New Insights for Fractal Zeta Functions Polyhedral Neighborhoods vs Tubular Neighborhoods

Claire David, Michel L Lapidus

Comptes Rendus
Mathématique
Draft

New Insights for Fractal Zeta Functions

Polyhedral Neighborhoods vs Tubular Neighborhoods

Claire David [a] and Michel L. Lapidus [b]

[a] Sorbonne Université – CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, 4, place Jussieu 75005, Paris, France

[b] University of California, Riverside, Department of Mathematics, Skye Hall, 900 University Ave., Riverside, CA 92521-0135, USA

E-mails: Claire.David@sorbonne-universite.fr (C. David), lapidus@math.ucr.edu (M. L. Lapidus)

Abstract.

In this note, which is an announcement of the long paper [8], we introduce new fractal zeta functions, associated with *polyhedral neighborhoods*, better suited to fractals when exact expressions for the volume of tubular neighborhoods cannot be computed. Accordingly, in the model – and very significant – case of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, we give exact expressions for the volume of *polyhedral neighborhoods* for the sequence of prefractal graphs which converge to $\Gamma_{\mathcal{W}}$ – the so-called *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs). Those IFDs are associated with a suitable (and geometrically meaningful) sequence of small parameters tending to zero, also known as *the cohomology infinitesimals*, due to their connections with fractal cohomology. We also introduce the associated *local* and *global polyhedral fractal zeta functions*. The local fractal zeta functions consist in the sequence of zeta functions associated with the sequence of polyhedral neighborhoods, and satisfy a recurrence relation, which enables us to prove that the poles of the limit fractal zeta function – the global zeta function, associated with the limit fractal object – are exactly the same as the Complex Dimensions of the Weierstrass function itself. This result makes the connection with fractal cohomology, where, for any nonnegative integer m, the m^{th} cohomology group is comprised of continuous functions which possess a generalized Taylor expansion, with fractional derivatives of orders the underlying – and actual – Complex Dimensions.

By using the aforementioned exact expressions of the polyhedral neighborhoods, we also revisit the computation of the box–counting (or Minkowski) dimension of the Weierstrass Curve, in a fully rigorous manner and therefore prove part of Mandelbrot's conjecture concerning the fractal dimension of the Weierstrass Curve.

2020 Mathematics Subject Classification. 00X99.

Funding. The research of M. L. L. was supported by the Burton Jones Endowed Chair in Pure Mathematics, as well as by grants from the U. S. National Science Foundation.

Electronic supplementary material. Supplementary material for this article is supplied as a separate archive available from the journal's website under article's URL or from the author.

This article is a draft (not yet accepted!)

Résumé.

Dans cette note, qui annonce l'article [8], nous introduisons de nouvelles fonctions zeta fractales, associées à des *voisinages polyhedraux*, qui sont les mieux indiqués lorsque l'on ne peut pas obtenir d'expressions exactes pour le volume des voisinages tubulaires des objets fractaux considérés, ce qui est le cas ici. Dans le cas modèle et très significatif de la courbe de Weierstrass $\Gamma_{\mathcal{W}}$, nous donnons les expressions exactes des volumes de la suite de graphes préfractaux qui converge vers $\Gamma_{\mathcal{W}}$. Ces graphes, appelés *tambours fractaux itérés*, sont associés à une suite de petits paramètres tendant vers 0 : les infinitésimaux cohomologiques. Nous introduisons ensuite les fonctions zêta fractales locales et globale correspondantes. Les fonctions zêta fractales locales sont les fonctions zêta associées à la suite de voisinages polyhédraux. Elles vérifient une relation de récurrence, ce qui nous permet de prouver que les pôles de la fonction zêta limite, c'est-à-dire la fonction zêta globale, associée à la courbe fractale elle-même, sont exactement les mêmes que les dimensions complexes de la fonction de Weierstrass. Ce résultat est un écho direct à la cohomologie fractale où, pour tout entier naturel m, le $m^{ième}$ groupe de cohomologie est composé de fonctions continues possédant un développement de Taylor gérénalisé, avec des dérivées fractionnaires dont les ordres sont les dimensions complexes.

En utilisant les expressions exactes des volumes des voisinages polyhedraux, nous révisitons ensuite de façon pleinement rigoureuse le calcul de la dimension de Minkowski de la courbe de Weierstrass. Nous donnons ainsi une démonstration complète d'une partie de la conjecture de Mandelbrot concernant les dimensions fractales de la courbe de Weierstrass.

1. Introduction

Up to now, tubular neighborhhoods were a compulsory and unavoidable step when it came to computing fractal zeta functions (see [19], [24], [25]). However, one cannot always obtain the exact expression of the associated volumes. For instance, this occurs when the fractals involved are defined by means of nonlinear and noncontractive iterated function systems (i.f.s.). A model – and very significant – object in this case is the (nowhere differentiable) Weierstrass Curve $\Gamma_{\mathcal{W}}$, where, in addition, nonlinearity makes the geometry extremely complicated.

A careful look at the sequence of prefractal, polygonal graph approximations which converge to $\Gamma_{\mathcal{W}}$ shows that not only polyhedral neighborhoods appear as perfectly suited to this fractal curve, but, even more important, we can obtain the exact expressions of the associated volumes. This comes from the fact that we know the coordinates of the vertex points which constitute the prefractal approximations. More precisely, those coordinates involve values of the Weierstrass function $\mathcal{W}$.

A very interesting – and rather *a priori* unexpected – feature then arises. Indeed, the Weierstrass function $\mathcal{W}$ itself possess a Complex Dimensions series expansion. Recall that the theory of Complex Dimensions, developed for many years now by Michel L. Lapidus and his collaborators in [14], [15], [22], [20], [21], [16], [19], [23], [24], [17], [11], [18], makes the connection between the geometry of an object and its differentiability properties, by means of *geometric (or fractal) zeta functions*, which stand for the trace of a differential operator at a complex order s. The poles of those fractal zeta functions are called *the Fractal Complex Dimensions*. The existence of *nonreal Complex Dimensions* is a characteristic of fractality and gives rise (via explicit formulas) to *the oscillations* that are intrinsic to fractal geometries.

By considering the fractal zeta functions associated with the sequence of polygonal neighborhoods, we came across the recurrence relation between consecutive fractal zeta functions. To our knowledge, this is the first time that such a result is obtained. It is all the more important that it enables us to prove the existence of the limit fractal zeta function – the global zeta function, associated with the limit fractal object, the Weierstrass Curve here, or rather, the Weierstrass iterated fractal drum (IFD). Moreover, this global zeta function has the same (possible) poles as the Complex Dimensions involved in the aforementioned series expansion of the Weierstrass function.

Our main results in the present setting can be found in the following places:

i. In Theorem 20, on page 11, where we obtain recurrence relations satisfied by the coefficients of the exact fractal (or Complex Dimensions) expansion for the complexified and the ordinary Weierstrass functions.

ii. In Definition 23, on page 12, where we introduce the sequence of polyhedral neighborhoods.

iii. In Theorem 24, on page 12, where we show that the polyhedral and tubular neighborhoods are nested.

iv. In Theorem 28, on page 16, where we obtain an exact fractal power series expansion for the volume of suitable m^{th} polyhedral neighborhoods of the Weierstrass Curve.

v. In Theorem 32, on page 20, where we introduce *the local and global effective polyhedral zeta function* and show that the global zeta function is well defined, meromorphic in all of $\mathbb{C}$, and is given by an explicit fractal power series.

vi. In Theorem 33, on page 24, where we obtain the exact Complex Dimensions of the Weierstrass Curve (or of the Weierstrass IFD), precisely of the same form as the possible Complex Dimensions occurring in the fractal cohomology theory developed in [6].

vii. In Theorem 37, on page 26, where we introduce a new and completely rigorous proof of the computation of the box–counting dimension (or, equivalently, of the Minkowski dimension) of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, by simply using the covers of $\Gamma_{\mathcal{W}}$ by polyhedral neighborhoods – therefore fully establishing part of Mandelbrot's conjecture [26] about the fractal dimensions of $\Gamma_{\mathcal{W}}$.

2. Geometric Framework

Henceforth, we place ourselves in the Euclidean plane, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by (x, y). The horizontal and vertical axes are respectively referred to as $(x'x)$ and $(y'y)$. We also let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^\star = \{1, 2, \ldots\}$. Given a, b with $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, we write, e.g., $[a, b[= [a, b)$ and similarly for other intervals.

Notation 1 (Weierstrass Parameters). In the sequel, λ and N_b are two real numbers such that

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^\star \quad \text{and} \quad \lambda\, N_b > 1 . \qquad (\clubsuit)$$

Note that this implies that $N_b > 1$ (i.e., $N_b \geq 2$).

As explained in [3], we deliberately made the choice to introduce the notation N_b which replaces the initial number b, in so far as, in Hardy's paper [10] (in contrast to Weierstrass's original article [28]), b is any positive real number satisfying $\lambda\, b > 1$, whereas we deal here with the specific case of a natural integer, which accounts for the natural notation N_b.

Definition 2 (Weierstrass Function, Weierstrass Curve). *We consider the Weierstrass function $\mathcal{W}$ (also called, in short, the $\mathcal{W}$-*function*) defined, for any real number x, by*

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\pi N_b^n x\right). \tag{1}$$

We call the associated graph the Weierstrass Curve*, and denote it by* $\Gamma_{\mathcal{W}}$.

Due to the one–periodicity of the $\mathcal{W}$–function, since N_b is an integer, from now on, and without loss of generality, we restrict our study to the interval $[0,1[=[0,1)$. We also identify the points $(0,\mathcal{W}(0))$ and $(1,\mathcal{W}(1)) = (1,\mathcal{W}(0))$.

Definition 3 (Weierstrass Complexified Function). *We introduce the Weierstrass Complexified function $\mathcal{W}_{comp}$, defined, for any real number x, by*

$$\mathcal{W}_{comp}(x) = \sum_{n=0}^{\infty} \lambda^n e^{2i\pi N_b^n x}.$$

Notation 4 (Logarithm). Given $y > 0$, $\ln y$ denotes the natural logarithm of y, while, given $a > 1$, $\ln_a y = \dfrac{\ln y}{\ln a}$ denotes the logarithm of y in base a; so that, in particular, $\ln = \ln_e$.

Notation 5 (Minkowski Dimension and Hölder Exponent). For the parameters λ and N_b satisfying condition $(\clubsuit)$ (see Notation 1, on page 3), we denote by

$$D_{\mathcal{W}} = 2 + \frac{\ln\lambda}{\ln N_b} = 2 - \ln_{N_b}\frac{1}{\lambda} \in \,]1,2[$$

the box–counting dimension (or Minkowski dimension) of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, which happens to be equal to its Hausdorff dimension [12], [1], [27], [13]. We point out that the results in our present paper – and its long version [8] – also provide a direct geometric proof of the fact that $D_{\mathcal{W}}$, the Minkowski dimension (or box–counting dimension) of $\Gamma_{\mathcal{W}}$, exists and takes the above values, and that our results in our previous paper [5] show that $\mathcal{W}$ is Hölder continuous with *optimal* Hölder exponent $2 - D_{\mathcal{W}} = -\dfrac{\ln\lambda}{\ln N_b} = \ln_{N_b}\dfrac{1}{\lambda}$.

Note that the latter result was also obtained by G. H. Hardy in [10], although new key information is obtained in [5] where we establish a new discrete analog of the *reverse* Hölder estimates (of order $D_{\mathcal{W}}$) which is used in an essential way in the present work; see Proposition 16, on page 8 and Theorem 17, on page 9.

