))\right) \\ &= \int_{W\cap D_r} J_m(\pi\circ F)(x)\mathrm{d}\mathscr{H}^m_{|\cdot|}(x)\end{aligned}$$

and

$$(6.3)\quad \mathscr{H}^m_{|\cdot|}\left(\pi(D_r)\right) = \int_{W\cap D_r} J_m\pi(x)\mathrm{d}\mathscr{H}^m_{|\cdot|}(x)\,.$$

Indeed if $\pi \in \mathrm{Hom}^{\sim}(X,W)$ then the two sides of both equations obviously vanish. Furthermore the second equation is in fact valid for every $\pi \in \mathrm{Hom}_{\mathrm{inv}}(X,W)$ as well, because in that case $\pi|_W$ is injective. Therefore it remains only to establish the first equation in case $\pi \in \mathrm{Hom}_{\mathrm{inv},\kappa}(X,W)$. The reason for which it holds true is that $(\pi \circ F)|_{D_r}$ is injective. Indeed if $\pi(F(x)) = \pi(F(x'))$ for $x, x' \in D_r$, then $\pi(\iota_W(x)) - \pi(\iota_W(x')) = \pi(\iota_{W^\perp}(f(x'))) - \pi(\iota_{W^\perp}(f(x)))$ and in turn

$$\begin{aligned}\kappa^{-1}|x-x'| &\leqslant \left(\mathrm{Lip}_{|\cdot|}\,\pi|_W^{-1}\right)^{-1}|x-x'| \\ &\leqslant |\pi(x)-\pi(x')| = |\pi(f(x))-\pi(f(x'))| \\ &\leqslant \left(\mathrm{Lip}_{|\cdot|}\,\pi\right)\left(\mathrm{Lip}_{|\cdot|}\,f\right)|x-x'| \leqslant \kappa\check{\varepsilon}|x-x'|\end{aligned}$$

so that $x = x'$ because $\check{\varepsilon} \leqslant \kappa^{-2}/2$.

From this ensues that if $\pi \in H_\kappa$ then

$$\begin{aligned}\text{(6.4)}\quad &\left|\mathscr{H}^m_{|\cdot|}\left(\pi(M \cap \mathbf{B}_{\|\cdot\|}(0,r))\right) - \mathscr{H}^m_{|\cdot|}\left(\pi(D_r)\right)\right| \\ &\qquad\qquad \leqslant \int_{W\cap D_r} |J_m(\pi\circ F)(x) - J_m\pi(x)|\,\mathrm{d}\mathscr{H}^m_{|\cdot|}(x)\,.\end{aligned}$$

We choose an orthonormal basis $e_1,\dots,e_n$ of X such that $e_1,\dots,e_m$ is a basis of W, and we let $A \in M_{m\times n}(\mathbf{R}^n)$ be the matrix of π with respect to these bases. Letting $B \in M_{n\times m}(\mathbf{R}^n)$ be defined as in Lemma 6.1 it is clear that AB is the matrix of $\pi|_W$ with respect to the basis $e_1,\dots,e_m$ of W, thus $J_m\pi = |\det(AB)|$. Given $x \in D_r$ we notice that $DF(x) = \iota_W + \iota_{W^\perp}\circ Df(x)$ whence the matrix B'_x of $DF(x)$ with respect to $e_1,\dots,e_m$ and $e_1,\dots,e_n$ is of the form of B' in Lemma 6.1 with E_x being the matrix of $Df(x)$ with respect to the obvious bases. In particular $\|E_x\|_\infty \leqslant \mathrm{Lip}_{|\cdot|} f \leqslant \hat{\varepsilon}$. Moreover $J_m(\pi\circ F)(x) = |\det(AB'_x)|$. Thus it follows from Lemma 6.1 that

$$\begin{aligned}\text{(6.5)}\quad |J_m(\pi\circ F)(x) - J_m\pi(x)| &= \left||\det(AB'_x)| - |\det(AB)|\right| \\ &\leqslant \boldsymbol{c}_{\text{Lemma 6.1}}(n,m)\hat{\varepsilon}\sum_{\substack{\lambda\in\Lambda(n,m)\\ \lambda\neq\{1,\dots,m\}}} |\det(A_\lambda)|\,.\end{aligned}$$

Now given $\lambda \in \Lambda(n,m)$ we let $W_\lambda = \mathrm{span}\{e_{\lambda(1)},\dots,e_{\lambda(m)}\}$ and we select $O_\lambda : X \to X$ be a isometric linear isomorphism such that $O_\lambda(W) = W_\lambda$. It then becomes clear that A_λ is the matrix of $(\pi\circ O_\lambda)|_W$ with respect to the basis $e_1,\dots,e_m$ of W. Thus $|\det(A_\lambda)| = J_m(\pi\circ O_\lambda)|_W$ and in turn

$$\int_{W\cap D_r} |\det(A_\lambda)|\mathrm{d}\mathscr{H}^m_{|\cdot|} = \mathscr{H}^m_{|\cdot|}\left(\pi(O_\lambda(D_r))\right)$$

Together with (6.4) and (6.5) this yields

$$\left|\mathscr{H}^m_{|\cdot|}\left(\pi(M\cap \mathbf{B}_{\|\cdot\|}(0,r))\right)-\mathscr{H}^m_{|\cdot|}\left(\pi(D_r)\right)\right| \leqslant \boldsymbol{c}_{\text{Lemma 6.1}}(n,m)\widehat{\varepsilon}\sum_{\substack{\lambda\in\Lambda(n,m)\\ \lambda\neq\{1,\dots,m\}}}\mathscr{H}^m_{|\cdot|}\left(\pi(O_\lambda(D_r))\right)$$

Since all sets whose measure appear in the above inequality are subsets of W, multiplying both sides by $\psi(W)$ we obtain at once that

$$\left|\mathscr{H}^m_{\|\cdot\|}\left(\pi(M\cap \mathbf{B}_{\|\cdot\|}(0,r))\right)-\mathscr{H}^m_{\|\cdot\|}\left(\pi(D_r)\right)\right| \leqslant \boldsymbol{c}_{\text{Lemma 6.1}}(n,m)\widehat{\varepsilon}\sum_{\substack{\lambda\in\Lambda(n,m)\\ \lambda\neq\{1,\dots,m\}}}\mathscr{H}^m_{\|\cdot\|}\left(\pi(O_\lambda(D_r))\right)$$

Next we integrate both sides with respect to $\boldsymbol{\mu}_W$ on the set $H_\kappa = \operatorname{Hom}^\sim(X,W)\cup\operatorname{Hom}_{\mathrm{inv},\kappa}(X,W)$:

$$\begin{aligned}(6.6)\quad &\left|\int_{H_\kappa}\mathscr{H}^m_{\|\cdot\|}(\pi(D_r))\mathrm{d}\boldsymbol{\mu}_W(\pi)-\int_{H_\kappa}\mathscr{H}^m_{\|\cdot\|}\left(\pi(M\cap\mathbf{B}_{\|\cdot\|}(0,r))\right)\mathrm{d}\boldsymbol{\mu}_W(\pi)\right|\\ &\leqslant \boldsymbol{c}_{\text{Lemma 6.1}(n,m)}\widehat{\varepsilon}\sum_{\substack{\lambda\in\Lambda(n,m)\\ \lambda\neq\{1,\dots,m\}}}\int_{\operatorname{Hom}_m(X,X)}\mathscr{H}^m_{\|\cdot\|}\left(\pi(O_\lambda(D_r))\right)\mathrm{d}\boldsymbol{\mu}(\pi)\\ &\leqslant \boldsymbol{c}_{\text{Lemma 6.1}(n,m)}\widehat{\varepsilon}\sum_{\substack{\lambda\in\Lambda(n,m)\\ \lambda\neq\{1,\dots,m\}}}\mathscr{H}^m_{\|\cdot\|}\left(O_\lambda(D_r)\right).\end{aligned}$$

