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Letter

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We introduce manifestly crossing-symmetric expansions for arbitrary systems of 1D CFT correlators. These expansions are given in terms of certain Polyakov blocks which we define and show how to compute efficiently. Equality of operator product expansion and Polyakov block expansions leads to sets of sum rules that any mixed correlator system must satisfy. The sum rules are diagonalized by correlators in tensor product theories of generalized free fields. We show that it is possible to do a change of a basis that diagonalizes instead mixed correlator systems involving elementary and composite operators in a single field theory. As an application, we find the first nontrivial examples of optimal bounds, saturated by the mixed correlator system ϕ, ϕ^2 in the theory of a single generalized free field.

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Introduction and setup. Conformal theories in one dimension are interesting both in theory and in practice. On the one hand, they have a wide range of applications: from conformal defects to boundary conditions in 2D conformal field theory (CFT), passing through 2D QFTs in anti–de Sitter (AdS) space and long range spin models [1–10]. On the other, being confined to a line and lacking spin, such theories have greatly simplified kinematics, while being far from elementary toy models, as there is no known nontrivial example which has been exactly solved. Thus these systems offer both a challenge as well as opportunity for significant progress in the conformal bootstrap program.

At its heart this program is about understanding how crossing equations constrain sets of CFT data, both analytically and numerically. As it turns out, working with such equations in their original formulation in position space is far from optimal for the purpose of deriving such constraints. Instead, work in recent years suggests one should apply a certain transform from position space onto an auxiliary functional space [11,12]. By construction, this mapping is done in such a way that crossing equations become translated into a completely equivalent set of sum rules which are now transparently solved by particular sparse sets of CFT data, something which is completely obscured in position space. Since all sets of CFT data, sparse or not, obey a measure of universality for large scaling dimensions [13,14], these sum rules naturally give rise to a decoupling between low and high energy data, leading to rapidly convergent bounds and constraints [15].

Up to now constructions of such functional spaces have been limited to systems of correlation functions of operators lying in the same symmetry multiplet [16]. This is a major restriction, as it is known that many CFTs of interest can only be effectively bootstrapped by considering systems of correlations functions involving distinct operator multiplets, the most famous example being the celebrated 3D Ising island [17–19]. In this paper we will resolve this shortcoming for 1D CFTs, by constructing new sets of sum rules valid for arbitrarily large systems of bootstrap equations. Our approach is to propose a generalization of the so-called Polyakov bootstrap to a multicorrelator setup [11,12,16,20–23]. Contrary to previous work, this allows us to bypass the laborious construction of functional kernels which implement the above mentioned transform, obtaining instead the relevant sum rules directly. The price to pay is that one cannot rigorously prove that these sum rules are really equivalent to the constraints of crossing. In the present work we will settle for checking that our sum rules pass several highly nontrivial consistency checks, giving us enough confidence to begin using them for both analytic and numeric explorations. In both cases we show they significantly outperform traditional approaches, leading to new qualitative and quantitative insights into the structure of crossing equations and the systematics of bootstrapping 1D CFTs.

Setup: After these remarks, let us begin by recalling some basic facts and establishing notation. We are interested in

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1D CFTs with primary operators labeled generically as Φ_i. Conformal invariance dictates a correlation function $\langle\Phi_l(\infty)\Phi_k(1)\Phi_j(z)\Phi_i(0)\rangle$ can be expressed as a function $\mathcal{G}^{ijkl}(z)$ of a single cross-ratio z. Using the operator product expansion (OPE) $\Phi_i\times\Phi_j$ (or simply ij), we obtain an expansion for $\mathcal{G}$:

$$\mathcal{G}^{ij,kl}(z)=\sum_{\mathcal{O}}\lambda^{ij}_{\mathcal{O}}\lambda^{kl}_{\mathcal{O}}G^{ij,kl}_{\Delta_{\mathcal{O}}}(z). \tag{1}$$

The sum runs over all primary operators $\mathcal{O}$ labeled by their scaling dimension $\Delta_{\mathcal{O}}$ and spacetime parity $P_{\mathcal{O}}=\pm$. Here $\lambda^{ij}_{\mathcal{O}^\pm}=\langle\Phi_i(\infty)|\Phi_j(1)|\mathcal{O}^\pm(0)\rangle=\pm\lambda^{ji}_{\mathcal{O}^\pm}$ and $G^{ij,kl}_{\Delta_{\mathcal{O}}}(z)$ is the 1D conformal block, see Eq. (6) in the Supplemental Material [24].