Proposition 6 (Nonlinear and Noncontractive Iterated Function System (IFS)). *Following our previous work* [2]*, we approximate the restriction $\Gamma_{\mathcal{W}}$ to $[0,1[\times\mathbb{R}$, of the Weierstrass Curve, by a sequence of graphs, built via an iterative process. For this purpose, we use the nonlinear (and noncontractive) iterated function system (IFS) of the family of C^∞ maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ denoted by*

$$\mathcal{T}_{\mathcal{W}} = \left\{T_0, \cdots, T_{N_b-1}\right\},$$

where, for any integer i belonging to $\{0, \cdots, N_b - 1\}$ and any point (x,y) of $\mathbb{R}^2$,

$$T_i(x,y) = \left(\frac{x+i}{N_b}, \lambda\, y + \cos\left(2\pi\left(\frac{x+i}{N_b}\right)\right)\right).$$

Theorem 7 (Attractor of the IFS [2], [3]). *The Weierstrass Curve $\Gamma_{\mathcal{W}}$ is the attractor of the IFS $\mathcal{T}_{\mathcal{W}}$, and hence, is the unique nonempty compact subset $\mathcal{K}$ of $\mathbb{R}^2$ satisfying $\mathcal{K} = \bigcup_{i=0}^{N_b-1} T_i(\mathcal{K})$; in particular, we have that $\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$.*

Notation 8 (Fixed Points). For any integer i belonging to $\{0, \cdots, N_b-1\}$, we denote by $P_i = (x_i, y_i) = \left(\dfrac{i}{N_b-1}, \dfrac{1}{1-\lambda}\cos\left(\dfrac{2\pi i}{N_b-1}\right)\right)$ the unique fixed point of the map T_i; see [3].

Definition 9 (Sets of Vertices, Prefractals). *We denote by V_0 the ordered set (according to increasing abscissae) of the points*

$$\left\{P_0, \cdots, P_{N_b-1}\right\}.$$

The set of points V_0 – where, for any integer i in $\{0, \cdots, N_b-2\}$, the point P_i is linked to the point P_{i+1} – constitutes an oriented finite graph, ordered according to increasing abscissae, which we will denote by $\Gamma_{\mathcal{W}_0}$. Then, V_0 is called the set of vertices *of the graph $\Gamma_{\mathcal{W}_0}$.*

For any positive integer m, i.e., for $m \in \mathbb{N}^\star$, we set $V_m = \bigcup_{i=0}^{N_b-1} T_i(V_{m-1})$.

*The set of points V_m, where two consecutive points are linked by an edge, is an oriented finite graph, ordered according to increasing abscissa, called the **m^{th} order $\mathcal{W}$-prefractal**. Then, V_m is called* the set of vertices *of the prefractal $\Gamma_{\mathcal{W}_m}$; see Figure 2, on page 7. We call* Weierstrass Iterated Fractal Drums (IFD) *the sequence of prefractal graphs which converge to the Weierstrass Curve.*

Proposition 10 (Density of the Set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ in the Weierstrass Curve [5]). *The set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ is dense in the Weierstrass Curve $\Gamma_{\mathcal{W}}$.*

Definition 11 (Adjacent Vertices, Edge Relation). *For any $m \in \mathbb{N}$, the prefractal graph $\Gamma_{\mathcal{W}_m}$ is equipped with an edge relation $\underset{m}{\sim}$, as follows: two vertices X and Y of $\Gamma_{\mathcal{W}_m}$ (i.e. two points belonging to V_m) are said to be* adjacent *(i.e.,* neighboring *or* junction points*) if and only if the line segment $[X, Y]$ is an edge of $\Gamma_{\mathcal{W}_m}$; we then write $X \underset{m}{\sim} Y$. Note that this edge relation depends on m, which means that points adjacent in V_m might not remain adjacent in V_{m+1}.*

Proposition 12. [2] *For any $m \in \mathbb{N}$, the following statements hold*:

i. $V_m \subset V_{m+1}$.

ii. $\#V_m = (N_b-1)\,N_b^m + 1$, *where $\#V_m$ denotes the number of elements in the finite set V_m.*

iii. *The prefractal graph $\Gamma_{\mathcal{W}_m}$ has exactly $(N_b-1)\,N_b^m$ edges.*

iv. *The consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$ are the vertices of N_b^m simple nonregular polygons $\mathcal{P}_{m,k}$ with N_b sides (or N_b-gons). For any strictly positive integer m, the junction point between two consecutive polygons is the point*

$$\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 1 \le k \le N_b^m - 1.$$

Hence, the total number of junction points is $N_b^m - 1$. For instance, in the case $N_b = 3$, the polygons are all triangles; see Figure 1.

*In the sequel, we will denote by $\mathcal{P}_0$ **the initial polygon**, whose vertices are the fixed points of the maps T_i, $0 \le i \le N_b - 1$, introduced in Notation 8 and Definition 9, on page 5, i.e., $\left\{P_0, \cdots, P_{N_b-1}\right\}$; see, again, Figure 1, on page 6.*

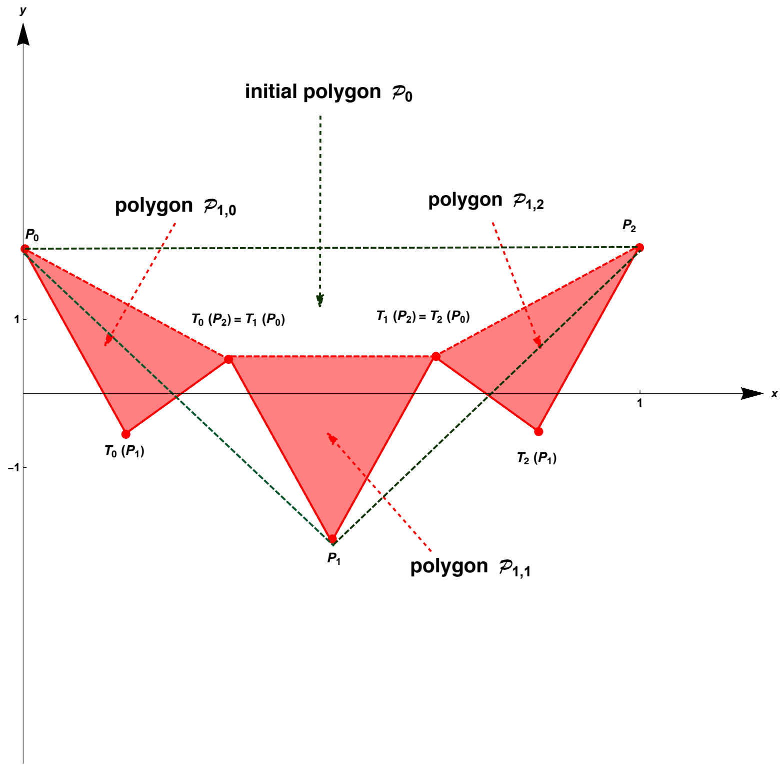

Figure 1. The initial polygon $\mathcal{P}_0$, and the polygons $\mathcal{P}_{1,0}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$. (See also Figure 2, on page 7.)

Definition 13 (Polygonal Sets [4]). *For any $m \in \mathbb{N}$, we introduce the following* polygonal sets

$$\mathcal{P}_m = \left\{\mathcal{P}_{m,k}\,,\, 0 \le k \le N_b^m - 1\right\} \quad \textit{and} \quad \mathcal{Q}_m = \left\{\mathcal{Q}_{m,k}\,,\, 0 \le k \le N_b^m - 2\right\}.$$

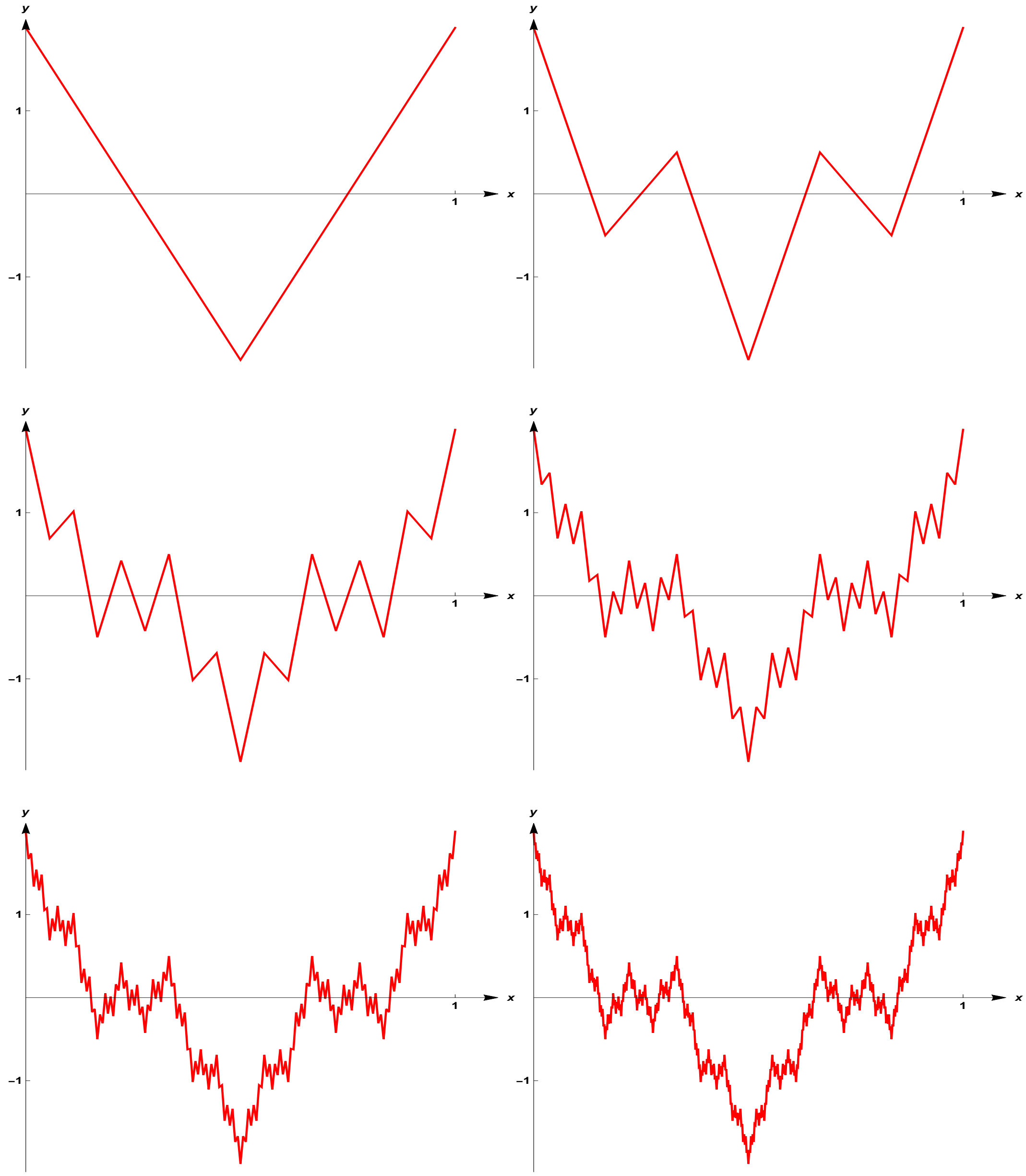

Figure 2. The prefractal graphs $\Gamma_{\mathcal{W}_0}, \Gamma_{\mathcal{W}_1}, \Gamma_{\mathcal{W}_2}, \Gamma_{\mathcal{W}_3}, \Gamma_{\mathcal{W}_4}, \Gamma_{\mathcal{W}_5}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$. For example, $\Gamma_{\mathcal{W}_1}$ is on the right side of the top row, while $\Gamma_{\mathcal{W}_4}$ is on the left side of the bottom row.

