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Asymptotic behavior for textiles with loose contact

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The paper is dedicated to the modeling of the elasticity problem for a textile structure. The textile is made of long and thin fibers, crossing each other in a periodic pattern, forming a woven canvas of a square domain. The textile is partially clamped. The fibers cannot penetrate each other but can slide with respect to each other in the in-plane directions. The sliding is bounded by a contact function, which is chosen loose. The partial clamp and the loose contact lead to a domain partitioning, with different expected behaviors on each of the four subdomains. The homogenization is made via the periodic unfolding method, with an additional dimension reduction. The macroscopic limit problem results in a Leray–Lions problem with only macroconstraints in the plane.

KEYWORDS

contact, dimension reduction, homogenization, Leray–Lions problem, linear elasticity, non-penetration condition, periodic unfolding method, plates, structure of beams, variational inequality

MSC CLASSIFICATION

35B27, 35J50, 47H05, 74B05, 74K10, 74K20

1 | INTRODUCTION

This work is dedicated to the homogenization and dimension reduction of an elasticity problem of a textile structure via the periodic unfolding method. For the homogenization in elasticity, we first refer to Oleinik et al. [1], and for the periodic unfolding, also for elasticity, to previous studies [2–5]. For decompositions and limits for dimension reduction of plates and beams, we refer to previous studies [6–11]. The combination of homogenization and dimension reduction can be found in Chapter 11 of Cioranescu et al. [4] and in Griso et al. [5].

We refer to the homogenization of a periodically perforated shell to Griso et al. [12], of a textile structure made of beams glued to each other, to Griso et al. [13], and to three-dimensional periodic lattice structures made of beams to previous studies [14–18].

We would like to mention works in analysis, homogenization and numerical analysis of periodic structures with a sliding contact on the structural components with Tresca and Coulomb friction [19–22].

Finally, we refer to the simulation works for textiles with a contact sliding [21–24], which inspired this work. Here, we consider a textile structure made of long beams. These beams are not glued to each other, and we allow for their small in-plane sliding. This fact prohibits their extension to a perforated shell. The sliding feature due to contact was already taken into account in Griso et al. [25] in a stricter way. There, the sliding was very small, and the associated energy dissipation was smaller than the elastic energy of the beams. In this paper, we allow the beams to slide more (hence the name "loose contact"), and the dissipation is visible. The sliding feature raised the problem's complexity level, compared to Griso et al. [25], and led to a set of issues or intermediate tasks for which new ideas and mathematical tools were needed. We first developed such tools and illustrated them on simple examples in Falconi et al. [17, 18]. Now, the mathematical investigation of the textile problem becomes possible, and we can describe the effective textile behavior.

Similar to Griso et al. [25], after some notations in Section 2, in Section 3, we describe a structure consisting of long oscillating beams of length L with a small square cross-section of width $2r$. We first use the decomposition of displacements, proposed in Griso [9], to adopt this decomposition to the curved beams. And then, we propose the decomposition of the long-beam displacements. In Section 4, we proceed to the construction of a woven canvas in the square $\Omega = (0, L)^2$, a structure $\mathcal{S}_\varepsilon$, as depicted in Figure 1. We set a small parameter ε and put the long beams ε-periodically in direction $\mathbf{e}_1$ and $\mathbf{e}_2$. For the sake of simplicity, we assume that the cross-section and the distance between fibers are asymptotically related ($r \sim \varepsilon$).

For every displacement on the textile structure $u_\varepsilon \in H^1(\mathcal{S}_\varepsilon)$, we set the following constraints:

(i) Clamp conditions: on a partial segment of the left boundary, the displacement in direction $\mathbf{e}_1$ vanishes, while on a partial segment of the bottom boundary, the displacement in direction $\mathbf{e}_2$ vanishes;
(ii) In-plane contact conditions: the in-plane sliding is bounded by a gap function $g_\varepsilon = \varepsilon^2 g$, where $g \in C(\overline{\Omega})$.
(iii) Outer-plane non-penetration condition: in the outer-plane component, the displacements are not allowed to penetrate each other.

In condition (ii), order 2 with respect to ε shows the strength of the contact. The exponents of order 3 and higher have already been studied in Griso et al. [25]. Order 1 is not interesting for the applications since the yarns no longer hold as a textile shell. The combination of the loose contact and the partial clamp leads to a partition of the domain in the subdomains, Ω_1–Ω_4 (see Figure 1), where we expect the displacement to behave differently. In Section 5, we define the set of admissible displacements:

$$\mathcal{X}_\varepsilon = \left\{ v_\varepsilon \in H^1(\mathcal{S}_\varepsilon) \,|\, v_\varepsilon \text{ satisfies conditions (i)–(iii)} \right\}.$$

Due to condition (ii), the elasticity problem is set via variational inequality, similar as in previous studies [19, 20, 25]:

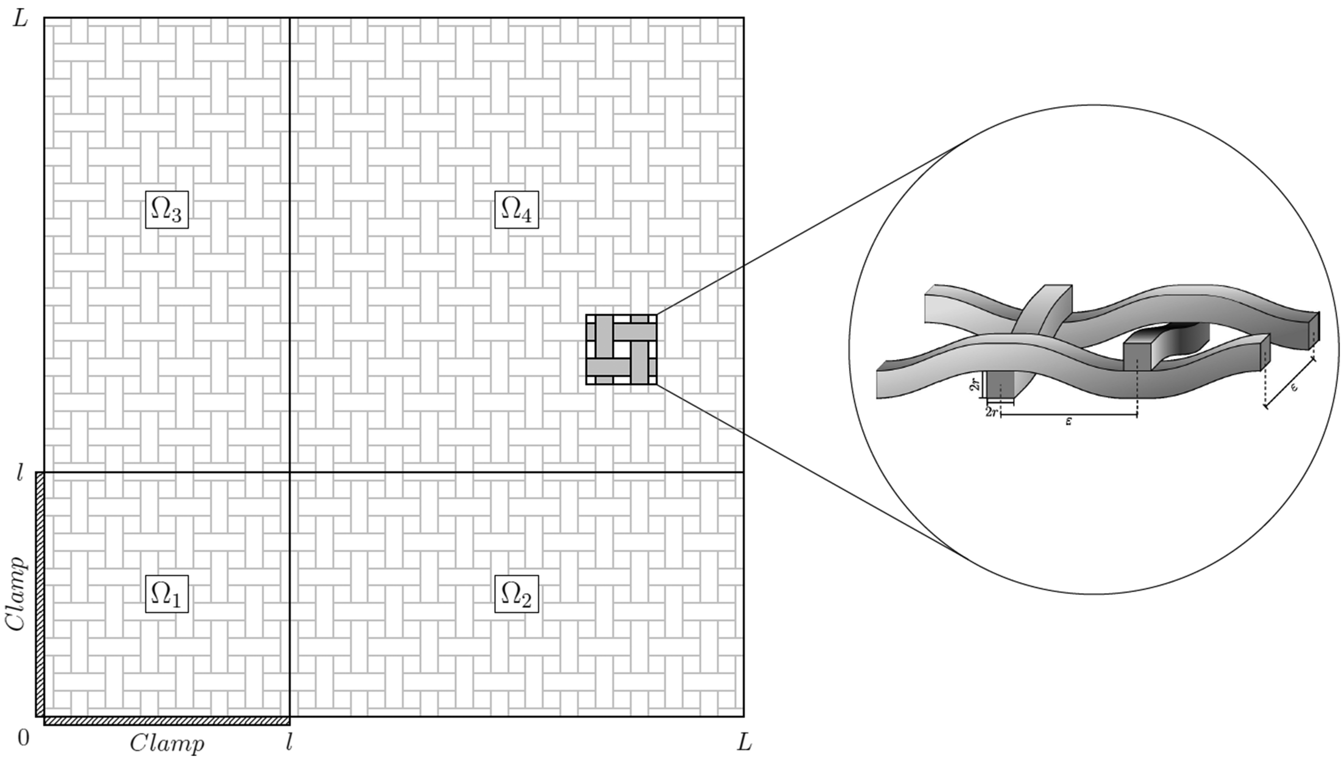

FIGURE 1 The textile domain partition according to the partial clamp on the left and bottom boundary. Each cell has a 2ε periodic pattern. The distance between fibers is ε, and their cross-section is $r \sim \varepsilon$. The complete structure can be seen as almost 2D.

$$\text{Find } u_\varepsilon \in \mathcal{X}_\varepsilon \text{ such that for every } v_\varepsilon \in \mathcal{X}_\varepsilon : \int_{\mathcal{S}_\varepsilon} a_{ijkl,\varepsilon}\, e_{ij}(u_\varepsilon) e_{kl}(u_\varepsilon - v_\varepsilon)\, dx \le \int_{\mathcal{S}_\varepsilon} f_\varepsilon \cdot (u_\varepsilon - v_\varepsilon)\, dx, \tag{1}$$

where a_ε is the fourth-order strain tensor, describing the material law, and f_ε is the applied stress. The problem itself admits solution by the Stampacchia lemma (see Kinderlehrer and Stampacchia [26]), a version of the Lax–Milgram lemma, formulated for closed convex subsets of Hilbert spaces. Uniqueness is ensured by (i). Before going to the limit, we show the compactness of sequences solving (1) in Section 6. Here, we first use the beam decomposition, clamping condition on one of the extremities, and the Poincaré inequalities. Then, we extend these results to the non-clamped subdomains, using contact conditions and the Trace theorem. In this frame, we want to draw attention to the improved estimate, (32), for the outer-plane direction. Here, we skip the requirement for the gap in the third direction to be bounded from above (present in Griso et al. [25]), since we will later prove that we can obtain sufficiently good estimates by the mere oscillatory manner of fibers. Estimates from Section 6 suggest a better decomposition of the displacements, which we use to pass to the limit in Section 7. Here, we find bounds to the new fields, apply the unfolding, and find the limit convergences. As in Griso et al. [25], we choose the applied forces on the right-hand side of the problem so to keep the stain tensor in a linear elasticity regime, where deformations are small:

$$\|e(u_\varepsilon)\|_{L^2(\mathcal{S}_\varepsilon)} \sim \varepsilon^{5/2}.$$

Then, compactness results ensure the weak convergence of the displacement fields. The convergence via unfolding for these fields is not straightforward: It is based on the extension of the periodic unfolding from Cioranescu et al. [2, 4] to anisotropically bounded functions (see Falconi et al. [18]) and to lattice domains (see Falconi et al. [17]). We define three operators:

$\mathcal{T}_\varepsilon^{\mathfrak{G}}$: which is the most important, unfolds the middle line of displacement fields and gives all the limit fields;
Π_ε: which extends fields to the whole three-dimensional textile structure and is used to express the limit strain tensor;
$\mathcal{T}_\varepsilon^{\mathbf{C}}$: which gives the unfolding of functions in the contact domains and is used for getting the limit contact conditions.

Once the displacement fields' weak limits, the strain tensors' form, and the contact conditions are found, we can define the limit set of admissible displacements $\mathcal{X}$. However, in order to go to the limit with problem (1), we need the strong convergence of the test functions in the variational formulation. Their construction is a matter of Section 8, where we require them to meet the following requirements:

- ensure strong convergence via unfolding;
- satisfy contact conditions in original and in the limit;
- yield the same strain tensor in the limit.

In Section 9, we employ the results of the previous sections to go to the limit via unfolding with problem (1) for $\varepsilon \to 0$. The limit problem (93) is derived, and the Stampacchia lemma again ensures the existence of its solution. According to the procedure in Chapter 5.6 of Cioranescu et al. [4], we split the microscopic scale from the macroscopic one. The correctors, solutions of the cell problems, are just coupled in the third direction. Such coupling is reasoned by the limit non-penetration condition and preserves the inequality-kind in the microscopic scale. Analogously, the only macroscopic in-plane contact conditions preserve the inequality of the macroscopic limit problem. This result differs from Griso et al. [25], where the coupling concerned all three directions and the inequality was present only in the microscopic scale. The macroscopic problem appears in a Leray–Lions form (see [27]).

Finally, we make the following conclusions:

- the approximation of the initial displacements, according to the limit fields, gives an idea of how the displacements behave in the different textile subdomains;
- in the limit, as a consequence of the presence of the contact function g, uniqueness is not preserved on both scales;
- the limit contact constraints bound the in-plane rotations in subdomains with non-clamped parts of yarns. The obtained textile structure behavior is similar to that of the right Figure 2.

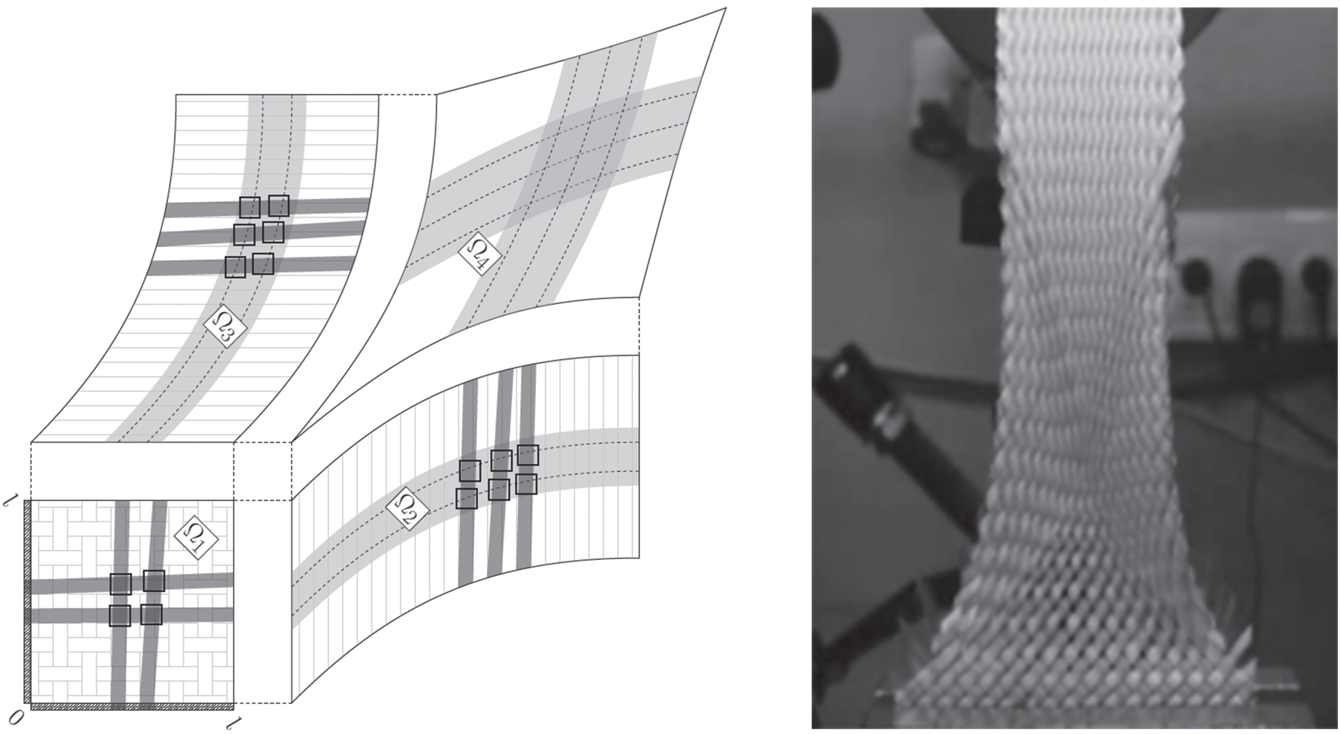

FIGURE 2 Left figure gives a sketch obtained by analyzing yarn's deformations in each textile part. On the right-hand side, a real experiment for the tension of the textile with 45° with respect to the yarn directions.

The investigations done here and in Griso et al. [25] show the impact of sliding between fibers according to the strength (from glued to moderately loose) on the effective textile plate behavior in the frame of linear elasticity.

Forthcoming works will aim to extend the sliding features to nonlinear regimes. For now, we can only recommend Griso et al. [13], the study of an elasticity problem in von-Kármán regime on a textile structure with glued fibers.

2 | NOTATIONS

Throughout the paper, the following notations will be used:

- $\Omega \doteq (0,L)^2$, $l > 0$ is a constant. For simplicity we assume that $\frac{L}{l} = \frac{a}{b}$ where a and b are integers such that $(a,b) = 1$;
- $\varepsilon \in \mathbb{R}$ is a small parameter such that $2\varepsilon N_\varepsilon = L$ and $2\varepsilon n_\varepsilon = l$, $N_\varepsilon = k_\varepsilon a$, $n_\varepsilon = k_\varepsilon b$ where k_ε is an integer;
- $\mathcal{K}_\varepsilon \doteq \{(p,q) \in \mathbb{N}\times\mathbb{N} \mid (p\varepsilon, q\varepsilon) \in \overline{\Omega}\} = \{0, \dots, 2N_\varepsilon\}^2$ is the set of nodes;
- κ is a fixed constant and r is a parameter related to ε via $r \doteq \kappa\varepsilon$;
- $\omega_\kappa \doteq (-\kappa,\kappa)^2$ is the reference beam cross-section, while the rescaled one is $\omega_r \doteq (-r,r)^2 = (-\kappa\varepsilon, \kappa\varepsilon)^2$;
- $\phi_i \doteq \phi \cdot \mathbf{e}_i$ for $i \in \{1,2,3\}$;
- $z' \doteq (z_1, z_2) \in \mathbb{R}^2$, $\partial_i \doteq \frac{\partial}{\partial z_i}$ denote the partial derivative w.r. to z_i, $i \in \{1,2\}$;
- we denote by $[z] \in \mathbb{Z}^2$ the integer part and $z \in (0,1)^2$ the fractional part given by the unique decomposition of $z \in \mathbb{R}^2$;
- $(\alpha,\beta) \in \{1,2\}^2$ and $(a,b,c) \in \{0,1\}^3$ (if not specified);
- C is a real strictly positive constant independent of ε (if not specified).

This paper uses the Einstein convention of summation over repeated indexes.

3 | PRELIMINARIES

In this section, we give the main results concerning the basic element that will later form the textile structure: a strongly oscillating beam. We will describe it with a mathematical model and define a displacement over it. We will decompose this displacement in a suitable way for the problem we are going to study and define the associated strain tensor. These results have already been proven in Section 3 of Griso et al. [25].

3.1 Parameterization of the oscillating beam

We start by defining the 2-periodic function

$$\Phi(t) \doteq \begin{cases} -\kappa & \text{if } t \in [0,\kappa], \\ \kappa\left(6\dfrac{(t-\kappa)^2}{(1-2\kappa)^2} - 4\dfrac{(t-\kappa)^3}{(1-2\kappa)^3} - 1\right) & \text{if } t \in [\kappa, 1-\kappa], \\ \kappa & \text{if } t \in [1-\kappa, 1], \\ \Phi(2-t) & \text{if } t \in [1,2] \end{cases} \tag{2}$$

and we rescale it to a 2ε-periodic function setting $\Phi_\varepsilon(t) = \varepsilon\Phi\left(\frac{t}{\varepsilon}\right)$, which is piece-wise $C^2(\mathbb{R})$ and overall $C^1(\mathbb{R})$. By definition, such a function satisfies

$$\varepsilon^2\|\partial_1^2\Phi_\varepsilon\|_{L^\infty(\mathbb{R})} + \varepsilon\|\partial_1\Phi_\varepsilon\|_{L^\infty(\mathbb{R})} + \|\Phi_\varepsilon\|_{L^\infty(\mathbb{R})} \le C\varepsilon.$$

When we deal with strongly oscillating beams, the centerline of a beam is parameterized by the function

$$M_\varepsilon(z_1) \doteq z_1\mathbf{e}_1 + \Phi_\varepsilon(z_1)\mathbf{e}_3, \qquad z_1 \in [0, L].$$

This curve has mean direction in $\mathbf{e}_1$ and oscillations in direction $\mathbf{e}_3$. We refer the beam to a mobile reference frame (Frenet–Serret), denoted by $(\mathbf{t}_\varepsilon, \mathbf{e}_2, \mathbf{n}_\varepsilon)$ and defined by

$$\mathbf{t}_\varepsilon \doteq \frac{\partial_1 M_\varepsilon}{|\partial_1 M_\varepsilon|} = \frac{1}{\boldsymbol{\gamma}_\varepsilon}(\mathbf{e}_1 + \partial_1\Phi_\varepsilon\mathbf{e}_3), \qquad \mathbf{n}_\varepsilon \doteq \mathbf{t}_\varepsilon \wedge \mathbf{e}_2 = \frac{1}{\boldsymbol{\gamma}_\varepsilon}(-\partial_1\Phi_\varepsilon\mathbf{e}_1 + \mathbf{e}_3), \tag{3}$$

where $\boldsymbol{\gamma}_\varepsilon \doteq \sqrt{1+(\partial_1\Phi_\varepsilon)^2}$. The unit vector fields $\mathbf{t}_\varepsilon$ and $\mathbf{n}_\varepsilon$ belong to $C^1([0,L])^3$. Their derivatives are

$$\frac{d\mathbf{t}_\varepsilon}{dz_1} = \mathbf{c}_\varepsilon\boldsymbol{\gamma}_\varepsilon\mathbf{n}_\varepsilon, \qquad \frac{d\mathbf{n}_\varepsilon}{dz_1} = -\mathbf{c}_\varepsilon\boldsymbol{\gamma}_\varepsilon\mathbf{t}_\varepsilon,$$

where the piece-wise continuous function $\mathbf{c}_\varepsilon(z_1) \doteq \frac{\partial_1^2\Phi_\varepsilon(z_1)}{\boldsymbol{\gamma}_\varepsilon^3(z_1)}$ is the curvature. We denote

$$\mathbf{C}_\varepsilon \doteq \begin{pmatrix} \mathbf{t}_\varepsilon & \mathbf{e}_2 & \mathbf{n}_\varepsilon \end{pmatrix} \in SO(3),$$

the basis transformation matrix from the fixed frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to the mobile one $(\mathbf{t}_\varepsilon, \mathbf{e}_2, \mathbf{n}_\varepsilon)$. We set the straight reference beam of length L and cross-section ω_r

$$P_r \doteq (0,L) \times \omega_r.$$

The oscillating beam results to be

$$\mathcal{P}_\varepsilon \doteq \psi_\varepsilon(P_r),$$

where the function $\psi_\varepsilon : [0,L] \times \mathbb{R}^2 \to \mathbb{R}^3$ is the transition map from the straight to an oscillating beam, it is defined by

$$\psi_\varepsilon(z_1, y_2, y_3) \doteq M_\varepsilon(z_1) + y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon(z_1).$$

We have

$$\nabla\psi_\varepsilon = \begin{pmatrix} \partial_1\psi_\varepsilon & \partial_{y_2}\psi_\varepsilon & \partial_{y_3}\psi_\varepsilon \end{pmatrix} = \mathbf{C}_\varepsilon \begin{pmatrix} \eta_\varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

where η_ε is the Jacobian for the changing of coordinates

$$\eta_\varepsilon(z_1, y_2, y_3) \doteq \det(\nabla\psi_\varepsilon(z_1, y_2, y_3)) = \boldsymbol{\gamma}_\varepsilon(z_1)(1 - y_3\mathbf{c}_\varepsilon(z_1)), \ \forall (z_1, y_2, y_3) \in \overline{P_r}.$$

Remark A.1 of Griso et al. [25] shows that if $\kappa \in (0, 1/3]$, then the Jacobian η_ε of ψ_ε is bounded from below and above. Therefore, the transformation ψ_ε from P_r onto $\mathcal{P}_\varepsilon$ results to be a diffeomorphism. In particular, there exist two constants C_0, C_1 such that for every $\phi \in L^2(\mathcal{P}_\varepsilon)$:

$$C_0\|\phi\circ\psi_\varepsilon\|_{L^2(P_r)} \le \|\phi\|_{L^2(\mathcal{P}_\varepsilon)} \le C_1\|\phi\circ\psi_\varepsilon\|_{L^2(P_r)}. \tag{5}$$

This means that the L^2 estimates for a function computed on the straight beam and the estimates computed on the oscillating one will only differ by a constant.

From now on, we will simply denote ϕ the function $\phi\circ\psi_\varepsilon$.

3.2 | Decomposition of a beam displacement

Let $u \in H^1(\mathcal{P}_\varepsilon)^3$ be a displacement. From Theorem 3.1 of Griso [11] and Lemma 3.2 of Griso [10], we have the following decomposition for a curved beam:

$$u = U_{e\ell} + \overline{u}, \qquad \text{a.e. in } \mathcal{P}_\varepsilon \text{ or equivalently in } P_r. \tag{6}$$

The first quantity $U_{e\ell} \in H^1(P_r)^3$ is called elementary displacement, and it is defined by

$$U_{e\ell}(z_1, y_2, y_3) \doteq \underbrace{\mathbb{U}(z_1) + \mathcal{R}(z_1) \wedge \Phi_\varepsilon(z_1)\mathbf{e}_3}_{\text{middle line displacement}} + \mathcal{R}(z_1) \wedge (y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon(z_1)),$$

where the fields $\mathbb{U}$ and $\mathcal{R}$ belong to $H^1(0, L)^3$. It consists of the beam's middle line displacement and the cross-section's rotation, according to the vector $\mathcal{R}$. The further split of the middle line displacement is done in the same fashion as Section 3.3 of Griso et al. [25], and its main purpose is to simplify the estimates of the fields that appear in the form of the symmetric gradient associated with the elementary displacement. In this sense, we have

$$\|\partial_1\mathcal{R}\|_{L^2(0,L)} \le \frac{C}{\varepsilon^2}\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \ \|\partial_1\mathbb{U} - \mathcal{R} \wedge \mathbf{e}_1\|_{L^2(0,L)} \le \frac{C}{\varepsilon}\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \tag{7}$$

where C only depends on κ.

The second quantity $\overline{u} \in H^1(P_r)^3$ is called warping, and it consists of the remainder term of the displacement. It satisfies (see also Griso [11])

$$\int_{\omega_r} \overline{u}(z_1, y_2, y_3)dy_2dy_3 = 0, \qquad \int_{\omega_r} \overline{u}(z_1, y_2, y_3) \wedge (y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon(z_1))\, dy_2dy_3 = 0, \qquad \text{for a.e. } z_1 \in (0, L),$$

and the estimates

$$\|\overline{u}\|_{L^2(\mathcal{P}_\varepsilon)} \le C\varepsilon\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \qquad \|\nabla_x\overline{u}\|_{L^2(\mathcal{P}_\varepsilon)} \le C\|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}. \tag{8}$$

We also remind that if a beam is clamped at one extremity, for example, $z_1 = 0$, then

$$\mathbb{U}(0) = \mathcal{R}(0) = 0, \qquad \overline{u} = 0 \text{ a.e. in } \{0\} \times \omega_r. \tag{9}$$

In order to simplify the problem we are going to study, we want to define a new decomposition of the displacement that better suits our purposes. In order to be sure that the new decomposition is close enough to the old one, we briefly recall some results about the approximation of functions by linear and cubic interpolations.

Remark 1. The map $X \in \mathbb{R}^3 \mapsto X + \mathcal{R}(z_1) \wedge X$ (z_1 fixed) represents a “small rotation” of the cross-section $\{z_1\} \times \omega_r$, whose axis is directed by $\mathcal{R}(z_1)$ and whose angle is approximately $|\mathcal{R}(z_1)|$.

3.3 | Preliminary results

Let $\mathbf{A} = (A_0, \dots, A_{2N_\varepsilon})$ and $\mathbf{B} = (B_0, \dots, B_{2N_\varepsilon})$ be two vectors in $\mathbb{R}^{2N_\varepsilon+1}$. We define the functions $\phi_{\mathbf{A},\mathbf{B}} \in W^{2,\infty}(0,L)$ and $\psi_{\mathbf{B}} \in W^{1,\infty}(0,L)$ by

$$\begin{aligned}
\phi_{\mathbf{A},\mathbf{B}}(z_1) =& A_p \left(\frac{2z_1-(2p-1)\varepsilon}{\varepsilon}\right)\left(\frac{z_1-(p+1)\varepsilon}{\varepsilon}\right)^2 + A_{p+1}\left(\frac{(3+2p)\varepsilon-2z_1}{\varepsilon}\right)\left(\frac{z_1-p\varepsilon}{\varepsilon}\right)^2 \\
&+ \frac{(z_1-p\varepsilon)(z_1-(p+1)\varepsilon)}{\varepsilon^2}\left(B_{p+1}(z_1-p\varepsilon)+B_p(z_1-(p+1)\varepsilon)\right), \\
\psi_{\mathbf{B}}(z_1) =& \left(B_{p+1}\frac{z_1-p\varepsilon}{\varepsilon}+B_p\frac{(p+1)\varepsilon-z_1}{\varepsilon}\right) + \frac{(z_1-p\varepsilon)(z_1-(p+1)\varepsilon)(2z_1-(2p+1)\varepsilon)}{\varepsilon^3}\left(\mathbf{B}_{p+1}-B_p\right), \\
&\forall z_1 \in [p\varepsilon,(p+1)\varepsilon], \qquad \forall p \in \{0,\dots,2N_\varepsilon-1\}.
\end{aligned}$$

Note that the function $\phi_{\mathbf{A},\mathbf{B}}$ represents the cubic interpolation of a function with values $\mathbf{A}$ on the nodes and first-order derivative with values $\mathbf{B}$ on the nodes. A straightforward calculation gives

$$\begin{aligned}
\partial_1\phi_{\mathbf{A},\mathbf{B}}(z_1) = \left(B_{p+1}\frac{z_1-p\varepsilon}{\varepsilon}+B_p\frac{(p+1)\varepsilon-z_1}{\varepsilon}\right) + 6\frac{(z_1-p\varepsilon)((p+1)\varepsilon-z_1)}{\varepsilon^2}\left(\frac{A_{p+1}-A_p}{\varepsilon}-\frac{B_{p+1}+B_p}{2}\right), \\
\forall z_1 \in [p\varepsilon,(p+1)\varepsilon], \qquad \forall p \in \{0,\dots,2N_\varepsilon-1\}.
\end{aligned}$$

Lemma 1. *We have the following estimates:*

$$\begin{aligned}
&\|\phi_{\boldsymbol{A},\boldsymbol{B}}\|^2_{L^2(0,L)} \le C\varepsilon\left(\sum_{p=0}^{2N_\varepsilon}(A_p^2+\varepsilon^2B_p^2)+\sum_{p=0}^{2N_\varepsilon-1}\varepsilon^2\left|\frac{A_{p+1}-A_p}{\varepsilon}-\frac{B_{p+1}+B_p}{2}\right|^2\right), \qquad \|\psi_{\boldsymbol{B}}\|^2_{L^2(0,L)} \le C\varepsilon\sum_{p=0}^{2N_\varepsilon}|B_p|^2, \\
&\|\partial_1\phi_{\boldsymbol{A},\boldsymbol{B}}\|^2_{L^2(0,L)} \le C\varepsilon\left(\sum_{p=0}^{2N_\varepsilon}B_p^2+\sum_{p=0}^{2N_\varepsilon-1}\left|\frac{A_{p+1}-A_p}{\varepsilon}-\frac{B_{p+1}+B_p}{2}\right|^2\right), \qquad \|\partial_1\psi_{\boldsymbol{B}}\|^2_{L^2(0,L)} \le C\varepsilon\sum_{p=0}^{2N_\varepsilon}\left|\frac{B_{p+1}-B_p}{\varepsilon}\right|^2, \\
&\|\partial^2_{11}\phi_{\boldsymbol{A},\boldsymbol{B}}\|^2_{L^2(0,L)} \le \frac{C}{\varepsilon}\sum_{p=0}^{2N_\varepsilon-1}\left(|B_{p+1}-B_p|^2+\left|\frac{A_{p+1}-A_p}{\varepsilon}-\frac{B_{p+1}+B_p}{2}\right|^2\right).
\end{aligned}$$

3.4 | The decomposition of a beam displacement via a Bernoulli–Navier displacement

In this subsection, we again decompose every displacement as a sum of a Bernoulli–Navier displacement and a residual one (warping). This new decomposition includes equalities that must otherwise be proven in the limit and simplifies the way to obtain estimates and later asymptotic behaviors of the displacements.

Let u be a displacement in $H^1(\mathcal{P}_\varepsilon)^3$ decomposed as (6) and recall the 3-vector fields $\mathbb{U}$ and $\mathcal{R}$. We define $\mathbb{U}'$ and $\mathcal{R}'$ by

$$\begin{aligned}
&\mathbb{U}'_1(z_1) = \mathbb{U}_1((p+1)\varepsilon)\frac{z_1-p\varepsilon}{\varepsilon}+\mathbb{U}_1(p\varepsilon)\frac{(p+1)\varepsilon-z_1}{\varepsilon}, \\
&\mathbb{U}'_2(z_1) = \phi_{\mathbf{A},\mathbf{B}} \text{ with } \mathbf{A} = (\mathbb{U}_2(0),\dots,\mathbb{U}_2(2N_\varepsilon\varepsilon)), \ \mathbf{B} = -(\mathcal{R}_3(0),\dots,\mathcal{R}_3(2N_\varepsilon\varepsilon)), \\
&\mathbb{U}'_3(z_1) = \phi_{\mathbf{A},\mathbf{B}} \text{ with } \mathbf{A} = (\mathbb{U}_3(0),\dots,\mathbb{U}_3(2N_\varepsilon\varepsilon)), \ \mathbf{B} = (\mathcal{R}_2(0),\dots,\mathcal{R}_2(2N_\varepsilon\varepsilon)), \\
&\qquad \forall z_1 \in [p\varepsilon,(p+1)\varepsilon], \qquad \forall p \in \{0,\dots,2N_\varepsilon-1\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}'_1(z_1) = \psi_{\mathbf{B}}(z_1),\ \mathbf{B} = (\mathcal{R}_1(0),\dots,\mathcal{R}_1(2N_\varepsilon\varepsilon)), \qquad \mathcal{R}'_2(z_1) = -\partial_1\mathbb{U}'_3(z_1), \qquad \mathcal{R}'_3(z_1) = \partial_1\mathbb{U}'_2(z_1), \\
\forall z_1 \in [p\varepsilon,(p+1)\varepsilon], \qquad \forall p \in \{0,\dots,2N_\varepsilon-1\}.
\end{aligned} \tag{10}$$

By construction, we get the following relation that will later simplify the strain tensor form:

$$\partial_1\mathbb{U}' - \mathcal{R}' \wedge \mathbf{e}_1 = \partial_1\mathbb{U}'_1\mathbf{e}_1, \ \text{a.e. in } (0,L). \tag{11}$$

We define U'_{BN} and $\overline{u}'$ by

$$\begin{aligned} U'_{BN} &\doteq \mathbb{U}' + \mathcal{R}' \wedge (\Phi_\varepsilon \mathbf{e}_3 + y_2 \mathbf{e}_2 + y_3 \mathbf{n}_\varepsilon), \\ u &= U'_{BN} + \overline{u}', \end{aligned} \quad \text{a.e. in } (0, L) \times \omega_r.$$

Note that the relation (11) means that U'_{BN} is a Bernoulli–Navier displacement.