It follows from (d) above and (6.1) that

$$\begin{aligned}(6.7)\quad &\int_{H_\kappa}\mathscr{H}^m_{\|\cdot\|}\left(\pi(D_r)\right)\mathrm{d}\boldsymbol{\mu}_W(\pi)\\ &\geqslant\left(1-C^2\widehat{\varepsilon}\right)^m r^m\int_{H_\kappa}\mathscr{H}^m_{\|\cdot\|}\left(\pi(W\cap\mathbf{B}_{\|\cdot\|}(0,1))\right)\mathrm{d}\boldsymbol{\mu}_W(\pi)\\ &\geqslant(1-\widehat{\varepsilon})\left(1-C^2\widehat{\varepsilon}\right)^m\boldsymbol{\alpha}(m)r^m.\end{aligned}$$

It further follows from (d) above and H. Busemann's identity (2.1) that

$$\mathscr{H}^m_{\|\cdot\|}(D_r)\leqslant\left(1-C^2\widehat{\varepsilon}\right)^{-m}\boldsymbol{\alpha}(m)r^m,$$

and in turn

$$\begin{aligned}\mathscr{H}^m_{\|\cdot\|}(O_\lambda(D_r)) &= \psi(W_\lambda)\mathscr{H}^m_{|\cdot|}(O_\lambda(D_r))=\psi(W_\lambda)\mathscr{H}^m_{|\cdot|}(D_r)\\ &=\psi(W_\lambda)\psi(W)^{-1}\mathscr{H}^m_{\|\cdot\|}(D_r)\leqslant\boldsymbol{\alpha}(m)C^2\left(1-C^2\widehat{\varepsilon}\right)^{-m}r^m\end{aligned}$$

Using these and (6.6) we see that

$$\begin{aligned}&\int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}\left(\pi(M\cap \mathbf{B}_{\|\cdot\|}(0,r))\right) \mathrm{d}\mu(\pi)\\ &\geqslant \left((1-\widehat{\varepsilon})(1-C^2\widehat{\varepsilon})^m - \widehat{\varepsilon}\boldsymbol{c}_{\text{Lemma 6.1}}(n,m)\left(\operatorname{card}\Lambda(n,m)\right)C^2\left(1-C^2\right)^{-m}\right)\boldsymbol{\alpha}(m)r^m.\end{aligned}$$

Furthermore it follows from B. Kirchheim's area formula [17] that there exists $r_2>0$ such that for every $0<r\leqslant r_2$ one has

$$(1+\widehat{\varepsilon})\boldsymbol{\alpha}(m)r^m \geqslant \mathscr{H}^m_{\|\cdot\|}\left(M\cap \mathbf{B}_{\|\cdot\|}(0,r)\right).$$

Finally we let $r_0=\min\{r_1/C,r_2\}$ and it should now be obvious how to choose $\widehat{\varepsilon}$ according to ε, n, m and C so that the conclusion holds. □

Lemma 6.3. — *$\zeta_{\boldsymbol{\mu}}(A)\leqslant \mathscr{G}^m_{\boldsymbol{\mu}}(A)$ whenever $A\subseteq X$ is Borel.*

Proof. — If $\{B_j : j\in J\}$ is a countable family of Borel subsets of X so that $A\subseteq \cup_{j\in J}B_j$ then $\mathscr{H}^m_{\|\cdot\|}(\pi(A))\leqslant \sum_{j\in J}\mathscr{H}^m_{\|\cdot\|}(\pi(B_j))$ for every $\pi\in \mathrm{Hom}_m(X,X)$. Thus for each $W\in \mathbf{G}_m(X)$ one has

$$\int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}(\pi(A))\mathrm{d}\boldsymbol{\mu}_W(\pi)\leqslant \sum_{j\in J}\int_{\mathrm{Hom}_m(X,X)}\mathscr{H}^m_{\|\cdot\|}(\pi(B_j))\mathrm{d}\boldsymbol{\mu}_W(\pi).$$

Taking the supremum over $W\in\mathbf{G}_m(X)$ on both sides yields $\zeta_{\boldsymbol{\mu}}(A)\leqslant \sum_{j\in J}\zeta_{\boldsymbol{\mu}}(B_j)$. The conclusion follows from the arbitrariness of $\{B_j : j\in J\}$. □

Theorem 6.4. — *If $A\subseteq X$ is Borel and countably $(\mathscr{H}^m_{\|\cdot\|},m)$ rectifiable then $\mathscr{G}^m_{\boldsymbol{\mu}}(A)=\mathscr{H}^m_{\|\cdot\|}(A)$.*

Proof. — We start by showing that $\mathscr{G}^m_{\boldsymbol{\mu}}(A)\leqslant \mathscr{H}^m_{\|\cdot\|}(A)$. Given $\delta>0$ choose a countable Borel partition $\{B_j : j\in J\}$ of A with $\operatorname{diam}B_j<\delta$, $j\in J$ (for instance the B_j are the intersection of A with dyadic semicubes of some generation). Since each B_j is countably $(\mathscr{H}^m_{\|\cdot\|},m)$ rectifiable it follows from Theorem 5.12 that $\zeta_{\boldsymbol{\mu}}(B_j)\leqslant \mathscr{H}^m_{\|\cdot\|}(B_j)$. Therefore

$$\mathscr{G}^m_{\boldsymbol{\mu},\delta}(A)\leqslant \sum_{j\in J}\zeta_{\boldsymbol{\mu}}(B_j)\leqslant \sum_{j\in J}\mathscr{H}^m_{\|\cdot\|}(B_j)=\mathscr{H}^m_{\|\cdot\|}(A)$$

and it remains to let $\delta\to 0^+$.

In order to establish the reverse inequality we start with the case when $A=M$ is an m dimensional submanifold of X of class C^1 such that $\mathscr{H}^m_{\|\cdot\|}(M)<\infty$. Let $\varepsilon>0$ and $a\in M$. It follows from Proposition 6.2 that there exists $r(a)>0$ and $W\in\mathbf{G}_m(X)$ such that for every $0<r\leqslant r(a)$

one has

$$\int_{\mathrm{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}\left(\pi(M\cap \mathbf{B}_{\|\cdot\|}(a,r))\right) \mathrm{d}\boldsymbol{\mu}_W(\pi) \\ \geqslant (1-\varepsilon)\mathscr{H}^m_{\|\cdot\|}\left(M\cap \mathbf{B}_{\|\cdot\|}(a,r)\right) .$$

Thus also

$$\zeta_{\boldsymbol{\mu}}\left(M\cap \mathbf{B}_{\|\cdot\|}(a,r)\right) \geqslant (1-\varepsilon)\mathscr{H}^m_{\|\cdot\|}\left(M\cap \mathbf{B}_{\|\cdot\|}(a,r)\right) .$$

It follows from [13, 2.8.9 and 2.8.18] that $\{\mathbf{B}_{\|\cdot\|}(a,r) : a \in M \text{ and } 0 < r \leqslant r(a)\}$ is an $\mathscr{H}^m_{\|\cdot\|} \llcorner M$ Vitali relation. Therefore there exists a countable subset $\{a_j : j \in J\}$ of M and corresponding $0 < r_j \leqslant r(a_j)$, $j \in J$, such that, upon abbreviating $B_j = \mathbf{B}_{\|\cdot\|}(a_j, r_j)$, the family $\{B_j : j \in J\}$ is disjointed and $\mathscr{H}^m_{\|\cdot\|}(M \sim \cup_{j\in J} B_j) = 0$. Thus,

$$\begin{aligned}(1-\varepsilon)\mathscr{H}^m_{\|\cdot\|}(M) &= \sum_{j\in J}(1-\varepsilon)\mathscr{H}^m_{\|\cdot\|}(M\cap B_j) \\ &\leqslant \sum_{j\in J}\zeta_{\boldsymbol{\mu}}(M\cap B_j) \leqslant \sum_{j\in J}\mathscr{G}^m_{\boldsymbol{\mu}}(M\cap B_j) \leqslant \mathscr{G}^m_{\boldsymbol{\mu}}(M),\end{aligned}$$

