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# Characterization of extreme Gibbs measures for a Categorical Approach to Statistical Mechanics

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## Abstract

In recent work we proposed to follow a categorical approach to Statistical Mechanics. In this paper we continue in this direction and give a categorical formulation of the decomposition of Gibbs measures by extending the characterization of extreme Gibbs measures to the categorical setting.

Keywords: Statistical mechanics, category theory, Extreme Gibbs measures, Zero-one laws

MSC: 18D99,60F20,60A99

## 1 Introduction

Statistical Mechanics is the field interested in the statistical study of interacting particles ( $X_i \in E_i, i \in \mathbb{N}$ ). A landmark rigorous formulation of Statistical Mechanics can be found in Georgii's *Gibbs Measures and Phase Transition* [1]. Such a framework is needed to define phases or, more generally, *Gibbs measures of statistical systems*. Phases only exist for infinitely many interacting particles and do not exist for finite-size systems. The main constructions of such a framework rely on the necessity to have a *universe*, denoted  $\Omega$ , that encompasses all possible configurations of the system, i.e., all the possible joint configurations of the particles  $\Omega := \prod_{i \in \mathbb{N}} E_i$ . Not referring to a global space is problematic as exhibited by Girya's seminal work on *A Categorical Approach to Probability Theory* [2] and can be summarized by the fact that the Girya monad,  $\mathbb{P}$ , does not commute to limits. We are interested in describing statistical systems for which it is not possible to have

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complete knowledge of the states of the particles at the same time; we want to forget about the ‘ $\Omega$ ’ and we want to study statistical systems ‘locally’.

In [3, 4], we proposed to reformulate Statistical Mechanics and Gibbs measures introducing an appropriate category. Among others, in [5], we gave a characterization of independent variables in terms of projective objects in this category and an easy-to-verify condition that characterizes such objects (for the injective case see [6–9]). In [10], we proposed an Entropy functional for the categorical version of statistical systems. In this article, we show how the characterization of extreme Gibbs measures (Theorem 7.7 [1]), one of the steps for proving a zero-one law for the extreme Gibbs measures, transfers in the categorical setting. In the classical theory of rigorous statistical mechanics, the tail  $\sigma$ -algebra generates the observables for which a ‘generalized’ law of large numbers (zero-one law) holds. In this article, we give a candidate for such a  $\sigma$ -algebra in the categorical setting and show the associated extreme Gibbs measures decomposition. We will discuss in a follow-up paper the generalization of the zero-one law of specifications to  $\mathcal{A}$ -specifications.

## 2 $\mathcal{A}$ -specifications and their Gibbs measures

In classical theory, a statistical system is defined by a specification (Definition 1.23 [1]). We proposed in [3, 4] to reformulate such constructions as a couple of a presheaf and a functor over a poset, denoted as  $\mathcal{A}$ ; we call such a couple an  $\mathcal{A}$ -specification. Let us recall their definition. Let us denote **Mes** the category that has as objects measurable spaces and as morphisms measurable maps, and **Kern** the category that has as objects measurable spaces and as morphisms Markov kernels.

**Definition 2.1** ( $\mathcal{A}$ -specifications (Definition 7 [3])). Let  $\mathcal{A}$  be a poset. An  $\mathcal{A}$ -specification is a couple  $(G, F)$  of a presheaf and a functor  $G : \mathcal{A}^{op} \rightarrow \mathbf{Mes}$  and  $F : \mathcal{A} \rightarrow \mathbf{Kern}$  such that for any  $a, b \in \mathcal{A}$  with  $b \leq a$ ,

$$G_b^a \circ F_a^b = \text{id} \quad (1)$$

We denote  $\alpha \circ \beta$  simply as  $\alpha\beta$  in what follows.

Let us now recall the categorical generalization of the classical Gibbs measures (Definition 1.23 [1]) that we propose.