One has the following estimates for the fields of the new decomposition.

Theorem 1. *The fields $\mathbb{U}'$ and $\mathcal{R}'$ satisfy the following estimates:*

$$\|\partial_1 \mathcal{R}'\|_{L^2(0,L)} \le \frac{C}{\varepsilon^2} \|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \ \|\partial_1 \mathbb{U}'_1\|_{L^2(0,L)} \le \frac{C}{\varepsilon} \|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \tag{12}$$

$$\|\partial^2_{11} \mathbb{U}'_2\|_{L^2(0,L)} + \|\partial^2_{11} \mathbb{U}'_3\|_{L^2(0,L)} \le \frac{C}{\varepsilon^2} \|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \tag{13}$$

$$\|\mathcal{R}' - \mathcal{R}\|_{L^2(0,L)} + \|\partial_1(\mathbb{U}' - \mathbb{U})\|_{L^2(0,L)} \le \frac{C}{\varepsilon} \|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \ \|\mathbb{U}' - \mathbb{U}\|_{L^2(0,L)} \le C \|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}, \tag{14}$$

$$\|\overline{u}'\|_{L^2(P_r)} \le C\varepsilon \|e_x(u)\|_{L^2(P_r)}, \qquad \|\nabla \overline{u}'\|_{L^2(\mathcal{P}_\varepsilon)} \le C \|e_x(u)\|_{L^2(\mathcal{P}_\varepsilon)}. \tag{15}$$

Proof. The Poincaré and Poincaré–Wirtinger inequalities lead to

$$\begin{aligned} &\sum_{p=0}^{2N_\varepsilon - 1} \varepsilon \left| \frac{\mathbb{U}((p+1)\varepsilon) - \mathbb{U}(p\varepsilon)}{\varepsilon} - \frac{1}{2} \left(\mathcal{R}((p+1)\varepsilon) + \mathcal{R}(p\varepsilon) \right) \wedge \mathbf{e}_1 \right|^2 \le C \left(\|\partial_1 \mathbb{U} - \mathcal{R} \wedge \mathbf{e}_1\|^2_{L^2(0,L)} + \varepsilon^2 \|\partial_1 \mathcal{R}\|^2_{L^2(0,L)} \right), \\ &\sum_{p=0}^{2N_\varepsilon - 1} \varepsilon \left| \frac{\mathcal{R}_1((p+1)\varepsilon) - \mathcal{R}_1(p\varepsilon)}{\varepsilon} \right|^2 \le C \|\partial_1 \mathcal{R}_1\|^2_{L^2(0,L)}. \end{aligned}$$

From Lemma 1 estimates, the estimates (7) and the above ones, we obtain (12) and (13). Then, again, the Poincaré inequality yields (14). The first and second estimates in (15) are the consequences of writing

$$\overline{u} - \overline{u}' = \left(\mathbb{U}' - \mathbb{U} \right) + \left(\mathcal{R}' - \mathcal{R} \right) \wedge (\Phi_\varepsilon \mathbf{e}_3 + y_2 \mathbf{e}_2 + y_3 \mathbf{n}_\varepsilon) \ \text{a.e. in } \mathcal{P}_\varepsilon,$$

and for the first, estimates (7), (8), and (14) are used, while the second comes from (7), (8), and (12)–(14). □

3.5 | The symmetric gradient

Now, after setting a suitable decomposition, we are interested in the form of the associated symmetric gradient, which will later enter the left-hand side of the elasticity problem as a strain tensor.

Given a beam displacement $u \in H^1(\mathcal{P}_\varepsilon)^3$, equality (4) yields

$$\left(\partial_{z_1} u \ \ \partial_{y_2} u \ \ \partial_{y_3} u \right) = \nabla_x u \nabla \psi_\varepsilon = \nabla_x u \, \mathbf{C}_\varepsilon \begin{pmatrix} \eta_\varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since we will later state the problem in the straight reference frame, we want to express the symmetric gradient in such a frame. Hence, we first note that the above equality implies that

$$\mathbf{C}_\varepsilon^T \nabla_x u \, \mathbf{C}_\varepsilon = \begin{pmatrix} \frac{1}{\eta_\varepsilon} \partial_{z_1} u \cdot \mathbf{t}_\varepsilon & \partial_{y_2} u \cdot \mathbf{t}_\varepsilon & \partial_{y_3} u \cdot \mathbf{t}_\varepsilon \\ \frac{1}{\eta_\varepsilon} \partial_{z_1} u \cdot \mathbf{e}_{y_2} & \partial_{y_2} u \cdot \mathbf{e}_{y_2} & \partial_{y_3} u \cdot \mathbf{e}_{y_2} \\ \frac{1}{\eta_\varepsilon} \partial_{z_1} u \cdot \mathbf{n}_\varepsilon & \partial_{y_2} u \cdot \mathbf{n}_\varepsilon & \partial_{y_3} u \cdot \mathbf{n}_\varepsilon \end{pmatrix},$$

which, together with the definition of symmetric gradient, leads to the quantity we are interested in

$$\mathbf{C}_\varepsilon^T e_x(u)\mathbf{C}_\varepsilon = \begin{pmatrix} \frac{1}{\eta_\varepsilon}\partial_{z_1} u \cdot \mathbf{t}_\varepsilon & * & * \\ \frac{1}{2}\left(\frac{1}{\eta_\varepsilon}\partial_{z_1} u \cdot \mathbf{e}_{y_2} + \partial_{y_2} u \cdot \mathbf{t}_\varepsilon\right) & \partial_{y_2} u \cdot \mathbf{e}_{y_2} & * \\ \frac{1}{2}\left(\frac{1}{\eta_\varepsilon}\partial_{z_1} u \cdot \mathbf{n}_\varepsilon + \partial_{y_3} u \cdot \mathbf{t}_\varepsilon\right) & \frac{1}{2}\left(\partial_{y_2} u \cdot \mathbf{n}_\varepsilon + \partial_{y_3} u \cdot \mathbf{e}_{y_2}\right) & \partial_{y_3} u \cdot \mathbf{n}_\varepsilon \end{pmatrix}.$$

Hence, straightforward calculations on the gradient of the decomposition in the previous section, together with equality (11) imply that the symmetric gradient of a beam displacement in the straight reference frame, which we will call $\widetilde{\mathbf{e}}(u)$, is given by

$$\widetilde{\mathbf{e}}(u) = \widetilde{\mathbf{e}}(U'_{BN}) + \widetilde{\mathbf{e}}(\overline{u}'),$$

where the first quantity is the symmetric gradient of the Bernoulli–Navier displacement

$$\widetilde{\mathbf{e}}(U'_{BN}) \doteq \begin{pmatrix} \frac{1}{\eta_\varepsilon}\left(\partial_1 \mathbb{U}'_1 \mathbf{e}_1 + \partial_1 \mathcal{R}' \wedge (\Phi_\varepsilon \mathbf{e}_3 + y_2 \mathbf{e}_2 + y_3 \mathbf{n}_\varepsilon)\right) \cdot \mathbf{t}_\varepsilon & * & * \\ \frac{1}{2\eta_\varepsilon}\left(\partial_1 \mathbb{U}'_1 \mathbf{e}_1 + \partial_1 \mathcal{R}' \wedge (\Phi_\varepsilon \mathbf{e}_3 + y_2 \mathbf{e}_2 + y_3 \mathbf{n}_\varepsilon)\right) \cdot \mathbf{e}_2 & 0 & * \\ \frac{1}{2\eta_\varepsilon}\left(\partial_1 \mathbb{U}'_1 \mathbf{e}_1 + \partial_1 \mathcal{R}' \wedge (\Phi_\varepsilon \mathbf{e}_3 + y_2 \mathbf{e}_2 + y_3 \mathbf{n}_\varepsilon)\right) \cdot \mathbf{n}_\varepsilon & 0 & 0 \end{pmatrix}, \tag{16}$$

and the second one is the symmetric gradient of the warping

$$\widetilde{\mathbf{e}}(\overline{u}') \doteq \begin{pmatrix} \frac{1}{\eta_\varepsilon}\partial_{z_1} \overline{u}' \cdot \mathbf{t}_\varepsilon & * & * \\ \frac{1}{2}\left(\frac{1}{\eta_\varepsilon}\partial_{z_1} \overline{u}' \cdot \mathbf{e}_{y_2} + \partial_{y_2} \overline{u}' \cdot \mathbf{t}_\varepsilon\right) & \partial_{y_2} \overline{u}' \cdot \mathbf{e}_{y_2} & * \\ \frac{1}{2}\left(\frac{1}{\eta_\varepsilon}\partial_{z_1} \overline{u}' \cdot \mathbf{n}_\varepsilon + \partial_{y_3} \overline{u}' \cdot \mathbf{t}_\varepsilon\right) & \frac{1}{2}\left(\partial_{y_2} \overline{u}' \cdot \mathbf{n}_\varepsilon + \partial_{y_3} \overline{u}' \cdot \mathbf{e}_{y_2}\right) & \partial_{y_3} \overline{u}' \cdot \mathbf{n}_\varepsilon \end{pmatrix}. \tag{17}$$

4 | THE TEXTILE STRUCTURE AND NATURAL ASSUMPTIONS

In this section, we will first define our problem's domain, which is our textile structure's mathematical model. Then, we set the natural assumptions that the woven fibers should satisfy: the boundary conditions to ensure the well-posedness of the elasticity problem, the contact conditions that allow shear between beams in the areas where they are one above the other, and the non-penetration conditions, which do not allow fibers to penetrate one into the other. These assumptions will shape the admissible set of displacements we will study.

4.1 | The textile structure

In Section 3, we studied a single oscillating beam of direction $\mathbf{e}_1$. Now, to define the textile structure, we need to do the same, but for alternate oscillating beams of direction $\mathbf{e}_1$ or $\mathbf{e}_2$, to obtain a woven canvas.

We denote $\mathfrak{G}_\varepsilon$ the reference lattice structure

$$\mathfrak{G}_\varepsilon \doteq \mathfrak{G}_\varepsilon^{(1)} \cup \mathfrak{G}_\varepsilon^{(2)}, \qquad \mathfrak{G}_\varepsilon^{(1)} = \bigcup_{q=0}^{2N_\varepsilon} [0, L] \times \{q\varepsilon\}, \qquad \mathfrak{G}_\varepsilon^{(2)} = \bigcup_{p=0}^{2N_\varepsilon} \{p\varepsilon\} \times [0, L].$$

This grid represents the domain of the beams' center lines in both directions: For every $(z_1, q\varepsilon) \in \mathfrak{G}^{(1)}$ and every $(p\varepsilon, z_2) \in \mathfrak{G}^{(2)}$, the middle lines of the beams become

$$M_\varepsilon^{(1)}(z_1, q\varepsilon) \doteq z_1 \mathbf{e}_1 + q\varepsilon \mathbf{e}_2 + \Phi_\varepsilon^{(1)}(z_1, q\varepsilon)\mathbf{e}_3, \qquad \Phi_\varepsilon^{(1)}(z_1, q\varepsilon) = (-1)^{q+1}\Phi_\varepsilon(z_1),$$
$$M_\varepsilon^{(2)}(p\varepsilon, z_2) \doteq p\varepsilon \mathbf{e}_1 + z_2 \mathbf{e}_2 + \Phi_\varepsilon^{(2)}(p\varepsilon, z_2)\mathbf{e}_3, \qquad \Phi_\varepsilon^{(2)}(p\varepsilon, z_2) = (-1)^{p}\Phi_\varepsilon(z_2).$$

Here, the quantities $(-1)^{q+1}$ and $(-1)^p$ denote the fact that the oscillations between parallel beams are alternate, so to allow the crossing between beams of different directions. Accordingly, we denote the Frenet–Serret mobile frames derived from (3) in the respective direction by

$$\left(\mathbf{t}^{(1)}, \mathbf{e}_2, \mathbf{n}^{(1)}\right),\ \text{where } \mathbf{t}^{(1)}(z_1, q\varepsilon) = \frac{1}{\gamma_\varepsilon(z_1)}\left(\mathbf{e}_1 + \partial_1 \Phi^{(1)}_\varepsilon(z_1, q\varepsilon)\mathbf{e}_3\right),\ \mathbf{n}^{(1)}_\varepsilon \doteq \mathbf{t}^{(1)}_\varepsilon \wedge \mathbf{e}_2,$$
$$\left(\mathbf{e}_1, \mathbf{t}^{(2)}, \mathbf{n}^{(2)}\right),\ \text{where } \mathbf{t}^{(2)}(p\varepsilon, z_2) = \frac{1}{\gamma_\varepsilon(z_2)}\left(\mathbf{e}_2 + \partial_2 \Phi^{(2)}_\varepsilon(p\varepsilon, z_2)\mathbf{e}_3\right),\ \mathbf{n}^{(2)}_\varepsilon \doteq \mathbf{t}^{(2)}_\varepsilon \wedge \mathbf{e}_1.$$

In such mobile frames, the diffeomorphisms become

$$\psi^{(1)}_\varepsilon(z_1, q\varepsilon, y_2, y_3) \doteq M^{(1)}_\varepsilon(z_1, q\varepsilon) + y_2\mathbf{e}_2 + y_3\mathbf{n}^{(1)}_\varepsilon(z_1, q\varepsilon),\ \text{for a.e. } (z_1, q\varepsilon) \in \mathfrak{G}^{(1)}_\varepsilon \text{ and } (y_2, y_3) \in \omega_r,$$
$$\psi^{(2)}_\varepsilon(p\varepsilon, z_2, y_1, y_3) \doteq M^{(2)}_\varepsilon(p\varepsilon, z_2) + y_1\mathbf{e}_1 + y_3\mathbf{n}^{(2)}_\varepsilon(p\varepsilon, z_2),\ \text{for a.e. } (p\varepsilon, z_2) \in \mathfrak{G}^{(2)}_\varepsilon \text{ and } (y_1, y_3) \in \omega_r,$$

and, thus, the whole textile results to be

$$\mathcal{S}_\varepsilon \doteq \mathcal{S}^{(1)}_\varepsilon \cup \mathcal{S}^{(2)}_\varepsilon,\ \text{where } \mathcal{S}^{(1)}_\varepsilon \doteq \psi^{(1)}_\varepsilon\left(\mathfrak{G}^{(1)}_\varepsilon \times \omega_r\right),\ \mathcal{S}^{(2)}_\varepsilon \doteq \psi^{(2)}_\varepsilon\left(\mathfrak{G}^{(2)}_\varepsilon \times \omega_r\right). \tag{18}$$

Every displacement, defined on such structures, is a couple $(u^{(1)}, u^{(2)})$, which belongs to $H^1(\mathcal{S}^{(1)}_\varepsilon)^3 \times H^1(\mathcal{S}^{(2)}_\varepsilon)^3$ or, equivalently (due to (5)), to $H^1\left(\mathfrak{G}^{(1)}_\varepsilon \times \omega_r\right)^3 \times H^1\left(\mathfrak{G}^{(2)}_\varepsilon \times \omega_r\right)^3$. For simplicity, a function defined on $\mathfrak{G}^{(\alpha)}_\varepsilon$ is considered as an element, defined in $\mathcal{S}^{(\alpha)}_\varepsilon$, constant in the cross-sections ω_r, and this is the main reason why we name z the beam center line variable and y the cross-section variable.

Let $C(\mathfrak{G}_\varepsilon)$ be the space of continuous functions, defined on $\mathfrak{G}_\varepsilon$. We denote the spaces of functions, defined on the lattice, by $(\alpha \in \{1, 2\})$

$$H^1(\mathfrak{G}^{(\alpha)}_\varepsilon) \doteq \left\{\phi \in L^2\left(\mathfrak{G}^{(\alpha)}_\varepsilon\right) \mid \partial_\alpha\phi \in L^2\left(\mathfrak{G}^{(\alpha)}_\varepsilon\right)\right\}, \qquad H^1(\mathfrak{G}_\varepsilon) \doteq \left\{\phi \in C(\mathfrak{G}_\varepsilon) \mid \partial_\alpha\phi \in L^2\left(\mathfrak{G}^{(\alpha)}_\varepsilon\right),\ \text{for } \alpha \in \{1, 2\}\right\},$$

and

$$H^2(\mathfrak{G}^{(\alpha)}_\varepsilon) \doteq \left\{\phi \in H^1\left(\mathfrak{G}^{(\alpha)}_\varepsilon\right) \mid \partial_\alpha\phi \in H^1\left(\mathfrak{G}^{(\alpha)}_\varepsilon\right)\right\}, \qquad H^2(\mathfrak{G}_\varepsilon) \doteq \left\{\phi \in H^1(\mathfrak{G}_\varepsilon) \mid \partial_\alpha\phi \in H^1\left(\mathfrak{G}^{(\alpha)}_\varepsilon\right),\ \text{for } \alpha \in \{1, 2\}\right\}.$$

We endow these spaces with the following norms:

$$\|\phi\|^2_{L^2(\mathfrak{G}^{(1)}_\varepsilon)} \doteq \sum_{q=0}^{2N_\varepsilon} \|\phi(\cdot, q\varepsilon)\|^2_{L^2(0,L)},\quad \forall\phi \in L^2(\mathfrak{G}^{(1)}_\varepsilon), \qquad \|\psi\|^2_{L^2(\mathfrak{G}^{(2)}_\varepsilon)} \doteq \sum_{p=0}^{2N_\varepsilon} \|\psi(p\varepsilon, \cdot)\|^2_{L^2(0,L)},\quad \forall\psi \in L^2(\mathfrak{G}^{(2)}_\varepsilon)$$
$$\|\cdot\|_{L^2(\mathfrak{G}_\varepsilon)} \doteq \sqrt{\|\cdot\|^2_{L^2(\mathfrak{G}^{(1)}_\varepsilon)} + \|\cdot\|^2_{L^2(\mathfrak{G}^{(2)}_\varepsilon)}},$$
$$\|\cdot\|_{H^1(\mathfrak{G}^{(\alpha)}_\varepsilon)} \doteq \sqrt{\|\cdot\|^2_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon)} + \|\partial_\alpha(\cdot)\|^2_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon)}}, \qquad \|\cdot\|_{H^1(\mathfrak{G}_\varepsilon)} \doteq \sqrt{\|\cdot\|^2_{H^1(\mathfrak{G}^{(1)}_\varepsilon)} + \|\cdot\|^2_{H^1(\mathfrak{G}^{(2)}_\varepsilon)}},$$
$$\|\cdot\|_{H^2(\mathfrak{G}^{(\alpha)}_\varepsilon)} \doteq \sqrt{\|\cdot\|^2_{H^1(\mathfrak{G}^{(\alpha)}_\varepsilon)} + \|\partial_{\alpha\alpha}(\cdot)\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon)}}, \qquad \|\cdot\|_{H^2(\mathfrak{G}_\varepsilon)} \doteq \sqrt{\|\cdot\|^2_{H^2(\mathfrak{G}^{(1)}_\varepsilon)} + \|\cdot\|^2_{H^2(\mathfrak{G}^{(2)}_\varepsilon)}}.$$

4.2 | Boundary conditions

These conditions are fundamental for the well-posedness of the problem.

We assume that the displacement is equal to zero in the left and bottom boundary of the domain Ω. According to the notation in Section 2 (n_ε is such that $2\varepsilon n_\varepsilon = l$, with $l < L$), we set

$$\text{Clamp condition } \begin{cases} u^{(1)}(0, q\varepsilon, \cdot) = 0 \text{ for every } q \in \{0, \dots, 2n_\varepsilon\}, \\ u^{(2)}(p\varepsilon, 0, \cdot) = 0 \text{ for every } p \in \{0, \dots, 2n_\varepsilon\}, \end{cases} \text{ a.e. in } \omega_r. \tag{19}$$

As we can see in Figure 1, this partial clamp leads to a natural partition of the domain

$$\Omega = \text{int}\left(\overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3 \cup \overline{\Omega}_4\right),$$

where the four subdomains are defined by

$$\Omega_1 \doteq (0,l)^2,\ \Omega_2 \doteq (l,L)\times(0,l),\ \Omega_3 \doteq (0,l)\times(l,L),\ \Omega_4 \doteq (l,L)^2.$$

To conclude, it is worth mentioning that, since the clamp takes place only partially on the left and bottom boundary of Ω_1, it affects the behavior of the displacement $u^{(1)}$ by inheritance (since the fibers are the same) solely in the subdomain $\Omega_1 \cup \Omega_2$, while $u^{(2)}$ inherits the clamp condition solely in $\Omega_1 \cup \Omega_3$.

4.3 | The contact and non-penetration condition

Now, we proceed to what we call contact conditions, which determine how the fibers interact in the internal part of the domain. Here, the fibers are not glued and can shear quite loosely, one with respect to another, in the in-plane directions. As we will see, this little assumption gives rise to some issues.

The contact is restricted to the portions where the beams are right above each other. We define such contact domains in the straight reference frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by setting $((p,q) \in \mathcal{K}_\varepsilon)$

$$\mathbf{C}_\varepsilon \doteq \bigcup_{(p,q)\in\mathcal{K}_\varepsilon} \mathbf{C}_{pq,\varepsilon}, \qquad \mathbf{C}_{pq,\varepsilon} \doteq \left(C_{pq,\varepsilon} \cap \Omega\right) \times \{0\}, \qquad C_{pq,\varepsilon} \doteq (p\varepsilon, q\varepsilon) + \omega_r. \tag{20}$$

In the case $(p,q) \in \{1, \dots, 2N_\varepsilon - 1\}^2$, these areas correspond to

$$\left(\mathfrak{G}_\varepsilon^{(1)} \times \omega_r\right)_{|C_{pq,\varepsilon}} \doteq \left\{\left(p\varepsilon + y_1, q\varepsilon, y_2, (-1)^{p+q+1}\kappa\varepsilon\right) \in \mathbb{R}^3 \mid (y_1, y_2) \in \omega_r\right\},$$
$$\left(\mathfrak{G}_\varepsilon^{(2)} \times \omega_r\right)_{|C_{pq,\varepsilon}} \doteq \left\{\left(p\varepsilon, q\varepsilon + y_2, y_1, (-1)^{p+q}\kappa\varepsilon\right) \in \mathbb{R}^3 \mid (y_1, y_2) \in \omega_r\right\}.$$

In the in-plane directions, the sliding between the fibers is characterized by the nonnegative gap functions $g_{\varepsilon,\alpha}$. We assume

$$g_{\varepsilon,\alpha} = \varepsilon^2 g_\alpha,\ g_\alpha \in C(\overline{\Omega}). \tag{21}$$

Moreover, while later constructing the test functions, we need the additional assumption that there is no zone in the inner part of Ω, in which the fibers could be glued

$$\exists C_3 > 0 \text{ such that } g_\alpha \geq C_3 \text{ a.e. in } \Omega. \tag{22}$$

Now, let $(u^{(1)}, u^{(2)}) \in H^1\left(\mathfrak{G}_\varepsilon^{(1)} \times \omega_r\right)^3 \times H^1\left(\mathfrak{G}_\varepsilon^{(2)} \times \omega_r\right)^3$ be a displacement on the textile. We define the in-plane contact conditions by setting

$$|u_\alpha^{(1)} - u_\alpha^{(2)}| \leq g_{\varepsilon,\alpha}, \qquad \text{a.e in } \mathbf{C}_{pq,\varepsilon},\ \ \forall (p,q) \in \mathcal{K}_\varepsilon, \tag{23}$$

and the outer plane ones

$$0 \leq (-1)^{p+q}\left(u_3^{(1)} - u_3^{(2)}\right) \qquad \text{a.e in } \mathbf{C}_{pq,\varepsilon},\ \ \forall (p,q) \in \mathcal{K}_\varepsilon. \tag{24}$$

Concerning the in-plane contact conditions, the physical meaning is that, in the contact areas, the displacements can slide one with respect to the other from a value more than zero to a maximum given by the L^∞ norm of g in that direction.

Concerning the outer-plane direction, the only required restriction is that the fibers cannot penetrate each other. As we will see later, this condition, together with the oscillating manner results, will create a “natural constraint” that keeps the outer-plane displacements so close to each other, as we were assuming an outer-plane artificial bound.

5 | SET OF THE ELASTICITY PROBLEM

In this section, we proceed to the definition of the elasticity problem that we will then investigate through homogenization via the periodic unfolding method.

5.1 | Set of admissible displacements for a textile structure with loose contact

Given the structure, the clamp, and the contact conditions, we can finally define the set of displacements as the closed convex set

$$\mathcal{X}_\varepsilon \doteq \left\{ \left(u^{(1)}, u^{(2)}\right) \in H^1(\mathfrak{G}_\varepsilon^{(1)} \times \omega_r)^3 \times H^1(\mathfrak{G}_\varepsilon^{(2)} \times \omega_r)^3 \;\middle|\; \left(u^{(1)}, u^{(2)}\right) \text{ satisfies (19), (23), and (24)} \right\}.$$

We endow the product space $H^1(\mathfrak{G}_\varepsilon^{(1)} \times \omega_r)^3 \times H^1(\mathfrak{G}_\varepsilon^{(2)} \times \omega_r)^3$ with the semi-norm

$$\|u\|_{S_\varepsilon} \doteq \sqrt{\|\widetilde{\mathbf{e}}^{(1)}(u^{(1)})\|^2_{L^2(\mathfrak{G}_\varepsilon^{(1)}\times\omega_r)} + \|\widetilde{\mathbf{e}}^{(2)}(u^{(2)})\|^2_{L^2(\mathfrak{G}_\varepsilon^{(2)}\times\omega_r)}}.$$

Note that $\mathcal{X}_\varepsilon$ is a closed convex subset of $H^1\left(\mathfrak{G}_\varepsilon^{(1)} \times \omega_r\right)^3 \times H^1\left(\mathfrak{G}_\varepsilon^{(2)} \times \omega_r\right)^3$.

5.2 | The elasticity problem

Let $f_\varepsilon^{(\alpha)} \in L^2(S_\varepsilon^{(\alpha)})^3$ be some applied forces and let $a_\varepsilon^{(\alpha)}$ be the fourth-order strain tensor describing the elasticity of the material. We will study the following elasticity problem:

$$\begin{gathered}\text{Find } \left(u_\varepsilon^{(1)}, u_\varepsilon^{(2)}\right) \in \mathcal{X}_\varepsilon \text{ such that for every } \left(v_\varepsilon^{(1)}, v_\varepsilon^{(2)}\right) \in \mathcal{X}_\varepsilon : \\ \sum_{\alpha=1}^{2} \int_{S_\varepsilon^{(\alpha)}} a_{ijkl,\varepsilon}^{(\alpha)}\, e_{ij}\left(u_\varepsilon^{(\alpha)}\right) e_{kl}\left(u_\varepsilon^{(\alpha)} - v_\varepsilon^{(\alpha)}\right) dx \le \sum_{\alpha=1}^{2} \int_{S_\varepsilon} f_\varepsilon^{(\alpha)} \cdot \left(u_\varepsilon^{(\alpha)} - v_\varepsilon^{(\alpha)}\right) dx.\end{gathered} \tag{25}$$

We state the problem as a variational inequality; it is also possible to give an equivalent problem in a minimization form. We also find it convenient to consider this problem in the straight reference frame:

$$\begin{gathered}\text{Find } \left(u_\varepsilon^{(1)}, u_\varepsilon^{(2)}\right) \in \mathcal{X}_\varepsilon \text{ such that for every } \left(v_\varepsilon^{(1)}, v_\varepsilon^{(2)}\right) \in \mathcal{X}_\varepsilon : \\ \sum_{\alpha=1}^{2} \int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} A_{ijkl,\varepsilon}^{(\alpha)}\, \widetilde{e}_{ij}^{(\alpha)}\left(u_\varepsilon^{(\alpha)}\right) \widetilde{e}_{kl}^{(\alpha)}\left(u_\varepsilon^{(\alpha)} - v_\varepsilon^{(\alpha)}\right) \eta_\varepsilon^{(\alpha)}\, dz_\alpha dy_{3-\alpha} dy_3 \le \sum_{\alpha=1}^{2} \int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} F_\varepsilon^{(\alpha)} \cdot \left(u_\varepsilon^{(\alpha)} - v_\varepsilon^{(\alpha)}\right) \eta_\varepsilon^{(\alpha)}\, dz_\alpha dy_{3-\alpha} dy_3.\end{gathered} \tag{26}$$

Since we assume the linear elasticity of the problem, Hooke's law implies for the fourth-order tensors $A_\varepsilon^{(\alpha)}$ to satisfy the following properties:

(i) $A_\varepsilon^{(\alpha)}$ is bounded: $A_{\varepsilon,ijkl}^{(\alpha)} \in L^\infty(\mathfrak{G}_\varepsilon^{(\alpha)} \times \omega_r)$;
(ii) $A_\varepsilon^{(\alpha)}$ is symmetric: $A_{\varepsilon,ijkl}^{(\alpha)} = A_{\varepsilon,jikl}^{(\alpha)} = A_{\varepsilon,klij}^{(\alpha)}$;
(iii) $A_\varepsilon^{(\alpha)}$ is elliptic: there exist $C_0, C_1 > 0$ independent of ε, such that $C_0|||\xi|||_F^2 \le A_{ijkl,\varepsilon}^{(\alpha)}\, \xi_{ij}\, \xi_{kl} \le C_1|||\xi|||_F^2$ a.e. in $\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r$, for all symmetric 3×3 matrices ξ (we denote $|||\cdot|||_F$ the Frobenius norm).

Stampacchia lemma (see Kinderlehrer and Stampacchia [26]) ensures the existence of solutions for the problem.

6 | FIELDS ESTIMATES

In order to pass to the limit in problem (26), we need to bound the fields and their derivatives present on the left- and right-hand sides.

6.1 Estimates, given by the new displacements decomposition

Let $(u^{(1)}, u^{(2)})$ be a displacement in $\mathcal{X}_\varepsilon$. Applying the decomposition, introduced in Section 3.4, we can write

$$\begin{aligned} u^{(1)}(z_1, q\varepsilon, y_2, y_3) &= \mathbb{U}'^{(1)}(z_1, q\varepsilon) + \mathcal{R}'^{(1)}(z_1, q\varepsilon) \wedge \left(\Phi_\varepsilon^{(1)}(z_1, q\varepsilon)\mathbf{e}_3 + y_2\mathbf{e}_2 + y_3\mathbf{n}^{(1)}(z_1, q\varepsilon)\right) \\ &\quad + \overline{u}'^{(1)}(z_1, q\varepsilon, y_2, y_3), \text{ for a.e. } (z_1, q\varepsilon, y_2, y_3) \in \mathfrak{G}_\varepsilon^{(1)} \times \omega_r, \\ u^{(2)}(p\varepsilon, z_2, y_1, y_3) &= \mathbb{U}'^{(2)}(p\varepsilon, z_2) + \mathcal{R}'^{(2)}(p\varepsilon, z_2) \wedge \left(\Phi_\varepsilon^{(2)}(p\varepsilon, z_2)\mathbf{e}_3 + y_1\mathbf{e}_1 + y_3\mathbf{n}^{(2)}(p\varepsilon, z_2)\right) \\ &\quad + \overline{u}'^{(2)}(p\varepsilon, z_2, y_1, y_3), \text{ for a.e. } (p\varepsilon, z_2, y_1, y_3) \in \mathfrak{G}_\varepsilon^{(2)} \times \omega_r. \end{aligned} \tag{27}$$

From the results of Theorem 1, the fields of this decomposition satisfy ($\alpha \in \{1, 2\}$)

$$\begin{aligned} &\varepsilon\|\partial_{\alpha\mathcal{R}}'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \|\partial_\alpha \mathbb{U}_\alpha'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon}\|u\|_{S_\varepsilon}, \\ &\|\partial_{\alpha\alpha}^2 \mathbb{U}_{3-\alpha}'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \|\partial_{\alpha\alpha}^2 \mathbb{U}_3'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon^2}\|u\|_{S_\varepsilon}, \\ &\|\overline{u}'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)} \times \omega_r)} + \varepsilon\|\nabla\overline{u}'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le C\varepsilon\|u\|_{S_\varepsilon}. \end{aligned} \tag{28}$$

However, in order to obtain the field estimates from the above, a few additional tools will be needed, which we will introduce in the following sections.

6.2 Estimates, given by the contact, non-penetration, and boundary conditions

To estimate the fields themselves, starting from the estimates on their derivatives, we use the Poincaré inequality and the clamp conditions. However, since the domain is clamped only partially, some fields are defined on the subdomains that do not inherit the clamp. For this reason, the contact and non-penetration conditions will furnish the necessary relations to transfer the estimates for the fields defined on clamped subdomains to the fields defined in the not clamped ones.

Starting from (27), we note that for a.e. $(t_1, t_2) \in \omega_r$, the displacements in the contact areas reduce to

$$\begin{aligned} u^{(1)}(p\varepsilon + t_1, q\varepsilon, t_2, (-1)^{p+q+1}r) &= \mathbb{U}'^{(1)}(p\varepsilon + t_1, q\varepsilon) + \mathcal{R}'^{(1)}(p\varepsilon + t_1, q\varepsilon) \wedge t_2\mathbf{e}_2 + \overline{u}'^{(1)}(p\varepsilon + t_1, q\varepsilon, t_2, (-1)^{p+q+1}r), \\ u^{(2)}(p\varepsilon, q\varepsilon + t_2, t_1, (-1)^{p+q}r) &= \mathbb{U}'^{(2)}(p\varepsilon, q\varepsilon + t_2) + \mathcal{R}'^{(2)}(p\varepsilon, q\varepsilon + t_2) \wedge t_1\mathbf{e}_1 + \overline{u}'^{(2)}(p\varepsilon, q\varepsilon + t_2, t_1, (-1)^{p+q}r). \end{aligned} \tag{29}$$

We start by giving the warping estimates in the contact areas.

Lemma 2. *The warping terms in the contact areas satisfy*

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \left(\|\overline{u}'^{(1)}\|^2_{L^2(\boldsymbol{C}_{pq,\varepsilon})} + \|\overline{u}'^{(2)}\|^2_{L^2(\boldsymbol{C}_{pq,\varepsilon})}\right) \le C\varepsilon\|u\|^2_{S_\varepsilon}. \tag{30}$$

Proof. It is a direct consequence of the third estimate in (28) of the remainder displacements $\overline{u}'^{(\alpha)}$ and the trace theorem. □

Now, set

$$\|g\|_{L^\infty(\Omega)} \doteq \|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}.$$

We have the following.

