Understanding Fractality: A Polyhedral Approach to the Koch Curve and its Complex Dimensions

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Abstract

We extend our results about the Weierstrass Curve to the Koch Curve and provide exact expressions of the volume of *polyhedral neighborhoods* for the sequence of prefractal graphs which converge to the Koch Curve. We also introduce the associated *local* and *global polyhedral fractal zeta functions*. The actual poles of the global polyhedral fractal zeta function, which are all simple, yield the set of exact Complex Dimensions of the Koch Curve, a result which had never been obtained before.

MSC Classification: 11M41, 28A12, 28A75, 28A80.

Keywords: Koch Curve, prefractal approximations, iterated fractal drum (IFD), Complex Dimensions of an IFD, box–counting (or Minkowski) dimension, fractal tube formula, effective local and global tube zeta function, effective local and global distance zeta function.

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Contents

1 Introduction

" Une courbe sans tangente où l'apparence géométrique fut *en accord* avec le fait dont il s'agit."

" A curve without any tangent where the geometric appearance would be *in agreement* with the fact in question."

Helge Von Koch [vK04]

The pathological, everywhere continuous, while nowhere differentiable function introduced by Karl Weierstrass [KH16], [Wei75], has struck the mind and the imagination of many mathematicians. And even the numerous examples and variants that later emerged – from the work of very famous mathematicians such as Jean Gaston Darboux [Dar75], [Dar79] and Ulisse Dini [Din77], [Din78], to name just a few – visibly did not seem satisfactory from the geometer's point of view of Nils Fabian Helge von Koch. This might explain the introduction, in 1904, of the so-called Koch curve [vK04].

The initial idea seems very simple: take the unit line segment, divide it intro three equal parts; then, replace the middle segment by the other two sides of an equilateral triangle. You then have a total of four line segments, to which you apply the same process: divide each segment into three equal parts, and replace the middle segment by the other two sides of an equilateral triangle; at each step, the sidelength of each of the line segments is divided by 3. And you iterate endlessly and infinitely ... . So that, in the end, you obtain a nowhere differentiable curve! Unlike in the case of the Weierstrass Curve, or of the Darboux Curve, or else of the Dini Curve, the construction process is purely geometric. You do not have to rely on any complicated analytic expression – " une expression analytique qui cache la nature géométrique de la courbe correspondante, de sorte qu'on ne voit pas, en se plaçant à ce point de vue, pourquoi la courbe n'a pas de tangente" ("an analytic expression which hides the geometric nature of the corresponding curve, in such a way that one does not see, from this point of view, why the curve does not have any tangent"), as was explained by Helge von Koch in [vK04], at a time when many mathematicians *écrivaient en français* – wrote in French.

Since the tangent to a geometric curve is given by the limiting position of the *secant*, one easily understands why this curve does not – and cannot, have any tangent.

Things could have remained at that stage without Benoît Mandelbrot. In [Man77], [Man83], B. Mandelbrot highlighted the fractal nature of the Koch Curve. He also described the resulting variants which can be obtained by slightly changing the geometric process in the construction of the curve: for instance, the rectangular Koch curve is obtained by dividing the initial line segment into four equal parts – instead of three; then, the two middle segments are replaced by the three other sides of a square, in the construction of the curve, so that the resulting squares are opposite to each other on either side of the initial line. And so on . . . .

Certainly thanks to Mandelbrot, the fractal Koch Curve became increasingly popular among scientists, while also gaining interest in the mathematical community. Analysis was back: how could one define function spaces in the case of this everywhere singular boundary? Those questions were particularly studied by Alf Jonsson and Hans Wallin in [JW84], [Wal91], for instance, who obtained trace theorems which are still very useful (see, for example, our work on the Weierstrass Curve [DL24b]).

Nearly at the same time, physicists came accross anomalous diffusion, and unusual vibration modes in disordered media; see the works of Rammal Rammal and Gérard Toulouse in [RT83], Samuel H. Liu in [Liu86], or of Shlomo Havlin and Daniel Ben–Avraham [HBA87], [bA91]. As is explained in, [bA91], “ the Koch curve can serve as a model for a linear polymer chain.” It goes without saying that, as a result, the Koch Curve gained in popularity; more and more people began to work on the subject, a phenomenon which went hand in hand with an exponential increase in the number of published research articles.

In this ocean of papers, our interest focuses on the very specific vibrational properties of the Koch Curve. As is pointed out in [LP95], Koch–like, fractal shaped coastlines, exhibit very powerful damping properties – as if fractal-shaped cells captured sound waves. This phenomena was experimentally verified by Bernard Sapoval and his collaborators; see, for instance, [Sap89], [SGM91], [SG93], [HS98]. It was later rigorously established in [LP95]. Further results along these lines have since been obtained by Nizar Riane in [Ria22].

From a theoretical point of view, our intuition is that those properties are directly connected to the Complex Dimensions of the Koch Curve. Recall that the theory of Complex Dimensions, developed for many years now by Michel L. Lapidus and his collaborators in [Lap91], [Lap93], [LP93], [LM95], [LP06], [Lap08], [LvF13], [LRŽ17a], [LRŽ17b], [Lap19], [HL21], [Lap24], makes the connection between the geometry of an object and its differentiability properties, by means of *geometric (or fractal) zeta functions*, which stand for the trace of a differential operator at a complex order s. The poles of those fractal zeta functions are called *the Fractal Complex Dimensions*. The existence of *nonreal Complex Dimensions* is a characteristic of fractality and gives rise (via explicit formulas) to *the oscillations* that are intrinsic to fractal geometries. However, in the case of the Koch Curve, the determination of the associated Complex Dimensions still remains an open problem. In [LP06], preliminary results give the approximate expression for the tubular volume, but one cannot deduce the values of the exact Complex Dimensions.

In [DL23b] (announced in [DL23a]), and building upon [DL22], [DL24c], [DL24b], we introduced new fractal zeta functions, associated with *polyhedral neighborhoods*, better suited to fractals when exact expressions of tubular neighborhoods cannot be computed (see also [DL24a]). In the model – and significant – case of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, we therefore gave exact expressions of the volume of *polyhedral neighborhoods* for the sequence of prefractal graphs which converge to $\Gamma_{\mathcal{W}}$ – the so-called *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs). Those IFDs are associated with a suitable (and geometrically meaningful) sequence of small parameters tending to zero, also known as *the cohomology infinitesimal*, due to their connections with fractal cohomology [DL24c]. It happens that the associated *local* fractal (or polyhedral) zeta functions, which consist in the sequence of zeta functions associated with the sequence of polyhedral neighborhoods, satisfy a recurrence relation, which enables us to obtain the exact values of the poles of the limit fractal zeta function – called the *global* (polyhedral) zeta function, associated with the limit fractal object – and hence, to determine the exact (intrinsic) Complex Dimensions of the Weierstrass Curve.

We hereafter extend those results to the Koch Curve, thereby pursuing, in a completely different and new way, the seminal work of E. P. J. Pearse and the second author in [LP06] (well before the development in [LRŽ17b] of the higher-dimensional theory of Complex Dimensions), where they obtained a corresponding (approximate) fractal tube formula for the Koch Curve. (We note that by using the tube zeta functions introduced in [LRŽ17b] and [LRŽ17a], one can nowadays deduce from the results of [LP06] the *possible* Complex Dimensions of the Koch Curve, in the sense of [LRŽ17b], and as given in Remark 3.3, on page 21 below.) For this purpose, given a sufficiently small positive parameter $\varepsilon \to 0^+$, they determined an approximation of the inner ε-neighborhood of the Koch Curve, consisting in rectangles, wedges and triangles. Due to the specific geometry of the Koch Curve, which involves what can be considered as *a fringe* – a region slightly larger than the exact ε-neighborhood had to be considered. Hence, it was not possible to obtain the exact value of the associated two-dimensional area. However, the authors gave very precise estimates for the corresponding error. Note that when this work was carried out, the use of tubular neighborhoods was a compulsory and unavoidable step when it came to computing fractal zeta functions (see [LvF13], [LRŽ17b], [LRŽ18]). They were thus the very first to provide values for *the possible* Complex Dimensions of the Koch Curve; see the above comment. As for the values of *the actual* Complex Dimensions of the Koch Curve, it has remained, up to now, an open problem to determine them precisely. As a matter of fact, we are now able to shed a new light on the values obtained by E. P. J. Pearse and the second author in [LP06] for the possible Complex Dimensions associated with the Minkowski dimension $D_{\mathfrak{KC}}$ of the Koch Curve $\mathfrak{KC}$. Our results go even further, since we obtain the full set of (actual) Complex Dimensions of the Koch Curve, as well as of its prefractal approximations.

In order to establish these results, we rely on the same method as the one we used in the case of the Weierstrass Curve [DL23b], [DL23a] – where things were at the same time more difficult – due to the very complicated geometry, and the nonlinear IFS involved – and easier, since we had to our disposal the Weierstrass function, which possesses an exact expansion expressed in terms of the cohomology infinitesimal and of the Complex Dimensions. More precisely, once we had determined the sequence of polyhedral neighborhoods, it was then possible to obtain the expression of the Lebesgue measure (i.e., the area) of each polyhedral neighborhood. Nothing like that can be done for the present case of the Koch Curve, since the only available analytic expression – the one given by Hans Sagan in [Sag94], does not (explicitly at least) involve any *Complex Dimensions*-like complex numbers.

More specifically, we prove in Theorem 3.5, on page 20, that the set of intrinsic Complex Dimensions is

$$\{D_{\mathfrak{KC}} - m\,(2 - D_{\mathfrak{KC}}) - i\,m\,\mathbf{p}\,,\, D_{\mathfrak{KC}} - m\,(2 - D_{\mathfrak{KC}}) + i\,m\,\mathbf{p}\,, \text{with } m \in \mathbb{N}\}\,, \tag{$\mathcal{R}1$}$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$, $D_{\mathfrak{KC}} = \dfrac{\ln 4}{\ln 3}$ is the Minkowski dimension of the Koch Curve $\mathfrak{KC}$ (which also coincides with the Hausdorff dimension of $\mathfrak{KC}$), while $\mathbf{p} = \dfrac{\pi}{3 \ln 3}$ is the oscillatory period of $\mathfrak{KC}$. Each Complex Dimension in relation ($\mathcal{R}1$) is simple and exact; i.e., it is a simple and actual pole of the global polyhedral effective zeta function of the Koch iterated fractal drum (Koch IFD, in short).