Definition 14 (Vertices of the Prefractals, Elementary Lengths, Heights and Angles). *Given a strictly positive integer m, we denote by $\left(M_{j,m}\right)_{0\leq j\leq (N_b-1)\,N_b^m}$* the set of vertices *of the prefractal graph $\Gamma_{\mathcal{W}_m}$. One thus has, for any integer j in $\left\{0,\cdots,(N_b-1)\,N_b^m\right\}$:*

$$M_{j,m}=\left(\frac{j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right).$$

We also introduce, for any integer j in $\left\{1,\cdots,(N_b-1)\,N_b^m-1\right\}$:

i. <u>*the elementary horizontal lengths:*</u>

$$L_m=\frac{j}{(N_b-1)\,N_b^m};$$

ii. <u>*the elementary heights:*</u>

$$h_{j-1,j,m}=\left|\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)-\mathcal{W}\left(\frac{j-1}{(N_b-1)\,N_b^m}\right)\right|;$$

iii. <u>*the geometric angles:*</u>

$$\theta_{j-1,j,m}=\left((y'y),\left(M_{j-1,m}M_{j,m}\right)\right),$$

which yield ***the following value of the geometric angle between consecutive edges****, namely, $\left[M_{j-1,m}\,M_{j,m},M_{j,m}\,M_{j+1,m}\right]$, with $\arctan=\tan^{-1}$:*

$$\theta_{j-1,j,m}+\theta_{j,j+1,m}=\arctan\frac{L_m}{\left|h_{j-1,j,m}\right|}+\arctan\frac{L_m}{\left|h_{j,j+1,m}\right|}.$$

Proposition 15 (Scaling Properties of the Weierstrass Function, and Consequences [5]). *Since, for any real number x, $\mathcal{W}(x)=\sum_{n=0}^{\infty}\lambda^n\cos\left(2\pi N_b^n x\right)$, one also has*

$$\mathcal{W}(N_b\,x)=\sum_{n=0}^{\infty}\lambda^n\cos\left(2\pi N_b^{n+1}x\right)=\frac{1}{\lambda}\sum_{n=1}^{\infty}\lambda^n\cos\left(2\pi N_b^n x\right)=\frac{1}{\lambda}\left(\mathcal{W}(x)-\cos(2\pi x)\right),$$

which yields, for any strictly positive integer m and any j in $\{0,\ldots,\#V_m\}$,

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)=\lambda\,\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}\right)+\cos\left(\frac{2\pi j}{(N_b-1)\,N_b^m}\right).$$

By induction, one then obtains that

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)=\lambda^m\,\mathcal{W}\left(\frac{j}{(N_b-1)}\right)+\sum_{k=0}^{m-1}\lambda^k\cos\left(\frac{2\pi N_b^k\,j}{(N_b-1)\,N_b^m}\right).$$

Proposition 16 (Lower Bound and Upper Bound for the Elementary Heights [5]). *For any strictly positive integer m and any j in $\left\{0,\cdots,(N_b-1)\,N_b^m\right\}$, we have the following estimates:*

$$C_{inf}\,L_m^{2-D_{\mathcal{W}}}\leq\left|\mathcal{W}\left((j+1)\,L_m\right)-\mathcal{W}\left(j\,L_m\right)\right|\leq C_{sup}\,L_m^{2-D_{\mathcal{W}}}\quad,\quad m\in\mathbb{N},\,0\leq j\leq(N_b-1)\,N_b^m$$

where the finite and positive constants C_{inf} and C_{sup} are given by

$$C_{inf}=(N_b-1)^{2-D_{\mathcal{W}}}\min_{0\leq j\leq N_b-1,\,\mathcal{W}\left(\frac{j+1}{N_b-1}\right)\neq\mathcal{W}\left(\frac{j}{N_b-1}\right)}\left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right|$$

and

$$C_{sup}=(N_b-1)^{2-D_{\mathcal{W}}}\left(\max_{0\leq j\leq N_b-1}\left|\mathcal{W}\left(\frac{j+1}{N_b-1}\right)-\mathcal{W}\left(\frac{j}{N_b-1}\right)\right|+\frac{2\pi}{(N_b-1)\,(\lambda\,N_b-1)}\right).$$

One should note, in addition, that these constants C_{inf} and C_{sup} depend on the initial polygon $\mathcal{P}_0$.

Theorem 17 (Sharp Local Discrete Reverse Hölder Properties of the Weierstrass Function).

For any natural integer m, let us consider a pair of real numbers (x,x') such that

$$x=\frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m}=((N_b-1)\,k+j)\,L_m\quad,\quad x'=\frac{(N_b-1)\,k+j+\ell}{(N_b-1)\,N_b^m}=((N_b-1)\,k+j+\ell)\,L_m\,,$$

where $0\leq k\leq N_b-1^m-1$, and

i. *if the integer N_b is odd,*

$$0\leq j<\frac{N_b-1}{2}\quad and\quad 0<j+\ell\leq\frac{N_b-1}{2}$$

or

$$\frac{N_b-1}{2}\leq j<N_b-1\quad and\quad \frac{N_b-1}{2}<j+\ell\leq N_b-1\,;$$

ii. *if the integer N_b is even,*

$$0\leq j<\frac{N_b}{2}\quad and\quad 0<j+\ell\leq\frac{N_b}{2}$$

or

$$\frac{N_b}{2}+1\leq j<N_b-1\quad and\quad \frac{N_b}{2}+1<j+\ell\leq N_b-1\cdot$$

This means that the points $(x,\mathcal{W}(x))$ and $\left(x',\mathcal{W}(x')\right)$ are vertices of the polygon $\mathcal{P}_{m,k}$ (see Property 12, on page 5 above), both located on the left-side of the polygon, or both located on the right-side; see Figure 3, on page 9.

Then, one has the following (discrete, local) reverse–Hölder inequality, *with sharp Hölder exponent $-\dfrac{\ln\lambda}{\ln N_b}=2-D_{\mathcal{W}}$,*

$$C_{inf}\,|x'-x|^{2-D_{\mathcal{W}}}\leq\left|\mathcal{W}(x')-\mathcal{W}(x)\right|\cdot$$

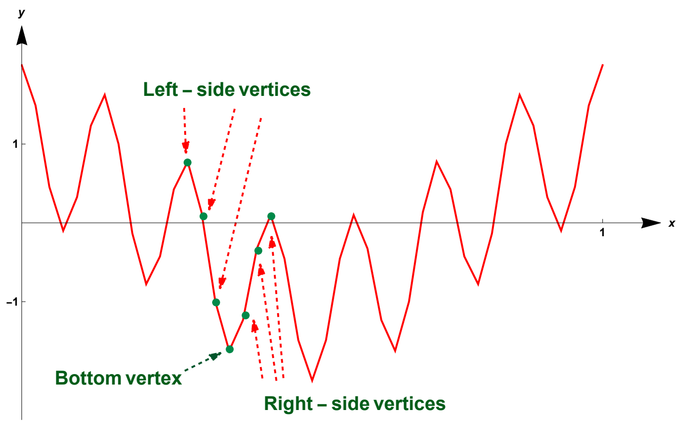

Figure 3. The left-side and right-side vertices.

Sketch of the proof.

We simply use Proposition 16, on page 8.

3. Cohomology Infinitesimal – Complex Dimensions Series Expansion of the Weierstrass Function

Definition 18 (m^{th} Cohomology Infinitesimal [5], [6] and m^{th} Intrinsic Cohomology Infinitesimal). *From now on, given any $m \in \mathbb{N}$, we will call m^{th}* cohomology infinitesimal *(of $\Gamma_{\mathcal{W}}$) the number $\varepsilon_m^m > 0$ which also corresponds to the elementary horizontal length introduced in part i. of Definition 14, on page 7; i.e., $\varepsilon_m^m = (\varepsilon_m)^m = \frac{1}{N_b - 1}\,\frac{1}{N_b^m}$.*

Observe that, clearly, ε_m itself – and not just ε_m^m – depends on m.

In addition, since $N_b > 1$, ε_m^m satisfies the following asymptotic behavior,

$$\varepsilon_m^m \to 0\,,\ as\ m \to \infty,$$

which, naturally, results in the fact that the larger m, the smaller ε_m^m. It is for this reason that we call ε_m^m – or rather, the infinitesimal sequence $\left(\varepsilon_m^m\right)_{m=0}^{\infty}$ *of positive numbers tending to zero as $m \to \infty$, with $\varepsilon_m^m = (\varepsilon_m)^m$, for each $m \in \mathbb{N}$ – an* infinitesimal. *Note that this m^{th} cohomology infinitesimal is the one naturally associated to the scaling relation of Proposition 15, on page 8.*

In the sequel, it is also useful to keep in mind that the sequence of positive numbers $(\varepsilon_m)_{m=0}^{\infty}$ itself satisfies

$$\varepsilon_m \sim \frac{1}{N_b}\,,\ as\ m \to \infty\ \ ;$$

i.e., $\varepsilon_m \to \frac{1}{N_b}$, as $m \to \infty$. In particular, $\varepsilon_m \not\to 0$, as $m \to \infty$, but, instead, ε_m tends to a strictly positive and finite limit.

We also introduce, given any $m \in \mathbb{N}$, the m^{th} intrinsic cohomology infinitesimal, *denoted by $\varepsilon^m > 0$, such that*

$$\varepsilon^m = \frac{1}{N_b^m}\,,$$

where

$$\varepsilon = \frac{1}{N_b}\,.$$

We call ε the intrinsic scale, *or* intrinsic subdivision scale.

Note that

$$\varepsilon_m^m = \frac{\varepsilon^m}{N_b - 1}$$

and that the m^{th} intrinsic cohomology infinitesimal ε^m is asymptotic equal (when m tends to ∞) to the m^{th} cohomology infinitesimal ε_m^m.

Remark 19. We note that the choice of the m^{th} intrinsic cohomology infinitesimal, instead of the m^{th} cohomology infinitesimal, as is done in [5] and [6], will significantly help the computation of the polyhedral prefractal volumes, the polyhedral fractal zeta functions and hence also, of the Complex Dimensions, without any loss of information.