For a system of mixed correlators we find it useful to introduce the "OPE orientation vector" $r^{ij}_{\mathcal{O}}$, a new set of quantum numbers such that $\lambda^{ij}_{\mathcal{O}}=|\lambda_{\mathcal{O}}|r^{ij}_{\mathcal{O}}$ and $\sum_{i,j}(r^{ij}_{\mathcal{O}})^2=1$, which describes how $\mathcal{O}$ couples to different pairs of operators. We may then rewrite (1) as

$$\mathcal{G}^{ij,kl}(z)=\sum_{\mathcal{O}}\lambda^2_{\mathcal{O}}G^{ij,kl}_{\mathcal{O}}(z), \tag{2}$$

with $G^{ij,kl}_{\mathcal{O}}(z)\coloneqq r^{ij}_{\mathcal{O}}r^{kl}_{\mathcal{O}}G^{ij,kl}_{\Delta_{\mathcal{O}}}(z)$.

An important property of a four-point correlator is crossing symmetry. The OPE decomposition given by (2) is not manifestly crossing symmetric, as conformal blocks are associated to a particular OPE channel. To remedy this we introduce the *Polyakov block expansion* [12,20,21]:

$$\mathcal{G}^{ij,kl}(z)=\sum_{\mathcal{O}}\lambda^2_{\mathcal{O}}\mathcal{P}^{ij,kl}_{\mathcal{O}}(z). \tag{3}$$

By construction, the Polyakov blocks $\mathcal{P}^{ij,kl}_{\mathcal{O}}$ are built to manifestly satisfy crossing. In particular, while in (2) the only operators which give a nonzero contribution are those in the s-channel OPE $\Phi_i\times\Phi_j$, the Polyakov block sum receives contributions also from the OPE channels ik and il. The price to pay for this representation is that term by term the OPE contains contributions from unphysical states which must decouple in the full sum. Concretely, the *Polyakov bootstrap* is the following statement:

$$\sum_{\mathcal{O}}\lambda^2_{\mathcal{O}}[G^{ij,kl}_{\mathcal{O}}(z)-\mathcal{P}^{ij,kl}_{\mathcal{O}}(z)]=0. \tag{4}$$

This should be thought of as a reformulation of the constraints of crossing symmetry, which has to be satisfied by any system of correlators in any CFT. We will shortly show how these bootstrap equations may be turned into a more useful discrete set of sum rules on the CFT data by using the OPE decomposition of the Polyakov blocks.

As functions, Polyakov blocks can be computed as sums of Witten diagrams. Starting off with the theory of N decoupled free fields Ψ_i in AdS_2, the associated boundary correlators correspond to those of the tensor product theory of N generalized free fields (GFF) Φ_i. Introduce now a new spin-0 bulk field χ_Δ with mass $m^2=\Delta(\Delta-d)$ and dual operator $\mathcal{O}_\Delta$ of positive parity [32], with the following couplings:

$$\mathcal{L}^{\mathrm{int}}\propto\int_{\mathrm{AdS}}\sum_{i,j}^{N}r^{ij}_{\mathcal{O}}\chi_\Delta\Psi_i\Psi_j. \tag{5}$$

Then to leading order, the connected correlators in this theory are essentially proportional to the Polyakov block with the right quantum numbers, including the OPE orientation $r_{\mathcal{O}}$. We emphasize that this construction is simply a convenient recipe for computing the Polyakov blocks *as functions*: the constraints (4) are meant to hold for all CFTs and not just those arising from weakly coupled fields in AdS_2.

