From Weierstrass to Riemann: The Frobenius Pass When Fractal Flows come into Play with Zeta Functions

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When Fractal Flows come into Play

with Zeta Functions

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Abstract

We establish new connections between the local and global polyhedral zeta functions associated with a fractal – in our present context, the Weierstrass (fractal) Curve – and differential operators. First, we exhibit Lie brackets (or commutators), associated with this global polyhedral zeta function.

We then introduce a (natural) transfer operator, which acts on the underlying fractal cohomology, and we extend, to our fractal setting, the classical Lefschetz operator. This new operator, a bigraded operator, of bigrading $(1,1)$, induces a Hodge Star relation on the functions defined on the Weierstrass Curve and on all the higher-order differential forms.

Moreover, we obtain an analog, in our fractal context, of the classical Hodge theory – and of the associated (orthogonal) decomposition of the fractal cohomology groups. This decomposition is presently obtained by means of an inner product involving the (specifically constructed for fractals) polyhedral measure introduced in our previous work [DL24b]. This inner product is a fractal counterpart of the classical polarization operator, in the sense of Deligne. These results enable us, in particular, to obtain fractal analogs of Poincaré Duality, the Hard Lefschetz Theorem, and the Hodge–Riemann (Bilinear) Relations, that are key to classical Hodge theory in algebraic and arithmetic geometry.

Finally, we introduce and study the (differential) operator induced by the global zeta function, which enables us to obtain the functional equation satisfied by this zeta function and its *dual* zeta function.

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We also show that the global zeta function – viewed as a differential operator acting on the fractal cohomology – enables us to obtain the operator which acts as the Frobenius operator in this context, since the spectrum of this operator essentially coincides with the Complex Codimensions. In fact, the spectrum of a small modification of this operator precisely coincides with the underlying Complex Dimensions.

Our work provides convincing strong evidence for a future unification between key aspects of fractal and arithmetic geometries.

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Keywords: Weierstrass Curve, iterated fractal drum (IFD), fractal zeta functions, Complex Dimensions of an IFD, box-counting (or Minkowski) dimension, cohomology infinitesimal, intrinsic scale, Toda-like system, Hodge Diamond Star relation, Poincaré duality, polyhedral measure, polyhedral (or polygonal) neighborhoods, effective local and global polyhedral zeta functions, prefractal cohomology, fractal cohomology, transfer operator, Lefschetz operator, Hodge decomposition, functional equation, dual zeta function and Weierstrass Curve, fractal flow, Frobenius operator, Complex Dimensions.

Contents

1 Introduction

This is a story of two mathematicians, who lived nearly at the same time, in the same country. We can say that the work of Bernhard Riemann was among the ones that inspired a lot Karl Weierstrass. For instance, we could wonder how the pathological, continuous everywhere, while nowhere differentiable, Weierstrass function [Wei75], came to Weierstrass's mind. As is mentioned in [Dav22], we note that some mathematicians, like J.-P. Kahane [Kah64], suggest that it could be attributed to the Riemann function, for which Karl Weierstrass did not know how to prove the non-differentiable feature.

Eventually, it was claimed by Godfrey Harold Hardy and John E. Littlewood in [HL14] that Riemann's function, defined, for all $x \in \mathbb{R}$, by

$$\mathcal{R}(x) = \sum_{n=1}^{+\infty} \frac{\sin n^2 x}{n^2} \,,$$

was nowhere differentiable. Later, Joseph Gerver, in [Ger70], proved this was false and determined the precise points at which it is differentiable.

A priori, no connection can be established between nowhere differentiable functions, and the Riemann Hypothesis. None. Unless we take into account the fractality of the corresponding curves. Then, things change drastically. Indeed, building on the theory of Complex Dimensions, developed for many years now by M. L. Lapidus and his collaborators, for example in [Lap91], [Lap92], [Lap93], [LP93], [LM95], [LP06], [Lap08], [LPW11], [LvF06], [LvF13], [LRŽ17a], [LRŽ17b], [LRŽ18], [Lap19], [HL21], [Lap24], [LR24], which makes the connection between the fractality of an object and its differentiability properties, we have at our disposal those *geometric (or fractal) zeta functions* – which stand for the trace of a differential operator at a complex order s. Thus far, however, this differential operator had not yet been identified. We hereafter propose two different, but convergent methods in order to characterize and study it.

The poles of those fractal (or geometric) zeta functions – i.e., the (fractal) Complex Dimensions – are of the highest interest, since they provide us with a range of specific informations, which enable us to characterize *fractality*. Recall that, for a long time, mathematicians have avoided defining fractality, after failed attempts (see [Fal97]), especially the wrong one by Benoît Mandelbrot himself, who claimed that a geometric object was fractal if its fractal dimension exceeded its topological dimension, which is not correct, since many actual fractals have the same fractal and topological dimension, including, for example, the Devil's staircase (the graph of the Cantor–Lebesgue function) and all plane (or space) filling curves (including the Peano and the Hilbert Curves). In this light, and for a long time, the consensus – among mathematicians (see again [Fal97]) – was that a set was fractal "if it has almost all or most of the following features: "it has a fine structure, that is, irregular details at arbitrarily small scales " ; or/and "it is too irregular to be described by calculus or traditional geometric language, either locally or globally"; or/and it has some self-similarity or self-affinity, perhaps in a statistical or approximate sense" ; "often" it has " a natural appearance" (the quotes are in *loc. cit.*). Each of these definitions attempts to convey the concept of a fractal, but only *informally.* A fractal for instance, while having details appearing at arbitrary scales, is not always self-similar. One had to wait until the work of the second author (see, among other references, [LvF13], [LRŽ17b], [Lap19]) for a proper and sound definition of fractality: a geometric object is said to be fractal if it admits at least one nonreal Complex Dimension.

If our recent work provides the exact values of the Complex Dimensions of the Weierstrass Curve – via new (local and global) fractal zeta functions and the new concept of iterated fractal drums (IFDs) associated with the underlying prefractal (polygonal or graph) approximations (see [DL23b] – things become even more complicated when the zeros are concerned. A natural question is the existence of an underlying functional equation, which would enable us to obtain (or, at least, to better understand) the zeros of those (fractal) zeta functions, as well as the poles.

In this light, the Weierstrass Curve appears as an incredibly rich source of results and inspiration. For instance, in our previous works [DL22b], [DL24a], [DL24b], [DL23b], [DL23a], a natural symmetry – $s \mapsto 2 - s$ – interchanges the abscissa of convergence $D_{\mathcal{W}}$ of the global fractal effective zeta function $\tilde{\zeta}^e_{\mathcal{W}}(s)$ and $D^\star_{\mathcal{W}} = 2 - D_{\mathcal{W}}$. It was then natural to expect the corresponding functional equation to be, for all $s \in \mathbb{C}$,

$$\tilde{\zeta}^{e,\star}_{\mathcal{W}}(s) = \tilde{\zeta}^e_{\mathcal{W}}(2 - s)\,,$$

which is the case, as will be shown in this paper, since the associated residues are the same. This functional equation connects the fractal zeta functions of $\mathcal{W}$ and of its *dual* $\mathcal{W}^\star$, where $\mathcal{W}^\star$ is a Weierstrass function which is both smooth and fractal, which is another novel feature of the theory.

To our knowledge, it is he first time that a functional equation is obtained for any kind of nontrivial fractal zeta function. Furthermore – especially in the self-dual case, when $\mathcal{W} = \mathcal{W}^\star$ and hence, $\tilde{\zeta}^e_{\mathcal{W}} = \tilde{\zeta}^{e,\star}_{\mathcal{W}}$ – this functional equation is eerily similar to the one satisfied by the (completed or global) Riemann zeta function $\xi(s)$, namely, $\xi(s) = \xi(1 - s)$, for all $s \in \mathbb{C}$.

At the same time, as is also shown in this paper, building on [DL22a], [DL22b], [DL23b], that the Weierstrass function $\mathcal{W}$ has a fractal power series, Taylor-like expansion, taken over its Complex Dimensions. Indeed, even though $\mathcal{W}$ is not differentiable, we can define associated fractional derivatives, by connecting each term of the aforementioned Taylor-like expansion to a differential operator. Insofar as our fractal is approximated by a sequence of finite discrete graphs – the prefractal graphs, the so-called *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs) – an interesting analogy was to seach for an equivalent lattice model. The Toda-like system, introduced by the Japanese physicist Morikazu Toda (see, for instance, [Tod89]), appeared as a good candidate. Indeed, the Toda model, initially used in the case of a one-dimensional crystal, enables us to describe the motion of chains of particles, taking into account neighbor interactions. In our present context, we already highlighted (in our work on fractal cohomology [DL22b]), a quasiperiodic geometric property (reminiscent of, but not identical to the one established in Chapter 3 of [LvF13] for nonlattice self-similar strings), which could possibly be connected to the structure of a (generalized) quasicrystal (see [LvF13], Problem 3.22, page 89, and [Lap08], especially, Chapter 5 and Appendix F).

At this stage, it is interesting to make a few comments about fractals and their mathematical representation. Often, fractals are apprehended as the limits of the aforementioned sequences of prefractal graphs. Those graphs are considered as *static graphs*, meaning that they do not evolve with time. Bear in mind that, since their first introduction by Benoît Mandelbrot in [Man83], fractals were meant to model and represent "the irregular and fragmented patterns around us" (the quote is in *loc. cit.*). To name a few, ferns, cauliflowers (see the recent work of the biomathematician Christophe Godin, the biologist François Parcy and their collaborators in [ATM+21]), trees, clouds, mountains, coastlines, rivers, lungs, networks of blood vessels, etc... In this light, things take a different direction. Indeed, in nature, growth is a continuous process – which means that fractal-shaped living forms cannot, reasonably, be modelled without taking into account the underlying dynamical expansions.

By relying on the aforementioned analogy – with quasicrystals – we thus exhibit a differential operator – in the form of a Lie bracket (or commutator), associated with our sequence of prefractal graphs. This enables us to make preliminary steps in the understanding of fractal dynamics, in terms of the connection with the Taylor-like expansions of functions belonging to the cohomology groups; see [DL22b]. Moreover, we provide an equivalent – but even much more meaningful – result, which concerns the fractal zeta functions introduced in [DL23b], since we prove that the Taylor-like expansions obtained in [DL22b] can also be obtained as the sums of traces of differentiable operators, and thus involve fractional derivatives.

This is not all. The aforementioned quasiperiodic geometric property suggests the existence of a natural *transfer operator* (as well as of a Frobenius operator), under the action of which our fractal – the Weierstrass Curve – along with its cohomology, both remain invariant. More precisely, our quest is to determine, given $m \in \mathbb{N}$, this operator which enables us to switch from each m^{th} graph approximation $\Gamma_{\mathcal{W}_m}$ to the $(m+1)^{th}$ graph approximation $\Gamma_{\mathcal{W}_{m+1}}$. At the same time, we also aim at *tracking* the evolution of the family of (fractal) differential operators when switching from a scale to the next or the previous one. If we, again, make an analogy with nature and fractal shaped living forms, this is in direct connection with *the growth* or *the retraction* phenomena; recall that there exists fractal-shaped living organisms which both expand and contract, as is the case for slime molds; see [TGE$^+$17].

Back to a purely mathematical point of view, we aim at a better understanding of the fractal cohomology obtained in our previous work [DL22b]. For this purpose, we first introduce our own (fractal, natural) transfer operator, which acts on the fractal cohomology; more precisely, given $m \in \mathbb{N}$, this transfer operator enables us to switch from the m^{th} cohomology group to the $(m+1)^{th}$ one. Going further, we extend, to our fractal setting, the classical Lefschetz operator which enables us to go directly from the m^{th} cohomology group to the $(m+2)^{th}$ one. This new operator, a bigraded operator of bigrading $(1,1)$, is defined in terms of the aforementioned transfer operator. A very interesting feature is that the Lefschetz operator induces a Hodge Star relation on the functions defined on the Weierstrass Curve and on all the higher-order differential forms.

The next step was to obtain an analog, in our fractal context, of the classical Hodge theory – and the associated (orthogonal) decomposition of the cohomology groups. This decomposition is obtained here by means of an inner product involving the polyhedral measure introduced in our previous work [DL24b] (and specifically associated to fractals). This inner product is an extension of the classical polarization operator, in the sense of Deligne, within the context of pure Hodge theory. We thereby obtain a fractal counterpart of many of the key classical theorems from Hodge theory in algebraic geometry (see, e.g., [Voi02], [Voi07], [Kon08]), including Poincaré Duality, the Hard Lefschetz Theorem and the Riemann–Hodge (Bilinear) Relations, along with various geometric and analytic forms of Hodge's Orthogonal Decompositions.

Things go even deeper if we envision the (differential) operator induced by the global zeta function. In an echo to the functional equation also unveiled in this paper, the Hodge star relation induced by our (fractal) Lefschetz operator enables us to recover, in a completely different manner, the same functional equation, thereby going further in the understanding of the connections between the zeros and the poles of the global zeta function.

The global zeta function – viewed as an operator acting on the underlying fractal cohomology – can be seen in our present context as a suitable counterpart of the Frobenius operator, which plays a key role in several aspects of number theory, algebraic geometry and arithmetic geometry, whose spectrum (when it acts on the underlying fractal cohomology) yields the zeros and the poles of the corresponding zeta functions. Consistent with this philosophy, we show that the spectrum of our version of the Frobenius operator essentially coincides with the Complex Codimensions of the Weierstrass Curve. We also define a slight modification of this operator whose spectrum precisely coincides with the set of Complex Codimensions.

Recall that the Frobenius operator, at least in the context of curves – or, more generally, varieties over finite fields – was used in a successful manner by André Weil [Wei40], [Wei41], [Wei46], [Wei48] and Pierre Deligne [Del74], [Del80] in order to establish the analog of the Riemann Hypothesis. In the case of number fields (and associated L-functions), however, it was – and still is – a great challenge to achieve such a goal and prove the associated Riemann hypothesis (including the celebrated 1859 original Riemann hypothesis [Rie60] which corresponds to the number field $\mathbb{Q}$ of rational numbers and to the completion ξ of the classic Riemann zeta function).

Along these lines, Alexander Grothendieck [Gro60], [Gro66], [Gro69], proposed a set of conjectures and a possible plan of attack which, thus far, has remained unfulfilled. His then conjectural motivic cohomology, along with its associated and partly mythical notion of motive, has remained to this day an unrealized, although quite attractive, dream.

More recently, Christopher Deninger has proposed a largely heuristic but quite interesting cohomological approach to number theory (see, e.g., [Den92], [Den93], [Den94]).

Within the context of fractal geometry, M. L. Lapidus and Machiel van Frankenhuisjen [LvF00], [LvF06], [LvF13], suggested a possible analogy between aspects of fractal geometry and arithmetic geometry, via in particular, the use of a Frobenius operator – viewed as a differential operator acting on a suitable Hilbert space – and a yet to be constructed fractal cohomology associated with the Complex Dimensions of the underlying fractal space. Part of this program was further extended by the second author in his book [Lap08], by means of a conjectural fractal flow acting on a moduli space of fractal strings and of its quantization, the moduli space of fractal membranes. In particular, a conjectural functional equation connecting a fractal membrane and its dual membrane, played a key role in that fractal setting.

In the recent book of Hafedh Herichi and the second author on quantized number theory [HL21], the real version of the differentiation operator (or *infinitesimal shift*) proposed in [LvF06], [LvF13], was made precise and studied in detail, while a complex version (based, in particular, on suitable weighted Bergman spaces) enabled Tim Cobler and the second author [CL17] to develop a first rigorous version of fractal cohomology and of a Frobenius operator acting on it, the spectrum of which coincides with the zeros and the poles of any given (appropriate) meromorphic function – including the Riemann zeta function and other L-functions. (See also the second author's forthcoming book, [Lap24].)

What was missing, however, in this analytic approach, was a direct connection with an underlying geometric space. Building on the new h-cohomology developed by the first author and Gilles Lebeau in [DL23d], along with an extension of the theory of Complex Dimensions introduced in [DL22a] (and later completed in [DL23b], see also [DL24a], [DL24b]), the authors of the present paper proposed for the first time in [DL22b] a geometric fractal cohomology theory, which they applied to the Weierstrass Curve in order to explicitly calculate the corresponding local and global (i.e., total) fractal cohomology groups and the associated sets of potential Complex Dimensions, which turn out to be subsets of their counterparts obtained in [DL22a] and [DL23b].

In the present work, we will show that the resulting local and global fractal cohomology spaces and their orthogonal decompositions provide a very useful geometric map of the Weierstrass Curve and its prefractal graph approximations (as well as of the vertices of its associated polygons), respectively, and that the corresponding decompositions of the Frobenius operator (viewed as a specific operator defined by means of the local and global zeta functions and acting on the corresponding fractal cohomology spaces) yield a significant amount of new information – including finite and infinite products of the characteristic determinants of Frobenius.

In the process, we obtain entirely new connections between fractal geometry, complex differential geometry, Kähler geometry and Hodge theory, as well as algebraic and arithmetic geometry. Consequently, those developments constitute an important step towards a future unification of many aspects of fractal geometry, algebraic topology and geometry, differential geometry, number theory and arithmetic geometry.

Our main results in the present paper can be found in the following places:

i. In Theorem 3.1, on page 30, along with Theorem 3.2, on page 33, where we establish the connection between the Weierstrass function, our fractal zeta functions and suitable differential operators. Note that the first result also makes the connection with the sequence of prefractal graphs which approximate the Weierstrass Curve, because the coordinates of the vertices are, of course, obtained by means of the values taken by the Weierstrass function.

ii. In Definition 4.14, on page 46, and Theorem 4.10, on page 46, where we introduce *the natural transfer operator*, $\mathcal{L}_{\mathcal{W}}$, and its dual $\mathcal{L}_{\mathcal{W}}^{\#}$, respectively associated with the Weierstrass Curve $\Gamma_{\mathcal{W}}$ and defined in terms of the underlying iterated function system (IFS) (or of the dual IFS).

iii. In Theorem 4.24, on page 59, where we give the generators of the $\mathbb{C}$-tensored (i.e., complex) prefractal (or local) cohomology.

iv. In Theorem 4.30, on page 68, where we obtain a fractal analog of Hodge's orthogonal decomposition for the total (or global) cohomology space.

v. In Theorem 4.27, on page 63, where we obtain a fractal analog of Poincaré Duality, the Hard Lefschetz Theorem (with respect to our (fractal) Lefschetz Operator, also introduced in Proposition 4.32, on page 69), respectively, and in Theorem 4.25, on page 60, for the Hodge–Riemann Relations.

vi. In Theorem 4.35, on page 72, where we give the explicit expression of the resolvent of the (differential) operator induced by the global zeta function, thereby going further in the understanding of the connections between the zeros and the poles of the global zeta function.

vii. In Theorem 4.36, on page 78, where we unveil the Frobenius operator and determine its spectrum, which essentially consists of the underlying Complex Dimensions of the Weierstrass Curve.

viii. In Theorem 4.37, on page 79, where we obtain the functional equation satisfied by the global fractal effective zeta function $\tilde{\zeta}_{\mathcal{W}}^{e}$, associated with the Weierstrass function $\mathcal{W}$, and by the global fractal effective zeta function $\tilde{\zeta}_{\mathcal{W}}^{e,\star}$ associated with the dual Weierstrass function $\mathcal{W}^{\star}$.

2 Preliminaries: Geometry of the Weierstrass Curve and Fractal Zeta Functions

We begin by reviewing the main geometric properties of the Weierstrass Curve (and of the associated IFD; i.e., the sequence of prefractal graphs, the so-called *Weierstrass Iterated Fractal Drums* (in short, Weierstrass IFDs)), which will be needed in this paper.

We point out that this discussion of the geometric properties of the Weierstrass largely overlaps with its counterparts in [DL22a], [DL22b], [DL23b], [DL24b], but is essential for understanding and laying out the ground for the results obtained in the remainder of this paper.

2.1 Geometry of the Weierstrass Curve

Henceforth, we place ourselves in the Euclidean plane, equipped with a direct orthonormal frame. The usual Cartesian coordinates are denoted by (x, y). The horizontal and vertical axes will be respectively referred to as $(x'x)$ and $(y'y)$.

Notation 1 (Set of all Natural Numbers, and Intervals).

As in Bourbaki [Bou04] (Appendix E. 143), we denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of all natural numbers, and set $\mathbb{N}^\star = \mathbb{N} \setminus \{0\}$.

Given a, b with $-\infty \leqslant a \leqslant b \leqslant \infty$, $]a, b[= (a, b)$ denotes an open interval, while, for example, $]a, b] = (a, b]$ denotes a half-open, half-closed interval.

Notation 2 (Wave Inequality Symbol (see [Tao06], Preface, page xiv)).

Given two positive-valued functions f and g, defined on a subset $\mathcal{I}$ of $\mathbb{R}$, we use the following notation, for all $x \in \mathcal{I}$: $f(x) \lesssim g(x)$ when there exists a strictly positive constant C such that, for all $x \in \mathcal{I}$, $f(x) \leqslant C\, g(x)$, which is equivalent to $f = \mathcal{O}(g)$. Note that in our forthcoming context, we will often use $\mathcal{O}(1)$ to denote terms which depend on $m \in \mathbb{N}$, but are bounded away from 0 and ∞, uniformly in $m \in \mathbb{N}$; more precisely, those terms will always satisfy bounds of the following form,

$$0 < \text{Constant}_{inf} \leqslant \mathcal{O}(1) \leqslant \text{Constant}_{sup} < \infty\,, \qquad (\mathcal{R}\,1)$$

where Constant_{inf} and Constant_{sup} denote strictly positive and finite constants (independent of $m \in \mathbb{N}$).

Notation 3 (Weierstrass Parameters).

In the sequel, λ and N_b are two real numbers such that

$$0 < \lambda < 1 \quad , \quad N_b \in \mathbb{N}^\star \quad \text{and} \quad \lambda\, N_b > 1 \cdot \qquad (\clubsuit) \qquad (\mathcal{R}\,2)$$

Note that this implies that $N_b > 1$; i.e., $N_b \geqslant 2$, if $N_b \in \mathbb{N}^\star$, as will be the case in this paper.

As is explained in [Dav19], we deliberately made the choice to introduce the notation N_b which replaces the initial number b, in so far as, in Hardy's paper [Har16] (in contrast to Weierstrass' original article [Wei75]), b is any positive real number satisfying $\lambda\, b > 1$, whereas we deal here with the specific case of a nonnegative integer, which accounts for the natural notation N_b.

Definition 2.1 (Weierstrass Function, Weierstrass Curve).

We consider the *Weierstrass function* $\mathcal{W}$ (also called, in short, the $\mathcal{W}$-*function*) defined, for any real number x, by

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\, N_b^n\, x\right) \cdot \qquad (\mathcal{R}\,3)$$

We call the associated graph the *Weierstrass Curve*, and denote it by $\Gamma_{\mathcal{W}}$.

Due to the one-periodicity of the Weierstrass function (since $N_b \in \mathbb{N}^\star$), from now on, and without loss of generality, we restrict our study to the interval $[0,1[=[0,1)$. Note that $\mathcal{W}$ is continuous, and hence, bounded on all of $\mathbb{R}$. In particular, $\Gamma_{\mathcal{W}}$ – which can equivalently be defined as the graph of the restriction of $\mathcal{W}$ to $[0,1[$ or to $[0,1]$ – is a (nonempty) compact and connected subset of $\mathbb{R}^2$.

Definition 2.2 (**Complexified Weierstrass Function**).

We introduce the *Complexified Weierstrass function* $\mathcal{W}_{comp}$, defined, for any real number x, by

$$\mathcal{W}_{comp}(x) = \sum_{n=0}^{\infty} \lambda^n \, e^{2\,i\,\pi\,N_b^n\,x} \,.$$

Clearly, $\mathcal{W}_{comp}$ is also a continuous and 1-periodic function on $\mathbb{R}$.

Notation 4 (**Logarithm**).

Given $y > 0$, $\ln y$ denotes the natural logarithm of y, while, given $a > 0$, $a \neq 1$, $\ln_a y = \dfrac{\ln y}{\ln a}$ denotes the logarithm of y in base a; so that, in particular, $\ln = \ln_e$.

Notation 5 (**Minkowski Dimension and Hölder Exponent**).

For the parameters λ and N_b satisfying condition $(\clubsuit)$ (see Notation 3, on page 8), we denote by

$$D_{\mathcal{W}} = 2 + \frac{\ln \lambda}{\ln N_b} = 2 - \ln_{N_b} \frac{1}{\lambda} \in \,]1,2[$$

the box-counting dimension (or Minkowski dimension) of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, which happens to be equal to its Hausdorff dimension [KMPY84], [BBR14], [She18], [Kel17]. We point out that, in [DL23b] (announced in [DL23a]), and in [DL22a] respectively, is provided a direct geometric and fully rigorous proof of the fact that $D_{\mathcal{W}}$, the Minkowski dimension (or box-counting dimension) of $\Gamma_{\mathcal{W}}$, exists and takes the above values, as well as of the fact that $\mathcal{W}$ is Hölder continuous with *optimal* Hölder exponent

$$2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b} = \ln_{N_b} \frac{1}{\lambda} \,.$$

Convention (**The Weierstrass Curve as a Cyclic Curve**).

In the sequel, we identify the points $(0,\mathcal{W}(0))$ and $(1,\mathcal{W}(1)) = (1,\mathcal{W}(0))$. This is justified by the fact that the Weierstrass function $\mathcal{W}$ is 1-periodic, since N_b is an integer.

Remark 2.1. The above convention makes sense, because, in addition to the periodicity property of the $\mathcal{W}$-function, the points $(0,\mathcal{W}(0))$ and $(1,\mathcal{W}(1))$ have the same vertical coordinate.

Property 2.1 (**Symmetry with Respect to the Vertical Line $x = \frac{1}{2}$**).

Since, for any $x \in [0,1]$,

$$\mathcal{W}(1-x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\pi N_b^n - 2\pi N_b^n x\right) = \mathcal{W}(x)\,,$$

the Weierstrass Curve is symmetric with respect to the vertical straight line $x = \frac{1}{2}$.

In the sequel, we will denote by $\mathcal{S}$ the symmetry with respect to the vertical straight line $x = \frac{1}{2}$.

Proposition 2.2 (**Nonlinear and Noncontractive Iterated Function System (IFS)**).

Following our previous work [Dav18]*, we approximate the restriction $\Gamma_{\mathcal{W}}$ to $[0,1[\times\mathbb{R}$, of the Weierstrass Curve, by a sequence of finite graphs, built via an iterative process. For this purpose, we use the nonlinear iterated function system (IFS) consisting of a finite family of C^∞ bijective maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ and denoted by*

$$\mathcal{T}_{\mathcal{W}} = \{T_0, \ldots, T_{N_b-1}\}\ ,$$

where, for any integer i belonging to $\{0,\ldots,N_b-1\}$ and any point (x,y) of $\mathbb{R}^2$,

$$T_i(x,y) = \left(\frac{x+i}{N_b}, \lambda\, y + \cos\left(2\pi \left(\frac{x+i}{N_b}\right)\right)\right).$$

Note that unlike in the classical situation, these maps are not contractions. Nevertheless, $\Gamma_{\mathcal{W}}$ can be recovered from this IFS in the usual way, as we next explain.

Property 2.3 (**Attractor of the IFS [Dav18], [Dav19]**).

The Weierstrass Curve $\Gamma_{\mathcal{W}}$ is the attractor of the IFS $\mathcal{T}_{\mathcal{W}}$, and hence, is the unique nonempty compact subset $\mathcal{K}$ of $\mathbb{R}^2$satisfying $\mathcal{K} = \bigcup_{i=0}^{N_b-1} T_i(\mathcal{K})$; in particular, we have that $\Gamma_{\mathcal{W}} = \bigcup_{i=0}^{N_b-1} T_i(\Gamma_{\mathcal{W}})$.

Notation 6 (**Fixed Points**).