Lemma 3. *The in-plane contact conditions lead to the following estimate:*

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \left(|(\mathbb{U}_\alpha'^{(1)} - \mathbb{U}_\alpha'^{(2)})(p\varepsilon, q\varepsilon)|^2 + \varepsilon^2|(\mathcal{R}_3'^{(1)} - \mathcal{R}_3'^{(2)})(p\varepsilon, q\varepsilon)|^2\right) \le C\left(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|^2_{S_\varepsilon}\right). \tag{31}$$

The outer-plane non-penetration conditions lead to the following estimates:

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\left(|(\mathbb{U}_3'^{(1)}-\mathbb{U}_3'^{(2)})(p\varepsilon,q\varepsilon)|^2+\varepsilon^2|(\mathcal{R}_\alpha'^{(1)}-\mathcal{R}_\alpha'^{(2)})(p\varepsilon,q\varepsilon)|^2\right)\le\frac{C}{\varepsilon}\|u\|^2_{S_\varepsilon}. \tag{32}$$

Proof. Estimate (31) is done in the same fashion as Lemma 5.6 in Griso et al. [25], while the proof of (32) is shifted to Appendix A.1 due to the heavy computations. □

As we already mentioned, note that the estimate in the outer-plane direction does not depend on the contact function: Since the displacement alternatively switches vertical position and two fibers cannot penetrate one into the other, it gives a sufficiently good estimate for the distance between fibers in the outer-plane component, without assuming an additional upper bound contact function $g_{\varepsilon,3}$ in (24) (as done in Griso et al. [25]).

The clamp conditions (9) and (19) give

$$\mathbb{U}'^{(1)}(0,q\varepsilon)=\mathcal{R}'^{(1)}(0,q\varepsilon)\ q\in\{0,\dots,2n_\varepsilon\},\qquad \mathbb{U}'^{(2)}(p\varepsilon,0)=\mathcal{R}'^{(2)}(p\varepsilon,0)=0\ p\in\{0,\dots,2n_\varepsilon\}. \tag{33}$$

6.3 | Final decomposition of the displacements

Comparing the estimates for each field and the ones concerning their difference (31) and (32), we find it convenient to define the final displacements fields, such that they combine both directions and that take into account the clamp conditions.

We start with the outer-plane component. Proceeding as in Section 3.4, we define the field $\mathbb{U}_3\in H^2(\mathfrak{G}_\varepsilon)$ by

$$\begin{aligned}
\mathbb{U}_3(z_1,q\varepsilon)&\doteq\phi_{\mathbf{A},\mathbf{B}}(z_1), &&\text{with } \mathbf{A}=\frac{1}{2}\left((\mathbb{U}_3'^{(1)}+\mathbb{U}_3'^{(2)})(0,q\varepsilon),\dots,(\mathbb{U}_3'^{(1)}+\mathbb{U}_3'^{(2)})(2N_\varepsilon\varepsilon,q\varepsilon)\right),\\
& &&\mathbf{B}=-\frac{1}{2}\left((\mathcal{R}_2'^{(1)}+\mathcal{R}_2'^{(1)})(0,q\varepsilon),\dots,(\mathcal{R}_2'^{(1)}+\mathcal{R}_2'^{(1)})(2N_\varepsilon\varepsilon,q\varepsilon)\right),\\
& &&\forall z_1\in[p\varepsilon,(p+1)\varepsilon],\ \forall q\in\{0,\dots,2N_\varepsilon\},\\
\mathbb{U}_3(p\varepsilon,z_2)&\doteq\phi_{\mathbf{A},\mathbf{B}}(z_2), &&\text{with } \mathbf{A}=\frac{1}{2}\left((\mathbb{U}_3'^{(1)}+\mathbb{U}_3'^{(2)})(p\varepsilon,0),\dots,(\mathbb{U}_3'^{(1)}+\mathbb{U}_3'^{(2)})(p\varepsilon,2N_\varepsilon\varepsilon)\right),\\
& &&\mathbf{B}=\frac{1}{2}\left((\mathcal{R}_1'^{(1)}+\mathcal{R}_1'^{(1)})(p\varepsilon,0),\dots,(\mathcal{R}_1'^{(1)}+\mathcal{R}_1'^{(1)})(2N_\varepsilon\varepsilon,q\varepsilon)\right),\\
& &&\forall z_2\in[q\varepsilon,(q+1)\varepsilon],\ \forall p\in\{0,\dots,2N_\varepsilon\},
\end{aligned}$$

and the fields $\mathcal{R}_1,\mathcal{R}_2\in H^1(\mathfrak{G}_\varepsilon)$ by

$$\begin{aligned}
&\mathcal{R}_2(z_1,q\varepsilon)\doteq-\partial_1\mathbb{U}_3(z_1,q\varepsilon),\ \forall z_1\in[0,L],\ \forall q\in\{0,\dots,2N_\varepsilon\},\\
&\mathcal{R}_2(p\varepsilon,z_2)\doteq\psi_{\mathbf{B}}(z_2)\text{ with }\mathbf{B}=\frac{1}{2}\left((\mathcal{R}_2'^{(1)}+\mathcal{R}_2'^{(2)})(p\varepsilon,0),\dots,(\mathcal{R}_2'^{(1)}+\mathcal{R}_2'^{(2)})(p\varepsilon,2N_\varepsilon\varepsilon)\right),\\
&\qquad\forall z_2\in[q\varepsilon,(q+1)\varepsilon],\ \forall p\in\{0,\dots,2N_\varepsilon\},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{R}_1(z_1,q\varepsilon)\doteq\psi_{\mathbf{B}}(z_1)\text{ with }\mathbf{B}=\frac{1}{2}\left((\mathcal{R}_1'^{(1)}+\mathcal{R}_1'^{(2)})(p\varepsilon,0),\dots,(\mathcal{R}_1'^{(1)}+\mathcal{R}_1'^{(2)})(p\varepsilon,2N_\varepsilon\varepsilon)\right),\\
&\qquad\forall z_2\in[q\varepsilon,(q+1)\varepsilon],\ \forall p\in\{0,\dots,2N_\varepsilon\},\\
&\mathcal{R}_1(p\varepsilon,z_2)\doteq\partial_2\mathbb{U}_3(p\varepsilon,z_2),\qquad\forall z_2\in[0,L],\ \forall p\in\{0,\dots,2N_\varepsilon\},
\end{aligned}$$

where, due to the clamp conditions (33), we replace

$$\begin{aligned}
&(\mathbb{U}_3'^{(1)}+\mathbb{U}_3'^{(2)})(0,q\varepsilon)\ \text{ and }\ (\mathcal{R}_2'^{(1)}+\mathcal{R}_2'^{(1)})(0,q\varepsilon)\ \text{by}\ 0\ \text{ if }\ q\in\{0,\dots,2n_\varepsilon\},\\
&(\mathbb{U}_3'^{(1)}+\mathbb{U}_3'^{(2)})(p\varepsilon,0)\ \text{ and }\ (\mathcal{R}_1'^{(1)}+\mathcal{R}_1'^{(1)})(p\varepsilon,0)\ \text{by}\ 0\ \text{ if }\ p\in\{0,\dots,2n_\varepsilon\}.
\end{aligned}$$

Note that these fields satisfy equalities $\mathcal{R}_2 = -\partial_1 \mathbb{U}_3$ a.e. in $\mathfrak{G}_\varepsilon^{(1)}$, $\mathcal{R}_1 = \partial_2 \mathbb{U}_3$ a.e. in $\mathfrak{G}_\varepsilon^{(2)}$ and vanish on the clamped points of $\mathfrak{G}_\varepsilon$.

Proposition 1. *One has the following estimates:*

$$\|\mathcal{R}_1\|_{H^1(\mathfrak{G}_\varepsilon)} + \|\mathcal{R}_2\|_{H^1(\mathfrak{G}_\varepsilon)} + \|\mathbb{U}_3\|_{H^2(\mathfrak{G})} \le \frac{C}{\varepsilon^2}\|u\|_{S_\varepsilon}.$$

Proof.

Step 1. We prove the estimates of $\mathcal{R}_1$ and $\mathcal{R}_2$.
From the definitions of $\mathcal{R}_2$ and $\mathcal{R}_2'^{(1)}$, $\mathcal{R}_2'^{(2)}$, estimates (32), equalities (33), and Lemma 1, we get

$$\|\mathcal{R}_2 - \mathcal{R}_2'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \varepsilon\|\partial_\alpha\left(\mathcal{R}_2 - \mathcal{R}_2'^{(\alpha)}\right)\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon}\|u\|_{S_\varepsilon}. \tag{34}$$

Hence, the first estimate in (28) and the above one lead to

$$\|\partial_1\mathcal{R}_2\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})} + \|\partial_2\mathcal{R}_2\|_{L^2(\mathfrak{G}_\varepsilon^{(2)})} \le \frac{C}{\varepsilon^2}\|u\|_{S_\varepsilon}.$$

By the fact that $\mathcal{R}_2(0, q\varepsilon) = 0$ for all $q \in \{0, \dots, 2n_\varepsilon\}$, the above estimate and the Poincaré inequality imply

$$\sum_{q=0}^{2n_\varepsilon}\|\mathcal{R}_2(\cdot, q\varepsilon)\|^2_{L^2(0,L)} \le \frac{C}{\varepsilon^4}\|u\|^2_{S_\varepsilon}.$$

One has

$$\sum_{q=0}^{2n_\varepsilon}\sum_{p=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_2(p\varepsilon, q\varepsilon)|^2 \le C\sum_{q=0}^{2n_\varepsilon}\left(\|\mathcal{R}_2(\cdot, q\varepsilon)\|^2_{L^2(0,L)} + \varepsilon^2\|\partial_1\mathcal{R}_2(\cdot, q\varepsilon)\|^2_{L^2(0,L)}\right) \le \frac{C}{\varepsilon^4}\|u\|^2_{S_\varepsilon},$$

and then,

$$\|\mathcal{R}_2\|^2_{L^2(\mathfrak{G}_\varepsilon^{(2)})} \le C\sum_{q=0}^{2n_\varepsilon}\sum_{p=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_2(p\varepsilon, q\varepsilon)|^2 + C\varepsilon^2\|\partial_2\mathcal{R}_2\|^2_{L^2(\mathfrak{G}_\varepsilon^{(2)})} \le \frac{C}{\varepsilon^4}\|u\|^2_{S_\varepsilon}.$$

By a symmetrical argumentation, we prove that $\|\mathcal{R}_2\|_{L^2(\mathfrak{G}_\varepsilon^{(1)})} \le C/\varepsilon^2\|u\|_{S_\varepsilon}$ and thus $\|\mathcal{R}_2\|_{H^1(\mathfrak{G}_\varepsilon)} \le C/\varepsilon^2\|u\|_{S_\varepsilon}$.
In the same fashion, we prove $\|\mathcal{R}_1\|_{H^1(\mathfrak{G}_\varepsilon)} \le C/\varepsilon^2\|u\|_{S_\varepsilon}$.

Step 2. We prove the estimates of $\mathbb{U}_3$.
First, from estimates (32), equalities (33), and Lemma 1, we have

$$\|\mathbb{U}_3 - \mathbb{U}_3'^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \varepsilon\|\partial_\alpha\left(\mathbb{U}_3 - \mathbb{U}_3'^{(\alpha)}\right)\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \varepsilon^2\|\partial^2_{\alpha\alpha}\left(\mathbb{U}_3 - \mathbb{U}_3'^{(\alpha)}\right)\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le C\|u\|_{S_\varepsilon}. \tag{35}$$

Then, the second estimate in (28) and the above one lead to

$$\|\partial^2_{\alpha\alpha}\mathbb{U}_3^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon^2}\|u\|_{S_\varepsilon}.$$

In Section 6.1 of Falconi et al. [17], we show that $\mathbb{U}_3$ can be characterized as the restriction to $\mathfrak{G}_\varepsilon$ of a function $\mathbb{U}_3$, belonging to $H^2(\Omega)$. In particular, in every small cell $Y_{pq,\varepsilon} = (p\varepsilon, q\varepsilon) + \varepsilon^2$, the function $\mathbb{U}_3$ is a polynomial with respect to z_1 and z_2, which depends only on the values of $\mathbb{U}_3$, $\mathcal{R}_1$, and $\mathcal{R}_2$ at the vertices of the cell $Y_{pq,\varepsilon}$ and satisfies

$$\begin{aligned}&\mathbb{U}_{3|\mathfrak{G}_\varepsilon} = \mathbb{U}_3, \qquad (\partial_1\mathbb{U}_3)_{|\mathfrak{G}_\varepsilon} = -\mathcal{R}_2, \qquad (\partial_2\mathbb{U}_3)_{|\mathfrak{G}_\varepsilon} = \mathcal{R}_1,\\ &\mathbb{U}_3 = 0,\ \nabla\mathbb{U}_3 = 0 \ \text{ a.e. on } \{0\}\times(0,l)\cup(0,l)\times\{0\}.\end{aligned} \tag{36}$$

Thus, one has the estimate

$$\|\mathbb{U}_3\|_{H^2(\Omega)} \le C\|\nabla^2\mathbb{U}_3\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\left(\|\partial^2_{\alpha\alpha}\mathbb{U}_3^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \|\mathcal{R}_1\|_{H^1(\mathfrak{G}_\varepsilon)} + \|\mathcal{R}_2\|_{H^1(\mathfrak{G}_\varepsilon)}\right) \le \frac{C}{\varepsilon^{3/2}}\|u\|_{S_\varepsilon}. \tag{37}$$

This first gives

$$\|\mathbb{U}_3\|_{L^2(\mathfrak{G}_\varepsilon)} \le \frac{C}{\varepsilon^2}\|u\|_{S_\varepsilon}.$$

Then, we obtain $\|\partial_\alpha\mathbb{U}_3\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le C/\varepsilon^2\|u\|_{S_\varepsilon}$.

□

We now consider the in-plane components. We define the functions $\mathbb{U}_1^{(\alpha)}$, $\mathbb{U}_2^{(\alpha)}$, and $\mathcal{R}_3^{(\alpha)}$ by

$$\mathbb{U}_1^{(\alpha)} = \mathbb{U}_1'^{(\alpha)},\ \mathbb{U}_2^{(\alpha)} = \mathbb{U}_2'^{(\alpha)},\ \mathcal{R}_3^{(\alpha)} = \mathcal{R}_3'^{(\alpha)} \text{ a.e. } \mathfrak{G}_\varepsilon^{(\alpha)}.$$

Due to (10) in the respective direction, these fields satisfy the equalities $\partial_1\mathbb{U}_2^{(1)} = \mathcal{R}_3^{(1)}$ a.e. in $\mathfrak{G}_\varepsilon^{(1)}$ and $\partial_2\mathbb{U}_1^{(2)} = -\mathcal{R}_3^{(2)}$ a.e. in $\mathfrak{G}_\varepsilon^{(2)}$. Moreover, note that from (28), we have

$$\|\partial_\alpha\mathcal{R}_3^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon^2}\|u\|_{S_\varepsilon}, \qquad \|\partial_\alpha\mathbb{U}_\alpha^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon}\|u\|_{S_\varepsilon}, \qquad \|\partial^2_{\alpha\alpha}\mathbb{U}_{3-\alpha}^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le \frac{C}{\varepsilon^2}\|u\|_{S_\varepsilon}. \tag{38}$$

Below, in the proof of Propositions 2 and 3, we will use the following classical result:

$$\left.\begin{aligned} &\sum_{k=0}^{2N_\varepsilon}\varepsilon|\phi(k\varepsilon)|^2 \le 2\|\phi\|^2_{L^2(0,L)} + \varepsilon^2\|\partial\phi\|_{L^2(0,L)},\\ &\|\phi\|^2_{L^2(0,L)} \le 2\varepsilon\sum_{k=0}^{2N_\varepsilon}|\phi(k\varepsilon)|^2 + \varepsilon^2\|\partial\phi\|_{L^2(0,L)}, \end{aligned}\right\} \quad \forall\phi\in H^1(0,L). \tag{39}$$

Proposition 2. *One has ($\alpha\in\{1,2\}$)*

$$\|\mathcal{R}_3^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le C\left(\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|_{S_\varepsilon}\right), \qquad \|\mathbb{U}_\alpha^{(3-\alpha)}\|_{H^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le C\left(\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|_{S_\varepsilon}\right). \tag{40}$$

Proof. We prove $(40)_1$ for $\alpha=2$. A similar argumentation will show this inequality for $\alpha=1$.
Equalities (33), together with $(38)_1$, $(39)_2$ imply

$$\sum_{q=0}^{2n_\varepsilon}\left(\|\mathcal{R}_3^{(1)}(\cdot,q\varepsilon)\|^2_{L^2(0,L)} + \|\partial_1\mathcal{R}_3^{(1)}(\cdot,q\varepsilon)\|^2_{L^2(0,L)}\right) \le \frac{C}{\varepsilon^4}\|u\|^2_{S_\varepsilon}.$$

One has (see 39_1)

$$\sum_{q=0}^{2n_\varepsilon}\sum_{p=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_3^{(1)}(p\varepsilon,q\varepsilon)|^2 \le C\sum_{q=0}^{2n_\varepsilon}\left(\|\mathcal{R}_3^{(1)}(\cdot,q\varepsilon)\|^2_{L^2(0,L)} + \varepsilon^2\|\partial_1\mathcal{R}_3^{(1)}(\cdot,q\varepsilon)\|^2_{L^2(0,L)}\right) \le \frac{C}{\varepsilon^4}\|u\|^2_{S_\varepsilon}.$$

Besides, from the in-plane contact estimates (31) and the above one, we obtain

$$\begin{aligned}\sum_{q=0}^{2n_\varepsilon}\sum_{p=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_3^{(2)}(p\varepsilon,q\varepsilon)|^2 &\le \sum_{q=0}^{2n_\varepsilon}\sum_{p=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_3^{(2)}(p\varepsilon,q\varepsilon) - \mathcal{R}_3^{(1)}(p\varepsilon,q\varepsilon)|^2 + \sum_{q=0}^{2n_\varepsilon}\sum_{p=0}^{2N_\varepsilon}\varepsilon|\mathcal{R}_3^{(1)}(p\varepsilon,q\varepsilon)|^2\\ &\le C\left(\varepsilon\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|^2_{S_\varepsilon} + \frac{1}{\varepsilon^4}\|u\|^2_{S_\varepsilon}\right) \le C\left(\varepsilon\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^4}\|u\|^2_{S_\varepsilon}\right).\end{aligned}$$

Hence,

$$\sum_{p=0}^{2N_\varepsilon} \|\mathcal{R}_3^{(2)}(p\varepsilon,\cdot)\|^2_{L^2(0,l)} \le C\sum_{p=0}^{2N_\varepsilon}\left(\sum_{q=0}^{2n_\varepsilon} \varepsilon|\mathcal{R}_3^{(2)}(p\varepsilon,q\varepsilon)|^2 + \varepsilon^2\|\partial_2\mathcal{R}_3^{(2)}(p\varepsilon,\cdot)\|^2_{L^2(0,l)}\right) \le C\left(\varepsilon\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^4}\|u\|^2_{S_\varepsilon}\right),$$

and thus,

$$\|\mathcal{R}_3^{(2)}\|^2_{L^2(\mathfrak{G}_\varepsilon^{(2)})} \le C\left(\sum_{p=0}^{2N_\varepsilon}\|\mathcal{R}_3^{(2)}(p\varepsilon,\cdot)\|^2_{L^2(0,l)} + \|\partial_2\mathcal{R}_3^{(2)}\|^2_{L^2(\mathfrak{G}^{(2)})}\right) \le C\left(\varepsilon\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^4}\|u\|^2_{S_\varepsilon}\right),$$

which proves the first estimate in $(40)_1$ for $\alpha = 2$.

Then, from estimate $(38)_3$, $(40)_1$ and the fact that $\partial_1\mathbb{U}_2^{(1)} = \mathcal{R}_3^{(1)}$, $\partial_2\mathbb{U}_1^{(2)} = -\mathcal{R}_3^{(2)}$, we obtain

$$\|\partial_\alpha\mathbb{U}_\alpha^{(3-\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} \le C\left(\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|_{S_\varepsilon}\right).$$

Now, proceeding as to obtain the L^2 norm of $\mathcal{R}_3^{(\alpha)}$, we obtain the L^2 norm of $\mathbb{U}_\alpha^{(3-\alpha)}$, so $(40)_2$ is proved. □

Now that the fields are set, we can construct the Bernoulli–Navier displacements $\mathbb{U}_{BN}^{(\alpha)}$ by

$$\begin{aligned}
&U_{BN}^{(1)}(z_1,q\varepsilon,y_2,y_3) = \left(\mathbb{U}_1^{(1)}\mathbf{e}_1 + \mathbb{U}_2^{(1)}\mathbf{e}_2 + \mathbb{U}_3\mathbf{e}_3\right)(z_1,q\varepsilon)\\
&+\left(\mathcal{R}_1\mathbf{e}_1 + \mathcal{R}_2\mathbf{e}_2 + \mathcal{R}_3^{(1)}\mathbf{e}_3\right)(z_1,q\varepsilon)\wedge\left(\Phi_\varepsilon^{(1)}(z_1,q\varepsilon)\mathbf{e}_3 + y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon^{(1)}(z_1,q\varepsilon)\right) \text{ for a.e. } (z_1,q\varepsilon,y_2,y_3)\in\mathfrak{G}^{(1)}\times\omega_r,\\
&U_{BN}^{(2)}(p\varepsilon,z_2,y_1,y_3) = \left(\mathbb{U}_1^{(2)}\mathbf{e}_1 + \mathbb{U}_2^{(2)}\mathbf{e}_2 + \mathbb{U}_3\mathbf{e}_3\right)(p\varepsilon,z_2)\\
&+\left(\mathcal{R}_1\mathbf{e}_1 + \mathcal{R}_2\mathbf{e}_2 + \mathcal{R}_3^{(1)}\mathbf{e}_3\right)(p\varepsilon,z_2)\wedge\left(\Phi_\varepsilon^{(2)}(p\varepsilon,z_2)\mathbf{e}_3 + y_1\mathbf{e}_1 + y_3\mathbf{n}_\varepsilon^{(2)}(p\varepsilon,z_2)\right) \text{ for a.e. } (p\varepsilon,z_2,y_1,y_3)\in\mathfrak{G}^{(1)}\times\omega_r.
\end{aligned} \tag{41}$$

As a consequence, the residual displacements are

$$\overline{u}^{(\alpha)} = u^{(\alpha)} - \mathbb{U}_{BN}^{(\alpha)} \in H^1(S_\varepsilon^{(\alpha)}),$$

and, due to the third estimate in (28), estimates (34), (35), they satisfy

$$\|\overline{u}^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r)} + \varepsilon\|\nabla\overline{u}^{(\alpha)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r)} \le C\varepsilon\|u\|_{S_\varepsilon}. \tag{42}$$

Note that this last estimate is of the same order as the residual displacement in the prime decomposition (15) and as the classical one (8). Hence, it justifies the choice of the final decomposition, being close enough to both the prime and the classical one.

6.4 | The final split of the in-plane "centerline" displacements

In this subsection, we operate the final split of the in-plane centerlines fields $\mathbb{U}_\alpha^{(1)}$ and $\mathbb{U}_\alpha^{(2)}$.

A preliminary remark before the split.

Remark 2. We remind that a function ϕ, belonging to $H^1(\mathfrak{G}_\varepsilon^{(\alpha)})$, could be extended (if necessary) to a function (still denoted ϕ) belonging to $H^1(\mathfrak{G}_\varepsilon)$, affine between two consecutive nodes of the lines in $\mathfrak{G}_\varepsilon^{(3-\alpha)}$. In particular, we have

$$\|\phi\|_{L^2(\mathfrak{G}_\varepsilon)} + \varepsilon\|\partial_{3-\alpha}\phi\|_{L^2(\mathfrak{G}_\varepsilon^{(3-\alpha)})} \le C\sqrt{\sum_{(p,q)\in\mathcal{K}_\varepsilon}\varepsilon|\phi(p\varepsilon,q\varepsilon)|^2} \le C\left(\|\phi\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})} + \varepsilon\|\partial_\alpha\phi\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})}\right). \tag{43}$$

Proposition 3. *There exist $\mathbb{U}_\alpha \in H^2(0,L)$ and $\mathbb{U}_\alpha^{(\boldsymbol{S})} \in H^1(\mathfrak{G}_\varepsilon^{(\alpha)})$, and $\mathbb{U}_\alpha^{(\boldsymbol{B})} \in H^2(\mathfrak{G}_\varepsilon^{(3-\alpha)})$, such that*

$$\begin{aligned}
&\|\mathbb{U}_\alpha\|_{H^2(0,L)} \le C\left(\varepsilon\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^{3/2}}\|u\|_{S_\varepsilon}\right), \qquad \|\mathbb{U}_\alpha^{(\boldsymbol{S})}\|_{H^1(\mathfrak{G}_\varepsilon^{(\alpha)})} \le C\left(\varepsilon^{3/2}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{S_\varepsilon}\right),\\
&\|\mathbb{U}_\alpha^{(\boldsymbol{B})}\|_{H^2(\mathfrak{G}_\varepsilon^{(3-\alpha)})} \le C\left(\sqrt{\varepsilon}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2}\|u\|_{S_\varepsilon}\right).
\end{aligned} \tag{44}$$

Moreover, we have the following improvements of some L^2 norms:

$$\|\mathbb{U}_\alpha\|_{L^2(0,l)} \le C\varepsilon\left(\varepsilon\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^{3/2}}\|u\|_{S_\varepsilon}\right), \\ \|\mathbb{U}_\alpha^{(\boldsymbol{B})}\|_{L^2(\mathfrak{G}_\varepsilon^{(3-\alpha)})} \le C\left(\varepsilon^{3/2}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{S_\varepsilon}\right). \tag{45}$$

Proof. We prove the proposition for $\alpha = 1$, for $\alpha = 2$, the proof is similar.

Step 1. We define the fields $\mathbb{U}_1$, $\mathbb{U}_1^{(\mathbf{B})}$, and $\mathbb{U}_1^{(\mathbf{S})}$.

From the estimates (31) and $(38)_2$ and the definition of $\mathbb{U}_1^{(\alpha)}$, we have

$$\sum_{p=0}^{2N_\varepsilon}\left(\sum_{q=0}^{2N_\varepsilon} |\left(\mathbb{U}_1^{(1)} - \mathbb{U}_1^{(2)}\right)(p\varepsilon, q\varepsilon)|^2 + \varepsilon\|\mathbb{U}_1^{(2)}(p\varepsilon,\cdot)\|^2_{H^2(0,L)}\right) \le C\left(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{S_\varepsilon}\right).$$

Then, there exists $p_\varepsilon \in \{0, \dots, 2N_\varepsilon\}$ such that

$$\begin{aligned}&\sum_{q=0}^{2N_\varepsilon} |\left(\mathbb{U}_1^{(1)} - \mathbb{U}_1^{(2)}\right)(p_\varepsilon\varepsilon, q\varepsilon)|^2 + \varepsilon\|\mathbb{U}_1^{(2)}(p_\varepsilon\varepsilon,\cdot)\|^2_{H^2(0,L)} \\ &\le \frac{1}{2N_\varepsilon+1}\sum_{p=0}^{2N_\varepsilon}\left(\sum_{q=0}^{2N_\varepsilon} |\left(\mathbb{U}_1^{(1)} - \mathbb{U}_1^{(2)}\right)(p\varepsilon, q\varepsilon)|^2 + \varepsilon\|\mathbb{U}_1^{(2)}(p\varepsilon,\cdot)\|^2_{H^2(0,L)}\right) \\ &\le C\varepsilon\left(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{S_\varepsilon}\right).\end{aligned} \tag{46}$$

We set

$$\begin{aligned}&\mathbb{U}_1(z_2) \doteq \mathbb{U}_1^{(2)}(p_\varepsilon\varepsilon, z_2) \text{ for a.e. } z_2 \in (0,L),\ \mathbb{U}_1^{(\mathbf{B})} \\ &\doteq \begin{cases}\mathbb{U}_1^{(2)}(p\varepsilon, z_2) - \mathbb{U}_1(z_2) & \text{for a.e. } z_2 \in (0,L),\quad p \in \{2n_\varepsilon, \dots, 2N_\varepsilon\}, \\ \mathbb{U}_1^{(2)}(p\varepsilon, z_2) & \text{for a.e. } z_2 \in (0,L),\quad p \in \{0, \dots, 2n_\varepsilon - 1\},\end{cases} \\ &\mathbb{U}_1^{(\mathbf{S})} \doteq \begin{cases}\mathbb{U}_1^{(1)}(z_1, q\varepsilon) - \mathbb{U}_1(q\varepsilon) & \text{for a.e. } z_1 \in (0,L),\quad q \in \{2n_\varepsilon, \dots, 2N_\varepsilon\}, \\ \mathbb{U}_1^{(1)}(z_1, q\varepsilon) & \text{for a.e. } z_1 \in (0,L),\quad q \in \{0, \dots, 2n_\varepsilon - 1\}.\end{cases}\end{aligned}$$

By construction, we have $\mathbb{U}_1 \in H^2(0,L)$, $\mathbb{U}_1^{(\mathbf{B})} \in H^2(\mathfrak{G}_\varepsilon^{(2)})$, and $\mathbb{U}_1^{(\mathbf{S})} \in H^1(\mathfrak{G}_\varepsilon^{(1)})$.

In a similar way, we define $\mathbb{U}_2 \in H^2(0,L)$ (function of z_1), $\mathbb{U}_2^{(\mathbf{B})} \in H^2(\mathfrak{G}_\varepsilon^{(1)})$, and $\mathbb{U}_2^{(\mathbf{S})} \in H^1(\mathfrak{G}_\varepsilon^{(2)})$.

Step 2. We prove the estimates (44).

By definition of $\mathbb{U}_1$ and due to (46), we first have $(44)_1$. Then, again from (46), we have

$$\sum_{q=0}^{2N_\varepsilon} |\mathbb{U}_1^{(1)}(p_\varepsilon\varepsilon, q\varepsilon) - \mathbb{U}_1(q\varepsilon)|^2 \le C\varepsilon\left(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{S_\varepsilon}\right). \tag{47}$$

The above inequality, together with $(44)_1$ and $(39)_1$, yields

$$\sum_{q=0}^{2N_\varepsilon} \varepsilon|\mathbb{U}_1^{(1)}(p_\varepsilon\varepsilon, q\varepsilon)|^2 \le C\varepsilon\left(\varepsilon^2\|g\|^2_{L^\infty(\Omega)} + \frac{1}{\varepsilon^3}\|u\|^2_{S_\varepsilon}\right),$$

which in turn with $(38)_2$ and $(39)_2$ leads to

$$\|\mathbb{U}_1^{(1)} - \mathbb{U}_1\|_{H^1(\mathfrak{G}_\varepsilon^{(1)})} \le C\left(\varepsilon^{3/2}\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon}\|u\|_{S_\varepsilon}\right). \tag{48}$$

Now, from (40) and $(44)_1$, one obtains

$$\begin{aligned}\|\mathbb{U}_1^{(2)}-\mathbb{U}_1\|^2_{H^2(\mathfrak{G}_\varepsilon^{(2)})} &\le 2\sum_{p=0}^{2N_\varepsilon}\left(\|\mathbb{U}_1^{(2)}(p\varepsilon,\cdot)\|^2_{H^2(0,L)}+\|\mathbb{U}_1\|^2_{H^2(0,L)}\right)\\ &\le C\left(\varepsilon\|g\|^2_{L^\infty(\Omega)}+\frac{1}{\varepsilon^4}\|u\|^2_{\mathcal{S}_\varepsilon}\right)+(2N_\varepsilon+1)C\left(\varepsilon^2\|g\|^2_{L^\infty(\Omega)}+\frac{1}{\varepsilon^3}\|u\|^2_{\mathcal{S}_\varepsilon}\right).\end{aligned} \tag{49}$$

This gives $(44)_3$.

Step 3. We prove the estimates (45).