where the second inequality is a consequence of Lemma 6.3. It follows from the arbitrariness of $\varepsilon > 0$ that the inequality is established in case $A = M$ is an m dimensional C^1 submanifold. We consider the Borel finite measures $\phi_{\mathscr{H}} = \mathscr{H}^m_{\|\cdot\|} \llcorner M$ and $\phi_{\mathscr{G}} = \mathscr{G}^m_{\boldsymbol{\mu}} \llcorner M$ (that $\phi_{\mathscr{G}}$ be finite follows from the fact that so is $\phi_{\mathscr{H}}$ – by assumption – and the first part of this proof). Now if $U \subseteq X$ is open then $M\cap U$ is also an m dimensional C^1 submanifold and therefore $\phi_{\mathscr{H}}(U) \leqslant \phi_{\mathscr{G}}(U)$. Since $\phi_{\mathscr{G}}$ is outer regular (being a finite Borel measure in a metric space) we infer that $\phi_{\mathscr{H}}(A) \leqslant \phi_{\mathscr{G}}(A)$ for all Borel subsets $A \subseteq M$. Assuming now that A is a Borel countable $(\mathscr{H}^m_{\|\cdot\|}, m)$ rectifiable subset of X, the conclusion follows from the fact that A admits a partition into Borel sets $A_0, A_1, \dots$ such that $\mathscr{H}^m_{\|\cdot\|}(A_0) = 0$ and each A_j, $j \geqslant 1$, is a subset of some m dimensional submanifold of X of class C^1, [13, 3.2.29]. □

We close this section by showing that there exists a universally measurable choice of density contractors.

Proposition 6.5. — *Assume that $W, W_1, W_2, \dots$ are members of $\mathbf{G}_m(X)$ and that $d(W, W_k) \to 0$ as $k \to \infty$. For every $n = 1, 2, \dots$ the following holds:*

$$\lim_k \sup_{\substack{\pi\in\mathrm{Hom}_m(X,X)\\ |||\pi|||\leqslant n}} \left| \mathscr{H}^m_{\|\cdot\|}\left(\pi(W_k\cap B_{|\cdot|})\right) - \mathscr{H}^m_{\|\cdot\|}\left(\pi(W\cap B_{|\cdot|})\right) \right| = 0 .$$

Proof. — We show how this is a consequence of Steiner's formula. Let $\pi \in \mathrm{Hom}_m(X,X)$ and choose $V \in \mathbf{G}_m(X)$ such that $\mathrm{im}\,\pi \subseteq V$. Through the choice of an orthonormal basis we identify V with ℓ_2^m. For any convex set $C \subseteq V$ and $\delta > 0$ we have

$$\mathscr{H}^m_{|\cdot|}(\mathbf{B}_{|\cdot|}(C,\delta)) = \mathscr{H}^m_{|\cdot|}(C) + \sum_{k=0}^{m-1} \delta^{m-k}\boldsymbol{\alpha}(m-k)\zeta^k(C) \tag{6.8}$$

where

$$\zeta^k(C) = \boldsymbol{\beta}_1(m,k)^{-1}\int_{\mathbf{O}^*(m,k)} \mathscr{L}^k(p(C))\mathrm{d}\boldsymbol{\theta}^*_{m,k}(p)\,,$$

see for instance [13, 3.2.35]. Now if n is an integer so that $C \subseteq V \cap \mathbf{B}_{|\cdot|}(0,n)$ then $p(C) \subseteq \mathbf{R}^k \cap \mathbf{B}(0,n)$ for all $p \in \mathbf{O}^*(m,k)$ and therefore $\zeta^k(C) \leqslant \boldsymbol{\beta}_1(m,k)^{-1}\boldsymbol{\alpha}(k)n^k$. Thus if $C, C' \subseteq V \cap \mathbf{B}_{|\cdot|}(0,n)$ are compact convex and $\delta = \mathrm{dist}_{\mathscr{H}}(C,C') \leqslant 1$ then it follows from (6.8) that

$$\left|\mathscr{H}^m_{|\cdot|}(C) - \mathscr{H}^m_{|\cdot|}(C')\right| \leqslant \delta\boldsymbol{c}(m)$$

where

$$\boldsymbol{c}(m) = \sum_{k=0}^{m-1} \boldsymbol{\beta}_1(m,k)^{-1}\boldsymbol{\alpha}(m-k)\boldsymbol{\alpha}(k)n^k\,.$$

Now we let $C_k = \pi(W_k \cap B_{|\cdot|})$, $C = \pi(W \cap B_{|\cdot|})$, $\delta_k = \mathrm{dist}_{\mathscr{H}}(C_k, C)$ and we assume that $|||\pi||| \leqslant n$. Since $d(W_k, W) \to 0$ we have $\delta_k \leqslant n\,\mathrm{dist}_{\mathscr{H}}(W_k \cap B_{|\cdot|}, W \cap B_{|\cdot|}) \to 0$ as well. Finally,

$$\begin{aligned}\left|\mathscr{H}^m_{\|\cdot\|}\left(\pi(W_k \cap B_{|\cdot|})\right) - \mathscr{H}^m_{\|\cdot\|}\left(\pi(W \cap B_{|\cdot|})\right)\right|& \\ = \psi(W)\left|\mathscr{H}^m_{|\cdot|}\left(\pi(W_k \cap B_{|\cdot|})\right) - \mathscr{H}^m_{|\cdot|}\left(\pi(W \cap B_{|\cdot|})\right)\right|& \\ \leqslant C\boldsymbol{c}(m)n\,\mathrm{dist}_{\mathscr{H}}(W_k \cap B_{|\cdot|}, W \cap B_{|\cdot|})&\end{aligned}$$

if k is large enough (for $n\delta_k \leqslant 1$), where

$$C = \max\{\psi(W) : W \in \mathbf{G}_m(X)\}. \qquad \square$$

6.4. Space of probability measures

We recall that $\mathrm{Hom}_m(X,X)$, the collection of linear maps $X \to X$ of rank at most m, is a closed subspace of $\mathrm{Hom}(X,X)$ and therefore a Polish space. The set $\mathscr{M}_1(\mathrm{Hom}_m(X,X))$ consisting of those probability Borel measures on $\mathrm{Hom}_m(X,X)$ is itself a Polish space, equipped with the so called topology of weak convergence of measures, [19, Chapter II §6].

Theorem 6.6. — *The set*

$$\mathscr{E} = \mathbf{G}_m(X) \times \mathscr{M}_1(\operatorname{Hom}_m(X,X)) \cap \left\{ (W,\mu) : \begin{array}{l} \mu \textit{ is a density contractor} \\ \textit{on } W \textit{ with respect to } \mathscr{H}^m_{\|\cdot\|} \end{array} \right\}$$

is Borel.

Proof. — We define

$$\mathscr{E}_a = \{(W,\mu) : \mu \text{ is supported in } \operatorname{Hom}(X,W)\}$$

and

$$\mathscr{E}_b = \left\{ (W,\mu) : \begin{array}{c} \text{for every } V \in \mathbf{G}_m(X) \text{ one has} \\ \displaystyle\int_{\operatorname{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}\left(\pi(V \cap B_{|\cdot|})\right) \mathrm{d}\mu(\pi) \leqslant \mathscr{H}^m_{\|\cdot\|}\left(V \cap B_{|\cdot|}\right) \end{array} \right\}$$

and

$$\mathscr{E}_c = \left\{ (W,\mu) : \int_{\operatorname{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}\left(\pi(W \cap B_{|\cdot|})\right) \mathrm{d}\mu(\pi) = \mathscr{H}^m_{\|\cdot\|}\left(W \cap B_{|\cdot|}\right) \right\}.$$

It follows from Proposition 5.7 that $\mathscr{E} = \mathscr{E}_a \cap \mathscr{E}_b \cap \mathscr{E}_c$.