For any integer i belonging to $\{0,\ldots,N_b-1\}$, we denote by

$$P_i = (x_i, y_i) = \left(\frac{i}{N_b-1}, \frac{1}{1-\lambda}\cos\left(\frac{2\pi i}{N_b-1}\right)\right)$$

the unique fixed point of the map T_i; see [Dav19].

Definition 2.3 (**Sets of Vertices, Prefractals**).

We denote by V_0 the ordered set (according to increasing abscissae) of the points

$$\{P_0, \ldots, P_{N_b-1}\}\ .$$

The set of points V_0 – where, for any integer i in $\{0, \ldots, N_b - 2\}$, the point P_i is linked to the point P_{i+1} – constitutes an oriented finite graph, ordered according to increasing abscissae, which we will denote by $\Gamma_{\mathcal{W}_0}$. Then, V_0 is called *the set of vertices* of the graph $\Gamma_{\mathcal{W}_0}$.

For any nonnegative integer m, i.e., for any $m \in \mathbb{N}$, we set $V_m = \bigcup_{i=0}^{N_b-1} T_i(V_{m-1})$.

The set of points V_m, where two consecutive points are linked (to form an edge), is an oriented finite graph, ordered according to increasing abscissae, called the $\boldsymbol{m^{th}}$ **order $\mathcal{W}$-prefractal**. Then, V_m is called *the set of vertices* of the prefractal $\Gamma_{\mathcal{W}_m}$; see Figure 2, on page 15.

Property 2.4 (**Density of the Set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ in the Weierstrass Curve** [**DL22b**]).

The set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ is dense in the Weierstrass Curve $\Gamma_{\mathcal{W}}$.

Definition 2.4 (**Adjacent Vertices, Edge Relation**).

For any $m \in \mathbb{N}$, the prefractal graph $\Gamma_{\mathcal{W}_m}$ is equipped with an edge relation $\underset{m}{\sim}$, as follows: two vertices X and Y of $\Gamma_{\mathcal{W}_m}$ (i.e., two points belonging to V_m) will be said to be *adjacent* (i.e., *neighboring* or *junction points*) if and only if the line segment $[X, Y]$ is an edge of $\Gamma_{\mathcal{W}_m}$; we then write $X \underset{m}{\sim} Y$. Note that this edge relation depends on m, which means that points adjacent in V_m might not remain adjacent in V_{m+1}. This simple fact will play a crucial role in this paper, especially when discussing the orthogonal decomposition of the (complex) fractal cohomology spaces and their consequences, in Section 4.

Property 2.5 (**Scaling Properties of the Weierstrass Function, and Consequences** [**DL22a**]).

Since, for any real number x, $\mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\pi N_b^n x\right)$, we have that

$$\mathcal{W}(N_b\, x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\pi N_b^{n+1} x\right) = \frac{1}{\lambda} \sum_{n=1}^{\infty} \lambda^n \cos\left(2\pi N_b^n x\right) = \frac{1}{\lambda}\left(\mathcal{W}(x) - \cos(2\pi x)\right),$$

this yields, for any strictly positive integer m and any j in $\{0, \ldots, \#V_m - 1\}$,

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \lambda\,\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}}\right) + \cos\left(\frac{2\pi j}{(N_b-1)\,N_b^m}\right).$$

By induction, one then obtains that

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \lambda^m\,\mathcal{W}\left(\frac{j}{(N_b-1)}\right) + \sum_{k=0}^{m-1} \lambda^k \cos\left(\frac{2\pi N_b^k\, j}{(N_b-1)\,N_b^m}\right).$$

We refer to part *iv.* of Property 2.6, along with Figure 1, for the definition of the polygons $\mathcal{P}_{m,k}$ and $\mathcal{Q}_{m,k}$ associated with the Weierstrass Curve and which will play an important role in the sequel.

Property 2.6. [Dav18] *For any $m \in \mathbb{N}$, the following statements hold:*

i. $V_m \subset V_{m+1}$.

ii. $\#V_m = (N_b - 1)\, N_b^m + 1$, *where $\#V_m$ denotes the number of elements in the finite set V_m.*

iii. The prefractal graph $\Gamma_{\mathcal{W}_m}$ has exactly $(N_b - 1)\, N_b^m$ edges.

iv. The consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$ are the vertices of N_b^m simple nonregular polygons $\mathcal{P}_{m,k}$ with N_b sides. For any strictly positive integer m, the junction point between two consecutive polygons $\mathcal{P}_{m,k}$ and $\mathcal{P}_{m,k+1}$ is the point

$$\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 1 \leqslant k \leqslant N_b^m - 1\,.$$

Hence, the total number of junction points is $N_b^m - 1$. For instance, in the case $N_b = 3$, the polygons are all triangles; see Figure 1, on page 13.

We call extreme vertices *of the polygon $\mathcal{P}_{m,k}$ the junction points*

$$\mathcal{V}_{initial}\left(\mathcal{P}_{m,k}\right) = \left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 0 \leqslant k \leqslant N_b^m - 1\,,$$

and

$$\mathcal{V}_{end}\left(\mathcal{P}_{m,k}\right) = \left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 0 \leqslant k \leqslant N_b^m - 2\,.$$

*In the sequel, we will denote by $\mathcal{P}_0$ **the initial polygon**, whose vertices are the fixed points of the maps T_i, $0 \leqslant i \leqslant N_b - 1$, introduced in Notation 6, on page 10 and Definition 2.3, on page 10, i.e., $\left\{P_0, \ldots, P_{N_b-1}\right\}$; see, again, Figure 1, on page 13.*

In the same way, the consecutive vertices of the prefractal graph $\Gamma_{\mathcal{W}_m}$, distinct from the fixed points P_0 and P_{N_b-1} (see Notation 6, on page 10), are also the vertices of $N_b^m - 1$ simple nonregular polygons $\mathcal{Q}_{m,j}$, for $1 \leqslant j \leqslant N_b^m - 2$, again with N_b sides. For any integer j such that $1 \leqslant j \leqslant N_b^m - 2$, one obtains each polygon $\mathcal{Q}_{m,j}$ by connecting the point number j (i.e., with the notation of Property 2.6 below, on page 13, the vertex $M_{j,m}$) to the point number $j+1$ (i.e., the vertex $M_{j+1,m}$) if $j \equiv i \bmod N_b$, for $1 \leqslant i \leqslant N_b - 1$, and the point number j to the point number $j - N_b + 1$ if $j \equiv 0 \bmod N_b$.

As previously, we call extreme vertices *of the polygon $\mathcal{Q}_{m,k}$ the points*

$$\mathcal{V}_{initial}\left(\mathcal{Q}_{m,k}\right) = \left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,k}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 1 \leqslant k \leqslant N_b^m - 1\,,$$

and

$$\mathcal{V}_{end}\left(\mathcal{Q}_{m,k}\right) = \left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,(k+1)}{(N_b-1)\,N_b^m}\right)\right) \quad , \quad 1 \leqslant k \leqslant N_b^m - 2\,.$$

Definition 2.5 (**Polygonal Set [DL24b]**).

For any $m \in \mathbb{N}$, we introduce the following *polygonal sets*,

$$\mathcal{P}_m = \left\{\mathcal{P}_{m,k} \, , \, 0 \leqslant k \leqslant N_b^m - 1\right\} \quad \text{and} \quad \mathcal{Q}_m = \left\{\mathcal{Q}_{m,k} \, , \, 0 \leqslant k \leqslant N_b^m - 2\right\} \, .$$

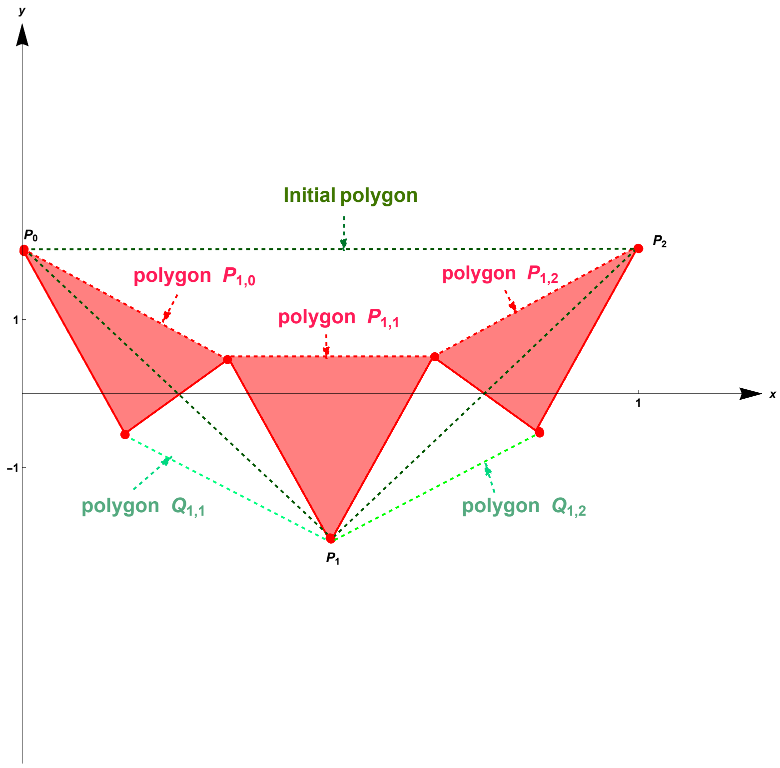


Figure 1: **The initial polygon $\mathcal{P}_0$, and the respective polygons $\mathcal{P}_{0,1}$, $\mathcal{P}_{1,1}$, $\mathcal{P}_{1,2}$, $\mathcal{Q}_{1,1}$, $\mathcal{Q}_{1,2}$, in the case when $\lambda = \dfrac{1}{2}$ and $N_b = 3$. (See also Figure 2, on page 15.)**

Notation 7. For any $m \in \mathbb{N}$, we denote by:

ii. $X \in \mathcal{P}_m$ (resp., $X \in \mathcal{Q}_m$) a vertex of a polygon $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$ (resp., a vertex of a polygon $\mathcal{Q}_{m,k}$, with $1 \leqslant k \leqslant N_b^m - 2$).

ii. $\mathcal{P}_m \bigcup \mathcal{Q}_m$ the reunion of the polygonal sets $\mathcal{P}_m$ and $\mathcal{Q}_m$, which consists in the set of all the vertices of the polygons $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$, along with the vertices of the polygons $\mathcal{Q}_{m,k}$, with $1 \leqslant k \leqslant N_b^m - 2$. In particular, $X \in \mathcal{P}_m \bigcup \mathcal{Q}_m$ simply denotes a vertex in $\mathcal{P}_m$ or $\mathcal{Q}_m$.

iii. $\mathcal{P}_m \bigcap \mathcal{Q}_m$ the intersection of the polygonal sets $\mathcal{P}_m$ and $\mathcal{Q}_m$, which consists in the set of all the vertices of both a polygon $\mathcal{P}_{m,k}$, with $0 \leqslant k \leqslant N_b^m - 1$, and a polygon $\mathcal{Q}_{m,k'}$, with $1 \leqslant' k \leqslant N_b^m - 2$.

Definition 2.6 (**Vertices of the Prefractals, Elementary Lengths, Heights and Angles [DL22a]**).

Given a strictly positive integer m, we denote by $\left(M_{j,m}\right)_{0 \leqslant j \leqslant (N_b - 1)\, N_b^m}$ *the set of vertices* of the prefractal graph $\Gamma_{\mathcal{W}_m}$. One thus has, for any integer j in $\left\{0, \ldots, (N_b - 1)\, N_b^m\right\}$:

$$M_{j,m} = \left(\frac{j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m} \right) \right) .$$

We also introduce, for any integer j in $\{0,\ldots,(N_b-1)\,N_b^m-1\}$:

i. <u>the elementary horizontal lengths</u>:

$$L_m = \frac{j}{(N_b-1)\,N_b^m} \, ; \qquad (\mathcal{R}\,4)$$

ii. <u>the elementary lengths</u>:

$$l_{j,j+1,m} = d\left(M_{j,m}, M_{j+1,m}\right) = \sqrt{L_m^2 + h_{j,j+1,m}^2} \, ,$$

where $h_{j,j+1,m}$ is defined in *iii.* just below.

iii. <u>the elementary heights</u>:

$$h_{j,j+1,m} = \left| \mathcal{W}\left(\frac{j+1}{(N_b-1)\,N_b^m} \right) - \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m} \right) \right| \, ;$$

iv. <u>the minimal height</u>:

$$h_m^{inf} = \inf_{0\leqslant j\leqslant (N_b-1)\,N_b^m-1} h_{j,j+1,m} \, , \qquad (\mathcal{R}\,5)$$

along with the <u>the maximal height</u>:

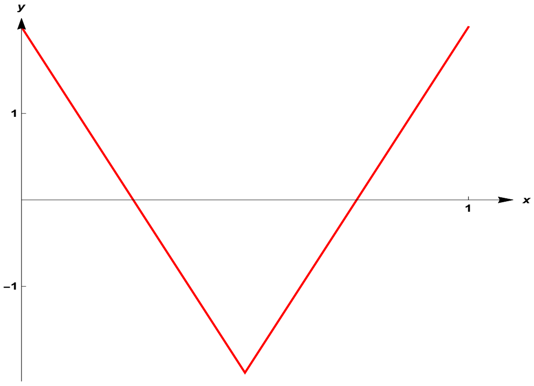
$$h_m = \sup_{0\leqslant j\leqslant (N_b-1)\,N_b^m-1} h_{j,j+1,m} \, ; \qquad (\mathcal{R}\,6)$$

v. <u>the geometric angles</u>:

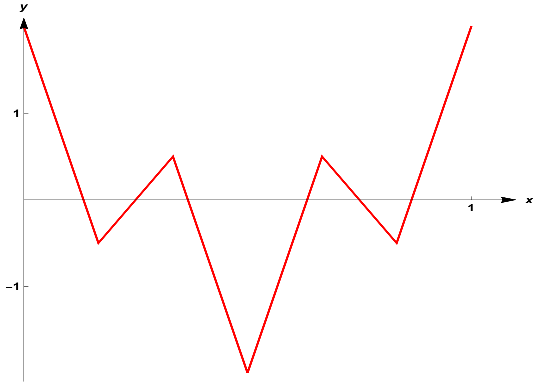
$$\theta_{j-1,j,m} = \left((y'y), \left(\widehat{M_{j-1,m} M_{j,m}}\right) \right) \quad , \quad \theta_{j,j+1,m} = \left((y'y), \left(\widehat{M_{j,m} M_{j+1,m}}\right) \right) ,$$

which yield **the value of the geometric angle between consecutive edges**, namely, $\left[M_{j-1,m}\,M_{j,m}, M_{j,m}\,M_{j+1,m}\right]$:

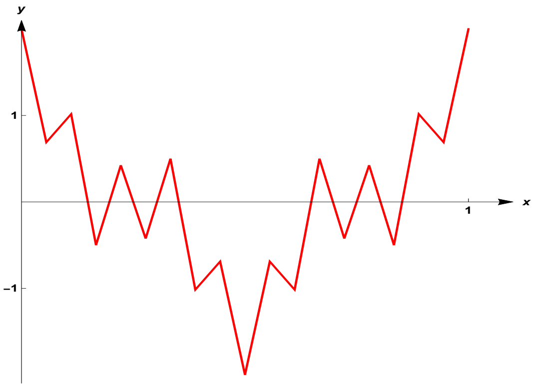
$$\theta_{j-1,j,m} + \theta_{j,j+1,m} = \arctan \frac{L_m}{\left|h_{j-1,j,m}\right|} + \arctan \frac{L_m}{\left|h_{j,j+1,m}\right|} \, .$$



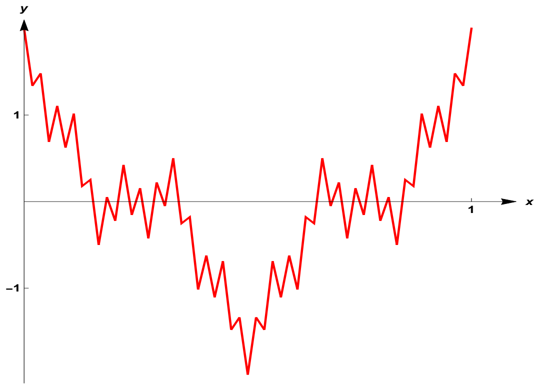
(a) **The prefractal graph $\Gamma_{\mathcal{W}_0}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**



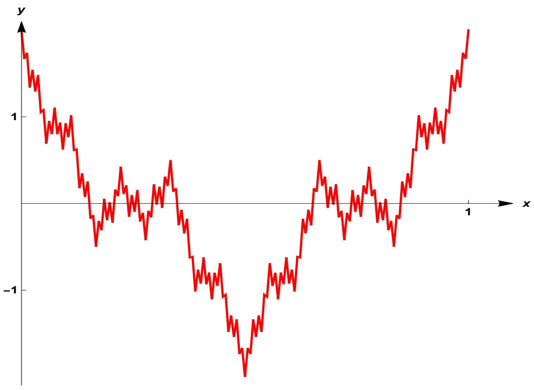
(b) **The prefractal graph $\Gamma_{\mathcal{W}_1}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**



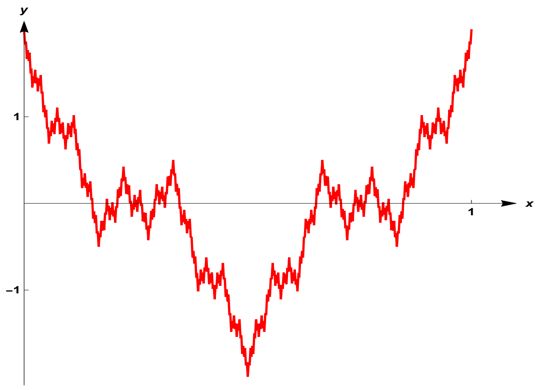
(c) **The prefractal graph $\Gamma_{\mathcal{W}_2}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**



(d) **The prefractal graph $\Gamma_{\mathcal{W}_3}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**



(e) **The prefractal graph $\Gamma_{\mathcal{W}_4}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**



(f) **The prefractal graph $\Gamma_{\mathcal{W}_5}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

Figure 2: **The prefractal graphs $\Gamma_{\mathcal{W}_0}$, $\Gamma_{\mathcal{W}_1}$, $\Gamma_{\mathcal{W}_2}$, $\Gamma_{\mathcal{W}_3}$, $\Gamma_{\mathcal{W}_4}$, $\Gamma_{\mathcal{W}_5}$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

Property 2.7. *For the geometric angle $\theta_{j-1,j,m}$, $0 \leqslant j \leqslant (N_b-1)\,N_b^m$, $m \in \mathbb{N}$, we have the following relation:*

$$\tan\theta_{j-1,j,m} = \frac{h_{j-1,j,m}}{L_m}\,.$$

The following property and definition play an important role in our previous work (especially, [DL22a], [DL24a], provided in this context in [DL22b], including for our interpretation of Poincaré duality).

Property 2.8 (A Consequence of the Symmetry with Respect to the Vertical Line $x = \frac{1}{2}$).

For any strictly positive integer m and any j in $\{0,\ldots,\#V_m-1\}$, we have that

$$\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right) = \mathcal{W}\left(\frac{(N_b-1)\,N_b^m-j}{(N_b-1)\,N_b^m}\right),$$

which means that the points

$$\left(\frac{(N_b-1)\,N_b^m-j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{(N_b-1)\,N_b^m-j}{(N_b-1)\,N_b^m}\right)\right) \quad \textit{and} \quad \left(\frac{j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right)$$

are symmetric with respect to the vertical line $x = \frac{1}{2}$; see Figure 3, on page 17.

Definition 2.7 (Left-Side and Right-Side Vertices).

Given nonnegative integers m, k such that $0 \leqslant k \leqslant N_b^m - 1$, and a polygon $\mathcal{P}_{m,k}$, we define:

i. The set of its *left-side vertices* as the set of the first $\left[\frac{N_b-1}{2}\right]$ vertices, where $[y]$ denotes the integer part of the real number y.

ii. The set of its *right-side vertices* as the set of the last $\left[\frac{N_b-1}{2}\right]$ vertices.

When the integer N_b is odd, we define the bottom vertex as the $\left(\frac{N_b-1}{2}\right)^{th}$ one; see Figure 4, on page 17.

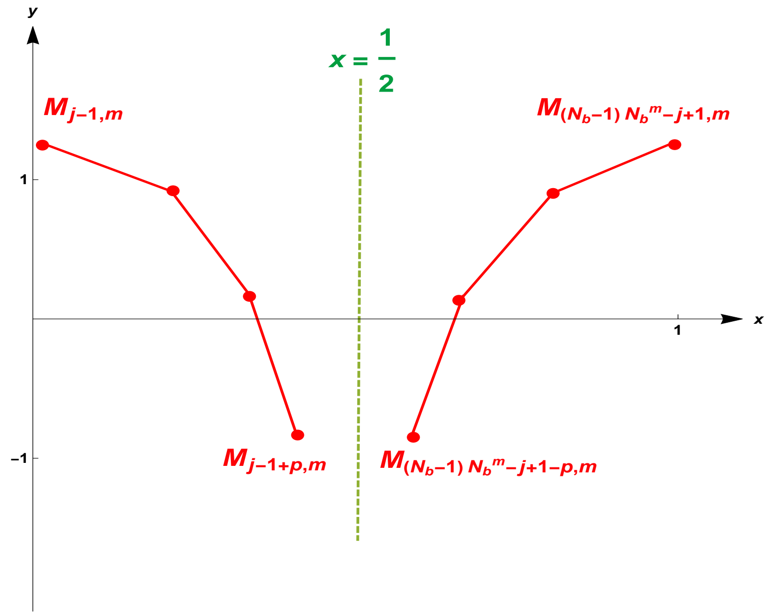


Figure 3: **Symmetric points with respect to the vertical line $x = \frac{1}{2}$.**

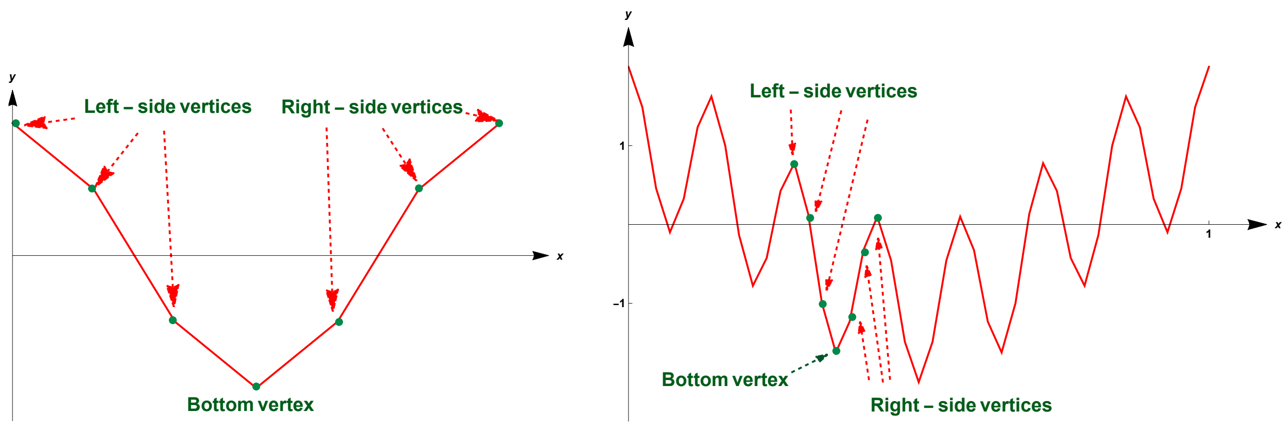


Figure 4: **The Left and Right–Side Vertices.**

Property 2.9 ([DL22a])**.**

For any integer j in $\{0, \ldots, N_b - 1\}$:

$$\mathcal{W}\left(\frac{j}{N_b-1}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\,N_b^n\,\frac{j}{(N_b-1)}\right) = \sum_{n=0}^{\infty} \lambda^n \cos\left(\frac{2\,\pi\,j}{N_b-1}\right) = \frac{1}{1-\lambda}\cos\left(\frac{2\,\pi\,j}{N_b-1}\right).$$

Property 2.10 ([DL22a])**.**

For $0 \leqslant j \leqslant \frac{(N_b-1)}{2}$ (resp., for $\frac{(N_b-1)}{2} \leqslant j \leqslant N_b - 1$), we have that

$$\mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \leqslant 0 \quad \left(\textit{resp.},\ \mathcal{W}\left(\frac{j+1}{N_b-1}\right) - \mathcal{W}\left(\frac{j}{N_b-1}\right) \geqslant 0\right).$$

Notation 8 (Signum Function).

The *signum function* of a real number x is defined by

$$\mathrm{sgn}\,(x) = \begin{cases} -1, & \text{if} \quad x < 0, \\ 0, & \text{if} \quad x = 0, \\ +1, & \text{if} \quad x > 0\,. \end{cases}$$

Property 2.11 ([DL22a]).

Given any strictly positive integer m, we have the following properties:

i. *For any j in $\{0, \ldots, \#V_m - 1\}$, the point*

$$\left(\frac{j}{(N_b - 1)\, N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1)\, N_b^m} \right) \right)$$

is the image of the point

$$\left(\frac{j}{(N_b - 1)\, N_b^{m-1}} - i, \mathcal{W}\left(\frac{j}{(N_b - 1)\, N_b^{m-1}} - i \right) \right) = \left(\frac{j - i\,(N_b - 1)\, N_b^{m-1}}{(N_b - 1)\, N_b^{m-1}}, \mathcal{W}\left(\frac{j - i\,(N_b - 1)\, N_b^{m-1}}{(N_b - 1)\, N_b^{m-1}} \right) \right)$$

under the map T_i, where $i \in \{0, \ldots, N_b - 1\}$ is arbitrary.

*Consequently, for $0 \leqslant j \leqslant N_b - 1$, **the j^{th} vertex of the polygon** $\mathcal{P}_{m,k}$, $0 \leqslant k \leqslant N_b^m - 1$, i.e., the point*

$$\left(\frac{(N_b - 1)\, k + j}{(N_b - 1)\, N_b^m}, \mathcal{W}\left(\frac{(N_b - 1)\, k + j}{(N_b - 1)\, N_b^m} \right) \right),$$

is the image of the point

$$\left(\frac{(N_b - 1)\,\left(k - i\,(N_b - 1)\, N_b^{m-1}\right) + j}{(N_b - 1)\, N_b^{m-1}}, \mathcal{W}\left(\frac{(N_b - 1)\,\left(k - i\,(N_b - 1)\, N_b^{m-1}\right) + j}{(N_b - 1)\, N_b^{m-1}} \right) \right)$$

*under the map T_i, where $i \in \{0, \ldots, N_b - 1\}$ is again arbitrary. It is also **the j^{th} vertex of the polygon** $\mathcal{P}_{m-1,k-i\,(N_b-1)\,N_b^{m-1}}$. Therefore, there is an exact correspondence between vertices of the polygons at consecutive steps $m - 1$, m.*

ii. *Given j in $\{0, \ldots, N_b - 2\}$ and k in $\left\{0, \ldots, N_b^m - 1\right\}$, we have that*

$$sgn\left(\mathcal{W}\left(\frac{k\,(N_b - 1) + j + 1}{(N_b - 1)\, N_b^m} \right) - \mathcal{W}\left(\frac{k\,(N_b - 1) + j}{(N_b - 1)\, N_b^m} \right) \right) = sgn\left(\mathcal{W}\left(\frac{j + 1}{N_b - 1} \right) - \mathcal{W}\left(\frac{j}{N_b - 1} \right) \right).$$

Proof.

i. Given $m \in \mathbb{N}^\star$, let us consider $i \in \{0, \ldots, N_b - 1\}$. The image of the point

$$\left(\frac{j}{(N_b-1)\,N_b^{m-1}} - i, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}} - i \right) \right)$$

under the map T_i is obtained by applying the analytic expression given in Property 2.2, on page 10, to the coordinates of this point, which, thanks to Property 2.5, on page 11 above, yields the expected result, namely,

$$\left(\frac{j}{(N_b-1)\,N_b^{m}}, \lambda \underbrace{\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}} - i \right)}_{\substack{\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m-1}} \right) \\ \text{(by 1-periodicity)}}} + \cos \frac{2\pi j}{(N_b-1)\,N_b^{m}} \right) = \left(\frac{j}{(N_b-1)\,N_b^{m}}, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^{m}} \right) \right).$$

ii. See [DL22a].