Sum rules. Since Polyakov blocks correspond to deformations of generalized free correlators, their OPE content consists of double trace operators, whose schematic form and scaling dimensions are given as

$$\begin{aligned}(ij)_n&\equiv\Phi_i\Box^n\Phi_j, &\Delta_{(ij)_n}&=\Delta_i+\Delta_j+2n,\\ [ij]_n&\equiv\Phi_i\overleftrightarrow{\partial}\Box^n\Phi_j, &\Delta_{[ij]_n}&=\Delta_i+\Delta_j+2n+1.\end{aligned} \tag{6}$$

At this point it is convenient to introduce some notational shorthand. We will denote a set of external fields ij,kl by a letter E (for external). We will also denote by I_c (for internal channel) a generic double trace operator appearing in the OPE channel c. A given I_c is always with respect to some particular E. As an example, for $E=\{ij,kl\}$ we have

$$I^+_s\in\mathcal{I}^{E,+}_s\coloneqq\{(ij)_n\}_{n=0}^{\infty}\cup\{(kl)_n\}_{n=0}^{\infty}. \tag{7}$$

Similarly we denote I^+_t and I^+_u for the double traces $(il)_n$, $(jk)_n$ and $(ik)_n$, $(jl)_n$, respectively. The negative parity channels are obtained from the above by swapping round and square brackets. We will also denote $I_s\in\mathcal{I}^{E,+}_s\cup\mathcal{I}^{E,-}_s$, and so on. Finally, we set

$$r^{ij,kl;s}_{\mathcal{O}}\coloneqq r^{ij}_{\mathcal{O}}r^{kl}_{\mathcal{O}} \tag{8}$$

and similarly $r^{ij,kl;t}_{\mathcal{O}}=r^{il}_{\mathcal{O}}r^{jk}_{\mathcal{O}}$ and $r^{ij,kl;u}_{\mathcal{O}}=r^{ik}_{\mathcal{O}}r^{jl}_{\mathcal{O}}$.

After these notational preliminaries we are ready to discuss the OPE for Polyakov blocks [24]. As mentioned, they can be written as sums of Witten diagrams in AdS_2, which include both exchanges and contact diagrams. Exchange diagrams have conformal block decompositions as follows [22,33]:

$$W^{E,c}_{\Delta,P}(z)=\delta_{c,s}G^E_\Delta(z)-\sum_{I_s}\alpha^{E,c}_{I_s}(\Delta,P)G^E_{\Delta_{I_s}}(z), \tag{9}$$

where parity P of the exchanged operator (even, odd) corresponds to the bulk spin $\ell = 0, 1$, respectively. The coefficients appearing above satisfy the orthogonality properties

$$\alpha^{E,c}_{I_s^\pm}(\Delta_{I_c'^\pm}, P) = \delta_{c,s}\delta_{P,\pm}\delta_{I_s^\pm, I_c'^\pm}. \tag{10}$$

As for contact diagrams, their block expansions take the form

$$W^{E,\mathrm{con}}(z) = \sum_{I_s} \alpha^{E,\mathrm{con}}_{I_s} G^E_{\Delta_{I_s}}(z). \tag{11}$$

The Polyakov blocks are then given as

$$\mathcal{P}^E_{\mathcal{O}}(z) = \sum_c r^{E,c}_{\mathcal{O}} W^{E,c}_{\Delta,P}(z) + \text{contact diagrams}. \tag{12}$$

We will fix the contribution of contact diagrams momentarily. Using the OPE and commuting sums, the Polyakov bootstrap equations become a set of conditons on OPE data which can be written as follows:

Functional sum rules: For all $I_s \in \mathcal{I}^E_s$ there is a functional $\alpha^E_{I_s}$ with the sum rule

$$\begin{aligned}\sum_{\mathcal{O}} \lambda^2_{\mathcal{O}} \alpha^E_{I_s}[\mathcal{O}] \coloneqq \sum_{\Delta,P=\pm} \lambda^2_{\Delta,P} \Big[& r^{ij,kl;s}_{\Delta,P} \alpha^{ij,kl;s}_{I_s}(\Delta,P) \\ & + r^{ij,kl;t}_{\Delta,P} \alpha^{ij,kl;t}_{I_s}(\Delta,P) \\ & + r^{ij,kl;u}_{\Delta,P} \alpha^{ij,kl;u}_{I_s}(\Delta,P) \Big] = 0.\end{aligned} \tag{13}$$

The coefficients appearing in these sum rules are the *functional actions*. They satisfy a set of duality properties following the orthogonality relations (10). A slight issue arises when we have $\Delta_i + \Delta_j = \Delta_k + \Delta_l$, for instance in a correlator $\langle \phi_1\phi_2\phi_1\phi_2\rangle$. In that case some coefficients in the Witten exchange diagram OPE expansion are degenerate [24], and we must make the replacements

$$\{\alpha_{(ij)_n}, \alpha_{(kl)_n}\} \to \{\alpha_{(ij)_n}, \beta_{(ij)_n}\}, \tag{14}$$

and similarly for $[ij]_n$. The duality properties are then [writing $\alpha^E_{I_s}[I_c'] \coloneqq \alpha^{E,c}_{I_s^{P'}}(\Delta_{I_c'^{P'}}, P)$, etc.]

$$\begin{aligned}\alpha^E_{I_s}[I_c'] &= \delta_{c,s}\delta_{I_s,I_c'}, & \partial_\Delta \alpha^E_{I_s}[I_c'] &= 0, \\ \partial_\Delta \beta^E_{I_s}[I_c'] &= \delta_{c,s}\delta_{I_s,I_c'}, & \beta^E_{I_s}[I_c'] &= 0.\end{aligned} \tag{15}$$

The sum rules and duality relations have an implicit dependence on the contact diagrams as described below. Ignoring this part, there is one sum rule per label $I_s \in \mathcal{I}_s$ and per choice of E.

Finally, let us discuss how to fix contact diagrams. Our guiding principle is that the sum rules should bootstrap solutions to crossing with the same UV, or Regge behavior, as ordinary CFT correlators [12,34]. In AdS_2 such solutions can be contact diagrams arising from *relevant* deformations in the bulk theory i.e. four-point interactions with at most two bulk derivatives. So for each choice of external states we include all such independent contact diagrams in the Polyakov block (12). We also require that we lose as many sum rules from (13) as there are contact diagrams. For example, for a certain choice of external states E if there is only one independent contact diagram for all the possible permutations, then we would redefine

$$\begin{aligned}\mathcal{P}^E_{\mathcal{O}}(z) &\to \mathcal{P}^E_{\mathcal{O}}(z) + \frac{\alpha^{\hat{E}}_{I_s}[\mathcal{O}]}{\alpha^{\hat{E},\mathrm{con}}_{\hat{I}_s}} W^{E,\mathrm{con}}(z), \\ \alpha^E_{I_s}[\mathcal{O}] &\to \alpha^E_{I_s}[\mathcal{O}] - \frac{\alpha^{E,\mathrm{con}}_{I_s}}{\alpha^{\hat{E},\mathrm{con}}_{\hat{I}_s}} \alpha^{\hat{E}}_{\hat{I}_s}[\mathcal{O}],\end{aligned} \tag{16}$$

where $\hat{E}$ corresponds to some definite permutation of the indices in E. The above eliminates the functional $\alpha^{\hat{E}}_{\hat{I}_s}$, as well as the corresponding duality relations. This procedure is such that by construction, contact diagrams are now manifest solutions to the sum rules. For notational clarity, below we will leave the subtraction procedure implicit.