For this reason, we had to think of something different here. The solution is, at each step of the prefractal approximation, to consider the triangular patterns which constitute the sequence of polyhedral neighborhoods, as the images of the initial triangular pattern. This enables us to involve the Complex Codimensions, in the expression of the associated sequence of two-dimensional areas. As was already the case in our previous work regarding the Weierstrass Curve, as shown in [DL23b], [DL23a], by considering the fractal zeta functions associated with the sequence of polygonal neighborhoods, called *the local polyhedral zeta functions*, we establish the recurrence relation between consecutive fractal zeta functions. This enables us to prove the existence of the limit fractal zeta function – *the global polyhedral zeta function*, associated with the limit fractal object, as well as to deduce from it the *exact* Complex Dimensions of the Koch Curve and establish an associated fractal power series expansion.

Our main results in the present setting can be found in the following places:

i. In Definition 3.1, on page 9, where we introduce the sequence of polyhedral neighborhoods.

ii. In Theorem 3.1, on page 11, where we give, for every $m \in N$, the exact expression for the volume of the m^{th} polyhedral neighborhood.

iii. In Theorem 3.3, on page 15, where we introduce *the local and global effective polyhedral zeta function* and show that the global zeta function is well defined, meromorphic in all of $\mathbb{C}$, and is given by an explicit fractal power series, expressed in terms of the underlying intrinsic scale (or cohomology infinitesimal).

iii. In Theorem 3.5, on page 20, where we give the values of the actual Complex Dimensions of the Koch Curve $\mathfrak{KC}$ (or of the Koch IFD). See also Theorem 3.6, on page 21, providing the exact Complex Dimensions of the prefractal approximations $\mathfrak{KC}_m$ of $\mathfrak{KC}$, for all $m \geqslant m_0$, with m_0 large enough, as well as Corollary 3.7, on page 21, regarding the fractality of $\mathfrak{KC}$ and $\mathfrak{KC}_m$, again for all $m \geqslant m_0$.

2 Geometric Framework

Notation 1 (**Minkowski Dimension of the Koch Curve**).

Hereafter, we denote by $D_{\mathfrak{KC}} = \dfrac{\ln 4}{\ln 3} = \ln_3 4 \in]1,2[$ the Minkowski dimension of the Koch Curve $\mathfrak{KC}$ (which also coincides with the Hausdorff dimension of $\mathfrak{KC}$).

Remark 2.1 (**Best Hölder Exponent for the Koch Curve**).

The codimension $2 - D_{\mathfrak{KC}} = 2 - \dfrac{\ln 4}{\ln 3} \in]0,1[$ is also the optimalt Hölder exponent of the Koch Curve $\mathfrak{KC}$ (viewed as a continuous parametrized curve).

Notation 2 (**Rotation Matrix**).

For $\theta \in \mathbb{R}$, we denote by $\mathcal{R}_{O,\theta}$ the following rotation matrix,

$$\mathcal{R}_{O,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Property 2.1 (**The Koch IFS; see, e.g., [HOP92]**).

Following our previous work on the Weierstrass Curve [Dav18]*, we will approximate the Koch Curve* $\mathfrak{KC}$ *by a sequence of graphs, built via an iterative process. For this purpose, we use the linear* *iterated function system (IFS) of the family of C^∞ maps – here, contracting similitudes – from $\mathbb{R}^2$ to $\mathbb{R}^2$ denoted by*

$$\mathcal{T}^{\mathfrak{KC}} = \left\{ T_0^{\mathfrak{KC}}, \dots, T_3^{\mathfrak{KC}} \right\} ,$$

where, for any point (x,y) of $\mathbb{R}^2$,

$$T_0^{\mathfrak{KC}}(x,y) = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{3} \end{pmatrix} \quad ; \quad T_1^{\mathfrak{KC}}(x,y) = \frac{1}{3}\,\mathcal{R}_{O,\frac{\pi}{3}}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} \quad ; \quad T_2^{\mathfrak{KC}}(x,y) = \frac{1}{3}\,\mathcal{R}_{O,\frac{-\pi}{3}}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} ;$$

$$T_3^{\mathfrak{KC}}(x,y) = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{3} \end{pmatrix}$$

and where, for $\theta \in \mathbb{R}$, the rotation matrix $\mathcal{R}_{O,\theta}$ has been introduced in Notation 2, on page 5.

Note that the contracting similitudes $\left\{ T_0^{\mathfrak{KC}}, \dots, T_3^{\mathfrak{KC}} \right\}$ can also be viewed as (complex) maps from $\mathbb{C}$ to $\mathbb{C}$; namely, for all $s \in \mathbb{C}$, we have that

$$T_0^{\mathfrak{KC}}(s) = \frac{1}{3}\,e^{i\,\theta_0}\,s + s_0^{\mathfrak{KC}} \quad ; \quad T_1^{\mathfrak{KC}}(s) = \frac{1}{3}\,e^{i\,\theta_1}\,s + s_1^{\mathfrak{KC}} = \varepsilon_{\mathfrak{KC}}{}^{\frac{ip}{2}}\,s + s_1^{\mathfrak{KC}} ;$$

$$T_2^{\mathfrak{KC}}(s) = \frac{1}{3}\,e^{i\,\theta_2}\,s + s_2^{\mathfrak{KC}} ;$$

$$T_3^{\mathfrak{KC}}(s) = \frac{1}{3}\,e^{i\,\theta_3}\,s + s_3^{\mathfrak{KC}} = \varepsilon_{\mathfrak{KC}}{}^{-\frac{ip}{2}}\,s + s_3^{\mathfrak{KC}} ,$$

where

$$\theta_0 = 0 \quad ; \quad \theta_1 = -\frac{\pi}{3}; \quad \theta_2 = \frac{\pi}{3} \quad ; \quad \theta_3 = 0 ; \tag{$\mathcal{R}$ 2}$$

along with

$$s_0^{\mathfrak{KC}} = -\frac{1}{\sqrt{3}} + \frac{i}{3} \quad ; \quad s_1^{\mathfrak{KC}} = s_2^{\mathfrak{KC}} = \frac{2}{3} \quad ; \quad s_3^{\mathfrak{KC}} = \frac{1}{\sqrt{3}} + \frac{i}{3} . \tag{$\mathcal{R}$ 3}$$

The respective fixed points (in $\mathbb{C}$) of the similarities $\left\{ T_0^{\mathfrak{KC}}, \dots, T_3^{\mathfrak{KC}} \right\}$ are denoted by $\left\{ P_0^{\mathfrak{KC}}, \dots, P_3^{\mathfrak{KC}} \right\}$. Note that

$$P_0^{\mathfrak{KC}} = -\frac{\sqrt{3}}{2} + \frac{i}{2} \quad ; \quad P_1^{\mathfrak{KC}} = -\frac{\sqrt{3}}{7} + \frac{5\,i}{7} \quad ; \quad P_2^{\mathfrak{KC}} = \frac{\sqrt{3}}{7} + \frac{5\,i}{7} \quad ; \quad P_3^{\mathfrak{KC}} = \frac{\sqrt{3}}{2} + \frac{i}{2} .$$

Following standard terminology, $\mathcal{T}^{\mathfrak{KC}} = \left\{ T_0^{\mathfrak{KC}}, \dots, T_3^{\mathfrak{KC}} \right\}$ is called the iterated function system *(IFS, in short) determining the Koch Curve; see Property 2.2, on page 6.*

Property 2.2 (**Attractor of the Koch IFS**).

The Koch Curve $\mathfrak{KC}$ is the attractor of the IFS $\mathcal{T}^{\mathfrak{KC}}$: $\mathfrak{KC} = \bigcup_{i=0}^{3} T_i^{\mathfrak{KC}}\,(\mathfrak{KC})$.

Definition 2.1 (**Sets of Vertices, Prefractals**).

We define the *initial points*, respectively denoted by I and J, of respective coordinates

$$I = (0,0) \quad , \quad J = (1,0) \, .$$

We set

$$V_0 = \{I, J\} \quad \text{and} \quad \mathfrak{KC}_0 = [IJ] \, .$$

The line segment $[IJ]$ is called *the initial segment*, while V_0 is called *the set of vertices* of the initial prefractal graph $\mathfrak{KC}_0$.

For any positive integer m (i.e., for all $m \in \mathbb{N}^\star$), we set $V_m = \bigcup\limits_{i=0}^{3} T_i^{\mathfrak{KC}} (V_{m-1})$.

The set of points V_m, where two consecutive points are linked, is an oriented finite graph, denoted by $\mathfrak{KC}_m$ and called *the m^{th} order Koch prefractal* (also called *the m^{th} prefractal approximation* of $\mathfrak{KC}$). Then, V_m is called *the set of vertices* of the prefractal $\mathfrak{KC}_m$; see Figure 1, on page 7. By analogy with our work in [DL22], [DL23b], [DL23a], we call *Koch Iterated Fractal Drums (IFD)* the sequence of prefractal graphs which converge to the Koch Curve.

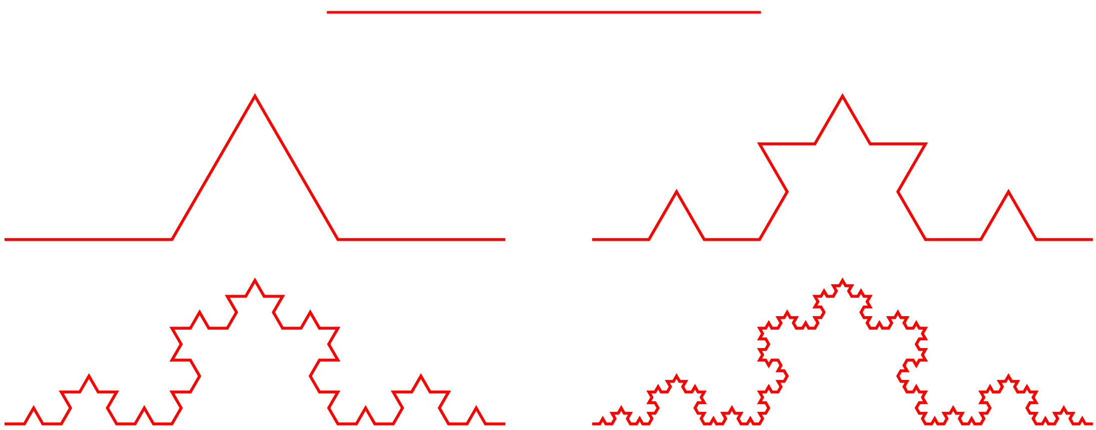

Figure 1: **The prefractal graphs $\mathfrak{KC}_0$, $\mathfrak{KC}_1$, $\mathfrak{KC}_2$, $\mathfrak{KC}_3$, $\mathfrak{KC}_4$. For example, $\mathfrak{KC}_1$ is on the left side of the second row, while $\mathfrak{KC}_4$ is on the right side of the bottom row.**

Definition 2.2 (**Adjacent Vertices, Edge Relation**).