From $(38)_2$, we get (remind that $\mathbb{U}_1^{(1)}(0,q\varepsilon)=0$ if $q\in\{0,\dots,2n_\varepsilon\}$)

$$\sum_{q=0}^{2n_\varepsilon}|\mathbb{U}_1^{(1)}(p_\varepsilon\varepsilon,q\varepsilon)|^2\le L\sum_{q=0}^{2n_\varepsilon}\|\partial_1\mathbb{U}_1^{(1)}(\cdot,q\varepsilon)\|^2_{L^2(0,L)}\le C\frac{1}{\varepsilon^2}\|u\|^2_{\mathcal{S}_\varepsilon}.$$

This, together with (47), yields

$$\sum_{q=0}^{2n_\varepsilon}|\mathbb{U}_1(q\varepsilon)|^2\le C\varepsilon\left(\varepsilon^2\|g\|^2_{L^\infty(\Omega)}+\frac{1}{\varepsilon^3}\|u\|^2_{\mathcal{S}_\varepsilon}\right).$$

The above and $(39)_2$, $(44)_1$ lead to

$$\|\mathbb{U}_1\|^2_{L^2(0,l)}\le C\left(\sum_{q=0}^{2n_\varepsilon}\varepsilon|\mathbb{U}_1(q\varepsilon)|^2+\varepsilon^2\|\partial_2\mathbb{U}_1\|^2_{L^2(0,l)}\right)\le C\varepsilon^2\left(\varepsilon^2\|g\|^2_{L^\infty(\Omega)}+\frac{1}{\varepsilon^3}\|u\|^2_{\mathcal{S}_\varepsilon}\right),$$

which proves $(45)_1$. The above, together with (48) (resp. 49), leads to $(44)_2$ (resp. 44_3). We have $\left(\mathbb{U}_1^{(\mathbf{B})}-\mathbb{U}_1^{(\mathbf{S})}\right)(p\varepsilon,q\varepsilon)=\left(\mathbb{U}_1^{(2)}-\mathbb{U}_1^{(1)}\right)(p\varepsilon,q\varepsilon)$ for all $(p,q)\in\mathcal{K}_\varepsilon$. Hence, from the in-plane contact estimates (31), we obtain

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}|(\mathbb{U}_1^{(\mathbf{B})}-\mathbb{U}_1^{(\mathbf{S})})(p\varepsilon,q\varepsilon)|^2\le C\left(\varepsilon^2\|g\|^2_{L^\infty(\Omega)}+\frac{1}{\varepsilon^3}\|u\|^2_{\mathcal{S}_\varepsilon}\right).$$

Then, the estimate $(44)_2$ of $\mathbb{U}_1^{(\mathbf{S})}$ and $(39)_1$ yield

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\varepsilon|\mathbb{U}_1^{(\mathbf{S})}(p\varepsilon,q\varepsilon)|^2\le C\|\mathbb{U}_1^{(\mathbf{S})}\|^2_{H^1(\mathfrak{G}_\varepsilon^{(1)})}\le C\left(\varepsilon^3\|g\|^2_{L^\infty(\Omega)}+\frac{1}{\varepsilon^2}\|u\|^2_{\mathcal{S}_\varepsilon}\right).$$

So

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\varepsilon|\mathbb{U}_1^{(\mathbf{B})}(p\varepsilon,q\varepsilon)|^2\le C\varepsilon\left(\varepsilon^2\|g\|^2_{L^\infty(\Omega)}+\frac{1}{\varepsilon^3}\|u\|^2_{\mathcal{S}_\varepsilon}\right).$$

Finally, we obtain $(45)_2$ from $(39)_2$ and $(44)_3$. □

We end this section by giving the final decomposition of the Bernoulli–Navier displacements (41):

$$\begin{aligned}&U_{BN}^{(1)}(z_1,q\varepsilon,y_2,y_3)=\left((\mathbb{U}_1+\mathbb{U}_1^{(\mathbf{S})})\mathbf{e}_1+(\mathbb{U}_2+\mathbb{U}_2^{(\mathbf{B})})\mathbf{e}_2+\mathbb{U}_3\mathbf{e}_3\right)(z_1,q\varepsilon)\\ &+\left(\mathcal{R}_1\mathbf{e}_1+\mathcal{R}_2\mathbf{e}_2+\mathcal{R}_3^{(1)}\mathbf{e}_3\right)(z_1,q\varepsilon)\wedge\left(\Phi_\varepsilon^{(1)}(z_1,q\varepsilon)\mathbf{e}_3+y_2\mathbf{e}_2+y_3\mathbf{n}_\varepsilon^{(1)}(z_1,q\varepsilon)\right)\ \text{for a.e. }(z_1,q\varepsilon,y_2,y_3)\in\mathfrak{G}^{(1)}\times\omega_r,\\ &U_{BN}^{(2)}(p\varepsilon,z_2,y_1,y_3)=\left((\mathbb{U}_1+\mathbb{U}_1^{(\mathbf{B})})\mathbf{e}_1+(\mathbb{U}_2+\mathbb{U}_2^{(\mathbf{S})})\mathbf{e}_2+\mathbb{U}_3\mathbf{e}_3\right)(z_2,p\varepsilon)\\ &+\left(\mathcal{R}_1\mathbf{e}_1+\mathcal{R}_2\mathbf{e}_2+\mathcal{R}_3^{(1)}\mathbf{e}_3\right)(z_2,p\varepsilon)\wedge\left(\Phi_\varepsilon^{(2)}(p\varepsilon,z_2)\mathbf{e}_3+y_1\mathbf{e}_1+y_3\mathbf{n}_\varepsilon^{(2)}(p\varepsilon,z_2)\right)\ \text{for a.e. }(p\varepsilon,z_2,y_1,y_3)\in\mathfrak{G}^{(2)}\times\omega_r.\end{aligned} \tag{50}$$

7 | WEAK CONVERGENCE VIA UNFOLDING OF THE DISPLACEMENT FIELDS

In this section, we first give sufficient assumptions on the right-hand side of the elasticity problem in order to give a bound for the displacement fields. Then, by compactness results, we show the weak convergences via the unfolding operator. We will then get the weak limits of the displacement fields, their associated strain tensors, the limit contact constraints, and the limit set.

7.1 | Assumption on the applied forces

Note that the estimates for the fields in Propositions 1 and 3 still depend on the norm of the strain tensor, and we need to eliminate such dependence.

We can bound the norm of the strain tensor by assuming sufficient forces to apply on the right-hand side of problem (26). From property (iii) of tensor $A_\varepsilon^{(\alpha)}$, applied to problem (25) with $v_\varepsilon^{(\alpha)} = 0$, we first have

$$C_0 \|u_\varepsilon\|_{S_\varepsilon}^2 \le \sum_{\alpha=1}^{2} \int_{\mathfrak{G}_\varepsilon^{(\alpha)} \times \omega_r} A_{ijkl,\varepsilon}^{(\alpha)} \, \widetilde{e}_{ij}^{(\alpha)}\left(u_\varepsilon^{(\alpha)}\right) \widetilde{e}_{kl}^{(\alpha)}\left(u_\varepsilon^{(\alpha)}\right) \eta_\varepsilon^{(\alpha)} dz_\alpha dy_{3-\alpha} dy_3 \le \sum_{\alpha=1}^{2} \left| \int_{\mathfrak{G}_\varepsilon^{(\alpha)} \times \omega_r} F_\varepsilon^{(\alpha)} \cdot u_\varepsilon^{(\alpha)} \eta_\varepsilon^{(\alpha)} dz_\alpha dy_{3-\alpha} dy_3 \right|. \tag{51}$$

Now, let $f^{(\alpha)} \in H^1(\Omega)^3$ and $\widetilde{f}^{(\alpha)} \in H^1(\Omega)^2$, such that

$$\widetilde{f}_1^{(\alpha)} = 0 \text{ a.e. in } \Omega_3 \cup \Omega_4, \qquad \widetilde{f}_2^{(\alpha)} = 0 \text{ a.e. in } \Omega_2 \cup \Omega_4.$$

We choose the forces for the right-hand side of problem (26) by setting

$$F_\varepsilon^{(1)} \doteq \varepsilon^{3/2} \begin{pmatrix} \widetilde{f}_1^{(1)} + \varepsilon f_1^{(1)} \\ \widetilde{f}_2^{(1)} + \varepsilon f_2^{(1)} \\ \varepsilon f_3^{(1)} \end{pmatrix} \text{ a.e. in } \mathfrak{G}_\varepsilon^{(1)}, \qquad F_\varepsilon^{(2)} \doteq \varepsilon^{3/2} \begin{pmatrix} \widetilde{f}_1^{(2)} + \varepsilon f_1^{(2)} \\ \widetilde{f}_2^{(2)} + \varepsilon f_2^{(2)} \\ \varepsilon f_3^{(2)} \end{pmatrix} \text{ a.e. in } \mathfrak{G}_\varepsilon^{(2)}. \tag{52}$$

The Hölder inequality, straightforward computation, and estimates in Propositions 2 and 3 lead to

$$\sum_{\alpha=1}^{2} \int_{\mathfrak{G}_\varepsilon^{(\alpha)} \times \omega_r} |F_\varepsilon^{(\alpha)}| \, |u_\varepsilon^{(\alpha)} \eta_\varepsilon^{(\alpha)}| \, dz_\alpha dy_{3-\alpha} dy_3 \le C\varepsilon^5 \sum_{\alpha=1}^{2} \left(\|\widetilde{f}^{(\alpha)}\|_{H^1(\Omega)} + \|f^{(\alpha)}\|_{H^1(\Omega)} \right) \left(\|g\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2 \sqrt{\varepsilon}} \|u_\varepsilon\|_{S_\varepsilon} \right),$$

which, together with (51), gives the following bound for the strain tensor:

$$\|u_\varepsilon\|_{S_\varepsilon} \le C\varepsilon^{5/2} \left(\|g\|_{L^\infty(\Omega)} + \sum_{\alpha=1}^{2} \left(\|\widetilde{f}^{(\alpha)}\|_{H^1(\Omega)} + \|f^{(\alpha)}\|_{H^1(\Omega)} \right) \right) \le C\varepsilon^{5/2}. \tag{53}$$

This estimation is completely determined by the assumption on the forces (52), which is the minimum amount of stress we apply to the problem. We choose the order so that the oscillations remain small and study the problem on a linear regime.

Finally, we can apply (53) to the estimates in Propositions 1 and 3 and extend the ones defined on lines to the whole grid by the meanings of (43). The explicit estimates for the final fields are

$$\begin{aligned} &\|\mathbb{U}_{\varepsilon,3}\|_{H^2(\mathfrak{G}_\varepsilon)} \le C\sqrt{\varepsilon}, \qquad \|\mathcal{R}_{\varepsilon,\alpha}\|_{H^1(\mathfrak{G}_\varepsilon)} + \|\mathcal{R}_{\varepsilon,3}^{(\alpha)}\|_{H^1(\mathfrak{G}_\varepsilon^{(\alpha)})} \le C\sqrt{\varepsilon}, \\ &\|\mathbb{U}_{\varepsilon,\alpha}\|_{H^2(0,L)} \le C\varepsilon, \ \|\mathbb{U}_{\varepsilon,\alpha}\|_{L^2(0,l)} \le C\varepsilon^2, \ \|\mathbb{U}_{\varepsilon,\alpha}^{(\mathbf{S})}\|_{H^1(\mathfrak{G}_\varepsilon^{(\alpha)})} \le C\varepsilon\sqrt{\varepsilon}, \\ &\|\mathbb{U}_{\varepsilon,\alpha}^{(\mathbf{B})}\|_{L^2(\mathfrak{G}_\varepsilon)} + \varepsilon\|\partial_\beta \mathbb{U}_{\varepsilon,\alpha}^{(\mathbf{B})}\|_{L^2(\mathfrak{G}_\varepsilon)} + \varepsilon\|\partial_{3-\alpha\,3-\alpha}^2 \mathbb{U}_{\varepsilon,\alpha}^{(\mathbf{B})}\|_{L^2(\mathfrak{G}_\varepsilon^{(3-\alpha)})} \le C\varepsilon\sqrt{\varepsilon}, \end{aligned} \tag{54}$$

while the ones for the residual terms come from (42):

$$\|\overline{u}^{(\alpha)}\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon\times\omega_r)} + \varepsilon\|\nabla\overline{u}^{(\alpha)}\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon\times\omega_r)} \le C\varepsilon^3\sqrt{\varepsilon}. \tag{55}$$

It is known that by compactness, these fields weakly converge in the space. In the next subsections, we will introduce the unfolding operators and go to the limit via unfolding.

7.2 | Preliminaries on the unfolding operators

The convergence via unfolding is done through three different unfolding operators, all related to each other:

- the middle line unfolding operator $\mathcal{T}^{\mathfrak{G}}_\varepsilon$, which unfolds the functions defined on the one-dimensional lattice $\mathfrak{G}_\varepsilon$, given by the middle lines of the displacements;
- the global unfolding operator Π_ε, which unfolds the functions defined on the whole three-dimensional textile structure $\mathcal{S}_\varepsilon$;
- the contact unfolding operator $T^{\mathbf{C}_{ab}}_\varepsilon$, which unfolds the functions defined on the two-dimensional contact domains $\mathbf{C}_{ab}$ for $(a,b)\in\{0,1\}^2$.

In this subsection, we introduce the first and most important middle line unfolding operator, which will give all the unfolded convergences of the displacement fields.

Define the reference lattice grid by ($(a,b)\in\{1,2\}^2$)

$$\mathfrak{G}^{(1)} \doteq (0,2)\times\{0,1\}, \qquad \mathfrak{G}^{(2)} \doteq \{0,1\}\times(0,2), \qquad \mathfrak{G} \doteq \mathfrak{G}^{(1)}\cup\mathfrak{G}^{(2)}.$$

Definition 1 (Middle line unfolding operator). For every measurable function ϕ on $\mathfrak{G}_\varepsilon$, one defines the measurable function $\mathcal{T}^{\mathfrak{G}}_\varepsilon(\phi)$ in $\Omega\times\mathfrak{G}$ by

$$\mathcal{T}^{\mathfrak{G}}_\varepsilon(\phi)(z',Y_1,Y_2) \doteq \phi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon(Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2)\right) \qquad \text{for a.e. } (z',Y_1,Y_2)\in\Omega\times\mathfrak{G}.$$

This operator is a more specific example of unfolding operators, defined on lattice domains, which have been widely investigated in Falconi et al. [17]. From Proposition 4.2 in Falconi et al. [17], we recall that such operator satisfies

$$\|\mathcal{T}^{\mathfrak{G}}_\varepsilon(\phi)\|_{L^2(\Omega\times\mathfrak{G})} \le C\sqrt{\varepsilon}\|\phi\|_{L^2(\mathfrak{G}_\varepsilon)}, \qquad \forall\phi\in L^2(\mathfrak{G}_\varepsilon).$$

Below, we recall and complete the main results obtained in Falconi et al. [17] in the context of this lattice structure.

Lemma 4 (Lemma 5.3 in Falconi et al. [17]). *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $H^1(\mathfrak{G}_\varepsilon)$, satisfying*

$$\|\phi_\varepsilon\|_{L^2(\mathfrak{G}_\varepsilon)} + \varepsilon\left(\|\partial_1\phi_\varepsilon\|_{L^2(\mathfrak{G}^{(1)}_\varepsilon)} + \|\partial_2\phi_\varepsilon\|_{L^2(\mathfrak{G}^{(2)}_\varepsilon)}\right) \le \frac{C}{\sqrt{\varepsilon}}.$$

There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$ and $\widehat{\phi}\in L^2(\Omega;H^1_{per}(\mathfrak{G}))$, such that

$$\mathcal{T}^{\mathfrak{G}}_\varepsilon(\phi_\varepsilon) \rightharpoonup \widehat{\phi} \text{ weakly in } L^2(\Omega;H^1(\mathfrak{G})).$$

Lemma 5 (Lemma 5.7 in Falconi et al. [17]). *Let $\{\phi_\varepsilon\}_\varepsilon$ be a sequence in $H^1(\mathfrak{G}_\varepsilon)$, satisfying ($\alpha\in\{1,2\}$)*

$$\|\phi_\varepsilon\|_{L^2(\mathfrak{G}_\varepsilon)} + \|\partial_\alpha\phi_\varepsilon\|_{L^2(\mathfrak{G}^{(\alpha)}_\varepsilon)} + \varepsilon\|\partial_{3-\alpha}\phi_\varepsilon\|_{L^2(\mathfrak{G}^{(3-\alpha)}_\varepsilon)} \le \frac{C}{\sqrt{\varepsilon}}.$$

There exist a subsequence of $\{\varepsilon\}$, *still denoted* $\{\varepsilon\}$, $\widetilde{\phi} \in L^2(\Omega, \partial_\alpha; H^1_{per}(\mathfrak{G}^{(3-\alpha)}))$ *and* $\widehat{\phi} \in L^2(\Omega; H^1_{per}(\mathfrak{G}))$, *such that*

$$\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi_\varepsilon) \rightharpoonup \widetilde{\phi} \text{ weakly in } L^2(\Omega; H^1(\mathfrak{G})),$$
$$\mathcal{T}_\varepsilon^{\mathfrak{G}}(\partial_\alpha \phi_\varepsilon) \rightharpoonup \partial_\alpha \widetilde{\phi} + \partial_{Y_\alpha} \widehat{\phi} \text{ weakly in } L^2(\Omega \times \mathfrak{G}^{(\alpha)}).$$

We recall that

$$L^2(\Omega, \partial_\alpha; H^1_{per}(\mathfrak{G}^{(3-\alpha)})) = \left\{ \phi \in L^2(\Omega \times \mathfrak{G}^{(3-\alpha)}) \;\middle|\; \partial_\phi \in L^2(\Omega \times \mathfrak{G}^{(3-\alpha)}) \text{ and } \phi \in L^2(\Omega; H^1_{per}(\mathfrak{G}^{(3-\alpha)})) \right\}.$$

Lemma 6 (Corollary 5.8 in Falconi et al. [17]). *Let* $\{\phi_\varepsilon\}_\varepsilon$ *be a sequence in* $H^1(\mathfrak{G}_\varepsilon)$, *satisfying*

$$\|\phi_\varepsilon\|_{H^1(\mathfrak{G}_\varepsilon)} \le \frac{C}{\sqrt{\varepsilon}}.$$

There exist a subsequence of $\{\varepsilon\}$, *still denoted* $\{\varepsilon\}$, *and* $\phi \in H^1(\Omega)$, *and* $\widehat{\phi} \in L^2(\Omega; H^1_{per}(\mathfrak{G}))$, *such that* ($\alpha \in \{1,2\}$)

$$\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi_\varepsilon) \to \phi \text{ strongly in } L^2(\Omega; H^1(\mathfrak{G})),$$
$$\mathcal{T}_\varepsilon^{\mathfrak{G}}(\partial_\alpha \phi_\varepsilon) \rightharpoonup \partial_\alpha \phi + \partial_{Y_\alpha} \widehat{\phi} \text{ weakly in } L^2(\Omega \times \mathfrak{G}^{(\alpha)}).$$

Theorem 2 (Theorem 6.3 in Falconi et al. [17]). *Let* $\{\phi_\varepsilon\}_\varepsilon$ *be a sequence in* $H^2(\mathfrak{G}_\varepsilon)$, *satisfying*

$$\|\phi_\varepsilon\|^2_{H^2(\mathfrak{G}_\varepsilon)} + \sum_{p=0}^{2N_\varepsilon} \sum_{q=0}^{2N_\varepsilon-1} \varepsilon \left| \frac{\partial_1 \phi_\varepsilon(p\varepsilon, q\varepsilon+\varepsilon) - \partial_1 \phi_\varepsilon(p\varepsilon, q\varepsilon)}{\varepsilon} \right|^2 + \sum_{p=0}^{2N_\varepsilon-1} \sum_{q=0}^{2N_\varepsilon} \varepsilon \left| \frac{\partial_2 \phi_\varepsilon(p\varepsilon+\varepsilon, q\varepsilon) - \partial_2 \phi_\varepsilon(p\varepsilon, q\varepsilon)}{\varepsilon} \right|^2 \le \frac{C}{\varepsilon}.$$

There exist a subsequence of $\{\varepsilon\}$, *still denoted* $\{\varepsilon\}$, *and functions* $\phi \in H^2(\Omega)$ *and* $\widehat{\phi} \in L^2(\Omega; H^2_{per}(\mathfrak{G}))$, *such that* ($\alpha \in \{1,2\}$)

$$\mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi_\varepsilon) \to \phi \;\text{ strongly in } L^p(\Omega; H^2(\mathfrak{G})),$$
$$\mathcal{T}_\varepsilon^{\mathfrak{G}}(\partial_\alpha \phi_\varepsilon) \to \partial_\alpha \phi \;\text{ strongly in } L^2(\Omega; H^1(\mathfrak{G}^{(\alpha)})),$$
$$\mathcal{T}_\varepsilon^{\mathfrak{G}}(\partial_{\alpha\alpha} \phi_\varepsilon) \rightharpoonup \partial^2_{\alpha\alpha} \phi + \partial^2_{Y_\alpha Y_\alpha} \widehat{\phi} \text{ weakly in } L^2(\Omega \times \mathfrak{G}^{(\alpha)}).$$

Below, we recall the definition of the periodic unfolding operator for functions defined a.e. in Ω (for its properties, see Cioranescu et al. [4]).

Definition 2. For every measurable function ϕ in Ω, one defines the measurable function $\mathcal{T}_\varepsilon(\phi)$ in $\Omega \times \mathcal{Y}$, $\mathcal{Y} \doteq (0,2)^2$, by

$$\mathcal{T}_\varepsilon(z', Y_1, Y_2) \doteq \phi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon(Y_1 \mathbf{e}_1 + Y_2 \mathbf{e}_2)\right) \qquad \text{for a.e. } (z', Y_1, Y_2) \in \Omega \times \mathcal{Y}.$$

Note that if $\phi \in H^1(\Omega)$, then

$$\mathcal{T}_\varepsilon(\phi)_{|\Omega \times \mathfrak{G}} = \mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi_{|\mathfrak{G}_\varepsilon}). \tag{56}$$

7.3 | Limit displacement fields via the middle line unfolding operator.

We first define the limit boundary condition

$$\mathbf{\Gamma} \doteq \{0\} \times (0,l) \cup (0,l) \times \{0\}.$$

Then, we set the limit spaces

$$\begin{aligned} H^1_{\mathbf{\Gamma}}(\Omega) &\doteq \left\{ \phi \in H^1(\Omega) \;|\; \phi = 0 \text{ a.e. on } \mathbf{\Gamma} \right\}, \\ H^2_{\mathbf{\Gamma}}(\Omega) &\doteq \left\{ \phi \in H^2(\Omega) \;|\; \phi = 0 \;\text{ and }\; \nabla\phi = 0 \text{ a.e. on } \mathbf{\Gamma} \right\}, \end{aligned} \tag{57}$$

and

$$
\begin{aligned}
L^2_{(0,l)}(0,L) &\doteq \left\{\phi \in L^2(0,L) \;\mid\; \phi = 0 \;\text{ a.e. in }\; (0,l)\right\},\\
H^k_{(0,l)}(0,L) &\doteq \left\{\phi \in H^1(0,L) \;\mid\; \phi = 0 \;\text{ a.e. in }\; (0,l)\right\},\; k \in \{1,2\},\\
L^2(\Omega \times \{0,1\}, \partial_\alpha) &\doteq \left\{\phi \in L^2(\Omega \times \{0,1\}) \mid \partial_\alpha \phi \in L^2(\Omega \times \{0,1\})\right\},\\
\mathbf{L}^2(\Omega \times \{0,1\}, \partial_1) &\doteq \left\{\phi \in L^2(\Omega \times \{0,1\}, \partial_1) \mid \phi = 0 \text{ a.e. on }\; \{0\} \times (0,l)\right\},\\
\mathbf{L}^2(\Omega \times \{0,1\}, \partial_2) &\doteq \left\{\phi \in L^2(\Omega \times \{0,1\}, \partial_2) \mid \phi = 0 \text{ a.e. on }\; (0,l) \times \{0\}\right\}.
\end{aligned}
\tag{58}
$$

We are ready to give the asymptotic behavior of our unfolded sequences.

Lemma 7. *There exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions:*

- $\mathbb{U}_3 \in H^2_{\boldsymbol{\Gamma}}(\Omega)$, $\widehat{\mathbb{U}}_3 \in L^2(\Omega; H^2_{per}(\mathfrak{G}))$ *such that*

$$
\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\mathbb{U}_{\varepsilon,3}\right) \to \mathbb{U}_3 \;\textit{ strongly in }\; L^2(\Omega; H^2(\mathfrak{G})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\partial_\alpha \mathbb{U}_{\varepsilon,3}\right) \to \partial_\alpha \mathbb{U}_3 \;\textit{ strongly in }\; L^2(\Omega; H^1(\mathfrak{G}^{(\alpha)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\partial^2_{\alpha\alpha} \mathbb{U}_{\varepsilon,3}\right) \rightharpoonup \partial^2_{\alpha\alpha}\mathbb{U}_3 + \partial^2_{Y_\alpha Y_\alpha}\widehat{\mathbb{U}}_3 \;\textit{ weakly in }\; L^2(\Omega \times \mathfrak{G}^{(\alpha)}).
\end{aligned}
\tag{59}
$$

- $\widehat{\mathcal{R}}_\alpha \in L^2(\Omega; H^1_{per,0}(\mathfrak{G}))$ *such that*

$$
\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\mathcal{R}_{\varepsilon,1}\right) \to \partial_2 \mathbb{U}_3 \;\textit{ strongly in }\; L^2(\Omega; H^1(\mathfrak{G})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\partial_\alpha \mathcal{R}_{\varepsilon,1}\right) \rightharpoonup \partial_{\alpha 2}\mathbb{U}_3 + \partial_{Y_\alpha}\widehat{\mathcal{R}}_1 \;\textit{ weakly in }\; L^2(\Omega; H^1(\mathfrak{G}^{(\alpha)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\mathcal{R}_{\varepsilon,2}\right) \to -\partial_1 \mathbb{U}_3 \;\textit{ strongly in }\; L^2(\Omega; H^1(\mathfrak{G})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\partial_\alpha \mathcal{R}_{\varepsilon,2}\right) \rightharpoonup -\partial_{\alpha 1}\mathbb{U}_3 + \partial_{Y_\alpha}\widehat{\mathcal{R}}_2 \;\textit{ weakly in }\; L^2(\Omega; H^1(\mathfrak{G}^{(\alpha)})).
\end{aligned}
\tag{60}
$$

Moreover, we have

$$
\partial_{Y_1}\widehat{\mathbb{U}}_3 = -\widehat{\mathcal{R}}_2 \;\textit{ a.e. in }\; \Omega \times \mathfrak{G}^{(1)}, \qquad \partial_{Y_2}\widehat{\mathbb{U}}_3 = \widehat{\mathcal{R}}_1 \;\textit{ a.e. in }\; \Omega \times \mathfrak{G}^{(2)}.
\tag{61}
$$

- $\mathbb{U}_\alpha \in H^2_{(0,l)}\left((0,L)_{z_{3-\alpha}}\right)$, $\widehat{\mathbb{U}}_\alpha \in L^2((0,L); H^2_{per}((0,2)_{Y_{3-\alpha}})))$ ($\widehat{\mathbb{U}}_\alpha(z_{3-\alpha},\cdot) = 0$ *a.e. in* $(0,l)\times(0,2)$) *such that*

$$
\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\mathbb{U}_{\varepsilon,\alpha}\right) \to \mathbb{U}_\alpha \;\textit{ strongly in }\; L^2(\Omega; H^2(\mathfrak{G}^{(3-\alpha)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\partial_{3-\alpha} \mathbb{U}_{\varepsilon,\alpha}\right) \to \partial_{3-\alpha} \mathbb{U}_\alpha \;\textit{ strongly in }\; L^2(\Omega; H^1(\mathfrak{G}^{(3-\alpha)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\partial^2_{3-\alpha 3-\alpha} \mathbb{U}_{\varepsilon,\alpha}\right) \rightharpoonup \partial^2_{3-\alpha 3-\alpha}\mathbb{U}_\alpha + \partial^2_{Y_{3-\alpha} Y_{3-\alpha}}\widehat{\mathbb{U}}_\alpha \;\textit{ weakly in }\; L^2(\Omega \times \mathfrak{G}^{(3-\alpha)}),
\end{aligned}
\tag{62}
$$

and

$$
\begin{aligned}
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\mathcal{R}^{(1)}_{\varepsilon,3}\right) \to \partial_1 \mathbb{U}_2 \;\textit{ strongly in }\; L^2(\Omega; H^1(\mathfrak{G}^{(1)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\partial_1 \mathcal{R}^{(1)}_{\varepsilon,3}\right) \rightharpoonup \partial_{11}\mathbb{U}_2 + \partial^2_{Y_1 Y_1}\widehat{\mathbb{U}}_2 \;\textit{ weakly in }\; L^2(\Omega; H^1(\mathfrak{G}^{(1)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\mathcal{R}^{(2)}_{\varepsilon,3}\right) \to -\partial_2 \mathbb{U}_1 \;\textit{ strongly in }\; L^2(\Omega; H^1(\mathfrak{G}^{(2)})),\\
&\frac{1}{\varepsilon}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\partial_2 \mathcal{R}^{(2)}_{\varepsilon,3}\right) \rightharpoonup -\partial_{22}\mathbb{U}_1 - \partial^2_{Y_2 Y_2}\widehat{\mathbb{U}}_1; \;\textit{ weakly in }\; L^2(\Omega; H^1(\mathfrak{G}^{(2)})).
\end{aligned}
\tag{63}
$$

- $\mathbb{U}^{(\boldsymbol{B})}_\alpha \in L^2(\Omega; H^2_{per}((0,2)_{Y_{3-\alpha}}))$, *such that*

$$
\frac{1}{\varepsilon^2}\mathcal{T}^{\mathfrak{G}}_\varepsilon\left(\mathbb{U}^{(\boldsymbol{B})}_{\varepsilon,\alpha}\right) \rightharpoonup \mathbb{U}^{(\boldsymbol{B})}_\alpha \;\textit{ weakly in }\; L^2(\Omega; H^2(\mathfrak{G}^{(3-\alpha)})) \cap L^2(\Omega; H^1(\mathfrak{G})).
\tag{64}
$$

- $\mathbb{U}_\alpha^{(\boldsymbol{S})} \in L^2(\Omega, \partial_\alpha; H^1_{per}(\mathfrak{G}))$, $\widehat{\mathbb{U}}_\alpha^{(\boldsymbol{S})} \in L^2(\Omega; H^1_{per}(\mathfrak{G}))$ *such that*

$$\begin{aligned} &\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^{\mathfrak{G}}\left(\mathbb{U}_{\varepsilon,\alpha}^{(\boldsymbol{S})}\right) \rightharpoonup \mathbb{U}_\alpha^{(\boldsymbol{S})} \text{ weakly in } L^2(\Omega; H^1(\mathfrak{G})),\\ &\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^{\mathfrak{G}}\left(\partial_\alpha\mathbb{U}_{\varepsilon,\alpha}^{(\boldsymbol{S})}\right) \rightharpoonup \partial_\alpha\mathbb{U}_\alpha^{(\boldsymbol{S})} + \partial_{Y_\alpha}\widehat{\mathbb{U}}_\alpha^{(\boldsymbol{S})} \text{ weakly in } L^2(\Omega\times\mathfrak{G}^{(\alpha)}). \end{aligned} \tag{65}$$

Proof.

Step 1. We prove convergences (59) and (60).
By construction, we have that $(\partial_1\mathbb{U}_{\varepsilon,3}, \partial_2\mathbb{U}_{\varepsilon,3}) = (-\mathcal{R}_{\varepsilon,2}, \mathcal{R}_{\varepsilon,1})$. So, by estimates (54) and (37), we have

$$\|\mathbb{U}_{\varepsilon,3}\|_{H^2(\Omega)} \le C\varepsilon.$$

In the proof of Falconi et al. [17, Theorem 6.3], we showed that

$$\frac{1}{\varepsilon}\mathbb{U}_{3,\varepsilon} \rightharpoonup \mathbb{U}_3 \text{ weakly in } H^2(\Omega).$$

Thus, $\mathbb{U}_3 \in H^2_{\boldsymbol{\Gamma}}(\Omega)$. Moreover, we have $((i,j)\in\{1,2\}^2)$

$$\frac{1}{\varepsilon}\mathcal{T}_\varepsilon(\partial^2_{ij}\mathbb{U}_{3,\varepsilon}) \rightharpoonup \partial^2_{ij}\mathbb{U}_3 + \partial^2_{Y_iY_j}\widehat{\mathbb{U}}_3 \text{ weakly in } L^2(\Omega\times\mathcal{Y}),$$

where $\widehat{\mathbb{U}}_3 \in L^2(\Omega; H^2_{per}(\mathcal{Y}))$. Then, convergences (59) follow from those above with $\mathbb{U}_3 = \mathbb{U}_3$ due to equality (56) and Theorem 2, while convergence (60) is an easy consequence of Lemma 6, since at the limit, we have $(\partial_1\mathbb{U}_3, \partial_2\mathbb{U}_3) = (-\mathcal{R}_2, \mathcal{R}_1)$. Note that we have $\widehat{\mathbb{U}}_3 = \widehat{\mathbb{U}}_{3|\Omega\times\mathfrak{G}}$, and from the equalities (36), we get (61).

Step 2. We prove the convergences (62)–(64).
Convergences (62)–(64) are the consequences of the estimates (54) with $\mathbb{U}_\alpha^{(\mathbf{B})} \in L^2(\Omega, H^1_{per}(\mathfrak{G})) \cap L^2(\Omega, H^2_{per}(\mathfrak{G}^{(3-\alpha)}))$. Due to $(45)_1$, $\mathbb{U}_1$ and $\mathbb{U}_2$ vanish on $(0,l)$.
From $(54)_2$ and Lemma 5, we first have ($\mathcal{R}_3^{(1)} \in L^2(\Omega, \partial_1; H^1_{per}(\mathfrak{G}^{(2)}))$ and $\widehat{\mathcal{R}}_3^{(1)} \in L^2(\Omega; H^1_{per}(\mathfrak{G}))$)

$$\begin{aligned} &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^{\mathfrak{G}}\left(\mathcal{R}_{\varepsilon,3}^{(1)}\right) \to \mathcal{R}_3^{(1)} \text{ strongly in } L^2(\Omega; H^1(\mathfrak{G})),\\ &\frac{1}{\varepsilon}\mathcal{T}_\varepsilon^{\mathfrak{G}}\left(\partial_1\mathcal{R}_{\varepsilon,3}^{(1)}\right) \rightharpoonup \partial_1\mathcal{R}_3^{(1)} + \partial_{Y_1}\widehat{\mathcal{R}}_3^{(1)} \text{ weakly in } L^2(\Omega; H^1(\mathfrak{G}^{(1)})). \end{aligned} \tag{66}$$

Besides, we have $\partial_1\mathbb{U}_{\varepsilon,2}(\cdot, q\varepsilon) + \partial_1\mathbb{U}_{\varepsilon,2}^{(\mathbf{B})}(\cdot, q\varepsilon) = \mathcal{R}_{\varepsilon,3}^{(1)}(\cdot, q\varepsilon)$ a.e. in $(0,L)\times\{2n_\varepsilon, \dots, 2N_\varepsilon\}$ and $\partial_1\mathbb{U}_{\varepsilon,2}^{(\mathbf{B})}(\cdot, q\varepsilon) = \mathcal{R}_{\varepsilon,3}^{(1)}(\cdot, q\varepsilon)$ a.e. in $(0,L)\times\{0, \dots, 2n_\varepsilon - 1\}$. Then, transforming by $\mathcal{T}_\varepsilon^{\mathfrak{G}}$ and, passing to the limit (from estimates 54 and Lemmas 4 and 5), lead to $\partial_1\mathbb{U}_2 + \partial_{Y_1}\mathbb{U}_2^{(\mathbf{B})} = \mathcal{R}_3^{(1)}$ a.e. in $\Omega\times\mathfrak{G}^{(1)}$. Since $\partial_1\mathbb{U}_2$ and $\mathcal{R}_3^{(1)}$ does not depend on Y_1 and $\mathbb{U}_2^{(\mathbf{B})}$ belongs to $L^2\left(\Omega; H^2_{per}\left(\mathfrak{G}^{(1)}\right)\right)$, this gives

$$\partial_1\mathbb{U}_2(z_1) = \mathcal{R}_3^{(1)}(z', b) \text{ for a.e. } (z', b)\in\Omega\times\{0,1\} \text{ and } \partial_{Y_1}\mathbb{U}_2^{(\mathbf{B})} = 0 \text{ a.e. in } \Omega\times\mathfrak{G}^{(1)}.$$

As a consequence, the function $\mathbb{U}_2^{(\mathbf{B})}$ does not depend on Y_1. Then, due to the convergences $(62)_3$, $(66)_2$, and (64), at the limit, we obtain

$$\partial^2_{Y_1Y_1}\widehat{\mathbb{U}}_2 + \partial^2_{Y_1Y_1}\mathbb{U}_2^{(\mathbf{B})} = \partial^2_{Y_1Y_1}\widehat{\mathbb{U}}_2 = \partial_{Y_1}\widehat{\mathcal{R}}_3^{(1)},$$

since $\partial^2_{11}\mathbb{U}_2 = \partial_1\mathcal{R}_3^{(1)}$. This gives $(63)_2$. Similarly, we show that $\partial_2\mathbb{U}_1 = -\mathcal{R}_3^{(2)}$ and convergence $(63)_3$–$(63)_4$.
Convergence (65) is a consequence of the estimates (54), Lemma 5, and Remark 2.