Basis change and optimal bounds. The ϕ, ϕ^2 system: The duality relations (10) imply that the functional sum rules trivialize on a set of correlators of decoupled GFF scalars [24]. As a more interesting application, here we will examine how to bootstrap the ϕ, ϕ^2 system in a single GFF theory. In our language, this is the following system of unitary CFT correlators of fields Φ_1 and Φ_2 with dimensions $\Delta_\phi \equiv \Delta_1 = \frac{1}{2}\Delta_2$:

$$\begin{aligned}&\mathcal{G}^{11,11}(z) = \mathcal{G}^{\mathrm{gff}}_{\Delta_\phi}(z), \quad \mathcal{G}^{12,12}(z) = 1 + a_1^{12,12}\big(\mathcal{G}^{\mathrm{gff}}_{\Delta_\phi}(z) - 1\big), \\ &\mathcal{G}^{22,22}(z) = \mathcal{G}^{\mathrm{gff}}_{2\Delta_\phi}(z) + \frac{a_2^{22,22}}{2} \frac{z^{2\Delta_\phi} + (1-z)^{2\Delta_\phi} + 1}{z^{2\Delta_\phi}(1-z)^{2\Delta_\phi}}.\end{aligned} \tag{17}$$

with $\mathcal{G}^{\mathrm{gff}}$ the GFF correlator [see (25) in [24]]. There is a Z_2 symmetry under which Φ_1 and Φ_2 are odd $(-)$ and even $(+)$, respectively. It is straightforward to expand the above in conformal blocks to extract the CFT data.

Let us discuss how the same data is reproduced using our sum rules. We include in all OPE channels all possible single and double trace operators consistent with the symmetry. To allow for possible degeneracies we introduce the notation $a^{ij,kl}_{\mathcal{O}_\pm} = \sum_{\Delta=\Delta_\pm} \lambda^{ij}_{\Delta_\pm}\lambda^{kl}_{\Delta_\pm}$. We then must impose the nondegeneracy conditions:

$$a_1^{12,12} = a^{11,11}_{(11)_0}, \quad a_2^{22,22} = (a^{11,22}_{(11)_0})^2 / a^{11,11}_{(11)_0}. \tag{18}$$

These tell us that there are unique operators with dimensions Δ_ϕ or $2\Delta_\phi$, respectively. Note that these conditions

cannot be derived, but rather must be imposed as inputs which fix the solution to be bootstrapped (i.e. there are solutions to crossing where they are not true [24]). We can now determine all other OPE coefficients. In particular, the $\alpha^{11,22}_{(11)_n}$, $\alpha^{12,12}_{(12)_n}$ $\alpha^{12,12}_{[12]_n}$ sum rules determine the OPE coefficients $a^{11,22}_{(11)_n}$, $a^{12,12}_{(12)_n}$, and $a^{12,12}_{[12]_n}$, respectively, in terms of $a^{12,12}_{\Delta_\phi}$. Similarly, using $\alpha^{11,22}_{(22)_n}$ the $a^{11,22}_{(22)_n}$ are also determined and come out zero as expected. At this point we use the nondegeneracy condition $a^{22,22}_{(11)_n} = (a^{11,22}_{(11)_n})^2/a^{11,11}_{(11)_n}$ and now the remaining OPE data can be solved for using the $\alpha^{11,11}_{(11)_n}$ and $\alpha^{22,22}_{(22)_n}$ sum rules. Our results match those extracted from the correlators above, giving a nontrivial check of our sum rules.

Basis change: A dissatisfying feature of the last computation was that the basis of sum rules was not entirely diagonal with respect to the solution. Indeed the equations in the 22,22 channel involve an infinite number of variables, and have to be solved after the 11,11 and 11,22 channels. Here we will construct a new functional basis that does not suffer from this problem. Consider the functionals $\alpha^{22,22}_{(22)_n}$. We will construct modified versions, $\hat{\alpha}^{22,22}_{(22)_n}$ satisfying the following duality conditions:

$$\begin{aligned}
&\hat{\alpha}^{22,22}_{(22)_n}[(22)_m] = \delta_{n,m}, && \partial_\Delta \hat{\alpha}^{22,22}_{(22)_n}[(22)_m] = 0,\\
&\hat{\alpha}^{22,22}_{(22)_n}[(11)_m] = 0 && \partial_\Delta \hat{\alpha}^{22,22}_{(22)_n}[(11)_m] = 0,\\
&\partial_r \hat{\alpha}^{22,22}_{(22)_n}[(22)_m] = 0, && \partial_r \hat{\alpha}^{22,22}_{(22)_n}[(11)_m] = 0.
\end{aligned} \tag{19}$$