For any $m \in \mathbb{N}$, two vertices X and Y belonging to V_m will be said to be *adjacent* (i.e., *neighboring* or *junction points*) if and only if the edge XY belongs to $\mathfrak{KC}_m$; we then write $X \underset{m}{\sim} Y$. Note that this edge relation depends on m, which means that points adjacent in V_m might not remain adjacent in V_{m+1}.

Property 2.3. *For any $m \in \mathbb{N}$, the following statements hold:*

i. $V_m \subset V_{m+1}$.

ii. *When $m \geqslant 1$, $\# V_m = 4^m + 1$, where $\# V_m$ denotes the number of elements in the finite set V_m.*

iii. *The prefractal graph $\mathfrak{KC}_m$ has exactly 4^m edges.*

iv. *The consecutive vertices of $V_{m+1} \setminus V_m$ are the vertices of 4^m equilateral triangles $\mathcal{P}^{\mathfrak{KC}}_{j,m}$, with $1 \leqslant j \leqslant 4^m$; see Figure 3, on page 8. As for $\mathcal{P}^{\mathfrak{KC}}_0 = \mathfrak{KC}_0$, it can be considered as* a flat (equilateral) triangle. The triangle $\mathcal{P}^{\mathfrak{KC}}_{1,0}$ is called *the initial triangle*; see Figure 2, on page 8.

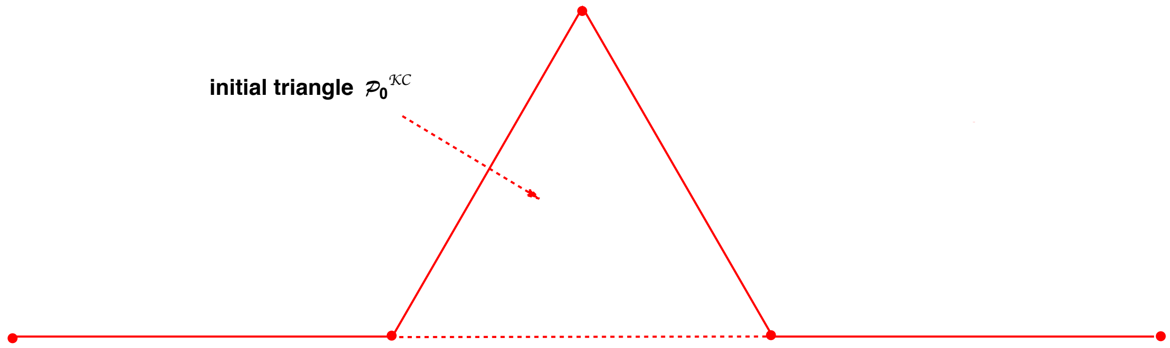

Figure 2: **The initial triangle $\mathcal{P}^{\mathfrak{KC}}_{1,0}$.**

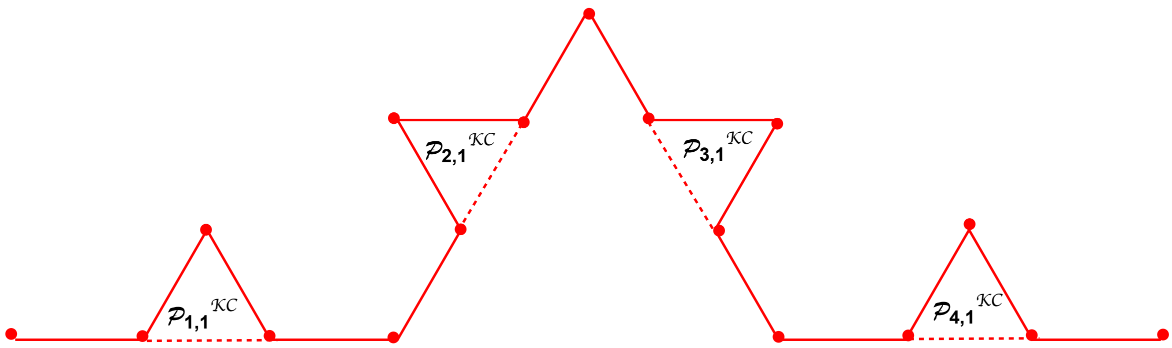

Figure 3: **The triangles $\mathcal{P}^{\mathfrak{KC}}_{1,1}$, $\mathcal{P}^{\mathfrak{KC}}_{2,1}$, $\mathcal{P}^{\mathfrak{KC}}_{3,1}$ and $\mathcal{P}^{\mathfrak{KC}}_{4,1}$.**

Proof.

i. By construction, the sequence of sets of vertices $(V_m)_{m \in \mathbb{N}}$ is an increasing sequence.

ii. The initial line segment $V_0 = [IJ]$ consists of an edge, and two points. By applying the maps $\left\{T_0^{\mathfrak{K}\mathfrak{C}}, \ldots, T_3^{\mathfrak{K}\mathfrak{C}}\right\}$ of the IFS given in Property 2.1, on page 5, we then obtain $2 + (2-1) \times 3 = 5$ points in V_1. By induction, we have that, for any integer $m \geqslant 2$,

$$\# V_m = (\# V_{m-1} - 1) \times 3 + \# V_{m-1} = 4\,(\# V_{m-1}) - 3\,.$$

The sequence $(\# V_m)_{m \in \mathbb{N}^\star}$ is an arithmetico-geometric sequence. We then have that, for any integer $m \geqslant 1$,

$$\# V_m = 4^m\,(\# V_0 - 1) + 1 = 4^m + 1\,.$$

iii. We immediately deduce from *ii*. just above that the prefractal graph $\mathfrak{K}\mathfrak{C}_m$ has exactly 4^m edges.

iv. By construction, each edge of V_m gives birth to an equilateral triangle, whose vertices belong to $V_{m+1} \setminus V_m$. This immediately ensures that the consecutive vertices of $V_{m+1} \setminus V_m$ are the vertices of 4^m equilateral triangles $\mathcal{P}_{j,m+1}^{\mathfrak{K}\mathfrak{C}}$, with $1 \leqslant j \leqslant 4^m$.

$\square$

3 Polyhedral Neighborhoods – Cohomology Infinitesimal

We first introduce the notion of polyhedral neighborhoods of the Koch Curve $\mathfrak{K}\mathfrak{C}$, which will play a key role in the remainder of this paper – much as is the case for the polyhedral neighborhoods of the Weierstrass Curve in our earlier works [DL24b], [DL23b] [DL23a].

Definition 3.1 (**Sequence of Polyhedral Neighborhoods of the Koch Curve**).

We introduce *the sequence of polygonal neighborhoods of the Koch Curve*, as the sequence $(\mathcal{D}(\mathfrak{K}\mathfrak{C}_m))_{m \in \mathbb{N}}$ of open, connected polygonal sets such that $\mathcal{D}(\mathfrak{K}\mathfrak{C}_0) = \mathcal{P}_0^{\mathfrak{K}\mathfrak{C}}$ and for each $m \in \mathbb{N}^\star$,

$$\mathcal{D}(\mathfrak{K}\mathfrak{C}_m) = \bigcup_{j=1}^{4^m} \mathcal{P}_{j,m}^{\mathfrak{K}\mathfrak{C}} = \bigcup_{k_{0,m}+k_{1,m}+k_{2,m}+k_{3,m} \leqslant m} \left(T_0^{\mathfrak{K}\mathfrak{C}}\right)^{k_{0,m}} \left(T_1^{\mathfrak{K}\mathfrak{C}}\right)^{k_{1,m}} \left(T_2^{\mathfrak{K}\mathfrak{C}}\right)^{k_{2,m}} \left(T_3^{\mathfrak{K}\mathfrak{C}}\right)^{k_{3,m}} \left(\mathcal{P}_0^{\mathfrak{K}\mathfrak{C}}\right), \quad (\mathcal{R}\,4)$$

where $\mathcal{P}_0^{\mathfrak{K}\mathfrak{C}}$ is the initial prefractal graph introduced in part *iv*. of Property 2.3, on page 7 just above, $\left\{\mathcal{P}_{j,m}^{\mathfrak{K}\mathfrak{C}},\, 1 \leqslant j \leqslant 4^m\right\}$ is the family of 4^m equilateral triangles also introduced in part *iv*. of Property 2.3, on page 7, and where

$$\left\{T_0^{\mathfrak{K}\mathfrak{C}}, \ldots, T_3^{\mathfrak{K}\mathfrak{C}}\right\} = \mathcal{T}^{\mathfrak{K}\mathfrak{C}}$$

is the Koch IFS introduced in Property 2.1, on page 5. Furthermore, here (in relation ($\mathcal{R}4$)) and thereafter, $k_{0,m}$, …, $k_{3,m}$ are arbitrary nonnegative integers satisfying the indicated inequality.

Given $m \in \mathbb{N}$, we call $\mathcal{D}(\mathfrak{K}\mathfrak{C}_m)$ the m^{th} *polyhedral neighborhood* (of the Koch Curve $\mathfrak{K}\mathfrak{C}$).

Note that, for each integer $m \in \mathbb{N}^\star$, $\mathcal{D}(\mathfrak{K}\mathfrak{C}_m)$ is then comprised of the union of 4^m open equilateral triangles, of sidelength equal to $\dfrac{1}{3^{m+1}}$; accordingly, to each edge of the prefractal graph $\mathfrak{K}\mathfrak{C}_m$ corresponds an open triangle $\mathcal{P}_{j,m}^{\mathfrak{K}\mathfrak{C}}$. Indeed, the vertices of those triangles belong to $V_{m+1} \setminus V_m$, it thus makes sense to consider them as the polyhedral neighborhood of $\mathfrak{K}\mathfrak{C}_m$. Also, it is because the vertices of those triangles belong to $V_{m+1} \setminus V_m$ that the sidelength of all equilateral triangles is equal to $\dfrac{1}{3^{m+1}}$.

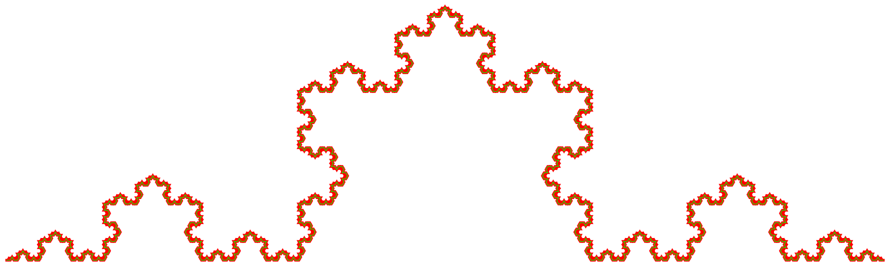

Figure 4: **The polygonal neigborhood $\mathcal{D}\left(\mathfrak{K}\mathfrak{C}_5\right)$ (in red) of the prefractal approximation $\mathfrak{K}\mathfrak{C}_5$ (in green).**

In our previous work – i.e., in the case of the Weierstrass Curve – things were at the same time more difficult – due to the very complicated geometry, and the nonlinear IFS involved – and easier, since we had to our disposal the Weierstrass function, which possesses an exact expansion in function of the cohomology infinitesimal and of the Complex Dimensions; see [DL23b] (announced in [DL23a]). More precisely, once we had determined the sequence of polyhedral neighborhoods, it was thus possible to obtain the expression of the Lebesgue measure (i.e., the area) of each polyhedral neighborhood. Nothing like that can be done with the Koch Curve, since the only available analytic expression – the one given by Hans Sagan in [Sag94], does not (explicitly at least) involve any *Complex Dimensions*–like complex numbers.