Theorem 20 (Complex Dimensions Series Expansion of the Weierstrass Complexified Function $\mathcal{W}_{comp}$ [6], [7], [8] and of the Weierstrass function $\mathcal{W}$). *For any sufficiently large positive integer m and any j in $\{0,\cdots,\#V_m-1\}$, we have the following* exact *expansion, indexed by the Complex Codimensions $k\,(D_{\mathcal{W}}-2)+i\,k\,\ell_{k,j,m}\,\mathbf{p}$, with $0\leq k\leq m$,*

$$\mathcal{W}_{comp}\left(j\,\varepsilon_m^m\right)=\mathcal{W}_{comp}\left(\frac{j\,\varepsilon^m}{N_b-1}\right)=\sum_{k=0}^{m}c_{k,j,m}\,\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_{k,j,m}\,\mathbf{p}}\,, \tag{2}$$

where, for $0\leq k\leq m$, ε^k is the k^{th} intrinsic cohomology infinitesimal, introduced in Definition 18, on page 10, with $\mathbf{p}=\dfrac{2\pi}{\ln N_b}$ denoting the oscillatory period of the Weierstrass Curve, as introduced in [5] and where:

i. $\ell_{k,j,m}\in\mathbb{Z}$ *is arbitrary.*

ii. $c_{m,j,m}=\mathcal{W}_{comp}\left(\dfrac{j}{N_b-1}\right)$ *and, for $0\leq k\leq m-1$, $c_{k,j,m}\in\mathbb{C}$ is given by*

$$c_{k,j,m}=exp\left(\frac{2\,i\,\pi}{N_b-1}\,j\,\varepsilon^{m-k}\right). \tag{3}$$

For any $m\in\mathbb{N}$, the complex numbers $\left\{c_{0,j,m+1},\cdots,c_{m+1,j,m+1}\right\}$ and the integers $\left\{\ell_{0,j,+1},\cdots,\ell_{m+1,j,m+1}\right\}$ respectively satisfy the following recurrence relations:

$$c_{m+1,j,m+1}=\mathcal{W}\left(\frac{j}{N_b-1}\right)=c_{m,j,m} \tag{4}$$

and

$$\forall\,k\in\{1,\ldots,m\}:\quad c_{k,j,m+1}=c_{k,j,m}\quad,\quad \ell_{k,j,m+1}=\ell_{k,j,m}\,, \tag{5}$$

In addition, since relation (2) is valid for any $m\in\mathbb{N}^\star$, we note that the associated Complex Dimensions (i.e., in fact, the Complex Dimensions associated with the Weierstrass function) are

$$D_{\mathcal{W}}-k\,(2-D_{\mathcal{W}})+i\,\ell_{k,j,m}\,\mathbf{p}$$

where $0\leq j\leq\#V_m-1$ and $0\leq k\leq m$.

This immediately ensures, for the Weierstrass function (i.e., the real part of the Weierstrass complexified function $\mathcal{W}_{comp}$), that, for any strictly positive integer m and any j in $\{0,\cdots,\#V_m\}$,

$$\mathcal{W}\left(j\,\varepsilon_m^m\right)=\frac{1}{2}\sum_{k=0}^{m}\varepsilon^{k\,(2-D_{\mathcal{W}})}\left(c_{k,j,m}\,\varepsilon_k^{\,i\,\ell_{k,j,m}\,\mathbf{p}}+\overline{c_{k,j,m}}\,\varepsilon_k^{\,-i\,\ell_{k,j,m}\,\mathbf{p}}\right). \tag{6}$$

Remark 21. The recurrence relations (4) and (5) stated in part ii. of Theorem 20 just above are new and will play a key role in order to establish the recurrence relations satisfied by the local fractal zeta functions; see relation (13) in Theorem 32, on page 20 below.

4. Polyhedral Neighborhoods

Definition 22 ((m,ε_m^m)-Neighborhood [5]). *Given $m \in \mathbb{N}$ sufficiently large (so that ε_m^m be a sufficiently small positive number), we define the (m,ε_m^m)-neighborhood of the m^{th} prefractal approximation $\Gamma_{\mathcal{W}_m}$ as follows:*

$$\mathcal{D}\left(\Gamma_{\mathcal{W}_m},\varepsilon_m^m\right) = \left\{M = (x,y) \in \mathbb{R}^2 \,,\, d\left(M,\Gamma_{\mathcal{W}_m}\right) \leq \varepsilon_m^m\right\}.$$

Definition 23 (Sequence of Domains Delimited by the Weierstrass IFD – Polyhedral Neighborhood of the Weierstrass Curve [4]).

We introduce the sequence of domains delimited by the Weierstrass IFD*, or* polygonal neighborhood of the Weierstrass Curve*, as the sequence $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ of open, connected polygonal sets $(\mathcal{P}_m \cup \mathcal{Q}_m)_{m\in\mathbb{N}}$, where, for each $m \in \mathbb{N}$, $\mathcal{P}_m$ and Q_m respectively denote the polygonal sets introduced in Definition 13, on page 6.*

Given $\in \mathbb{N}$, we call $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ the m^{th} polyhedral neighborhood *(of the Weierstrass Curve $\Gamma_{\mathcal{W}}$); see Figure 4, on page 12.*

Theorem 24 (The Nested Neighborhoods). *i. Given $m \in \mathbb{N}$ sufficiently large, there exists $k_1 \in \mathbb{N}$ such that, for all $k \geq k_1$, the polygonal neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ introduced in Definition 23, on page 12 contains, but for a finite number of wedges, the $(m+k,\varepsilon_{m+k}^{m+k})$-neighborhood; see Figures 4–7, on pages 12–14.*

ii. Given $m \in \mathbb{N}$ sufficiently large, there exists $k_2 \in \mathbb{N}$ such that, for all $k \geq k_2$, the (m,ε_m^m)-neighborhood contains the polygonal neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_{m+k}}\right)$ introduced in Definition 23, on page 12; see Figure 8, on page 14 and Figure 9, on page 15.

iii. Given $m \in \mathbb{N}$ sufficiently large, there exists $k_3 \in \mathbb{N}$ such that, for all $k \geq k_3$, the polygonal neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ introduced in Definition 23, on page 12 contains the polygonal neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_{m+k}}\right)$.

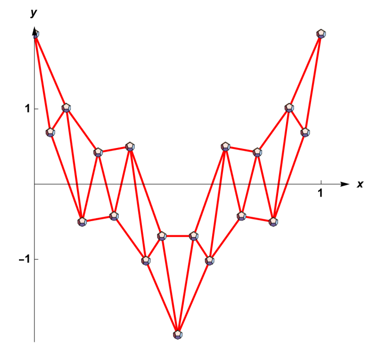

Figure 4. The polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_2}\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.

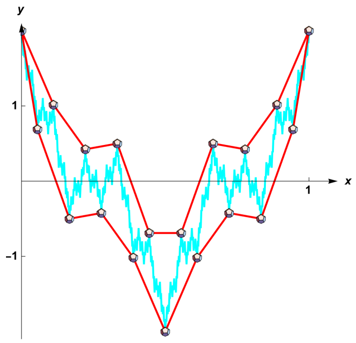

Figure 5. The exterior boundary of the polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_2}\right)$ (in red), and the tubular neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_7}, \varepsilon_7^7\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.

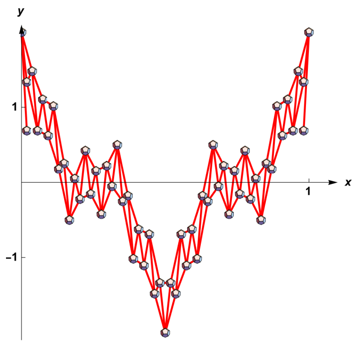

Figure 6. The polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_3}\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.

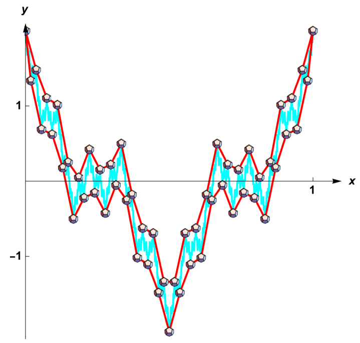

Figure 7. The exterior boundary of the polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_3}\right)$ (in red), and the tubular neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_7}, \varepsilon_7^7\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.

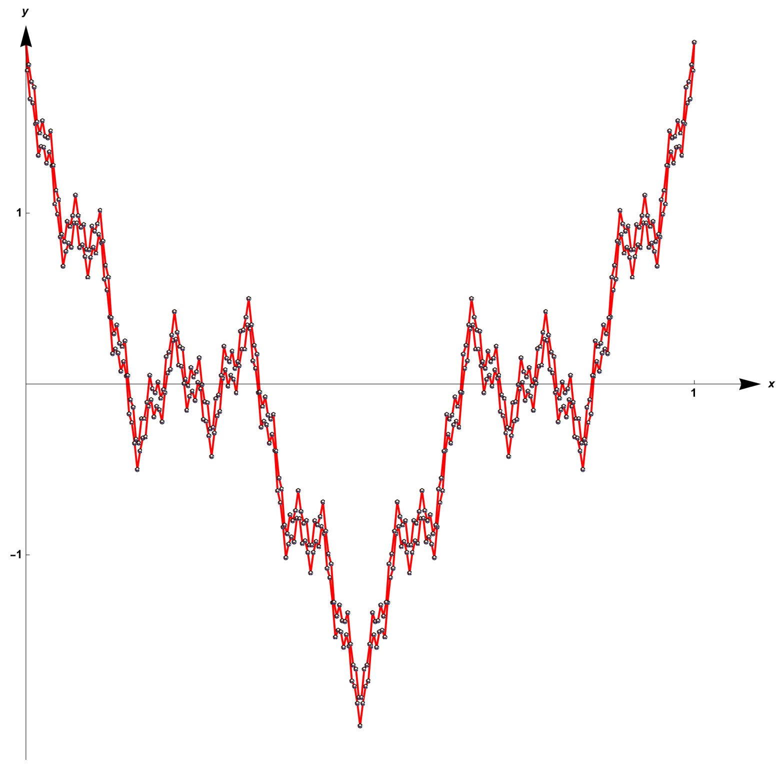

Figure 8. The polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_5}\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.

Corollary 25 (of Theorem 24, given on page 12).

We immediately deduce from Theorem 24 that, given any $m \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, the polyhedral neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ introduced in Definition 23, on page 12,

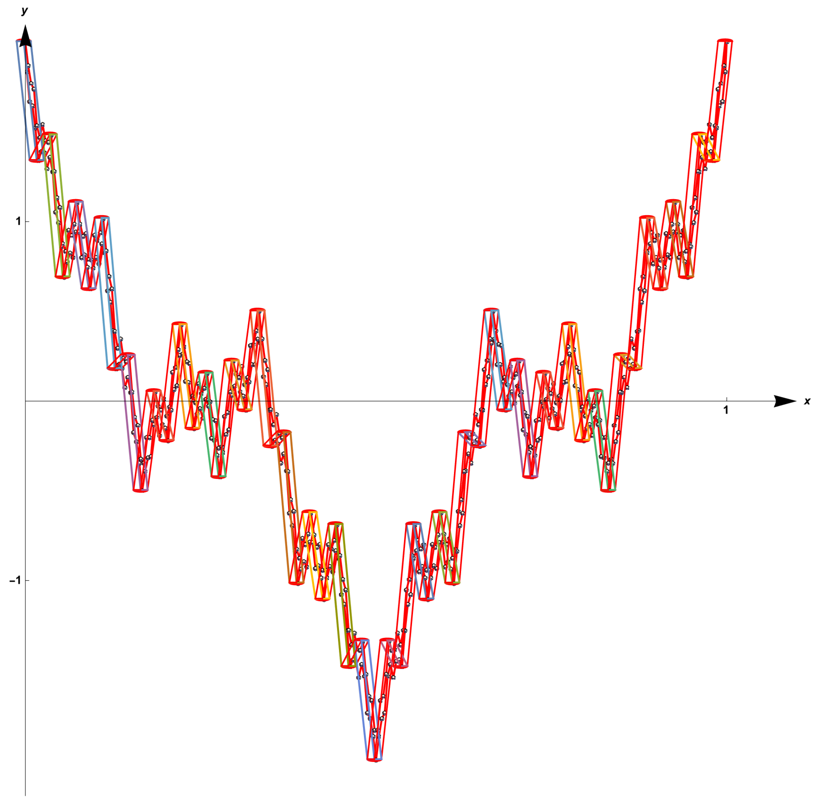

Figure 9. The tubular neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_3}, \varepsilon_3^3\right)$ and the polygonal neigborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_5}\right)$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$. The size of the vertex points of $\mathcal{D}\left(\Gamma_{\mathcal{W}_5}\right)$ has intentionally been magnified, in order to obtain an illustrative and understandable figure.

contains the set of vertices V_{m+k}. In particular, the density property 10 (Proposition 10, on page 5), also ensures that the Weierstrass Curve is contained in $\Gamma_{\mathcal{W}_m}$:

$$\forall\, m \geq k_0 : \quad \Gamma_{\mathcal{W}} \subset \mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right).$$

Notation 26 (Lebesgue Measure on $\mathbb{R}^2$). In the sequel, we denote by $\mu_{\mathcal{L}}$ the Lebesgue measure on $\mathbb{R}^2$.