Similar equations can be written for $\beta^{22,22}_{(22)_n}$, by moving the Kronecker delta to the right column. These conditions are understood as follows. The first line are the original conditions satisfied by the functionals. Adding the second line implies that the new functionals now have (double) zeros acting on the $(11)_m$ operators—recall this means evaluating the functionals on the right dimensions and OPE orientation, and in this case we mean the $(11)_m$ operator for the ϕ, ϕ^2 system. Finally the last line guarantees that these vanishing conditions are still true under small deformations of the OPE orientation. Essentially the second and third lines ensure that the functional does not change sign in the neighbourhood of its zeros. While these two extra sets of conditions are not necessary for diagonalizing the bootstrap equations for the mixed GFF solution, they do allow us to have diagonal equations even slightly away from this solution.

In practice the new duality conditions can be satisfied by setting

$$\hat{\alpha}^{22,22}_{(22)_n} = \alpha^{22,22}_{(22)_n} - \sum_{m=0}^{\infty}\left[c^m_n \alpha^{11,11}_{(11)_m} + d^m_n \alpha^{11,22}_{(11)_m} + e^m_n \beta^{11,11}_{(11)_m}\right], \tag{20}$$

and tuning coefficients appropriately. The same procedure can be applied to $\beta^{22,22}_{(22)_n}$.

An optimal bound: We will now show that using a functional closely related to the above we can obtain an optimal bound satisfied by GFF. Let us consider the same setup as before, with operators labeled by their Z_2 ($\pm$) and spacetime parity (0,1 for even/odd) quantum numbers, e.g. 0^+ etc. We set $\phi \equiv \Phi_1$ and $\phi^2 \equiv \Phi_2$ with dimensions and Z_2 charges as before. We assume that ϕ and ϕ^2 are unique and also the leading nonidentity operators, and fix the OPE data of the ϕ^2 operator (i.e. $a^{ij,kl}_{\phi^2}$) to the GFF values, which with the uniqueness assumption i.e. (18) also fixes $a^{12,12}_{\phi}$. Our analytic bound is more simply stated for $\Delta_\phi \leq 1$, which we do here, leaving further discussion to [24]. In this case our last assumption is that in the 0^+ sector there should be no other operators below a gap no smaller than $\simeq 2\sqrt{2}\Delta_\phi$, apart from ϕ^2.

Under these assumptions, consider a 0^+ operator denoted ϕ^4 and of dimension $4\Delta_\phi$. Then we claim there is an upper bound on $a^{22,22}_{\phi^4}$ which is saturated by the GFF solution. To prove this bound we first define $\Omega \coloneqq \alpha^{22,22}_{(22)_0} + a\alpha^{11,11}_{(11)_0} + b\alpha^{12,12}_{[12]_0} + c\beta^{12,12}_{[12]_0}$. For any a, b, c we can now "dress" this functional so that it is orthogonal to states in the 11 OPE. Specifically, we add to Ω an infinite sum as in (20) and impose most of the duality conditions (19) except the one involving the state $(11)_0$ to obtain a new dressed functional $\hat{\Omega}$. We find that for suitable choices of a, b, c in some range $\hat{\Omega}$ is positive semidefinite for scaling dimensions consistent with the gap assumptions above, see Fig. 1. Thus it leads to a bound

$$a^{22,22}_{\phi^4} \leq -\left(\hat{\Omega}[\mathrm{Id}] + a^{12,12}_{\phi}\hat{\Omega}[\phi] + a^{22,22}_{\phi^2}\hat{\Omega}[\phi^2]\right) = a^{22,22}_{\phi^4,\mathrm{gff}}. \tag{21}$$