The set $\mathscr{E}_a$ is closed. Let $(W_1,\mu_1),(W_2,\mu_2),\dots$ be members of $\mathscr{E}_a$ such that $(W_k,\mu_k) \to (W,\mu)$ in $\mathbf{G}_m(X) \times \mathscr{M}_1(\operatorname{Hom}_m(X,X))$ as $k \to \infty$. We observe that $\operatorname{Hom}_m(X,X) \sim \operatorname{Hom}(X,W) = \cup_{j=1}^{\infty} \mathcal{U}_j$ where

$$\mathcal{U}_j = \operatorname{Hom}_m(X,X) \cap \left\{ \pi : \operatorname{dist}(\pi, \operatorname{Hom}(X,W)) > \frac{|||\pi|||}{j} \right\}.$$

Since $|||\pi_{W_k} - \pi_W||| \to 0$ by assumption, it is easily seen that for each fixed $j = 1,2,\dots$ there exists an integer $k(j)$ such that $\operatorname{Hom}(X,W_k) \cap \mathcal{U}_j = \emptyset$ whenever $k \geqslant k(j)$. For such $k \geqslant k(j)$ it follows that $\mu_k(\mathcal{U}_j) = 0$. Since $\mathcal{U}_j$ is open we have $\mu(\mathcal{U}_j) \leqslant \liminf_k \mu_k(\mathcal{U}_j) = 0$. It follows that $\operatorname{supp}\mu \subseteq \operatorname{Hom}(X,W)$.

The set $\mathscr{E}_b$ is closed. Given $V \in \mathbf{G}_m(X)$ we define

$$\mathscr{E}_{b,V} = \left\{ (W,\mu) : \int_{\operatorname{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}\left(\pi(V \cap B_{|\cdot|})\right) \mathrm{d}\mu(\pi) \leqslant \mathscr{H}^m_{\|\cdot\|}\left(V \cap B_{|\cdot|}\right) \right\}.$$

Since clearly $\mathscr{E}_b = \cap_{V \in \mathbf{G}_m(X)} \mathscr{E}_{b,V}$ it suffices to show that each $\mathscr{E}_{b,V}$ is closed. Define $\Upsilon_V : \mathbf{G}_m(X) \times \mathscr{M}_1(\operatorname{Hom}_m(X,X)) \to \mathbf{R}$ by the formula

$$\Upsilon_V(W,\mu) = \int_{\operatorname{Hom}_m(X,X)} \mathscr{H}^m_{\|\cdot\|}\left(\pi(V \cap B_{|\cdot|})\right) \mathrm{d}\mu(\pi)$$

(thus Υ_V does depend upon its first variable W). As before define $f_{V \cap B_{|\cdot|}} : \operatorname{Hom}_m(X,X) \to \mathbf{R}$ by $f_{V \cap B_{|\cdot|}}(\pi) = \mathscr{H}^m_{\|\cdot\|}\left(\pi(V \cap B_{|\cdot|})\right)$. It follows from Proposition 5.1(2) that for each $n = 1,2,\dots$ the function $\min\{n, f_{V \cap B_{|\cdot|}}\}$ is bounded and continuous. Given $(W_1,\mu_1),(W_2,\mu_2),\dots$ members of $\mathbf{G}_m(X)\times\mathscr{M}_1(\mathrm{Hom}_m(X,X))$ such that $(W_k,\mu_k)\to(W,\mu)$ we infer that, for each $n=1,2,\dots$,

$$\begin{aligned}\int_{\mathrm{Hom}_m(X,X)}\min\{n,f_{V\cap B_{|\cdot|}}\}\mathrm{d}\mu&=\lim_k\int_{\mathrm{Hom}_m(X,X)}\min\{n,f_{V\cap B_{|\cdot|}}\}\mathrm{d}\mu_k\\&\leqslant\liminf_k\int_{\mathrm{Hom}_m(X,X)}f_{V\cap B_{|\cdot|}}\mathrm{d}\mu_k=\liminf_k\Upsilon_V(W_k,\mu_k)\,.\end{aligned}$$

Letting $n\to\infty$ it then follows from the monotone convergence theorem that

$$\begin{aligned}\Upsilon_V(W,\mu)&=\int_{\mathrm{Hom}_m(X,X)}f_{V\cap B_{|\cdot|}}\mathrm{d}\mu\\&=\lim_n\int_{\mathrm{Hom}_m(X,X)}\min\{n,f_{V\cap B_{|\cdot|}}\}\mathrm{d}\mu\leqslant\liminf_k\Upsilon_V(W_k,\mu_k)\,.\end{aligned}$$

This shows that Υ_V is lower semicontinuous and in turn that $\mathscr{E}_{b,V}$ is closed.

The set $\mathscr{E}_c$ is Borel. We define $\Upsilon:\mathbf{G}_m(X)\times\mathscr{M}_1(\mathrm{Hom}_m(X,X))\to\mathbf{R}$ by the formula

$$\Upsilon(W,\mu)=\int_{\mathrm{Hom}_m(X,X)}\mathscr{H}^m_{\|\cdot\|}\left(\pi(W\cap B_{|\cdot|})\right)\mathrm{d}\mu(\pi)$$

so that

$$\mathscr{E}_c=\mathbf{G}_m(X)\times\mathscr{M}_1(\mathrm{Hom}_m(X,X))\cap\Big\{(W,\mu):\Upsilon(W,\mu)=\mathscr{H}^m_{\|\cdot|}\left(W\cap B_{|\cdot|}\right)\Big\}\,.$$

Since $\mathscr{H}^m_{\|\cdot|}\left(W\cap B_{|\cdot|}\right)=\psi(W)\boldsymbol{\alpha}(m)$ is continuous according to Proposition 2.1 the conclusion will follow from the lower semicontinuity of Υ which we now establish. We choose $\chi_1,\chi_2,\dots$ a nondecreasing sequence of cut-off functions on $\mathrm{Hom}_m(X,X)$ as in (5.4). With each $n=1,2,\dots$ we then associate

$$\Upsilon_n(W,\mu)=\int_{\mathrm{Hom}_m(X,X)}\chi_n(\pi)\mathscr{H}^m_{\|\cdot\|}\left(\pi(W\cap B_{|\cdot|})\right)\mathrm{d}\mu(\pi)\,.$$

As $\Upsilon=\sup_{n=1,2,\dots}\Upsilon_n$ according to the monotone convergence theorem, it is enough to show that each Υ_n is continuous. Fix $n=1,2,\dots$ Let $(W_1,\mu_1),(W_2,\mu_2),\dots$ be members of $\mathbf{G}_m(X)\times\mathscr{M}_1(\mathrm{Hom}_m(X,X))$ such that $(W_k,\mu_k)\to(W,\mu)$. It follows from Proposition 5.1(2) that $\chi_nf_{W_1\cap B_{|\cdot|}},\chi_nf_{W_2\cap B_{|\cdot|}},\dots$ is a sequence in $C_b(\mathrm{Hom}_m(X,X))$ and it follows from Proposition 6.5 that it converges uniformly to $\chi_nf_{W\cap B_{|\cdot|}}$. The weak* convergence of $\mu_1,\mu_2,\dots$ clearly implies its uniform convergence on compact subsets of $C_b(\mathrm{Hom}_m(X,X))$, thus

$$\begin{aligned}\Upsilon_n(W,\mu) &= \int_{\mathrm{Hom}_m(X,X)} \chi_n f_{W\cap B_{|\cdot|}} \mathrm{d}\mu \\ &= \lim_k \int_{\mathrm{Hom}_m(X,X)} \chi_n f_{W_k\cap B_{|\cdot|}} \mathrm{d}\mu_k = \lim_k \Upsilon_n(W_k,\mu_k) \qquad \square\end{aligned}$$

Theorem 6.7. — *Under the assumption Section 6.2 there exists a choice of density contractors* $\boldsymbol{\mu} : \mathbf{G}_m(X) \to \mathscr{M}_1(\mathrm{Hom}_m(X,X))$ *which is universally measurable.*

Proof. — Since $\mathbf{G}_m(X)$ and $\mathscr{M}_1(\mathrm{Hom}_m(X,X))$ are Polish spaces, and $\mathscr{E}$ is Borel according to Theorem 6.6, the result is a consequence of J. von Neumann's selection theorem, [21, 5.5.2]. $\square$

7. New type of Gross mass

7.1. Gross mass

In this section we work again under the same assumption as Section 6.2: $\boldsymbol{\mu}$ is a choice of density contractors of dimension m. The corresponding Gross measure $\mathscr{G}^m_{\boldsymbol{\mu}}$ is defined in the previous section. Given $T \in \mathscr{R}_m(X,G)$ we define its *Gross mass* as

$$\mathscr{M}_G(T) = \int_{\mathrm{set}_m \|T\|} \big| \boldsymbol{g}(x) \big| \, \mathrm{d}\mathscr{G}^m_{\boldsymbol{\mu}}(x)\,.$$