□

The function $\mathbb{U}^{(\mathbf{S})}_\alpha$ belongs to $L^2(\Omega; H^1_{per}(\mathfrak{G}))$, it is affine with respect to $Y_{3-\alpha}$ in $\Omega \times \mathfrak{G}^{(3-\alpha)}$ and is independent of Y_α in $\Omega \times \mathfrak{G}^{(\alpha)}$, so we will consider it as a function, belonging to $L^2(\Omega \times \{0,1\}, \partial_\alpha)$. Since $\mathbb{U}_{1,\varepsilon}(0, k\varepsilon) = 0$ (resp. $\mathbb{U}_{2,\varepsilon}(k\varepsilon, 0) = 0$) for every $k \in \{0, \dots, 2n_\varepsilon\}$, so the function $\mathbb{U}^{(\mathbf{S})}_1$ (resp. $\mathbb{U}^{(\mathbf{S})}_2$) vanishes on $\{0\} \times (0, l)$ (resp. $(0, l) \times \{0\}$). Hence, $\mathbb{U}^{(\mathbf{S})}_\alpha$ belongs to $\mathbf{L}^2(\Omega \times \{0,1\}, \partial_\alpha)$.

7.4 Limit of the strain tensor fields via global unfolding operator

Since this operator takes functions that live in the three-dimensional textile structure, it is the one with which we will go to the limit in problem (26) and, therefore, the one that gives the limit of the strain tensors.

We define the reference cell in the respective direction by setting

$$Cyl^{(1)} \doteq \mathfrak{G}^{(1)} \times \omega_\kappa = (0,2) \times \{0,1\} \times (-\kappa, \kappa)^2,$$
$$Cyl^{(2)} \doteq \mathfrak{G}^{(2)} \times \omega_\kappa = \{0,1\} \times (0,2) \times (-\kappa, \kappa)^2.$$

Definition 3 (Global unfolding operator). For every measurable function Φ on $\mathfrak{G}^{(1)}_\varepsilon \times \omega_r$ and Ψ on $\mathfrak{G}^{(1)}_\varepsilon \times \omega_r$, one defines the measurable functions $\Pi^{(1)}_\varepsilon(\Phi)$ on $\Omega \times Cyl^{(1)}$ and $\Pi^{(2)}_\varepsilon(\Psi)$ on $\Omega \times Cyl^{(2)}$, respectively, by

$$\Pi^{(1)}_\varepsilon(\Phi)(z', Y_1, b, Y_2, Y_3) \doteq \Phi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon Y_1 \mathbf{e}_1 + \varepsilon b \mathbf{e}_2 + \varepsilon(Y_2\mathbf{e}_2 + Y_3\mathbf{e}_3)\right) \text{ for a.e. } (z', Y_1, b, Y_2, Y_3) \in \Omega \times Cyl^{(1)},$$
$$\Pi^{(2)}_\varepsilon(\Psi)(z', a, Y_2, Y_1, Y_3) \doteq \Psi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon a \mathbf{e}_1 + \varepsilon Y_2 + +\varepsilon(Y_1\mathbf{e}_1 + Y_3\mathbf{e}_3)\right) \text{ for a.e. } (z', a, Y_2, Y_1, Y_3) \in \Omega \times Cyl^{(2)}.$$

We have the following.

Lemma 8. *For every $\phi \in L^1(\mathfrak{G}^{(1)}_\varepsilon \times \omega_r)$ and $\psi \in L^1(\mathfrak{G}^{(2)}_\varepsilon \times \omega_r)$, one has*

$$\left|\sum_{q=0}^{2N_\varepsilon-1} \int_{(0,L)\times\omega_r} \phi(z_1, q\varepsilon, y_2, y_3)dz_1dy_2dy_3 - \frac{\varepsilon}{2}\int_\Omega\int_{Cyl^{(1)}} \Pi^{(1)}_\varepsilon(\phi)(z', Y_1, b, Y_2, Y_3)dz'dY\right| \leq \int_{(0,L)\times\omega_r} |\phi(z_1, L, y_2, y_3)|dz_1dy_2dy_3,$$
$$\left|\sum_{p=0}^{2N_\varepsilon-1} \int_{(0,L)\times\omega_r} \psi(p\varepsilon, z_2, y_1, y_3)dz_2dy_1dy_3 - \frac{\varepsilon}{2}\int_\Omega\int_{Cyl^{(2)}} \Pi^{(2)}_\varepsilon(\psi)(z', a, Y_2, Y_1, Y_3)dz'dY\right| \leq \int_{(0,L)\times\omega_r} |\psi(L, z_2, y_1, y_3)|dz_2dy_1dy_3.$$

As a direct consequence, we get

$$\sum_{\alpha=1}^{2} \|\Pi^{(\alpha)}_\varepsilon(\phi)\|_{L^2(\Omega\times Cyl^{(\alpha)})} \leq \frac{C}{\sqrt{\varepsilon}}\|\phi\|_{L^2(\mathcal{S}_\varepsilon)}, \qquad \forall \phi \in L^2(\mathcal{S}_\varepsilon). \tag{67}$$

As one can expect, the global unfolding $\Pi^{(\alpha)}_\varepsilon$ and the middle line unfolding operator $\mathcal{T}^{\mathfrak{G}}_\varepsilon$ for every measurable function $\boldsymbol{\phi}$ defined on $\mathfrak{G}_\varepsilon$, the middle line unfolding operator $\mathcal{T}^{\mathfrak{G}}_\varepsilon$ and the global unfolding operators $\Pi^{(\alpha)}_\varepsilon$ are related in the following way:

$$\begin{aligned}
\Pi^{(1)}_\varepsilon(\boldsymbol{\phi})(z', Y_1, b, 0, 0) &= \boldsymbol{\phi}\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon Y_1\mathbf{e}_1 + \varepsilon b\mathbf{e}_2\right) = \mathcal{T}^{\mathfrak{G}}_\varepsilon(\boldsymbol{\phi})(z', Y_1, b), \qquad \text{for a.e. } (z', Y_1, b) \in \Omega \times \mathfrak{G}^{(1)},\\
\Pi^{(2)}_\varepsilon(\boldsymbol{\phi})(z', a, Y_2, 0, 0) &= \boldsymbol{\phi}\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon a\mathbf{e}_1 + \varepsilon Y_2\mathbf{e}_2\right) = \mathcal{T}^{\mathfrak{G}}_\varepsilon(\boldsymbol{\phi})(z', a, Y_2), \qquad \text{for a.e. } (z', a, Y_2) \in \Omega \times \mathfrak{G}^{(2)}.
\end{aligned} \tag{68}$$

Hence, unfolding functions restricted to the beams' middle lines via $\Pi^{(\alpha)}_\varepsilon$ is equivalent to unfolding them via $\mathcal{T}^{\mathfrak{G}}_\varepsilon$ and therefore we can use the convergence results of the previous subsection to express the strain tensor convergences of the whole structure.

Lemma 9. *The following convergences hold:*

$$\frac{1}{\varepsilon}\Pi_\varepsilon^{(1)}\left(\partial_1\mathcal{R}_\varepsilon^{(1)}\right) \rightharpoonup \begin{pmatrix}\partial_{12}\mathbb{U}_3\\ -\partial_{11}\mathbb{U}_3\\ \partial_{11}\mathbb{U}_2\end{pmatrix}+\begin{pmatrix}\partial_{Y_1}\widehat{\mathcal{R}}_1\\ \partial_{Y_1}\widehat{\mathcal{R}}_2\\ \partial^2_{Y_1Y_1}\widehat{\mathbb{U}}_2\end{pmatrix} \textit{ weakly in } L^2(\Omega\times Cyl^{(1)})^3,$$
$$\frac{1}{\varepsilon}\Pi_\varepsilon^{(2)}\left(\partial_2\mathcal{R}_\varepsilon^{(2)}\right) \rightharpoonup \begin{pmatrix}\partial_{22}\mathbb{U}_3\\ -\partial_{12}\mathbb{U}_3\\ -\partial_{22}\mathbb{U}_1\end{pmatrix}+\begin{pmatrix}\partial_{Y_2}\widehat{\mathcal{R}}_1\\ \partial_{Y_2}\widehat{\mathcal{R}}_2\\ -\partial^2_{Y_2Y_2}\widehat{\mathbb{U}}_1\end{pmatrix} \textit{ weakly in } L^2(\Omega\times Cyl^{(2)})^3, \tag{69}$$

and ($\alpha=\{1,2\}$)

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(\alpha)}\left(\partial_\alpha\mathbb{U}_{\varepsilon,\alpha}^{(\alpha)}\right) \rightharpoonup \partial_\alpha\mathbb{U}_\alpha^{(\boldsymbol{S})}+\partial_{Y_\alpha}\widehat{\mathbb{U}}_\alpha^{(\boldsymbol{S})} \textit{ weakly in } L^2(\Omega\times Cyl^{(\alpha)}). \tag{70}$$

Proof. First, for every function $\phi\in L^2(\mathfrak{G}_\varepsilon^{(\alpha)})$, we have the following change of convergence rate:

$$\|\Pi_\varepsilon^{(\alpha)}(\phi)\|_{L^2(\Omega\times Cyl^{(\alpha)})}\le C\sqrt{\varepsilon}\|\phi\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)})}.$$

Hence, convergence (69) follows from the above inequality, equality (68), and the convergences in Lemma 7. Convergence (70) is proven by the same meanings of (69), together with equality (11). □

7.5 | Form of the limit strain tensors for the warping

Define the space ($\alpha\in\{1,2\}$)

$$\mathbf{W}^{(\alpha)}\doteq\left\{\overline{w}^{(\alpha)}\in H^1(Cyl^{(\alpha)})^3 \;\middle|\; 2\text{ periodic with respect to } Y_\alpha\right\}. \tag{71}$$

In the lemma below, we show the warping convergences. The proof is done in the same way as in Lemma 7.7 of Griso et al. [25].

Lemma 10. *There exist a subsequence of* $\{\varepsilon\}$*, still denoted* $\{\varepsilon\}$*, and* $\overline{u}^{(1)}\in L^2(\Omega;\mathbf{W}^{(1)})$, $\overline{u}^{(2)}\in L^2(\Omega;\mathbf{W}^{(2)})$ *such that*

$$\frac{1}{\varepsilon^3}\Pi_\varepsilon^{(\alpha)}(\overline{u}_\varepsilon^{(\alpha)})\rightharpoonup\overline{u}^{(\alpha)} \qquad \textit{weakly in } L^2(\Omega;H^1(Cyl^{(\alpha)}))^3.$$

In the same fashion as in Subsection 7.3 of Griso et al. [25], we go to the limit with the strain tensor for the warping (17) and from the above convergence and the convergence results for the mobile reference frames in Appendix A.2, we get that

$$\frac{1}{\varepsilon^2}\Pi_\varepsilon^{(\alpha)}\left(\widetilde{\mathbf{e}}(\overline{u}_\varepsilon)\right)\rightharpoonup\mathcal{E}_Y^{(\alpha)}(\overline{u}^{(\alpha)}) \text{ weakly in } L^2(\Omega\times Cyl^{(\alpha)})^{3\times3},$$

where for every $\Psi^{(1)}\in H^1(Cyl^{(1)})^3$ and every $\Psi^{(2)}\in H^1(Cyl^{(2)})^3$, we have

$$\mathcal{E}_Y^{(1)}(\Psi^{(1)})=\begin{pmatrix}\frac{1}{\boldsymbol{\eta}^{(1)}}\partial_{Y_1}\Psi^{(1)}\cdot\mathbf{t}^{(1)} & * & *\\ \frac12\left(\frac{1}{\boldsymbol{\eta}^{(1)}}\partial_{Y_1}\Psi^{(1)}\cdot\mathbf{e}_2+\partial_{Y_2}\Psi^{(1)}\cdot\mathbf{t}^{(1)}\right) & \partial_{Y_2}\Psi^{(1)}\cdot\mathbf{e}_2 & *\\ \frac12\left(\frac{1}{\boldsymbol{\eta}^{(1)}}\partial_{Y_1}\Psi^{(1)}\cdot\mathbf{n}^{(1)}+\partial_{Y_3}\Psi^{(1)}\cdot\mathbf{t}^{(1)}\right) & \frac12\left(\partial_{Y_2}\Psi^{(1)}\cdot\mathbf{n}^{(1)}+\partial_{Y_3}\Psi^{(1)}\cdot\mathbf{e}_2\right) & \partial_{Y_3}\Psi^{(1)}\cdot\mathbf{n}^{(1)}\end{pmatrix}, \tag{72}$$

and

$$\mathcal{E}_Y^{(2)}(\Psi^{(2)})=\begin{pmatrix}\partial_{Y_1}\Psi^{(2)}\cdot\mathbf{e}_1 & * & *\\ \frac12\left(\partial_{Y_1}\Psi^{(2)}\cdot\mathbf{t}^{(2)}+\frac{1}{\boldsymbol{\eta}^{(2)}}\partial_{Y_2}\Psi^{(2)}\cdot\mathbf{e}_1\right) & \frac{1}{\boldsymbol{\eta}^{(2)}}\partial_{Y_2}\Psi^{(2)}\cdot\mathbf{t}^{(2)} & *\\ \frac12\left(\partial_{Y_1}\Psi^{(2)}\cdot\mathbf{n}^{(2)}+\partial_{Y_3}\Psi^{(2)}\cdot\mathbf{e}_1\right) & \frac12\left(\frac{1}{\boldsymbol{\eta}^{(2)}}\partial_{Y_2}\Psi^{(2)}\cdot\mathbf{n}^{(2)}+\partial_{Y_3}\Psi^{(2)}\cdot\mathbf{t}^{(2)}\right) & \partial_{Y_3}\Psi^{(2)}\cdot\mathbf{n}^{(2)}\end{pmatrix}. \tag{73}$$

7.6 Form of the limit strain tensors

We denote $\mathbf{\Theta}$ ($\ddot{\mathbf{\Theta}}$ its derivative) the following function belonging to $W^{1,\infty}_{per}(0,1)$:

$$\mathbf{\Theta}(t)=\frac{1}{2}\begin{cases} t^2 & \text{if } t\in[0,\kappa],\\ \kappa^2 & \text{if } t\in[\kappa,1-\kappa],\\ (t-1)^2 & \text{if } t\in[1-\kappa,1],\end{cases}\qquad \ddot{\mathbf{\Theta}}(t)=\begin{cases} t & \text{if } t\in[0,\kappa),\\ 0 & \text{if } t\in(\kappa,1-\kappa),\\ t-1 & \text{if } t\in(1-\kappa,1],\end{cases}$$

Below, we define the notation that will characterize the strain tensors in the limit.

Notation 1. We define two maps, $\mathfrak{F}^{(1)}$ and $\mathfrak{F}^{(2)}$, from $\mathbb{R}^5$ into $\mathbb{R}^3\times\mathfrak{G}^{(\alpha)}$ by

$$X=(X_0,X_{00},X_1,X_2,X_3)\mapsto\mathfrak{F}^{(1)}(X)\doteq\begin{cases} X_0\mathbf{e}_1-\ddot{\mathbf{\Theta}}X_2\mathbf{e}_3+(X_1\mathbf{e}_1-X_2\mathbf{e}_2+X_3\mathbf{e}_3)\wedge\left(\Phi^{(1)}\mathbf{e}_3+Y_2\mathbf{e}_2+Y_3\mathbf{n}^{(1)}\right) & \text{if } b=0,\\ X_{00}\mathbf{e}_1-\ddot{\mathbf{\Theta}}X_2\mathbf{e}_3+(X_1\mathbf{e}_1-X_2\mathbf{e}_2+X_3\mathbf{e}_3)\wedge\left(\Phi^{(1)}\mathbf{e}_3+Y_2\mathbf{e}_2+Y_3\mathbf{n}^{(1)}\right) & \text{if } b=1,\end{cases}$$

and

$$X=(X_0,X_{00},X_1,X_2,X_3)\mapsto\mathfrak{F}^{(2)}(X)\doteq\begin{cases} X_0\mathbf{e}_2-\ddot{\mathbf{\Theta}}X_1\mathbf{e}_3+(X_1\mathbf{e}_1-X_2\mathbf{e}_2-X_3\mathbf{e}_3)\wedge\left(\Phi^{(2)}\mathbf{e}_3+Y_1\mathbf{e}_1+Y_3\mathbf{n}^{(2)}\right) & \text{if } a=0,\\ X_{00}\mathbf{e}_2-\ddot{\mathbf{\Theta}}X_1\mathbf{e}_3+(X_1\mathbf{e}_1-X_2\mathbf{e}_2-X_3\mathbf{e}_3)\wedge\left(\Phi^{(2)}\mathbf{e}_3+Y_1\mathbf{e}_1+Y_3\mathbf{n}^{(2)}\right) & \text{if } a=1.\end{cases}$$

Accordingly, we set

$$\mathcal{E}^{(1)}(X)=\begin{pmatrix}\frac{1}{\boldsymbol{\eta}^{(1)}}\mathfrak{F}^{(1)}(X)\cdot\mathbf{t}^{(1)} & * & *\\ \frac{1}{2\boldsymbol{\eta}^{(1)}}\mathfrak{F}^{(1)}(X)\cdot\mathbf{e}_2 & 0 & *\\ \frac{1}{2\boldsymbol{\eta}^{(1)}}\mathfrak{F}^{(1)}(X)\cdot\mathbf{n}^{(1)} & 0 & 0\end{pmatrix},\qquad \mathcal{E}^{(2)}(X)=\begin{pmatrix}0 & * & *\\ \frac{1}{2\boldsymbol{\eta}^{(2)}}\mathfrak{F}^{(2)}(X)\cdot\mathbf{e}_1 & \frac{1}{\boldsymbol{\eta}^{(2)}}\mathfrak{F}^{(2)}(X)\cdot\mathbf{t}^{(2)} & *\\ 0 & \frac{1}{2\boldsymbol{\eta}^{(2)}}\mathfrak{F}^{(2)}(X)\cdot\mathbf{n}^{(2)} & 0\end{pmatrix}. \tag{74}$$

Before going to the limit, we must prove that the unfolded strain tensor is bounded. From the change of convergence rate (67) and (53), we have

$$\left\|\frac{1}{\varepsilon^2}\Pi^{(\alpha)}_\varepsilon\left(\widetilde{\mathbf{e}}(u_\varepsilon)\right)\right\|_{L^2(\Omega\times Cyl^{(\alpha)})}\le\frac{1}{\varepsilon^{5/2}}\|u_\varepsilon\|_{S_\varepsilon}\le C.$$

We first consider the direction $\mathbf{e}_1$. Due to the representation of the strain tensors (16) and (17), the convergences in Lemmas 9 and 10 together with the limit mobile reference frame given in Appendix A.2, we obtain that

$$\frac{1}{\varepsilon^2}\Pi^{(1)}_\varepsilon\left(\widetilde{\mathbf{e}}(u_\varepsilon)\right)\rightharpoonup\mathcal{E}^{(1)}(\partial\mathbf{U}^{(1)})+\mathcal{E}^{(1)}_Y(\hat{u}^{(1)})\qquad\text{weakly in}\quad L^2(\Omega;H^1(Cyl^{(1)}))^{3\times3}, \tag{75}$$

where the first quantity is given by (74) with $\partial\mathbf{U}^{(1)}=\left(\partial_1\mathbb{U}^{(\mathbf{S})}_1(\cdot,0),\partial_1\mathbb{U}^{(\mathbf{S})}_1(\cdot,1),\partial_{12}\mathbb{U}_3,\partial_{11}\mathbb{U}_3,\partial_{11}\mathbb{U}_2\right)$ and, where the second quantity is given by (72) and is the symmetric gradient of the displacement $\hat{u}^{(1)}$ defined by

$$\hat{u}^{(1)}\doteq\widehat{\mathbb{U}}^{(\mathbf{S})}_1\mathbf{e}_1+\mathbb{U}^{(\mathbf{B})}_2\mathbf{e}_2+(\widehat{\mathbb{U}}_3+\mathbf{\Theta}\partial^2_{11}\mathbb{U}_3)\mathbf{e}_3+\left(\widehat{\mathcal{R}}_1\mathbf{e}_1+\widehat{\mathcal{R}}_2\mathbf{e}_2+\partial_{Y_1}\widehat{\mathbb{U}}_2\mathbf{e}_3\right)\wedge\left(\Phi^{(1)}\mathbf{e}_3+Y_3\mathbf{n}^{(1)}+Y_2\mathbf{e}_2\right)+\overline{u}^{(1)}. \tag{76}$$

We have $\hat{u}^{(1)}\in L^2(\Omega;\mathbf{W}^{(1)})$.

Concerning direction $\mathbf{e}_2$, the same argumentation applies and the limit strain tensor becomes

$$\frac{1}{\varepsilon^2}\Pi^{(2)}_\varepsilon\left(\widetilde{\mathbf{e}}(u_\varepsilon)\right)\rightharpoonup\mathcal{E}^{(2)}(\partial\mathbf{U}^{(2)})+\mathcal{E}^{(2)}_Y(\hat{u}^{(2)})\qquad\text{weakly in}\quad L^2(\Omega;H^1(Cyl^{(2)}))^{3\times3}, \tag{77}$$

where again the first quantity is given by (74) with $\partial \mathbf{U}^{(2)} = \left(\partial_2 \mathbb{U}_2^{(\mathbf{S})}(\cdot,0), \partial_2 \mathbb{U}_2^{(\mathbf{S})}(\cdot,1), \partial_{22}\mathbb{U}_3, \partial_{12}\mathbb{U}_3, \partial_{22}\mathbb{U}_1\right)$ and where the second quantity is given by (73) and is the symmetric gradient of the displacement $\hat{u}^{(2)}$, defined by

$$\hat{u}^{(2)} \doteq \mathbb{U}_1^{(\mathbf{B})}\mathbf{e}_1 + \widehat{\mathbb{U}}_2^{(\mathbf{S})}\mathbf{e}_2 + (\widehat{\mathbb{U}}_3 + \mathbf{\Theta}\partial^2_{22}\mathbb{U}_3)\mathbf{e}_3 + \left(\widehat{\mathcal{R}}_1\mathbf{e}_1 + \widehat{\mathcal{R}}_2\mathbf{e}_2 - \partial_{Y_2}\widehat{\mathbb{U}}_1\mathbf{e}_3\right) \wedge \left(\Phi^{(2)}\mathbf{e}_3 + Y_3\mathbf{n}^{(2)} + Y_1\mathbf{e}_1\right) + \overline{u}^{(2)}. \tag{78}$$

We have $\hat{u}^{(2)} \in L^2(\Omega; \mathbf{W}^{(2)})$.

In the expressions of $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$, given above, the terms $\mathbf{\Theta}\partial^2_{11}\mathbb{U}_3$ and $\mathbf{\Theta}\partial^2_{22}\mathbb{U}_3$ do not come from the asymptotic behavior of the strain tensors. The role of these terms will appear to simplify the non-penetration limit condition (see Lemma 12).

7.7 | Unfold of the contact conditions via contact unfolding operator

The main purpose of this unfolding operator is to unfold functions on the two-dimensional contact areas $\mathbf{C}_\varepsilon$, defined in (20), in order to find out the unfolded limit contact conditions.

We define the limit reference contact domains by

$$\mathbf{C}_{ab} \doteq ((a,b) + \omega_\kappa) \cap \Omega, \qquad \text{for } (a,b) \in \{0,1\}^2.$$

Definition 4 (Contact unfolding operator)**.** For every measurable function ϕ in $\mathbf{C}_{ab}$, we define the measurable functions $T_\varepsilon^{\mathbf{C}_{ab}}(\phi)$ in $\Omega \times \omega_\kappa$ by $((a,b) \in \{0,1\}^2)$[1]:

$$T_\varepsilon^{\mathbf{C}_{ab}}(\phi)(z', Y_1, Y_2) \doteq \phi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon(a\mathbf{e}_1 + b\mathbf{e}_2) + \varepsilon(Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2)\right) \text{ for a.e. } (z', Y_1, Y_2) \in \Omega \times \omega_\kappa.$$

Note that for every ϕ belonging to $L^2(\mathfrak{G}^{(1)})$ (resp. $\psi \in L^2(\mathfrak{G}^{(2)})$), $T_\varepsilon^{\mathbf{C}_{ab}}(\phi)$ (resp. $T_\varepsilon^{\mathbf{C}_{ab}}(\psi)$) is given by

$$T_\varepsilon^{\mathbf{C}_{ab}}(\phi)(z', Y_1, 0) \doteq \phi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon(a\mathbf{e}_1 + b\mathbf{e}_2) + \varepsilon Y_1\mathbf{e}_1\right) \quad \text{for a.e. } (z', Y_1) \in \Omega \times (-\kappa, \kappa),$$

$$\left(\text{resp. } T_\varepsilon^{\mathbf{C}_{ab}}(\psi)(z', 0, Y_2) \doteq \phi\left(2\varepsilon\left[\frac{z'}{2\varepsilon}\right] + \varepsilon(a\mathbf{e}_1 + b\mathbf{e}_2) + \varepsilon Y_2\mathbf{e}_2\right) \quad \text{for a.e. } (z', Y_2) \in \Omega \times (-\kappa, \kappa)\right).$$

Let ϕ be in $L^2(\mathfrak{G}_\varepsilon^{(1)})$. The operator $T_\varepsilon^{\mathbf{C}_{ab}}$ is related to the previous operator $\mathcal{T}_\varepsilon^{\mathfrak{G}}$ via the equalities

$$\begin{aligned} T_\varepsilon^{\mathbf{C}_{0b}}(\phi)(z', Y_1, 0) &= \begin{cases} \mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi)(z', Y_1, b) & \text{for a.e. } (z', Y_1, b) \in \Omega \times (0,\kappa) \times \{0,1\}, \\ \mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi)(z' - 2\varepsilon\mathbf{e}_1, 2 + Y_1, b) & \text{for a.e. } (z', Y_1, b) \in (\Omega \cap (\Omega + 2\varepsilon\mathbf{e}_1)) \times (-\kappa, 0) \times \{0,1\}, \end{cases} \\ T_\varepsilon^{\mathbf{C}_{1b}}(\phi)(z', Y_1, 0) &= \mathcal{T}_\varepsilon^{\mathfrak{G}}(\phi)(z', 1 + Y_1, b) \qquad \text{for a.e. } (z', Y_1, b) \in \Omega \times (-\kappa, \kappa) \times \{0,1\}. \end{aligned} \tag{79}$$

One can easily give similar equalities if ϕ belongs to $L^2(\mathfrak{G}_\varepsilon^{(2)})$ or ψ in $L^2(S_\varepsilon)$. We have

$$\begin{aligned} \|T_\varepsilon^{\mathbf{C}_{ab}}(\varphi)\|_{L^2(\Omega\times\omega_\kappa)} &\le C\sqrt{\varepsilon}\|\varphi\|_{L^2(\mathfrak{G}_\varepsilon)}, & \forall \varphi \in L^2(\mathfrak{G}_\varepsilon), \\ \|T_\varepsilon^{\mathbf{C}_{ab}}(\psi)\|_{L^2(\Omega\times\omega_\kappa)} &\le \frac{C}{\sqrt{\varepsilon}}\left(\|\psi\|_{L^2(S_\varepsilon^{(\alpha)})} + \varepsilon\|\nabla\psi\|_{L^2(S_\varepsilon^{(\alpha)})}\right), & \forall \psi \in H^1(S_\varepsilon^{(\alpha)}). \end{aligned} \tag{80}$$

Recall the form of the displacements in the contact areas (29).

We start with the in-plane components. Note that due to the contact conditions (23), we have the following bound for the difference between the displacements in the contact areas:

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|u^{(1)}_{\varepsilon,\alpha} - u^{(2)}_{\varepsilon,\alpha}\|^2_{L^2(\mathbf{C}_{pq,\varepsilon})} \le C\varepsilon^4.$$

[1]Observe that we obtain four operators, since $(a,b) \in \{0,1\}^2$.

Hence, the unfolded sequence is bounded, and we can go to the limit in the in-plane components.

Lemma 11. *The in-plane limit contact conditions are* $((a,b)\in\{0,1\}^2)$

$$\begin{aligned} &|\mathbb{U}_1^{(\boldsymbol{S})}(\cdot,b)-\mathbb{U}_1^{(\boldsymbol{B})}(\cdot,a)|+\kappa|\partial_2\mathbb{U}_1+\partial_1\mathbb{U}_2|\le g_1 \quad \textit{a.e. in } \Omega,\\ &|\mathbb{U}_2^{(\boldsymbol{S})}(\cdot,a)-\mathbb{U}_2^{(\boldsymbol{B})}(\cdot,b)|+\kappa|\partial_2\mathbb{U}_1+\partial_1\mathbb{U}_2|\le g_2 \quad \textit{a.e. in } \Omega. \end{aligned} \tag{81}$$

Proof. We prove only the first inequality in (81), since the second one follows the same lines. We split the proof into two steps.

Step 1. A preliminary convergence.
For a.e. (z',Y_2) in $\Omega\times(-\kappa,\kappa)$, we define the function $((a,b)\in\{0,1\}^2)$

$$\ddot{\mathbb{U}}_{\varepsilon,1}(z',Y_2,a,b)\doteq T_\varepsilon^{\mathbf{C}_{ab}}(\mathbb{U}_{\varepsilon,1})\left(z',0,0\right)-T_\varepsilon^{\mathbf{C}_{ab}}(\mathbb{U}_{\varepsilon,1})\left(z',0,Y_2\right),$$

which does not depend on z_1 by definition of $\mathbb{U}_{\varepsilon,1}$. It belongs to $L^2(\Omega;H^1(-\kappa,\kappa))$, and the following relation holds:

$$\partial_{Y_2}\ddot{\mathbb{U}}_{\varepsilon,1}=-\varepsilon T_\varepsilon^{\mathbf{C}_{ab}}(\partial_2\mathbb{U}_{\varepsilon,1}).$$

From the Poincaré inequality, the first inequality in (80) and estimates (54), we have

$$\|\ddot{\mathbb{U}}_{\varepsilon,1}\|_{L^2(\Omega;H^1(-\kappa,\kappa))}\le C\varepsilon^2.$$

This, together with the third convergence in (62) and (79), implies that there exist a function $\ddot{\mathbb{U}}_1\in L^2(\Omega;H^1(-\kappa,\kappa))$ such that

$$\begin{aligned} &\frac{1}{\varepsilon^2}\ddot{\mathbb{U}}_{\varepsilon,1}\ \rightharpoonup\ \ddot{\mathbb{U}}_1 \quad \text{weakly in } L^2(\Omega;H^1(-\kappa,\kappa)),\\ &\frac{1}{\varepsilon^2}\partial_{Y_2}\ddot{\mathbb{U}}_{\varepsilon,1}\ \rightharpoonup\ \partial_{Y_2}\ddot{\mathbb{U}}_2=-\partial_2\mathbb{U}_1 \quad \text{weakly in } L^2(\Omega\times(-\kappa,\kappa)). \end{aligned}$$

As a consequence, we get a.e. in $\Omega\times(-\kappa,\kappa)$ the equality $\ddot{\mathbb{U}}_1(z',Y_2,a,b)=-Y_2\partial_2\mathbb{U}_1(z_2)$.