To understand the equality, note that by construction $\hat{\Omega}$ annihilates every operator appearing in the GFF solution, except for ϕ, ϕ^2, and ϕ^4. This implies that for that solution the above inequality becomes an identity, i.e. the bound is saturated. To show this it is important to note that in the GFF solution there is no operator $[12]_0$ (it is $\partial\phi^3$, a descendant), otherwise $\alpha^{12,12}_{[12]_0}$ would be sensitive to it.

To conclude, let us make a few comments on our assumptions. It may seem surprising that we need to impose a gap in the Z_2 even sector even after completely fixing the data of ϕ^2. Indeed, a nontrivial but true fact is that fixing this data to its GFF values already implies that the *entire* 11 OPE must be that of a GFF [12]. It is easy to see that this means that the 1111 and 1122 four-point functions are automatically the same as the GFF ones, and that the $(11)_n$ double traces appear in the 22 OPE with their GFF

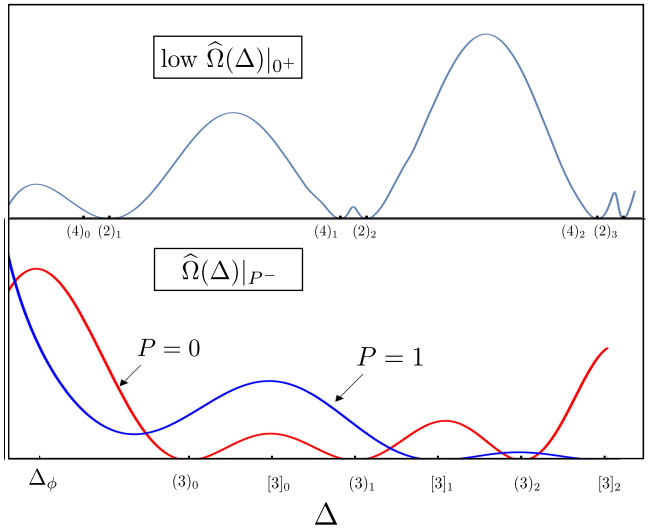

FIG. 1. The $\hat{\Omega}$ functional action for $\Delta_\phi = 9/10$ [height rescaled for clarity; $(n)_k$ and $[n]_k$ denote $\Delta = n\Delta_\phi + 2k$ and $\Delta = n\Delta_\phi + 2k + 1$, respectively]. It is a 2×2 matrix corresponding to 11 and 22 OPE channels. Top (Z_2 even sector): we plot its lowest eigenvalue which is non-negative for $\Delta \geq 2\Delta_\phi + 2$. At zeros the corresponding eigenvector is proportional to $\{\lambda^{11}_{\mathcal{O}}, \lambda^{22}_{\mathcal{O}}\}$ of the GFF solution. Bottom (Z_2 odd sector): we see positivity is achieved for any gap, but optimal bound requires compatibility with GFF.