□

Property 2.12 (**Lower Bound and Upper Bound for the Elementary Heights [DL22a]**).

For any strictly positive integer m and any j in $\left\{0, \ldots, (N_b-1)\,N_b^m\right\}$, we have the following estimates, where L_m is the elementary horizontal length introduced in part i. of Definition 2.6, on page 13:

$$C_{inf}\, L_m^{2-D_\mathcal{W}} \leqslant \underbrace{|\mathcal{W}((j+1)\,L_m) - \mathcal{W}(j\,L_m)|}_{h_{j,j+1,m}} \leqslant C_{sup}\, L_m^{2-D_\mathcal{W}} \quad , \quad m \in \mathbb{N},\, 0 \leqslant j \leqslant (N_b-1)\,N_b^m, \qquad (✠)$$

where the finite and positive constants C_{inf} and C_{sup} are given by

$$C_{inf} = (N_b-1)^{2-D_\mathcal{W}} \min_{0 \leqslant j \leqslant N_b-1,\, \mathcal{W}\left(\frac{j+1}{N_b-1}\right) \neq \mathcal{W}\left(\frac{j}{N_b-1}\right)} \left| \mathcal{W}\left(\frac{j+1}{N_b-1} \right) - \mathcal{W}\left(\frac{j}{N_b-1} \right) \right|$$

and

$$C_{sup} = (N_b-1)^{2-D_\mathcal{W}} \left(\max_{0 \leqslant j \leqslant N_b-1} \left| \mathcal{W}\left(\frac{j+1}{N_b-1} \right) - \mathcal{W}\left(\frac{j}{N_b-1} \right) \right| + \frac{2\pi}{(N_b-1)\,(\lambda\,N_b-1)} \right).$$

One should note, in addition, that these constants C_{inf} and C_{sup} depend on the initial polygon $\mathcal{P}_0$, but are independent of $m \in \mathbb{N}$ sufficiently large.

As a consequence, we also have that

$$C_{inf}\, L_m^{2-D_{\mathcal{W}}} \leqslant h_m^{inf} \leqslant C_{sup}\, L_m^{2-D_{\mathcal{W}}} \quad and \quad C_{inf}\, L_m^{2-D_{\mathcal{W}}} \leqslant h_m \leqslant C_{sup}\, L_m^{2-D_{\mathcal{W}}} \,,$$

where h_m^{inf} and h_m respectively denote the minimal and maximal heights introduced in part iv. of Definition 2.6, on page 13.

Theorem 2.13 (**Sharp Local Discrete Reverse Hölder Properties of the Weierstrass Function [DL22a]**).

For any nonnegative integer m (i.e., for any $m \in \mathbb{N}$), let us consider a pair of real numbers (x, x') such that

$$x = \frac{(N_b-1)\,k+j}{(N_b-1)\,N_b^m} = ((N_b-1)\,k+j)\;L_m \quad , \quad x' = \frac{(N_b-1)\,k+j+\ell}{(N_b-1)\,N_b^m} = ((N_b-1)\,k+j+\ell)\;L_m \,,$$

where $0 \leqslant k \leqslant N_b^m - 1$. We then have the following (discrete, local) reverse-Hölder inequality*, with sharp Hölder exponent $-\dfrac{\ln\lambda}{\ln N_b} = 2 - D_{\mathcal{W}}$:*

$$C_{inf}\,|x'-x|^{2-D_{\mathcal{W}}} \leqslant \left|\mathcal{W}(x') - \mathcal{W}(x)\right| \,,$$

where $(x, \mathcal{W}(x))$ and $\left(x', \mathcal{W}(x')\right)$ are adjacent vertices of the same m^{th} prefractal approximation, $\Gamma_{\mathcal{W}_m}$, with $m \in \mathbb{N}$ arbitrary. Here, C_{inf} is given as in Property 2.12, on page 19 just above.

Corollary 2.14 (**Optimal Hölder Exponent for the Weierstrass Function (see [DL22a])**).

The local reverse Hölder property of Theorem 2.13, on page 20 just above – in conjunction with the Hölder condition satisfied by the Weierstrass function (see also [Zyg02], Chapter II, Theorem 4.9, page 47) – shows that the Codimension $2 - D_{\mathcal{W}} = -\dfrac{\ln\lambda}{\ln N_b} \in \,]0,1[$ is the best (i.e., optimal) Hölder exponent for the Weierstrass function (as was originally shown, by a completely different method, by G. H. Hardy in [Har16]).

Note that, as a consequence, since the Hölder exponent is strictly smaller than one, it follows that the Weierstrass function $\mathcal{W}$ is nowhere differentiable. Indeed, if $\mathcal{W}$ were differentiable at some point $x_0 \in \mathbb{R}$, it would have to be locally Lipschitz at x_0.

Corollary 2.15 (**of Property 2.12 (see [DL22a])**).

Thanks to Property 2.12, on page 19, one may now write, for any strictly positive integer m and any integer j in $\left\{0, \ldots, (N_b-1)\,N_b^m - 1\right\}$, and with C_{inf} and C_{supf} defined as in Property 2.12, on page 19:

i. <u>for the elementary heights:</u>

$$h_{j-1,j,m} = L_m^{2-D_{\mathcal{W}}}\,\mathcal{O}(1) \;; \tag{$\mathcal{R}\,7$}$$

ii. <u>*for the elementary quotients:*</u>

$$\frac{h_{j-1,j,m}}{L_m} = L_m^{1-D_{\mathcal{W}}}\,\mathcal{O}(1) \ , \tag{$\mathcal{R}\,8$}$$

and where

$$0 < C_{inf} \leqslant \mathcal{O}(1) \leqslant C_{sup} < \infty$$

ad where, as is explained in Notation 2, on page 8, the notation $\mathcal{O}(1)$ denotes terms which depend on $m \in \mathbb{N}$, but are bounded away from 0 and ∞, uniformly in $m \in \mathbb{N}$.

Corollary 2.16 (**Nonincreasing Sequence of Geometric Angles** (**Coming from Property** 2.11; see [DL22a])).

*For the **geometric angles** $\theta_{j-1,j,m}$, $0 \leqslant j \leqslant (N_b-1)\,N_b^m$, $m \in \mathbb{N}$, introduced in part v. of Definition 2.6, on page 13, we have the following result:*

$$\tan\theta_{j-1,j,m} = \frac{L_m}{h_{j-1,j,m}}(N_b-1) > \tan\theta_{j-1,j,m+1} \,,$$

which yields

$$\theta_{j-1,j,m} > \theta_{j-1,j,m+1} \quad \textit{and} \quad \theta_{j-1,j,m+1} \lesssim L_m^{D_{\mathcal{W}}-1} \,.$$

Definition 2.8 (**m^{th} Cohomology Infinitesimal [DL22a], [DL22b] and m^{th} Intrinsic Cohomology Infinitesimal [DL23b], [DL23a]**)**.** From now on, given any $m \in \mathbb{N}$, we will call m^{th} *cohomology infinitesimal* the number $\varepsilon_m^m > 0$ which also corresponds to the elementary horizontal length introduced in part *i.* in Definition 2.6, on page 13; i.e., $\varepsilon_m^m = (\varepsilon_m)^m = \frac{1}{N_b-1}\,\frac{1}{N_b^m}$.

Observe that, clearly, ε_m itself – and not just ε_m^m – depends on m.

In addition, since $N_b > 1$, ε_m^m satisfies the following asymptotic behavior,

$$\varepsilon_m^m \to 0\,, \text{ as } m \to \infty,$$

which, naturally, results in the fact that the larger m, the smaller ε_m^m. It is for this reason that we call ε_m^m – or rather, the *sequence* $\left(\varepsilon_m^m\right)_{m=0}^{\infty}$ of positive numbers tending to zero as $m \to \infty$, with $\varepsilon_m^m = (\varepsilon_m)^m$, for each $m \in \mathbb{N}$ – an *infinitesimal.* Note that this m^{th} cohomology infinitesimal is the one naturally associated to the scaling relation of Proposition 2.5, on page 11.

In the sequel, it is also useful to keep in mind that the sequence of positive numbers $(\varepsilon_m)_{m=0}^{\infty}$ itself satisfies

$$\varepsilon_m \sim \frac{1}{N_b}\,, \text{ as } m \to \infty \ ;$$

i.e., $\varepsilon_m \to \frac{1}{N_b}$, as $m \to \infty$. In particular, $\varepsilon_m \not\to 0$, as $m \to \infty$, but, instead, ε_m tends to a strictly positive and finite limit.

We also introduce, given any $m \in \mathbb{N}$, the m^{th} *intrinsic cohomology infinitesimal*, denoted by $\varepsilon^m > 0$, such that

$$\varepsilon^m = \frac{1}{N_b^m} ,$$

where

$$\varepsilon = \frac{1}{N_b} .$$

We call ε *the intrinsic scale*, or *intrinsic subdivision scale*.

Note that

$$\varepsilon_m^m = (\varepsilon_m)^m = \frac{\varepsilon^m}{N_b - 1}$$

and that $\varepsilon_m \to \varepsilon$, as $m \to \infty$.

Remark 2.2 (**Connection Between the Parameter λ and the Minkowski Dimension $D_{\mathcal{W}}$**).

Note that since

$$2 - D_{\mathcal{W}} = -\frac{\ln \lambda}{\ln N_b} = \ln_{N_b} \frac{1}{\lambda} ,$$

(see Notation 5, on page 9), we also have that

$$\lambda = N_b^{D_{\mathcal{W}}-2} = \varepsilon^{2-D_{\mathcal{W}}} ,$$

where $\varepsilon = \dfrac{1}{N_b}$ is the intrinsic scale introduced in Definition 2.8 just above, on page 21.

Definition 2.9 (**Cohomological Vertex Integer [DL24b]**).

Given $m \in \mathbb{N}$, and a vertex $M_{j,m} = M_{(N_b-1)\,k'+k",m} \in V_m$, of abscissa $\left((N_b - 1)\,k' + k"\right) \varepsilon_m^m$, where $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k" \leqslant N_b - 1$, we introduce the *cohomological vertex integer* $\ell_{j,m}$ associated to the vertex $M_{j,m}$ (which is also the $(k")^{th}$ vertex of the polygon $\mathcal{P}_{m,k'}$; see part *iv.* of Property 2.6, on page 12), as

$$\ell_{j,m} = \ell_{k',k",m} = (N_b - 1)\,k' + k" . \qquad (\mathcal{R}\,9)$$

$$\left((N_b - 1)\,k' + k"\right) \varepsilon_m^m = \left((N_b - 1)\,k'bis + k"bis\right) \varepsilon_{m+1}^{m+1} .$$

Depending on the context; that is,

i. when the cohomological vertex integer enables one to locate the vertex $M_{j,m}$.

ii. When it is used in a more general framework, i.e., in order to describe the generators of cohomology groups;

we will use the best suited notation between $\ell_{j,m}$, in case *i.*, or $\ell_{k',k",m}$, in case *ii.*

Proposition 2.17 (Cross-Scales Paths, and Associated Sequence of Vertex Integers).

Given $m \in \mathbb{N}$, $0 \leqslant j \leqslant \#V_m - 1$ *and a vertex* $M_{j,m} = M_{(N_b-1)\,k'+k",m}$ *in* V_m, *with* $0 \leqslant k' \leqslant N_b^m - 1$ *and* $0 \leqslant k" \leqslant N_b - 1$, *we introduce* the cross-scales path $\mathcal{P}ath\left(P_{k"}, M_{j,m}\right)$, *where* $P_{k'}$ *is the* $(k')^{th}$ *fixed point of the map* $T_{k'}$ *(see Proposition 2.2, on page 10, along with Notation 6, on page 10), as the ordered set* $\left(M_{j_{k,m},k}\right)_{0\leqslant k\leqslant m}$ *such that:*

i. For $0 \leqslant k \leqslant m$, *each vertex* $M_{j_{k,m},k}$ *is in* $V_k \setminus V_k \cap V_m$ *(which means that* $M_{j_{k,m},k}$ *strictly belongs to* V_k, *i.e., it is in the* k^{th} *prefractal approximation* $\Gamma_{\mathcal{W}_k}$, *and not in* $\Gamma_{\mathcal{W}_{k+1}}$*).*

ii. For $1 \leqslant k \leqslant m$, *each vertex* $M_{j_{k,m},k} = M_{(N_b-1)\,k'_{k,m}+k",k}$, *with* $0 \leqslant k'_{k,m} \leqslant N_b^k - 1$, *is the image of the point* $M_{j_{k-1,m},k-1}$ *under the map* T_i *(see again Proposition 2.2, on page 10), where* $i \in \{0,\ldots,N_b-1\}$ *is the smallest admissible value. We thus also have that*

$$M_{j_{k-1,m},k-1} = \left(\frac{(N_b-1)\left(k'_{k,m} - i\,(N_b-1)\,N_b^{k-1}\right)+k"}{(N_b-1)\,N_b^{k-1}}, \mathcal{W}\left(\frac{(N_b-1)\left(k'_{k,m} - i\,(N_b-1)\,N_b^{k-1}\right)+k"}{(N_b-1)\,N_b^{k-1}}\right)\right).$$

This latter point is also ***the*** $\boldsymbol{(k")^{th}}$ ***vertex of the polygon*** $k'_{k,m} - i\,(N_b-1)\,N_b^{k-1}$ *(see part iv. of Property 2.6, on page 12).*

The sequence of vertex integers associated with the cross-scales path $\mathcal{P}ath\left(P_{k"}, M_{j,m}\right)$ *(or, in short, and equivalently, also refered to as the sequence of vertex integers associated with* $M_{j,m}$*) is the sequence* $\left(\ell_{j_{k,m},k}\right)_{0\leqslant k\leqslant m}$, *where, for* $0 \leqslant k \leqslant m$, $\ell_{j_{k,m},k}$ *is the cohomological vertex integer associated with the vertex* $M_{j_{k,m},k}$ *(see Definition 2.9, on page 22).*

Proof. We simply use the results of Property 2.11, on page 18.

□

We now recall the following key result, obtained in [DL22b], and extended in [DL23b].

Theorem 2.18 (Complex Dimensions Series Expansion of the Complexified Weierstrass function $\mathcal{W}_{comp}$ [DL22b], [DL23b], and of the Weierstrass function $\mathcal{W}$).

For any sufficiently large positive integer m *and any* j *in* $\{0,\ldots,\#V_m-1\}$, *we have the following* exact *expansion, indexed by the Complex Codimensions* $k\,(D_{\mathcal{W}}-2) + i\,k\,\ell_{j_{k,m},k}\,\mathbf{p}$, *with* $0 \leqslant k \leqslant m$,

$$\begin{aligned}
\mathcal{W}_{comp}\left(j\,\varepsilon_m^m\right) &= \mathcal{W}_{comp}\left(\frac{j\,\varepsilon^m}{N_b-1}\right) \\
&= \varepsilon^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}_{comp}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} c_{k,j,m}\,\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_{k,j,m}\,\mathbf{p}} \qquad (\mathcal{R}\,10) \\
&= \sum_{k=0}^{m} c_{k,j,m}\,\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_{j_{k,m},k}\,\mathbf{p}},
\end{aligned}$$

where, for $0 \leqslant k \leqslant m$, ε^k *is the* k^{th} *intrinsic cohomology infinitesimal, introduced in Definition 2.8, on page 21, with* $\mathbf{p} = \dfrac{2\pi}{\ln N_b}$ *denoting the oscillatory period of the Weierstrass Curve, as introduced in [DL22a] and where:*

i. $\ell_{j_{k,m},k} \in \mathbb{Z}$ *is the cohomological vertex integer associated with the vertex* $M_{j_{k,m},k}$ *(see Definition 2.9, on page 22);*

ii. $c_{m,j,m} = \mathcal{W}_{comp}\left(\dfrac{j}{N_b-1}\right)$ *and, for* $0 \leqslant k \leqslant m-1$, $c_{k,j,m} \in \mathbb{C}$ *is given by*

$$c_{k,j,m} = exp\left(\frac{2\,i\,\pi}{N_b-1}\,j\,\varepsilon^{m-k}\right). \quad (\diamond\diamond) \tag{$\mathcal{R}$ 11}$$

for $0 \leqslant k \leqslant m$, *the coefficient* $c_{k,j,m}$ *will also be referred to as* the k^{th} Weierstrass coefficient associated with the vertex $M_{j_{k,m},k} \in V_k$.

For any $m \in \mathbb{N}$, *the complex numbers* $\{c_{0,j,m+1}, \ldots, c_{m+1,j,m+1}\}$ *satisfy the following recurrence relations:*

$$c_{m+1,j,m+1} = \mathcal{W}\left(\frac{j}{N_b-1}\right) = c_{m,j,m} \tag{$\mathcal{R}$ 12}$$

and

$$\forall\, k \in \{1,\ldots,m\} : \quad c_{k,j,m+1} = c_{k-1,j,m}\,. \tag{$\mathcal{R}$ 13}$$

In addition, since relation ($\mathcal{R}$10) is valid for any $m \in \mathbb{N}^\star$ *(and since, clearly, relation ($\mathcal{R}$11) implies that the coefficients* $c_{k,j,m}$ *are nonzero for* $0 \leqslant k \leqslant m$*), we deduce that the associated Complex Dimensions (i.e., in fact, the Complex Dimensions associated with the Weierstrass function) are*

$$D_{\mathcal{W}} - k\,(2-D_{\mathcal{W}}) + i\,\ell_{j_{k,m},k}\,\mathbf{p}$$

$0 \leqslant k \leqslant m$ *and* $\ell_{j_{k,m},k} \in \mathbb{Z}$ *is the cohomological vertex integer associated with the vertex* $M_{j_{k,m},k}$ *(see Definition 2.9, on page 22).*

This immediately ensures, for the Weierstrass function (i.e., the real part of the Complexified Weierstrass function $\mathcal{W}_{comp}$*), that, for any strictly positive integer* m *and for any* j *in* $\{0,\ldots,\#V_m-1\}$,

$$\begin{aligned}
\mathcal{W}\left(j\,\varepsilon_m^m\right) &= \varepsilon^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}_{comp}\left(\frac{j}{N_b-1}\right) + \sum_{k=0}^{m-1} \varepsilon^{k\,(2-D_{\mathcal{W}})}\,\mathcal{R}e\left(c_{k,j,m}\,\varepsilon_k^{\,i\,\ell_{j_{k,m},k}\,\mathbf{p}}\right) \\
&= \varepsilon^{m\,(2-D_{\mathcal{W}})}\,\mathcal{W}_{comp}\left(\frac{j}{N_b-1}\right) + \frac{1}{2}\sum_{k=0}^{m-1} \varepsilon^{k\,(2-D_{\mathcal{W}})}\left(c_{k,j,m}\,\varepsilon^{\,i\,\ell_{j_{k,m},k}\,\mathbf{p}} + \overline{c_{k,j,m}}\,\varepsilon^{-\,i\,\ell_{j_{k,m},k}\,\mathbf{p}}\right) \\
&= \frac{1}{2}\sum_{k=0}^{m} \varepsilon^{k\,(2-D_{\mathcal{W}})}\left(c_{k,j,m}\,\varepsilon^{\,i\,\ell_{j_{k,m},k}\,\mathbf{p}} + \overline{c_{k,j,m}}\,\varepsilon^{-\,i\,\ell_{j_{k,m},k}\,\mathbf{p}}\right),
\end{aligned} \tag{$\mathcal{R}$ 14}$$

where $\bar{z}$ *denotes the complex conjugate of* $z \in \mathbb{C}$.

More generally, for any strictly positive integer m and for any integer $j \in \mathbb{N}$ (and not only any j in $\{0, \ldots, \#V_m - 1\}$),

$$\mathcal{W}_{comp}\left(j\,\varepsilon^m\right) = \sum_{k=0}^{\infty} \varepsilon^{k\,(2-D_{\mathcal{W}})}\, c_{k,j,m}\, \varepsilon^{k\,(2-D_{\mathcal{W}})}\, \varepsilon^{i\,\ell_{j_{k,m},k}\,\mathbf{p}}\,, \qquad (\mathcal{R}\,15)$$

where, for all $k \in \mathbb{N}$,

$$c_{k,j,m} = \varepsilon^{2\,i\,\pi\,N_b^k\,j\,\varepsilon^m}\,,. \qquad (\mathcal{R}\,16)$$

Due to the density of the set $V^\star = \bigcup_{n\in\mathbb{N}} V_n$ in the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (see Property 2.4, on page 11), for each point $X \in \Gamma_{\mathcal{W}}$:

$$X = \lim_{m\to\infty} M_{j,m}\,,$$

where $M_{j,m} \in V_m$. We also note that, if a vertex $M_{j,m} = M_{j',m+m'}$ is in $V_m \cap V_{m+m'}$, for $m' \in \mathbb{N}$, we of course have that, for $0 \leqslant k \leqslant m$

$$c_{k,j,m} = c_{k,j',m+m'}\,, \qquad (\mathcal{R}\,17)$$

along with

$$\varepsilon^{i\,\ell_{j_{k,m},k}} = \varepsilon^{i\,\ell_{j_{k,m+m''},k}}\,. \qquad (\mathcal{R}\,18)$$

For $m+1 \leqslant k \leqslant m+m'$, we have that

$$c_{k,j,m} = c_{k,j',m+m'} = 0\,. \qquad (\mathcal{R}\,19)$$

Property 2.19 (**A Hodge Diamond Star Relation [DL23b]**).

For any $m \in \mathbb{N}^\star$, any k in $\{1, \ldots, m\}$ and any j in $\{0, \ldots, \#V_m - 1\}$, we have the following Hodge Diamond Star *relation*

$$c_{k,(N_b-1)\,N_b^m-j,m}\, \varepsilon^{i\,\ell_{k,(N_b-1)\,N_b^m-j,m}\,\mathbf{p}} = \overline{c_{k,j,m}\, \varepsilon^{i\,\ell_{k,j,m}\,\mathbf{p}}}\,, \qquad (\mathcal{R}\,20)$$

which is directly connected to the symmetry with respect to the vertical line $x = \frac{1}{2}$, stated in Property 2.8, on page 16, since the points

$$\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{(N_b-1)\,N_b^m - j}{(N_b-1)\,N_b^m}\right)\right) \quad \text{and} \quad \left(\frac{j}{(N_b-1)\,N_b^m}, \mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right)$$

are symmetric with respect to the vertical line $x = \frac{1}{2}$; see Figure 3, on page 17. It is also reminiscent of Poincaré duality (see our previous work [DL22b], and the corresponding result in Theorem 4.27, on page 63).

2.2 Fractal Zeta Functions

Theorem 2.20 (**Local and Global Polyhedral Effective Zeta Functions [DL23b]**).

Given any $m \in \mathbb{N}^\star$ sufficiently large, we introduce the m^{th} local polyhedral effective zeta function $\tilde{\zeta}^e_m$, *such that, for all $s \in \mathbb{C}$,*

$$\begin{aligned}\tilde{\zeta}^e_m(s) &= \frac{1}{2}\,\varepsilon^m \sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^{m} \frac{c_{k,(N_b-1)\,j+q,m}\,\varepsilon^{s-2+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-2+k\,(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}} \\ &+\frac{1}{2}\,\varepsilon^m \sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^{m} \frac{\overline{c_{k,(N_b-1)\,j+q,m}}\,\varepsilon^{s-2+k\,(2-D_{\mathcal{W}})-i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-2+k\,(2-D_{\mathcal{W}})-i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}\,,\end{aligned} \qquad (\mathcal{R}\,21)$$

or, equivalently,

$$\begin{aligned}\tilde{\zeta}^e_m(s) &= \frac{1}{2}\,\varepsilon^m \sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^{m} \frac{c_{k,(N_b-1)\,j+q,m}\,\varepsilon^{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}} \\ &+\frac{1}{2}\,\varepsilon^m \sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b} \alpha_q(N_b) \sum_{k=1}^{m} \frac{\overline{c_{k,(N_b-1)\,j+q,m}}\,\varepsilon^{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})-i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})-i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}\,,\end{aligned} \qquad (\mathcal{R}\,22)$$

where the complex coefficients $c_{k,(N_b-1)\,j+q,m}$ are given by relation ($\mathcal{R}$11), on page 24, $\left(\ell_{k,(N_b-1)\,j+q,m}\right)_{0\leqslant k\leqslant m}$ denotes the sequence of cohomological vertex integers associated with the vertex $M_{(N_b-1)\,j+q,m} \in V_m$ (see Definition 2.17, on page 23), and where ε is the intrinsic scale introduced in Definition 2.8, on page 21), and where, for $0 \leqslant j \leqslant N_b^m-1$ and $0 \leqslant q \leqslant N_b$,

a. <u>*When the integer N_b is odd:*</u>

$$\alpha_0(N_b) = \alpha_{N_b-1}(N_b) = -\alpha_{\frac{N_b-1}{2}}(N_b) = -\alpha_{\frac{N_b-1}{2}+N_b-1}(N_b) = \frac{N_b-2}{2\,(N_b-1)}$$

and for $1 \leqslant q \leqslant N_b-2$,

$$\alpha_{\frac{N_b-1}{2}+q}(N_b) = -\alpha_q(N_b) = -\alpha_{N_b^m,(N_b-1)\,N_b^m-(N_b-1)+q}(N_b) = \frac{1}{N_b-1}\,,$$

along with

$$\alpha_{N_b^m,(N_b-1)\,N_b^m-(N_b-1)}(N_b) = \alpha_{N_b^m,1}(N_b) = \frac{N_b-2}{2\,(N_b-1)}\,.$$

b. <u>*When the integer N_b is even:*</u>

$$\alpha_0(N_b) = \alpha_{N_b-1}(N_b) = -\alpha_{\frac{N_b}{2}}(N_b) = -\alpha_{\frac{N_b}{2}+N_b-1}(N_b) = \frac{N_b-2}{2\,(N_b-1)}$$

and for $1 \leqslant q \leqslant N_b-2$,

$$\alpha_{\frac{N_b}{2}+q}(N_b) = -\alpha_q(N_b) = -\alpha_{N_b^m,(N_b-1)\,N_b^m-(N_b-1)+q}(N_b) = -\frac{1}{N_b-1}\,,$$

along with

$$\alpha_{N_b^m,(N_b-1)\,N_b^m-(N_b-1)}(N_b) = \alpha_{N_b^m,1}(N_b) = \frac{N_b-2}{2\,(N_b-1)}\,.$$

More specifically, still for all $m \in \mathbb{N}^\star$ sufficiently large, the function $\tilde{\zeta}_m^e$ is well defined and meromorphic in all of $\mathbb{C}$. Furthermore, its (necessarily unique) meromorphic extension (still denoted by $\tilde{\zeta}_m^e$) is given, for all $s \in \mathbb{C}$ by the expressions given in relation ($\mathcal{R}$21) above.