For this reason, we had to think of something different here. The solution is, at each step $m \in \mathbb{N}^\star$ of the prefractal approximation, to consider the triangular patterns $\mathcal{P}_{j,m}^{\mathfrak{K}\mathfrak{C}}$, for $1 \leqslant j \leqslant 4^m$, which constitute the polyhedral neighborhood, as the images of the initial triangular pattern $\mathcal{P}_0^{\mathfrak{K}\mathfrak{C}}$, under the IFS given in Proposition 2.1, on page 5.

Definition 3.2 (m^{th} **Intrinsinc Koch Cohomology Infinitesimal).**

We introduce *the intrinsic scale* as

$$\varepsilon_{\mathfrak{K}\mathfrak{C}} = \frac{1}{3}\,.$$

From now on, given any $m \in \mathbb{N}$, we call m^{th} *intrinsic Koch cohomology infinitesimal* the number $\varepsilon_{\mathfrak{K}\mathfrak{C}}^m = \left(\varepsilon_{\mathfrak{K}\mathfrak{C}}\right)^m = \dfrac{1}{3^m}$. Obviously, $\varepsilon_{\mathfrak{K}\mathfrak{C}}^m$ satisfies the following asymptotic behavior,

$$\varepsilon_{\mathfrak{K}\mathfrak{C}}^m \to 0\,,\ \text{as}\ m \to \infty\ \ .$$

Remark 3.1. Note that our definition is in perfect agreement with the one we gave in the case of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, where the intrinsic scale was equal to $\dfrac{1}{N_b}$, with $N_b \geqslant 3$ denoting the number of divisions which occurred in the corresponding iterative process giving birth, also by means of the sequence of prefractals, to $\Gamma_{\mathcal{W}}$.

Theorem 3.1 (**Exact Expression for the Volume of the m^{th} Polyhedral Neighborhood (or m^{th} Natural Polyhedral Volume)**).

Much as in [LP06], we introduce the oscillatory period *of the Koch Curve as*

$$\mathbf{p} = \frac{\pi}{3 \ln 3}, \tag{$\mathcal{R}$5}$$

which, however, differs from the one $\left(\tilde{\mathbf{p}} = \dfrac{2\pi}{\ln 3}\right)$ in [LP06].

Then, for any $m \in \mathbb{N}$, the two-dimensional Lebesgue measure (or area) of the m^{th} polyhedral neighborhood $\mathcal{D}(\mathfrak{KC}_m)$ is given by

$$\begin{aligned}\mathcal{V}_m(\varepsilon_{\mathfrak{KC}}) &= \frac{\sqrt{3}}{8}\,\varepsilon_{\mathfrak{KC}}^{2(m+1)} \sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\,-j\leqslant\ell_{j,m}\leqslant j}\left({\varepsilon_{\mathfrak{KC}}}^{-i\,\ell_{j,m}\,\mathbf{p}}+{\varepsilon_{\mathfrak{KC}}}^{i\,\ell_{j,m}\,\mathbf{p}}\right)\right)\\ &= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{KC}}^{2(m+1)}\,\mathcal{R}e\left(\sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\,-j\leqslant\ell_{j,m}\leqslant j}{\varepsilon_{\mathfrak{KC}}}^{-i\,\ell_{j,m}\,\mathbf{p}}\right)\right),\end{aligned} \tag{$\mathcal{R}$6}$$

where $\varepsilon_{\mathfrak{KC}}$ is the intrinsic scale introduced in Definition 3.2, on page 10.

Given the form of the expression in relation ($\mathcal{R}$6) just above, it is natural to introduce the associated m^{th} complex natural polyhedral volume $\mathcal{V}_{m,comp}(\varepsilon_{\mathfrak{KC}})$*, such that*

$$\mathcal{V}_{m,comp}(\varepsilon_{\mathfrak{KC}}) = \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{KC}}^{2(m+1)} \sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\,-j\leqslant\ell_{j,m}\leqslant j}\left({\varepsilon_{\mathfrak{KC}}}^{-i\,\ell_{j,m}\,\mathbf{p}}+{\varepsilon_{\mathfrak{KC}}}^{i\,\ell_{j,m}\,\mathbf{p}}\right)\right). \tag{$\mathcal{R}$7}$$

For any integer $m \in \mathbb{N}$, we have the following recurrence relation:

$$\mathcal{V}_{m+1,comp}(\varepsilon_{\mathfrak{KC}}) = \frac{1}{9}\,\mathcal{V}_{m,comp}(\varepsilon_{\mathfrak{KC}}) + \frac{3\sqrt{3}}{64}\,{\varepsilon_{\mathfrak{KC}}}^{2-D_{\mathfrak{KC}}}\,\varepsilon_{\mathfrak{KC}}^{(m+1)(2-D_{\mathfrak{KC}})}\left({\varepsilon_{\mathfrak{KC}}}^{-i(m+1)\,\mathbf{p}}+{\varepsilon_{\mathfrak{KC}}}^{i(m+1)\,\mathbf{p}}\right). \tag{$\mathcal{R}$8}$$

Remark 3.2. Note that the complex exponents corresponding to the right-hand side of relation ($\mathcal{R}$7) of Theorem 3.1, on page 11, should be respectively interpreted as follows:

$$(m+1)\,(2-D_{\mathfrak{KC}}) - i\,\ell_{j,m}\,\mathbf{p} = 2 - \left(D_{\mathfrak{KC}} - m\,(2-D_{\mathfrak{KC}}) + i\,\ell_{j,m}\,\mathbf{p}\right)$$

and

$$(m+1)\,(2-D_{\mathfrak{KC}}) + i\,\ell_{j,m}\,\mathbf{p} = 2 - \left(D_{\mathfrak{KC}} - m\,(2-D_{\mathfrak{KC}}) - i\,\ell_{j,m}\,\mathbf{p}\right)$$

Accordingly, those exponents should be viewed as Complex Codimensions, respectively associated with the Complex Dimensions

$$D_{\mathfrak{KC}} - m\,(2-D_{\mathfrak{KC}}) + i\,\ell_{j,m}\,\mathbf{p}$$

and

$$D_{\mathfrak{KC}} - m\,(2-D_{\mathfrak{KC}}) - i\,\ell_{j,m}\,\mathbf{p}\,.$$

Proof. (of Theorem 3.1, given on page 11)

Recall that the angles $\theta_0, \ldots, \theta_3$ and the complex numbers $s_0^{\mathfrak{KC}}, \ldots, s_3^{\mathfrak{KC}}$ have respectively been introduced in relation ($\mathcal{R}2$), on page 6, and in relation ($\mathcal{R}3$) (see Property 2.1, on page 5).

Furthermore, recall that, for any $m \in \mathbb{N}$,

$$\mathcal{D}\left(\mathfrak{KC}_m\right) = \bigcup_{j=1}^{4^m} \mathcal{P}_{j,m}^{\mathfrak{KC}} = \bigcup_{k_{0,m}+k_{1,m}+k_{2,m}+k_{3,m}\leqslant m} \left(T_0^{\mathfrak{KC}}\right)^{k_{0,m}} \left(T_1^{\mathfrak{KC}}\right)^{k_{1,m}} \left(T_2^{\mathfrak{KC}}\right)^{k_{2,m}} \left(T_3^{\mathfrak{KC}}\right)^{k_{3,m}} \left(\mathcal{P}_0^{\mathfrak{KC}}\right).$$

Now, since we are only concerned with the computation of the area – and thanks to the translation invariance of the two-dimensional Lebesgue measure – the terms coming from the translations involved in the maps

$$\mathcal{T}^{\mathfrak{KC}} = \left\{T_0^{\mathfrak{KC}}, \ldots, T_3^{\mathfrak{KC}}\right\},$$

do not play any role; so that we do not have to take them into account.

Thus, we simply have to compute the two-dimensional area of 4^m equilateral triangles, of sidelength equal to $\dfrac{1}{3^{m+1}}$ and of associated height

$$\sqrt{\frac{1}{9^{m+1}} - \frac{1}{4}\,\frac{1}{9^{m+1}}} = \frac{\sqrt{3}}{2}\,\frac{1}{3^{m+1}}.$$

We obtain that

$$\begin{aligned}
\mu_{\mathcal{L}}\left(\bigcup_{j=1}^{4^m} \mathcal{P}_{j,m}^{\mathfrak{KC}}\right) &= \mathcal{R}e\left(\sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+k_{1,j,m}+k_{2,j,m}+k_{3,j,m}=j} \mu_{\mathcal{L}}\left(\left(T_0^{\mathfrak{KC}}\right)^{k_{0,j,m}} \left(T_1^{\mathfrak{KC}}\right)^{k_{1,j,m}} \left(T_2^{\mathfrak{KC}}\right)^{k_{1,j,m}} \left(T_3^{\mathfrak{KC}}\right)^{k_{3,j,m}} \mathcal{P}_0^{\mathfrak{KC}}\right)\right)\right)\\
&= \frac{\sqrt{3}}{4}\,\frac{1}{9^{m+1}}\,\mathcal{R}e\left(\sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+k_{1,m}+k_{2,j,m}\,+k_{3,j,m}=j} e^{i\left(k_{1,j,m}\,\theta_1+k_{2,j,m}\,\theta_2\right)}\right)\right)\\
&= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{KC}}^{2\,(m+1)}\,\mathcal{R}e\left(\sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+k_{1,j,m}+,k_{2,j,m}+k_{3,j,m}=j} e^{\frac{i\,\pi\left(k_{2,j,m}-k_{1,j,m}\right)}{3}}\right)\right)\\
&= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{KC}}^{2\,(m+1)}\,\mathcal{R}e\left(\sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\,-j\leqslant\ell_{j,m}\leqslant j} e^{\frac{i\,\pi\,\ell_{j,m}}{3}}\right)\right)\\
&= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{KC}}^{2\,(m+1)} \sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\,-j\leqslant\ell_{j,m}\leqslant j} \mathcal{R}e\left({\varepsilon_{\mathfrak{KC}}}^{-i\,\ell_{j,m}\,\mathbf{p}}\right)\\
&= \frac{\sqrt{3}}{8}\,\varepsilon_{\mathfrak{KC}}^{2\,(m+1)} \sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\,-j\leqslant\ell_{j,m}\leqslant j} \left({\varepsilon_{\mathfrak{KC}}}^{-i\,\ell_{j,m}\,\mathbf{p}} + {\varepsilon_{\mathfrak{KC}}}^{i\,\ell_{j,m}\,\mathbf{p}}\right).
\end{aligned} \tag{$\mathcal{R}\,9$}$$