**Definition 2.2** (Gibbs measure for  $\mathcal{A}$ -specifications Definition 8 [3]). Let  $\gamma := (G, F)$  be an  $\mathcal{A}$ -specification,

$$\mathcal{G}(\gamma) := \{(p_a \in \mathbb{P}(G(a)), a \in \mathcal{A}) \mid \forall b \leq a, F_a^b p_b = p_a\} \quad (2)$$

with  $\mathbb{P}(G(a))$  the (measurable) space of probability measures over  $G(a)$ .

### 3 Categorical Version of the Tail $\sigma$ -Algebra

Measurable maps are particular Markov kernels. Let  $F : E_b \rightarrow E_a$  be a Markov kernel; let us denote it also as  $F(\omega_a|\omega_b)$ , where  $\omega_b \in E_b$  and  $\omega_a \in E_a$ . One associates to a Markov kernel  $F : E_b \rightarrow E_a$  a linear map  $\pi : L^\infty(E_a) \rightarrow L^\infty(E_b)$  defined as follows: for any  $f \in L^\infty(E_a)$ ,

$$\forall \omega_b \in E_b, \quad \pi(f)(\omega_b) = \int f(\omega_a)F(d\omega_a|\omega_b) \quad (3)$$

This association is ‘functorial’, we may denote the underlying functor  $L^\infty : \mathbf{Kern}^{op} \rightarrow \mathbf{Vect}$  which is presheaf from the category of Markov kernels to the category of vector spaces. It is the presheaf that associates spaces to their space of observables. Let us denote  $L^\infty \circ G : \mathcal{A} \rightarrow \mathbf{Vect}$  as  $i$  and  $L^\infty \circ F : \mathcal{A}^{op} \rightarrow \mathbf{Vect}$  as  $\pi$ . In these notation one has that for any  $a, b \in \mathcal{A}$  such that  $b \leq a$  then  $\pi_b^a \circ i_a^b = \text{id}$ .

Let us recall the definition of the tail  $\sigma$ -algebra in the classical formulation. Let  $\Omega = \prod_{i \in \mathbb{N}} E_i$ , where  $E_i$  are measurable spaces; let us denote  $\mathcal{F}_{\geq k}$  as the  $\sigma$ -algebra generated by the cylinders  $\prod_{n \geq k} E_n$ . The tail  $\sigma$ -algebra is defined as  $\bigcap_{k \in \mathbb{N}} \mathcal{F}_{\geq k}$ . A more general definition holds when  $I$  is any set and  $\mathcal{F}_{\supseteq a}$  is indexed by a subset  $a \subseteq I$  that is co-finite, i.e., which has a finite complement. For a functor from  $\mathcal{A}$  to  $\mathbf{Mes}$ , let us denote  $\sigma(G(a))$  as the sigma algebra of the measurable space  $G(a)$ , where  $a \in \mathcal{A}$ , and  $\sigma(G)$  as the underlying functor defined as  $\sigma(G)_a^b A_b := G_b^{a-1} A_b$ , with  $b \leq a$ . We propose that one candidate that plays the role of the tail  $\sigma$ -algebra for a given specification  $\gamma = (G, F)$  is  $\lim \sigma(G)$  defined as,

$$\lim \sigma(G) := \{(A_a \in \sigma(G(a)), a \in \mathcal{A}) | \forall a, b \in \mathcal{A}, \quad A_a = G_b^{a-1} A_b\} \quad (4)$$

Let us denote  $1_A : E \rightarrow \{0, 1\}$  the indicator function over the set  $A$  that sends  $\omega \in A$  to 1 and  $\omega \notin A$  to 0. Let us remark that  $1_{A_b} \circ G_b^a = 1[G_b^a(\omega_a) \in A_b] = 1_{G_b^{a-1} A_b}$ . Remark that  $A \in \lim \sigma(G)$  is equivalent to  $1_A \in \lim i$ ; in other words,  $\lim \sigma(G)$  is the restriction of  $\lim i$  to indicator functions of the form  $1_{A_a}$ ,  $a \in \mathcal{A}$ .

Finally, we also need to recall that for any  $f \in L^\infty(E)$  and  $\mu \in \mathbb{P}(E)$ , one can define a measure  $f \cdot \mu$  as  $f \cdot \mu(d\omega) = f(\omega)d\omega$ .