Definition 27 (The Weierstrass Complex Curve). *We place ourselves in the Riemann sphere (or complex projective line) $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$. We define the* Weierstrass Complex Curve *as the graph in $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$ associated with the Weierstrass complefixied function $\mathcal{W}_{comp}$; i.e., the set denoted by $\Gamma_{\mathcal{W},comp}$ such that*

$$\Gamma_{\mathcal{W},comp} = \left\{\left(x, \mathcal{W}_{comp}(x)\right), \, x \in [0,1]\right\} = \Gamma_{\mathcal{W}} + i\, \Gamma_{\mathcal{W},\mathcal{I}},$$

where $\Gamma_{\mathcal{W},\mathcal{I}}$ is the graph (in $\mathbb{R}^2$) associated with the imaginary part $\mathcal{W}_{\mathcal{I}}$ of the Weierstrass complefixied function $\mathcal{W}_{comp}$ and defined, for any real number x, by

$$\mathcal{W}_{\mathcal{I}}(x) = \sum_{n=0}^{\infty} \lambda^n \sin\left(2\pi N_b^n x\right). \tag{7}$$

Theorem 28 (Exact Expression for the Volume of the m^{th} Polyhedral Neighborhood (or m^{th} Natural Polyhedral Volume)). *Given $m \in \mathbb{N}^\star$ sufficiently large, the volume (or two–dimensional Lebesgue measure) $\mathcal{V}_m(\varepsilon_m^m)$ of the m^{th} polygonal neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$, or m^{th}* natural polyhedral volume*, is given by*

$$\begin{aligned}
\mathcal{V}_m(\varepsilon_m^m) = \quad & \mu_{\mathcal{L}}\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right) \\
= \quad & \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b)\, \mathcal{W}\left(\frac{(N_b-1)\, j+q}{(N_b-1)\, N_b^m}\right) \\
= \; & \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m} \varepsilon^{k(2-D_{\mathcal{W}})}\, \mathcal{R}e\left(c_{m,k,(N_b-1)\,j+q}\, \varepsilon^{i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}\right),
\end{aligned} \tag{8}$$

where $\varepsilon^m = (N_b-1)\,\varepsilon_m^m = \dfrac{1}{N_b^m}$ is the m^{th} intrinsic cohomology infinitesimal introduced in Definition 18, on page 10, while the real numbers $\alpha_q(N_b)$, with $0 \le q \le N_b$, are coefficients which depend on N_b and where the complex numbers $c_{m,k,(N_b-1)\,j+q} \in \mathbb{C}$ have been introduced in part ii. of Theorem 20, on page 11, while the integers $\ell_{m,k,(N_b-1)\,j+q} \in \mathbb{Z}$ are arbitrary (see also Theorem 20, on page 11), with

a. *<u>When the integer N_b is odd:</u>*

$$\alpha_0(N_b) = \alpha_{N_b-1}(N_b) = -\alpha_{\frac{N_b-1}{2}}(N_b) = -\alpha_{\frac{N_b-1}{2}+N_b-1}(N_b) = \frac{N_b-2}{2\,(N_b-1)}$$

and for $1 \le q \le N_b-2$,

$$\alpha_{\frac{N_b-1}{2}+q}(N_b) = -\alpha_q(N_b) = -\alpha_{N_b^m,(N_b-1)\,N_b^m-(N_b-1)+q}(N_b) = \frac{1}{N_b-1},$$

along with

$$\alpha_{N_b^m,(N_b-1)\,N_b^m-(N_b-1)}(N_b) = \alpha_{N_b^m,1}(N_b) = \frac{N_b-2}{2\,(N_b-1)}.$$

b. *<u>When the integer N_b is even:</u>*

$$\alpha_0(N_b) = \alpha_{N_b-1}(N_b) = -\alpha_{\frac{N_b}{2}}(N_b) = -\alpha_{\frac{N_b}{2}+N_b-1}(N_b) = \frac{N_b-2}{2\,(N_b-1)}$$

and for $1 \le q \le N_b-2$,

$$\alpha_{\frac{N_b}{2}+q}(N_b) = -\alpha_q(N_b) = -\alpha_{N_b^m,(N_b-1)\,N_b^m-(N_b-1)+q}(N_b) = -\frac{1}{N_b-1},$$

along with

$$\alpha_{N_b^m,(N_b-1)\,N_b^m-(N_b-1)}(N_b) = \alpha_{N_b^m,1}(N_b) = \frac{N_b-2}{2\,(N_b-1)}.$$

Given the form of the expression in relation (8) just above, it is natural to introduce, for any $m \in \mathbb{N}^\star$ sufficiently large, the associated m^{th} complex natural polyhedral volume $\mathcal{V}_{m,comp}(\varepsilon_m^m)$*, such that*

$$\mathcal{V}_{m,comp}(\varepsilon_m^m) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m} c_{m,k,(N_b-1)\,j+q}\, \varepsilon^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}} . \tag{9}$$

By considering, in the same manner, given $m \in \mathbb{N}^\star$ sufficiently large, the volume (or two–dimensional Lebesgue measure) $\mathcal{V}_{\mathcal{I},m}(\varepsilon_m^m)$ of the m^{th} polygonal neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W},I_m}\right)$, associated with the m^{th} prefractal approximation to the graph $\Gamma_{\mathcal{W},\mathcal{I}}$ of the imaginary part $\mathcal{W}_{\mathcal{I}}$ of the Weierstrass complefixied function $\mathcal{W}_{comp}$ (see Definition 27, on page 15 above), we note that the complex volume $\mathcal{V}_{m,comp}(\varepsilon_m^m)$ *can be envisioned as the volume associated with the Weierstrass complex curve $\Gamma_{\mathcal{W},comp}$ (see again Definition 27, on page 15) in the Riemann sphere (or complex projective line) $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \infty$.*

Observe that much as for its Euclidean counterpart obtained in our earlier work, [5], the m^{th} polyhedral fractal formula in Theorem 28, on page 16, is expressed at the m^{th} cohomology infinitesimal ε_m^m, for all $m \in \mathbb{N}$ sufficiently large (instead of being expressed at any real number $\epsilon > 0$ sufficiently small).

Notation 29 (Minimal and Maximal Values of the Weierstrass Function $\mathcal{W}$ on $[0,1]$).

We set

$$m_{\mathcal{W}} = \min_{t\in[0,1]} \mathcal{W}(t) = -\frac{1}{1-\lambda} \quad , \quad M_{\mathcal{W}} = \max_{t\in[0,1]} \mathcal{W}(t) = \frac{1}{1-\lambda} .$$

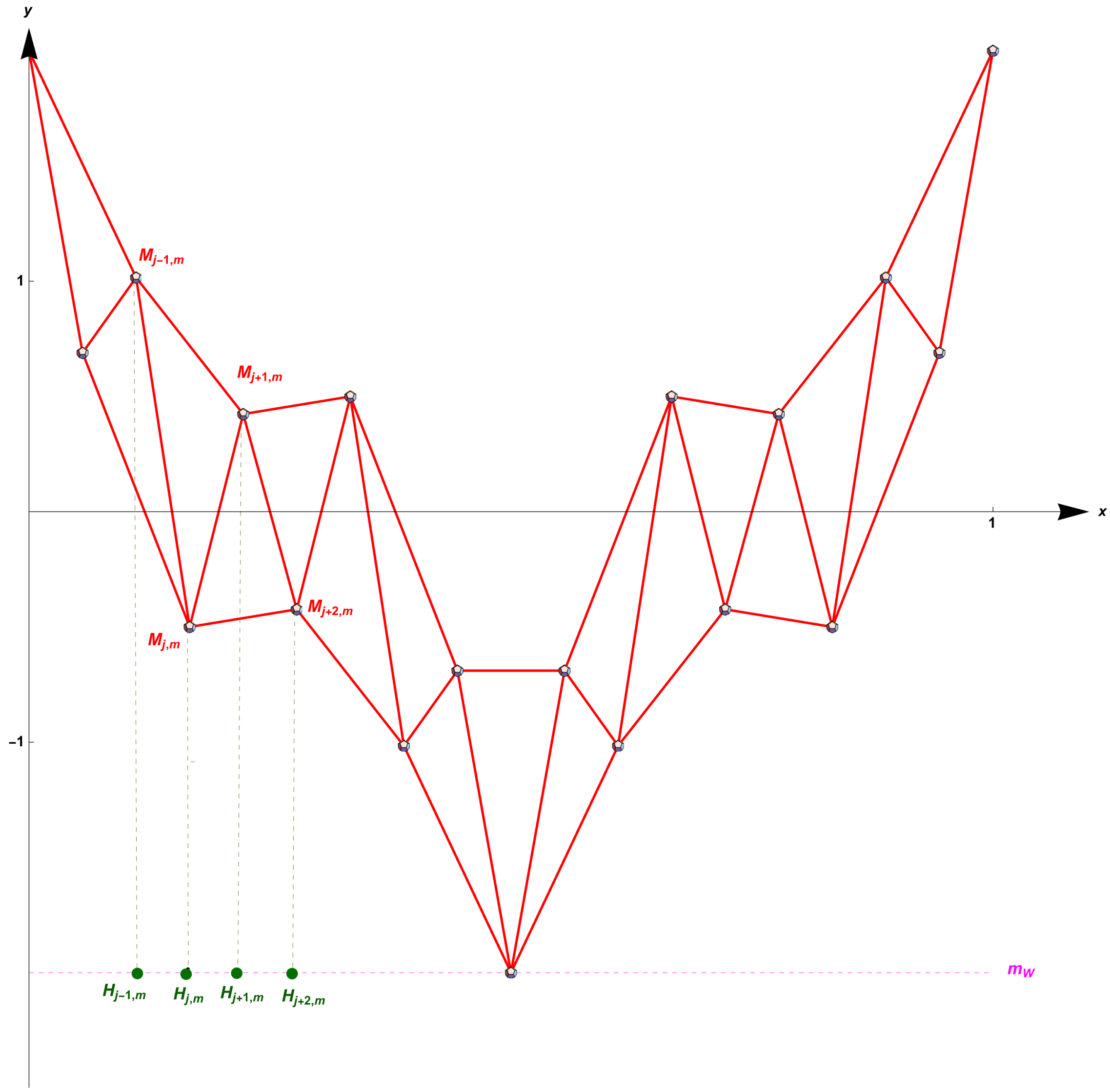

Figure 10. The orthogonal projections $H_{j,m}$ of the vertices $M_{j,m}$, for $0 \leq j \leq \# V_m - 1$, onto the horizontal line $y = m_{\mathcal{W}}$, where $m_{\mathcal{W}}$ is introduced in Notation 29, on page 17.

Sketch of the proof. (See Figure 10, on page 18.)