It follows at once from Theorem 6.4, approximation by simple functions and the monotone convergence theorem that

$$\mathscr{M}_G(T) = \mathscr{M}_H(T)\,. \tag{7.1}$$

We also define

$$\zeta_{\boldsymbol{\mu}}(T) = \sup_{W\in\mathbf{G}_m(X)} \int_{\mathrm{Hom}_m(X,X)} \mathscr{M}_H(\pi_\# T) \mathrm{d}\boldsymbol{\mu}_W(\pi)\,.$$

We recall that integrand in the above formula is Borel measurable, and that

$$\zeta_{\boldsymbol{\mu}}(T) \leqslant \mathscr{M}_H(T)\,, \tag{7.2}$$

both according to Theorem 5.11. We also notice that $\zeta_{\boldsymbol{\mu}}(T_1+T_2) \leqslant \zeta_{\boldsymbol{\mu}}(T_1)+\zeta_{\boldsymbol{\mu}}(T_2)$ and $\zeta_{\boldsymbol{\mu}}(T) = \zeta_{\boldsymbol{\mu}}(-T)$.

Proposition 7.1. — $\zeta_{\boldsymbol{\mu}} : \mathscr{R}_m(X,G) \to \mathbf{R}$ *is lower semicontinuous with respect to* $\mathscr{F}$ *convergence.*

Proof. — Let $T, T_1, T_2, \dots$ be members of $\mathscr{R}_m(X,G)$ such that $\mathscr{F}(T - T_j) \to 0$ as $j \to \infty$. Fix $W \in \mathbf{G}_m(X)$ and $\pi \in \operatorname{Hom}(X,W)$. Since $\mathscr{F}(\pi_\# T - \pi_\# T_j) \to 0$ and $\mathscr{F}(S) = \mathscr{M}_H(S)$ for each $S \in \mathscr{R}_m(X,G)$ we infer that $\mathscr{M}_H(\pi_\# T) = \lim_j \mathscr{M}_H(\pi_\# T_j)$. It thus ensues from Fatou's Lemma that

$$\int_{\operatorname{Hom}_m(X,X)} \mathscr{M}_H(\pi_\# T) \mathrm{d}\boldsymbol{\mu}_W(\pi) \leqslant \liminf_j \int_{\operatorname{Hom}_m(X,X)} \mathscr{M}_H(\pi_\# T_j) \mathrm{d}\boldsymbol{\mu}_W(\pi) .$$

Taking the supremum over $W \in \mathbf{G}_m(X)$ on both sides yields the conclusion. □

Proposition 7.2. — *Let $T \in \mathscr{R}_m(X,G)$ and $0 < \varepsilon < 1$. For $\mathscr{H}^m_{\|\cdot\|}$ almost every $x \in \operatorname{set}_m \|T\|$ there then exist $r(x) > 0$ with the following property. For every $0 < r \leqslant r(x)$ one has*

$$\zeta_{\boldsymbol{\mu}} \left(T \llcorner \mathbf{B}_{\|\cdot\|}(x,r) \right) \geqslant (1-\varepsilon) \mathscr{M}_H \left(T \llcorner \mathbf{B}_{\|\cdot\|}(x,r) \right) .$$

Proof. — Part of the proof is very similar to that of Proposition 6.2. We indicate how to modify the argument and leave out the details. We start with the case when $\operatorname{supp} T \subseteq M$ for some m dimensional submanifold $M \subseteq X$ of class C^1. We restrict to $\mathscr{H}^m_{\|\cdot\|}$ almost every $x \in \operatorname{set}_m \|T\|$ by considering only those which are Lebesgue points of $\left| \boldsymbol{g} \right|$, [13, 2.9.9], *i.e.*

$$0 = \lim_{r \to 0^+} \frac{1}{\boldsymbol{\alpha}(m) r^m} \int_{\mathbf{B}_{\|\cdot\|}(x,r)} \left| \left| \boldsymbol{g}(\xi) \right| - \left| \boldsymbol{g}(x) \right| \right| \mathrm{d}\mathscr{H}^m_{\|\cdot\|} \llcorner M(\xi) .$$

Assuming without loss of generality that $x = 0$ and that $\left| \boldsymbol{g}(0) \right| > 0$ the proof then proceeds as that of Proposition 6.2 with a few changes. We keep the same notation as there. If $\pi \in H_\kappa$ then for each $y \in \pi(M \cap \mathbf{B}_{\|\cdot\|}(0,r))$ there is a unique $\xi = \pi^{-1}\{y\} \cap M \cap \mathbf{B}_{\|\cdot\|}(0,r)$ and one has (here $\boldsymbol{\chi} : \mathscr{R}_0(X,G) \to G$ denotes the augmentation map)

$$\begin{aligned} \boldsymbol{g}_{\pi_\#(T \llcorner \mathbf{B}_{\|\cdot\|}(0,r))}(y) &= \boldsymbol{\chi} \left(\langle T \llcorner \mathbf{B}_{\|\cdot\|}(0,r), \pi, y \rangle \right) \\ &= \boldsymbol{\chi} \left(\boldsymbol{\delta}_\xi \, \boldsymbol{g}_{T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)}(\xi) \right) \\ &= \boldsymbol{g}_{T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)}(\xi) . \end{aligned}$$

Now (6.2) becomes

$$\begin{aligned} (7.3) \quad & \mathscr{M}_{H,|\cdot|} \left(\pi_\# \left(T \llcorner \mathbf{B}_{\|\cdot\|}(0,r) \right) \right) \\ &= \int_{\pi(F(D_r))} \left| \boldsymbol{g}_{\pi_\#(T \llcorner \mathbf{B}_{\|\cdot\|}(0,r))}(y) \right| \mathrm{d}\mathscr{H}^m_{|\cdot|}(y) \\ &= \int_{W \cap D_r} \left| \boldsymbol{g}_{T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)}(F(x)) \right| J_m(\pi \circ F)(x) \mathrm{d}\mathscr{H}^m_{|\cdot|}(x) . \end{aligned}$$

We henceforth abbreviate $\boldsymbol{g} = \boldsymbol{g}_{T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)}$. Combining (7.3) and (6.3), we obtain the following replacement for (6.4)

$$\begin{aligned}
&\left| \mathscr{M}_{H,|\cdot|}\left(\pi_\#\left(T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)\right)\right) - \left\| \boldsymbol{g}(0) \right\| \mathscr{H}^m_{|\cdot|}\left(\pi(D_r)\right) \right| \\
&\leqslant \int_{W\cap D_r} \left| \left\| \boldsymbol{g}(F(x)) \right\| J_m(\pi\circ F)(x) - \left\| \boldsymbol{g}(0) \right\| J_m\pi(x) \right| \mathrm{d}\mathscr{H}^m_{|\cdot|}(x) \\
&\leqslant \int_{W\cap D_r} \left| \left\| \boldsymbol{g}(F(x)) \right\| - \left\| \boldsymbol{g}(0) \right\| \right| J_m(\pi\circ F)(x) \mathrm{d}\mathscr{H}^m_{|\cdot|}(x) \\
&\qquad\qquad + \left\| \boldsymbol{g}(0) \right\| \int_{W\cap D_r} \left| J_m(\pi\circ F)(x) - J_m\pi(x) \right| \mathrm{d}\mathscr{H}^m_{|\cdot|}(x) \\
&= \mathrm{I} + \mathrm{II}\,.
\end{aligned} \tag{7.4}$$

The second term II is bounded from above in the exact same way as in the proof of Proposition 6.2, see (6.5), (6.6) and the lines thereafter:

$$\begin{aligned}
&\int_{H_\kappa} \mathrm{d}\boldsymbol{\mu}(\pi) \int_{W\cap D_r} \left| J_m(\pi\circ F)(x) - J_m\pi(x) \right| \mathrm{d}\mathscr{H}^m_{|\cdot|}(x) \\
&\qquad \leqslant \widehat{\varepsilon}\boldsymbol{c}_{\text{Lemma 6.1}}(n,m)C^2(1-C^2\widehat{\varepsilon})^{-m}(\operatorname{card}\Lambda(n,m))\boldsymbol{\alpha}(m)r^m\,.
\end{aligned} \tag{7.5}$$