The key proposition of this document is Proposition 3.1; the proof of that proposition is given for  $G(a)$ ,  $a \in \mathcal{A}$  finite measurable sets. Therefore, we assume in what follows that the measurable sets are finite. However, there is no finiteness constraint on  $\mathcal{A}$ . We will say that  $F > 0$  when for any  $a, b \in \mathcal{A}$ , such that  $b \leq a$ ,  $F(\omega_a|\omega_b) > 0$  for any  $\omega_b$  such that  $G_b^a(\omega_a) = \omega_b$ ;  $G \circ F = \text{id}$  requires that  $F(\omega_a|\omega_b) = 0$  when  $G_b^a(\omega_a) \neq \omega_b$ .

The following lemma is an extension of the classical result that states that conditioning over a  $\sigma$ -subalgebra  $\mathcal{F}_1 \subseteq \mathcal{F}$  defines a morphism of modules when finer ( $\mathcal{F}$  measurable) observables are seen as modules over the coarser ( $\mathcal{F}_1$  measurable observables).

**Lemma 3.1.** *Let  $E_1, E_2$  be two measurable spaces, let  $g : E_2 \rightarrow E_1$  be a measurable map and  $f : E_1 \rightarrow E_2$  be a Markov kernel so that,  $f \circ g = \text{id}$ . Let us denote respectively  $i$  and  $\pi$  the induced linear maps on  $L^\infty(E_1), L^\infty(E_2)$ . Let  $h \in L^\infty(E_2)$  and  $k \in L^\infty(E_1)$ , then,*

$$\pi(h).k = \pi(h.i(k)) \quad (5)$$

*Proof.* Let us first prove the result in the particular case when  $h = 1_B$  with  $B \in \sigma(E_2)$  and  $k = 1_A$  with  $A \in \sigma(E_1)$ . Let us denote  $\bar{A}$  the complement of  $A$ , then  $1_A + 1_{\bar{A}} = 1$  and  $1_A.1_{\bar{A}} = 0$ . Furthermore  $i(1_A) = g^{-1}A$

$$\pi(1_B) = \pi(1_B.1_{g^{-1}A}) + \pi(1_B.1_{\overline{g^{-1}A}}) \quad (6)$$

and

$$\pi(1_B.1_{g^{-1}A}) \leq \pi(1_{g^{-1}A}) = \pi \circ i(1_A) = 1_A \quad (7)$$

$$\pi(1_B.1_{\overline{g^{-1}A}}) \leq \pi(1_{\overline{g^{-1}A}}) = 1_{\bar{A}} \quad (8)$$

Therefore,

$$\pi(1_B)1_A = \pi(1_B.1_{g^{-1}A})1_A + \pi(1_B.1_{\overline{g^{-1}A}})1_A \quad (9)$$

But,  $\pi(1_B.1_{\overline{g^{-1}A}})1_A \leq 1_{\bar{A}}.1_A = 0$  so,  $\pi(1_B)1_A = \pi(1_B.1_{g^{-1}A})1_A$ . Furthermore,  $\pi(1_B.1_{g^{-1}A}) = \pi(1_B.1_{g^{-1}A})1_A + \pi(1_B.1_{g^{-1}A})1_{\bar{A}}$  therefore,

$$\pi(1_B.1_{g^{-1}A})1_{\bar{A}} \leq \pi(i(1_A)).1_{\bar{A}} = 0 \quad (10)$$

We just showed that,

$$\pi(1_B.1_{g^{-1}A}) = \pi(1_B.1_{g^{-1}A})1_{\bar{A}} \quad (11)$$

So  $\pi(1_B).1_A = \pi(1_B.i(1_A))$ . The result then extends by linearity directly to  $h = \sum_{k \leq n} 1_{B_k}$  and  $k = \sum_{k \leq n_1} 1_{A_k}$ , which ends the proof.  $\square$

Let us remark that if  $A \in \lim \sigma(G)$  then  $\bar{A} := (\bar{A}_a, a \in \mathcal{A})$  is also in  $\lim i$ .