Step 2. We prove the first statement of the lemma.
By the form of the displacement in the contact areas (29) and the final decomposition (50), we need to go to the limit for the following expressions $(p,q)\in\{0,\dots,2N_\varepsilon\}\times\{2n_\varepsilon,\dots,2N_\varepsilon\}$):

$$\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^{\mathbf{C}_{ab}}\left(\mathbb{U}_{\varepsilon,1}^{(\mathbf{S})}(p\varepsilon+y_1,q\varepsilon)-\mathbb{U}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon,q\varepsilon+y_2)-y_2\mathcal{R}_{\varepsilon,3}^{(1)}(p\varepsilon+y_1,q\varepsilon)+\overline{u}_{\varepsilon,1}^{(1)}-\overline{u}_{\varepsilon,1}^{(2)}+\mathbb{U}_{\varepsilon,1}(q\varepsilon)-\mathbb{U}_{\varepsilon,1}(q\varepsilon+y_2)\right), \tag{82}$$

and $(p,q)\in\{0,\dots,2N_\varepsilon\}\times\{0,\dots,2n_\varepsilon\}$),

$$\frac{1}{\varepsilon^2}\mathcal{T}_\varepsilon^{\mathbf{C}_{ab}}\left(\mathbb{U}_{\varepsilon,1}^{(\mathbf{S})}(p\varepsilon+y_1,q\varepsilon)-\mathbb{U}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon,q\varepsilon+y_2)-y_2\mathcal{R}_{\varepsilon,3}^{(1)}(p\varepsilon+y_1,q\varepsilon)+\overline{u}_{\varepsilon,1}^{(1)}-\overline{u}_{\varepsilon,1}^{(2)}\right). \tag{83}$$

Concerning the warping terms, from estimate (55) and the second inequality in (80), we obtain

$$\left\|T_\varepsilon^{\mathbf{C}_{ab}}\left(\overline{u}_{\varepsilon,\alpha}^{(1)}-\overline{u}_{\varepsilon,\alpha}^{(2)}\right)\right\|_{L^2(\Omega\times\omega_\kappa)}\le\frac{C}{\sqrt{\varepsilon}}\|\overline{u}_{\varepsilon,\alpha}^{(1)}-\overline{u}_{\varepsilon,\alpha}^{(2)}\|_{L^2(\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r)}\le C\varepsilon^3.$$

Hence, applying the contact unfolding operator to the in-plane warping quantity leads to

$$\frac{1}{\varepsilon^2}T_\varepsilon^{\mathbf{C}_{ab}}\left(\overline{u}_{\varepsilon,\alpha}^{(1)}-\overline{u}_{\varepsilon,\alpha}^{(2)}\right)\to 0 \text{ strongly in } L^2(\Omega\times\omega_\kappa). \tag{84}$$

Using convergences (63), (64), (65), (84), and the ones in Step 1, we get

$$\frac{1}{\varepsilon^2}T_\varepsilon^{\mathbf{C}_{ab}}\left(\left(\mathbb{U}^{(\mathbf{S})}_{\varepsilon,1}(p\varepsilon+y_1,q\varepsilon)-\mathbb{U}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon,q\varepsilon+y_2)-y_2\mathcal{R}^{(1)}_{\varepsilon,3}(p\varepsilon+y_1,q\varepsilon)\right)+\left(\overline{u}^{(1)}_{\varepsilon,1}-\overline{u}^{(2)}_{\varepsilon,1}\right)\right)+\frac{1}{\varepsilon^2}\ddot{\mathbb{U}}_{\varepsilon,1}(\cdot,Y_1,a,b)$$
$$\rightharpoonup\ \mathbb{U}^{(\mathbf{S})}_1(\cdot,b)-\mathbb{U}^{(\mathbf{B})}_1(\cdot,a)-Y_2\partial_1\mathbb{U}_2-Y_2\partial_2\mathbb{U}_1\qquad\text{weakly in } L^2((\Omega_3\cup\Omega_4)\times(-\kappa,\kappa))^2,$$

and

$$\frac{1}{\varepsilon^2}T_\varepsilon^{\mathbf{C}_{ab}}\left(\left(\mathbb{U}^{(\mathbf{S})}_{\varepsilon,1}(p\varepsilon+y_1,q\varepsilon)-\mathbb{U}^{(\mathbf{B})}_{\varepsilon,1}(p\varepsilon,q\varepsilon+y_2)-y_2\mathcal{R}^{(1)}_{\varepsilon,3}(p\varepsilon+y_1,q\varepsilon)\right)+\left(\overline{u}^{(1)}_{\varepsilon,1}-\overline{u}^{(2)}_{\varepsilon,1}\right)\right)$$
$$\rightharpoonup\ \mathbb{U}^{(\mathbf{S})}_1(\cdot,b)-\mathbb{U}^{(\mathbf{B})}_1(\cdot,a)-Y_2\partial_1\mathbb{U}_2\qquad\text{weakly in } L^2((\Omega_1\cup\Omega_2)\times(-\kappa,\kappa))^2.$$

So, for a.e $(z',Y_2)\in\Omega\times(-\kappa,\kappa)$, we get

$$|\mathbb{U}^{(\mathbf{S})}_1(z',b)-\mathbb{U}^{(\mathbf{B})}_1(z',a)-Y_2(\partial_2\mathbb{U}_1(z_2)+\partial_1\mathbb{U}_2(z_1))|\le g_1(z').$$

The statement follows easily. □

Now, we look at the outer-plane component. From equalities (29), estimates (30), (32) (a consequence of the non-penetration condition), (34), (35), and (53) lead to the following estimate in the contact areas (see also Griso et al. [25]):

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\|u^{(1)}_{\varepsilon,3}-u^{(2)}_{\varepsilon,3}\|^2_{L^2(\mathbf{C}_{pq,\varepsilon})}\le C\varepsilon^6.$$

We are ready to go to the limit in the outer-plane component.

Lemma 12. *The outer-plane limit contact conditions are ($(a,b)\in\{0,1\}^2$)*

$$0\le(-1)^{a+b}\left(\hat{u}^{(1)}_3(\cdot,a+Y_1,b,Y_2,(-1)^{a+b+1}\kappa)-\hat{u}^{(2)}_3(\cdot,a,b+Y_2,Y_1,(-1)^{a+b}\kappa)\right),\ \text{a.e. in } \Omega\times\omega_\kappa. \tag{85}$$

Proof. We split the proof into two steps.

Step 1. Preliminary convergences.

For a.e. (z',Y_1) in $\Omega\times(-\kappa,\kappa)$, we consider the function ($(a,b)\in\{0,1\}^2$)

$$\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3}(z',Y_1,a,b)\doteq T_\varepsilon^{\mathbf{C}_{ab}}(\mathbb{U}_{\varepsilon,3})\left(z',Y_1,0\right)-T_\varepsilon^{\mathbf{C}_{ab}}(\mathbb{U}_{\varepsilon,3})\left(z',0,0\right)-\varepsilon Y_1T_\varepsilon^{\mathbf{C}_{ab}}(\partial_1\mathbb{U}_{\varepsilon,3})\left(z',0,0\right).$$

This function belongs to $L^2(\Omega;H^2(-\kappa,\kappa))$, and we have

$$\partial^2_{Y_1Y_1}\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3}=\varepsilon^2T_\varepsilon^{\mathbf{C}_{ab}}(\partial^2_{11}\mathbb{U}_{\varepsilon,3}).$$

From the Poincaré inequality, the first inequality in (80) and estimates (54), we obtain

$$\|\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3}\|_{L^2(\Omega;H^2(-\kappa,\kappa))}\le C\varepsilon^3.$$

This, together with the third convergence in (59) and equalities (79), imply that there exists a function $\ddot{\mathbb{U}}^{(1)}_3\in L^2(\Omega;H^2(-\kappa,\kappa))$ such that

$$\frac{1}{\varepsilon^3}\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3}\ \rightharpoonup\ \ddot{\mathbb{U}}^{(1)}_3\quad\text{weakly in } L^2(\Omega;H^2(-\kappa,\kappa)),$$
$$\frac{1}{\varepsilon^3}\partial^2_{Y_1Y_1}\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3}\ \rightharpoonup\ \partial^2_{Y_1Y_1}\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3}=\partial_{11}\mathbb{U}_3+\partial^2_{Y_1Y_1}\widehat{\mathbb{U}}_3(\cdot,a+Y_1,b)\quad\text{weakly in } L^2(\Omega\times(-\kappa,\kappa)).$$

As a consequence, we get a.e. in $\Omega\times\omega_\kappa$

$$\ddot{\mathbb{U}}^{(1)}_3(z',Y_1,a,b)=\frac{1}{2}Y_1^2\partial_{11}\mathbb{U}_3(z')+\widehat{\mathbb{U}}_3(z',a+Y_1,b)-\widehat{\mathbb{U}}_3(z',a,b)-Y_1\partial_{Y_1}\widehat{\mathbb{U}}_3(z',a,b).$$

Now, for a.e. (z', Y_1) in $\Omega \times (-\kappa, \kappa)$, we consider the function ($(a,b) \in \{0,1\}^2$)

$$\ddot{\mathcal{R}}^{(1)}_{\varepsilon,1}(z', Y_1, a, b) \doteq T^{\mathbf{C}_{ab}}_\varepsilon(\mathcal{R}_{\varepsilon,1})(z', Y_1, 0) - T^{\mathbf{C}_{ab}}_\varepsilon(\mathcal{R}_{\varepsilon,1})(z', 0, 0).$$

This function belongs to $L^2(\Omega; H^1(-\kappa,\kappa))$. Proceeding as in the proof of Lemma 11, we show that there exists a function $\ddot{\mathcal{R}}^{(1)}_1 \in L^2(\Omega; H^1(-\kappa,\kappa))$, such that

$$\frac{1}{\varepsilon^2}\ddot{\mathcal{R}}^{(1)}_{\varepsilon,1} \rightharpoonup \ddot{\mathcal{R}}^{(1)}_1 \text{ weakly in } L^2(\Omega; H^1(-\kappa,\kappa)),$$

where

$$\ddot{\mathcal{R}}^{(1)}_1(z', Y_1, a, b) = Y_1\partial_{12}\mathbb{U}_3(z') + \widehat{\mathcal{R}}_1(z', a+Y_1, b) - \widehat{\mathcal{R}}_1(z', a, b) \text{ for a.e.} (z', Y_1) \text{in}\, \Omega \times (-\kappa,\kappa).$$

Regarding the direction $\mathbf{e}_2$, we set for a.e. (z', Y_2) in $\Omega \times (-\kappa, \kappa)$ the functions

$$\begin{aligned}
\ddot{\mathbb{U}}^{(2)}_{\varepsilon,3}(z', Y_2, a, b) &\doteq T^{\mathbf{C}_{ab}}_\varepsilon(\mathbb{U}_{\varepsilon,3})\left(z', 0, Y_2\right) - T^{\mathbf{C}_{ab}}_\varepsilon(\mathbb{U}_{\varepsilon,3})\left(z', 0, 0\right) - \varepsilon Y_2 T^{\mathbf{C}_{ab}}_\varepsilon(\partial_1\mathbb{U}_{\varepsilon,3})\left(z', 0, 0\right),\\
\ddot{\mathcal{R}}^{(2)}_{\varepsilon,2}(z', Y_2, a, b) &\doteq T^{\mathbf{C}_{ab}}_\varepsilon(\mathcal{R}_{\varepsilon,2})(z', 0, Y_2) - T^{\mathbf{C}_{ab}}_\varepsilon(\mathcal{R}_{\varepsilon,2})(z', 0, 0).
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{1}{\varepsilon^3}\ddot{\mathbb{U}}^{(2)}_{\varepsilon,3} \rightharpoonup \ddot{\mathbb{U}}^{(2)}_3 \text{ weakly in } L^2(\Omega; H^2(-\kappa,\kappa)),\\
&\frac{1}{\varepsilon^2}\ddot{\mathcal{R}}^{(2)}_{\varepsilon,2} \rightharpoonup \ddot{\mathcal{R}}^{(2)}_2 \text{ weakly in } L^2(\Omega; H^1(-\kappa,\kappa)),
\end{aligned}$$

where

$$\begin{aligned}
\ddot{\mathbb{U}}^{(2)}_3(z', Y_2, a, b) &= \frac{1}{2}Y_2^2\partial_{22}\mathbb{U}_3(z') + \widehat{\mathbb{U}}_3(z', a, b+Y_2) - \widehat{\mathbb{U}}_3(z', a, b) - Y_2\partial_{Y_2}\widehat{\mathbb{U}}_3(z', a, b),\\
\ddot{\mathcal{R}}^{(2)}_2(z', Y_2, a, b) &= -Y_2\partial_{12}\mathbb{U}_3(z') + \widehat{\mathcal{R}}_2(z', a, b+Y_2) - \widehat{\mathcal{R}}_2(z', a, b) \text{ for a.e.} (z', Y_2) \text{in}\, \Omega \times (-\kappa,\kappa).
\end{aligned}$$

Step 2. We prove the statement.
We transform the difference $u^{(1)}_{\varepsilon,3} - u^{(2)}_{\varepsilon,3}$, using $T^{\mathbf{C}_{ab}}_\varepsilon$ and taking into account the functions introduced in the first step that gives

$$\frac{1}{\varepsilon^3}T^{\mathbf{C}_{ab}}_\varepsilon\left(u^{(1)}_{\varepsilon,3} - u^{(2)}_{\varepsilon,3}\right) = \frac{1}{\varepsilon^3}\left(\ddot{\mathbb{U}}^{(1)}_{\varepsilon,3} - \ddot{\mathbb{U}}^{(2)}_{\varepsilon,3} + \varepsilon Y_1\ddot{\mathcal{R}}^{(2)}_{\varepsilon,2} + \varepsilon Y_2\ddot{\mathcal{R}}^{(1)}_{\varepsilon,1} + T^{\mathbf{C}_{ab}}_\varepsilon\left(\overline{u}^{(1)}_{\varepsilon,3} - \overline{u}^{(2)}_{\varepsilon,3}\right)\right) \rightharpoonup \Delta \text{ weakly in } L^2(\Omega\times\omega_\kappa),$$

where

$$\begin{aligned}
\Delta(z', Y_1, Y_2, a, b) = &\frac{1}{2}Y_1^2\partial_{11}\mathbb{U}_3(z') + \widehat{\mathbb{U}}_3(z', a+Y_1, b) - \widehat{\mathbb{U}}_3(z', a, b) - Y_1\partial_{Y_1}\widehat{\mathbb{U}}_3(z', a, b)\\
&- \left(\frac{1}{2}Y_2^2\partial_{22}\mathbb{U}_3(z') + \widehat{\mathbb{U}}_3(z', a, b+Y_2) - \widehat{\mathbb{U}}_3(z', a, b) - Y_2\partial_{Y_2}\widehat{\mathbb{U}}_3(z', a, b)\right)\\
&+ Y_1\left(-Y_2\partial_{12}\mathbb{U}_3(z') + \widehat{\mathcal{R}}_2(z', a, b+Y_2) - \widehat{\mathcal{R}}_2(z', a, b)\right)\\
&+ Y_2\left(Y_1\partial_{12}\mathbb{U}_3(z') + \widehat{\mathcal{R}}_1(z', a+Y_1, b) - \widehat{\mathcal{R}}_1(z', a, b)\right)\\
&+ \overline{u}^{(1)}_3\left(z', a+Y_1, b, Y_2, (-1)^{a+b+1}\kappa\right) - \overline{u}^{(2)}_3\left(z', a, b+Y_2, Y_1, (-1)^{a+b}\kappa\right).
\end{aligned}$$

Taking into account the expressions of $\hat{u}^{(1)}$ and $\hat{u}^{(2)}$, given by (76)–(78), and the equalities (see 61)

$$\partial_{Y_1}\widehat{\mathbb{U}}_3(z', a, b) = -\widehat{\mathcal{R}}_2(z', a, b), \qquad \partial_{Y_2}\widehat{\mathbb{U}}_3(z', a, b) = \widehat{\mathcal{R}}_1(z', a, b),$$

the outer-plane contact condition (85) is proved. □

7.8 | The displacements limit set

Now, since all the fields involved in the limit strain tensor, limit displacement, and limit contact conditions have been found, we, finally, can define the limit set of admissible displacements.

We set (see 57,58, and 71)

- $\mathcal{X}_M \doteq H^2_{(0,l)}\left((0,L)_{z_2}\right) \times H^2_{(0,l)}\left((0,L)_{z_1}\right) \times H^2_{\Gamma}(\Omega)$ the space of macroscopic functions;
- $\mathcal{X}_S \doteq \mathbf{L}^2(\Omega \times \{0,1\}, \partial_1) \times \mathbf{L}^2(\Omega \times \{0,1\}, \partial_2)$ the space of the relative macroscopic stretching functions;
- $\mathcal{X}_B \doteq L^2(\Omega \times \{0,1\})^2$ the space of the relative macroscopic bending functions;
- $\mathcal{X}_m \doteq L^2(\Omega; \mathbf{W}^{(1)}) \times L^2(\Omega; \mathbf{W}^{(2)})$ the space of all the microscopic functions.

In particular, the functions belonging to their respective spaces are defined by

$$\mathbb{V} \doteq (\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3) \in \mathcal{X}_M, \qquad \mathbb{V}^{(\mathbf{S})} \doteq \left(\mathbb{V}^{(\mathbf{S})}_1, \mathbb{V}^{(\mathbf{S})}_2\right) \in \mathcal{X}_S, \qquad \mathbb{V}^{(\mathbf{B})} \doteq \left(\mathbb{V}^{(\mathbf{B})}_1, \mathbb{V}^{(\mathbf{B})}_2\right) \in \mathcal{X}_B, \qquad \hat{v} \doteq \left(\hat{v}^{(1)}, \hat{v}^{(2)}\right) \in \mathcal{X}_m. \tag{86}$$

Including the limit contact conditions (81) and (85), the limit set of admissible displacements is defined by

$$\begin{aligned}\mathcal{X} \doteq \Big\{ \left(\mathbb{V}, \mathbb{V}^{(\mathbf{S})}, \mathbb{V}^{(\mathbf{B})}, \hat{v}\right) \in \mathcal{X}_M \times \mathcal{X}_S \times \mathcal{X}_B \times \mathcal{X}_m \;\Big|\; & |\mathbb{V}^{(\mathbf{S})}_1(\cdot, b) - \mathbb{V}^{(\mathbf{B})}_1(\cdot, a)| + \kappa|\partial_2\mathbb{V}_1 + \partial_1\mathbb{V}_2 \,| \le g_1 \text{ a.e. in } \Omega, \\ & |\mathbb{V}^{(\mathbf{S})}_2(\cdot, a) - \mathbb{V}^{(\mathbf{B})}_2(\cdot, b)| + \kappa|\partial_2\mathbb{V}_1 + \partial_1\mathbb{V}_2| \le g_2 \text{ a.e. in } \Omega, \\ 0 \le (-1)^{a+b}\left(\hat{v}^{(1)}_3(\cdot, a+Y_1, b, Y_2, (-1)^{a+b+1}\kappa) - \hat{v}^{(2)}_3(\cdot, a, b+Y_2, Y_1, (-1)^{a+b}\kappa)\right) & \text{ a.e. in } \Omega \times \omega_\kappa,\ (a,b) \in \{0,1\}^2 \Big\}.\end{aligned}$$

Note that $\mathcal{X}$ is a closed convex subset of the Hilbert space $\mathcal{X}_M \times \mathcal{X}_S \times \mathcal{X}_B \times \mathcal{X}_m$, endowed with the product norm. We set

$$\partial\mathbf{V}^{(1)} = \left(\partial_1\mathbb{V}^{(\mathbf{S})}_1(\cdot,0), \partial_1\mathbb{V}^{(\mathbf{S})}_1(\cdot,1), \partial_{12}\mathbb{V}_3, \partial_{11}\mathbb{V}_3, \partial_{11}\mathbb{V}_2\right), \qquad \partial\mathbf{V}^{(2)} = \left(\partial_2\mathbb{V}^{(\mathbf{S})}_2(\cdot,0), \partial_2\mathbb{V}^{(\mathbf{S})}_2(\cdot,1), \partial_{22}\mathbb{V}_3, \partial_{12}\mathbb{V}_3, \partial_{22}\mathbb{V}_1\right). \tag{87}$$

8 | STRONG CONVERGENCE OF THE TEST FUNCTIONS VIA UNFOLDING

We construct the test functions in order to to be sufficiently regular to be dense in the set of displacements. Moreover, they must ensure strong convergence via unfolding, have the same strain tensor as in the limit, and match the contact condition before and after the limit.

8.1 | Construction of the test functions

Consider the spaces

$$\mathcal{C}_M \doteq C^3(\overline{\Omega})^3 \cap \mathcal{X}_M,\ \mathcal{C}_S \doteq C^2(\overline{\Omega} \times \{0,1\})^2 \cap \mathcal{X}_S,\ \mathcal{C}_{\mathbf{B}} \doteq C^2_c(\Omega \times \{0,1\})^2 \cap \mathcal{X}_B,\ \mathcal{C}_m \doteq C^1_c(\Omega; \mathbf{W}^{(1)}) \times C^1_c(\Omega; \mathbf{W}^{(2)}).$$

Accordingly to (86), we take $(\mathbb{V}, \mathbb{V}^{(\mathbf{S})}, \mathbb{V}^{(\mathbf{B})}, \hat{v}) \in \mathcal{C}_M \times \mathcal{C}_S \times \mathcal{C}_B \times \mathcal{C}_m$.

First, the function $\mathbb{V}^{(1)}_\varepsilon$ is defined by

$$\mathbb{V}^{(1)}_\varepsilon(z_1, q\varepsilon) \doteq \mathbb{V}_1(q\varepsilon)\mathbf{e}_1 + \mathbb{V}_2(z_1)\mathbf{e}_2 + \mathbb{V}_3(z_1, q\varepsilon)\mathbf{e}_3,$$

then $\mathbb{V}^{(\mathbf{S})}_{\varepsilon,1}$ and $\mathbb{V}^{(\mathbf{B})}_{\varepsilon,2}$

$$\mathbb{V}^{(\mathbf{S})}_{\varepsilon,1}(z_1, q\varepsilon) \doteq \mathbb{V}^{(\mathbf{S})}_1\left(z_1, 2\left[\frac{q}{2}\right]\varepsilon, 2\left\{\frac{q}{2}\right\}\right), \qquad \mathbb{V}^{(\mathbf{B})}_{\varepsilon,2}(z_1, q\varepsilon) \doteq \mathbb{V}^{(\mathbf{B})}_2\left(z_1, 2\left[\frac{q}{2}\right]\varepsilon, 2\left\{\frac{q}{2}\right\}\right),$$

and $\widehat{v}_\varepsilon^{(1)}$

$$\widehat{v}_\varepsilon^{(1)}(z_1, q\varepsilon, y_2, y_3) \doteq \begin{cases} \widehat{v}^{(1)}\left(p\varepsilon, 2\left[\frac{q}{2}\right]\varepsilon, 2\left\{\frac{z_1}{2\varepsilon}\right\}, 2\left\{\frac{q}{2}\right\}, \frac{y_2}{\varepsilon}\frac{y_3}{\varepsilon}\right) & \text{if } z_1 \in [p\varepsilon - r, p\varepsilon + r], \\ \text{linear interpolated with respect to} z_1 & \text{if } z_1 \in [p\varepsilon + r, (p+1)\varepsilon - r]. \end{cases} \quad (y_2, y_3) \in \omega_r.$$

Now, we define the different test functions on the reference beams of direction $\mathbf{e}_2$.

The function $\mathbb{V}_\varepsilon^{(2)}$ is defined by

$$\mathbb{V}_\varepsilon^{(2)}(p\varepsilon, z_2) \doteq \mathbb{V}_1(z_2)\mathbf{e}_1 + \mathbb{V}_2(p\varepsilon)\mathbf{e}_2 + \mathbb{V}_3(p\varepsilon, z_2)\mathbf{e}_3,$$

then $\mathbb{V}_{\varepsilon,2}^{(\mathbf{S})}$ and $\mathbb{V}_{\varepsilon,1}^{(\mathbf{B})}$

$$\mathbb{V}_{\varepsilon,2}^{(\mathbf{S})}(p\varepsilon, z_2) \doteq \mathbb{V}_{\varepsilon,2}^{(\mathbf{S})}\left(2\left[\frac{p}{2}\right]\varepsilon, z_2, 2\left\{\frac{p}{2}\right\}\right), \qquad \mathbb{V}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon, z_2) \doteq \mathbb{V}_1^{(\mathbf{B})}\left(2\left[\frac{p}{2}\right]\varepsilon, z_2, 2\left\{\frac{p}{2}\right\}\right),$$

and $\widehat{v}_\varepsilon^{(2)}$

$$\widehat{v}_\varepsilon^{(2)}(p\varepsilon, z_2, y_1, y_3) \doteq \begin{cases} \widehat{v}^{(2)}\left(2\left[\frac{p}{2}\right]\varepsilon, q\varepsilon, 2\left\{\frac{p}{2}\right\}, 2\left\{\frac{z_2}{2\varepsilon}\right\}, \frac{y_1}{\varepsilon}\frac{y_3}{\varepsilon}\right) & \text{if } z_2 \in [q\varepsilon - r, q\varepsilon + r], \\ \text{linear interpolated with respect to} z_2 & \text{if } z_2 \in [q\varepsilon + r, (q+1)\varepsilon - r], \end{cases} \quad (y_1, y_3) \in \omega_r.$$

We compose the test displacements v_ε in the directions $\mathbf{e}_1$ and $\mathbf{e}_2$ by

$$v_\varepsilon^{(\alpha)} = V_{\varepsilon,BN}^{(\alpha)} + \varepsilon^3 \widehat{v}_\varepsilon^{(\alpha)}, \tag{88}$$

where the Bernoulli–Navier displacements are given by

$$V_{\varepsilon,BN}^{(1)}(z_1, q\varepsilon, y_2, y_3) \doteq \varepsilon\mathbb{V}_\varepsilon^{(1)}(z_1, q\varepsilon) + \begin{pmatrix} \varepsilon^2\mathbb{V}_{\varepsilon,1}^{(\mathbf{S})}(z_1, q\varepsilon) \\ \varepsilon^2\mathbb{V}_{\varepsilon,2}^{(\mathbf{B})}(z_1, q\varepsilon) \\ 0 \end{pmatrix} + \begin{pmatrix} \varepsilon\partial_2\mathbb{V}_{\varepsilon,3}(z_1, q\varepsilon) \\ -\varepsilon\partial_1\mathbb{V}_{\varepsilon,3}(z_1, q\varepsilon) \\ \varepsilon\partial_1\mathbb{V}_{\varepsilon,2}(z_1) + \varepsilon^2\partial_1\mathbb{V}_{\varepsilon,2}^{(\mathbf{B})}(z_1, q\varepsilon) \end{pmatrix}$$
$$\wedge\left(\varepsilon(-1)^{q+1}\Phi\left(2\left\{\frac{z_1}{2\varepsilon}\right\}\right)\mathbf{e}_3 + y_2\mathbf{e}_2 + y_3(-1)^{q+1}\mathbf{n}\left(2\left\{\frac{z_1}{2\varepsilon}\right\}\right)\right),$$

and

$$V_{\varepsilon,BN}^{(2)}(p\varepsilon, z_2, y_1, y_3) \doteq \varepsilon\mathbb{V}_\varepsilon^{(2)}(p\varepsilon, z_2) + \begin{pmatrix} \varepsilon^2\mathbb{V}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon, z_2) \\ \varepsilon^2\mathbb{V}_{\varepsilon,2}^{(\mathbf{S})}(p\varepsilon, z_2) \\ 0 \end{pmatrix} + \begin{pmatrix} \varepsilon\partial_2\mathbb{V}_{\varepsilon,3}(p\varepsilon, z_2) \\ -\varepsilon\partial_1\mathbb{V}_{\varepsilon,3}(p\varepsilon, z_2) \\ -\varepsilon\partial_2\mathbb{V}_{\varepsilon,1}(z_2) - \varepsilon^2\partial_2\mathbb{V}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon, z_2) \end{pmatrix}$$
$$\wedge\left(\varepsilon(-1)^{p}\Phi\left(2\left\{\frac{z_2}{2\varepsilon}\right\}\right)\mathbf{e}_3 + y_1\mathbf{e}_1 + y_3(-1)^{p}\mathbf{n}\left(2\left\{\frac{z_2}{2\varepsilon}\right\}\right)\right).$$

8.2 | The limit strain tensors for the test functions

The limit of the unfolded strain tensor is an immediate consequence of (16) and (17), the unfolding operator properties and the regularity of the test functions (see also Lemma 8.1 in Griso et al. [25]). We obtain

$$\frac{1}{\varepsilon^2}\Pi^{(\alpha)}\left(\widetilde{\mathbf{e}}\left(v_\varepsilon^{(\alpha)}\right)\right) \to \mathcal{E}^{(\alpha)}\left(\partial\mathbf{V}^{(\alpha)}\right) + \mathcal{E}_Y^{(\alpha)}\left(\widehat{v}^{(\alpha)}\right) \quad \text{strongly in } L^2(\Omega \times Cyl^{(\alpha)})^{3\times3}, \tag{89}$$

where $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ are, respectively, given by (74) with the fields $\partial\mathbf{V}^{(1)}$, $\partial\mathbf{V}^{(2)}$ given by (87).

8.3 | The initial contact conditions for the test functions

First, by construction of the test displacements, the boundary conditions are satisfied.

Now, we check the in-plane contact conditions (23). We set

$$\mathbf{N} \doteq \sum_{\alpha=1}^{2} \left(\|\nabla \mathbb{V}_{\alpha}^{(\mathbf{S})}\|_{L^{\infty}(\Omega\times\{0,1\})} + \|\nabla^2 \mathbb{V}_{\alpha}^{(\mathbf{B})}\|_{L^{\infty}(\Omega\times\{0,1\})} + \|\hat{v}^{(\alpha)}\|_{L^{\infty}(\Omega\times Cyl^{(\alpha)})} \right).$$

Below, we replace the components $v_{\varepsilon,\alpha}^{(1)}$ and $v_{\varepsilon,\alpha}^{(2)}$ of the test displacement by $\lambda_{\varepsilon}^{*} v_{\varepsilon,\alpha}^{(1)}$ and $\lambda_{\varepsilon}^{*} v_{\varepsilon,\alpha}^{(2)}$ in order to satisfy the contact conditions (23). We will choose $\lambda_{\varepsilon}^{*} \doteq 1 - C^{*}\varepsilon$, where C^{*} is a nonnegative constant that will be assigned later.

We start with the first component. We take the difference between the displacements and get (remind that $\kappa < 1$)

$$\begin{aligned} & v_{\varepsilon,1}^{(1)}(y_1 + p\varepsilon, q\varepsilon, y_2, (-1)^{a+b+1}\kappa\varepsilon) - v_{\varepsilon,1}^{(2)}(p\varepsilon, q\varepsilon + y_2, y_1, (-1)^{a+b}\kappa\varepsilon) \\ & = \varepsilon \left(\mathbb{V}_1(q\varepsilon) - \mathbb{V}_1(q\varepsilon + y_2) + y_2 \partial_1 \mathbb{V}_2(p\varepsilon + y_1) \right) + \varepsilon^2 \left(\mathbb{V}_{\varepsilon,1}^{(\mathbf{S})}(p\varepsilon + y_1, q\varepsilon) - \mathbb{V}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon, q\varepsilon + y_2) + y_2 \partial_1 \mathbb{V}_{\varepsilon,2}^{(\mathbf{B})}(p\varepsilon + y_1, q\varepsilon) \right) \\ & \quad + \varepsilon^3 \left(\hat{v}_{\varepsilon,1}^{(1)}(y_1 + p\varepsilon, q\varepsilon, y_2, (-1)^{a+b+1}\kappa\varepsilon) - \hat{v}_{\varepsilon,1}^{(2)}(p\varepsilon, q\varepsilon + y_2, y_1, (-1)^{a+b}\kappa\varepsilon) \right). \end{aligned}$$

Besides, we have for a.e. $(y_1, y_2) \in \omega_r$

$$|\mathbb{V}_1(q\varepsilon) - \mathbb{V}_1(q\varepsilon + y_2) + y_2 \partial_2 \mathbb{V}_1(q\varepsilon + y_2)| \le \kappa^2 \varepsilon^2 \|\partial_{22}^2 \mathbb{V}_1\|_{L^{\infty}(\Omega)},$$

and

$$\begin{aligned} & \left| \left(\mathbb{V}_{\varepsilon,1}^{(\mathbf{S})}(p\varepsilon + y_1, q\varepsilon) - \mathbb{V}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon, q\varepsilon + y_2) + y_2 \partial_1 \mathbb{V}_{\varepsilon,2}^{(\mathbf{B})}(p\varepsilon + y_1, q\varepsilon) \right) - \left(\mathbb{V}_{\varepsilon,1}^{(\mathbf{S})}(p\varepsilon + y_1, q\varepsilon + y_2) - \mathbb{V}_{\varepsilon,1}^{(\mathbf{B})}(p\varepsilon + y_1, q\varepsilon + y_2) \right) \right| \\ & \quad \le \kappa\varepsilon \left(\|\nabla \mathbb{V}_1^{(\mathbf{S})}\|_{L^{\infty}(\Omega)} + \|\nabla^2 \mathbb{V}_1^{(\mathbf{B})}\|_{L^{\infty}(\Omega\times\{0,1\})} \right) \\ & \left| \hat{v}_{\varepsilon,1}^{(1)}(y_1 + p\varepsilon, q\varepsilon, y_2, (-1)^{a+b+1}\kappa\varepsilon) - \hat{v}_1^{(1)} \left(p\varepsilon + y_1, q\varepsilon + y_2, 2\left\{ \frac{q}{2} \right\}, 2\left\{ \frac{p\varepsilon + y_1}{2\varepsilon} \right\}, \frac{y_2}{\varepsilon}, (-1)^{a+b+1}\kappa \right) \right| \le C \|\partial_2 \hat{v}_1^{(1)}\|_{L^{\infty}(\Omega\times Cyl^{(1)})}, \\ & \left| \hat{v}_{\varepsilon,1}^{(2)}(p\varepsilon, q\varepsilon + y_2, y_1, (-1)^{a+b}\kappa\varepsilon) - \hat{v}_1^{(2)} \left(p\varepsilon + y_1, q\varepsilon + y_2, 2\left\{ \frac{p}{2} \right\}, 2\left\{ \frac{q\varepsilon + y_2}{2\varepsilon} \right\}, \frac{y_1}{\varepsilon}, (-1)^{a+b}\kappa \right) \right| \le C \|\partial_2 \hat{v}_1^{(1)}\|_{L^{\infty}(\Omega\times Cyl^{(1)})}. \end{aligned}$$

Regarding the in-plane contact in original, we have a.e. in C_{pq} that

$$\left| \left(v_{\varepsilon,1}^{(1)}(z_1, q\varepsilon, y_2, (-1)^{a+b+1}\kappa) - v_{\varepsilon,1}^{(2)}(p\varepsilon, z_2, y_1, (-1)^{a+b}\kappa) \right) - \varepsilon^2 (\mathbb{V}_1^{(\mathbf{S})}(z', b) - \mathbb{V}_1^{(\mathbf{B})}(z', a) - \frac{y_2}{\varepsilon} \left(\partial_2 \mathbb{V}_1(z_2) + \partial_1 \mathbb{V}_2(z_1) \right) \right| \le C^{\diamond} \varepsilon^3 \mathbf{N},$$

where $C^{\diamond}$ does not depend on ε. So a.e. in $C_{pq,\varepsilon}$, we have

$$|v_{\varepsilon,1}^{(1)} - v_{\varepsilon,1}^{(2)}| \le \varepsilon^2 g_1 + C^{\diamond}\varepsilon^3 \mathbf{N}.$$

Taking into account the property (22) of g_{α}, we take the value $C^{*} = C^{\diamond}\mathbf{N}/C_3$. Hence, the in-plane contact conditions (23) are satisfied, since

$$|\lambda_{\varepsilon}^{*} v_{\varepsilon,1}^{(1)} - \lambda_{\varepsilon}^{*} v_{\varepsilon,1}^{(2)}| \le \varepsilon^2 \lambda_{\varepsilon}^{*} g_1 + \lambda_{\varepsilon}^{*} C^{\diamond}\varepsilon^3 \mathbf{N} \le \varepsilon^2 g_1 - C^{*}\varepsilon^3 g_1 + C^{\diamond}\varepsilon^3 \mathbf{N} \le \varepsilon^2 g_1 - \varepsilon^3 (C^{*} C_3 - C^{\diamond}\mathbf{N}) = \varepsilon^2 g_1.$$

This proves that the contact conditions are satisfied for the first component. We do the same in the direction $\mathbf{e}_2$.

By construction of the test displacement in the direction $\mathbf{e}_3$, the non-penetration conditions (24) are satisfied (see also Griso et al. [25]).

The clamping conditions are also satisfied.

9 STUDY OF THE LIMIT PROBLEM

In this section, we employ all the results developed in the previous ones to go to the limit for problem (26). We will then proceed to its investigation and draw conclusions.

9.1 The unfolded limit problem

We start with the technical lemma below.