Moreover, the associated sequence $\left(\tilde{\zeta}_m^e\right)_{m\in\mathbb{N}}$ satisfies the following recurrence relation, for all values of the positive integer m sufficiently large, and for all $s \in \mathbb{C}$:

$$\begin{aligned}\tilde{\zeta}_{m+1}^e(s) = {} & \varepsilon\,\tilde{\zeta}_m^e(s)\\ & +\frac{1}{2}\,\varepsilon^{m+1}\sum_{j=0}^{N_b^{m+1}-1}\sum_{q=1}^{N_b-1}\sum_{k=1}^{m+1}\alpha_q(N_b)\,c_{k,(N_b-1)\,j+q,m+1}\,\frac{\varepsilon^{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})+i\,((N_b-1)\,j+q)\,\mathbf{p}}}{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})+i\,((N_b-1)\,j+q)\,\mathbf{p}}\\ & +\frac{1}{2}\,\varepsilon^{m+1}\sum_{j=0}^{N_b^{m+1}-1}\sum_{q=1}^{N_b-1}\sum_{k=1}^{m+1}\alpha_q(N_b)\,\overline{c_{k,(N_b-1)\,j+q,m+1}}\,\frac{\varepsilon^{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})-i\,((N_b-1)\,j+q)\,\mathbf{p}}}{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})-i\,((N_b-1)\,j+q)\,\mathbf{p}}\,,\end{aligned} \qquad (\mathcal{R}\,23)$$

where the complex coefficients $c_{k,(N_b-1)\,j+q,m+1}$ are given by relation ($\mathcal{R}$11), on page 24.

This ensures the existence of the limit fractal zeta function $\tilde{\zeta}_{\mathcal{W}}^e$, i.e., the fractal zeta function associated with the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (or, rather, with the Weierstrass IFD), called the global polyhedral effective zeta function *and given by*

$$\tilde{\zeta}_{\mathcal{W}}^e = \lim_{m\to\infty}\tilde{\zeta}_m^e\,,$$

where the convergence is locally uniform on $\mathbb{C}$, along with the existence of an integer $m_0 \in \mathbb{N}$ such that, for any integer integer $m \geqslant m_0$, the set of poles of $\tilde{\zeta}_{\mathcal{W}}^e$ is a subset of the set of poles of $\tilde{\zeta}_{m+1}^e$ – and hence also, of that of $\tilde{\zeta}_{\mathcal{W}}^e$. More specifically, $\tilde{\zeta}_{\mathcal{W}}^e$ is meromorphic in all of $\mathbb{C}$ and its meromorphic extension (still denoted $\tilde{\zeta}_{\mathcal{W}}^e$) is given, for all $s \in \mathbb{C}$ by

$$\begin{aligned}\tilde{\zeta}_{\mathcal{W}}^e(s) = {} & \frac{1}{2}\,\frac{1}{1-\varepsilon}\lim_{m\to\infty}\varepsilon^{m+1}\sum_{j=0}^{N_b^{m+1}-1}\sum_{q=1}^{N_b-1}\sum_{k=1}^{m+1}\alpha_q(N_b)\,c_{k,(N_b-1)\,j+q,m+1}\,\frac{\varepsilon^{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})+i\,((N_b-1)\,j+q)\,\mathbf{p}}}{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})+i\,((N_b-1)\,j+q)\,\mathbf{p}}\\ & +\frac{1}{2}\,\frac{1}{1-\varepsilon}\lim_{m\to\infty}\varepsilon^{m+1}\sum_{j=0}^{N_b^{m+1}-1}\sum_{q=1}^{N_b-1}\sum_{k=1}^{m+1}\alpha_q(N_b)\,\overline{c_{k,(N_b-1)\,j+q,m+1}}\,\frac{\varepsilon^{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})-i\,((N_b-1)\,j+q)\,\mathbf{p}}}{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})-i\,((N_b-1)\,j+q)\,\mathbf{p}}\,,\end{aligned} \qquad (\mathcal{R}\,24)$$

where the complex coefficients $c_{k,(N_b-1)\,j+q,m+1}$ are given by relation ($\mathcal{R}$11), on page 24.

In this statement, the meromorphic functions $\tilde{\zeta}_m^e$ and $\tilde{\zeta}_{\mathcal{W}}^e$ are viewed as continuous functions with values in the Riemann sphere In this statement, the meromorphic functions $\mathbb{P}^1(\mathbb{C}) = \mathbb{C}\cup\infty$, equipped with the chordal metric, as in [LvF13], [LRŽ17b], [LvF00], [HL21].

In other words, $\tilde{\zeta}_{\mathcal{W}}^e$ admits a (necessarily unique) meromorphic continuation, given, for all $s \in \mathbb{C}$, by relation ($\mathcal{R}$24) just above.

As was mentioned in the introduction, we note that our result is stronger than the one previously obtained in [DL22a], where, in particular, the values of the possible Complex Dimensions of the Weierstrass IFD included -2, 0 *and* $1-2\,k$, *with* $k\in\mathbb{N}$ *arbitrary. As we can see in relation* ($\mathcal{R}24$) *just above, the poles of the limit effective fractal zeta function* $\tilde{\zeta}^{e}_{\mathcal{W}}$ *are exactly the same as the Complex Dimensions of the Weierstrass function itself; see Theorem 2.18, on page 23. Note that, in [DL22a], the Complex Dimensions are defined in terms of the volume of the tubular (rather than polyhedral) neighborhoods of the Weierstrass IFD.*

Corollary 2.21 (**Intrinsic Complex Dimensions of the Weierstrass Curve**).

The Intrinsic Complex Dimensions *(or* Complex Dimensions, *in short) of the Weierstrass Curve* $\Gamma_{\mathcal{W}}$ *– defined as the poles of the global polyhedral effective zeta function* $\tilde{\zeta}^{e}_{\mathcal{W}}$ *– are all exact (i.e., actual poles of* $\tilde{\zeta}^{e}_{\mathcal{W}}$*), simple and given by*

$$\omega_{j_{k,m},k}=D_{\mathcal{W}}-k\,(2-D_{\mathcal{W}})\pm i\,\ell_{j_{k,m},k}\,\mathbf{p}\quad,\quad \textit{with } m\in\mathbb{N}\textit{ arbitrary and } 0\leqslant k\leqslant m\,, \qquad (\mathcal{R}\,25)$$

where the integers $\ell_{j_{k,m},k}\in\mathbb{N}$ *(which depend on* k *and* m*) are given in Theorem 2.18, on page 23) and where* $D_{\mathcal{W}}=2-\ln_{N_b}\dfrac{1}{\lambda}$, $2-D_{\mathcal{W}}=\ln_{N_b}\dfrac{1}{\lambda}$ *and* $\mathbf{p}=\dfrac{2\,\pi}{\ln N_b}$ *are, respectively, the Minkowski Dimension, the optimal Hölder exponent (as well as the Minkoswki Codimension) and the oscillatory period of the Weierstrass Curve. (Note that, for notational simplicity, we use the notation* $\ell_{j_{k,m},k}$ *of Theorem 2.18, on page 23 – associated to the integer* $0\leqslant k\leqslant m$ *– instead of the notation* $\ell_{k,(N_b-1)\,j+q,m}$ *– associated to the integer* $k-1\geqslant 0$ *of Theorem 2.20, on page 26. Of course, both notations are equivalent.)*

Consequently, the Weierstrass Curve $\Gamma_{\mathcal{W}}$ *has (countably) infinitely many nonreal Complex Dimensions and is fractal. More specifically, in the terminology of [LRŽ17b], [Lap19],* $\Gamma_{\mathcal{W}}$ *is fractal in (countably) infinitely many dimensions* $d_k=D_{\mathcal{W}}-k\,(2-D_{\mathcal{W}})$, *with* $k\in\mathbb{N}$ *arbitrary. Furthermore, for any* $k\in\mathbb{N}$, *on the vertical line with abscissa* $d_k=D_{\mathcal{W}}-k\,(2-D_{\mathcal{W}})\in\mathbb{R}$, *there are (countably) infinitely many nonreal Complex Dimensions, in complex conjugate pairs.*

Remark 2.3 (**Intrinsic Prefractal Complex Dimensions of the Weierstrass Curve**).

Similarly, for all $m\in\mathbb{N}^\star$ sufficiently large (i.e., for all $m\geqslant m_0$, for some $m_0\in\mathbb{N}^\star$), the m^{th} prefractal Complex Dimensions of the Weierstrass Curve – defined as the poles of the m^{th} local polyhedral effective zeta function $\tilde{\zeta}^{e}_{m}$ – are given as in Corollary 2.21 just above, except for the fact that $m\in\mathbb{N}^\star$ is fixed (equal to this value of m) and hence, is no longer arbitrary. Accordingly, for a given $k\in\mathbb{N}$, with $0\leqslant k\leqslant m$, there are only finitely many m^{th} prefractal Complex Dimensions with real part $d_k=D_{\mathcal{W}}-k\,(2-D_{\mathcal{W}})$ – namely $\omega_k\in\mathbb{R}$ itself and also, finitely many (nonreal) complex conjugate pairs with real part ω_k.

In particular, for all $m\geqslant m_0$, the m^{th} prefractal approximation $\Gamma_{\mathcal{W}_m}$ to the Weierstrass Curve $\Gamma_{\mathcal{W}}$ is fractal in finitely many dimensions $d_k=D_{\mathcal{W}}-k\,(2-D_{\mathcal{W}})$, as above, with $0\leqslant k\leqslant m$.

Corollary 2.22 (A Resulting Hodge Diamond Star Relation Satisfied by the Coefficients Involved in the Expression for the Local and Global Polyhedral Effective Zeta Functions).

For any $m \in \mathbb{N}^\star$, any k in $\{1,\ldots,m\}$ and any j in $\{0,\ldots,\#V_m-1\}$, we also have the following Hodge Diamond Star *relation,*

$$c_{k,(N_b-1)\,N_b^m-(N_b-1)\,j-q,m} = \overline{c_{k,(N_b-1)\,j+q,m}}\,. \qquad (\mathcal{R}\,26)$$

Again, as in Property 2.19 above, on page 25, it is also reminiscent of Poincaré duality (see our previous work [DL22b], and the corresponding result inTheorem 4.27, on page 63).

Proof. This directly follows from Property 2.19, on page 25, since the coefficients involved in the expression for the local polyhedral effective zeta functions, in relation ($\mathcal{R}$21), on page 26, or, equivalently, in relation ($\mathcal{R}$22), on page 26, are themselves coefficients of the Complex Dimensions series expansion of the Weierstrass function, and automatically satisfy the Hodge Diamond star relation of Property 2.19, on page 25.

As for the coefficients involved in the expression for the global polyhedral effective zeta functions, in relation ($\mathcal{R}$24), on page 27, they are given by countably infinite linear combinations of the coefficients involved in the expression for the local polyhedral effective zeta functions. Hence, they also automatically satisfy the Hodge Diamond star relation of Property 2.19, on page 25.

□

3 A Toda-Like System

In this section, our main aim is to determine whether there could exist a differential operator $\mathcal{L}$ – in a matrix form – such that, for any strictly positive integer m and any j in $\{0,\ldots,\#V_m-1\}$, each term $\mathcal{W}_{comp}\left(j\,\varepsilon_m^m\right)$ could be connected to $\mathcal{L}$. More precisely, we would like to obtain $\mathcal{L}$ in the form of *a Lie bracket* (or *a commutator*), i.e., $\mathcal{L} = [\mathcal{M},\mathcal{J}] = \mathcal{M}\,\mathcal{J} - \mathcal{J}\,\mathcal{M}$, where $(\mathcal{J},\mathcal{M})$ is *a Lax pair.* In such a context, it is known that, with suitable assumptions on the initial conditions, the time-dependent differential system

$$\frac{d\,(\mathcal{J}\,X)}{dt} = [\mathcal{M},\mathcal{J}]\;X(t) \qquad (\mathcal{R}\,27)$$

can be solved due to *the isospectral properties* (i.e., here, the eigenvalues of $\mathcal{J}$ are independent of time, or of any equivalent parameter that would play the role of time). More precisely, In the case of the time variable, given an eigenvalue λ of the Jacobi matrix $\mathcal{J}$, along with an eigenvector X associated with λ, we have that

$$\frac{d\,(\mathcal{J}\,X)}{dt} = \frac{d\,(\lambda\,X)}{dt} = \underbrace{\frac{d\,(\lambda)}{dt}}_{0}\,X + \lambda\,\frac{d\,(X)}{dt} = \lambda\,\frac{d\,X}{dt}\,,$$

while, at the same time,

$$[\mathcal{M},\mathcal{J}]\;X = \lambda\,\mathcal{M}\,X - \mathcal{J}\,\mathcal{M}\,X = (\lambda\,Id - \mathcal{J})\;\mathcal{M}\,X\,,$$

which ensures that

$$\lambda \frac{dX}{dt} = \lambda \dot{X} = (\lambda Id - \mathcal{J})\, \mathcal{M}\, X \,.$$

Usually, the knowledge of the eigenvectors of the matrix $\mathcal{J}$ – or, even better, of an Hilbert basis of eigenvectors of $\mathcal{J}$ – enables us to solve an inverse spectral problem; i.e., in our context, to obtain the solutions of the differential system ($\mathcal{R}$27) as functions of the (eigen)vectors of the Hilbert basis. However, the determination of this Hilbert basis is not a compulsory step, especially, in our present context, where the sole fact that, for any strictly positive integer m and any j in $\{0, \ldots, \#V_m - 1\}$, each term $\mathcal{W}_{comp}\left(j\, \varepsilon_m^m\right)$ can be connected to a differential operator is already a very important result, which will enable us to make the connection with the Taylor-like expansions of functions belonging to the cohomology groups; see [DL22b].

Notation 9 (**k^{th} Truncated Trace of an Infinite (Square) Matrix**).

Given a countably infinite (square) matrix

$$\mathcal{A}_\infty = \left(a_{ij}\right)_{1\leqslant i,\, 1\leqslant j} \,,$$

along with an integer $k \geqslant 1$, we denote by $\mathrm{tr}_k\left(A_\infty\right)$ *the* k^{th} *truncated trace*

$$\mathrm{tr}_k\left(A_\infty\right) = \sum_{k'=0}^{k} a_{k',k'} \,.$$

Theorem 3.1 (**The Weierstrass Function as the Sum of Traces of a Differential Operator**).

We set, for all $m \in \mathbb{N}$, $0 \leqslant j \leqslant \#V_m - 1$, $0 \leqslant k' \leqslant k \leqslant m$, $0 \leqslant k" \leqslant N_b^m - 1$ *and* $0 \leqslant j' \leqslant N_b - 1$,

$$\beta_{k',j}^2 = c_{k',j,m}\, \varepsilon^{k'(2-D_{\mathcal{W}})+i\,\ell_{k",j,'m}\,\mathbf{p}} \,, \qquad (\mathcal{R}\,28)$$

where $\beta_{k',j} \in \mathbb{C}$ *and we choose the standard determination of the positive square root,* ε *is the intrinsic scale, introduced in Definition 2.8, on page 21, while the coefficient* $c_{k',j,m}$ *is given by relation ($\mathcal{R}$11), on page 24, and* $\ell_{k",j',m} = j = (N_b - 1)\, k" + j'$ *(see Definition 2.9 above, on page 22).*

Let us consider the following countably infinite matrices,

$$\mathcal{J}_{\mathcal{W},j,\infty} = \begin{pmatrix} 1 & \beta_{0,j} & 0 & \ldots & \ldots & \ldots & 0 \\ \beta_{0,j} & 1 & \beta_{1,j} & \ldots & \ldots & \vdots & \\ 0 & \beta_{1,j} & 1 & \beta_{2,j} & 0 & \ldots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ldots & \ddots & \ddots & \ddots & 0 & \ddots \end{pmatrix} , \quad \mathcal{M}_{\mathcal{W},j,\infty} = \frac{1}{2} \begin{pmatrix} 1 & \beta_{0,j} & 0 & \ldots & \ldots & \ldots & 0 \\ -\beta_{0,j} & 1 & \beta_{1,j} & \ldots & \ldots & \vdots & \\ 0 & -\beta_{1,j} & 1 & \beta_{2,j} & 0 & \ldots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ldots & \ddots & \ddots & \ddots & 0 & \ddots \end{pmatrix} ,$$

and, as a result, the Lie bracket *(or* commutator*) of these two matrices, which is the diagonal matrix*

$$\left[\mathcal{M}_{\mathcal{W},j,\infty},\mathcal{J}_{\mathcal{W},j,\infty}\right]=\begin{pmatrix} \beta_{1,j}^2-\beta_{0,j}^2 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \beta_{2,j}^2-\beta_{1,j}^2 & 0 & \dots & \dots & \vdots & \\ 0 & 0 & \beta_{3,j}^2-\beta_{2,j}^2 & 0 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 0 & \beta_{k,j}^2-\beta_{k-1}^2 & 0 \\ 0 & \dots & \dots & 0 & 0 & 0 & \ddots \end{pmatrix}.$$

Note that the symmetric, tridiagonal matrix $\mathcal{J}_{\mathcal{W},j,\infty}$ is a Jacobi matrix.

We then have that, for each integer j in $\{0,\dots,\#V_m-1\}$,

$$\mathcal{W}_{comp}\left(j\,\varepsilon_m^m\right)=\sum_{k=0}^{m}\sum_{k'=0}^{k} tr_{k'}\left(\left[\mathcal{M}_{\mathcal{W},j,\infty},\mathcal{J}_{\mathcal{W},j,\infty}\right]\right). \qquad (\mathcal{R}\,29)$$

where, for all $0\leqslant k'\leqslant k$, $tr_{k'}\left(\left[\mathcal{M}_{\mathcal{W},j,\infty},\mathcal{J}_{\mathcal{W},j,\infty}\right]\right)$ is the k'^{th} truncated trace of the commutator $\left[\mathcal{M}_{\mathcal{W},j,\infty},\mathcal{J}_{\mathcal{W},j,\infty}\right]$, as introduced in Notation 9, on page 30.

Note that $\left(\mathcal{J}_{\mathcal{W},j,\infty},\mathcal{M}_{\mathcal{W},j,\infty}\right)$ forms a Lax pair*, in the sense of Lax's approach to integrable Hamiltonian systems; see [Lax68].*

Proof. Thanks to Theorem 2.18, on page 23, we have that, for any strictly positive integer m and any j in $\{0,\dots,\#V_m-1\}$,

$$\mathcal{W}\left(j\,\varepsilon_m^m\right)=\frac{1}{2}\sum_{k=0}^{m}\varepsilon^{k\,(2-D_{\mathcal{W}})}\left(c_{k,j,m}\,\varepsilon^{i\,\ell_{k,j,m}\,\mathbf{p}}+\overline{c_{k,j,m}}\,\varepsilon^{-i\,\ell_{k,j,m}\,\mathbf{p}}\right), \qquad (\mathcal{R}\,30)$$

or, equivalently,

$$\mathcal{W}_{comp}\left(j\,\varepsilon_m^m\right)=\sum_{k=0}^{m}c_{k,j,m}\,\varepsilon^{k\,(2-D_{\mathcal{W}})+i\,\ell_{k,j,m}\,\mathbf{p}}, \qquad (\mathcal{R}\,31)$$

where ε is the intrinsic scale, introduced in Definition 2.8, on page 21, while the coefficient $c_{k',j,m}$ is given by relation ($\mathcal{R}$11), on page 24, and where $\ell_{k",j',m}=j=(N_b-1)\,k"+j'$ (see Definition 2.9 above, on page 22), that we can also write in the following form:

$$\mathcal{W}_{comp}\left(j\,\varepsilon_m^m\right) = \sum_{k=0}^{m}\sum_{k'=0}^{k}\left(c_{k',j,m}\,\varepsilon^{k'\,(2-D_{\mathcal{W}})+i\,\ell_{k',j,m}\,\mathbf{p}}-c_{k'-1,j,m}\,\varepsilon^{(k'-1)\,(2-D_{\mathcal{W}})+i\,\ell_{k',j,m}\,\mathbf{p}}\right), \qquad (\mathcal{R}\,32)$$

with the additional convention that $c_{-1,j,m}=0$.

In our present context, we have that, for any strictly positive integer m and any j in $\{0,\dots,\#V_m-1\}$,

$$\mathcal{W}_{comp}\left(j\,\varepsilon_m^m\right) = \sum_{k=0}^{m}\sum_{k'=0}^{k}\left(c_{k',j,m}\,\varepsilon^{k'\,(2-D_{\mathcal{W}})+i\,\ell_{k',j,m}\,\mathbf{p}}-c_{k'-1,j,m}\,\varepsilon^{(k'-1)\,(2-D_{\mathcal{W}})+i\,\ell_{k',j,m}\,\mathbf{p}}\right). \qquad (\mathcal{R}\,33)$$

For each fixed $k\in\{0,\dots,m\}$, we introduce the following finite, $k\times k$ matrices where, for each $j\in\{0,\dots,\#V_m-1\}$,

$$
\mathcal{J}_{\mathcal{W},j} = \begin{pmatrix} 1 & \beta_{0,j} & 0 & \dots & \dots & \dots & 0 \\ \beta_{0,j} & 1 & \beta_{1,j} & 0 & \dots & \dots & \vdots \\ 0 & \beta_{1,j} & 1 & \ddots & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \beta_{k-1,j} & 1 & \beta_{k,j} \end{pmatrix}
$$

and

$$
\mathcal{M}_{\mathcal{W},j} = \frac{1}{2} \begin{pmatrix} 1 & \beta_{0,j} & 0 & \dots & \dots & \dots & 0 \\ -\beta_{0,j} & 1 & -\beta_{1,j} & 0 & \dots & \dots & \vdots \\ 0 & -\beta_{1,j} & 0 & \beta_{2,j} & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \beta_k \\ 0 & \dots & \dots & 0 & 0 & -\beta_k & 1 \end{pmatrix},
$$

where, for every $k' \in \{0, \dots, k\}$, $\beta_{k',j}$ is given by relation ($\mathcal{R}$28), on page 30. Note that the symmetric, tridiagonal matrix $\mathcal{J}_{\mathcal{W},j}$ is a Jacobi matrix.

It follows that the commutator of these two matrices is the countably infinite diagonal matrix

$$
\left[\mathcal{M}_{\mathcal{W},j}, \mathcal{J}_{\mathcal{W},j}\right] = \mathcal{M}_{\mathcal{W},j}\, \mathcal{J}_{\mathcal{W},j} - \mathcal{J}_{\mathcal{W},j}\, \mathcal{M}_{\mathcal{W},j} = \begin{pmatrix} \beta_{1,j}^2 - \beta_{0,j}^2 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \beta_{2,j}^2 - \beta_{1,j}^2 & 0 & \dots & \dots & \vdots & \\ 0 & 0 & \beta_{3,j}^2 - \beta_{2,j}^2 & 0 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 0 & 0 & \beta_{k,j}^2 \end{pmatrix}.
$$

Note that we are interested here in the associated countably infinite matrices

$$
\mathcal{J}_{\mathcal{W},j,\infty} = \begin{pmatrix} 1 & \beta_{0,j} & 0 & \dots & \dots & \dots & 0 \\ \beta_{0,j} & 1 & \beta_{1,j} & \dots & \dots & \vdots & \\ 0 & \beta_{1,j} & 1 & \beta_{2,j} & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & 0 & \ddots \end{pmatrix}, \quad \mathcal{M}_{\mathcal{W},j,\infty} = \frac{1}{2} \begin{pmatrix} 1 & \beta_{0,j} & 0 & \dots & \dots & \dots & 0 \\ -\beta_{0,j} & 1 & \beta_{1,j} & \dots & \dots & \vdots & \\ 0 & -\beta_{1,j} & 1 & \beta_{2,j} & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & 0 & \ddots \end{pmatrix}
$$

and, as a result, in their commutator, given by the following countably infinite diagonal matrix

$$
\left[\mathcal{M}_{\mathcal{W},j,\infty}, \mathcal{J}_{\mathcal{W},j,\infty}\right] = \begin{pmatrix} \beta_{1,j}^2 - \beta_{0,j}^2 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \beta_{2,j}^2 - \beta_{1,j}^2 & 0 & \dots & \dots & \vdots & \\ 0 & 0 & \beta_{3,j}^2 - \beta_{2,j}^2 & 0 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 0 & \beta_{k,j}^2 - \beta_{k-1,j}^2 & 0 \\ 0 & \dots & \dots & 0 & 0 & 0 & \ddots \end{pmatrix}.
$$

Observe that the trace of the $k \times k$ matrix $\left[\mathcal{M}_{\mathcal{W},j}, \mathcal{J}_{\mathcal{W},j}\right]$ is equal to $\beta_{k,j}^2 - \beta_{0,j}^2$; see relation ($\mathcal{R}$28), on page 30. It is also equal to the k^{th} truncated trace of $\left[\mathcal{M}_{\mathcal{W},j,\infty}, \mathcal{J}_{\mathcal{W},j,\infty}\right]$.

In light of relation ($\mathcal{R}29$), on page 31, combined with relation ($\mathcal{R}28$), on page 30, we therefore precisely obtain $\mathcal{W}_{comp}\left(j\,\varepsilon_m^m\right)$ as the sum, from $k=0$ to m, of the k^{th} truncated trace of the commutator $\left[\mathcal{M}_{\mathcal{W},j,\infty},\mathcal{J}_{\mathcal{W},j,\infty}\right]$, as desired and as stated by relation ($\mathcal{R}29$), on page 31 .

□

This amounts to envisioning the set of points $M_{j,m}$ of the m^{th} prefractal set V_m, with $j\in\{0,\ldots,\#V_m-1\}$, as a Toda-like system, where each point $M_{j,m}$ plays the part of a particle. We assume that the order of the particles does not depend on the variable that plays the role of time. We will identify a natural order further on in this paper. Note that in the discussion here, just above, we have used the notation $M_{j,m}$ for the points of V_m introduced in Property 2.6 above, on page 12.

For the moment, we have an equivalent – but even much more meaningful – result, which concerns the polyhedral, effective, fractal zeta functions introduced in [DL23b] (announced in [DL23a]), since it will enable us to prove that the Taylor-like expansions obtained in [DL22b] can also be obtained as the sums of traces of differentiable operators, and thus involve actual fractional derivatives.

Theorem 3.2 (**The Fractal Zeta Functions as the Sum of Traces of Differential Operators**).