The proof is based on the fact that for $0 \leq j \leq N_b^m - 1$, the two–dimensional Lebesgue measure (i.e., area) of the N_b–gon $\mathcal{P}_{m,j+1}$, with (consecutive) vertices

$$M_{(N_b-1)\,j,m}, \ldots, M_{(N_b-1)\,j+N_b-1,m}$$

is obtained by substracting from the area of the trapezoid

$$H_{(N_b-1)\,j,m} M_{(N_b-1)\,j,m} M_{(N_b-1)\,j+N_b-1,m} H_{(N_b-1)\,j+N_b-1,m}$$

the respective areas of the following $N_b - 1$ trapezoids

$$H_{(N_b-1)\,j,m} M_{(N_b-1)\,j,m} M_{(N_b-1)\,j+1,m} H_{(N_b-1)\,j+1,m}$$

$$\vdots$$

$$H_{(N_b-1)\,j+N_b-2,m} M_{(N_b-1)\,j+N_b-2,m} M_{(N_b-1)\,j+N_b-1,m} H_{(N_b-1)\,j+N_b-1,m}\,;$$

i.e.,

$$\begin{aligned}
\mu_{\mathcal{L}}\left(\mathcal{P}_{m,j+1}\right) &= \frac{(N_b-1)\,\varepsilon_m^m}{2}\left(\mathcal{W}\left(\frac{(N_b-1)\,j}{(N_b-1)\,N_b^m}\right)+\mathcal{W}\left(\frac{(N_b-1)\,j+N_b-1}{(N_b-1)\,N_b^m}\right)\right) \\
&\quad -\frac{\varepsilon_m^m}{2}\sum_{q=0}^{N_b-2}\left(\mathcal{W}\left(\frac{(N_b-1)\,j+q}{(N_b-1)\,N_b^m}\right)+\mathcal{W}\left(\frac{(N_b-1)\,j+q+1}{(N_b-1)\,N_b^m}\right)\right) \\
&= \frac{(N_b-1)\,\varepsilon_m^m}{2}\left(\mathcal{W}\left(\frac{(N_b-1)\,j}{(N_b-1)\,N_b^m}\right)+\mathcal{W}\left(\frac{(N_b-1)\,j+N_b-1}{(N_b-1)\,N_b^m}\right)\right) \\
&\quad -\frac{\varepsilon_m^m}{2}\sum_{q=1}^{N_b-2} 2\,\mathcal{W}\left(\frac{(N_b-1)\,j+q}{(N_b-1)\,N_b^m}\right)-\frac{\varepsilon_m^m}{2}\,\mathcal{W}\left(\frac{(N_b-1)\,j}{(N_b-1)\,N_b^m}\right).
\end{aligned}$$

The two–dimensional Lebesgue measure (i.e., area) of the non necessarily convex N_b–gon $\mathcal{Q}_{m,j+1}$ is obtained in a similar manner.

Lemma 30 (Natural Polyhedral Volume Extension Formula). *Given $m \geq 1$ sufficiently large, we set*

$$\mathcal{V}_{m,comp}(\varepsilon_m^m) = \varepsilon^m\,\mathcal{V}_{partial,m,comp}\left(\varepsilon^m\right), \tag{10}$$

where

$$\mathcal{V}_{partial,m,comp}\left(\varepsilon^m\right) = \sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b}\sum_{k=0}^{m}\alpha_q(N_b)\,c_{m,k,(N_b-1)\,j+q}\,\varepsilon^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}$$

and where $\varepsilon^m = (N_b-1)\,\varepsilon_m^m = \dfrac{1}{N_b^m}$ is the m^{th} intrinsic cohomology infinitesimal introduced in Definition 18, on page 10, while the real numbers $\alpha_q(N_b)$, with $0 \leq q \leq N_b$, have been introduced in Proposition 28, on page 16, and where the complex numbers $c_{m,k,(N_b-1)\,j+q} \in \mathbb{C}$ have been introduced in part ii. of Theorem 20, on page 11, while the integers $\ell_{m,k,(N_b-1)\,j+q} \in \mathbb{Z}$ are arbitrary (see Theorem 20, on page 11). We have used the notation $\mathcal{V}_{partial,m,comp}\left(\varepsilon^m\right)$, since, as mentioned previously, the sum

$$\sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b}\sum_{k=0}^{m}\alpha_q(N_b)\,c_{m,k,(N_b-1)\,j+q}\,\varepsilon^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}$$

can be considered as a function of the m^{th} intrinsic cohomology infinitesimal ε^m.

We then introduce, for all sufficiently large $m \in \mathbb{N}^\star$, $\tilde{\mathcal{V}}_{partial,m,comp}$ as the continuous function defined for all $t \in [0,\varepsilon]$ by substituting t for ε in the right-hand side of the expression for $\mathcal{V}_{partial,m,comp}(\varepsilon^m)$, in relation (10).

As is explained in [5] (in the case of the ordinary Euclidean volume), one can think of $\varepsilon^m \tilde{\mathcal{V}}_{partial,m,comp}(t^m)$ as being the effective polyhedral volume *of the m^{th} prefractal approximation to the Weierstrass Curve.*

Notation 31 (Natural Polyhedral Complex Volume Extension).

For the sake of simplicity, given $m \in \mathbb{N}$ sufficiently large, we will from now on call *the m^{th} natural polyhedral complex volume extension*, the volume extension function $\tilde{\mathcal{V}}_{m,comp} = \varepsilon^m \tilde{\mathcal{V}}_{partial,m,comp}$ associated with the m^{th} natural polyhedral complex volume $\mathcal{V}_{m,comp}$ introduced in Theorem 28, on page 16. Alternatively, $\tilde{\mathcal{V}}_{m,comp}$ will be called *the m^{th} effective polyhedral complex volume.*

In the same way, as is done in [5], and given $m \in \mathbb{N}$, we call *the m^{th} natural volume extension*, the volume extension function $\tilde{\mathcal{V}}_m^{tube}$ associated with the *the m^{th} tubular volume $\mathcal{V}_m^{tube}$*. Alternatively, $\tilde{\mathcal{V}}_m^{tube}$ will be called *the m^{th} effective tubular volume.*

Theorem 32 (Local and Global Polyhedral Effective Zeta Functions: Fractal Power Expansions and Recurrence Relations). *Given $m \in \mathbb{N}$ sufficiently large, we introduce the m^{th}* local polyhedral effective zeta function $\tilde{\zeta}^e_m$*, such that, for all $s \in \mathbb{C}$ with $\mathcal{R}e(s) > D_{\mathcal{W}}$,*

$$\tilde{\zeta}^e_m(s) = \int_0^\varepsilon t^{s-3}\,\varepsilon^m\,\tilde{\mathcal{V}}_{m,comp}(t)\,dt\,, \tag{11}$$

where $\tilde{\mathcal{V}}_m = \varepsilon^m \tilde{\mathcal{V}}_{partial,m,comp}$ is the m^{th} effective polyhedral complex volume, introduced in Notation 31, on page 20 above, with ε denoting the m^{th} intrinsic cohomology infinitesimal introduced in Definition 18, on page 10.

Then, for all $m \in \mathbb{N}$ sufficiently large, $\tilde{\zeta}^e_m$ admits a (necessarily unique) meromorphic extension to all of $\mathbb{C}$, given, for all $s \in \mathbb{C}$, by the following explicit expression:

$$\tilde{\zeta}^e_m(s) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m} \frac{c_{k,(N_b-1)\,j+q,m}\,\varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-2+k(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}\,, \tag{12}$$

where ε is the intrinsic scale introduced in Definition 18, on page 10), and where, for $0 \le j \le N_b^m - 1$ and $0 \le q \le N_b$, the coefficients $\alpha_q(N_b) \in \mathbb{R}$ have been introduced in Theorem 28, on page 16, while the coefficients $c_{k,(N_b-1)\,j+q,m} \in \mathbb{C}$, along with the arbitrary integers $\ell_{k,(N_b-1)\,j+q,m} \in \mathbb{Z}$, have been introduced in Theorem 20, on page 11.

More specifically, still for all $m \in \mathbb{N}^\star$ sufficiently large, the function $\tilde{\zeta}^e_m$ is well defined and meromorphic in all of $\mathbb{C}$. Furthermore, its (necessarily unique) meromorphic extension (still denoted by $\tilde{\zeta}^e_m$) is given, for all $s \in \mathbb{C}$ by the expressions given in the last two equalities of relation (12) just above.

Moreover, the associated sequence $\left(\tilde{\zeta}^e_m\right)_{m\in\mathbb{N}}$ *– initially given (for* $\mathcal{R}e(s) > D_{\mathcal{W}}$*) by the truncated Mellin transform in relation (11), on page 20 – satisfies the following recurrence relation, for all values of the positive integer m sufficiently large, and for all* $s \in \mathbb{C}$*,*

$$\begin{aligned}\tilde{\zeta}^e_{m+1}(s) = \varepsilon\,\tilde{\zeta}^e_m(s) + \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b)\, \frac{c_{m+1,(N_b-1)\,j+q,m+1}\, \varepsilon^{s-2+(m+1)\,(2-D_{\mathcal{W}})+i\,\ell_{m+1,(N_b-1)\,j+q,m+1}\,\mathbf{p}}}{s+2\,(m+1-1)-(m+1)\,D_{\mathcal{W}}+i\,\ell_{m+1,(N_b-1)\,j+q,m+1}\,\mathbf{p}} \\ +\varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)\,j+q,m+1}\, \varepsilon^{s-2+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}}}{s+2\,(k-1)-k\,D_{\mathcal{W}}+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}}\,.\end{aligned} \tag{13}$$

This ensures the existence of the limit fractal zeta function $\tilde{\zeta}^e_{\mathcal{W}}$*, i.e., the fractal zeta function associated with the Weierstrass Curve* $\Gamma_{\mathcal{W}}$ *(or, rather, with the Weierstrass IFD), called the* global polyhedral effective zeta function *and given by*

$$\tilde{\zeta}^e_{\mathcal{W}} = \lim_{m\to\infty} \tilde{\zeta}^e_m\,,$$

where the convergence is locally uniform on $\mathbb{C}$*, along with the existence of an integer* $m_0 \in \mathbb{N}$ *such that, for all* $m \geq m_0$*, the set of poles of* $\tilde{\zeta}^e_{\mathcal{W}}$ *consists of simple poles and contains the poles of the* m^{th} *fractal effective polyhedral zeta function* $\tilde{\zeta}^e_m$*. More specifically,* $\tilde{\zeta}^e_{\mathcal{W}}$ *is meromorphic in all of* $\mathbb{C}$ *and its meromorphic extension (still denoted* $\tilde{\zeta}^e_{\mathcal{W}}$*) is given, for all* $s \in \mathbb{C}$*, by*

$$\begin{aligned}\tilde{\zeta}^e_{\mathcal{W}}(s) = & \sum_{m=m_0}^{\infty} \varepsilon^{m-m_0+1}\,\tilde{\zeta}^e_m(s) \\ & + \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b)\, \frac{c_{m+1,(N_b-1)\,j+q,m+1}\, \varepsilon^{s-2+(m+1)\,(2-D_{\mathcal{W}})+i\,\ell_{m+1,(N_b-1)\,j+q,m+1}\,\mathbf{p}}}{s+2\,m-(m+1)\,D_{\mathcal{W}}+i\,\ell_{m+1,(N_b-1)\,j+q,m+1}\,\mathbf{p}} \\ & + \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)\,j+q,m+1}\, \varepsilon^{s-2+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}}}{s+2\,(k-1)-k\,D_{\mathcal{W}}+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}}\,.\end{aligned} \tag{14}$$

As was mentioned in the introduction, we note that our result is stronger than the one previously obtained in [5], where, in particular, the values of the possible Complex Dimensions of the Weierstrass IFD included -2, 0 *and* $1-2\,k$*, with* $k \in \mathbb{N}$ *arbitrary. As we can see in relation (14) just above, the poles of the limit effective fractal zeta function* $\tilde{\zeta}^e_{\mathcal{W}}$ *are exactly the same as the Complex Dimensions of the Weierstrass function itself; see Theorem 20, on page 11. Note that, in [5], the Complex Dimensions are defined in terms of the volume of the tubular (rather than polyhedral) neighborhood of the Weierstrass IFD.*

Sketch of the proof.

i. We first give the explicit expression for the m^{th} *local effective polyhedral zeta function* $\tilde{\zeta}^e_m$.