Lemma 13. *Let $X = (X_0, X_{00}, X_1, X_2, X_3)$ be in $\mathbb{R}^5$ and $\widehat{v}^{(\alpha)} \in \mathbf{W}^{(\alpha)}$ satisfying*

$$\mathcal{E}^{(\alpha)}(X) + \mathcal{E}_Y^{(\alpha)}\left(\widehat{v}^{(\alpha)}\right) = 0. \tag{90}$$

Then, $X = 0$ and $\widehat{v}^{(\alpha)}$ are periodic rigid displacements.
Moreover, there exist two strictly positive constants C_0, C_1 such that for every $X \in \mathbb{R}^4$ and every $\widehat{v}^{(\alpha)} \in \mathbf{W}^{(\alpha)}$,

$$C_0\left(|X|^2 + \|\mathcal{E}_Y^{(\alpha)}\left(\widehat{v}^{(\alpha)}\right)\|^2_{L^2(Cyl^{(\alpha)})}\right) \le \|\mathcal{E}^{(\alpha)}(X) + \mathcal{E}_Y^{(\alpha)}\left(\widehat{v}^{(\alpha)}\right)\|^2_{L^2(Cyl^{(\alpha)})} \le C_1\left(|X|^2 + \|\mathcal{E}_Y^{(\alpha)}\left(\widehat{v}^{(\alpha)}\right)\|^2_{L^2(Cyl^{(\alpha)})}\right). \tag{91}$$

Proof. We prove the lemma for $\alpha = 1$.
The solution of the Equation (90) is given by

$$\widehat{v}^{(1)} = \mathcal{A}^{(1)} + \mathcal{B}^{(1)} \wedge \left(\Phi^{(1)}\mathbf{e}_3 + Y_2\mathbf{e}_2 + Y_3\mathbf{n}^{(1)}\right),\ \mathcal{A}^{(1)},\ \mathcal{B}^{(1)} \in H^1(\mathfrak{G}^{(1)})^3,$$

with first (see 72–74) $\partial_{Y_1}\mathcal{B}^{(1)} = X_1\mathbf{e}_1 - X_2\mathbf{e}_2 + X_3\mathbf{e}_3$. Since $\mathcal{B}^{(1)}$ is periodic, this gives $X_1 = X_2 = X_3 = 0$ and $\mathcal{B}^{(1)}(Y_1, b) = \mathbf{B}^{(1)}(b)$ for a.e. $(Y_1, b) \in \mathfrak{G}^{(1)}$. Then, we get $\partial_{Y_1}\mathcal{A}^{(1)}(Y_1, 0) = \mathbf{B}^{(1)}(0)\wedge\mathbf{e}_1 - X_0\mathbf{e}_1$ (resp. $\partial_{Y_1}\mathcal{A}^{(1)}(Y_1, 1) = \mathbf{B}^{(1)}(1)\wedge\mathbf{e}_1 - X_{00}\mathbf{e}_1$), again since $\mathcal{A}^{(1)}$ is periodic, this gives $X_0 = X_{00} = 0$ and $\mathbf{B}^{(1)} = \mathbf{b}^{(1)}\mathbf{e}_1$. Hence, $\widehat{v}^{(1)}$ is a rigid periodic displacement

$$\widehat{v}^{(1)}(Y_1, b, Y_2, Y_3) = \mathbf{A}^{(1)}(b) + \mathbf{b}^{(1)}(b)\mathbf{e}_1 \wedge \left(\Phi^{(1)}\mathbf{e}_3 + Y_2\mathbf{e}_2 + Y_3\mathbf{n}^{(1)}\right),\ \mathbf{A}^{(1)}(b) \in \mathbb{R}^3,\ \mathbf{b}^{(1)}(b) \in \mathbb{R}^3.$$

The inequality in the right-hand side of (91) is obvious. The left-hand side inequality is proven by contradiction. □

We are now ready to show the limit elasticity problem.

Theorem 3. *Let $u_\varepsilon \in \mathcal{X}_\varepsilon$ be a solution of problem (26) and let f, $\widetilde{f}$ be as in Section 7.1. We assume that there exist $A^{(\alpha)} \in L^\infty(Cyl^{(\alpha)})^{6\times 6}$, such that*

$$\Pi_\varepsilon^{(\alpha)}\left(\boldsymbol{A}_\varepsilon^{(\alpha)}\left(\frac{\cdot}{\varepsilon}\right)\right)(z', Y) \to A^{(\alpha)}(Y) \text{ for a.e.} (z', Y) \in \Omega \times Cyl^{(\alpha)}. \tag{92}$$

Then, there exist a subsequence of $\{\varepsilon\}$, still denoted $\{\varepsilon\}$, and functions $(\mathbb{U}, \mathbb{U}^{(\boldsymbol{S})}, \mathbb{U}^{(\boldsymbol{B})}, \widehat{u}) \in \mathcal{X}$, such that a solution $(u_\varepsilon^{(1)}, u_\varepsilon^{(2)})$ of problem (26) converges. The unfolded limit problem admits solutions and has the following formulation:

$$\begin{aligned}
&\text{Find } (\mathbb{U}, \mathbb{U}^{(\boldsymbol{S})}, \mathbb{U}^{(\boldsymbol{B})}, \widehat{u}) \in \mathcal{X} \text{ such that for every } (\mathbb{V}, \mathbb{V}^{(\boldsymbol{B})}, \mathbb{V}^{(\boldsymbol{B})}, \widehat{v}) \in \mathcal{X},\\
&\sum_{\alpha=1}^{2} \int_{\Omega\times Cyl^{(\alpha)}} A_{ijkl}^{(\alpha)}\left(\mathcal{E}_{ij}^{(\alpha)}\left(\partial\mathbf{U}^{(\alpha)}\right) + \mathcal{E}_{Y,ij}^{(\alpha)}\left(\widehat{u}^{(\alpha)}\right)\right)\left(\mathcal{E}_{kl}^{(\alpha)}(\partial\mathbf{U}^{(\alpha)} - \partial\mathbf{V}^{(\alpha)}) + \mathcal{E}_{Y,kl}^{(\alpha)}(\widehat{u}^{(\alpha)} - \widehat{v}^{(\alpha)})\right)\boldsymbol{\eta}^{(\alpha)}dz'dY\\
&\le \boldsymbol{C}_0(\kappa)\sum_{\beta=1}^{2}\left(\sum_{\alpha=1}^{2}\int_\Omega f_\alpha^{(\beta)}(\mathbb{U}_\alpha - \mathbb{V}_\alpha)dz' + \int_\Omega f_3^{(\beta)}(\mathbb{U}_3 - \mathbb{V}_3)dz'\right)\\
&\quad + \frac{\boldsymbol{C}_0(\kappa)}{2}\int_\Omega \sum_{c=0}^{1}\left(\widetilde{f}_\alpha^{(\alpha)}\left(\mathbb{U}_\alpha^{(\boldsymbol{S})} - \mathbb{V}_\alpha^{(\boldsymbol{S})}\right)(\cdot, c) + \widetilde{f}_\alpha^{(3-\alpha)}\left(\mathbb{U}_\alpha^{(\boldsymbol{B})} - \mathbb{V}_\alpha^{(\boldsymbol{B})}\right)(\cdot, c)\right)dz' - \boldsymbol{C}_1(\kappa)\sum_{\alpha=1,\beta=1}^{2}\int_\Omega \widetilde{f}_\alpha^{(\beta)}(\partial_\alpha\mathbb{U}_3 - \partial_\alpha\mathbb{V}_3)dz',
\end{aligned} \tag{93}$$

where $\partial\mathbf{U}^{(\alpha)}$ and $\partial\mathbf{V}^{(\alpha)}$ are defined in (87) and where

$$\boldsymbol{C}_0(\kappa) \doteq 4\kappa^2 \int_0^2 \boldsymbol{\gamma}(t)dt, \qquad \boldsymbol{C}_1(\kappa) \doteq 4\kappa^2 \int_0^2 \Phi(t)\boldsymbol{\gamma}(t)dt. \tag{94}$$

Proof. First, from the weak convergence of the strain tensors (75)–(77), the strong convergence of the test functions (89), convergence (92), and Lemma 8, we get that

$$\begin{aligned}&\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} A_{ijkl,\varepsilon}^{(\alpha)}\widetilde{\mathbf{e}}_{ij}\left(u_\varepsilon^{(\alpha)}\right)\widetilde{\mathbf{e}}_{kl}\left(v_\varepsilon^{(\alpha)}\right)\,\eta^{(\alpha)}\,dz_\alpha dy_{3-\alpha}dy_3 \to \\ &\sum_{\alpha=1}^{2}\frac{1}{2}\int_{\Omega\times Cyl^{(\alpha)}} A_{ijkl}^{(\alpha)}\left(\mathcal{E}_{ij}^{(\alpha)}\left(\partial\mathbf{U}^{(\alpha)}\right)+\mathcal{E}_{Y,ij}^{(\alpha)}\left(\widehat{u}^{(\alpha)}\right)\right)\left(\mathcal{E}_{kl}^{(\alpha)}\left(\partial\mathbf{V}^{(\alpha)}\right)+\mathcal{E}_{Y,kl}^{(\alpha)}\left(\widehat{v}^{(\alpha)}\right)\right)\boldsymbol{\eta}^{(\alpha)}\,dz'dY,\end{aligned} \tag{95}$$

where $(u_\varepsilon^{(1)}, u_\varepsilon^{(2)})$ is a solution to problem (26) and $\left(v_\varepsilon^{(1)}, v_\varepsilon^{(2)}\right)$ is the test function defined in (88). Note that we have

$$\int_0^L\int_{\omega_r} |\widetilde{\mathbf{e}}_{ij}\left(v_\varepsilon^{(\alpha)}\right)(\cdot, 2N_\varepsilon\varepsilon, \cdot)|^2 dz_\alpha dy_{3-\alpha}dy_3 \le C\varepsilon^3.$$

The constant does not depend on ε, it depends on the test function. By the weak convergences (75)–(77), the weak lower semicontinuity of the convex functionals in problem (26), we have

$$\begin{aligned}&\frac{1}{2}\sum_{\alpha=1}^{2}\int_{\Omega\times Cyl^{(\alpha)}} A_{ijkl}^{(\alpha)}\left(\mathcal{E}_{ij}^{(\alpha)}\left(\partial\mathbf{U}^{(\alpha)}\right)+\mathcal{E}_{Y,ij}^{(\alpha)}\left(\widehat{u}^{(\alpha)}\right)\right)\left(\mathcal{E}_{kl}^{(\alpha)}\left(\partial\mathbf{U}^{(\alpha)}\right)+\mathcal{E}_{Y,kl}^{(\alpha)}\left(\widehat{u}^{(\alpha)}\right)\right)\boldsymbol{\eta}^{(\alpha)}\,dz'dY \\ &\le \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} A_{ijkl,\varepsilon}^{(\alpha)}\widetilde{\mathbf{e}}_{ij}\left(u_\varepsilon^{(\alpha)}\right)\widetilde{\mathbf{e}}_{kl}\left(u_\varepsilon^{(\alpha)}\right)\,\eta^{(\alpha)}dz_\alpha dy_{3-\alpha}dy_3 \le \liminf_{\varepsilon\to 0}\sum_{\alpha=1}^{2}\frac{1}{\varepsilon^5}\int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} F_\varepsilon^{(\alpha)}\cdot u_\varepsilon^{(\alpha)}\,\eta^{(\alpha)}dz_\alpha dy_{3-\alpha}dy_3.\end{aligned} \tag{96}$$

Now, determine the limit of the last term in (96). We transform this term, using $\Pi_\varepsilon^{(\alpha)}$ and Lemma 8. Due to the regularity of the applied forces, we easily get their strong convergences, then the strong convergences in Lemma 7, the mobile frame convergences (see Section A.2), we obtain for $\alpha = 1$

$$\begin{aligned}&\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\int_{\mathfrak{G}_\varepsilon^{(1)}\times\omega_r} F_\varepsilon^{(1)}\cdot u_\varepsilon^{(1)}\,\eta^{(1)}\,dz_1dy_2dy_3 \\ &\to \mathbf{C}_0(\kappa)\left(\sum_{\alpha=1}^{2}\int_\Omega f_\alpha^{(1)}\mathbb{U}_\alpha\,dz' + \int_\Omega f_3^{(1)}\mathbb{U}_3dz' + \frac{1}{2}\int_\Omega\sum_{a=0}^{1}\left(\widetilde{f}_1^{(1)}\mathbb{U}_1^{(\mathbf{S})}(\cdot,a)+\widetilde{f}_2^{(1)}\mathbb{U}_2^{(\mathbf{B})}(\cdot,a)\right)dz'\right) \\ &- \mathbf{C}_1(\kappa)\sum_{\alpha=1}^{2}\int_\Omega \widetilde{f}_\alpha^{(1)}\partial_\alpha\mathbb{U}_3dz'.\end{aligned} \tag{97}$$

At last, we get the limit of

$$\frac{1}{\varepsilon^5}\sum_{\alpha=1}^{2}\int_{\mathfrak{G}_\varepsilon^{(\alpha)}\times\omega_r} F_\varepsilon^{(\alpha)}\cdot v_\varepsilon^{(\alpha)}\,\eta^{(\alpha)}\,dz_\alpha dy_{3-\alpha}dy_3,$$

by replacing in (97) the functions $(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}, \mathbb{U}^{(\mathbf{B})})$ by the functions $(\mathbb{V}, \mathbb{V}^{(\mathbf{S})}, \mathbb{V}^{(\mathbf{B})})$. Hence, inequality (93) follows, due to (95)–(97). A density argument gives (93) for any test function in $\mathcal{X}$.

The existence of solutions for problem (93) is a direct consequence of the bilinearity, boundedness, and coercivity of $A^{(\alpha)}$ (inherited from the properties (i)–(iii) of $A_\varepsilon^{(\alpha)}$ in Section 5.2 through convergence 92) and the Stampacchia lemma. □

9.2 | The microscopic cell problem

Since the limit problem has been found, we can proceed to the split of the microscopic scale from the macroscopic one. In this subsection, we investigate the microscopic problem, or cell problem, whose solution, the correctors, will later form the homogenizing operator in the macroscopic scale.

We first define $\mathbf{W}$ as the convex subset of $\mathbf{W}^{(1)} \times \mathbf{W}^{(2)}$ by

$$\mathbf{W} \doteq \left\{ (\hat{w}^{(1)}, \hat{w}^{(2)}) \in \mathbf{W}^{(1)} \times \mathbf{W}^{(2)} \;\middle|\; 0 \le (-1)^{a+b} \left(\hat{w}_3^{(1)}(a+Y_1, b, Y_2, (-1)^{a+b+1}\kappa) - \hat{w}_3^{(2,a)}(a, b+Y_2, Y_1, (-1)^{a+b}\kappa) \right) \text{ a.e. on } \omega_\kappa,\ (a,b) \in \{0,1\}^2 \right\}.$$

Now, we introduce the corrector's problem. For every $X \in \mathbb{R}^9$, we denote

$$X^{(1)} = (X_1, X_2, X_5, X_6, X_7), \qquad X^{(2)} = (X_3, X_4, X_8, X_5, X_9).$$

We consider the following microscopic cell problems: The microscopic cell problems become

$$\begin{gathered} \text{For each} X^{(\alpha)} \in \mathbb{R}^5, \text{ find} \hat{\chi} \in \mathbf{W} \text{such that for every} \hat{v} \in \mathbf{W}: \\ \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A_{ijkl}^{(\alpha)} \left(\mathcal{E}_{ij}^{(\alpha)}(X^{(\alpha)}) + \mathcal{E}_{Y,ij}^{(\alpha)}(\hat{\chi}) \right) \mathcal{E}_{Y,kl}^{(\alpha)}(\hat{\chi} - \hat{v}) \boldsymbol{\eta}^{(\alpha)} \, dY \le 0. \end{gathered} \tag{98}$$

The existence of solutions follows by Stampacchia lemma.

Now, if $\hat{\chi}$ and $\tilde{\chi}$ are both solutions of (98), then we can first consider problem (98) with $\hat{\chi}$ as the solution and $\tilde{\chi}$ as the test function and then vice versa. Summing up both inequalities leads to

$$\sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A_{ijkl}^{(\alpha)} \mathcal{E}_{ij,Y}^{(\alpha)}(\tilde{\chi} - \hat{\chi}) \mathcal{E}_{kl,Y}^{(\alpha)}(\tilde{\chi} - \hat{\chi}) \boldsymbol{\eta}^{(\alpha)} \, dY \le 0,$$

from where we get that $\mathcal{E}_Y^{(\alpha)}(\hat{\chi}) = \mathcal{E}_Y^{(\alpha)}(\tilde{\chi})$, since by coercivity the above quantity is also nonnegative. Hence, Lemma 13 implies that there exist rigid displacements $\mathbf{r}^{(\alpha)} \in \mathbf{W}$, such that

$$\hat{\chi} = \mathbf{r}^{(\alpha)} + \tilde{\chi}, \qquad \text{a.e. in } Cyl^{(\alpha)}. \tag{99}$$

As the strain tensors of the solutions of (98) are uniquely determined, we will henceforth denote them $\mathcal{E}_Y^{(\alpha)}(\hat{\chi}(X^{(\alpha)}, \cdot))$.

9.3 | The homogenizing operator and the macroscopic cell problem

Now, since problem (98) has been investigated, we can define the homogenizing operators by integrating over the solutions of the cell problems.

Definition 5. We define the homogenizing operator A_{hom} by

$$\forall X \in \mathbb{R}^9, \qquad A_{hom,n}(X) \doteq \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A_{ijkl}^{(\alpha)} \left(\mathcal{E}_{ij}^{(\alpha)}(X^{(\alpha)}) + \mathcal{E}_{Y,ij}^{(\alpha)}(\hat{\chi}(X^{(\alpha)}, Y)) \right) \mathcal{E}_{kl}^{(\alpha)}(\mathbf{e}_n^{(\alpha)}) \boldsymbol{\eta}^{(\alpha)} \, dY,$$

where $\hat{\chi}(X^{(1)}, \cdot)$ (resp. $\hat{\chi}(X^{(2)}, \cdot)$) is a solution of problem (98) with $X^{(1)}$ (resp. $X^{(2)}$) and $(\mathbf{e}_1, \dots, \mathbf{e}_9)$ the usual basis of $\mathbb{R}^9$.

Now, to ensure the existence of solutions for the macroscopic problem, we need to prove some properties of the homogenizing operator to apply the Stampacchia lemma.

Proposition 4. *The operator A_{hom} is continuous (and thus of Carathéodory type), bounded, monotone, and coercive.*

Proof.

Step 1. We show that the map $X^{(\alpha)} \in \mathbb{R}^5 \mapsto \mathcal{E}_Y(\hat{\chi}(X^{(\alpha)},\cdot))$ is Lipschitz continuous for the strong topology of $L^2(Cyl^{(\alpha)})^6$. We will only prove the statement for $\alpha = 1$, since the proof for $\alpha = 2$ is analogous.
Let $X^{(1)}$, $Z^{(1)}$ be two vectors in $\mathbb{R}^5$ and $\hat{\chi}(X^{(1)},\cdot)$, $\hat{\chi}(Z^{(1)},\cdot)$ be the associated solutions, given by the cell problem (98). By the coercivity of the tensor $A^{(1)}$, we have

$$\begin{aligned}
&\left\| \mathcal{E}_Y^{(1)}(\hat{\chi}(X^{(1)},\cdot)) - \mathcal{E}_Y^{(1)}(\hat{\chi}(Z^{(1)},\cdot)) \right\|^2_{L^2(Cyl^{(1)})} \\
&\le \int_{Cyl^{(1)}} A^{(1)}_{ijkl} \left(\mathcal{E}^{(1)}_{Y,ij}(\hat{\chi}(X^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,ij}(\hat{\chi}(Z^{(1)},\cdot)) \right) \left(\mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(X^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(Z^{(1)},\cdot)) \right) \boldsymbol{\eta}^{(1)}\, dY \\
&\le \int_{Cyl^{(1)}} A^{(1)}_{ijkl} \mathcal{E}^{(1)}_{Y,ij}(\hat{\chi}(X^{(1)},\cdot)) \left(\mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(X^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(Z^{(1)},\cdot)) \right) \boldsymbol{\eta}^{(1)}\, dY \\
&\quad + \int_{Cyl^{(1)}} A^{(1)}_{ijkl} \mathcal{E}^{(1)}_{Y,ij}(\hat{\chi}(Z^{(1)},\cdot)) \left(\mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(Z^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(X^{(1)},\cdot)) \right) \boldsymbol{\eta}^{(1)}\, dY \\
&\le - \int_{Cyl^{(1)}} A^{(1)}_{ijkl} \mathcal{E}^{(1)}_{ij}(X^{(1)}) \left(\mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(X^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(Z^{(1)},\cdot)) \right) \boldsymbol{\eta}^{(1)}\, dY \\
&\quad - \int_{Cyl^{(1)}} A^{(1)}_{ijkl} \mathcal{E}^{(1)}_{ij}(Z^{(1)}) \left(\mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(Z^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(X^{(1)},\cdot)) \right) \boldsymbol{\eta}^{(1)}\, dY \\
&\le \int_{Cyl^{(1)}} A^{(1)}_{ijkl} \mathcal{E}^{(1)}_{ij}(X^{(1)} - Z^{(1)}) \left(\mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(X^{(1)},\cdot)) - \mathcal{E}^{(1)}_{Y,kl}(\hat{\chi}(Z^{(1)},\cdot)) \right) \boldsymbol{\eta}^{(1)}\, dY \\
&\le C \left\| \mathcal{E}^{(1)}(X^{(1)} - Z^{(1)}) \right\|_{L^2(Cyl^{(1)})} \left\| \mathcal{E}_Y^{(1)}(\hat{\chi}(X^{(1)},\cdot)) - \mathcal{E}_Y^{(1)}(\hat{\chi}(Z^{(1)},\cdot))) \right\|_{L^2(Cyl^{(1)})}.
\end{aligned}$$

Hence, the Lipschitz continuity is proven since

$$\left\| \mathcal{E}_Y^{(1)}(\hat{\chi}(X^{(1)},\cdot)) - \mathcal{E}_Y^{(1)}(\hat{\chi}(Z^{(1)},\cdot)) \right\|_{L^2(Cyl^{(1)})} \le C \left\| \mathcal{E}^{(1)}(X^{(1)} - Z^{(1)}) \right\|_{L^2(Cyl^{(1)})} \le C|X^{(1)} - Z^{(1)}|.$$

So the statement of this step is proven. As a consequence the map $X \in \mathbb{R}^9 \mapsto A_{hom}(X) \in \mathbb{R}^9$ is continuous.

Step 2. We prove the monotonicity.
Let X and Z be two vectors in $\mathbb{R}^9$. By the coercivity of the tensor $A^{(\alpha)}$, we have

$$\begin{aligned}
&(A_{hom}(X) - A_{hom}(Z)) \cdot (X - Z) \\
&= \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} \left(\mathcal{E}^{(\alpha)}_{ij}(X^{(\alpha)} - Z^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\hat{\chi}(X^{(\alpha)},\cdot) - \hat{\chi}(Z^{(\alpha)},\cdot)) \right) \mathcal{E}^{(\alpha)}_{kl}(X^{(\alpha)} - Z^{(\alpha)}) \boldsymbol{\eta}^{(\alpha)}\, dY \\
&\ge C \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} \left| \mathcal{E}^{(\alpha)}_{ij}(X^{(\alpha)} - Z^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\hat{\chi}(X^{(\alpha)},\cdot) - \hat{\chi}(Z^{(\alpha)},\cdot)) \right|^2 \boldsymbol{\eta}^{(\alpha)}\, dY \\
&\quad - \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} \left(\mathcal{E}^{(\alpha)}_{ij}(X^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\hat{\chi}(X^{(\alpha)},\cdot) \right) \mathcal{E}^{(\alpha)}_{Y,kl}(\hat{\chi}(X^{(\alpha)},\cdot) - \hat{\chi}(Z^{(\alpha)},\cdot)) \boldsymbol{\eta}^{(\alpha)}\, dY \\
&\quad - \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A^{(\alpha)}_{ijkl} \left(\mathcal{E}^{(\alpha)}_{ij}(Z^{(\alpha)}) + \mathcal{E}^{(\alpha)}_{Y,ij}(\hat{\chi}(Z^{(\alpha)},\cdot)) \right) \mathcal{E}^{(\alpha)}_{Y,kl}(\hat{\chi}(Z^{(\alpha)},\cdot) - \hat{\chi}(X^{(\alpha)},\cdot)) \boldsymbol{\eta}^{(\alpha)}\, dY \ge 0.
\end{aligned}$$

Thus, the monotonicity of A_{hom} is proved since the first integral is nonnegative, and the second and third integrals are nonnegative by definition of problem (98) with the choice of test functions $\hat{\chi}(X^{(1)},\cdot)$ and $\hat{\chi}(Z^{(1)},\cdot)$, respectively.

Step 3. We prove that A_{hom} is coercive.

From the first inequality of (91), we have

$$\int_{Cyl^{(1)}} \left|\mathcal{E}^{(1)}(X^{(1)}) + \mathcal{E}_Y^{(1)}\left(\hat{\chi}(X^{(1)},\cdot)\right)\right|^2 \boldsymbol{\eta}^{(1)}\, dY \geq C_0|X^{(1)}|^2 \qquad \forall X^{(1)} \in \mathbb{R}^5. \tag{100}$$

Hence, for every X in $\mathbb{R}^9$, we get

$$\begin{aligned}
A_{hom}(X)\cdot X &= \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A_{ijkl}^{(\alpha)} \left(\mathcal{E}_{ij}^{(\alpha)}(X^{(\alpha)}) + \mathcal{E}_{Y,ij}^{(\alpha)}(\hat{\chi}(X^{(\alpha)},\cdot))\right) \mathcal{E}_{kl}^{(\alpha)}(X^{(\alpha)})\boldsymbol{\eta}^{(\alpha)}\, dY \\
&= \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A_{ijkl}^{(\alpha)} \left(\mathcal{E}_{ij}^{(\alpha)}(X^{(\alpha)}) + \mathcal{E}_{Y,ij}^{(\alpha)}(\hat{\chi}(X^{(\alpha)},\cdot))\right)\left(\mathcal{E}_{kl}^{(\alpha)}(X^{(\alpha)}) + \mathcal{E}_{Y,kl}^{(\alpha)}(\hat{\chi}(X^{(\alpha)},\cdot))\right)\boldsymbol{\eta}^{(\alpha)}\, dY \\
&\quad - \sum_{\alpha=1}^{2} \int_{Cyl^{(\alpha)}} A_{ijkl}^{(\alpha)} \left(\mathcal{E}_{ij}^{(\alpha)}(X^{(\alpha)}) + \mathcal{E}_{Y,ij}^{(\alpha)}(\hat{\chi}(X^{(\alpha)},\cdot))\right) \mathcal{E}_{Y,kl}^{(\alpha)}(\hat{\chi}(X^{(\alpha)},\cdot))\boldsymbol{\eta}^{(\alpha)}\, dY \geq C_0|X|^2,
\end{aligned}$$

where the last passage follows from inequality (100) and the fact that the second integral is nonnegative by the definition of problem (98) with the choice of a zero test function. Hence, the coercivity of A_{hom} is proved.

□

We can, finally, write the macroscopic problem. Set

$$\mathcal{X}^H \doteq \left\{\mathbf{V} = \left(\mathbb{V}, \mathbb{V}^{(\mathbf{S})}\right) \in \mathcal{X}_M \times \mathcal{X}_S \;\middle|\; \frac{1}{2}\left|\mathbb{V}_\alpha^{(\mathbf{S})}(\cdot,1) - \mathbb{V}_\alpha^{(\mathbf{S})}(\cdot,0)\right| + \kappa|\partial_2\mathbb{V}_1 + \partial_1\mathbb{V}_2| \leq g_\alpha, \quad \text{a.e. in } \Omega,\ \alpha \in \{0,1\}\right\}.$$

For all $\mathbf{V}$ in $\mathcal{X}^H$, we denote

$$\partial\mathbf{V} \doteq \left(\partial_1\mathbb{V}_1^{(\mathbf{S})}(\cdot,0), \partial_1\mathbb{V}_1^{(\mathbf{S})}(\cdot,1), \partial_2\mathbb{V}_2^{(\mathbf{S})}(\cdot,0), \partial_2\mathbb{V}_2^{(\mathbf{S})}(\cdot,1), \partial_{12}\mathbb{V}_3, \partial_{11}\mathbb{V}_3, \partial_{11}\mathbb{V}_2, \partial_{22}\mathbb{V}_3, \partial_{22}\mathbb{V}_1\right).$$

Theorem 4. *The macroscopic homogenized problem has the following formulation:*

$$\begin{cases}
\text{Find } \mathbf{U} \in \mathcal{X}^H \text{ such that for every } \mathbf{V} \in \mathcal{X}^H: \\
\displaystyle\int_\Omega A_{hom}(\partial\mathbf{U})\cdot(\partial\mathbf{U} - \partial\mathbf{V})\, dz' \leq \boldsymbol{C}_0(\kappa)\sum_{\beta=1}^{2}\left(\sum_{\alpha=1}^{2}\int_\Omega f_\alpha^{(\beta)}(\mathbb{U}_\alpha - \mathbb{V}_\alpha)dz' + \int_\Omega f_3^{(\beta)}(\mathbb{U}_3 - \mathbb{V}_3)dz'\right) \\
\displaystyle + \frac{\boldsymbol{C}_0(\kappa)}{2}\left(\int_\Omega \sum_{c=0}^{1} \tilde{f}_\alpha^{(\alpha)}\left(\mathbb{U}_\alpha^{(\boldsymbol{S})} - \mathbb{V}_\alpha^{(\boldsymbol{S})}\right)(\cdot,c)dz' + \int_\Omega \tilde{f}_\alpha^{(3-\alpha)}\left(\left(\mathbb{U}_\alpha^{(\boldsymbol{S})}(\cdot,1) + \mathbb{U}_\alpha^{(\boldsymbol{S})}(\cdot,0)\right) - \left(\mathbb{V}_\alpha^{(\boldsymbol{S})}(\cdot,1) + \mathbb{V}_\alpha^{(\boldsymbol{S})}(\cdot,0)\right)\right)dz'\right) \\
\displaystyle - \boldsymbol{C}_1(\kappa)\sum_{\alpha=1,\beta=1}^{2}\int_\Omega \tilde{f}_\alpha^{(\beta)}(\partial_\alpha\mathbb{U}_3 - \partial_\alpha\mathbb{V}_3)dz'.
\end{cases} \tag{101}$$

It admits solutions, but we do not have uniqueness.

Proof. First, from (93), we obtain that $\left(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}, \mathbb{U}^{(\mathbf{B})}\right) \in \mathcal{X}_M \times \mathcal{X}_S \times \mathcal{X}_B$ satisfies

$$\begin{cases} \displaystyle\int_\Omega A_{hom}(\partial \mathbf{U}) \cdot (\partial \mathbf{U} - \partial \mathbf{V})\, dz' \le \mathbf{C}_0(\kappa) \sum_{\beta=1}^{2} \left(\sum_{\alpha=1}^{2} \int_\Omega f_\alpha^{(\beta)} (\mathbb{U}_\alpha - \mathbb{V}_\alpha) dz' + \int_\Omega f_3^{(\beta)} (\mathbb{U}_3 - \mathbb{V}_3) dz' \right) \\ \displaystyle + \frac{\mathbf{C}_0(\kappa)}{2} \int_\Omega \sum_{c=0}^{1} \left(\widetilde{f}_\alpha^{(\alpha)} \left(\mathbb{U}_\alpha^{(\mathbf{S})} - \mathbb{V}_\alpha^{(\mathbf{S})} \right)(\cdot, c) + \widetilde{f}_\alpha^{(3-\alpha)} \left(\mathbb{U}_\alpha^{(\mathbf{B})} - \mathbb{V}_\alpha^{(\mathbf{B})} \right)(\cdot, c) \right) dz' \\ \displaystyle - \mathbf{C}_1(\kappa) \sum_{\alpha=1,\beta=1}^{2} \int_\Omega \widetilde{f}_\alpha^{(\beta)} (\partial_\alpha \mathbb{U}_3 - \partial_\alpha \mathbb{V}_3) dz', \ \forall \left(\mathbb{V}, \mathbb{V}^{(\mathbf{S})}, \mathbb{V}^{(\mathbf{B})}\right) \in \mathcal{X}_M \times \mathcal{X}_S \times \mathcal{X}_B. \end{cases} \tag{102}$$

The existence of solutions to problem (102) is a direct consequence of the properties of the homogenizing operator A_{hom}, given in Proposition 4, together with the Stampacchia lemma.

Now, in the definitions of $\mathcal{X}$, the conditions, coming from (81), imply

$$\forall (a, b) \in \{0, 1\}^2, \qquad \begin{cases} |\mathbb{V}_1^{(\mathbf{S})}(\cdot, b) - \mathbb{V}_1^{(\mathbf{B})}(\cdot, a)| \ge \dfrac{1}{2} |\mathbb{V}_1^{(\mathbf{S})}(\cdot, 1) - \mathbb{V}_1^{(\mathbf{S})}(\cdot, 0)|, \\ |\mathbb{V}_2^{(\mathbf{S})}(\cdot, b) - \mathbb{V}_2^{(\mathbf{B})}(\cdot, a)| \ge \dfrac{1}{2} |\mathbb{V}_2^{(\mathbf{S})}(\cdot, 1) - \mathbb{V}_2^{(\mathbf{S})}(\cdot, 0)|, \end{cases} \qquad \text{a.e. in } \Omega.$$

So we have

$$\frac{1}{2} |\mathbb{V}_\alpha^{(\mathbf{S})}(\cdot, 1) - \mathbb{V}_\alpha^{(\mathbf{S})}(\cdot, 0)| + \kappa |\partial_2 \mathbb{V}_1 + \partial_1 \mathbb{V}_2| \le g_\alpha, \quad \text{a.e. in } \Omega, \ \alpha \in \{0, 1\}.$$

Hence, in problem (102), we have $\left(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}\right)$ and $\left(\mathbb{V}, \mathbb{V}^{(\mathbf{S})}\right)$ in the convex set $\mathcal{X}^H$. We set

$$\begin{cases} \mathbb{V}_\alpha^{(\mathbf{B})}(\cdot, 0) = \dfrac{1}{2} \left(\mathbb{V}_\alpha^{(\mathbf{S})}(\cdot, 1) + \mathbb{V}_\alpha^{(\mathbf{S})}(\cdot, 0) \right) + \widetilde{\mathbb{V}}_\alpha^{(\mathbf{B})}(\cdot, 0), \\ \mathbb{V}_\alpha^{(\mathbf{B})}(\cdot, 1) = \dfrac{1}{2} \left(\mathbb{V}_\alpha^{(\mathbf{S})}(\cdot, 1) + \mathbb{V}_\alpha^{(\mathbf{S})}(\cdot, 0) \right) - \widetilde{\mathbb{V}}_\alpha^{(\mathbf{B})}(\cdot, 1), \end{cases} \quad \text{for every } \ \mathbb{V}^{(\mathbf{B})} \in \mathcal{X}_B.$$

Choosing $\widetilde{\mathbb{V}}_\alpha^{(\mathbf{B})} = \widetilde{\mathbb{U}}_\alpha^{(\mathbf{B})}$ in problem (102), leads to problem (101). So $\left(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}\right) \in \mathcal{X}^H$ is a solution to problem (101).