**Proposition 3.1.** *Let  $\gamma = (G, F)$  be a specification, let  $G(a)$  be finite sets for any  $a \in \mathcal{A}$ , let  $F > 0$ . Let  $\mu \in \mathcal{G}(\gamma)$ , for any  $f \in \prod_{a \in \mathcal{A}} L^\infty(G(a))$ , such that  $\forall a \in \mathcal{A}, \mu_a(f_a) = 1$ ,*

$$f.\mu \in \mathcal{G}(\gamma) \iff \exists \tilde{f} \in \lim i, \text{ s.t. } f.\mu = \tilde{f}.\mu \quad (12)$$

*Proof.* Let us assume that  $f.\mu \in \mathcal{G}(\gamma)$ , then for any  $a, b \in \mathcal{A}$  such that  $b \leq a$ , and for any  $g_a \in L^\infty(G(a))$ , by hypothesis,  $(f.\mu)_b \pi_b^a(g_a) = (f.\mu)_a(g_a)$ ; it can be rewritten as,

$$\mu_b(\pi_b^a(g_a).f_b) = \mu_a(f_a.g_a) \quad (13)$$

By Lemma 3.1,  $\pi_b^a(g_a).f_b = \pi_b^a(g_a.i_a^b f_b)$ ; therefore,

$$\mu_b \pi_b^a(g_a.i_a^b f_b) = \mu_a(f_a.g_a) \quad (14)$$

Therefore  $f_a = i_a^b f_b$   $\mu_a$ -almost surely.

We will now show that there is  $\tilde{f} \in \lim i$  such that  $\tilde{f}.\mu = f.\mu$ . It is in this part of the proof that we assume that  $G(a)$  are finite sets and that  $F > 0$ . Let's call  $S_a := \text{supp} \mu_a$ , the support of  $\mu_a$ , i.e., the set  $\text{supp} \mu_a := \{\omega_a \in G(a) \mid \mu_a(\omega_a) > 0\}$ . Let us denote  $N_a := \overline{S_a}$  its complement and  $M_a = 1_{\overline{N_a}}$ . We will now show that  $(N_a, a \in \mathcal{A}) \in \lim \sigma(G)$ .

For any  $b, a \in \mathcal{A}$  such that  $b \leq a$ ,  $\mu_a i_a^b = \mu_b$ ; therefore, as  $\mu_b(M_b) = 0$ , one has that  $\mu_a i_a^b(M_b) = 0$ . Recall that  $i_a^b(M_b)$  is the indicator function of the set  $G_b^{a-1} N_b$ ; the previous remark implies that  $i_a^b(M_b) \leq M_a$ . Furthermore,  $\mu_b \pi_b^a(M_a) = \mu_a(M_a) = 0$ ; therefore,  $\pi_b^a(M_a) \leq M_b$ .

Hence, as  $\pi_b^a(i_a^b M_b) = M_b$  and  $i_a^b M_b \leq M_a$ , then by applying  $\pi_b^a$  on both sides,  $M_b \leq \pi_b^a(M_a)$ . And so  $\pi_b^a(M_a) = M_b$ .

Recall that we showed that  $\pi_b^a(M_a) = M_b$  and  $i_a^b(M_b) \leq M_a$ . In particular,  $M_a - i_a^b(M_b) \geq 0$ ; furthermore,  $\pi_b^a(M_a - i_a^b(M_b)) = 0$  so  $M_a = i_a^b(M_b)$ . To be more explicit:  $\forall \omega_b \in G(b)$ ,

$$\pi_b^a(M_a - i_a^b(M_b))(\omega_b) = \sum_{\omega_a \in G(a)} F(\omega_a \mid \omega_b) [M_a - i_a^b(M_b)](\omega_a) \quad (15)$$

As  $F(\omega_a \mid \omega_b) > 0$  by hypothesis, then  $M_a = i_a^b(M_b)$ . This implies that  $M \in \lim i$  and  $N \in \lim \sigma(G)$ . This also implies that  $S \in \lim \sigma(G)$ .