We set, for all $m\in N$, $0\leqslant j\leqslant \#V_m-1$, $1\leqslant k'\leqslant k\leqslant m$, $0\leqslant k"\leqslant N_b^m-1$, $0\leqslant j'\leqslant N_b-1$ *and all* $s\in\mathbb{C}$,

$$\begin{aligned}\gamma^2_{k',j,q}(s) &= \frac{1}{2}\,\frac{\alpha_q(N_b)\,c_{k',(N_b-1)\,j+q,m}\,\varepsilon^{s-D_{\mathcal{W}}-(k'-1)\,(2-D_{\mathcal{W}})+i\,\ell_{k',(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-D_{\mathcal{W}}-(k'-1)\,(2-D_{\mathcal{W}})+i\,\ell_{k",(N_b-1)\,j+q,m}\,\mathbf{p}} \\ &+\frac{1}{2}\,\frac{\alpha_q(N_b)\,\overline{c_{k',(N_b-1)\,j+q,m}}\,\varepsilon^{s-D_{\mathcal{W}}-(k'-1)\,(2-D_{\mathcal{W}})-o\,\ell_{k',(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-D_{\mathcal{W}}-(k'-1)\,(2-D_{\mathcal{W}})-i\,\ell_{k",(N_b-1)\,j+q,m}\,\mathbf{p}}\,.\end{aligned} \qquad (\mathcal{R}\,34)$$

where ε *is the intrinsic scale, introduced in Definition 2.8, on page 21, while the coefficient* $c_{k',j,m}$ *is given by relation (*$\mathcal{R}11$*), on page 24, and where* $\ell_{k",j',m}=j=(N_b-1)\,k"+j'$ *(see Definition 2.9 above, on page 22). Also, in relation (*$\mathcal{R}34$*),* $\gamma_{k',j,q}(s)\in\mathbb{C}$ *and as is done above (see relation (*$\mathcal{R}28$*) and the discussion preceding it), we choose the standard determination of the positive square root.*

Let us consider the following countably infinite matrices, given, for all $s\in\mathbb{C}$, *by*

$$\mathcal{J}_{\zeta^e_m,j,\infty}(s)=\begin{pmatrix} 1 & \gamma_{0,j,q}(s) & 0 & \ldots & \ldots & \ldots & 0 \\ \gamma_{0,j}(s) & 1 & \gamma_{1,j,q}(s) & \ldots & \ldots & \vdots & \\ 0 & \gamma_{1,j,q}(s) & 1 & \gamma_{2,j}(s) & 0 & \ldots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ldots & \ddots & \ddots & \ddots & 0 & \ddots \end{pmatrix},$$

$$\mathcal{M}_{\tilde{\zeta}^e_m,j,\infty}(s) = \frac{1}{2}\begin{pmatrix} 1 & \gamma_{0,j,q}(s) & 0 & \dots & \dots & \dots & 0 \\ -\gamma_{0,j}(s) & 1 & \gamma_{1,j,q}(s) & \dots & \dots & \vdots & \\ 0 & -\gamma_{1,j}(s) & 1 & \gamma_{2,j,q}(s) & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & 0 & \ddots \end{pmatrix}$$

and, as a result, their Lie bracket *(or* commutator*), given, for each fixed* $s \in \mathbb{C}$*, by the countably infinite diagonal matrix*

$$\left[\mathcal{M}_{\tilde{\zeta}^e_m,j,\infty}(s), \mathcal{J}_{\tilde{\zeta}^e_m,j,\infty}(s)\right] =$$

$$= \begin{pmatrix} \gamma^2_{1,j,q}(s) - \gamma_{0,j,q}(s)^2 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \gamma^2_{2,j,q}(s) - \gamma^2_{1,j,q}(s) & 0 & \dots & \dots & \vdots & \\ 0 & 0 & \gamma^2_{3,j,q}(s) - \gamma^2_{2,j,q}(s) & 0 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 0 & \gamma^2_{k,j,q}(s) - \gamma^2_{k-1,j,q}(s) & 0 \\ 0 & \dots & \dots & 0 & 0 & 0 & \ddots \end{pmatrix}.$$

Note that, for each fixed $s \in \mathbb{C}$*, the symmetric, tridiagonal matrix* $\mathcal{J}_{\tilde{\zeta}^e_m,j,\infty}(s)$ *is a countably infinite Jacobi matrix; furthermore,* $\left(\mathcal{J}_{\tilde{\zeta}^e_m,j,\infty}(s), \mathcal{M}_{\tilde{\zeta}^e_m,j,\infty}(s)\right)$ *forms a Lax pair.*

Still for all $m \in \mathbb{N}^\star$ *sufficiently large, we have that, for all* $s \in \mathbb{C}$*,*

$$\tilde{\zeta}^e_m(s) = \varepsilon^m \sum_{j=0}^{N_b^m-1} \sum_{q=0}^{N_b} \sum_{k=0}^{m} \sum_{k'=0}^{k} tr_{k'}\left(\left[\mathcal{M}_{\tilde{\zeta}^e_m,j,\infty}(s), \mathcal{J}_{\tilde{\zeta}^e_m,j,\infty}(s)\right]\right), \qquad (\mathcal{R}\,35)$$

where, for all $0 \leqslant k' \leqslant k$*,* $tr_{k'}\left(\left[\mathcal{M}_{\tilde{\zeta}^e_m,j,\infty}(s), \mathcal{J}_{\tilde{\zeta}^e_m,j,\infty}(s)\right]\right)$ *is the* k'^{th} *truncated trace of the commutator*

$$\left[\mathcal{M}_{\tilde{\zeta}^e_m,j,\infty}(s), \mathcal{J}_{\tilde{\zeta}^e_m,j,\infty}(s)\right].$$

The proof of Theorem 3.2 is given on page 34, just after Remark 3.1.

Remark 3.1. Note that, in classical approaches, the occurrence of the zeta function can be understood very intuitively, since it represents the trace of the differential operator at a complex order s – coming from the Selberg trace formula (see the very clear, detailed and enthusiastic point of view which is presented in the work of Dennis A. Hejhal [Hej76]). Our result, obtained by means of a different approach, can therefore be thought as being, in some sense, related to the Selberg trace formula.

Proof. We simply apply the same reasoning as in the proof of Theorem 3.1, stated on page 30. Indeed, we can check that

$$\left[\mathcal{M}_{\tilde{\zeta}^e_m,j,\infty}(s), \mathcal{J}_{\tilde{\zeta}^e_m,j,\infty}(s)\right] =$$

$$= \begin{pmatrix} \gamma_{1,j,q}^2(s) - \gamma_{0,j,q}^2(s) & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \gamma_{2,j}^2(s) - \gamma_{1,j,q}^2(s) & 0 & \dots & \dots & \vdots & \\ 0 & 0 & \gamma_{3,j}^2(s) - \gamma_{2,j}^2(s) & 0 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 0 & \gamma_{k,j,q}^2(s) - \gamma_{k-1,j,q}^2(s) & 0 \\ 0 & \dots & \dots & 0 & 0 & 0 & \ddots \end{pmatrix}.$$

□

Note that our context is somewhat different from classical contexts involving Lie brackets with Jacobi matrices. Indeed, such a decomposition (the Lie bracket) is used for integrating a differential system. We are in a totally different kind of situation, here: if, of course, we were looking for the equivalent of *a canonical flow*, the suitable adaptation to fractals of *a geodesic flow*, as is suggested in [Lap08], on top of page 185), we would already have at our disposal explicit expressions.

At the same time, recall that the determination of the eigenvectors of a Jacobi matrix is a well-known, and a frequently studied problem in the literature. In short, in the case of the countably infinite Jacobi matrix

$$\mathcal{J}_\infty = \begin{pmatrix} 1 & \gamma_{0,\infty}(s) & 0 & \dots & \dots & \dots & 0 \\ \gamma_{0,\infty}(s) & 1 & \gamma_{1,\infty}(s) & \dots & \dots & \vdots & \\ 0 & \gamma_{1,\infty}(s) & 1 & \gamma_{2,\infty}(s) & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & 0 & \ddots \end{pmatrix},$$

it amounts to determining a sequence of complex-valued polynomials $(P_k(z))_{k\in\mathbb{N}}$, of a complex variable z, where $P_0(z) = 1$ and such that

$$\mathcal{J}_{j,\infty}\,\mathbf{P} = z\,\mathbf{P}\,,$$

where

$$\mathbf{P} = \begin{pmatrix} P_0(z) \\ P_1(z) \\ P_2(z) \\ \vdots \end{pmatrix};$$

i.e.,

$$P_0(z) + \gamma_{0,\infty}\,P_1(z) = z\,P_0(z)$$

$$\gamma_{0,j}\,P_1(z) + P_1(z) + \gamma_{1,\infty}\,P_2(z) = z\,P_1(z)$$

$$\dots\dots\dots$$

$$\beta_{k,\infty}\,P_k(z) + P_{k+1}(z) + \gamma_{k+1,\infty}\,P_{k+2}(z) = z\,P_{k+1}(z)\,.$$

....

Those polynomials thus satisfy a three-term recurrence relation. By induction, we clearly obtain that each polynomial P_k is exactly of degree k.

Also, the family $(P_k(z))_{k\in\mathbb{N}}$ is orthonormal with respect to a specific measure. Note also that the family $(P_k(z))_{k\in\mathbb{N}}$ can be deduced from the canonical polynomial basis by means of the Gram–Schmidt process.

However, this method cannot be applied realistically in our present context. To begin with, there are several Jacobi matrices involved, and not only one. This results in awfully complicated expressions for the coefficients associated to the aforementioned three-term recurrence relation.

Moreover, we already have at our disposal Taylor-like expansions of the Weierstrass function, with underlying exponents the Complex Codimensions. This strongly suggests, instead of classical polynomials (i.e., with integer exponents, in $\mathbb{N}$), to consider *fractional polynomials*, with underlying exponents the Complex Codimensions; namely, of the following form,

$$\sum_{k=0}^{m} c_k\, z^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell\,\mathbf{p}} = \sum_{k=0}^{m} c_k\, \left(z^{(2-D_{\mathcal{W}})+i\,\ell\,\mathbf{p}}\right)^k ,$$

where $m \in \mathbb{N}$ and for $0 \leqslant k \leqslant m$, c_k denotes a complex coefficient, ℓ is an integer, while

$$\mathbf{p} = \frac{2\,\pi}{\ln N_b}$$

is the oscillatory period of the Weierstrass Curve introduced in [DL23b].

In fact, this simply amounts to considering complex polynomials of the form

$$\sum_{k=0}^{m} c_k\, Z^k$$

and to evaluate them at

$$Z = (2 - D_{\mathcal{W}}) + i\,\ell\,\mathbf{p}\,.$$

As for their orthogonality with respect to a specific measure, we will obtain it below in Theorem 4.25, on page 60, by means of *a polarization operator* (connected with *a transfer operator*) involving the polyhedral measure.

4 The Frobenius Operator – Connections with Fractal Cohomology

In this section, we unveil the natural transfer operator and the Frobenius operators associated with the Weierstrass Curve $\Gamma_{\mathcal{W}}$, in direct connection with the Toda-like system of Section 3.

For the sake of clarity, and in order to facilitate the reader's understanding, we provide a first subsection consisting in *a dictionary* of the main concepts and mathematical objects or results generalized to the fractal setting in Subsection 4.2, on page 44.

More precisely, we briefly discuss in Subsection 4.1 some of the basic elements of the classical complex geometry (including Kähler geometry and Hodge theory, establishing a key bridge between algebraic topology, algebraic geometry and differential geometry), and the associated basic results (especially, Poincaré Duality, the Hard Lefschetz Theorem and the Hodge–Riemann Bilinear Relations). In the remainder of this section (i.e., in Subsection 4.2), we develop fractal analogs of many of those notions and results, building, in particular, on our previous work on fractal cohomology [DL22b] and extended theory of Complex Dimensions in [DL22a], [DL23a], [DL23b], [DL24a], along with the notion of polyhedral measure (and associated function spaces) introduced and studied in [DL24b]. The latter measure will play a crucial role in order to define a suitable fractal counterpart of Deligne's polarization in pure Hodge theory that is key to many topics at the junction of algebraic and arithmetic geometry.

At the end of this section, i.e., in in Subsection 4.2.6, on page 84, we give a table providing a correspondence between the main concepts and results discussed in the classical setting (in Subsection 4.1) and their fractal counterparts introduced in Subsections 4.2.1–4.2.5.

4.1 Dictionary: The Classical Case

We inform the reader that very useful references on Hodge theory can be found in the transcription of the lectures given by Maxim Kontsevich in [Kon08], or in the book of Claire Voisin [Voi02]. As for Poincaré Duality, we refer to the book of Jean Gallier and Jocelyn Quaintance [GQ22].

4.1.1 Notation and General Framework

In the sequel, X denotes a smooth manifold, of dimension $n \in \mathbb{N}^\star$. We hereafter use the classical $\wedge$ notation both for the exterior product and exterior derivatives. Without further mention, throughout this subsection, X will be assumed to be compact and connected (and without boundary, i.e., closed).

Given a nonnegative integer p, we denote by $\Omega^p(X)$ the space of smooth p-forms on X.

Definition 4.1 (**Partial Derivatives**).

Given a strictly positive integer p, a smooth p-form f on X, and k in $\{0,\ldots,p\}$, the partial derivative $\partial_k f$ is defined, for any

$$(x_0,\ldots,x_p) = \left(\left(x_{0,i_0}\right)_{1\leqslant i_0\leqslant n},\ldots,\left(x_{p,i_p}\right)_{1\leqslant i_p\leqslant n}\right) \in X^{p+1}$$

by

$$\partial_k\, f\left(x_0,\ldots,x_p\right) \quad = \quad \sum_{i_k=1}^{n} \frac{\partial f}{\partial x_{k,i_k}} \left(\left(x_{0,i_0}\right)_{1\leqslant i_0\leqslant n},\ldots,\left(x_{p,i_p}\right)_{1\leqslant i_p\leqslant n}\right) dx^{k,i_k} \,.$$

Definition 4.2 (**Exterior Derivative**).

Given a smooth function f defined on X, the exterior derivative of f is the differential of f, i.e., the unique 1-form such that, for any smooth vector field $u = (du_1,\ldots,du_n)$, we have (in local coordinates),

$$d\,f(u) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\, du_k \,.$$

By induction, given $p \in \mathbb{N}$, the exterior derivative of a p-form (i.e., a differential form of degree p) is a $(p+1)$-form (i.e., a differential form of degree $p+1$).

Definition 4.3 (**De Rham Differential**).

Given a p-form $\omega \in \Omega^p(X)$, such that, for any $x = (x_1, \ldots, x_n) \in X$,

$$\omega(x) = \sum_{1 \leqslant i_1 < \ldots < i_p \leqslant n} f_{i_1,\ldots,i_p}(x)\, dx^{i_1} \wedge \ldots dx^{i_p} \,,$$

and where, for any $(i_1, \ldots, i_p) \in \{1, \ldots, n\}^p$, the coefficients $f_{i_1,\ldots,i_p}$ denote smooth functions on X, the de Rham differential $d\omega$ is defined (still in local coordinates) by

$$d\omega(x) = \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \ldots < i_p \leqslant n} \frac{\partial f_{i_1,\ldots,i_p}}{\partial x_k}(x)\, dx^k \wedge dx^{i_1} \wedge \ldots dx^{i_p} \,.$$

Definition 4.4 (**De Rham Complex on** X).

The de Rham Complex on X is the cochain complex of differential forms

$$0 \xrightarrow{d} \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \ldots$$

that we will denote by $\Omega^{\bullet,d}$.

4.1.2 Riemannian Metric

Given a smooth manifold X (i.e., a C^∞ manifold) of dimension $n \in \mathbb{N}^\star$, a *Riemannian metric* on X is the smooth (assignment) of a positive definite inner product (or bilinear form) defined on the tangent space $T_M X$ at each point $M \in X$.

4.1.3 Almost Complex Structure

Definition 4.5 (**Almost Complex Structure**).

Given a smooth manifold $X = X_{\mathbb{C}}$ of dimension $n \in \mathbb{N}^\star$, an *almost complex structure* on X is a linear map J acting on the tangent space TX such that, for all tangent vectors $u \in TX$,

$$J^2 u = -u \,.$$

Motre precisely, for $0 \leqslant k \leqslant n$, by using the (complex) coordinates

$$z_k = x_k + i\, y_k \,,$$

we have that

$$J\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k} \quad , \quad J\left(\frac{\partial}{\partial x_y}\right) = -\frac{\partial}{\partial x_k} \,.$$

Note that, necessarily, n must be an even integer, for an almost complex structure to exist on X.

4.1.4 Kähler Manifold – Kähler Form

Definition 4.6 (**Kähler Manifold**).

Given a complex manifold X (also denoted $X_{\mathbb{C}}$), a *Kähler structure* on X is a triple of tensors (g, ω, J), where g is a Riemannian metrix on X, ω an antisymmetric and nondegenerate 2-form on the tangent bundle TX and J an almost complex structure on X. Note that ω, called a *Kähler form*, is a closed differential form: $d\omega = 0$. Therefore, ω is a symplectic form on X. Furthermore, g, ω and T satisfy the following compatibility condition:

$$g\,(K_1, K_2) = \omega\,(K_1, J\,K_2)\ ,$$

for all tangent vectors $K_1\,, K_2$ to X at any $x\,\in\,X$.

The manifold X, equipped with such a Kähler structure, is then called a *Kähler manifold.* It is naturally endowed with the Hermitian metric

$$h = g - i\,\omega\,.$$

Definition 4.7 (**(p, q)-Form**).

Given a pair of nonnegative integers $(p, q) \in \{1, \ldots, n\}^2$, where n is the complex dimension of $X = X_{\mathbb{C}}$, a differential form on $X_{\mathbb{C}}$ is said to be of (p, q) *type* (or (p, q)-form, in short) if it is locally of the form (with $\bar{z}$ denoting the complex conjugate of $z\,\in\,\mathbb{C}$),

$$\sum_{(i_1,\ldots,i_p,j_1,\ldots,j_q)} dz_{i_1} \wedge \ldots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots d\bar{z}_{j_p}\ ,$$

Note that (p, q)-forms are of (total) degree $p + q$.

Definition 4.8 (**∂ and $\bar{\partial}$ Operators**).

The ∂ and $\bar{\partial}$ operators, which act on smooth functions f defined on the complex manifold $X_{\mathbb{C}}$ (of complex dimension n and hence, of real dimension $2\,n$), are defined via

$$\partial f = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}\, dz_i$$

and

$$\bar{\partial} f = \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i}\, d\bar{z}_i\,.$$

They are such that

$$\partial f + \bar{\partial} f = d\,,$$

where d is the exterior derivative (see Definition 4.2 above, on page 37).

Note that ∂ is a $(1, 0)$-form, while $\bar{\partial}$ is a $(0, 1)$-form.

Remark 4.1. In short, Kähler geometry extends results on real compact manifolds X of dimension $n \in \mathbb{N}^\star$ to compact, connected, complex manifolds $X_\mathbb{C} = X \otimes \mathbb{C} = X + i\,X$. If we denote by $(z_1, \ldots, z_n)$ local, holomorphic coordinates on $X_\mathbb{C}$, a local basis of the space of differential forms associated on $X_\mathbb{C}$ is thus

$$(dz_1, \ldots, dz_n, d\bar{z}_1, \ldots, d\bar{z}_n) \; .$$

4.1.5 Hodge Structure

The beginning of this subsection, which deals with the case of a real manifold, is an excerpt from the corresponding one in [DL23d].

Definition 4.9 (**Hodge Star Operator on a Finite-Dimensional Oriented Euclidean Space**).

Let E be a finite-dimensional oriented Euclidean space, endowed with a nondegenerate symmetric bilinear form $\wedge$. We set

$$\dim E = n \in \mathbb{N}^\star .$$

Given a nonnegative integer $p \leqslant n$, $\wedge^p E$ and $\wedge^{n-p} E$ respectively denote the subspaces of p and $n-p$ vectors. One trivially has

$$\dim \wedge^p E = \dim \wedge^{n-p} E = \binom{n}{p} = \binom{n-p}{p} ;$$

and similarly for $\wedge^{\star,p} E$, with p replaced by $n-p$. (The choice of a basis of $\wedge^p E$ amounts to choosing p vectors among the n elements of any basis of E.)

The Hodge star operator $\star$ is simply the natural isomorphism between $\wedge^p E$ and $\wedge^{n-p} E$. For any orthonormal basis $\{e_1, \ldots, e_n\}$, we have that

$$\star \left(e_1 \wedge \ldots \wedge e_p \right) = e_{p+1} \wedge \ldots \wedge e_n \, .$$

Property 4.1. *Given a nonnegative integer $p \leqslant n$, and a p-vector $\eta \in \wedge^p E$, we have that*

$$\star \star \eta = (-1)^{p\,(n-p)} \, \eta \, .$$

Remark 4.2. We thus have that

$$\begin{array}{ccc} \Lambda^p & \xrightarrow{\quad \star \quad} & \Lambda^{n-p} \\ & & \Big\downarrow d \\ \Lambda^{p-1} & \xleftarrow{\quad \star \quad} & \Lambda^{n-p+1} \end{array}$$

Now, in the case of our smooth manifold X, the above space E is simply the tangent space $T_M X$ at some point $M \in X$, as is given in Definition 4.10 just below, on page 41.

Definition 4.10 ($d^\star$ **Operator on the de Rham Complex).**

Given a strictly positive integer $p \leqslant n$, we define the codifferential $d^\star$ by

$$d^\star \,:\, \Omega^p \longrightarrow \Omega^{p-1}$$

via

$$d^\star = (-1)^{n\,(p-1)+1} \star d \star \cdot$$

$$\begin{array}{ccc}
\Omega^p & \xrightarrow{\quad\star\quad} & \Omega^{n-p} \\
{\scriptstyle d^\star}\downarrow & & \downarrow{\scriptstyle d} \\
\Omega^{p-1} & \xleftarrow{\quad\star\quad} & \Omega^{n-p+1}
\end{array}$$

Definition 4.11 (Hodge Laplacian and Space of Harmonic Forms).

The Hodge Laplacian on $\Omega^\bullet(X)$ is given by

$$\Box = \left(d + d^\star\right)^2 = d\,d^\star + d^\star\,d\,.$$

In the sequel, we will denote by $\mathcal{H}$ the space of harmonic forms (i.e., the kernel of $\Box$, with $\Box \,:\, \Omega = \bigoplus_{p=0}^{n} \Omega^p \to \Omega$), and for each nonnegative integer $p \leqslant n$, by $\mathcal{H}^p = \mathcal{H}_{|\Omega^p}$ the space of p-harmonic forms (i.e., the kernel of $\Box^p = \Box_{|\Omega^p} \,:\, \Omega^p \to \Omega^p$).

Theorem 4.2 (Hodge Decomposition Associated with a Compact Analytic Riemannian Manifold).

In the case where X is a compact analytic Riemannian manifold, then, for any strictly positive integer p, we have that

$$\Omega^{p-1} \xrightarrow{\;d\;} \Omega^p \xrightarrow{\;d\;} \Omega^{p+1}$$

and

$$\Omega^{p+1} \xrightarrow{\;d^\star\;} \Omega^p \xrightarrow{\;d^\star\;} \Omega^{p-1}$$

To facilitate understanding, the following diagram might be helpful:

$$\begin{array}{ccc}
\Omega^{p-1} & \xrightarrow{\quad d\quad} & \Omega^p \\
{\scriptstyle d^\star}\uparrow & \swarrow{\scriptstyle d\,d^\star} & \downarrow{\scriptstyle d} \\
\Omega^p & \xleftarrow{\quad d^\star\quad} & \Omega^{p+1}
\end{array}$$

Also, we have the following orthogonal, direct sum, decompositions, with respect to the inner product associated with the L^2 metric on differential forms (where $\mathcal{H}$ is the space of harmonic forms, see Definition 4.11, on page 41),

$$\begin{cases} \ker d_{|\Omega^p} & = & Im\, d_{|\Omega^{p-1}} \oplus \mathcal{H}_{|\Omega^p} \\ \ker d^\star_{|\Omega^p} & = & Im\, d^\star_{|\Omega^{p+1}} \oplus \mathcal{H}_{|\Omega^p} \end{cases},$$

and

$$\begin{cases} \Omega^p(X) & = & Im\, d_{|\Omega^{p-1}} \oplus \mathcal{H}_{|\Omega^p} \oplus \left(\ker d_{|\Omega^p}\right)^\perp \\ \Omega^p(X) & = & Im\, d^\star_{|\Omega^{p+1}} \oplus \mathcal{H}_{|\Omega^p} \oplus \left(\ker d^\star_{|\Omega^p}\right)^\perp \end{cases},$$

which naturally yields

$$\Box = \begin{pmatrix} d\, d^\star & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d^\star\, d \end{pmatrix},$$

and similarly for each $\Box^p = \Box_{|\Omega^p}$, *for* $0 \leqslant p \leqslant n$.

Theorem 4.3 (**Hodge Decomposition Associated with a Compact Kähler Manifold, as described in [Kon08]**).

Given a compact Kähler manifold $X_{\mathbb{C}}$, *of complex dimension* $n \in \mathbb{N}^\star$, *along with an integer* $k \in \{0, \dots, 2n\}$, *the de Rham cohomology associated with* $X_{\mathbb{C}}$, *denoted by* $H^k(X_{\mathbb{C}})$, *is such that*

$$H^k(X_{\mathbb{C}}) = \bigoplus_{p+q=k}^{\infty} H^{p,q}(X_{\mathbb{C}}),$$

where, for all pairs of nonnegative integers (p, q) *such that* $p + q = k$, $H^{p,q}(X_{\mathbb{C}})$ *is the subspace of* $H^k(X_{\mathbb{C}})$ *of cohomology classes of* (p, q) *type closed forms. Note that each (Dolbeault cohomology) space* $H^{p,q}(X_{\mathbb{C}})$ *satisfies the Hodge symmetry*

$$H^{p,q}(X_{\mathbb{C}}) = \overline{H^{q,p}(X_{\mathbb{C}})},$$

where, as before, the overbar indicates complex conjugation.

Here and elsewhere, we use the convention that $H^q(X_{\mathbb{C}}) = \{0\}$ *if* $q < 0$ *or if if* $q < n$.

4.1.6 Poincaré Duality – Lefschetz Operator, Hard Lefschetz Theorem

Theorem 4.4 (**Poincaré Duality**).

Given a compact Kähler manifold $X_{\mathbb{C}}$, *of dimension* n, *along with any integer* $k \in \{0, \dots, n\}$, *the* Poincaré Duality *consists in the isomorphism*

$$H^k(X_{\mathbb{C}}) \simeq H^{n-k}(X_{\mathbb{C}}).$$

Theorem 4.5 (**Lefschetz Operator [Voi07], Chapter 6, on page** 139).

Given a compact Kähler manifold $X_{\mathbb{C}}$, of real dimension $2n$, with associated Kähler (or fundamental form) ω (see Definition 4.6, on page 39), along with an integer $k \in \{0, \ldots, 2n-2\}$, the exterior product with ω induces a differential operator from $H^k(X_{\mathbb{C}})$ to $H^{k+2}(X_{\mathbb{C}})$, which is a bigraded operator, of bigrading $(1,1)$, called the Lefschetz operator, *denoted by $L = \mathcal{L}ef$.*

More specifically, $\mathcal{L}ef([\varphi]) = [\omega \wedge \varphi]$, for any φ in $H^{k-2}(X_{\mathbb{C}})$, and $[\eta]$ denotes the cohomology class of $\eta \in H^q(X_{\mathbb{C}})$, for any $q \in \{0, \ldots, n\}$.

Its adjoint, with respect to the Hermitian metric associated with X, is denoted by $\mathcal{L}ef^{\#}$ and such that, for $2 \leqslant k \leqslant 2n$,

$$\mathcal{L}ef^{\#} : \quad H^{k-2}(X_{\mathbb{C}}) \to H^k(X_{\mathbb{C}}) .$$

Theorem 4.6 (Hard Lefschetz Theorem).

Given a compact Kähler manifold $X_{\mathbb{C}}$, of real dimension $2n$ (i.e., of complex dimension n), along with an integer $k \in \{0, \ldots, n\}$, then, the k^{th} power of L (i.e., the linear operator induced by the exterior wedge product by ω^k), induces an isomorphism

$$L^k : \quad H^{n-k} \to H^{n+k} ,$$

given by $L^k([\varphi]) = \left[\omega^k \wedge \varphi\right]$, the cohomology class of the wedge product $\omega^k \wedge \varphi$, for any $[\varphi] \in H^{n-k}(X)$.

Note that the Hard Lefschetz Theorem implies Poincaré Duality with real coefficients.

Definition 4.12 (Primitive Harmonic Form [Voi02], on page 141).

Given a compact Kähler manifold $X_{\mathbb{C}}$, of real dimension $2n$, along with a strictly positive integer $k \leqslant n$, a harmonic form $\omega \in H^k(X_{\mathbb{C}})$ is said to be *a primitive form* if

$$\omega \in \ker \mathcal{L}ef^{n-k+1} ,$$

where $\mathcal{L}ef$ is the Lefschetz operator introduced in Theorem 4.5 above, on page 42.

Corollary 4.7 ((of Theorem 4.6) Lefschetz Decomposition [Voi02], on page 143).