We restrict ourselves to sufficienly large values of $m \in \mathbb{N}$, i.e., $m \geq m_0$, for some suitable integer $m_0 \in \mathbb{N}$.

We then have that, for $\mathcal{R}e(s) > D_{\mathcal{W}}$,

$$\begin{aligned}
\tilde{\zeta}^e_m(s) &= \int_0^\varepsilon t^{s-3}\, \tilde{\mathcal{V}}_{m,comp}(t)\, dt \quad (\text{for } \mathcal{R}e(s) > D_{\mathcal{W}}) \\
&= \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m} \frac{c_{k,(N_b-1)\,j+q,m}\, \varepsilon^{s-2+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s+2\,(k-1)-k\,D_{\mathcal{W}}+i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}\,.
\end{aligned}$$

We then have that, for any $m \in \mathbb{N}$,

$$\begin{aligned}
\tilde{\zeta}^e_{m+1}(s) &= \varepsilon^{m+1} \sum_{j=0}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)\,j+q,m+1}\, \varepsilon^{s-2+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}}}{s+2\,(k-1)-k\,D_{\mathcal{W}}+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}} \\
&= \varepsilon\,\tilde{\zeta}^e_m(s) + \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b)\, \frac{c_{m+1,(N_b-1)\,j+q,m+1}\, \varepsilon^{s-2+(m+1)\,(2-D_{\mathcal{W}})+i\,\ell_{m+1,(N_b-1)\,j+q,m+1}\,\mathbf{p}}}{s+2\,m-(m+1)\,D_{\mathcal{W}}+i\,\ell_{m+1,(N_b-1)\,j+q,m+1}\,\mathbf{p}} \\
&\quad + \varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)\,j+q,m+1}\, \varepsilon^{s-2+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}}}{s+2\,(k-1)-k\,D_{\mathcal{W}}+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}}\,.
\end{aligned} \tag{15}$$

ii. For any integer $m \geq m_0$, we denote by $\mathcal{P}\left(\tilde{\zeta}^e_m\right) \subset \mathbb{C}$ the set of poles of the zeta function $\tilde{\zeta}^e_m$.

We set $\mathcal{U} = \{s \in \mathbb{C},\, 1 < \mathcal{R}e(s) < 3\}$. We note that, for all $m \geq m_0$, we have that

$$\mathcal{P}\left(\tilde{\zeta}^e_{m_0}\right) \subset \mathcal{P}\left(\tilde{\zeta}^e_m\right) \subset \mathcal{U}\,.$$

The series

$$\begin{aligned}
&\sum_{m=m_0}^{\infty} \varepsilon^{m-m_0+1}\, \tilde{\zeta}^e_m(s) \\
&+ \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b)\, \frac{c_{m+1,(N_b-1)\,j+q,m+1}\, \varepsilon^{s-2+(m+1)\,(2-D_{\mathcal{W}})+i\,\ell_{m+1,(N_b-1)\,j+q,m+1}\,\mathbf{p}}}{s+2\,m-(m+1)\,D_{\mathcal{W}}+i\,\ell_{m+1,(N_b-1)\,j+q,m+1}\,\mathbf{p}} \\
&+ \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)\,j+q,m+1}\, \varepsilon^{s-2+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}}}{s+2\,(k-1)-k\,D_{\mathcal{W}}+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}}
\end{aligned} \tag{16}$$

is (locally) normally convergent, and, hence, uniformly convergent on $\mathcal{U}$. This ensures the existence of the limit effective global fractal zeta function, i.e., the fractal zeta function associated with the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (or with the Weierstrass IFD), given by

$$
\begin{aligned}
\tilde{\zeta}^e_{\mathcal{W}}(s) &= \lim_{m\to\infty} \tilde{\zeta}^e_m(s) \\
&= \sum_{m=m_0}^{\infty} \varepsilon^{m-m_0+1}\, \tilde{\zeta}^e_m(s) \\
&+ \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \alpha_q(N_b)\, \frac{c_{m+1,(N_b-1)\,j+q,m+1}\, \varepsilon^{s-2+(m+1)(2-D_{\mathcal{W}})+i\,\ell_{m+1,(N_b-1)\,j+q,m+1}\,\mathbf{p}}}{s+2\,m-(m+1)\,D_{\mathcal{W}}+i\,\ell_{m+1,(N_b-1)\,j+q,m+1}\,\mathbf{p}} \\
&+ \sum_{m=m_0}^{\infty} \varepsilon^{m+1} \sum_{j=N_b^m}^{N_b^{m+1}-1} \sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=0}^{m+1} \frac{c_{k,(N_b-1)\,j+q,m+1}\, \varepsilon^{s-2+k(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}}}{s+2\,(k-1)-k\,D_{\mathcal{W}}+i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}}\,.
\end{aligned}
$$

Here, and in the remainder of this proof, a (complex–valued) meromorphic function f is viewed as a continuous function with values in $\mathbb{P}^1(\mathbb{C})$, equipped with the chordal metric, and such that, for any pole ω of f, $f(\omega)$ takes the value ∞ (for instance, as in [19], Section 3.4 and Appendix C).

More precisely, if $\mathbb{P}^1(\mathbb{C}) = \mathbb{C}\cup\infty$ denotes the Riemann sphere (or complex projective line), we can show that, for *the chordal metric*, defined, for all $(z_1,z_2)\in\left(\mathbb{P}^1(\mathbb{C})\right)^2$ by

$$
||z_1,z_2|| = \frac{|z_1-z_2|}{\sqrt{1+|z_1^2|}\,\sqrt{1+|z_2^2|}}\,,\ \text{if } z_1\neq\infty \text{ and } z_2\neq\infty
$$

and

$$
||z_1,\infty|| = \frac{1}{\sqrt{1+|z_1^2|}}\,,\ \text{if } z_1\neq\infty\,,
$$

we have, thanks to the local uniform convergence of the series on $\mathbb{C}$,

$$
\lim_{m\to\infty} \left|\left|\tilde{\zeta}^e_m,\tilde{\zeta}^e_{\mathcal{W}}\right|\right| = 0\,.
$$

Indeed, for any $\eta>0$ and any compact set $\mathcal{K}\subset\mathbb{C}$, we can choose $m_0\in\mathbb{N}^\star$ such that, for all $m\geq m_0$ and $s\in\mathcal{K}$, we have that

$$
\left|\tilde{\zeta}^e_m(s)-\tilde{\zeta}^e_{\mathcal{W}}(s)\right|\leq\eta\,,
$$

and hence, still for all $m\geq m_0$ and $s\in\mathcal{K}$,

$$
\left|\left|\tilde{\zeta}^e_m(s),\tilde{\zeta}^e_{\mathcal{W}}(s)\right|\right|\leq\left|\tilde{\zeta}^e_m(s)-\tilde{\zeta}^e_{\mathcal{W}}(s)\right|\leq\eta\,.
$$

The sum of this series, i.e., the (uniform) limit fractal zeta function $\tilde{\zeta}^e_{\mathcal{W}}$, is holomorphic on $\mathcal{U}$. We can then deduce that, for all $m\geq m_0$, the zeta function $\tilde{\zeta}^e_m$ is meromorphic on $\mathbb{C}$.

Moreover, the counterpart of the results obtained in [5] for the sequence of tube zeta functions associated with the Weierstrass IFD, which admit a meromorphic continuation to all of $\mathbb{C}$, obviously hold for the sequence of polyhedral tube zeta functions: hence, $\tilde{\zeta}^e_m$ is meromorphic on $\mathbb{C}$, with only simple poles, as specified in Theorem 33, on page 24.

In the next new result, which is now an immediate corollary of Theorem 32, on page 20, we define implicitly the Complex Dimensions of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (or, rather, of the associated Weierstrass IFD) as the poles of the polyhedral global effective zeta function $\tilde{\zeta}^e_{\mathcal{W}}$. The phrase "possible Complex Dimension" then refers to $\omega \in \mathbb{C}$ which is either an actual Complex Dimension or appears as an ω-exponent in the fractal power series expansion (13), on page 21, but is such that the corresponding residue vanishes: $\operatorname{res}\left(\tilde{\zeta}^e_{\mathcal{W}},\omega\right)=0$.

Theorem 33 (Complex Dimensions of the Weierstrass Curve). *The possible Complex Dimensions of the Weierstrass Curve (or of the Weierstrass IFD) are given by Theorem 32, on page 20. Furthermore, if they are actual poles, they are simple poles. They are given as follows:*

$$D_{\mathcal{W}} - k\,(2-D_{\mathcal{W}}) + i\,\ell\,\mathbf{p}\,, \tag{17}$$

where the integers $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ are arbitrary (compare the corresponding result in Theorem 20, on page 11).

We note that they only partly coincide with the possible poles of the tube fractal zeta function $\tilde{\zeta}^{e,tube}_m$. In particular, we then deduce that the following possible Complex Dimensions previously obtained in [5], i.e., $1-2\,k+i\,\ell\,\mathbf{p}$, with $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$, along with -2 and 0, are not actual Complex Dimensions, in the sense introduced in the present paper.

Moreover, the Complex Dimensions associated with $D_{\mathcal{W}}$, i.e.,

$$D_{\mathcal{W}} - k\,(2-D_{\mathcal{W}}) + i\,\ell\,\mathbf{p}\,,$$

where $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ are arbitrary, are actual *Complex Dimensions of $\Gamma_{\mathcal{W}}$. Hence, all of the possible Complex Dimensions of $\Gamma_{\mathcal{W}}$, as given by relation (17) above, are exact and they are all simple.*

For the exceptional cases, we refer to [5] for a closely related discussion.

Remark 34. Note that, in Theorem 33 just above, the fact that – in light of relation (32) in Theorem 12, on page 20, and of the expression of the coefficients α_q and $c_{k,jm}$ given, respectively, in relation (8) in Theorem 28, on page 16 and in part ii. of Theorem 20, on page 11 – the possible Complex Dimensions are all actual Complex Dimensions of the Weierstrass IFD follows from the fact that the coefficients of the fractal power series for $\tilde{\zeta}^e_{\mathcal{W}}$ in relation (14) are nonzero, which implies that the residues of $\tilde{\zeta}^e_{\mathcal{W}}$ at each possible pole is nonzero.

In the theory of Complex Dimensions (see, e.g., [19], [24], [17]), a geometric object is said to be *fractal* if it admits at least one *nonreal* Complex Dimension (defined as a pole of the associated geometric or fractal zeta function.) [In the present context, the fractal zeta function is the global polyhedral effective zeta function in the case of the Weierstrass Curve $\Gamma_{\mathcal{W}}$. (resp., m^{th} local polyhedral effective zeta function of its m^{th} prefractal approximation, with $m \geq m_0$).]

In addition, given $d \in \mathbb{R}$, it is fractal in dimension d if it has at least one *nonreal* Complex Dimension with real part d. (Note that nonreal Complex Dimensions come in complex conjugate pairs.) In particular, it is *principally fractal* if it is fractal in dimension d (here, $D_{\mathcal{W}}$), the abscissa of convergence of the associated fractal zeta function, which is the largest possible value of d.

We can now state the following corollary of Theorems 32, on page 20 and 33, on page 24.

Corollary 35 (Fractality of the The Weierstrass Curve and of the Prefractal Approximations).