In order to bound I from above we use our restriction to 0 being a Lebesgue point of $\| g(\cdot) \|$. Choose $r_3 > 0$ small enough (according to $\widehat{\varepsilon}$ and κ) for

$$\int_{\mathbf{B}_{\|\cdot\|}(0,r)} \left| \left\| \boldsymbol{g}(\xi) \right\| - \left\| \boldsymbol{g}(0) \right\| \right| \mathrm{d}\mathscr{H}^m_{|\cdot|} \llcorner M(\xi) \leqslant \frac{\widehat{\varepsilon}}{2^m\kappa^m}\boldsymbol{\alpha}(m)r^m \tag{7.6}$$

whenever $0 < r \leqslant r_3$. We notice that for $x \in D_r$,

$$J_m(\pi\circ F)(x) \leqslant (\operatorname{Lip}\pi\circ F)^m \leqslant \kappa^m\left(1+\widehat{\varepsilon}\right)^m \leqslant \kappa^m\left(1+\widehat{\varepsilon}\right)^m J_mF(x)\,.$$

We now apply the area formula [13, 3.2.22],

$$\begin{aligned}
&\int_{W\cap D_r} \left| \left\| \boldsymbol{g}(F(x)) \right\| - \left\| \boldsymbol{g}(0) \right\| \right| J_m(\pi\circ F)(x) \mathrm{d}\mathscr{H}^m_{|\cdot|}(x) \\
&\leqslant \kappa^m\left(1+\widehat{\varepsilon}\right)^m \int_{W\cap D_r} \left| \left\| \boldsymbol{g}(F(x)) \right\| - \left\| \boldsymbol{g}(0) \right\| \right| J_mF(x) \mathrm{d}\mathscr{H}^m_{|\cdot|}(x) \\
&= \kappa^m\left(1+\widehat{\varepsilon}\right)^m \int_{M\cap\mathbf{B}_{\|\cdot\|}(0,r)} \left| \left\| \boldsymbol{g}(\xi) \right\| - \left\| \boldsymbol{g}(0) \right\| \right| \mathrm{d}\mathscr{H}^m_{|\cdot|}(\xi) \\
&\leqslant \widehat{\varepsilon}\boldsymbol{\alpha}(m)r^m\,.
\end{aligned} \tag{7.7}$$

Now multiplying both members of (7.4) by $\psi(W)$ it follows from (7.5) and (7.7) that

$$\left| \int_{H_\kappa} \mathscr{M}_{H,\|\cdot\|}\left(\pi_\#\left(T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)\right)\right) \mathrm{d}\boldsymbol{\mu}_W(\pi) \right.$$
$$\left. - \left\lvert \boldsymbol{g}(0) \right\rvert \int_{H_\kappa} \mathscr{H}^m_{\|\cdot\|}\left(\pi(D_r)\right) \mathrm{d}\boldsymbol{\mu}_W(\pi) \right| \leqslant \widehat{\varepsilon}\,\mathbf{C}\boldsymbol{\alpha}(m) r^m \tag{7.8}$$

where

$$\mathbf{C} = C\Big(\boldsymbol{c}_{\text{Lemma 6.1}}(n,m) C^2 (1 - C^2\widehat{\varepsilon})^{-m} (\operatorname{card}\Lambda(n,m)) + \left\lvert \boldsymbol{g}(0) \right\rvert C(1+C^2)^m + 1 \Big).$$

Recalling (6.7) it ensues from (7.8) that

$$\int_{\mathrm{Hom}_m(X,X)} \mathscr{M}_{H,\|\cdot\|}\left(\pi_\#\left(T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)\right)\right) \mathrm{d}\boldsymbol{\mu}_W(\pi)$$
$$\geqslant \left((1-\widehat{\varepsilon})\left(1 - C^2\widehat{\varepsilon}\right)^m - \mathbf{C} \left\lvert \boldsymbol{g}(0) \right\rvert^{-1} \widehat{\varepsilon} \right) \boldsymbol{\alpha}(m) r^m \left\lvert \boldsymbol{g}(0) \right\rvert .$$

Choosing $r_2 > 0$ according to B. Kirchheim's area formula as at the end of the proof of Proposition 6.2 we infer that if also $0 < r \leqslant r_2$ then

$$\begin{aligned}
\mathscr{M}_H\left(T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)\right) &= \int_{\mathbf{B}_{\|\cdot\|}(0,r)} \left\lvert \boldsymbol{g}(\xi) \right\rvert \mathrm{d}\left(\mathscr{H}^m_{\|\cdot\|} \llcorner M\right)(\xi) \\
&\leqslant \left\lvert \boldsymbol{g}(0) \right\rvert \mathscr{H}^m_{\|\cdot\|}\left(M \cap \mathbf{B}_{\|\cdot\|}(0,r)\right) \\
&\qquad + \int_{\mathbf{B}_{\|\cdot\|}(0,r)} \Big| \left\lvert \boldsymbol{g}(\xi) \right\rvert - \left\lvert \boldsymbol{g}(0) \right\rvert \Big| \mathrm{d}\mathscr{H}^m_{|\cdot|} \llcorner M(\xi) \\
&\leqslant (1+\widehat{\varepsilon}) \left\lvert \boldsymbol{g}(0) \right\rvert \boldsymbol{\alpha}(m) r^m + \widehat{\varepsilon} C^m \boldsymbol{\alpha}(m) r^m .
\end{aligned}$$

Letting $r_0 = \min\{r_1/C, r_2, r_3\}$ it becomes clear that

$$\int_{\mathrm{Hom}_m(X,X)} \mathscr{M}_H\left(\pi_\#\left(T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)\right)\right) \mathrm{d}\boldsymbol{\mu}_W(\pi) \geqslant (1-\varepsilon)\mathscr{M}_H\left(T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)\right)$$

whenever $0 < r \leqslant r_0$, provided $\widehat{\varepsilon}$ has been chosen small enough according to ε, n, m and $\left\lvert \boldsymbol{g}(0) \right\rvert$. This readily implies that

$$\zeta_{\boldsymbol{\mu}}\left(T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)\right) \geqslant (1-\varepsilon)\mathscr{M}_H\left(T \llcorner \mathbf{B}_{\|\cdot\|}(0,r)\right)$$

for such $0 < r \leqslant r_0$.

We now turn to the general case. Since T is concentrated on a countably $(\mathscr{H}^m_{\|\cdot\|}, m)$ rectifiable Borel set it follows from [13, 3.2.29] that there exists a disjointed sequence $A_1, A_2, \dots$ of Borel sets such that $T = T \llcorner \cup A_k$ and each A_k is contained in some m dimensional submanifold $M_k \subseteq X$ of class C^1. It suffices to show that conclusion holds for $\mathscr{H}^m_{\|\cdot\|}$ almost every $x \in A_k$, with k fixed. Defining $T_k = T \llcorner A_k$ and $R_k = T - T_k = T \llcorner E_k$ where $E_k = \cup_{j \neq k} A_j$ we claim that for almost every $x \in A_k$ the following holds. Given $0 < \hat{\varepsilon} < 1$ there exists $r(x, \hat{\varepsilon}) > 0$ such that the following three conditions are satisfied for all $0 < r \leqslant r(x, \hat{\varepsilon})$.

(A) $\zeta_{\boldsymbol{\mu}} \left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right) \geqslant (1-\hat{\varepsilon}) \mathscr{M}_H \left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right)$;
(B) $\mathscr{M}_H \left(T \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right) \geqslant (1-\hat{\varepsilon}) \left| \boldsymbol{g}_k(x) \right| \boldsymbol{\alpha}(m) r^m$;
(C) $\mathscr{M}_H \left(R_k \cap \mathbf{B}_{\|\cdot\|}(x,r)\right) \leqslant \hat{\varepsilon} \boldsymbol{\alpha}(m) r^m$.