Let  $\tilde{f} = f 1_S$ . Then  $f 1_S \in \lim F$  and for any  $a \in \mathcal{A}$ ,  $f_a = \tilde{f}_a$   $\mu_a$ -a.s., which ends the proof. □

One remarks that  $\lim i$  is a subset of  $\lim \pi$ : for any  $f \in \lim i$ , by definition for any  $a, b \in \mathcal{A}$  such that  $b \leq a$ ,  $i_a^b f_b = f_a$  and so  $\pi_b^a i_a^b f_b = \pi_b^a f_a$  so  $f_b = \pi_b^a f_a$ .

Let us also remark that for any  $b \leq a \in \mathcal{A}$ ,  $\mu_a(A_a) = \mu_b(A_b)$  for  $\mu \in \mathcal{G}(\gamma)$ ,  $A \in \lim G$ .

**Theorem 3.1** (Extreme measure characterisation (Generalisation of Theorem 7.7 [1])). *Let  $\gamma = (G, F)$  be a specification, let  $G(a)$  be finite sets for any  $a \in \mathcal{A}$ , let  $F > 0$ .  $G(\gamma)$  is a convex set. Each  $\mu \in \mathcal{G}(\gamma)$  is uniquely determined by its restriction to  $\lim \sigma(G)$ . Furthermore  $\mu$  is extreme in  $\mathcal{G}(\gamma)$  if and only if for any  $A \in \lim \sigma(G)$ ,  $\forall a \in \mathcal{A}$ ,  $\mu_a(A_a) = 0$  or 1.*

*Proof.* Let us denote  $\pi^*$  the functor from  $\mathcal{A}$  to  $\mathbf{Vect}$  for which for any  $b \leq a$ ,  $\pi_a^{*b} : (L^\infty F(b))^* \rightarrow (L^\infty F(a))^*$  is the dual of  $\pi_b^a$  that send linear forms to linear forms. Then  $\mathcal{G}(\gamma)$  is a subspace of the vector space  $\lim F^*$  and furthermore for any  $a \in \mathcal{A}$  and  $p \in [0, 1]$ ,  $p\mu_a + (1-p)\nu_a \in \mathbb{P}(G(a))$  whenever  $\mu_a, \nu_a \in \mathbb{P}(G(a))$ . Therefore  $\mathcal{G}(\gamma)$  is a convex set.

Proposition 3.1, allows us to apply a similar proof, when done with caution, to the one found of Theorem 7.7 in [1]. Let us recall the proof. Let  $\mu, \nu \in \mathbb{G}(\gamma)$  such that  $\mu|_{\lim i} = \nu|_{\lim i}$ . Let  $\bar{\mu} = \frac{\mu + \nu}{2}$ , then  $\bar{\mu} \in \mathcal{G}(\gamma)$ . But  $\mu$  and  $\nu$  are absolutely continuous with respect to  $\bar{\mu}$  therefore for any  $a \in \mathcal{A}$  there is  $f_a, g_a \in L^\infty(G(a))$  such that  $\mu_a = f_a \bar{\mu}_a$  and  $\nu_a = g_a \bar{\mu}_a$ . By Proposition 3.1,  $f, g \in \lim i$ . By hypothesis, for any  $h \in \lim i$   $\bar{\mu}_a(h_a) = \mu_a(h_a) = \nu_a(h_a)$ . Importantly  $i_a^b$  is a ring morphism of  $L^\infty(G(b))$ , i.e.  $i_a^b(k_b \cdot h_b) = i_a^b(k_b) \cdot i_a^b(h_b)$ . Therefore for any  $h, k \in \lim i$ ,  $k \cdot h \in \lim i$ ; as  $f - g \in \lim i$ , then it is also the case that  $(f - g)^2 \in \lim i$ ; but for any  $a \in \mathcal{A}$ ,

$$\bar{\mu}_a[(f_a - g_a)^2] = 0 \quad (16)$$

so  $f_a = g_a$   $\bar{\mu}_a$  - a.s. Therefore  $f\bar{\mu} = g\bar{\mu}$  and  $\mu = \nu$ .