We also have that

$$
\begin{aligned}
\mu_{\mathcal{L}}\left(\bigcup_{j=1}^{4^{m+1}} \mathcal{P}_{j,m}^{\mathfrak{K}\mathfrak{C}}\right) &= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,\mathcal{R}e\left(\sum_{j=0}^{m+1}\left(\sum_{k_{0,j,m+1}+k_{1,j,m+1}+k_{2,j,m+1}+k_{3,j,m+1}=j} e^{i\left(k_{1,j,m+1}\,\theta_1+k_{2,j,m+1}\,\theta_2\right)}\right)\right)\\
&= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,\mathcal{R}e\left(\sum_{j=0}^{m+1}\left(\sum_{k_{0,j,m+1}+k_{1,j,m+1}+k_{2,j,m+1}+k_{3,j,m+1}=j} e^{\frac{i\left(k_{2,j,m+1}-k_{1,j,m+1}\right)\pi}{3}}\right)\right)\\
&= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,\mathcal{R}e\left(\sum_{j=0}^{m}\left(\sum_{k_{0,j,m+1}+k_{1,j,m+1}+k_{2,j,m+1}+k_{3,j,m+1}=j} e^{\frac{i\left(k_{2,j,m+1}-k_{1,j,m+1}\right)\pi}{3}}\right)\right)\\
&\quad+\frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,\mathcal{R}e\left(\sum_{k_{0,j,m+1}+k_{1,j,m+1}+k_{2,j,m+1}+k_{3,j,m+1}=m+1} e^{\frac{i\left(k_{2,j,m+1}-k_{1,j,m+1}\right)\pi}{3}}\right)\\
&= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,\mathcal{R}e\left(\sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\,-j\leqslant\ell_{j,m}\leqslant j} e^{\frac{i\,\ell_{j,m}\,\pi}{3}}\right)\right)\\
&\quad+\frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,\mathcal{R}e\left(\sum_{k_{0,j,m+1}+2\,k_{1,j,m+1}+\ell_{j,m+1}+k_{3,j,m+1}=j+1,\,\ell_{j,m+1}\in\{-m-1,m+1\}} e^{\frac{i\,\ell_{j,m+1}\,\pi}{3}}\right)\\
&= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,\mathcal{R}e\left(\sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\,-j\leqslant\ell_{j,m}\leqslant j} {\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{-i\,\ell_{j,m}\,\mathbf{p}}\right)\right)\\
&\quad+\frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,\mathcal{R}e\left(4^{m+1}-4^{m}\right)\left({\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{-i\,(m+1)\,\mathbf{p}}+{\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{i\,(m+1)\,\mathbf{p}}\right)\\
&= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,\mathcal{R}e\left(\sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\,-j\leqslant\ell_{j,m}\leqslant j} {\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{-i\,\ell_{j,m}\,\mathbf{p}}\right)\right)\\
&\quad+\frac{3\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,4^{m}\,\mathcal{R}e\left({\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{-i\,(m+1)\,\mathbf{p}}+{\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{i\,(m+1)\,\mathbf{p}}\right)\\
&= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,\mathcal{R}e\left(\sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\,-j\leqslant\ell_{j,m}\leqslant j} {\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{-i\,\ell_{j,m}\,\mathbf{p}}\right)\right)\\
&\quad+\frac{3\sqrt{3}}{16}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+1)}\,4^{m+1}\,\mathcal{R}e\left({\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{-i\,(m+1)\,\mathbf{p}}+{\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{i\,(m+1)\,\mathbf{p}}\right)\\
&= \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+2)}\,\mathcal{R}e\left(\sum_{j=0}^{m}\left(\sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\,-j\leqslant\ell_{j,m}\leqslant j} {\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{-i\,\ell_{j,m}\,\mathbf{p}}\right)\right)\\
&\quad+\frac{3\sqrt{3}}{64}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2-D_{\mathfrak{K}\mathfrak{C}}}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{(m+1)\,(2-D_{\mathfrak{K}\mathfrak{C}})}\,\mathcal{R}e\left({\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{-i\,(m+1)\,\mathbf{p}}+{\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{i\,(m+1)\,\mathbf{p}}\right),
\end{aligned}
\tag{$\mathcal{R}\,10$}
$$

where, in the second sum of the last equality, we have used the easily verified identities

$$4^{m+1} = \varepsilon_{\mathfrak{K}\mathfrak{C}}^{-(m+1)\,D_{\mathfrak{K}\mathfrak{C}}} \quad \text{and} \quad \varepsilon_{\mathfrak{K}\mathfrak{C}}^{2} = \frac{\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2-D_{\mathfrak{K}\mathfrak{C}}}}{4}\,,$$

since

$$D_{\mathfrak{K}\mathfrak{C}} = \frac{\ln 4}{\ln 3}\,.$$

Furthermore, the triangles corresponding to $\ell_{j,m+1} = m+1$ (resp., $\ell_{j,m+1} = -m-1$) are obtained when $k_{1,j,m+1} = 0$ and $k_{2,j,m+1} = m+1$ (resp., when $k_{1,j,m+1} = m+1$ and $k_{2,j,m+1} = 0$). Hence, this amounts to the following number of triangles

$$4^{m+1} - 4^{m} = 3 \times 4^{m} = \frac{3}{4} \times 4^{m+1} = \frac{3}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{-(m+1)\,D_{\mathfrak{K}\mathfrak{C}}}\,.$$

By letting, as is stated in relation $(\mathcal{R}7)$, on page 11,

$$\mathcal{V}_{m,comp}\left(\varepsilon_{\mathfrak{K}\mathfrak{C}}\right) = \frac{\sqrt{3}}{4}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2\,(m+1)} \sum_{j=0}^{m} \left(\sum_{k_{0,j,m}+2\,k_{1,j,m}+\ell_{j,m}+k_{3,j,m}=j,\; -j\leqslant\ell_{j,m}\leqslant j} {\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{-i\,\ell_{j,m}\,\mathbf{p}} \right),$$

we then deduce that

$$\mathcal{V}_{m+1,comp}\left(\varepsilon_{\mathfrak{K}\mathfrak{C}}\right) = \varepsilon_{\mathfrak{K}\mathfrak{C}}^{2}\,\mathcal{V}_{m,comp}\left(\varepsilon_{\mathfrak{K}\mathfrak{C}}\right) + \frac{3\sqrt{3}}{64}\,{\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{2-D_{\mathfrak{K}\mathfrak{C}}}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{(m+1)\,(2-D_{\mathfrak{K}\mathfrak{C}})} \left({\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{-i\,(m+1)\,\mathbf{p}} + {\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{i\,(m+1)\,\mathbf{p}} \right),$$

as is claimed in relation $(\mathcal{R}8)$, on page 11.

This completes the proof of the theorem.

For the sake of the forthcoming computation of the associated fractal zeta functions, we note that it is better to proceed with the equivalent following formula (in a manner equivalent to what is done in our previous works [DL23b], [DL23a]),

$$\mathcal{V}_{m+1,comp}\left(\varepsilon_{\mathfrak{K}\mathfrak{C}}\right) = \frac{1}{9}\,\mathcal{V}_{m,comp}\left(\varepsilon_{\mathfrak{K}\mathfrak{C}}\right) + \frac{3\sqrt{3}}{64}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{2}\,\varepsilon_{\mathfrak{K}\mathfrak{C}}^{(m+1)\,(2-D_{\mathfrak{K}\mathfrak{C}})} \left({\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{-i\,(m+1)\,\mathbf{p}} + {\varepsilon_{\mathfrak{K}\mathfrak{C}}}^{i\,(m+1)\,\mathbf{p}} \right). \qquad (\mathcal{R}\,11)$$

$\square$

Note now that we are looking for a polyhedral neighborhood. Then, the 4^{m} aforementioned triangles constitute the upper polyhedral neighborhood of $\mathfrak{K}\mathfrak{C}_{m}$.

Lemma 3.2 (**Natural Polyhedral Volume Extension Formula**).

We introduce, for all sufficiently large $m \in \mathbb{N}^{\star}$, $\tilde{\mathcal{V}}_{m}$ as the continuous function defined for all $t \in [0, \varepsilon_{\mathfrak{K}\mathfrak{C}}]$ by substituting t for $\varepsilon_{\mathfrak{K}\mathfrak{C}}$ in the recurrence relation $(\mathcal{R}8)$, on page 11 (or, equivalently, in relation $(\mathcal{R}11)$, on page 14), between $\mathcal{V}_{m}\left(\varepsilon_{\mathfrak{K}\mathfrak{C}}\right)$ and $\mathcal{V}_{m+1}\left(\varepsilon_{\mathfrak{K}\mathfrak{C}}\right)$.

As is explained in [DL22] (in the case of the ordinary Euclidean tubular volume), one can think of $\tilde{\mathcal{V}}_{m}(t)$ as being the polyhedral effective volume *of the m^{th} prefractal approximation to the Koch Curve.*

Notation 3 (**Natural Polyhedral Complex Volume Extension**).

For the sake of simplicity, given $m \in \mathbb{N}^\star$ sufficiently large, we will from now on call *the* m^{th} *natural polyhedral complex volume extension*, the volume extension function $\widetilde{\mathcal{V}}_{m,comp}$ associated with the m^{th} natural polyhedral complex volume $\mathcal{V}_{comp}$ introduced in Theorem 3.1, on page 11. Alternatively, $\widetilde{\mathcal{V}}_{m,comp}$ will be called *the* m^{th} *polyhedral effective complex volume*.

In the same way, as is done in [DL22], and given $m \in \mathbb{N}^\star$ large enough, we call *the* m^{th} *natural volume extension*, the volume extension function $\widetilde{\mathcal{V}}_m^{tube}$ associated with the *the* m^{th} *tubular volume* $\mathcal{V}_m^{tube}$. Alternatively, $\widetilde{\mathcal{V}}_m^{tube}$ will be called *the* m^{th} *effective tubular volume*.

Theorem 3.3 (**Local and Global Polyhedral Effective Zeta Functions**).