Given a compact Kähler manifold $X_{\mathbb{C}}$, of real dimension $2n$, along with an integer $k \in \{0, \ldots, n\}$, we define the k^{th} primitive cohomology space *by*

$$P^k(X_{\mathbb{C}}) = \ker\left(\mathcal{L}ef^{n-k+1}_{|H^k(X_{\mathbb{C}})}\right) ,$$

where the notation $\mathcal{L}ef^{n-k+1}_{|H^k(X_{\mathbb{C}})}$ (the k^{th} primitive cohomology space*) means that we consider the restriction of the $(n-k+1)^{th}$ iterate of the Lefschetz operator $\mathcal{L}ef$ to $H^k(X_{\mathbb{C}})$.*

We then have the following (primitive) Lefschetz decomposition*:*

$$H^k(X_{\mathbb{C}}) = P^k(X_{\mathbb{C}}) \oplus \mathcal{L}ef P^{k-2}(X_{\mathbb{C}}) \oplus \mathcal{L}ef P^{k-4}(X_{\mathbb{C}}) \oplus \ldots .$$

Note that the linear map

$$L^{n-k+1} : \quad H^k(X_{\mathbb{C}}) \to H^{2n-k+2}(X_{\mathbb{C}})$$

need not be injective – and let alone an isomorphism. Indee, since $n - k' = k$ *implies that* $k' = n - k \neq n - k + 1$*, the hypotheses of the Hard Lefschetz Theorem 4.6, on page 43,* are not satisfied*, with* $k' = n - k$ *instead of* k*.*

Definition 4.13 (**Polarization Operator (in the Deligne Sense)**)**.**

Given a compact Kähler manifold $X_{\mathbb{C}}$, of real dimension $2n$, along with an integer $k \in \{0, \ldots, n\}$, a map

$$\mathbf{Q} : \quad H^k(X_{\mathbb{C}}) \times H^k(X_{\mathbb{C}}) \to \mathbb{C}$$

is said to be *a polarization operator* if it is $\mathbb{C}$-bilinear and positive definite on $H^m \otimes \mathbb{C}$, the complexification of H^m.

Note that for simplicity, here and thereafter, we no longer use the notation for cohomology classes.

Remark 4.3 (**Usual Polarization Operator**)).

Given a compact Kähler manifold $X_{\mathbb{C}}$ (of associated Kähler form ω; see Definition 4.6 above, on page 39), along with an integer $k \in \{0, \ldots, n\}$, the usual polarization operator is the $\mathbb{C}$-bilinear map

$$\begin{array}{rccl} \mathbf{Q} : & H^k(X_{\mathbb{C}}) \times H^k(X_{\mathbb{C}}) & \to & \mathbb{C} \\ & (\varphi, \psi) & \mapsto & \mathbf{Q}(\varphi, \psi) = \displaystyle\int_X \varphi \wedge \psi \wedge \omega^{n-k} . \end{array}$$

Theorem 4.8 (**Hodge–Riemann Bilinear Relations (see [Huh18])**)**.**

Given a compact Kähler manifold $X_{\mathbb{C}}$*, along with an integer* $k \in \{0, \ldots, 2n\}$ *and the usual polarization map* $\mathbf{Q}$*, then, for all pairs of integers* (p, q) *such that* $p + q = k$*, the Hermitian form*

$$\begin{array}{ccl} H^{p,q}(X_{\mathbb{C}}) \times P^k(X_{\mathbb{C}}) & \to & \mathbb{C} \\ (\varphi, \psi) & \mapsto & (-1)^{\frac{k(k-1)}{2}} \, i^{p-q} \, \mathbf{Q}(\varphi, \psi) , \end{array}$$

where $P^k(X_{\mathbb{C}}) = \ker\left(\mathcal{L}ef^{n-k+1}_{|H^k(X_{\mathbb{C}})}\right)$ *is the* k^{th} *primitive cohomology space which has been introduced in Corollary 4.7 above, on page 43, is positive definite.*

4.2 Main Results

We now return to the fractal setting considered earlier in the paper, that of the Weierstrass fractal curve $\Gamma_{\mathcal{W}}$ (or, equivalently, of the associated iterated fractal drum).

4.2.1 Conjugation

Theorem 4.9 (A Two-Level Conjugation).

In light of the symmetry of $\Gamma_{\mathcal{W}}$ *with respect to the vertical line* $x = \frac{1}{2}$ *(see Property 2.1, on page 10), it seems natural to consider the following change of variables from real to complex coordinates (with an obvious relabelling of the axes):*

$$Z = y + i\left(x - \frac{1}{2}\right), \text{ where } (x, y) \in \mathbb{R}^2 .$$

We then define the corresponding complex conjugation, which, to any point M *of the complex plane, associates its symmetric point* $M' = \mathcal{S}(M)$ *with respect to the vertical line* $x = \frac{1}{2}$,

$$\overline{M} = M' ,$$

with affix the complex number $\bar{Z}$ *defined by* $\bar{Z} = y - i\left(x - \frac{1}{2}\right)$.

This yields:

i. ([DL22b]) For any strictly positive integer m *and any integer* j *in* $\left\{0, \ldots, \frac{\#V_m - 1}{2}\right\}$ *(where* $\#V_m = (N_b - 1)\, N_b^m + 1$*; see part ii. of Property 2.6, on page 12),*

$$\overline{M_{\#V_m - 1 - j, m}} = \overline{M_{(N_b - 1)\, N_b^m - j, m}} = M_{j,m} ,$$

or

$$\overline{\left(\frac{j}{(N_b - 1)\, N_b^m}, \mathcal{W}\left(\frac{j}{(N_b - 1)\, N_b^m}\right)\right)} = \left(\frac{(N_b - 1)\, N_b^m - j}{(N_b - 1)\, N_b^m}, \mathcal{W}\left(\frac{(N_b - 1)\, N_b^m - j}{(N_b - 1)\, N_b^m}\right)\right) .$$

(Note that the numbering of the vertices of V_m *begins at 0, which accounts for the fact that the last vertex of* V_m *is the vertex number* $\#V_m - 1 = (N_b - 1)\, N_b^m$*.)*

ii. Also, since the sequence of sets of vertices $(V_m)_{m \in \mathbb{N}}$ *is increasing, for all pairs of integers* $(m, m') \in \mathbb{N}^2$ *and any* j *in* $\left\{0, \ldots, \frac{(N_b - 1)\, N_b^m}{2}\right\}$*, we have that*

$$\overline{M_{\#V_{m+m'} - 1 - (\#V_{m+m'} - 1 - j)\,(N_b - 1)^{m'} - j, m+m'}} = M_{j,m} .$$

Note that this conjugation relation establishes a connection between two different levels of the sequence of prefractal approximations of $\Gamma_{\mathcal{W}}$.

Proof.

i. See [DL22b].

ii. When switching from the m^{th} prefractal approximation (with associated set of vertices V_m), to the $(m+1)^{th}$ prefractal approximation (with associated set of vertices V_{m+1}), there are $N_b - 2$ new vertices (i.e., the vertices in $V_{m+1} \setminus V_m$) between consecutive vertices of V_m. By induction, we therefore obtain that when switching to the m^{th} prefractal approximation (with associated set of vertices V_m), to the $(m+m')^{th}$ prefractal approximation (with associated set of vertices $V_{m+m'}$), the number of new vertices (i.e., in $V_{m+m'} \setminus V_m$) between the vertices $M_{\#V_m-1,m}$ and $M_{\#V_m-1-j,m}$ is equal to $(\#V_{m+m'}-j)\,(N_b-1)^{m'}$, which results in the above conjugation relation.

$\square$

4.2.2 The Natural Transfer Operator

Definition 4.14 (The Natural Transfer Operator).

We introduce *the natural transfer operator* $\mathcal{L}_{\mathcal{W}}$ associated with the Weierstrass Curve $\Gamma_{\mathcal{W}}$ by the formal sum

$$\mathcal{L}_{\mathcal{W}} = T_0 + \ldots + T_{N_b-1}\,, \tag{$\mathcal{R}$ 36}$$

where $T_0, \ldots, T_{N_b-1}$ are the C^∞ bijective maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ of the nonlinear iterated function system $\mathcal{T}_{\mathcal{W}}$ introduced in Proposition 2.2, on page 10.

More precisely, the sum in formula ($\mathcal{R}$36) above has to be understood in the sense that, given any function f defined on $\Gamma_{\mathcal{W}}$, and a point $M \in \Gamma_{\mathcal{W}}$,

$$\mathcal{L}_{\mathcal{W}}(f)(M) = \sum_{j=0}^{N_b-1} f\left(T_j(M)\right)\,. \tag{$\mathcal{R}$ 37}$$

Note that the sum $\sum_{j=0}^{N_b-1} f\left(T_j(\cdot)\right)$ can be interpreted modulo N_b, if the set of integers $\{0,\ldots,N_b-1\}$ is identified with the ring $\mathbb{Z}/N_b\,\mathbb{Z}$.

Theorem 4.10 (The Action of the Nonlinear and Noncontractive Iterated Function System on the Symmetry $\mathcal{S}$ – Dual Transfer Operator).

For any integer j belonging to $\{0,\ldots,N_b-1\}$, we set

$$T_j^{\#} = T_j \circ \mathcal{S}\,,$$

where $\mathcal{T}_{\mathcal{W}} = \left\{T_0,\ldots,T_{N_b-1}\right\}$ is the nonlinear iterated function system (IFS) introduced in Proposition 2.2, on page 10, while $\mathcal{S}$ is the symmetry with respect to the vertical straight line $x = \frac{1}{2}$ (see Property 2.1, on page 10).

Hereafter, we will use the notation

$$\mathcal{T}_{\mathcal{W}}^{\#} = \left\{T_0^{\#},\ldots,T_{N_b-1}^{\#}\right\}$$

and call $\mathcal{T}_{\mathcal{W}}^{\#}$ the dual IFS *of $\mathcal{T}_{\mathcal{W}}$.*

Similarly, since $S^2 = Id$, $\mathcal{T}_{\mathcal{W}}$ is the dual IFS of $\mathcal{T}_{\mathcal{W}}^{\#}$.

We note that

$$\mathcal{T}_{\mathcal{W}}^{\#} = \mathcal{T}_{\mathcal{W}} \circ \mathcal{S} \quad \textit{and} \quad \mathcal{T}_{\mathcal{W}} = \mathcal{T}_{\mathcal{W}}^{\#} \circ \mathcal{S} \,.$$

We then introduce the dual natural transfer operator $\mathcal{L}_{\mathcal{W}}^{\#}$ *associated with the Weierstrass Curve* $\Gamma_{\mathcal{W}}$ *by the formal sum*

$$\mathcal{L}_{\mathcal{W}}^{\#} = T_0^{\#} + \ldots + T_{N_b-1}^{\#} \,. \qquad (\mathcal{R}\,38)$$

More precisely, the sum in formula ($\mathcal{R}$38) above has to be understood in the sense that, given any function f defined on $\Gamma_{\mathcal{W}}$, and a point $M \in \Gamma_{\mathcal{W}}$,

$$\mathcal{L}_{\mathcal{W}}^{\#}(f)(M) = \sum_{j=0}^{N_b-1} f\left(T_j^{\#}(M)\right) \,. \qquad (\mathcal{R}\,39)$$

Also, since the symmetry $\mathcal{S}$ is an isometry, this ensures that $\mathcal{T}_{\mathcal{W}}$ and its dual $\mathcal{T}_{\mathcal{W}}^{\#}$ have the same attractor, the Weierstrass Curve $\Gamma_{\mathcal{W}}$ (see Property 2.3, on page 10).

Proposition 4.11 (Joint Actions of the Natural Transfer Operator and the Conjugation).

The joint action of the natural transfer operator $\mathcal{L}_{\mathcal{W}}$ introduced in Definition 4.14 above, on page 46, and of the conjugation $Z \mapsto \bar{Z}$ (see Theorem 4.9, on page 45), can be summarized as follows: given any function f defined on $\Gamma_{\mathcal{W}}$, we have that

$$\mathcal{L}_{\mathcal{W}}\left(\bar{f}\right) = \mathcal{L}_{\mathcal{W}}^{\#}(f) \,. \qquad (\mathcal{R}\,40)$$

Proof. This simply comes from the fact that the conjugation map $Z \mapsto \bar{Z}$ and the symmetry $\mathcal{S}$ are equivalent, in the sense that, given a point M of affix Z, its image $\mathcal{S}(M)$ has affix $\bar{Z}$ (see Theorem 4.9, on page 45).

□

Remark 4.4. From the point of view of symbolic dynamics (see, e.g., [BKS91]), for any given j in $\{0, \ldots, N_b - 1\}$, the action of T_j can be identified with the j^{th}-shift (acting on code space $\Sigma = \mathcal{A}^{\mathbb{N}}$ and with alphabet $\mathcal{A} = \{0, \ldots, N_b - 1\}$), which appends the letter j to any given infinite word in Σ.

Therefore, both the transfer operator $\mathcal{L}_{\mathcal{W}}$ and its dual $\mathcal{L}_{\mathcal{W}}^{\#}$ can be reexpressed in terms of the simultaneous actions of the N_b j shifts (combined with the action of the symmetry $\mathcal{S}$ – or, equivalently, of the complex conjugation $Z \mapsto \bar{Z}$ – in the case of the dual transfer operator $\mathcal{L}_{\mathcal{W}}^{\#}$), for $j = 0, \ldots, N_b - 1$.

Proposition 4.12 (Joint Actions of the Natural Transfer Operator and its Dual).

The joint action of the natural transfer operator $\mathcal{L}_{\mathcal{W}}$ introduced in Definition 4.14 above, on page 46, and its dual $\mathcal{L}_{\mathcal{W}}^{\#}$ (see Theorem 4.10, on page 46), can be summarized as follows: given any function f defined on $\Gamma_{\mathcal{W}}$, we have that

$$\mathcal{L}_{\mathcal{W}}\,\mathcal{L}_{\mathcal{W}}^{\#}(f) = \mathcal{L}_{\mathcal{W}}^{2}(f\circ\mathcal{S}) \quad \textit{and} \quad \mathcal{L}_{\mathcal{W}}^{\#}(f)\,\mathcal{L}_{\mathcal{W}}(f) = \mathcal{L}_{\mathcal{W}}^{2}(f\circ\mathcal{S})\,. \tag{$\mathcal{R}$ 41}$$

Proof. Thanks to Proposition 4.11, on page 47, given a function f defined on $\Gamma_{\mathcal{W}}$, we have that

$$\mathcal{L}_{\mathcal{W}}^{\#}(f) = \mathcal{L}_{\mathcal{W}}(\bar{f}) = \mathcal{L}_{\mathcal{W}}(f\circ\mathcal{S})\,.$$

We then deduce that

$$\mathcal{L}_{\mathcal{W}}\,\mathcal{L}_{\mathcal{W}}^{\#}(f) = \mathcal{L}_{\mathcal{W}}^{2}(f\circ\mathcal{S}) \quad \text{and} \quad \mathcal{L}_{\mathcal{W}}^{\#}(f)\,\mathcal{L}_{\mathcal{W}}(f) = \mathcal{L}_{\mathcal{W}}^{2}(f\circ\mathcal{S})\,,$$

as desired; so that

$$\mathcal{L}_{\mathcal{W}}\,\mathcal{L}_{\mathcal{W}}^{\#} = \mathcal{L}_{\mathcal{W}}^{\#}\;\mathcal{L}_{\mathcal{W}}\,.$$

□

4.2.3 Fractal Cohomology

Now, we are particularly interested in the action of the transfer operators $\mathcal{L}_{\mathcal{W}}$ and $\mathcal{L}_{\mathcal{W}}^{\#}$ on the (fractal) cohomology associated with the Weierstrass Curve $\Gamma_{\mathcal{W}}$. We introduced this cohomology in our previous work [DL22b], that we recall below. However, in this form, the cohomology groups are not vector spaces (they are simply additive abelian groups). In order to apply our transfer operator, a suitably adapted structure is required, as is given below in Proposition 4.21, on page 56, and Theorem 4.24, on page 59.

Definition 4.15 (**(m,p)-Fermion [DL22b]**).

By analogy with particle physics, given a pair of integers (m,p), with $m \in \mathbb{N}$ and $p \in \mathbb{N}^{\star}$, we will call (m,p)-*fermion* on V_m, with values in $\mathbb{C}$, any antisymmetric map f from V_m^{p+1} to $\mathbb{C}$. (Here, V_m^{p+1} denotes the $(p+1)$-fold cartesian product of V_m by itself.) Note that these maps are not assumed to be multilinear (which would be meaningless here, anyway).

For any $m \in \mathbb{N}$, an $(m,0)$-*fermion* on V_m (or a 0-*fermion*, in short) is simply a map f from V_m to $\mathbb{C}$. We adopt the convention according to which a 0-fermion on V_m is a 0-*antisymmetric map* on V_m.

In the sequel, for any $(m,p) \in \mathbb{N}^2$, we will denote by $\mathcal{F}^p(V_m,\mathbb{C})$ the $\mathbb{C}$-module (i.e., the complex vector space) of (m,p)-fermions on V_m, with values in $\mathbb{C}$, which makes it an abelian group with respect to the addition, with an external law from $\mathbb{C}\times\mathcal{F}^p(V_m,\mathbb{C})$ to $\mathcal{F}^p(V_m,\mathbb{C})$, corresponding to the multiplication by a scalar.

Definition 4.16 (**$(m-1,m)$-Path [DL22b]**).

Given a strictly positive integer m, and two adjacent vertices $X_{m-1,k}$, $X_{m-1,k+1}$ in V_{m-1}, for $0 \leqslant k \leqslant \#V_{m-1}-2$, we call $(m-1,m)$-path between $X_{m-1,k}$, $X_{m-1,k+1}$ the ordered set of vertices

$$\mathcal{P}_{m-1,m}\left(X_{m-1,k},X_{m-1,k+1}\right) = \left\{X_{m,\ell}\,,\ 0 \leqslant \ell \leqslant N_b\right\}\,,$$

where

$$X_{m,\ell} = X_{m-1,k} \quad \text{and} \quad X_{m,\ell+N_b} = X_{m-1,k+1}\,;$$

see Figure 5, on page 49.

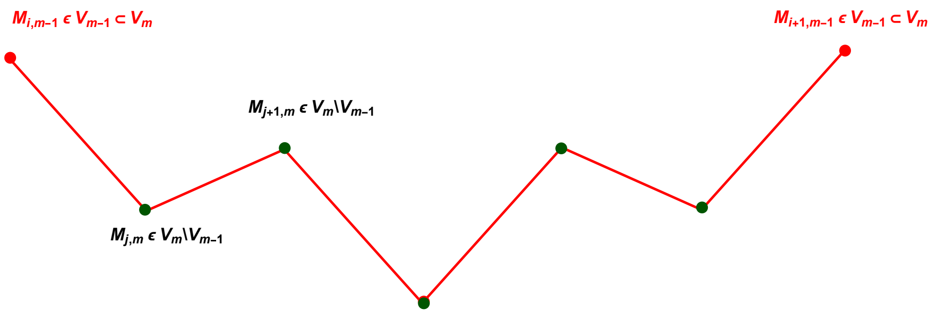


Figure 5: **In search of invariants, when switching from the initial prefractal graph, to the first one.**

Remark 4.5 (**On the Intrinsinc Meaning of our $(m-1,m)$-Paths**).

Note that our discrete paths enable us to mathematically represent the underlying dynamical evolution of the fractal, i.e., the switch/transition from a scale to the next or previous one.

Definition 4.17 ($(m-1,m)$-Differentials [DL22b]).

Given a strictly positive integer m, we define the $(m-1,m)$-*differential* $\delta_{m-1,m}$ from $\mathcal{F}^0\left(V_m,\mathbb{C}\right)$ to $\mathcal{F}^{N_b+1}\left(V_m,\mathbb{C}\right)$, for any f in $\mathcal{F}^0\left(\Gamma_{\mathcal{W}},\mathbb{C}\right)$ and any $\left(M_{i,m-1},M_{i+1,m-1},M_{j+1,m},\dots,M_{j+N_b-2,m}\right) \in V_m^{N_b+1}$ such that

$$M_{i,m-1} = M_{j,m} \quad \text{and} \quad M_{i+1,m-1} = M_{j+N_b,m}\,,$$

by

$$\delta_{m-1,m}(f)\left(M_{i,m-1},M_{i+1,m-1},M_{j+1,m},\dots,M_{j+N_b-1,m}\right) = c_{m-1,m}\left(\sum_{q=0}^{N_b}(-1)^q\,f\left(M_{j+q,m}\right)\right),$$

where $c_{m-1,m}$ denotes a suitable positive constant. Note that, given the way we deal with these differentials in the present paper, one does not need to know – or fix – the value of this constant. It becomes important, however, when operators involving the differentials, such as the Laplacian, are involved; see, for instance, Section 6 of [DL23d].

The spaces $\mathcal{F}^q\left(V_m,\mathbb{C}\right)$, equipped with the differentials, form a complex, in the standard sense of algebraic topology or homological algebra (see, e.g., [GQ22]), so that the definition of H^m is justified.

Proposition 4.13 (Prefractal Cohomology Groups [DL22b]).

In our present setting, with the differential introduced in Definition 4.17, on page 49, the cohomology groups are the quotient groups

$$H^m = \ker \delta_{m-1,m} / Im\, \delta_{m-2,m-1} \quad , \qquad \textit{for } m \geqslant 0 \ ,$$

which consist in maps the expression of which is obtained as the difference of an antisymmetric map with respect to the set of vertices $\left(M_{i,0}, M_{i+1,0}, M_{j+1,m}, \ldots, M_{j+N_b^m-1,m}\right)$ *and of an antisymmetric map with respect to the set of vertices* $\left(M_{i,0}, M_{i+1,0}, M_{j+1,m-1}, \ldots, M_{j+N_b^{m-1}-1,m-1}\right)$.

Hereafter, we will use the additional convention that $\delta_{-2,-1} = 0$ *and* $\delta_{-1,0} = 0$, *which ensures that* $H^0 = \{0\}$.

Definition 4.18 (Set of Functions of the Same Nature as the Weierstrass Function $\mathcal{W}$ [DL22b]).

i. We say that a continuous, complex-valued function f, defined on $\Gamma_{\mathcal{W}} \supset V^\star$, is *of the same nature as the Weierstrass function* $\mathcal{W}$, if it satisfies local Hölder and reverse-Hölder properties analogous to those satisfied by the Weierstrass function $\mathcal{W}$; i.e., for any pair of adjacent vertices (M, M') of respective affixes $(z, z') \in \mathbb{C}^2$ of the prefractal graph $\Gamma_{\mathcal{W}_m}$, with $m \in \mathbb{N}$ arbitrary,

$$\tilde{C}_{inf} \, |z' - z|^{2-D_{\mathcal{W}}} \leqslant \left| f(z') - f(z) \right| \leqslant \tilde{C}_{sup} \, |z' - z|^{2-D_{\mathcal{W}}} \ ,$$

where $\tilde{C}_{inf}$ and $\tilde{C}_{sup}$ denote positive and finite constants (but not necessarily the same ones as for the Weierstrass function $\mathcal{W}$ itself, in Proposition 2.12, on page 19). This can be written, equivalently, as

$$\left| z - z' \right|^{2-D_{\mathcal{W}}} \lesssim \left| f(z) - f(z') \right| \lesssim \left| z - z' \right|^{2-D_{\mathcal{W}}} \ . \qquad (\mathcal{R}\,42)$$

Hereafter, we will denote by $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ the complex vector space consisting of the continuous, complex-valued functions f, defined on $\Gamma_{\mathcal{W}} \supset V^\star$ and satisfying relation ($\mathcal{R}$42); see the discussion just above.

ii. Moreover, we will denote by $\mathcal{H}\ddot{o}ld_{geom}\,(\Gamma_{\mathcal{W}}) \subset \mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ the complex subspace of $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ consisting of the functions f of $\mathcal{H}\ddot{o}ld\,(\Gamma_{\mathcal{W}})$ which satisfy the following *additional geometric condition* ($\mathcal{R}$43), again, for any pair of adjacent vertices (M, M') with respective affixes $(z, z') \in \mathbb{C}^2$ of the prefractal graph V_m, with $m \in \mathbb{N}$ arbitrary; namely,

$$\left| \arg\,(f(z)) - \arg\left(f(z')\right) \right| \lesssim \left| z - z' \right|^{D_{\mathcal{W}}-1} \ . \qquad (\mathcal{R}\,43)$$

Theorem 4.14 (Complex Dimensions Series Expansion and Characterization of the Prefractal Cohomology Groups H^m [DL22b]).

Let $m \in \mathbb{N}$ *be arbitrary. Then, if the functions* f *belong to* $\mathcal{H}\ddot{o}ld_{geom}\,(\Gamma_{\mathcal{W}})$ *(see part ii. of Definition 4.18, on page 50 above), then, for any strictly positive integer* m, *and again with the convention* $H^0 = Im\, \delta_{-1,0} = \{0\}$, *the cohomology groups*

$$H^m = \ker \delta_{m-1,m} / Im\, \delta_{m-2,m-1}$$

are comprised of the restrictions to V_m of $(m, N_b^m + 1)$-fermions, i.e., the restrictions to $V_m^{N_b^m+1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of continuous functions f on $\Gamma_{\mathcal{W}}$, such that, for any vertex $M_{j,m} \in V_m$,

$$f\left(M_{j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \varepsilon_k^{k\,(2-D_{\mathcal{W}})}\, \varepsilon_k^{i\,k\,\ell_{k,j,m}\,\mathbf{p}} = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \varepsilon_k^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,j,m}\,\mathbf{p}} \quad , \quad M_{\star,m} \in V_m\,, \qquad (\mathcal{R}\,44)$$

where $\mathbf{p}$ denotes the oscillatory period introduced in [DL22a],

$$\mathbf{p} = \frac{2\,\pi}{\ln N_b}\,,$$

and where the coefficients $c_k(\star,\star)$ are complex numbers which still depend on the function f involved, and on the point at which they are evaluated. Here, in relation ($\mathcal{R}44$), for each integer k such that $0 \leqslant k \leqslant m$, $\ell_{k,j,m}$ denotes an integer (in $\mathbb{Z}$) satisfying the estimate

$$\left|\left\{k\,\ell_{k,j,m}\,\frac{\ln \varepsilon_k^k}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon_k^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi}\,. \qquad (\mathcal{R}\,45)$$

In expansion ($\mathcal{R}44$), the coefficients $c_k(f, M_\star)$, for $0 \leqslant k \leqslant m$, reflect the dependence of the value taken by the map f at the vertex $M_{j,m}$ on the values taken by f at previous steps – vertices – of the m^{th} prefractal graph approximation, in conjunction with values taken by f at neighboring vertices of $M_{j,m}$ at the same level (m) of the prefractal sequence and with vertices which, in addition, strictly belong to the same polygon $\mathcal{P}_{m,k}$ introduced in part iv. of Property 2.6, on page 12, with $1 \leqslant k \leqslant N_b^m - 1$ (by "strictly" here, we mean that the junction vertices are not included).

The expansion ($\mathcal{R}44$) could be interpreted as a kind of generalized Taylor expansion with corresponding complex derivatives of orders $-\omega_k = k\,(2 - D_{\mathcal{W}}) + i\,k\,\ell_{k,j,m}\,\mathbf{p}$, where $k \in \mathbb{N}$ is arbitrary, the coefficients $c_k(f, M_\star)$ can thus be interpreted as (discrete) derivatives of complex order $-\omega_k$ of the function f, evaluated at the point $M_\star$ of $V^\star \subset \Gamma_{\mathcal{W}}$.

Remark 4.6. Note that in the expansion ($\mathcal{R}44$), for $0 \leqslant k \leqslant m$, and by contrast with our previous work [DL22b], we have used the k^{th} intrinsic cohomology infinitesimal $\varepsilon^k = (N_b - 1)\,\varepsilon_k^k = \dfrac{1}{N_b^k}$, instead of the k^{th} intrinsic cohomology infinitesimal ε_k^k (see Definition 2.8, on page 21). Of course, the two expressions – ε^k and ε_k^k – are essentially equivalent, since they have the same asymptotic behavior as $k \to \infty$, up to a positive multiplicative constant.

Proposition 4.15 (The Cohomological Vertex Integers (see Definition 2.9, on page 22) Can be Used in the Estimate ($\mathcal{R}$50) below, on page 55).