Showing that  $\mu \in \mathcal{G}(\gamma)$  is extreme is equivalent to  $\mu$  being trivial on  $\lim i$  is a direct generalization of Corollary 7.4 [1] thanks to Proposition 3.1. Let  $\mu \in \mathcal{G}(\gamma)$  be not trivial on  $\lim \sigma(G)$  then there is  $A = (A_a, a \in \mathcal{A}) \in \lim \sigma(G)$  such that,

$$\forall a \in \mathcal{A}, \quad 0 < \mu_a(A_a) < 1 \quad (17)$$

Therefore for any  $a \in \mathcal{A}$ ,

$$\mu_a = \mu_a(A_a) \frac{1_{A_a} \cdot \mu_a}{\mu_a(A_a)} + \mu_a(\bar{A}_a) \frac{1_{\bar{A}_a} \cdot \mu_a}{\mu_a(\bar{A}_a)} \quad (18)$$

Furthermore, for any  $b \leq a$ ,

$$i_a^b \frac{1_{A_b}}{\mu_a(A_b)} = \frac{1_{A_a}}{\mu_a(A_b)} = \frac{1_{A_a}}{\mu_a(A_a)} \quad (19)$$

Therefore  $\frac{1_{A_a}}{\mu_a(A_a)}, a \in \mathcal{A}$  is in  $\lim i$  and so by Lemma 3.1,  $\left(\frac{1_{A_a} \cdot \mu_a}{\mu_a(A_a)}, a \in \mathcal{A}\right) \in \mathcal{G}(\gamma)$ . Similarly  $\left(\frac{1_{\bar{A}_a} \cdot \mu_a}{\mu_a(\bar{A}_a)}, a \in \mathcal{A}\right) \in \mathcal{G}(\gamma)$ . In particular there is  $0 < p < 1$  so that  $\mu = p\nu + (1-p)\nu_1$  with  $\nu, \nu_1 \in \mathcal{G}(\gamma)$ . Therefore  $\mu$  is not an extreme measure.

Assume now that  $\mu \in \mathcal{G}(\gamma)$  is such that for any  $A \in \lim \sigma(A)$ , and any  $a \in \mathcal{A}$ ,  $\mu_a(A_a) = 0$  or  $1$ . Suppose that there is  $0 < p < 1$  such that  $\mu = p\nu + (1-p)\nu_1$  with  $\nu, \nu_1 \in \mathcal{G}(\gamma)$ . Then for any  $a \in \mathcal{A}$ ,  $\nu_a, \nu_{1a}$  is absolutely continuous with respect to  $\mu_a$ . Therefore, there are  $(f_a \geq 0, a \in \mathcal{A}), (g_a \geq 0, a \in \mathcal{A})$  both in  $\prod_{a \in \mathcal{A}} L^\infty(G(a))$  such that  $\nu = f\mu$  and  $\nu_1 = g\mu$ . As  $\nu, \nu_1 \in \mathcal{G}(\gamma)$ , then by Lemma 3.1,  $f, g \in \lim i$ . Therefore, for all  $a \in \mathcal{A}$ ,

$\mu_a(f_a) = 0$  or for all  $a \in \mathcal{A}$ ,  $\mu_a(g_a) = 0$ . So,  $\mu = \nu$  or  $\mu = \nu_1$  and  $\mu$  is extreme in  $\mathcal{G}(\gamma)$ . □

Let us remark that if  $\mathcal{A}$  has only one connected component, then for  $A \in \lim \sigma(G)$ , satisfying  $\forall a \in \mathcal{A}, \mu_a(A_a) = 0$  or  $1$  is equivalent to  $\exists a \in \mathcal{A}, \mu_a(A_a) = 0$  or  $1$ . Indeed, if  $a, b$  are in the same connected component, i.e.,  $a \leq b$  or  $b \leq a$ , then  $\mu_a(A_a) = \mu_b(A_b)$ .

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