Given $m \in \mathbb{N}^\star$ sufficiently large (i.e., for all $m \geqslant m_0$, for some $m_0 \in \mathbb{N}^\star$), we introduce the m^{th} local polyhedral effective zeta function $\tilde{\zeta}_m^e$, *such that, for all $s \in \mathbb{C}$ with $\mathcal{R}e\,(s) > D_{\mathfrak{KC}}$,*

$$\tilde{\zeta}_m^e(s) = \int_0^{\varepsilon_{\mathfrak{KC}}} t^{s-3}\,\widetilde{\mathcal{V}}_{m,comp}(t)\,dt\,, \qquad (\mathcal{R}\,12)$$

where $\widetilde{\mathcal{V}}_{partial,m,comp}$ is the m^{th} polyhedral effective complex volume, introduced in Notation 3, on page 15 above, and where $\varepsilon_{\mathfrak{KC}}$ is the intrinsic scale introduced in Definition 3.2, on page 10 above.

Then, for all $m \in \mathbb{N}^\star$ sufficiently large (i.e., for all $m \geqslant m_0$, for some $m_0 \in \mathbb{N}^\star$), $\tilde{\zeta}_m^e$ admits a (necessarily unique) meromorphic extension to all of $\mathbb{C}$, given, for all $s \in \mathbb{C}$, by the following explicit expression, which is a convergent fractal power series*:*

$$\begin{aligned}\tilde{\zeta}_m^e(s) \quad = \quad & \frac{3\sqrt{3}}{64}\sum_{k=0}^{m}\frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+(k+1)\,(2-D_{\mathfrak{KC}})-i\,(k+1)\,\mathbf{p}}}{s-D_{\mathfrak{KC}}+(k+1)\,(2-D_{\mathfrak{KC}})-i\,(k+1)\,\mathbf{p}} \\ & +\frac{3\sqrt{3}}{64}\sum_{k=0}^{m}\frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+(k+1)\,(2-D_{\mathfrak{KC}})+i\,(k+1)\,\mathbf{p}}}{s-D_{\mathfrak{KC}}+(k+1)\,(2-D_{\mathfrak{KC}})+i\,(k+1)\,\mathbf{p}}\,,\end{aligned} \qquad (\mathcal{R}\,13)$$

where $\varepsilon_{\mathfrak{KC}}$ is the intrinsic scale introduced in Definition 3.2, on page 10).

More specifically, still for all $m \in \mathbb{N}^\star$ sufficiently large, the function $\tilde{\zeta}_m^e$ is well defined and meromorphic in all of $\mathbb{C}$. Furthermore, its unique meromorphic extension (still denoted by $\tilde{\zeta}_m^e$) is given, for all $s \in \mathbb{C}$ by the expressions given in relation ($\mathcal{R}$13) above.

Moreover, the associated sequence $\left(\tilde{\zeta}_m^e\right)_{m\in\mathbb{N}}$ – initially given (for $\mathcal{R}e(s) > D_{\mathcal{W}}$) by the truncated Mellin transform in relation ($\mathcal{R}$12), on page 15 – satisfies the following recurrence relation, for all values of the positive integer m sufficiently large, and for all $s \in \mathbb{C}$:

$$
\begin{aligned}
\tilde{\zeta}^e_{m+1}(s) &= \varepsilon^2_{\mathfrak{KC}}\,\tilde{\zeta}^e_m(s) \\
&\quad + \frac{3\sqrt{3}}{64}\,\frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})-i\,(m+1)\,\mathbf{p}}}{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})-i\,(m+1)\,\mathbf{p}} \\
&\quad + \frac{3\sqrt{3}}{64}\,\frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})+i\,(m+1)\,\mathbf{p}}}{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})+i\,(m+1)\,\mathbf{p}} \qquad (\mathcal{R}\,14) \\
&= \varepsilon^2_{\mathfrak{KC}}\,\tilde{\zeta}^e_m(s) \\
&\quad + \frac{3\sqrt{3}}{32}\,\mathcal{R}e\left(\frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})-i\,(m+1)\,\mathbf{p}}}{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})-i\,(m+1)\,\mathbf{p}}\right).
\end{aligned}
$$

The expression for the m^{th} local polyhedral effective zeta function $\tilde{\zeta}^e_m$ given in relation ($\mathcal{R}$13) above ensures the existence of the limit fractal zeta function $\tilde{\zeta}^e_{\mathfrak{KC}}$, i.e., the fractal zeta function associated with the Koch Curve $\mathfrak{KC}$ (or, rather, with the Koch IFD), called the global polyhedral effective zeta function *of $\mathfrak{KC}$, and given by* the following convergent (and locally convergent) fractal power series*:*

$$\tilde{\zeta}^e_{\mathfrak{KC}} = \lim_{m\to\infty} \tilde{\zeta}^e_m\,,$$

where the convergence is locally uniform on $\mathbb{C}$, along with the existence of an integer $m_0 \in \mathbb{N}$ such that, for all $m \geqslant m_0$, the set of poles of $\tilde{\zeta}^e_{\mathfrak{KC}}$ consists of simple poles and contains the poles of the m^{th} fractal polyhedral effective zeta function $\tilde{\zeta}^e_m$. More specifically, $\tilde{\zeta}^e_{\mathfrak{KC}}$ is meromorphic in all of $\mathbb{C}$ and its (necessarily unique) meromorphic extension (still denoted $\tilde{\zeta}^e_{\mathfrak{KC}}$) is given, for all $s \in \mathbb{C}$, by the following convergent fractal power series:

$$
\begin{aligned}
\tilde{\zeta}^e_{\mathfrak{KC}}(s) &= \frac{3\sqrt{3}}{64}\sum_{m=0}^{\infty}\frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})-i\,(m+1)\,\mathbf{p}}}{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})-i\,(m+1)\,\mathbf{p}} \\
&\quad + \frac{3\sqrt{3}}{64}\sum_{m=0}^{\infty}\frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})+i\,(m+1)\,\mathbf{p}}}{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})+i\,(m+1)\,\mathbf{p}}\,. \qquad (\mathcal{R}\,15)
\end{aligned}
$$

The following lemma will be used in part *ii.* of the proof of Theorem 3.3.

Lemma 3.4 (A Preliminary Result About the Sets of Poles of the Local Polyhedral Zeta Functions).

Let us denote by $\mathcal{P}^\star = \bigcup_{m\geqslant m_0} \mathcal{P}\left(\tilde{\zeta}^e_m\right)$ the union of the discrete sets of poles of the local polyhedral zeta functions $\tilde{\zeta}^e_m$, for all $m \in \mathbb{N}$ such that $m \geqslant m_0$. Then, the sequence $\left(\mathcal{P}\left(\tilde{\zeta}^e_m\right)\right)_{m\geqslant m_0}$ of subsets of $\mathbb{C}$ is increasing.

Furthermore, the set $\mathcal{P}^\star$ is discrete and countably infinite. It is, also, the set of poles of the global polyhedral zeta function $\tilde{\zeta}^e_{\mathcal{W}}$.

In addition, its complement $\mathcal{U}$ in $\mathbb{C}$ is an open, connected subset of $\mathbb{C}$ (since $\mathcal{P}^\star$ is necessarily closed).

Proof. (of Lemma 3.4)

As is done in [DL23b], the proof relies on the fact that, for all $m \in \mathbb{N}$ sufficiently large,

$$\mathcal{P}\left(\tilde{\zeta}^e_m\right) = \left\{D_{\mathcal{W}} - (k-1)\,(2 - D_{\mathcal{W}}) \pm i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}\,,\; 1 \leqslant k \leqslant m\right\} \subsetneq \mathcal{P}\left(\tilde{\zeta}^e_{m+1}\right)\,.$$

This ensures that, as a countable union of discrete sets, and without any accumulating points (in $\mathbb{C}$), the set $\mathcal{P}^\star$ is itself discrete and, hence, countably infinite. This ensures that its complement $\mathcal{U} \subset \mathbb{C}$ is an open, connected subset of $\mathbb{C}$. This implies that $\mathcal{P}^\star$ intersects each compact set $\mathcal{K} \subset \mathbb{C}$ at a finite number of points.

The fact that $\mathcal{P}^\star$ is the set of poles of the global polyhedral zeta function $\tilde{\zeta}^e_{\mathcal{W}}$ comes from the expression given in relation $(\mathcal{R}15)$, on page 16.

Also, the fact that, for all $m \geqslant m_0$, $\mathcal{P}\left(\tilde{\zeta}^e_m\right) \subseteq \mathcal{P}\left(\tilde{\zeta}^e_{m+1}\right)$, is established in the proof of Theorem 3.3 below.

□

Proof. (Of Theorem 3.3, given on page 15)

i. We first give the explicit expression for the m^{th} *local polyhedral effective zeta function* $\tilde{\zeta}^e_m$.

We restrict ourselves to sufficiently large values of $m \in \mathbb{N}^\star$, i.e., to all $m \geqslant m_0$, for some suitable integer $m_0 \in \mathbb{N}^\star$.

Thanks to the recurrence relation $(\mathcal{R}8)$, on page 11, or, equivalently, to relation $(\mathcal{R}11)$, on page 14, between $\mathcal{V}_m\left(\varepsilon_{\mathfrak{K}\mathfrak{C}}\right)$ and $\mathcal{V}_{m+1}\left(\varepsilon_{\mathfrak{K}\mathfrak{C}}\right)$, we then have that, for all $s \in \mathbb{C}$ with $\mathcal{R}e\,(s) > D_{\mathfrak{K}\mathfrak{C}}$,

$$
\begin{aligned}
\tilde{\zeta}^e_{m+1}(s) &= \int_0^{\varepsilon_{\mathfrak{KC}}} t^{s-3}\,\tilde{\mathcal{V}}_{m+1,comp}(t)\,dt \\
&= \frac{1}{9}\int_0^{\varepsilon_{\mathfrak{KC}}} t^{s-1-D_{\mathfrak{KC}}}\,\tilde{\mathcal{V}}_{m,comp}(t)\,dt \\
&\quad + \frac{3\sqrt{3}}{64}\int_0^{\varepsilon_{\mathfrak{KC}}} t^{s-1-D_{\mathfrak{KC}}}\,t^{(m+1)(2-D_{\mathfrak{KC}})}\left(t^{-i(m+1)\mathbf{p}}+t^{i(m+1)\mathbf{p}}\right)dt \\
&= \frac{1}{9}\,\tilde{\zeta}^e_m(s) \\
&\quad + \frac{3\sqrt{3}}{64}\left(\frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+(m+1)(2-D_{\mathfrak{KC}})-i(m+1)\mathbf{p}}}{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})-i\,(m+1)\,\mathbf{p}}\right) \\
&\quad + \frac{3\sqrt{3}}{64}\left(\frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+(m+1)(2-D_{\mathfrak{KC}})+i(m+1)\mathbf{p}}}{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})+i\,(m+1)\,\mathbf{p}}\right) \\
&= \varepsilon_{\mathfrak{KC}}^2\,\tilde{\zeta}^e_m(s) \\
&\quad + \frac{3\sqrt{3}}{64}\left(\frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+(m+1)(2-D_{\mathfrak{KC}})-i(m+1)\mathbf{p}}}{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})-i\,(m+1)\,\mathbf{p}}\right) \\
&\quad + \frac{3\sqrt{3}}{64}\left(\frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+(m+1)(2-D_{\mathfrak{KC}})+i(m+1)\mathbf{p}}}{s-D_{\mathfrak{KC}}+(m+1)\,(2-D_{\mathfrak{KC}})+i\,(m+1)\,\mathbf{p}}\right).
\end{aligned}
\tag{$\mathcal{R}$ 16}
$$