Given $m \in \mathbb{N}$, and a vertex $M_{j,m} = M_{(N_b-1)\,k'+k",m} \in V_m$, of abscissa $((N_b - 1)\,k + j)\,\varepsilon_m^m$, and associated cohomological vertex integer $\ell_{j,m} = (N_b - 1)\,k' + k"$, where $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k" \leqslant N_b - 1$, we have that

$$\left|\left\{\ell_{j,m}\,\frac{\ln \varepsilon_k^k}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi}\,. \qquad (\mathcal{R}\,46)$$

Proof. This simply comes from the fact that

$$0 < \varepsilon_k^{k\,(D_{\mathcal{W}}-1)} < 1\,.$$

We thus obviously have that

$$\left\{\ell_{j,m}\,\frac{\ln \varepsilon_k^k}{\ln N_b}\right\} \leqslant C_{k,j,m}\,\frac{\varepsilon_k^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi}\,,$$

where the positive constant $C_{k,j,m}$ can be choosen such that

$$C_{k,j,m} \geqslant 2\,\pi\,\varepsilon_k^{k\,(1-D_{\mathcal{W}})}\quad\left|\left\{\ell_{j,m}\,\frac{\ln \varepsilon_k^k}{\ln N_b}\right\}\right|\,.$$

□

Proposition 4.16 (Fixing the Integers in the Expansion ($\mathcal{R}$44) above, on page 51).

In the sequel, for the sake of the forthcoming Hodgde decomposition of the total (fractal) cohomology in Theorem 4.30 below, on page 68, given $m \in \mathbb{N}$, and a vertex $M_{j,m} = M_{(N_b-1)\,k'+k",m} \in V_m$, of abscissa $((N_b-1)\,k+j)\,\varepsilon_m^m$, and associated vertex integer $\ell_{j,m} = (N_b-1)\,k' + k"$, where $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k" \leqslant N_b - 1$ (see Definition 2.9, on page 22), we fix the integers $\ell_{k,j,m}$ in the expansion ($\mathcal{R}$44), on page 51, in the following way:

$$k\,\ell_{k,j,m} = k\,\ell_{j,m} = k\,\left((N_b-1)\,k' + k"\right)\,,$$

which yields

$$f\left(M_{j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right)\,\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,\varepsilon_k^{i\,k\,\ell_{k,j,m}\,\mathbf{p}} = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right)\,\varepsilon_k^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{j,m}\,\mathbf{p}}\,. \qquad (\mathcal{R}\,47)$$

Indeed, as seen in Proposition 4.15, on page 51, the cohomological vertex integers can be used in the estimate ($\mathcal{R}$50), on page 55. Also, it is important to bear in mind that the estimate ($\mathcal{R}$50), on page 55, comes from a generalized Hölder condition on the arguments of the involved functions f (see our previous work [DL22b]), which is a translation of the fact that, when switching from a vertex $M_{j',m-1}$, with $0 \leqslant j' \leqslant \#V_{m-1} - 1$, to the the adjacent neighboring vertices in V_m, the argument increases nearly arithmetically. The corresponding information is exactly carried by the cohomological vertex integers.

Proposition 4.17 (Generators of the Cohomology Groups [DL22b]).

For any integer $m \geqslant 1$, and with the convention $H^0 = Im\,\delta_{-1,0} = \{0\}$, the generators *of the (additive) cohomology groups*

$$H^m = \ker \delta_{m-1,m} / Im\,\delta_{m-2,m-1}$$

*are to be understood in the sense of the k^{th} (intrinsic) cohomology infinitesimals ε^k, for $0 \leqslant k \leqslant m$ (see Definition 2.8, on page 21, and recall that $\varepsilon = \frac{1}{N_b}$). Note that by "*generators*", here, we do not refer to the notion of group generators in the sense of group theory, but in a broader sense, instead. Indeed, in our present context, we express the values, at the vertices of V_m, of the continuous, complex-valued maps defined on the Weierstrass Curve $\Gamma_{\mathcal{W}} \supset \bigcup_{n\in\mathbb{N}} V_n$, which constitute the quotient groups H^m,* *by means of Taylor-like expansions involving fractional powers of the* k^{th} *(intrinsic) cohomology infinitesimals* ε^k*, for* $0 \leqslant k \leqslant m$*. So, in a sense, one could think of those* generators *as generalized* (fractional) *polynomial variables. Those* generators *are therefore of the following form:*

$$\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,\varepsilon_k^{i\,k\,\ell_{k,j,m}\,\mathbf{p}} = \varepsilon_k^{k\,((2-D_{\mathcal{W}})+i\,k\,\ell_{k,j,m}\,\mathbf{p})}$$

and

$$\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,\varepsilon_k^{-i\,k\,\ell_{k,j,m}\,\mathbf{p}} = \varepsilon_k^{k\,((2-D_{\mathcal{W}})-i\,k\,\ell_{k,j,m}\,\mathbf{p})},$$

or, equivalently, by using the cohomological vertex integers, introduced in Definition 2.9, on page 22,

$$\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,k\,\ell_{k',k",m}\,\mathbf{p}} = \varepsilon_k^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k',k",m}\,\mathbf{p}}$$

and

$$\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{-i\,k\,\ell_{k',k",m}\,\mathbf{p}} = \varepsilon_k^{k\,(2-D_{\mathcal{W}})-i\,k\,\ell_{k',k",m}\,\mathbf{p}},$$

with $0 \leqslant k \leqslant m$, $0 \leqslant k' \leqslant N_b^m - 1$, $0 \leqslant k" \leqslant N_b - 1$ *and where* $\ell_{j,m} = (N_b - 1)\,k' + k" \in \{0, \ldots, \#V_m - 1\}$ *is the the cohomological vertex integer associated with the point* $M_{j,m}$*; see also Theorem 4.14 above, on page 50, along with the aforementioned Definition 2.9, on page 22.*

In accordance with the expansion given in Theorem 4.14, in relation ($\mathcal{R}44$*), we can also write those generators in the equivalent following form,*

$$\varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{i\,\ell_{k,j,m}\,\mathbf{p}} \quad \text{and} \quad \varepsilon^{k\,(2-D_{\mathcal{W}})}\,\varepsilon^{-i\,\ell_{k,j,m}\,\mathbf{p}}.$$

We can now begin discussing our new results – starting with the connections between fractal cohomology, Taylor-like expansions and their coefficients, along with the traces of commutators of Lax pairs.

Theorem 4.18 (Fractional Taylor Expansions).

We note that the expansion in relation ($\mathcal{R}44$*) of Theorem 4.14, on page 51, is of the same form as the complex dimensions series expansion of the Complexified Weierstrass function given in Theorem 2.18, on page 23.*

We set, for all $m \in N$, $0 \leqslant j \leqslant \#V_m - 1$, $0 \leqslant k' \leqslant k \leqslant m$,

$$\beta_{k',j}^2(f) = \frac{1}{2}\,c_{k'}\left(f, M_{j,m}\right)\varepsilon_{k'}^{k'\,(2-D_{\mathcal{W}})+i\,k'\,\ell_{j,m}\,\mathbf{p}} + \frac{1}{2}\,\overline{c_{k'}\left(f, M_{j,m}\right)}\,\varepsilon_{k'}^{k'\,(2-D_{\mathcal{W}})-i\,k'\,\ell_{j,m}\,\mathbf{p}},$$

where ε *is the intrinsic scale, introduced in Definition 2.8, on page 21, while the numbers* $c_{k'}\left(f, M_{j,m}\right)$ *are the complex coefficients involved in expansion (*$\mathcal{R}44$*), on page 51, and where* $\ell_{j,m} = (N_b - 1)\,k' + k"$*, where* $0 \leqslant k' \leqslant N_b^m - 1$ *and* $0 \leqslant k" \leqslant N_b - 1$ *(see Definition 2.9 above, on page 22).*

By applying the same method as in Theorem 3.1, on page 30, we obtain that, for each continuous function f *on* $\Gamma_{\mathcal{W}}$ *satisfying, for all* $m \in \mathbb{N}$ *and each* $j \in \{0, \ldots, \#V_m - 1\}$*, relation (*$\mathcal{R}44$*),* $f\left(M_{j,m}\right)$ *can be expressed as the sum of traces of commutators, in the following form,*

$$f\left(M_{j,m}\right) = \sum_{k=m}^{k}\sum_{k'=0}^{k} tr_{k'}\left(\left[\mathcal{M}_{f,j,\infty}, \mathcal{J}_{f,j,\infty}\right]\right), \qquad (\mathcal{R}\,48)$$

with the countably infinite matrices $\mathcal{J}_{f,j,\infty}$ *and* $\mathcal{M}_{f,j,\infty}$ *(where* $\mathcal{J}_{f,j,\infty}$ *is a Jacobi matrix) given by*

$$\mathcal{J}_{f,j,\infty} = \begin{pmatrix} 1 & \beta_{0,j}(f) & 0 & \dots & \dots & \dots & 0 \\ \beta_{0,j}(f) & 1 & \beta_{1,j}(f) & \dots & \dots & \vdots & \\ 0 & \beta_{1,j}(f) & 1 & \beta_{2,j}(f) & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & 0 & \ddots \end{pmatrix},$$

$$\mathcal{M}_{f,j,\infty} = \frac{1}{2} \begin{pmatrix} 1 & \beta_{0,j}(f) & 0 & \dots & \dots & \dots & 0 \\ -\beta_{0,j}(f) & 1 & \beta_{1,j}(f) & \dots & \dots & \vdots & \\ 0 & -\beta_{1,j}(f) & 1 & \beta_{2,j}(f) & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & 0 & \ddots \end{pmatrix}$$

and, as a result, the Lie bracket *(or* commutator*), is given by the countably infinite diagonal matrix*

$$[\mathcal{M}_{f,j,\infty}, \mathcal{J}_{f,j,\infty}] = \begin{pmatrix} \beta_{1,j}^2(f) - \beta_{0,j}^2(f) & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \beta_{2,j}^2(f) - \beta_{1,j}^2(f) & 0 & \dots & \dots & \vdots & \\ 0 & 0 & \beta_{3,j}^2(f) - \beta_{2,j}^2(f) & 0 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 0 & \beta_{k,j}^2(f) - \beta_{k-1}^2(f) & 0 \\ 0 & \dots & \dots & 0 & 0 & 0 & \ddots \end{pmatrix},$$

and where, for all $0 \leqslant k' \leqslant k$, $tr_{k'}\left([\mathcal{M}_{f,j,\infty}, \mathcal{J}_{f,j,\infty}]\right)$ *is the* k'^{th} *truncated trace of the commutator* $[\mathcal{M}_{f,j,\infty}, \mathcal{J}_{f,j,\infty}]$, *as introduced in Notation 9, on page 30.*

This ensures that the expansion ($\mathcal{R}44$), on page 51, is an actual fractional Taylor expansion, since each term

$$\frac{1}{2}\, c_{k'}\left(f, M_{j,m}\right) \varepsilon_{k'}^{k'(2-D_{\mathcal{W}})+i\,k'\,\ell_{j,m}\,\mathbf{p}} + \frac{1}{2}\, \overline{c_{k'}\left(f, M_{j,m}\right)}\, \varepsilon_{k'}^{k'(2-D_{\mathcal{W}})-i\,k'\,\ell_{j,m}\,\mathbf{p}}$$

can be associated to the trace of a commutator.

Remark 4.7 (**A Forthcoming Structure of Vector Space**).

As is described above, the cohomology groups H^m are additive groups. However, as we will see below (in Theorem 4.24, on page 59), we can equip them with a vector space structure, by means of a tensor product with $\mathbb{C}$. Such an underlying structure is natural, since the condition ($\mathcal{R}42$), on page 50, which, in a sense, *governs* the cohomology groups (see the proof of Theorem 4.14 in our previous work [DL22b]) is still satisfied by a complex linear combination of functions f_1 and f_2 which themselves satisfy condition ($\mathcal{R}42$).

Theorem 4.19 (Fractal Cohomology of the Weierstrass Curve [DL22b]).

Within the set $\mathcal{H}\ddot{o}ld_{geom}(\Gamma_{\mathcal{W}})$ *of continuous, complex-valued functions* f*, defined on the Weierstrass Curve* $\Gamma_{\mathcal{W}} \supset V^\star = \bigcup_{m \in \mathbb{N}} V_m$ *(see part ii. of Definition 4.18, on page 50 above), let us consider the following Complex (which can be called* the Fractal Complex *of* $\Gamma_{\mathcal{W}}$*):*

$$H^\star = H^\bullet\left(\mathcal{F}^\bullet(\Gamma_{\mathcal{W}}, \mathbb{C}), \delta^\bullet\right) = \bigoplus_{m=0}^{\infty} H^m ,$$

where, for any integer $m \geqslant 1$*, and with the convention* $H^0 = Im\,\delta_{-1,0} = \{0\}$*,* H^m *is the cohomology group*

$$H^m = \ker \delta_{m-1,m} / Im\,\delta_{m-2,m-1} .$$

Then, $H^\star$ *is the set consisting of functions* f *on* $\Gamma_{\mathcal{W}}$*, viewed as* 0*-fermions (in the sense of Definition 4.15, on page 48), and, for any integer* $m \geqslant 1$*, of the restrictions to* V_m *of* $(m, N_b^m + 1)$*-fermions, i.e., the restrictions to (the Cartesian product space)* $V_m^{N_b^m+1}$ *of antisymmetric maps on* $\Gamma_{\mathcal{W}}$*, with* $N_b^m + 1$ *variables (corresponding to the vertices of* V_m*), involving the restrictions to* V_m *of the continuous, complex-valued functions* f *on* $\Gamma_{\mathcal{W}}$ *– like, naturally, the aforementioned* 0*-fermions – satisfying the following convergent (and even, absolutely convergent) Taylor-like expansions (with* $V^\star = \bigcup_{n \in \mathbb{N}} V_n$*),*

$$\forall\, M_{\star,\star} \in V^\star : \quad f\left(M_{\star,\star}\right) = \sum_{k=0}^{\infty} c_k\left(f, M_{\star,\star}\right) \varepsilon_k^{k\left((2-D_{\mathcal{W}})+i\,\ell_{k,j,m}\,\mathbf{p}\right)} , \qquad (\mathcal{R}\,49)$$

where, for each integer $k \geqslant 0$*, the coefficient* $c_k(\star,\star)$ *is a complex number which depends on the function* f *involved, and on the point at which it is evaluated. The number* $\varepsilon_k^k > 0$ *is the* k^{th} *component of the* k^{th} *cohomology infinitesimal introduced in Definition 2.8, on page 21, and* $\ell_{k,j,m}$ *denotes an integer (in* $\mathbb{Z}$*) such that, for all* $k \in \mathbb{N}$ *(with* $\{y\} \in [0,1[$ *denoting the fractional part of the real number* y*)*

$$\left|\left\{\ell_{k,j,m}\,\frac{\ln \varepsilon_k^k}{\ln N_b}\right\}\right| \lesssim \frac{\varepsilon_k^{k\,(D_{\mathcal{W}}-1)}}{2\,\pi} . \qquad (\mathcal{R}\,50)$$

Note that since the functions f *involved are uniformly continuous on the Weierstrass Curve* $\Gamma_{\mathcal{W}} \supset V^\star$*, and since the set* $V^\star$ *is dense in* $\Gamma_{\mathcal{W}}$*, they are uniquely determined by their restriction to* $V^\star$*, as given by (*$\mathcal{R}$*49). We caution the reader, however, that at this stage of our investigations, we do not know whether* $f(M)$ *is given by an expansion analogous to the one in (*$\mathcal{R}$*49), for every* $M \in \Gamma_{\mathcal{W}}$*, rather than just for all* $M \in V^\star$*.*

The convergence (or even, the absolute convergence) of the series $\sum_{k=0}^{\infty} c_k\left(f, M_{\star,\star}\right) \varepsilon_k^{k\left((2-D_{\mathcal{W}})+i\,\ell_{k,j,m}\,\mathbf{p}\right)}$ *directly follows from the fact that the coefficients* $c_k(\star,\star)$ *are uniformly bounded and that, for any* $k \in \mathbb{N}^\star$*,*

$$\left|\varepsilon_k^{k\left((2-D_{\mathcal{W}})+i\,\ell_{k,j,m}\,\mathbf{p}\right)}\right| = \varepsilon_k^{k\,(2-D_{\mathcal{W}})} = \left(\varepsilon_k^{2-D_{\mathcal{W}}}\right)^k , \text{ with } 0 < \varepsilon_k < 1 \text{ and } 2 - D_{\mathcal{W}} > 0 .$$

Finally, for each $M_\star = M_{\star,m} \in V^\star$*, the coefficients* $c_k(\star,\star)$ *(for any* $k \in \mathbb{N}$*) are the residues at the possible Complex Dimensions* $-\left(k\,(2 - D_{\mathcal{W}}) + i\,\ell_{k,j,m}\,\mathbf{p}\right)$ *of a suitable global scaling zeta function introuced in [DL22b].*

The group $H^\star = \bigoplus_{m=0}^{\infty} H^m$ *is called* the total fractal (or global) cohomology group *of the Weierstrass Curve* $\Gamma_{\mathcal{W}}$ *(or else, of the Weierstrass function* $\mathcal{W}$*).*

Note that $H^\star$ is to be understood in the sense of the inductive limit *of the sequence of cohomology groups $\left(H^m\right)_{m\in\mathbb{N}}$; namely, for each fermion $\varphi \in H^\star$, and each $m \in \mathbb{N}$, the restriction $\varphi_{|V_m}$ of φ to the set of vertices V_m belongs to H^m; the restriction $\left(\varphi_{|V_{m+1}}\right)_{|V_m}$ to V_m of the restriction $\varphi_{|V_{m+1}}$ of φ to the set of vertices V_{m+1} (which is itself in H_{m+1}), coïncides with the restriction $\varphi_{|V_m}$ of φ to V_m; i.e.,*

$$\forall\, m \in \mathbb{N} : \quad \varphi_{|V_m} \in H^m \quad \text{and} \quad \left(\varphi_{|V_{m+1}}\right)_{|V_m} = \varphi_{|V_m} \,.$$

This amounts, for each $\varphi \in H^\star$, to

$$\varphi = \left(\varphi_m\right)_{m\in\mathbb{N}} \,,$$

where, for each $m \in \mathbb{N}$, $\varphi_m \in H^m$, while, if we denote by $\pi_{m+1} : H^{m+1} \to H^m$ the projection from H^{m+1} to H^m, we have that $\pi_{m+1}\left(\varphi_{m+1}\right) \in H^m$ coincides with φ_m.

According to Proposition 4.15, on page 51, and Proposition 4.16, on page 52, the estimate ($\mathcal{R}$50) in Theorem 4.19, on page 55, is, in particular, satisfied by the cohomological vertex integers, introduced in our previous work [DL24b]; see Definition 2.9 just below. Not only do they satisfy the aforementioned estimate, they also carry the information associated to the vertex point involved. Also, as we will see below, in Theorem 4.20, on page 56, they satisfy a Hodge diamond star relation of the same form as the one given in Property 2.19, on page 25.

Theorem 4.20 (A Second Hodge Diamond Star Relation).

For any $m \in \mathbb{N}^\star$, any k in $\{0,\dots,m\}$ and any j in $\{0,\dots,\#V_m-1\}$, we have the following second Hodge Diamond Star *relation, which goes hand in hand with – and completes the previous one given in Property 2.19, on page 25,*

$$\varepsilon_k^{\,i\,k\,(\#V_m-1-\ell_{j,m})\,\mathbf{p}} = \overline{\varepsilon^{\,i\,k\,\ell_{j,m}\,\mathbf{p}}} \,, \qquad (\mathcal{R}\,51)$$

or, equivalently, with the second notation given in Definition 2.9, on page 22,

$$\varepsilon_k^{\,i\,k\left(\#V_m-1-\ell_{k',k'',m}\right)\mathbf{p}} = \varepsilon_k^{\,i\,k\left((N_b-1)\,N_b^m-(N_b-1)\,k'-k''\right)\mathbf{p}} = \overline{\varepsilon_k^{\,i\,k\,\ell_{k',k'',m}\,\mathbf{p}}} = \overline{\varepsilon_k^{\,i\,k\left((N_b-1)\,k'+k''\right)\mathbf{p}}} \,, \qquad (\mathcal{R}\,52)$$

where $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k'' \leqslant N_b - 1$, which is also directly connected to the symmetry $\mathcal{S}$ with respect to the vertical line $x = \dfrac{1}{2}$, stated in Property 2.8, on page 16, since the points

$$\left(\frac{(N_b-1)\,N_b^m-j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{(N_b-1)\,N_b^m-j}{(N_b-1)\,N_b^m}\right)\right) \quad \text{and} \quad \left(\frac{j}{(N_b-1)\,N_b^m},\mathcal{W}\left(\frac{j}{(N_b-1)\,N_b^m}\right)\right)$$

are symmetric with respect to the vertical line $x = \dfrac{1}{2}$; see Figure 3, on page 17.

Proposition 4.21 (The Prefractal Cohomology Groups as $\mathbb{Z}$-Modules).

For all $m \in \mathbb{N}$, the cohomology groups H^m – introduced in Proposition 4.13, on page 50, and characterized in Theorem 4.14, on page 50 – can also be viewed as $\mathbb{Z}$-modules (indeed, these groups are abelian); i.e., for all $(k,k') \in \mathbb{Z}^2$ and all $(\varphi,\psi) \in H^m \times H^m$,

$$k\,(\varphi+\psi) = k\,\varphi + k\,\psi \in H^m\,,$$

$$(k+k')\,\varphi = k\,\varphi + k'\,\varphi \in H^m\,.$$

4.2.4 The Polyhedral Measure

We now recall the results we have obtained in [DL24b], where we introduced and constructed a specific polyhedral measure μ (by means of a sequence of polygonal neighborhoods of the Weierstrass Curve), better suited to our polyhedral geometric context than the Hausdorff measure. This polyhedral measure is the weak limit as $m\to\infty$ of discrete measures (or Dirac Combs) μ_m, which will play a key role in obtaining the forthcoming orthogonal decomposition of the $\mathbb{C}$-tensored prefractal cohomology groups in Theorem 4.26, on page 63 below, this decomposition involving, of course, Hilbert spaces; namely, $L^2\left(\Gamma_{\mathcal{W}},\mu_m\right)$, the space of complex-valued functions f on the Weierstrass Curve $\Gamma_{\mathcal{W}}$ such that, for all $m\in\mathbb{N}$ and each vertex X in V_m, $f(X)$ exists.

Definition 4.19 (Sequence of Domains Delimited by the Weierstrass IFD – Polyhedral Neighborhoods of the Weierstrass Curve [DL24b]).

We introduce *the sequence of domains delimited by the Weierstrass IFD*, or *polygonal neighborhood of the Weierstrass Curve* as the sequence $\left(\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\right)_{m\in\mathbb{N}}$ of open, connected polygonal sets $(\mathcal{P}_m\cup\mathcal{Q}_m)_{m\in\mathbb{N}}$, where, for each $m\in\mathbb{N}$, $\mathcal{P}_m$ and $\mathcal{Q}_m$ respectively denote the polygonal sets introduced in Definition 2.5, on page 13; see also Notation 7, on page 13.

Given $m\in\mathbb{N}$, we call $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ the m^{th} *polyhedral neighborhood* (of the Weierstrass Curve $\Gamma_{\mathcal{W}}$).

Property 4.22 (Domain Delimited by the Weierstrass IFD [DL24b]).

We call domain, delimited by the Weierstrass IFD*, the set which is equal to the following limit,*

$$\mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \lim_{m\to\infty}\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)\,,$$

where the convergence is interpreted in the sense of the Hausdorff metric on $\mathbb{R}^2$*; see [DL24b]. In fact, we have that*

$$\mathcal{D}\left(\Gamma_{\mathcal{W}}\right) = \Gamma_{\mathcal{W}}\,.$$

Notation 10 (Lebesgue Measure on $\mathbb{R}^2$).

In the sequel, we denote by $\mu_{\mathcal{L}}$ the Lebesgue measure on $\mathbb{R}^2$.

Definition 4.20 (Power of a Vertex of the Prefractal Graph $\Gamma_{\mathcal{W}_m}$, $m\in\mathbb{N}^\star$, with Respect to the Polygonal Families $\mathcal{P}_m$ and $\mathcal{Q}_m$).

Given a strictly positive integer m, a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ will be said to be:

i. of power one relative to the polygonal family $\mathcal{P}_m$ if X belongs to (to be understood in the sense that X is a vertex of) one and only one N_b-gon $\mathcal{P}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 1$;

ii. of power $\frac{1}{2}$ *relative to the polygonal family* $\mathcal{P}_m$ if X is a common vertex to two consecutive N_b-gons $\mathcal{P}_{m,j}$ and $\mathcal{P}_{m,j+1}$, $0 \leqslant j \leqslant N_b^m - 2$;

iii. of power zero relative to the polygonal family $\mathcal{P}_m$ if X does not belong to (to be understood in the sense that X is not a vertex of) any N_b-gon $\mathcal{P}_{m,j}$,
$0 \leqslant j \leqslant N_b^m - 1$.

Similarly, given $m \in \mathbb{N}$, a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ is said to be:

i. of power one relative to the polygonal family $\mathcal{Q}_m$ if X belongs to (as above, to be understood in the sense that X is a vertex of) one and only one N_b-gon $\mathcal{Q}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 2$;

ii. of power $\frac{1}{2}$ *relative to the polygonal family* $\mathcal{Q}_m$ if X is a common vertex to two consecutive N_b–gons $\mathcal{Q}_{m,j}$ and $\mathcal{Q}_{m,j+1}$, $0 \leqslant j \leqslant N_b^m - 3$;

iii. of power zero relative to the polygonal family $\mathcal{Q}_m$ if X does not belong to (as previously, to be understood in the sense that X is not a vertex of) any N_b-gon $\mathcal{Q}_{m,j}$, $0 \leqslant j \leqslant N_b^m - 2$.

Notation 11. In the sequel, given a strictly positive integer m, the *power of a vertex X of the prefractal graph $\Gamma_{\mathcal{W}_m}$ relative to the polygonal families $\mathcal{P}_m$ and $\mathcal{Q}_m$* will be respectively denoted by

$$p(X, \mathcal{P}_m) \quad \text{and} \quad p(X, \mathcal{Q}_m)\,.$$

Notation 12 (**[DL24b]**).

For any $m \in \mathbb{N}$, and any vertex X of V_m, we set

$$\mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m) = \begin{cases} \dfrac{1}{N_b}\, p(X, \mathcal{P}_m) \displaystyle\sum_{0 \leqslant j \leqslant N_b^m - 1,\, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}(\mathcal{P}_{m,j}) \ , \text{ if } X \notin \mathcal{Q}_m \ , \\ \dfrac{1}{N_b}\, p(X, \mathcal{Q}_m) \displaystyle\sum_{1 \leqslant j \leqslant N_b^m - 2,\, X \text{ vertex of } \mathcal{Q}_{m,j}} \mu_{\mathcal{L}}(\mathcal{Q}_{m,j}) \ , \text{ if } X \notin \mathcal{P}_m \ , \\ \dfrac{1}{2\,N_b} \left\{ p(X, \mathcal{P}_m) \displaystyle\sum_{0 \leqslant j \leqslant N_b^m - 1,\, X \text{ vertex of } \mathcal{P}_{m,j}} \mu_{\mathcal{L}}(\mathcal{P}_{m,j}) + p(X, \mathcal{Q}_m) \displaystyle\sum_{1 \leqslant j \leqslant N_b^m - 2,\, X \text{ vertex of } \mathcal{Q}_{m,j}} \mu_{\mathcal{L}}(\mathcal{Q}_{m,j}) \right\}, \\ \text{if } X \in \mathcal{P}_m \cap \mathcal{Q}_m \,. \end{cases}$$

Theorem 4.23 (**Polyhedral Measure on the Weierstrass IFD [DL24b]**).