Now, suppose that $\left(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}\right)$ and $\left(\mathbb{U}', \mathbb{U}'^{(\mathbf{S})}\right)$, (resp. $\left(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}, \mathbb{U}^{(\mathbf{B})}\right)$ and $\left(\mathbb{U}', \mathbb{U}'^{(\mathbf{S})}, \mathbb{U}'^{(\mathbf{B})}\right)$) are both solutions of (101) (resp. 102). Then, by the two inequalities given by (101) (resp. 102) with $\left(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}\right)$ (resp. $\left(\mathbb{U}, \mathbb{U}^{(\mathbf{S})}, \mathbb{U}^{(\mathbf{B})}\right)$) as a solution and $\left(\mathbb{U}', \mathbb{U}'^{(\mathbf{S})}\right)$ (resp. $\left(\mathbb{U}', \mathbb{U}'^{(\mathbf{S})}, \mathbb{U}'^{(\mathbf{B})}\right)$) as a test function and vice versa, we get that

$$\int_\Omega \left(A_{hom}(\partial \mathbf{U}) - A_{hom}(\partial \mathbf{U}') \right) \cdot (\partial \mathbf{U} - \partial \mathbf{U}')\, dz' \le 0,$$

which together with the coercivity of the A_{hom} implies that $\partial \mathbf{U} = \partial \mathbf{U}'$. Then, we obtain $\mathbb{U} = \mathbb{U}'$ and $\partial_\alpha \mathbb{U}_\alpha^{(\mathbf{S})} = \partial_\alpha \mathbb{U}_\alpha^{(\mathbf{S})'}$. In $\Omega_1 \cup \Omega_2$ (resp. $\Omega_1 \cup \Omega_3$), we have $\mathbb{U}_1^{(\mathbf{S})} = \mathbb{U}_1'^{(\mathbf{S})}$ (resp. $\mathbb{U}_2^{(\mathbf{S})} = \mathbb{U}_2'^{(\mathbf{S})}$). In $\Omega_3 \cup \Omega_4$ (resp. $\Omega_2 \cup \Omega_4$) $\mathbb{U}_1^{(\mathbf{S})}$ (resp. $\mathbb{U}_2^{(\mathbf{S})}$) is determined up to a function of z_2 (resp z_1) belonging to $L^2(l, L)$. Hence, both problems (101) and (102) admit the same solutions. The homogenized problem is finally (101). □

The operator of the homogenized problem is a Leray–Lions operator.
Once $\mathbf{U}$ determined, we set

$$\mathcal{X}_{B,\mathbf{U}} \doteq \left\{ \widetilde{\mathbb{V}}^{(\mathbf{B})} \in \mathcal{X}_B \mid \left| \frac{1}{2}(\mathbb{U}_\alpha^{(\mathbf{S})}(\cdot,1) - \mathbb{U}_\alpha^{(\mathbf{S})}(\cdot,0)) - \widetilde{\mathbb{V}}_\alpha^{(\mathbf{B})}(\cdot,c) \right| \le g_\alpha - \kappa|\partial_2\mathbb{U}_1 + \partial_1\mathbb{U}_2| \text{ a.e. in } \Omega,\ (\alpha,c) \in \{1,2\}\times\{0,1\} \right\}.$$

So $\widetilde{\mathbb{U}}^{(\mathbf{B})}$ belongs to $\mathcal{X}_{B,\mathbf{U}}$ and satisfies ($(a,b) \in \{0,1\}^2$)

$$\begin{aligned} &\int_\Omega (-1)^a \widetilde{f}_2^{(1)} \left(\widetilde{\mathbb{U}}_2^{(\mathbf{B})} - \widetilde{\mathbb{V}}_2^{(\mathbf{B})} \right)(\cdot,a)dz' \ge 0, \\ &\int_\Omega (-1)^b \widetilde{f}_1^{(2)} \left(\widetilde{\mathbb{U}}_1^{(\mathbf{B})} - \widetilde{\mathbb{V}}_1^{(\mathbf{B})} \right)(\cdot,b)dz' \ge 0, \end{aligned} \qquad \forall \widetilde{\mathbb{V}}^{(\mathbf{B})} \in \mathcal{X}_{B,\mathbf{U}}.$$

$\widetilde{\mathbb{U}}^{(\mathbf{B})}$ is also a field which minimize the functional

$$\int_\Omega \left(\widetilde{f}_2^{(1)} |\widetilde{\mathbb{V}}_2^{(\mathbf{B})}(z',0)|^2 - \widetilde{f}_2^{(1)} |\widetilde{\mathbb{V}}_2^{(\mathbf{B})}(z',1))|^2 + \widetilde{f}_1^{(2)} |\widetilde{\mathbb{V}}_1^{(\mathbf{B})}(z',0)|^2 - \widetilde{f}_1^{(2)} |\widetilde{\mathbb{V}}_1^{(\mathbf{B})}(z',1)|^2 \right) dz'$$

on the convex set $\mathcal{X}_{B,\mathbf{U}}$.

10 | CONCLUSIONS

In this conclusive section, we summarize the most relevant results given by the limit problem's investigation and comment on them to have a clearer understanding and visualization.

- Starting from the form of the final decomposition of the displacement (50) and going to the limit, the cell problem (98) and the macroscopic problem (102) give the approximation of the limit displacements in the direction of beams $\mathbf{e}_1$ and $\mathbf{e}_2$ that are a.e. $z' \in \Omega$:

$$\begin{aligned} u^{(1)}(z_1, q\varepsilon, y_2, y_3) \approx\ & \underbrace{\begin{pmatrix} \varepsilon\mathbb{U}_1(q\varepsilon) + \varepsilon^2\mathbb{U}_1^{(\mathbf{S})}(z_1,q\varepsilon,b) \\ \varepsilon\mathbb{U}_2(z_1) + \varepsilon^2\mathbb{U}_2^{(\mathbf{B})}(z_1,q\varepsilon,b) \\ \varepsilon\mathbb{U}_3(z_1,q\varepsilon) \end{pmatrix} + \begin{pmatrix} \varepsilon\partial_2\mathbb{U}_3(z_1,q\varepsilon) \\ -\varepsilon\partial_1\mathbb{U}_3(z_1,q\varepsilon) \\ \varepsilon\partial_1\mathbb{U}_2(z_1) \end{pmatrix} \wedge \Phi_\varepsilon^{(1)}(z_1)\mathbf{e}_3}_{\text{middle line displacement}} \\ & + \begin{pmatrix} \varepsilon\partial_2\mathbb{U}_3(z_1,q\varepsilon) \\ -\varepsilon\partial_1\mathbb{U}_3(z_1,q\varepsilon) \\ \varepsilon\partial_1\mathbb{U}_2(z_1) \end{pmatrix} \wedge \left(y_2\mathbf{e}_2 + y_3\mathbf{n}_\varepsilon^{(1)}(z_1) \right), \\ u^{(2)}(p\varepsilon, z_2, y_1, y_3) \approx\ & \underbrace{\begin{pmatrix} \varepsilon\mathbb{U}_1(z_2) + \varepsilon^2\mathbb{U}_2^{(\mathbf{B})}(p\varepsilon,z_2,a) \\ \varepsilon\mathbb{U}_2(p\varepsilon) + \varepsilon^2\mathbb{U}_2^{(\mathbf{S})}(p\varepsilon,z_2) \\ \varepsilon\mathbb{U}_3(p\varepsilon,z_2) \end{pmatrix} + \begin{pmatrix} \varepsilon\partial_2\mathbb{U}_3(p\varepsilon,z_2) \\ -\varepsilon\partial_1\mathbb{U}_3(p\varepsilon,z_2) \\ \varepsilon\partial_2\mathbb{U}_1(z_2) \end{pmatrix} \wedge \Phi_\varepsilon^{(2)}(z_2)\mathbf{e}_3}_{\text{middle line displacement}} \\ & + \begin{pmatrix} \varepsilon\partial_2\mathbb{U}_3(p\varepsilon,z_2) \\ -\varepsilon\partial_1\mathbb{U}_3(p\varepsilon,z_2) \\ \varepsilon\partial_2\mathbb{U}_1(z_2) \end{pmatrix} \wedge \left(y_1\mathbf{e}_1 + y_3\mathbf{n}_\varepsilon^{(2)}(z_2) \right). \end{aligned}$$

- As we can see in the expressions of the limit displacements, given in A, the yarns' cross-sections in direction $\mathbf{e}_1$ rotate around axes of directions $\mathbf{e}_1$ and $\mathbf{e}_2$, due to the global bending $\mathbb{U}_3$ of the structure. Worthy to note that these yarns also rotate around the axes of direction $\mathbf{e}_3$, and the rotation angle is $\varepsilon\partial_1\mathbb{U}_2$. The cross-sections of the yarns of direction $\mathbf{e}_2$ also rotate around axes of directions $\mathbf{e}_3$, but with an angle of $\varepsilon\partial_2\mathbb{U}_1$. This behavior is represented in Figure 2.

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

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APPENDIX A

We gather here the heaviest computations presented in the paper.

A.1 | Proof of the second estimate in Lemma 3

The proof idea is the following: Due to the alternate sign of the difference between the displacements in the outer-plane direction, we can cleverly pair the terms and get a remainder with a good estimate. Then, we iterate the procedure until we note that the remainders have, in fact, a sufficiently good estimate.

Proof. First, to shorten the notation, a.e. (t_1, t_2) in ω_r, we set

$$\mathfrak{u}^{(1)}_{pq}(t_1,t_2) = u^{(1)}_3(t_1 + p\varepsilon, q\varepsilon, t_2, (-1)^{p+q+1}r), \qquad \mathfrak{u}^{(2)}_{pq}(t_1,t_2) = u^{(2)}_3(p\varepsilon, t_2 + q\varepsilon, t_1, (-1)^{p+q}r),$$
$$\overline{\mathfrak{u}}^{(1)}_{pq}(t_1,t_2) = \overline{u}^{(1)}_3(t_1 + p\varepsilon, q\varepsilon, t_2, (-1)^{p+q+1}r), \qquad \overline{\mathfrak{u}}^{(2)}_{pq}(t_1,t_2) = \overline{u}^{(2)}_3(p\varepsilon, t_2 + q\varepsilon, t_1, (-1)^{p+q}r).$$

From (29), the displacements become

$$\begin{aligned}\mathfrak{u}^{(1)}_{pq}(t_1,t_2) &= \mathbb{U}'^{(1)}_3(p\varepsilon + t_1, q\varepsilon) + \mathcal{R}'^{(1)}_1(p\varepsilon + t_1, q\varepsilon)t_2 + \overline{\mathfrak{u}}^{(1)}_{pq}(t_1,t_2),\\ \mathfrak{u}^{(2)}_{pq}(t_1,t_2) &= \mathbb{U}'^{(2)}_3(p\varepsilon, q\varepsilon + t_2) - \mathcal{R}'^{(2)}_2(p\varepsilon, q\varepsilon + t_2)t_1 + \overline{\mathfrak{u}}^{(2)}_{pq}(t_1,t_2).\end{aligned} \tag{A1}$$

We then organize the proof in steps.

Step 1. In this step, we rewrite the displacements in the contact areas as (for a.e. $z' = (t_1, t_2)$ in ω_r)

$$\begin{aligned}\mathfrak{u}^{(1)}_{pq}(t_1,t_2) &= \mathbb{U}'^{(1)}(p\varepsilon, q\varepsilon) + \mathcal{R}'^{(1)}_1(p\varepsilon, q\varepsilon)t_2 - \mathcal{R}'^{(1)}_2(p\varepsilon, q\varepsilon)t_1 + Q^{(1)}_{pq}(t_1,t_2),\\ \mathfrak{u}^{(2)}_{pq}(t_1,t_2) &= \mathbb{U}'^{(2)}(p\varepsilon, q\varepsilon) + \mathcal{R}'^{(2)}_1(p\varepsilon, q\varepsilon)t_2 - \mathcal{R}'^{(2)}_2(p\varepsilon, q\varepsilon)t_1 + Q^{(2)}_{pq}(t_1,t_2),\end{aligned} \tag{A2}$$

where the remainder terms $Q^{(\alpha)}_{p,q}$ are estimated by

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|Q^{(\alpha)}_{pq}\|^2_{L^2(\omega_r)} \le C\varepsilon \|u\|^2_{\mathcal{S}_\varepsilon}. \tag{A3}$$

From the form of the displacements in the contact areas (A1), for a.e. (t_1, t_2) in ω_r the remainder terms $Q^{(\alpha)}_{p,q}$ are defined by

$$Q^{(1)}_{pq}(t_1,t_2) \doteq \Big(\mathbb{U}'^{(1)}_3(p\varepsilon + t_1, q\varepsilon) - \mathbb{U}'^{(1)}_3(p\varepsilon, q\varepsilon) + \mathcal{R}'^{(1)}_2(p\varepsilon, q\varepsilon)t_1\Big) + \Big(\mathcal{R}'^{(1)}_1(p\varepsilon + t_1, q\varepsilon) - \mathcal{R}'^{(1)}_1(p\varepsilon, q\varepsilon)\Big)\,t_2 + \overline{\mathfrak{u}}^{(1)}_{pq}(t_1,t_2),$$
$$Q^{(2)}_{pq}(t_1,t_2) \doteq \Big(\mathbb{U}'^{(2)}_3(p\varepsilon, q\varepsilon + t_2) - \mathbb{U}'^{(2)}_3(p\varepsilon, q\varepsilon) - \mathcal{R}'^{(2)}_1(p\varepsilon, q\varepsilon)t_2\Big) - \Big(\mathcal{R}'^{(2)}_2(p\varepsilon, q\varepsilon + t_2) - \mathcal{R}'^{(2)}_2(p\varepsilon, q\varepsilon)\Big)\,t_1 + \overline{\mathfrak{u}}^{(2)}_{pq}(t_1,t_2).$$

We want now to prove (A3), and due to the symmetrical behavior, we will estimate $Q^{(1)}_{pq}$.
Remind that $\partial_1 \mathbb{U}'^{(1)}_3 = -\mathcal{R}'^{(1)}_2$. Then, the Poincaré inequality and the estimates (28)–(30) give

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|Q^{(1)}_{pq}\|^2_{L^2(\omega_r)} \le C\varepsilon^5 \sum_{q=0}^{2N_\varepsilon - 1} \|\partial_1 \mathcal{R}'^{(1)}(\cdot, q\varepsilon)\|^2_{L^2(0,L)^3} + C\varepsilon\|u\|^2_{\mathcal{S}_\varepsilon} \le C\varepsilon\|u\|^2_{\mathcal{S}_\varepsilon}.$$

So we get (A3).

Step 2. By the non-penetration condition (24) in the contact parts of the corners of the cell $(p\varepsilon, q\varepsilon) + \varepsilon Y$, we have

$$\begin{aligned} 0 \le (-1)^{p+q} &\left(\left(\mathbf{u}^{(1)}_{pq} - \mathbf{u}^{(2)}_{pq} \right) + \left(\mathbf{u}^{(1)}_{(p+1)(q+1)} - \mathbf{u}^{(2)}_{(p+1)(q+1)} \right) + \left(\mathbf{u}^{(2)}_{(p+1)q} - \mathbf{u}^{(1)}_{(p+1)q} \right) + \left(\mathbf{u}^{(2)}_{p(q+1)} - \mathbf{u}^{(1)}_{p(q+1)} \right) \right) \\ = (-1)^{p+q} &\left[\varepsilon \left(\mathcal{R}'^{(1)}_2(p\varepsilon, q\varepsilon) - \mathcal{R}'^{(2)}_1(p\varepsilon+\varepsilon, q\varepsilon) - \mathcal{R}'^{(1)}_2(p\varepsilon, q\varepsilon+\varepsilon) + \mathcal{R}'^{(2)}_1(p\varepsilon, q\varepsilon) \right) + \left(R^{(1)}_{pq} + R^{(2)}_{(p+1)q} + R^{(1)}_{p(q+1)} + R^{(2)}_{pq} \right) \right], \end{aligned} \tag{A4}$$

where the four remainder terms $R^{(\alpha)}_{pq}$, $R^{(1)}_{p(q+1)}$, and $R^{(2)}_{(p+1)q}$ are estimated by

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|R^{(\alpha)}_{pq}\|^2_{L^2(\omega_r)} + \|R^{(1)}_{p(q+1)}\|^2_{L^2(\omega_r)} + \|R^{(2)}_{(p+1)q}\|^2_{L^2(\omega_r)} \le C\varepsilon \|u\|^2_{S_\varepsilon}. \tag{A5}$$

Indeed, by the non-penetration condition (24) on the vertexes of the cell $(p\varepsilon, q\varepsilon) + \varepsilon Y$ and pairing the involved terms differently, we get a.e. in ω_r

$$\begin{aligned} 0 \le (-1)^{p+q} &\left(\left(\mathbf{u}^{(1)}_{pq} - \mathbf{u}^{(2)}_{pq} \right) + \left(\mathbf{u}^{(1)}_{(p+1)(q+1)} - \mathbf{u}^{(2)}_{(p+1)(q+1)} \right) + \left(\mathbf{u}^{(2)}_{(p+1)q} - \mathbf{u}^{(1)}_{(p+1)q} \right) + \left(\mathbf{u}^{(2)}_{p(q+1)} - \mathbf{u}^{(1)}_{p(q+1)} \right) \right) \\ = (-1)^{p+q} &\left(\left(\mathbf{u}^{(1)}_{pq} - \mathbf{u}^{(1)}_{(p+1)q} \right) + \left(\mathbf{u}^{(2)}_{(p+1)q} - \mathbf{u}^{(2)}_{(p+1)(q+1)} \right) + \left(\mathbf{u}^{(1)}_{(p+1)(q+1)} - \mathbf{u}^{(1)}_{p(q+1)} \right) + \left(\mathbf{u}^{(2)}_{p(q+1)} - \mathbf{u}^{(2)}_{pq} \right) \right). \end{aligned}$$

Then, the right-hand side of the above equation can be rewritten in the following way:

$$\begin{aligned} &\mathbf{u}^{(1)}_{pq} - \mathbf{u}^{(1)}_{(p+1)q} = \varepsilon \mathcal{R}'^{(1)}_2(p\varepsilon, q\varepsilon) + R^{(1)}_{pq}, && \mathbf{u}^{(2)}_{(p+1)q} - \mathbf{u}^{(2)}_{(p+1)(q+1)} = -\varepsilon \mathcal{R}'^{(2)}_1(p\varepsilon+\varepsilon, q\varepsilon) + R^{(2)}_{p+1,q}, \\ &\mathbf{u}^{(1)}_{(p+1)(q+1)} - \mathbf{u}^{(1)}_{p(q+1)} = -\varepsilon \mathcal{R}'^{(1)}_2(p\varepsilon, q\varepsilon+\varepsilon) + R^{(1)}_{p,q+1}, && \mathbf{u}^{(2)}_{p(q+1)} - \mathbf{u}^{(2)}_{pq} = \varepsilon \mathcal{R}'^{(2)}_1(p\varepsilon, q\varepsilon) + R^{(2)}_{p,q}, \end{aligned}$$

where $R^{(1)}_{pq}$ and $R^{(2)}_{pq}$ are equal to

$$\begin{aligned} R^{(1)}_{pq}(t_1,t_2) \doteq &\left(\mathbb{U}'^{(1)}_3(p\varepsilon, q\varepsilon) - \mathbb{U}'^{(1)}_3(p\varepsilon+\varepsilon, q\varepsilon) - \varepsilon \mathcal{R}'^{(1)}_2(p\varepsilon, q\varepsilon) \right) + \left(\mathcal{R}'^{(1)}_1(p\varepsilon, q\varepsilon) - \mathcal{R}'^{(1)}_1(p\varepsilon+\varepsilon, q\varepsilon) \right) t_2 \\ &- \left(\mathcal{R}'^{(1)}_2(p\varepsilon, q\varepsilon) - \mathcal{R}'^{(1)}_2(p\varepsilon+\varepsilon, q\varepsilon) \right) t_1 + (Q^{(1)}_{pq} - Q^{(1)}_{(p+1)q})(t_1,t_2), \\ R^{(2)}_{pq}(t_1,t_2) \doteq &\left(\mathbb{U}'^{(2)}_3(p\varepsilon, q\varepsilon) - \mathbb{U}'^{(2)}_3(p\varepsilon, q\varepsilon+\varepsilon) + \varepsilon \mathcal{R}'^{(2)}_1(p\varepsilon, q\varepsilon) \right) + \left(\mathcal{R}'^{(2)}_1(p\varepsilon, q\varepsilon) - \mathcal{R}'^{(2)}_1(p\varepsilon, q\varepsilon+\varepsilon) \right) t_2 \\ &- \left(\mathcal{R}'^{(2)}_2(p\varepsilon, q\varepsilon) - \mathcal{R}'^{(2)}_2(p\varepsilon, q\varepsilon+\varepsilon) \right) t_1 + (Q^{(2)}_{pq} - Q^{(2)}_{p(q+1)})(t_1,t_2), \end{aligned}$$

and $R^{(2)}_{(p+1)q}$, $R^{(1)}_{p(q+1)}$ are referred from the above defined. It is now left to prove estimate (A5), and due to the symmetrical behavior, we will estimate $R^{(1)}_{pq}$. Again Poincaré inequality leads to

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|R^{(1)}_{pq}\|^2_{L^2(\omega_r)} \le C \left(\varepsilon^5 \sum_{q=0}^{2N_\varepsilon - 1} \|\partial_1 \mathcal{R}'^{(1)}(\cdot, q\varepsilon)\|^2_{L^2(0,L)} + \sum_{(p,q)\in\mathcal{K}_\varepsilon} \|Q^{(1)}_{pq}\|^2_{L^2(\omega_r)} + \sum_{(p,q)\in\mathcal{K}_\varepsilon} \|Q^{(1)}_{(p+1)q}\|^2_{L^2(\omega_r)} \right).$$

Hence, by the first estimate in (28) and (A3), we get (A5) for $R^{(1)}_{pq}$.

Step 3. In this step, we prove that for a.e. $(t_1, t_2) \in \omega_r$

$$\begin{aligned} &\sum_{k=p}^{p+1} \sum_{\ell=q}^{q+1} \left| \left(\mathbb{U}'^{(1)}_3 - \mathbb{U}'^{(2)}_3 \right)(k\varepsilon, \ell\varepsilon) - t_1 \left(\mathcal{R}'^{(1)}_2 - \mathcal{R}'^{(2)}_2 \right)(k\varepsilon, \ell\varepsilon) + t_2 \left(\mathcal{R}'^{(1)}_1 - \mathcal{R}'^{(1)}_1 \right)(k\varepsilon, \ell\varepsilon) \right| \\ &\le (-1)^{p+q} \varepsilon \left(\mathcal{R}'^{(1)}_2(p\varepsilon, q\varepsilon) - \mathcal{R}'^{(2)}_1(p\varepsilon+\varepsilon, q\varepsilon) - \mathcal{R}'^{(1)}_2(p\varepsilon, q\varepsilon+\varepsilon) + \mathcal{R}'^{(2)}_1(p\varepsilon, q\varepsilon) \right) + S_{pq}(t_1,t_2), \end{aligned} \tag{A6}$$

where the remainder term S_{pq} is estimated by

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon} \|S_{p,q}\|^2_{L^2(\omega_r)} \le C\varepsilon \|u\|^2_{S_\varepsilon}. \tag{A7}$$

Note first that in the equality (A4), the left-hand side is positive. Hence, we replace the left-hand side with (A2) and take the modulus. Applying Step 1 on the left-hand side and Step 2 on the right-hand side, we get a.e $(t_1, t_2) \in \omega_r$ that

$$\sum_{k=p}^{p+1}\sum_{\ell=q}^{q+1}\left|\left(\mathbb{U}_3'^{(1)} - \mathbb{U}_3'^{(2)}\right)(k\varepsilon, \ell\varepsilon) - t_1\left(\mathcal{R}_2'^{(1)} - \mathcal{R}_2'^{(2)}\right)(k\varepsilon, \ell\varepsilon) + t_2\left(\mathcal{R}_1'^{(1)} - \mathcal{R}_1'^{(1)}\right)(k\varepsilon, \ell\varepsilon) + (Q^{(1)}_{k\ell,3} - Q^{(2)}_{k\ell,3})\right|$$
$$= (-1)^{p+q}\left[\varepsilon\left(\mathcal{R}_2'^{(1)}(p\varepsilon, q\varepsilon) - \mathcal{R}_1'^{(2)}(p\varepsilon+\varepsilon, q\varepsilon) - \mathcal{R}_2'^{(1)}(p\varepsilon, q\varepsilon+\varepsilon) + \mathcal{R}_1'^{(2)}(p\varepsilon, q\varepsilon)\right) + \left(R^{(1)}_{pq} + R^{(2)}_{(p+1)q} + R^{(1)}_{p(q+1)} + R^{(2)}_{pq}\right)\right].$$

Then, the above equation can be rewritten in the form (A6) with S_{pq}, defined by

$$S_{p,q} \doteq (-1)^{p+q}\left(R^{(1)}_{p,q} + R^{(2)}_{p+1,q} - R^{(1)}_{p,q+1} - R^{(2)}_{p,q}\right) + \sum_{k=p}^{p+1}\sum_{\ell=q}^{q+1}\left|(Q^{(1)}_{k\ell,3} - Q^{(2)}_{k\ell,3})\right|$$

Hence, estimate (A7) directly follows from the estimates (A3) and (A5).

Step 4. In this step, we prove the statement, that is, estimate (32).
Starting from inequality (A6), we replace (p, q) by $(2p, 2q)$, $(2p+1, 2q)$, $(2p, 2q+1)$, and $(2p+1, 2q+1)$. For a.e $(t_1, t_2) \in \omega_r$, we obtain

$$\begin{aligned}&\sum_{k=2p}^{2p+2}\sum_{\ell=2q}^{2q+2}\left|\left(\mathbb{U}_3'^{(1)} - \mathbb{U}_3'^{(2)}\right)(k\varepsilon, \ell\varepsilon) - t_1\left(\mathcal{R}_2'^{(1)} - \mathcal{R}_2'^{(2)}\right)(k\varepsilon, \ell\varepsilon) + t_2\left(\mathcal{R}_1'^{(1)} - \mathcal{R}_1'^{(2)}\right)(k\varepsilon, \ell\varepsilon)\right| \\ &\le \varepsilon\sum_{k=2p}^{2p+1}\sum_{\ell=2q}^{2q+1}(-1)^{k+\ell}\left(\mathcal{R}_2'^{(1)}(k\varepsilon, \ell\varepsilon) - \mathcal{R}_1'^{(2)}(k\varepsilon+\varepsilon, \ell\varepsilon) - \mathcal{R}_2'^{(1)}(k\varepsilon, \ell\varepsilon+\varepsilon) + \mathcal{R}_1'^{(2)}(k\varepsilon, \ell\varepsilon)\right) \\ &\quad + \left(S_{(2p)(2q)} + S_{(2p+1)(2q)} + S_{(2p)(2q+1)} + S_{(2p+1)(2q+1)}\right)(t_1, t_2).\end{aligned} \tag{A8}$$

Set

$$\begin{aligned}T_{pq} &\doteq \varepsilon\sum_{k=2p}^{2p+1}\sum_{\ell=2q}^{2q+1}(-1)^{k+\ell}\left(\mathcal{R}_2'^{(1)}(k\varepsilon, \ell\varepsilon) - \mathcal{R}_1'^{(2)}(k\varepsilon+\varepsilon, \ell\varepsilon) - \mathcal{R}_2'^{(1)}(k\varepsilon, \ell\varepsilon+\varepsilon) + \mathcal{R}_1'^{(2)}(k\varepsilon, \ell\varepsilon)\right) \\ &= \varepsilon\sum_{k=2p}^{2p+1}\sum_{\ell=2q}^{2q+1}(-1)^{k+\ell}\left(\left(\mathcal{R}_2'^{(1)}(k\varepsilon, \ell\varepsilon) - \mathcal{R}_2'^{(1)}(k\varepsilon, \ell\varepsilon+\varepsilon)\right) + \left(\mathcal{R}_1'^{(2)}(k\varepsilon, \ell\varepsilon) - \mathcal{R}_1'^{(2)}(k\varepsilon+\varepsilon, \ell\varepsilon)\right)\right).\end{aligned}$$

The Poincaré inequality and the first estimate in (28) yield

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\|T_{pq}\|^2_{L^2(\omega_r)} \le C\varepsilon^5\left(\sum_{q=0}^{2N_\varepsilon-1}\|\partial_1\mathcal{R}_2'^{(1)}(\cdot, q\varepsilon)\|^2_{L^2(0,L)} + \sum_{p=0}^{2N_\varepsilon-1}\|\partial_2\mathcal{R}_1'^{(2)}(p\varepsilon, \cdot)\|^2_{L^2(0,L)}\right) \le C\varepsilon\|u\|^2_{S_\varepsilon}. \tag{A9}$$

Taking the L^2-norm of the left-hand side of (A8) and using (A7)–(A9) in the right-hand side, we finally obtain

$$\sum_{(p,q)\in\mathcal{K}_\varepsilon}\left(\varepsilon^2\left|\left(\mathbb{U}_3^{(1)} - \mathbb{U}_3^{(2)}\right)(p\varepsilon, q\varepsilon)\right|^2 + \varepsilon^4\left|\left(\mathcal{R}_\alpha^{(1)} - \mathcal{R}_\alpha^{(2)}\right)(p\varepsilon, q\varepsilon)\right|^2\right) \le C\varepsilon\|u\|^2_{S_\varepsilon},$$

which divided by ε^2 gives estimate (32).

□

A.2 | Unfolded limit of the frame

This subsection is dedicated to the unfolding of the mobile reference frame, defined in Section 4.1, via the global unfolding operator and showing its limit form. Due to symmetry reasons, we will consider direction $\mathbf{e}_1$.

The unfolding of the oscillating function $\Phi_\varepsilon^{(1)}$ is $\frac{1}{\varepsilon}\Pi^{(1)}\left(\Phi_\varepsilon^{(1)}\right) = \Phi^{(1)}$ a.e. in $\Omega \times Cyl^{(1)}$, where Φ is given in (2). As a direct consequence, straightforward calculations show that

$$\Pi_\varepsilon^{(1)}(\boldsymbol{\gamma}_\varepsilon) \doteq \boldsymbol{\gamma} = \sqrt{1 + (\partial_1 \Phi_\varepsilon^{(1)})^2}, \quad \Pi_\varepsilon^{(1)}(\mathbf{t}_\varepsilon^{(1)}) = \mathbf{t}^{(1)} = \frac{1}{\boldsymbol{\gamma}}\left(\mathbf{e}_1 + \partial_{Y_1}\Phi^{(1)}\mathbf{e}_3\right), \quad \Pi_\varepsilon^{(1)}(\mathbf{n}_\varepsilon^{(1)}) = \mathbf{n}^{(1)} = \frac{1}{\boldsymbol{\gamma}}\left(-\partial_{Y_1}\Phi^{(1)}\mathbf{e}_1 + \mathbf{e}_3\right),$$

$$\varepsilon \Pi_\varepsilon^{(1)}(c_\varepsilon^{(1)}) = \boldsymbol{c}^{(1)} = \frac{\partial_{Y_1}^2 \Phi^{(1)}}{\boldsymbol{\gamma}^3}, \qquad \Pi_\varepsilon^{(1)}(\eta_\varepsilon^{(1)}) = \boldsymbol{\eta}^{(1)} = \boldsymbol{\gamma}\left(1 - Y_3 \boldsymbol{c}^{(1)}\right), \qquad \Pi_\varepsilon^{(1)}(\nabla \psi_\varepsilon^{(1)}) = \left(\boldsymbol{\eta}^{(1)}\mathbf{t}^{(1)} \ \mathbf{e}_2 \ \mathbf{n}^{(1)}\right).$$

Concerning the integration of the limit mobile frame on the reference cell $Cyl^{(1)}$, which appears in the proof of Theorem 3.

Lemma 14. *One has the following values for the integrals:*

$$\int_{Cyl^{(1)}} \boldsymbol{\eta}^{(1)} dX = 4\kappa^2 \int_0^2 \boldsymbol{\gamma}\, dY_1, \qquad \int_{Cyl^{(1)}} \left(\Phi^{(1)}\boldsymbol{e}_3 + Y_2 \boldsymbol{e}_2 + Y_3 \boldsymbol{n}^{(1)}\right)\boldsymbol{\eta}^{(1)} dX = 4\kappa^2 \left(\int_0^2 \boldsymbol{\gamma}\, \Phi\, dY_1\right)\boldsymbol{e}_3. \tag{A10}$$

Proof. From the definition of $\boldsymbol{\eta}^{(1)}$ and the symmetries of the cross-section with respect to the lines $Y_2 = 0$ and $Y_3 = 0$, the first equality in (A10) holds.

Concerning the second equality, from the symmetries of the cross-section with respect to the lines $Y_2 = 0$ and $Y_3 = 0$, we first get that

$$\int_{Cyl^{(1)}} \left(\Phi^{(1)}\mathbf{e}_3 + Y_2 \mathbf{e}_2 + Y_3 \mathbf{n}^{(1)}\right)\boldsymbol{\eta}^{(1)} dX = 4\kappa^2 \left(\int_0^2 \Phi^{(1)}\boldsymbol{\gamma}\, dY_1\right)\mathbf{e}_3 + \frac{4\kappa^4}{3}\left(\int_0^2 \partial_{Y_1}\Phi^{(1)}\boldsymbol{c}^{(1)} dY_1\right)\mathbf{e}_1 - \frac{4\kappa^4}{3}\left(\int_0^2 \boldsymbol{c}^{(1)} dY_1\right)\mathbf{e}_3$$

and then note that the second and third integral vanish since Φ is 2-periodic and satisfies $\partial_{Y_1}\Phi(0) = \partial_{Y_1}\Phi(1) = \partial_{Y_1}\Phi(2) = 0$. □