Indeed (A) follows from the first part of this proof and (B) follows from the fact that $\mathscr{M}_H \left(T \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right) \geqslant \mathscr{M}_H \left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right)$ and that the inequality with T replaced by T_k holds true at small scales according to B. Kirchheim's area formula as in the first part of this proof. Finally we leave it to the reader to check that (C) can be deduced from [13, 2.10.19(4)]. Finally, recalling (7.2) we infer that

$$\begin{aligned}
&\zeta_{\boldsymbol{\mu}} \left(T \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right) \\
&\qquad \geqslant \zeta_{\boldsymbol{\mu}} \left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right) - \zeta_{\boldsymbol{\mu}} \left(R_k \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right) \\
&\qquad \geqslant (1-\hat{\varepsilon}) \mathscr{M}_H \left(T_k \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right) - \mathscr{M}_H \left(R_k \cap \mathbf{B}_{\|\cdot\|}(x,r)\right) \\
&\qquad \geqslant (1-\hat{\varepsilon}) \mathscr{M}_H \left(T \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right) - (2+\hat{\varepsilon}) \mathscr{M}_H \left(R_k \cap \mathbf{B}_{\|\cdot\|}(x,r)\right) \\
&\qquad \geqslant \left(1 - \hat{\varepsilon}\left(1 + \frac{2+\hat{\varepsilon}}{1-\hat{\varepsilon}}\right) \left| \boldsymbol{g}_k(x) \right|^{-1}\right) \mathscr{M}_H \left(T \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right) .
\end{aligned}$$

For such good $x \in A_k$ it should now be clear how to initially choose $\hat{\varepsilon}$ according to ε and x so that the conclusion holds. ☐

THEOREM 7.3. — *Assume that $(X, \|\cdot\|)$ admits density contractors of dimension m and let $\boldsymbol{\mu}$ be a choice of density contractors. It follows that the Gross mass $\mathscr{G}^m_{\boldsymbol{\mu}}$ is lower semicontinuous with respect to $\mathscr{F}$ convergence on $\mathscr{R}_m(X,G)$.*

Proof. — Let $T, T_1, T_2, \dots$ be members of $\mathscr{R}_m(X,G)$ such that $\mathscr{F}(T - T_j) \to 0$ as $j \to \infty$. Given $\varepsilon > 0$ we apply Proposition 7.2 and [11, 5.2.3(2)] to T. It follows that for $\mathscr{H}^m_{\|\cdot\|}$ almost every $x \in \operatorname{set}_m \|T\|$ there exists $r(x) > 0$ with the the property that for $\mathscr{L}^1$ almost every $0 < r \leqslant r(x)$ one has

(A) $(1-\varepsilon)\mathscr{M}_H \left(T \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right) \leqslant \zeta_{\boldsymbol{\mu}} \left((T \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right)$;
(B) $\mathscr{F} \left(T \llcorner \mathbf{B}_{\|\cdot\|}(x,r) - T_j \llcorner \mathbf{B}_{\|\cdot\|}(x,r)\right) \to 0$ as $j \to \infty$.

The collection of those balls $\mathbf{B}_{\|\cdot\|}(x,r)$ for which both conditions (A) and (B) hold thus constitutes a fine covering of a set on which $\|T\|$ concentrates. It follows from [13, 2.8.9 and 2.8.15] that it contains a disjointed countable subcollection $B_k = \mathbf{B}_{\|\cdot\|}(x_k,r_k)$, $k=1,2,\dots$ such that $\mathscr{M}_G(T) = \sum_k \mathscr{M}_G(T \llcorner B_k)$. Therefore

$$(1-\varepsilon)\mathscr{M}_G(T) = (1-\varepsilon)\sum_k \mathscr{M}_G(T \llcorner B_k)$$

$$\leqslant \sum_k \zeta_{\boldsymbol{\mu}}(T \llcorner B_k)$$

(by (A))

$$\leqslant \sum_k \liminf_j \zeta_{\boldsymbol{\mu}}(T_j \llcorner B_k)$$

(by (B) and Proposition 7.1)

$$\leqslant \liminf_j \sum_k \zeta_{\boldsymbol{\mu}}(T_j \llcorner B_k)$$

$$\leqslant \liminf_j \sum_k \mathscr{M}_G(T_j \llcorner B_k)$$

(by (7.2) and (7.1))

$$\leqslant \liminf_j \mathscr{M}_G(T_j)\,.$$

Since $\varepsilon > 0$ is arbitrary the proof is complete. □

8. Mass minimizing chains of dimension 2 or of codimension 1

We are now able to dispense with hypothesis (C) of Theorem 3.1 in case of chains of dimension 2 or codimension 1.

Theorem 8.1. — *Assume that*

- (A) $(X,\|\cdot\|)$ *is a finite dimensional normed space and* $(G, |\cdot|)$ *is an Abelian normed locally compact White group;*
- (B) $m=2$ *or* $m=\dim X-1$*;*
- (D) $B \in \mathscr{R}_{m-1}(X,G)$ *and* $\partial B = 0$.

It follows that the Plateau problem

$$(\mathscr{P})\begin{cases}\textit{minimize } \mathscr{M}_H(T),\\ \textit{among } T \in \mathscr{R}^{\mathrm{loc}}_m(X,G) \textit{ such that } \partial T = B,\end{cases}$$

admits a solution. If one further assumes that

$$0 < \inf\{ \,|\!\!|\, g \,|\!\!|\, : g \in G \text{ and } g \neq 0_G\}$$

then each solution of $(\mathscr{P})$ *has compact support.*

Proof. — In view of Theorem 3.1 it suffices to show that the Hausdorff mass is lower semicontinuous with respect to $\mathscr{F}$ convergence, under assumption (B). We have in fact given two distinct proofs of this. First recall that $(X, \|\cdot\|)$ admits density contractors of dimension m with respect to $\mathscr{H}^m_{\|\cdot\|}$, Theorem 5.10, when $m = 2$ or $m = \dim X - 1$. The first proof then goes as follows: The triangle inequality for cycles is established in Theorem 5.3 and this in turn implies the sought for lower semicontinuity according to Theorem 4.5. The second proof refers to the lower semicontinuity of a Gross mass, Theorem 7.3 and the fact that Hausdorff mass and Gross mass coincide, Section 7.1. □

9. Convex hulls

In this final section we restrict to the case when $m = \dim X - 1$.

PROPOSITION 9.1. — *Assume* $W_1, \dots, W_Q$ *are affine hyperplanes and* $W_1^+, \dots, W_Q^+$ *are half spaces determined by these hyperplanes. Define* $C = \cap_{q=1}^{Q} W_q^+$. *There then exists a Lipschitz map* $f : X \to X$ *with the following properties.*

(1) $f|_C = \mathrm{id}_C$;
(2) *For every* $T \in \mathscr{R}_{\dim X-1}(X, G)$ *one has* $\mathscr{M}_H(f_\# T) \leqslant \mathscr{M}_H(T)$. *Furthermore if* $\operatorname{supp} \partial T \subseteq C$ *then* $\operatorname{supp} f_\# T \subseteq C$ *and* $\partial f_\# T = \partial T$.

Proof. — We each $q = 1, \dots, Q$ we associate a measure contracting projector $\pi_q : X \to X$ on W_q as in Theorem 5.5 (in fact, a translation of those) and we define

$$\rho_q : X \to X : x \mapsto \begin{cases} x & \text{if } x \in W_q^+, \\ \pi_q(x) & \text{if } x \notin W_q^+. \end{cases}$$

It is obvious that each ρ_q is Lipschitz, and so is $f = \rho_Q \circ \rho_{Q-1} \circ \dots \circ \rho_1$. Readily, $f|_C = \mathrm{id}_C$. Given $S \in \mathscr{R}_{\dim X-1}(X, G)$ and $q = 1, \dots, Q$ we observe that

$$\begin{aligned} \rho_{q\#} S &= \rho_{q\#}\Big(S \llcorner (\operatorname{Int} W_q^+)\Big) + \rho_{q\#}\Big(S \llcorner (\operatorname{Int} W_q^+)^c\Big) \\ &= S \llcorner (\operatorname{Int} W_q^+) + \pi_{q\#}\Big(S \llcorner (\operatorname{Int} W_q^+)^c\Big) \end{aligned}$$

according to [11, 5.5.2(1)]. It further follows from Theorem 5.11 (applied with the obvious density contractor $\mu = \delta_{\pi_q}$ on W_q) that

$$\mathscr{M}_H\left(\pi_{q\,\#}\left(S \llcorner \left(\operatorname{Int} W_q^+\right)^c\right)\right) \leqslant \mathscr{M}_H\left(S \llcorner \left(\operatorname{Int} W_q^+\right)^c\right).$$

We conclude that $\mathscr{M}_H(\rho_{q\,\#}S) \leqslant \mathscr{M}_H(S)$.