The Weierstrass Curve $\Gamma_{\mathcal{W}}$ (or the Weierstrass IFD) is fractal, and even principally fractal, as well as fractal in infinitely many Complex Dimensions d – namely, $d = D_{\mathcal{W}} - k\,(2-D_{\mathcal{W}})$, with $k \in \mathbb{N}$

arbitrary).

The same is true for the m^{th} prefractal approximation $\Gamma_{\mathcal{W}_m}$ to the Weierstrass Curve $\Gamma_{\mathcal{W}}$, with $m \in \mathbb{N}^\star$ sufficiently large, except for the fact that $\Gamma_{\mathcal{W}_m}$ is only fractal in finitely many dimensions – namely, $d = D_{\mathcal{W}} - k\,(2 - D_{\mathcal{W}})$, with $k \in \mathbb{N}$ such that $0 \leq k \leq m$. (Indeed, the Complex Dimensions of $\Gamma_{\mathcal{W}_m}$ are given by

$$d = D_{\mathcal{W}} - k\,(2 - D_{\mathcal{W}}) + i\,\ell\,\mathbf{p}\,,$$

where $k \in \mathbb{N}$, $0 \leq k \leq m$ and $\ell \in \mathbb{Z}$ is arbitrary.)

5. Revisiting the Computation of the Minkowski Dimension

Along the lines of our polyhedral fractal zeta functions, we hereafter revisit the computation of the box-counting (or box) dimension $D_{\mathcal{W}}$ of the Weierstrass Curve $\Gamma_{\mathcal{W}}$. Contrary to (classical) covers of the fractal under study by means of balls (in dimension 2, disks or squares; see Definition 36, on page 25 below, we simply use our polyhedral neighborhood (see Definition 23, on page 12), which is undoubtedly the most natural cover of $\Gamma_{\mathcal{W}}$.

We therefore announce a fully rigorous proof of the fact that the Minkowski dimension – which, as is well known, coincides with the box (or box–counting) dimension $D_{\mathcal{W}}$ of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ – is given by the expected formula, originally conjectured by Benoît Mandelbrot in [26] (in Chapter XI, on top of page 390). Note that B. Mandelbrot only mentioned he was talking about the *fractal dimension* and did not specify wether it was the Hausdorff or the Minkowski dimension of the curve. In our present context (see Theorem 37, on page 26 below), we have that $D_{\mathcal{W}} = 2 - \dfrac{\ln \frac{1}{\lambda}}{\ln N_b}$.

Note that earlier proofs of this fact were either not fully rigorous or complete. In addition, they rely on unexplicit estimates, whereas we work with explicit estimates that had not been obtained before, in relation with he local Hölder and reverse Hölder continuity of the Weierstrass function $\mathcal{W}$.

Definition 36 (Box–Counting Dimension). *As can be found, for instance, in [9], we recall that (when it exists) the* box–counting dimension *(or* box dimension, *in short), of $\Gamma_{\mathcal{W}}$, is given by*

$$D_{\mathcal{W}} = -\lim_{\delta \to 0^+} \frac{\ln N_\delta\,(\Gamma_{\mathcal{W}})}{\ln \delta}\,, \quad (\diamond)$$

where $N_\delta\,(\Gamma_{\mathcal{W}})$ stands for any of the following quantities:

i. *the smallest number of sets (here, subsets of $\mathbb{R}^2$) of diameter at most δ that cover $\Gamma_{\mathcal{W}}$ on $[0,1[$;*
ii. *the smallest number of closed balls (disks, here) of radius δ that cover $\Gamma_{\mathcal{W}}$ on $[0,1[$;*
iii. *the smallest number of cubes (squares, here) of side δ that cover $\Gamma_{\mathcal{W}}$ on $[0,1[$;*
iv. *the number of δ–mesh cubes (squares, here) that intersect $\Gamma_{\mathcal{W}}$ on $[0,1[$;*
v. *the largest number of disjoint balls (disks, here) of radius δ with centers in $\Gamma_{\mathcal{W}}$ on $[0,1[$.*

Furthermore, for the Weierstrass Curve $\Gamma_{\mathcal{W}}$ – as, more generally, for any bounded subset of Euclidean space – the box–counting dimension coincides with the Minkowski dimension.

Theorem 37. *The box–counting dimension – or, equivalently, the Minkowski dimension – of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ exists and is equal to $D_{\mathcal{W}} = 2 + \dfrac{\ln \lambda}{\ln N_b}$.*

Sketch of the proof.

We simply apply the result given in Corollary 25, on page 14. We then have the existence of an integer $m_0 \in \mathbb{N}$ such that

$$\forall\, m \geq m_0 : \quad \Gamma_{\mathcal{W}} \subset \mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right).$$

In other words, for all integers $m \geq m_0$, the polyhedral neighborhood $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ covers the Weierstrass Curve $\Gamma_{\mathcal{W}}$.

Note that we thus dispose, with the sequence of polyhedral domains $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\geq m_0}$, of a nonusual (but admissible, in the sense of Definition 36, on page 25 above) sequence of covers. However, $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\geq m_0}$ is the most natural and optimal sequence of covers of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, insofar that for any $m \geq m_0$, each domain $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ contains the Weierstrass Curve $\Gamma_{\mathcal{W}}$, while, at the same time, the sequence $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\geq m_0}$ converges to $\Gamma_{\mathcal{W}}$. This, in particular, means that when $m \to \infty$, the two–dimensional Lebesgue measure (i.e., area) of each polygon belonging to $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ tends to 0.

In our context, the cohomology infinitesimal ε_m^m plays the role of the elementary diameter δ in Definition 36, on page 25. Moreover, the area of each polygon belonging to $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ also corresponds to the area of the reunion of $\dfrac{(N_b - 1)\, h_m}{\varepsilon_m^m}$ elementary and smaller polygons, each of diameter ε_m^m, where *the diameter* of a polygon is to be understood in the sense of the largest distance between any pair of vertices of the considered polygon.

References

[1] K. Barańsky, B. Bárány, J. Romanowska, "On the dimension of the graph of the classical Weierstrass function", *Advances in Mathematics* **265** (2014), p. 791-800.

[2] C. David, "Bypassing dynamical systems: A simple way to get the box-counting dimension of the graph of the Weierstrass function", *Proceedings of the International Geometry Center* **11** (2018), no. 2, p. 1-16, https://journals.onaft.edu.ua/index.php/geometry/article/view/1028.

[3] ———, "On fractal properties of Weierstrass-type functions", *Proceedings of the International Geometry Center* **12** (2019), no. 2, p. 43-61, https://journals.onaft.edu.ua/index.php/geometry/article/view/1485.

[4] C. David, M. L. Lapidus, "Iterated fractal drums ~ Some new perspectives: Polyhedral measures, atomic decompositions and Morse theory", 2022, https://hal.sorbonne-universite.fr/hal-03946104v2.

[5] ———, "Weierstrass fractal drums - I - A glimpse of complex dimensions", 2022, https://hal.sorbonne-universite.fr/hal-03642326.

[6] ———, "Weierstrass fractal drums - II - Towards a fractal cohomology", 2022, https://hal.archives-ouvertes.fr/hal-03758820v3.

[7] ———, "New Insights for fractal zeta functions : Polyhedral neighborhoods vs tubular neighborhoods:", 2023, https://hal.science/hal-04153049.

[8] ———, "Polyhedral neighborhoods vs tubular neighborhoods: New insights for fractal zeta functions and complex dimensions", 2023, https://hal.science/hal-04153049.

[9] K. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1986.

[10] G. H. Hardy, "Weierstrass's Non-Differentiable Function", *Transactions of the American Mathematical Society* **17** (1916), no. 3, p. 301-325, https://www.ams.org/journals/tran/1916-017-03/S0002-9947-1916-1501044-1/S0002-9947-1916-1501044-1.pdf.

[11] H. Herichi, M. L. Lapidus, *Quantized Number Theory, Fractal Strings and the Riemann Hypothesis: From Spectral Operators to Phase Transitions and Universality*, World Scientific Publishing, Singapore and London, 2021.

[12] J. L. Kaplan, J. Mallet-Paret, J. A. Yorke, "The Lyapunov dimension of a nowhere differentiable attracting torus", *Ergodic Theory and Dynamical Systems* **4** (1984), p. 261-281.

[13] G. Keller, "A simpler proof for the dimension of the graph of the classical Weierstrass function", *Annales de l'Institut Henri Poincaré – Probabilités et Statistiques* **53** (2017), no. 1, p. 169-181.

[14] M. L. Lapidus, "Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture", *Transactions of the American Mathematical Society* **325** (1991), p. 465-529.

[15] ———, "Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media and the Weyl-Berry conjecture", in *Ordinary and Partial Differential Equations, Vol. IV (Dundee, 1992)*, Pitman Research Notes Mathematical Series, vol. 289, Longman Sci. Tech., Harlow, 1993, p. 126-209.

[16] M. L. Lapidus, *In Search of the Riemann Zeros: Strings, Fractal Membranes and Noncommutative Spacetimes*, American Mathematical Society, Providence, RI, 2008, xxx+558 pages.

[17] ———, "An overview of complex fractal dimensions: From fractal strings to fractal drums, and back", in *Horizons of Fractal Geometry and Complex Dimensions* (R. G. Niemeyer, E. P. J. Pearse, J. A. Rock and T. Samuel, eds.), Contemporary Mathematics, vol. 731, American Mathematical Society, Providence, RI, 2019, p. 143-265, https://arxiv.org/abs/1803.10399.

[18] ———, *From Complex Fractal Dimensions and Quantized Number Theory To Fractal Cohomology: A Tale of Oscillations, Unreality and Fractality*, World Scientific Publishing, Singapore and London, 2024.

[19] M. L. Lapidus, M. van Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*, Springer Monographs in Mathematics, Springer, New York, second revised and enlarged edition (of the 2006 edition), 2013, xxvi+567 pages.

[20] M. L. Lapidus, H. Maier, "The Riemann hypothesis and inverse spectral problems for fractal strings", *Journal of the London Mathematical Society. Second Series* **52** (1995), no. 1, p. 15-34.

[21] M. L. Lapidus, E. P. J. Pearse, "A tube formula for the Koch snowflake curve, with applications to complex dimensions", *Journal of the London Mathematical Society. Second Series* **74** (2006), no. 2, p. 397-414.

[22] M. L. Lapidus, C. Pomerance, "The Riemann zeta-function and the one-dimensional Weyl-Berry conjecture for fractal drums", *Proceedings of the London Mathematical Society. Third Series* **66** (1993), no. 1, p. 41-69.

[23] M. L. Lapidus, G. Radunović, D. Žubrinić, "Distance and tube zeta functions of fractals and arbitrary compact sets", *Advances in Mathematics* **307** (2017), p. 1215-1267.

[24] ———, *Fractal Zeta Functions and Fractal Drums: Higher-Dimensional Theory of Complex Dimensions*, Springer Monographs in Mathematics, Springer, New York, 2017, xl+655 pages.

[25] ———, "Fractal tube formulas for compact sets and relative fractal drums: Oscillations, complex dimensions and fractality", *Journal of Fractal Geometry* **5** (2018), no. 1, p. 1-119.

[26] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman & Co Ltd, San Francisco, 1982.

[27] W. Shen, "Hausdorff dimension of the graphs of the classical Weierstrass functions", *Mathematische Zeitschrift* **289** (2018), p. 223-266.

[28] K. Weierstrass, "Über continuirliche Funktionen eines reellen Arguments, die für keinen Werth des letzteren einen bestimmten Differential quotienten besitzen", *Journal fr die reine und angewandte Mathematik* **79** (1875), p. 29-31.