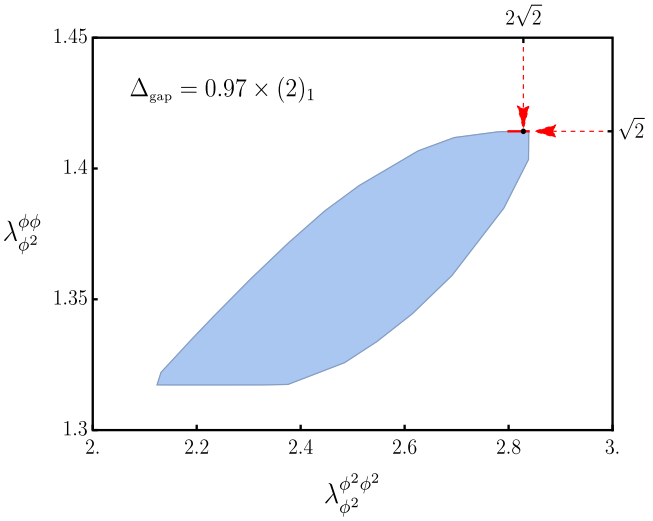

FIG. 2. Feasible region in $(\lambda^{\phi^2\phi^2}_{\phi^2}, \lambda^{\phi\phi}_{\phi^2})$ for $\Delta_\phi = \frac{7}{3}$ and a gap in the 0^+ sector equal to ~6.475. When the gap is set to $2\Delta_\phi + 2$ the region shrinks to a small line segment indicated in red whose rightmost tip seems to tend to the GFF solution.

OPE values. However, at this point the 22 OPE is still not fully constrained. Maximizing the OPE of the operator ϕ^4 fixes it to become that of the GFF solution, but only under our gap assumption in the 0^+ sector—OPE bounds generally require a minimal gap, otherwise the maximum is infinite.

Numerical explorations: Islands with GFF inhabitants and particle production. In this section we perform some preliminary numerical explorations using our setup, leaving more detailed computations for future work. In Fig. 2 we explore again the same mixed correlator system as before looking for the feasible region in the $\lambda^{11}_2, \lambda^{22}_2$ plane i.e. the OPE data of the operator ϕ^2 [35].

The island shown appears when we set gaps in the Z_2 odd sectors. In particular in the figure we have set these to be $3\Delta_\phi(3\Delta_\phi + 1)$ in parity even (odd) channels, respectively. When the gap is smaller, say Δ_ϕ in 0^-, we observe a region where the left direction of the plot is qualitatively similar, while on the right, it extends to a striplike shape without any upper bound for $\lambda^{\phi^2\phi^2}_{\phi^2}$, as found in [6]. Conversely, larger gaps further shrink the allowed region. The island displays two seemingly flat regions in the bottom and top, which are not numerical artifacts. In fact they are well described by single correlator bounds on $\lambda^{\phi\phi}_{\phi^2}$. This suggests that the 11 OPE is unchanged along these boundaries. On the top, the bound is simply $\sqrt{2}$ i.e. the GFF value. When we examine the spectrum of the extremal solution corresponding to the top right corner of the feasible region, we find a solution consistent with a noninteracting (GFF) 11 and 12 OPEs and an interacting 22, see Fig. 1 in the Supplemental Material [24]. As explained there, this can be thought of as a consequence of the existence of a Φ^8 deformation in AdS_2 which allows us to deform the 22 OPE while holding the 11 fixed. While in this special case the 11 OPE remains free, the general result is that it should not only become interacting but also manifest "particle production," i.e. couplings to the $(22)_n$ operators. An example is shown in Fig. 2 of the Supplemental Material [24].

Discussion and outlook. In this paper we have introduced a presumably complete reformulation of crossing symmetry constraints for an arbitrary set of 1D CFT correlators. Our sum rules are naturally adapted to study deformations of generalized free fields, allowing us to bridge the gap between analytic and numeric computations and promising to help us pinpoint desired theories to bootstrap. Our initial numerical explorations, which will be developed elsewhere, show not only greatly improved speed of convergence relative to traditional bootstrap methods, but much greater accuracy, allowing us to give precise spectra for interacting CFT solutions with features like "particle production" or lack thereof. Our methods thus promise to push the bootstrap into much larger sets of mixed correlators and probe the high energy asymptotics of CFT correlators. There are several interesting applications of the powerful functionals developed in this paper. One intriguing avenue is to consider RG flows in AdS, particularly focusing on the ϕ^4 flow. It would be amazing

to constrain the RG flow triggered by ϕ^4 using our functionals supplemented by local sum rules developed in [36,37]. Other applications involve various physical systems like 1D long-range Ising model and Z_2 twist defect in 3D Ising model. It would be rewarding to use our Polyakov blocks to constrain the CFT data of these theories. We will return to these questions in the very near future.

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