By induction, for all $m \geqslant m_0$, we then obtain the poles of $\tilde{\zeta}^e_m$ as

$$s = D_{\mathfrak{KC}} - k\,(2-D_{\mathfrak{KC}}) - i\,k\,\mathbf{p}$$

and

$$s = D_{\mathfrak{KC}} - k\,(2-D_{\mathfrak{KC}}) + i\,k\,\mathbf{p}\,,$$

where k is an arbitrary integer such that $0 \leqslant k \leqslant m$. These poles are all simple.

ii. We then place ourselves in a compact set $\mathcal{K} \subset \mathbb{C}$ contained in the connected open set $\mathcal{U} \subset \mathbb{C}$ which is the complement in $\mathbb{C}$ of the discrete (and hence, closed) set $\mathcal{P}^\star = \bigcup_{n\in\mathbb{N},\,n\geqslant m_0} \mathcal{P}\left(\tilde{\zeta}^e_n\right)$, the countable union of the sets of poles of the local polyhedral zeta functions $\tilde{\zeta}^e_n$, for all $n \geqslant m_0$ (see Lemma 3.4, on page 16).

The series of functions

$$\sum_{k=0}^{m} \frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+k(2-D_{\mathfrak{KC}})-i\,k\,\mathbf{p}}}{s-D_{\mathfrak{KC}}+k\,(2-D_{\mathfrak{KC}})-i\,k\,\mathbf{p}} \quad \left(\text{resp., } \sum_{k=0}^{m} \frac{\varepsilon_{\mathfrak{KC}}^{s-D_{\mathfrak{KC}}+k(2-D_{\mathfrak{KC}})+i\,k\,\mathbf{p}}}{s-D_{\mathfrak{KC}}+k\,(2-D_{\mathfrak{KC}})+i\,k\,\mathbf{p}}\right) \tag{$\mathcal{R}$ 17}$$

is (locally) normally convergent, and, hence, also (locally) uniformly convergent on $\mathcal{K}$. This ensures the existence of the limit effective fractal zeta function, i.e., the fractal zeta function associated with the Koch Curve $\mathfrak{KC}$ (or with the Koch IFD), given by

$$
\begin{aligned}
\tilde{\zeta}^{e}_{\mathfrak{K}\mathfrak{C}}(s) &= \lim_{m\to\infty} \tilde{\zeta}^{e}_{m}(s) \\
&= \frac{3\sqrt{3}}{64} \sum_{m=0}^{\infty} \frac{\varepsilon_{\mathfrak{K}\mathfrak{C}}^{s-D_{\mathfrak{K}\mathfrak{C}}+(m+1)\,(2-D_{\mathfrak{K}\mathfrak{C}})-i\,(m+1)\,\mathbf{p}}}{s-D_{\mathfrak{K}\mathfrak{C}}+(m+1)\,\left(2-D_{\mathfrak{K}\mathfrak{C}}\right)-i\,(m+1)\,\mathbf{p}} \qquad (\mathcal{R}\,18) \\
&\quad + \frac{3\sqrt{3}}{64} \sum_{m=0}^{\infty} \frac{\varepsilon_{\mathfrak{K}\mathfrak{C}}^{s-D_{\mathfrak{K}\mathfrak{C}}+(m+1)\,(2-D_{\mathfrak{K}\mathfrak{C}})+i\,(m+1)\,\mathbf{p}}}{s-D_{\mathfrak{K}\mathfrak{C}}+(m+1)\,\left(2-D_{\mathfrak{K}\mathfrak{C}}\right)+i\,(m+1)\,\mathbf{p}}\,.
\end{aligned}
$$

We note in passing that all the expressions involving the real parts in the statement of the theorem simply follow from the fact that for all $z \in \mathbb{C}$, $z+\bar{z}=2\,\mathcal{R}e\,(z)$.

The remainder of our proof – in the case when we place ourselves in a compact set $\mathcal{K} \subset \mathbb{C}$ containing some of the potential poles (necessarily, finitely many of them, since $\mathcal{K}$ is compact and the poles are isolated) of $\tilde{\zeta}^{e}_{m}$ – can be obtained by using the chordal metric.

For this purpose, a (complex-valued) meromorphic function f is viewed as a continuous function with values in $\mathbb{P}^1(\mathbb{C})$, equipped with the chordal metric, and such that, for any pole ω of f, $f(\omega)$ takes the value ∞ (for instance, as in [LvF13], Section 3.4 and Appendix C).

More precisely, if $\mathbb{P}^1(\mathbb{C})=\mathbb{C}\cup\infty$ denotes the Riemann sphere (or complex projective line), we can show that, for *the chordal metric*, defined, for all $(z_1,z_2)\in\left(\mathbb{P}^1(\mathbb{C})\right)^2$ by

$$
\|z_1,z_2\| = \frac{|z_1-z_2|}{\sqrt{1+|z_1^2|}\,\sqrt{1+|z_2^2|}}\,,\ \text{if } z_1\neq\infty \text{ and } z_2\neq\infty\,, \qquad (\mathcal{R}\,19)
$$

and

$$
\|z_1,\infty\| = \frac{1}{\sqrt{1+|z_1^2|}}\,,\ \text{if } z_1\neq\infty\,. \qquad (\mathcal{R}\,20)
$$

Since the discrete set $\mathcal{P}^\star=\bigcup\limits_{m\in\mathbb{N}}\mathcal{P}\left(\tilde{\zeta}^{e}_{m}\right)$ intersects each compact set $\mathcal{K}\subset\mathbb{C}$ at a countably finite number of points (see Lemma 3.4 above, on page 16), we can restrict ourselves to the case when $\mathcal{K}$ contains one single pole of a given $\tilde{\zeta}^{e}_{n}$, for some $n\geqslant m_0$.

The (uniform) limit fractal zeta function $\tilde{\zeta}^{e}_{\mathcal{W}}$, is holomorphic on $\mathcal{U}$, as the (locally) uniform limit of a sequence of holomorphic functions on $\mathcal{U}$ (the complement in $\mathbb{C}$ of the discrete set $\mathcal{P}^\star=\bigcup\limits_{n\in\mathbb{N},\,n\geqslant m_0}\mathcal{P}\left(\tilde{\zeta}^{e}_{n}\right)$, the countable union of the sets of poles of the local polyhedral zeta functions $\tilde{\zeta}^{e}_{n}$, for all $n\geqslant m_0$). By applying Weierstrass' Theorem, we then deduce that, for all $m\geqslant m_0$, the zeta function $\tilde{\zeta}^{e}_{m}$ is meromorphic on $\mathbb{C}$.

Moreover, the counterpart of the results obtained in [DL22] for the sequence of tube zeta functions associated with the Weierstrass IFD, which admit a meromorphic continuation to all of $\mathbb{C}$, obviously holds for the sequence of polyhedral tube zeta functions: hence, $\tilde{\zeta}^{e}_{m}$ is meromorphic on $\mathbb{C}$, with only simple poles, as specified in Theorem 3.5, on page 20 below.

More specifically, our earlier discussion (ending just after relation ($\mathcal{R}$18), on page 19, shows the uniform convergence of $\left(\tilde{\zeta}^{e}_{m}\right)_{m\geqslant m_0}$ on $\mathcal{K}$ – and, hence also, the existence of the global polyhedral effective zeta function $\tilde{\zeta}^{e}_{\mathcal{W}}$, whenever the compact set $\mathcal{K}\subset\mathbb{C}$ does not intersect $\mathcal{P}^\star$ (i.e., $\mathcal{K}\subseteq\mathcal{U}=\mathbb{C}\setminus\mathcal{P}^\star$). The fact that $\tilde{\zeta}^{e}_{\mathcal{W}}$ is holomorphic on the (connected) open set $\mathcal{U}\subseteq\mathbb{C}$ then follows at once from Weierstrass' Theorem about the holomorphicity of a (locally) uniformly convergent sequence of holomorphic functions (here, the sequence of holomorphic functions $\left(\tilde{\zeta}^e_m\right)_{m\geqslant m_0}$ on $\mathcal{U}$).

In the above case when $\mathcal{K}\cap\mathcal{P}^\star$ is nonempty – and, without loss of generality, when $\mathcal{K}\cap\mathcal{P}^\star=\{\omega\}$, where ω is a pole of $\tilde{\zeta}^e_{m_1}$, for some integer $m_1\geqslant m_0$ (and, thus, for any $m\geqslant m_1$), our above argument shows that $\tilde{\zeta}^e_{\mathcal{W}}$ is well defined and is continuous on $\mathcal{K}$ (as the uniform limit on the compact set $\mathcal{K}$ of of the sequence $\left(\tilde{\zeta}^e_m\right)_{m\geqslant m_0}$ of continuous functions on $\mathcal{K}$, with all the functions involved being viewed as taking their values in the compact – and thus, complete – metric space $\left(\mathbb{P}^1(\mathbb{C}),\|\cdot\|\right)$ (i.e., $\lim\limits_{m\to\infty}\sup\limits_{s\in\mathcal{K}}\left\|\tilde{\zeta}^e_m(s),\tilde{\zeta}^e_{\mathcal{W}}(s)\right\|=0$) and such that $\tilde{\zeta}^e_{\mathcal{W}}(\omega)=\infty$ since $\tilde{\zeta}^e_m(\omega)=\infty$, for all $m\geqslant m_1$. Consequently, $\tilde{\zeta}^e_{\mathcal{W}}$ is meromorphic on $\mathcal{U}$ and ω is also a pole of $\tilde{\zeta}^e_{\mathcal{W}}$ (i.e., $\omega\in\mathcal{P}\left(\tilde{\zeta}^e_{\mathcal{W}}\right)$).

In summary, we deduce that $\tilde{\zeta}^e_{\mathcal{W}}$ is well defined (by the locally uniform limit in relation ($\mathcal{R}15$), on page 16), is meromorphic on all of $\mathbb{C}$ and $\mathcal{P}\left(\tilde{\zeta}^e_{\mathcal{W}}\right)=\mathcal{P}^\star$.