Let $T \in \mathscr{R}_{\dim X - 1}(X,G)$. We define inductively $T_q = \rho_{q\,\#}T_{q-1}$, $q = 1,\dots,Q$, where $T_0 = T$. Thus $T_Q = f_\# T$. It follows from the preceding paragraph that $\mathscr{M}_H(T_q) \leqslant \mathscr{M}_H(T_{q-1})$ for each q, whence $\mathscr{M}_H(f_\# T) \leqslant \mathscr{M}_H(T)$. In the remaining part of this proof we assume that $\operatorname{supp}\partial T \subseteq C$. We will establish by induction that for every $q = 0,1,\dots,Q$ the following hold:

- $(A)_q$ $\partial T_q = \partial T$;
- $(B)_q$ $\operatorname{supp} T_q \subseteq \cap_{j=1}^q W_j^+$.

Condition $(A)_0$ is trivially true. Assuming $(A)_{q-1}$ holds we notice that $\partial T_q = \partial \rho_{q\,\#}T_{q-1} = \rho_{q\,\#}\partial T_{q-1} = \rho_{q\,\#}\partial T$. Since $\operatorname{supp}\partial T \subseteq C \subseteq W_j^+$ and $\rho_q|_{W_q^+} = \mathrm{id}_{W_q^+}$ it follows that $\partial T_q = \rho_{q\,\#}\partial T = \partial T$ according to [11, 5.5.2(1)].

Condition $(B)_1$ is verified because $\operatorname{supp} T_1 = \operatorname{supp}\rho_{1\,\#}T \subseteq \rho_1(X) = W_1^+$. We now assume $(B)_{q-1}$ holds true and we prove $(B)_q$. Abbreviate $E_q = \cap_{j=1}^q W_j^+$. One easily checks that

$$\rho_q(E_{q-1}) \subseteq W_q \cup E_q\,.$$

It then ensues from $(B)_{q-1}$ that $\operatorname{supp} T_q \subseteq W_q \cup E_q$ and it remains to show that $T_q' := T_q \llcorner (W_q \sim E_q) = 0$. Let U be a component of $W_q \sim E_q$. It is open, and unbounded (if $x \in U$ and $L \subseteq W_q$ is a line through x, then $L \cap E_q$ is convex, hence an interval and consequently one of the lines in L starting at x is included in U). Since $\operatorname{supp}\partial T_q \subseteq C \subseteq E_q$ according to $(A)_q$ we infer that $(\partial T_q') \llcorner U = 0$. It follows from the constancy theorem [12, Theorem 6.4] that $T_q' = g[\![U]\!]$ for some $g \in G$ and where an orientation of W_q has been chosen. Since $\operatorname{supp} T_q' \subseteq \operatorname{supp} T_q$ is compact and since U is unbounded it follows that $g = 0$. Accordingly $T_q' = 0$ and the proposition is proved. □

In the following $\operatorname{Conv}(\operatorname{supp} B)$ denotes the convex hull of $\operatorname{supp} B$.

Theorem 9.2. — *Assume that*

- *(A)* $(X, \|\cdot\|)$ *is a finite dimensional normed space and* $(G, |\cdot|)$ *is an Abelian normed locally compact White group;*
- *(D)* $B \in \mathscr{R}_{m-1}(X,G)$ *and* $\partial B = 0$.

It follows that the Plateau problem

$$(\mathscr{P})\begin{cases}\textit{minimize } \mathscr{M}_H(T),\\ \textit{among } T\in\mathscr{R}_{\dim X-1}(X,G) \textit{ such that } \partial T=B,\end{cases}$$

admits a minimizer T such that $\operatorname{supp} T\subseteq \operatorname{Conv}(\operatorname{supp} B)$.

Proof. — Let $C=\operatorname{Conv}(\operatorname{supp} B)$. For each $x\in \operatorname{Bdry} C$ there exists a half space $W_x^+\supseteq C$ according to Hahn's theorem. The separability of X^* guarantees that $C=\cap_{x\in D}W_x^+$ for some countable subset $D\subseteq \operatorname{Bdry} C$. Choose a numbering $D=\{x_1,x_2,\dots\}$. Choose also a compact convex polytope C_0 such that $C\subseteq C_0$. For each $n=1,2,\dots$ define $C_n=C_0\cap\left(\cap_{j=1}^n W_{x_j}^+\right)$. Given a minimizing sequence $T_1,T_2,\dots$ for problem $(\mathscr{P})$ we apply Proposition 9.1 to C_n and T_k. We obtain $T_{n,k}=f_{n\,\#}T_k$ and we notice that $T_{n,1},T_{n,2},\dots$ is another minimizing sequence with $\operatorname{supp} T_{n,k}\subseteq C_n$ for each $k=1,2,\dots$. As $C_n\subseteq C_0$ is compact, the compactness theorem guarantees the existence of a subsequence converging in the flat norm $\mathscr{F}$ to some $\widehat{T}_n\in\mathscr{R}_{\dim X-1}(X,G)$ with $\operatorname{supp}\widehat{T}_n\subseteq C_n$. Since $\mathscr{M}_H$ is $\mathscr{F}$ lower semicontinuous in codimension 1 we also infer that $\mathscr{M}_H(\widehat{T}_n)=\inf(\mathscr{P})$. Applying the compactness theorem to the sequence $\widehat{T}_1,\widehat{T}_2,\dots$ we obtain a subsequence $\widehat{T}_{\varphi(1)},\widehat{T}_{\varphi(2)},\dots$ converging in the flat norm to some $T\in\mathscr{R}_{\dim X-1}(X,G)$ such that $\partial T=B$ and $\mathscr{M}_H(T)=\inf(\mathscr{P})$. Finally $\operatorname{supp} T\subseteq C_{\varphi(n)}$ for every $n=1,2,\dots$ and since the sequence $C_1,C_2,\dots$ is decreasing we conclude that $\operatorname{supp} T\subseteq C$. ☐

Example 9.3. — We close this paper by observing that under the assumption of Theorem 9.2 there may also exist minimizers T of problem $(\mathscr{P})$ such that $\operatorname{supp} T\not\subseteq \operatorname{Conv}(\operatorname{supp} B)$. Consider $X=\ell_\infty^N$, $A=[-1,1]^{N-1}\subseteq\mathbf{R}^{N-1}$ and any $f:A\to\mathbf{R}$ with $|f(x)-f(x')|\leqslant\|x-x'\|_\infty$ for every $x,x'\in A$ and $f(x)=0$ whenever $x\in\operatorname{Bdry} A$. Define $F:\mathbf{R}^{N-1}\to\mathbf{R}^N$ by $F(x)=(x,f(x))$. Let also $f_0=0$ and $F_0=0$. The key point is that since $\operatorname{Lip} f\leqslant 1$ one has $\mathscr{H}^{N-1}_{\|\cdot\|_\infty}(F(A))=\mathscr{H}^{N-1}_{\|\cdot\|_\infty}(F_0(A))=\boldsymbol{\alpha}(N-1)$ independently of f, as the happy reader will easily check. Then for any G, any $g\in G\sim\{0\}$, letting $T_0=g[\![A]\!]\in\mathscr{R}_{N-1}(X,G)$ and $T=F_\#T_0$ we see that $\mathscr{M}_H(T)=\mathscr{M}_H(T_0)$ and $\partial T=\partial T_0$. Among those T only one, namely T_0, is supported in the convex hull of the support of its boundary.

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