This completes the proof of the theorem. $\square$

Definition 3.3 (**Intrinsic Complex Dimensions of the Koch Curve**).

Since they are obtained via the polyhedral neighborhoods (instead of via the Euclidean tubular neighborhoods) of the Koch IFD, we call the poles of $\tilde{\zeta}^e_{\mathfrak{KC}}$ (resp., for all $m\geqslant m_0$, of $\tilde{\zeta}^e_m$) *the intrinsic Complex Dimensions* of the Koch Curve (resp., of its m^{th} prefractal approximation).

In the sequel, a Complex Dimension is said to be *exact* (or *actual*) if it is a pole of $\tilde{\zeta}^e_{\mathfrak{KC}}$ (resp., for all $m\geqslant m_0$, of $\tilde{\zeta}^e_m$). Otherwise, it is said to be *a possible* Complex Dimension – which allows for the possibility that $\operatorname{res}\left(\tilde{\zeta}^e_{\mathfrak{KC}},\omega\right)=0$ (resp., that $\operatorname{res}\left(\tilde{\zeta}^e_m,\omega\right)=0$).

Theorem 3.5 (**Complex Dimensions of the Koch Curve**).

The intrinsic Complex Dimensions of the Koch Curve $\mathfrak{KC}$ (or of the Koch IFD), which are the poles of the global polyhedral effective zeta function $\tilde{\zeta}^e_{\mathfrak{KC}}$ given in relation ($\mathcal{R}15$), on page 16, are given as follows:

$$D_{\mathfrak{KC}}-m\,(2-D_{\mathfrak{KC}})-i\,m\,\mathbf{p}\,,$$

and

$$D_{\mathfrak{KC}}-m\,(2-D_{\mathfrak{KC}})+i\,m\,\mathbf{p}\,,$$

where the integer $m\in\mathbb{N}$ is arbitrary.

Furthermore, the intrinsic Complex Dimensions *of $\mathfrak{KC}$ are all simple and* exact.

The following result is an immediate corollary of Theorem 3.3, on page 15, and Theorem 3.5, on page 20, respectively, as well as of their proofs.

Theorem 3.6 (**Complex Dimensions of the Prefractal Approximations of $\mathfrak{KC}$**).

For all (positive) integers sufficiently large (i.e., for all $m \geqslant m_0$, for some $m_0 \in \mathbb{N}^\star$), the intrinsic Complex Dimensions *of the m^{th} approximation $\mathfrak{KC}_m$ of $\mathfrak{KC}$ are all* exact *and* simple. *Furthermore, they are given as follows:*

$$D_{\mathfrak{KC}} - k\,(2 - D_{\mathfrak{KC}}) - i\,k\,\mathbf{p}$$

and

$$D_{\mathfrak{KC}} - k\,(2 - D_{\mathfrak{KC}}) + i\,k\,\mathbf{p}\,,$$

where k is an arbitrary integer such that $0 \leqslant k \leqslant m$ (see relation ($\mathcal{R}13$) in Theorem 3.3, on page 15).

Recall that in the theory of Complex Dimensions (see, e.g., [LvF13], [LRŽ17b], [Lap19]), a geometric object is said to be *fractal* if it admits at least one *nonreal* Complex Dimension (defined as a pole of the associated geometric or fractal zeta function) – and hence, at least two nonreal complex conjugate poles. Furthermore, given a real number d, it is *fractal in dimension d* if it has at least one *nonreal* Complex Dimension with real part d.

Finally, it is said to be *principally fractal* if it is fractal in dimension $d_{\mathfrak{KC}}$, where $d_{\mathfrak{KC}}$ is the abscissa of convergence of the associated fractal zeta function (here, $\tilde{\zeta}^e_{\mathfrak{KC}}$ in the case of $\mathfrak{KC}$ or else, $\tilde{\zeta}^e_m$, with $m \geqslant m_0$, in the case of $\mathfrak{KC}_m$). Note that in either case, in light of Theorems 3.3 , on page 15 (or of Theorem 3.5, on page 20, and Theorem 3.6, on page 21), we have that

$$d_{\mathfrak{KC}} = D_{\mathfrak{KC}} = \frac{\ln 4}{\ln 3}\,,$$

the Minkowski dimension of the Koch Curve $\mathfrak{KC}$.

In light of the above definitions, we can now state the following immediate corollary of Theorem 3.5, on page 20 and Theorem 3.6, on page 21.

Corollary 3.7 (**Fractality of the Koch Curve**).

The Koch Curve $\mathfrak{KC}$ is fractal, *and even* principally fractal. *Somewhat surprisingly, so are all of its prefractal approximations $\mathfrak{KC}_m$, for any $m \geqslant m_0$.*

However, there is one important difference between $\mathfrak{KC}$ and any $\mathfrak{KC}_m$. Indeed, $\mathfrak{KC}$ is fractal in infinitely (and countably) many values of d (namely, $d = D_{\mathfrak{KC}} - m\,(2 - D_{\mathcal{W}})$, with $m \in \mathbb{N}$ arbitrary).

By contrast, for all $m \geqslant m_0$ sufficiently large, $\mathfrak{KC}_m$ is fractal in finitely many dimensions d (namely, $d = D_{\mathfrak{KC}} - k\,(2 - D_{\mathfrak{KC}})$, where k is an arbitrary integer such that $0 \leqslant k \leqslant m$).

Remark 3.3.

i. Our result in Theorem 3.5 extends the previous one of Erin P. J. Pearse and the second author in [LP06], where the possible Complex Dimensions of the Koch Curve were obtained as

$$\{D_{\mathfrak{KC}} + i\, n\, \tilde{\mathbf{p}}\,,\, n \in \mathbb{Z}\} \cup \{i\, n\, \tilde{\mathbf{p}}\,,\, n \in \mathbb{Z}\}\ ,$$

where a different oscillatory period $\tilde{\mathbf{p}} = \dfrac{2\,\pi}{\ln 3}$ was involved. The difference between our oscillatory period $\mathbf{p} = \dfrac{\pi}{3\,\ln 3}$ and $\tilde{\mathbf{p}} = \dfrac{2\,\pi}{\ln 3}$ comes from the fact that in [LP06], the oscillatory period is obtained via a Fourier series expansion involving the intrinsic scale $\varepsilon_{\mathfrak{KC}} = \dfrac{1}{3}$, whereas ours comes from the rotations, of respective angles $-\dfrac{\pi}{3}$ and $\dfrac{\pi}{3}$.

We also note that, in [LP06], no fractal zeta function was used, as the higher-dimensional theory of Complex Dimensions (as expounded in [LRŽ17b]) was not yet developed. Instead, an approximate fractal tube formula was used, deduced for the volume (i.e., area) of the Euclidean tubular neighborhoods (instead of the polyhedral neighborhoods) of the Koch Curve – and then, the *possible* Complex Dimensions were derived by analogy with the one-dimensional theory of Complex Dimensions (i.e., with the case of fractal strings studied in detail in [LvF13]).

It would be interesting to determine the precise possible Complex Dimensions of the Koch IFD, by using the method developed in our earlier work [DL22], where Euclidean tubular neighborhoods were also used (in the case of the Weierstrass Curve) and to decide, in particular, whether their real parts are of the form $D_{\mathfrak{KC}} - k\ (2 - D_{\mathfrak{KC}})$, with $k \in \mathbb{N}$ arbitrary, and 0.

ii. Our result is also in perfect agreement with the results obtained in our previous works [DL23b], [DL23a] about the Weierstrass Curve, in which case the exact Complex Dimensions of the Weierstrass Curve (or of the Weierstrass IFD) are given as follows:

$$D_{\mathcal{W}} - m\ (2 - D_{\mathcal{W}}) + i\,\ell\,\mathbf{p}\,, \qquad (\mathcal{R}\,21)$$

where the integers $m \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ are arbitrary.

Remark 3.4 (**Comparison with the Weierstrass Curve**).

In the case of the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (see our previous works [DL23b], [DL23a]), $\Gamma_{\mathcal{W}}$ is also fractal for infinitely many values of d, whereas, for all $m \geqslant m_0$, with $m_0 \in \mathbb{N}$ sufficiently large, the m^{th} prefractal approximation $\Gamma_{\mathcal{W}_m}$ to $\Gamma_{\mathcal{W}}$ is fractal for finitely many values of d (namely, $d = -m, \ldots, m$) (and not in just one value of d, as is the case for the m^{th} prefractal approximation $\mathfrak{KC}_m$ to the Koch Curve $\mathfrak{KC}$; see Corollary 3.7, on page 21 above). Indeed, the Complex Dimensions of $\Gamma_{\mathcal{W}_m}$ (for any $m \geqslant m_0$) are all simple and exact; furthermore, they are given by

$$d = D_{\mathcal{W}} - k\ (2 - D_{\mathcal{W}}) + i\,\ell_k\,\mathbf{p}\,,$$

where k is an arbitrary integer such that $0 \leqslant k \leqslant m$ and $\ell_k \in \mathbb{Z}$ is arbitrary.

This difference comes from the fact that the Weierstrass Curve $\Gamma_{\mathcal{W}}$ is not self-simlar, contrary to the Koch Curve $\mathfrak{K}C$. This results, in the case of the m^{th} prefractal approximation $\Gamma_{\mathcal{W}_m}$ to the Weierstrass Curve $\Gamma_{\mathcal{W}}$, in the kind of *memory effect* which keeps all the Complex Dimensions coming from all the previous prefractal approximations (namely, $\Gamma_{\mathcal{W}_0}$, …, $\Gamma_{\mathcal{W}_{m-1}}$).

4 Concluding Comments

The characterization of fractality by means of the Complex Dimensions now takes its full meaning: indeed, a unique and real value – in the fractal context, the sole Minkowski dimension – which only corresponds to the iterative division process, cannot represent the succession of geometric transformations which go hand in hand with this process. In this light, the Koch Curve is of particular significance, since it is obtained both by dividing line segments and by applying rotations. The information concerning both processes (the division process, and the geometric process associated with the rotations) is fully contained in the Complex Dimensions.

By using exact expressions for the polyhedral neighborhoods, we completely revisit the computation of the *exact Complex Dimensions* of a fractal curve, obtained by means of an i.f.s. Note that, with some effort, our method could clearly be extended not only to other Koch-type curves, but, also, to other planar curves, as the Devil's staircase (the graph of the Cantor–Lebesgue function), the Bedford–McMullen (self-affine) carpets, the Peano and Hilbert (space-filling) curves, as well as a variety of fractal trees, along with Brownian paths and other random fractals, in two or higher dimensions. We expect to address this issue in later work.

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