We introduce the polyhedral measure on the Weierstrass IFD, *denoted by μ, such that for any continuous function u on the Weierstrass Curve, with the use of Notation 12, on page 58,*

$$\int_{\Gamma_{\mathcal{W}}} u \, d\mu = \lim_{m \to \infty} \varepsilon_m^{m\,(D_{\mathcal{W}} - 2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}(X, \mathcal{P}_m, \mathcal{Q}_m)\, u(X) \ , \qquad (\star)$$

which, thanks to Definition 4.19, on page 57, can also be understood in the following way,

$$\int_{\Gamma_{\mathcal{W}}} u\, d\mu = \int_{\mathcal{D}(\Gamma_{\mathcal{W}})} u\, d\mu\,.$$

In addition, μ is the weak limit as $m \to \infty$ of the following discrete measures *(or Dirac Combs), given, for each $m \in \mathbb{N}$, by*

$$\mu_m = \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X \in \mathcal{P}_m \bigcup \mathcal{Q}_m} \mu^{\mathcal{L}}\,(X, \mathcal{P}_m, \mathcal{Q}_m)\;\delta_X\,, \qquad (\mathcal{R}\,53)$$

where ε_m^m denotes the m^{th} cohomology infinitesimal introduced in Definition 2.8, on page 21, δ_X is the Dirac measure concentrated at X, and we have used Notation 12, on page 58, for $\mu^{\mathcal{L}}\,(X, \mathcal{P}_m, \mathcal{Q}_m)$.

4.2.5 Fractal Hodge Decomposition

Theorem 4.24 (**The $\mathbb{C}$-Tensored Prefractal Cohomology Groups – Associated Generators**).

For all $m \in \mathbb{N}$, we consider the tensor product of the cohomology group H^m by $\mathbb{C}$:

$$H^m \otimes \mathbb{C} = \left\{ z\,\varphi \,:\, \varphi \in H^m,\, z \in \mathbb{C} \right\}\,.$$

Note that the tensor product $H^m \otimes \mathbb{C}$ is thus equipped with a vector space structure on the field $\mathbb{C}$; i.e., it is a complex vector space This enables us, in particular, to apply the natural transfer operator $\mathcal{L}_{\mathcal{W}}$ introduced in Definition 4.14, on page 46, to $H^m \otimes \mathbb{C}$.

From a strict algebraic point of view, we should really write $H^m \otimes_{\mathbb{Z}} \mathbb{C}$, and similarly $H^\star \otimes_{\mathbb{Z}} \mathbb{C}$ instead of $H^\star \otimes \mathbb{C}$ just below. For notational simplicity, we will not do so in the sequel.

We also introduce

$$H^\star \otimes \mathbb{C} = \bigoplus_{m=0}^{\infty} H^m \otimes \mathbb{C}\,,$$

which, as is the case at the end of Theorem 4.19, on page 55, has to be understood in the sense of the inductive limit *of the sequence of the $\mathbb{C}$-tensored cohomology groups $\left(H^m \otimes C\right)_{m \in \mathbb{N}}$.*

Along the lines of the generators of the (additive) cohomology groups introduced in Proposition 4.17, on page 52, we introduce, for any integer $m \geqslant 1$, the generators *of the tensor product $H^m \otimes \mathbb{C}$ as*

$$\varepsilon_k^{k\,((2-D_{\mathcal{W}})+i\,k\,\ell_{k,j,m}\,\mathbf{p})} = \varepsilon_k^{k\left((2-D_{\mathcal{W}})+i\,k\,\ell_{k',k",j,m}\,\mathbf{p}\right)}\,,$$

with $0 \leqslant k \leqslant m$, $0 \leqslant k' \leqslant N_b^m - 1$, $0 \leqslant k" \leqslant N_b - 1$ and $\ell_{k',k",j,m} = (N_b - 1)\,k' + k"$; see also Theorem 4.14 above, on page 50, and Definition 2.9, on page 22.

In our quest for an orthogonal decomposition of the complex cohomology – the natural extension to our fractal context, of the classical Hodge decomposition (see Theorem 4.3, on page 42), we next introduce a *polarization operator* on the $\mathbb{C}$-tensored cohomology groups; it will provide a suitable analog, in our context, of Deligne's polarization used in pure Hodge theory, for example.

Definition 4.21 (**Alternate (Tensor) Product of a Pair of Continuous Functions on** $\Gamma_{\mathcal{W}}$).

Given $m \in \mathbb{N}^\star$, along with a pair of continuous functions (f,g) on $\Gamma_{\mathcal{W}}$, satisfying relation ($\mathcal{R}$44), on page 51, we introduce their *alternate tensor product* $(f \otimes g)^a \in \mathbb{C}$, as the function defined, for all vertices $M_{j,m} \in V_m$, with $0 \leqslant j \leqslant \#V_m - 1$, by

$$(f \otimes g)^a \left(M_{j,m}\right) =$$

$$= \sum_{j'=0}^{\#V_m-1} \sum_{k'=0}^{N_b^m-1} \sum_{k=0}^{m} \left(1-\delta_{\ell_{j,m},(N_b-1)\,k'}\right) c_k\left(f, M_{j,m}\right) c_k\left(g, M_{\#V_m-1-j,m}\right) \varepsilon_k^{2\,k\,(2-D_{\mathcal{W}})+i\left(k\,\ell_{k,j,m}+k\,\ell_{k\#V_m-1-j,m}\right)\mathbf{p}},$$

($\mathcal{R}$ 54)

where, for $(q,q') \in, \mathbb{N}^2$, $\delta_{q,q'}$ denotes the Kronecker delta: $\delta_{q,q} = 1$ and $\delta_{q,q'} = 0$ if $q' \neq q$.

Remark 4.8 (**About the Alternate (Tensor) Product of Definition 4.21**).

The sum over the indices $k' \in \left\{0,\ldots,N_b^m - 1\right\}$ in relation ($\mathcal{R}$54) is to ensure that the vertices $M_{j,m}$, for $j \in \{0,\ldots,\#V_m - 1\}$, belong to $V_m \setminus V_{m-1}$ (i.e., they are not junction vertices; see part *iv*. of Property 2.6, on page 12, along with Definition 2.9, on page 22), which will play a key role in the forthoming orthogonal decomposition of the cohomology groups; see Theorem 4.26, on page 63.

Theorem 4.25 (**A Polarization Operator on** $\left(H^m \otimes \mathbb{C}\right)$**, and the Associated Fractal Hodge–Riemann Relations (Fractal Counterpart of the Operator Given in Remark 4.3, on page 44, and of the Classical Hodge-Riemann Relations, given in Theorem 4.8, on page 44)**).

For any $m \in \mathbb{N}^\star$, we introduce the polarization operator $\mathbf{Q}_m$ *from* $\left(H^m \otimes \mathbb{C}\right) \times \left(H^m \otimes \mathbb{C}\right)$ *to* $\mathbb{C}$ *such that, for all* $(\varphi,\psi) \in \left(H^m \otimes \mathbb{C}\right) \times \left(H^m \otimes \mathbb{C}\right)$ *– where φ and ψ respectively involve the restrictions to V_m of continuous functions f and g on $\Gamma_{\mathcal{W}}$, we have that, since μ_m is a bounded measure,*

$$\mathbf{Q}_m(\varphi,\psi) = \int_{\mathcal{D}(\Gamma_{\mathcal{W}_m})} (f\otimes g)^a \, d\mu_m = \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{X\in\mathcal{P}_m \bigcup \mathcal{Q}_m} (f\otimes g)^a(X)\, \mu^{\mathcal{L}}\left(X,\mathcal{P}_m,\mathcal{Q}_m\right) =$$

$$= \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{j=0}^{\#V_m-1} \mu^{\mathcal{L}}\left(M_{jm},\mathcal{P}_m,\mathcal{Q}_m\right) \times$$

$$\sum_{k'=0}^{N_b^m-1} \sum_{k=0}^{m} \left(1-\delta_{\ell_{j,m},(N_b-1)\,k'}\right) c_k\left(f, M_{j,m}\right) c_k\left(g, M_{\#V_m-1-j,m}\right) \varepsilon_k^{2\,k\,(2-D_{\mathcal{W}})+i\,k\left(\ell_{k,j,m}+\ell_{k,\#V_m-1-j,m}\right)\mathbf{p}},$$

($\mathcal{R}$ 55)

where $\mathcal{D}\left(\Gamma_{\mathcal{W}_m}\right)$ has been introduced in Definition 4.19 above, on page 57.

The alternate tensor product $(f\otimes g)^a$ has been introduced in Definition 4.21 above, on page 60.

Clearly, $\mathbf{Q}_m$ *is a* $\mathbb{C}$*-bilinear map on* $\left(H^m\otimes\mathbb{C}\right)\times\left(H^m\otimes\mathbb{C}\right)$*.*

The polarization operator $\mathbf{Q}_m$ *is* $\mathbb{C}$*-bilinear, positive definite on* $\left(H^m\otimes\mathbb{C}\right)^{\mathcal{H},\star}\times\left(H^m\otimes\mathbb{C}\right)^{\mathcal{H},\star}$*, where* $\left(H^m\otimes\mathbb{C}\right)^{\mathcal{H},\star}$ *is the subspace of* $\left(H^m\otimes\mathbb{C}\right)$ *comprised of antisymmetric maps which involve complex-valued continuous functions* f *on* $\Gamma_{\mathcal{W}}$ *such that, for any vertex* $M_{j,m}\in V_m$*,*

$$f\left(M_{j,m}\right)=\sum_{k=0}^{m}c_k\left(f,M_{j,m}\right)\varepsilon_k^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,j,m}\,\mathbf{p}}\,,$$

and where, for $0\leqslant k\leqslant m$*, the complex coefficients* $c_k\left(f,M_{j,m}\right)\varepsilon_k^{i\,k\,\ell_{k,j,m}\,\mathbf{p}}$ *satisfy the following Hodge Diamond star relation*

$$c_k\left(f,M_{(N_b-1)\,N_b^m-j,m}\right)\varepsilon_k^{i\,k\,\ell_{k,(N_b-1)\,N_b^m-j,m}\,\mathbf{p}}=\overline{c_k\left(f,M_{j,m}\right)\varepsilon_k^{i\,k\,\ell_{k,j,m}\,\mathbf{p}}}\,. \qquad (\mathcal{R}\,56)$$

This property constitutes a fractal analog *of the classital Hodge–Riemann relations, given in Theorem 4.8, on page 44.*

In addition, since, for all $m\in\mathbb{N}^\star$*,* $V_{m-1}\subseteq V_m$ *and hence also,* $H^{m-1}\subseteq H^m$ *(see [DL22b]), as well as* $H^{m-1}\otimes\mathbb{C}\subseteq H^m\otimes\mathbb{C}$*, we can extend* $\mathbf{Q}_m$ *to* $\left(H^m\otimes\mathbb{C}\right)\times\left(H^{m-1}\otimes\mathbb{C}\right)$ *by letting, for all* (φ,ψ) *in* $\left(H^m\otimes\mathbb{C}\right)\times\left(H^{m-1}\otimes\mathbb{C}\right)$ *– where, this time,* φ *and* ψ *respectively involve the restrictions to* V_m *and* V_{m-1} *of complex-valued continuous functions* f *and* g *on* $\Gamma_{\mathcal{W}}$*, and where, for all vertices* $X\in V_m\setminus V_{m-1}$*,*

$$g(X)=0\,.$$

This immediately ensures, for all $(\varphi,\psi)\in\left(H^m\otimes\mathbb{C}\right)\times\left(H^{m-1}\otimes\mathbb{C}\right)$*, the orthogonality relation*

$$\mathbf{Q}_m\,(\varphi,\psi)=0\,.$$

The proof of Theorem 4.25 is given on page 61, just after Remark 4.9.

Remark 4.9. The fact that the polarization operator $\mathbf{Q}_m$ is positive definite on the complex subspace $\left(H^m\otimes\mathbb{C}\right)^{\mathcal{H},\star}$ is essential, since our main interest, in the cohomology, is the operator induced by the global polyhedral effective zeta function, which belongs to $\left(H^m\otimes\mathbb{C}\right)^{\mathcal{H},\star}$; see also Corollary 2.22, on page 29.

Proof. (Of Theorem 4.25, on page 60)

The $\mathbb{C}$-bilinearity of the polarization operator $\mathbf{Q}_m$ directly follows from the expression in relation ($\mathcal{R}$55), on page 60.

We can check that $\mathbf{Q}_m$ is positive definite on the complex subspace $\left(H^m\otimes\mathbb{C}\right)^{\mathcal{H},\star}$ of antisymmetric maps which involve continuous functions f on $\Gamma_{\mathcal{W}}$ such that, for any vertex $M_{j,m}\in V_m$,

$$f\left(M_{j,m}\right)=\sum_{k=0}^{m}c_k\left(f,M_{j,m}\right)\varepsilon_k^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,j,m}\,\mathbf{p}}\quad,\quad M_{\star,m}\in V_m\,,$$

where, for $0\leqslant k\leqslant m$, the complex coefficients $c_k\left(f,M_{j,m}\right)\varepsilon_k^{i\,k\,\ell_{k,j,m}\,\mathbf{p}}$ satisfy the following Hodge Diamond star relation,

$$c_k\left(f, M_{(N_b-1)\,N_b^m-j,m}\right)\,\varepsilon_k^{\,i\,k\,\ell_{k,(N_b-1)\,N_b^m-j,m}\,\mathbf{p}} = \overline{c_k\left(f, M_{j,m}\right)\,\varepsilon_k^{\,i\,k\,\ell_{k,j,m}\,\mathbf{p}}}\,.$$

Indeed, for any $\varphi \in \left(H^m \otimes \mathbb{C}\right)^{\mathcal{H},\star}$, we have that

$$\begin{aligned}
\mathbf{Q}_m(\varphi,\varphi) &= \int_{\mathcal{D}(\Gamma_{\mathcal{W}_m})} (f\otimes f)^a\, d\mu_m \\
&= \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{j=0}^{\#V_m-1} \mu^{\mathcal{L}}\left(M_{j,m},\mathcal{P}_m,\mathcal{Q}_m\right)\times \\
&\quad \sum_{k'=0}^{N_b^m-1}\sum_{k=0}^{m}\left(1-\delta_{\ell_{j,m},(N_b-1)\,k'}\right) c_k\left(f,M_{j,m}\right) c_k\left(f,M_{\#V_m-1-j,m}\right)\varepsilon_k^{\,2\,k\,(2-D_{\mathcal{W}})+i\,k\,\left(\ell_{k,j,m}+\ell_{k,\#V_m-1-j,m}\right)\mathbf{p}} \\
&= \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{j=0}^{\#V_m-1} \mu^{\mathcal{L}}\left(M_{j,m},\mathcal{P}_m,\mathcal{Q}_m\right)\times \\
&\quad \sum_{k'=0}^{N_b^m-1}\sum_{k=0}^{m}\left(1-\delta_{\ell_{j,m},(N_b-1)\,k'}\right) \left|c_k\left(f,M_{j,m}\right)\right|^2 \varepsilon_k^{\,2\,k\,(2-D_{\mathcal{W}})+i\,k\,\left(\ell_{k,j,m}+\ell_{k,\#V_m-1-j,m}\right)\mathbf{p}} \\
&= \varepsilon_m^{m\,(D_{\mathcal{W}}-2)} \sum_{j=0}^{\#V_m-1} \mu^{\mathcal{L}}\left(M_{j,m},\mathcal{P}_m,\mathcal{Q}_m\right)\sum_{k'=0}^{N_b^m-1}\sum_{k=0}^{m}\left(1-\delta_{\ell_{j,m},(N_b-1)\,k'}\right) \left|c_k\left(f,M_{j,m}\right)\right|^2 \varepsilon_k^{\,2\,k\,(2-D_{\mathcal{W}})}
\end{aligned} \qquad (\mathcal{R}\,57)$$

since relation ($\mathcal{R}$51), on page 56, implies that

$$c_k\left(f, M_{(N_b-1)\,N_b^m-j,m}\right)\,\varepsilon_k^{\,i\,k\,\ell_{k,(N_b-1)\,N_b^m-j,m}\,\mathbf{p}} = \overline{c_k\left(f, M_{j,m}\right)\,\varepsilon_k^{\,i\,k\,\ell_{k,j,m}\,\mathbf{p}}}\,.$$

The last part of the theorem; i.e., for all $(\varphi,\psi) \in (H - \otimes\mathbb{C})\times(H`m-1\otimes\mathbb{C})$, the orthogonality relation

$$\mathbf{Q}_m(\varphi,\psi) = 0\,,$$

directly follows from the definition of the alternate tensor product; see Definition 4.21, on page 60, along with Remark 4.8, on page 60.

□

Notation 13. Henceforth, given $m \in \mathbb{N}^\star$, we will also use the following notation:

i. $\left(H^m \otimes \mathbb{C}\right) \setminus \left(H^{m-1} \otimes \mathbb{C}\right)$ to denote the complex vector space consisting of all functions φ in $\left(H^m \otimes \mathbb{C}\right)$, the expression of which only involves generators of the form

$$\varepsilon_m^{m\,(2-D_{\mathcal{W}})}\,\varepsilon^{\,i\,m\,\ell_{k',k",j,m}\,\mathbf{p}} = \varepsilon_m^{\,m\,(2-D_{\mathcal{W}})+i\,m\,\ell_{k',k",j,m}\,\mathbf{p}}\,,$$

with $0 \leqslant k' \leqslant N_b^m-1$, $0 \leqslant k" \leqslant N_b-1$.

In our present context, the notation $\left(H^m \otimes \mathbb{C}\right) \setminus \left(H^{m-1} \otimes \mathbb{C}\right)$ thus corresponds to the orthogonal of $H^{m-1} \otimes \mathbb{C}$ in $H_m \otimes \mathbb{C}$ with respect to the inner product $\mathbf{Q}_m$ on the Hilbert space $L^2\left(H^m \otimes \mathbb{C}, \mu_m\right)$, introduced in Theorem 4.25, on page 60. In particular, it is a (complex) Hilbert space itself.

ii. Given $0 \leqslant k \leqslant m$, $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k" \leqslant N_b - 1$, $\left(\left(H^m \otimes \mathbb{C}\right)\right)^{k',k"}$ to denote the complex vector space consisting of all functions φ in $\left(H^m \otimes \mathbb{C}\right)$, the expression of which only involves generators of the form

$$\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,\varepsilon_k^{i\,k\,\ell_{k',k",j,m}\,\mathbf{p}} = \varepsilon^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k',k",j,m}\,\mathbf{p}}\,.$$

iii. Given $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k" \leqslant N_b - 1$, $\left(\left(H^m \otimes \mathbb{C}\right) \setminus \left(H^{m-1} \otimes \mathbb{C}\right)\right)^{k',k"}$ to denote the complex vector space consisting of all functions φ in $\left(H^m \otimes \mathbb{C}\right)$, the expression of which only involves generators of the form

$$\varepsilon_m^{m\,(2-D_{\mathcal{W}})}\,\varepsilon_m^{i\,m\,\ell_{k',k",j,m}\,\mathbf{p}} = \varepsilon_m^{m\left((2-D_{\mathcal{W}})+i\,m\,\ell_{k',k",j,m}\,\mathbf{p}\right)}\,.$$

Here too, much as in *i.*, each of the complex vector space introduced in part *ii.* and *iii.* are complex Hilbert spaces.

Theorem 4.26 (A First Orthogonal Hodge Decomposition of the Cohomology Groups (Fractal Analog of Theorem 4.3, on page 42)).

Given $m \in \mathbb{N}^\star$, we have the following, orthogonal Hodge decomposition:

$$\left(H^m \otimes \mathbb{C}\right) = \bigoplus_{k=0}^{m}\left(\left(H^k \otimes \mathbb{C}\right) \setminus \left(H^{k-1} \otimes \mathbb{C}\right)\right) = \bigoplus_{k=0}^{m}\bigoplus_{k'=0}^{N_b^m-1}\bigoplus_{k"=0}^{N_b-1}\left(\left(H^k \otimes \mathbb{C}\right) \setminus \left(H^{k-1} \otimes \mathbb{C}\right)\right)^{k',k"}\,, \quad (\mathcal{R}\,58)$$

where, for each $k \in \{0,\dots,m\}$, $\left(H^k \otimes \mathbb{C}\right) \setminus \left(H^{k-1} \otimes \mathbb{C}\right)$ is the orthogonal of $H^{k-1} \otimes \mathbb{C}$ in $H^k \otimes \mathbb{C}$ with respect to $\mathbf{Q}_m$ (see Theorem 4.25, on page 60), and where, for $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k" \leqslant N_b - 1$, the notation $\left(\left(H^m \otimes \mathbb{C}\right) \setminus \left(H^{m-1} \otimes \mathbb{C}\right)\right)^{k',k"}$ has been introduced in part i. of Notation 13 above, on page 62.

Proof. This result directly follows from the properties of the polarization operator, given in Theorem 4.25, on page 60.

□

Theorem 4.27 (Poincaré Duality – Conjugation Relation Associated with $\left(H^m \otimes \mathbb{C}\right) \setminus \left(H^{m-1} \otimes \mathbb{C}\right)$ (see Notation 13 just above, on page 62) (Fractal Analog of Theorem 4.4, on page 42).

Note that, given any $m \in \mathbb{N}^\star$, we have that

$$\left(H^m \otimes \mathbb{C}\right) \setminus \left(H^{m-1} \otimes \mathbb{C}\right) = \bigoplus_{k'=0}^{N_b^m-1}\bigoplus_{k"=0}^{N_b-1}\left(\left(H^m \otimes \mathbb{C}\right) \setminus \left(H^{m-1} \otimes \mathbb{C}\right)\right)^{k',k"}\,. \quad (\mathcal{R}\,59)$$

Thanks to the Hodge Diamond Star relation given in Theorem 4.20, on page 56, still given $m \in \mathbb{N}^\star$, $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k" \leqslant N_b - 1$, we have that

$$\left(\left(H^m \otimes \mathbb{C}\right) \setminus \left(H^{m-1} \otimes \mathbb{C}\right)\right)^{\#V_m - 1 - (N_b - 1)\, k', -k"} = \overline{\left(\left(H^m \otimes \mathbb{C}\right) \setminus \left(H^{m-1} \otimes \mathbb{C}\right)\right)^{k', k"}} , \qquad (\mathcal{R}\, 60)$$

where the equality in relation ($\mathcal{R}$60) really stands for an isomorphism of complex Hilbert spaces.

Remark 4.10 (**Connection with the Geometry of the Prefractal Sequence**).

Also, $\left(H^m \otimes \mathbb{C}\right) \setminus \left(H^{m-1} \otimes \mathbb{C}\right)$ can be viewed as a $(\#V_m - \#V_{m-1} + 1)$-dimensional complex Hilbert space.

Moreover, the orthogonal decomposition in relation ($\mathcal{R}$59), on page 63, corresponds to the following disjoint decomposition of the set of vertices V_m, given by

$$V_m = V_{m_0} \bigcup \left(\bigcup_{k=m_0+1}^{m} V_k \setminus V_{k-1} \right) ,$$

where, for $m_0 + 1 \leqslant k \leqslant m$, $V_k \setminus V_{k-1}$ now truly denotes the set-theoretic difference

$$V_k \setminus V_{k-1} = V_k \bigcap V_{k-1}^c ,$$

with V_{k-1}^c standing for the complement of V_{k-1} in V_k. See also Figure 6, on page 65, along with Figure 7, on page 66.

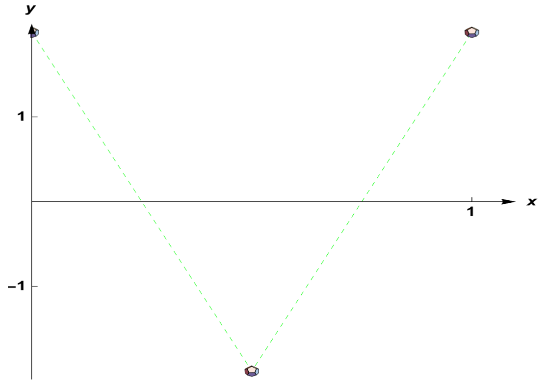
In addition, due to the exact correspondance between vertices of the polygons at consecutive steps $m-1$, m (since, for $0 \leqslant k'' \leqslant N_b - 1$, the $(k'')^{th}$ vertex of the polygon $\mathcal{P}_{m,k'}$, is the image of the $(k'')^{th}$ vertex of the polygon $\mathcal{P}_{m-1,(N_b-1)\,(k'-i\,(N_b-1)\,N_b^{m-1})}$ under the map T_i, where $i \in \{0, \ldots, N_b - 1\}$ is arbitrary (see Property 2.11, on page 18), we also have the associated correspondence (i.e., isomorphism) between $\left(H^{m-1} \otimes \mathbb{C}\right)^{(N_b-1)\left(k'-i\,(N_b-1)\,N_b^{m-1}\right),k"}$ and $\left(H^{m-1} \otimes \mathbb{C}\right)^{k',k"}$. This correspondence can be interpreted as a quasi-periodicity property of the cohomology.

Theorem 4.28 (**The Action of the Natural Transfer Operator $\mathcal{L}_{\mathcal{W}}$ on the Cohomology Groups**).

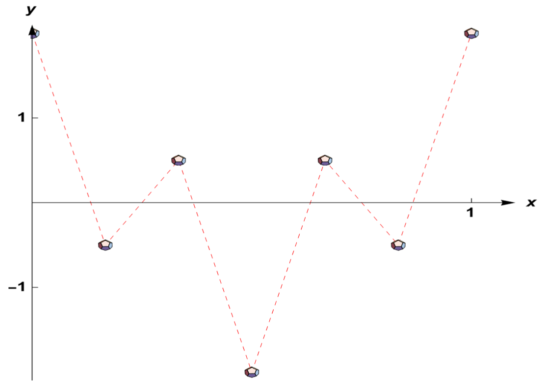
Given $m \in \mathbb{N}^\star$, the natural transfer operator $\mathcal{L}_{\mathcal{W}}$ introduced in Definition 4.14, on page 46, acts on the tensor product $H^m \otimes \mathbb{C}$ (see Theorem 4.24 above, on page 59), in the following way (see also Figure 8, on page 67),

$$\mathcal{L}_{\mathcal{W}}\left(H^m \otimes \mathbb{C}\right) = H^{m+1} \otimes \mathbb{C} . \qquad (\mathcal{R}\, 61)$$

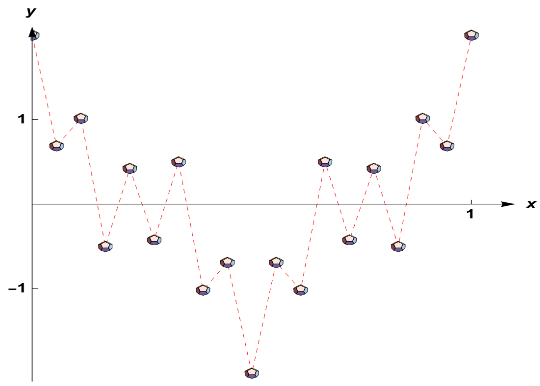
In this light, the natural transfer operator $\mathcal{L}_{\mathcal{W}}$ can be interpreted as a differential, *the natural extension to our fractal context of the classical (exterior) derivative d (see Definition 4.2 above, on page 37). In the same manner, the dual transfer operator $\mathcal{L}_{\mathcal{W}}^{\#}$ (see Theorem 4.10, on page 46, and recall that $\varepsilon = \dfrac{1}{N_b}$) can be interpreted as* the conjugate $\bar{d}$ *of the aforementioned classical (exterior) derivative d.*



(a) **The set of vertices V_0 (of the prefractal graph $\Gamma_{\mathcal{W}_0}$), in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**



(b) **The sets of vertices V_1 (of the prefractal graph $\Gamma_{\mathcal{W}_1}$), in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**



(c) **The set of vertices V_2 (of the prefractal graph $\Gamma_{\mathcal{W}_2}$), in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

Figure 6: **The sets of vertices V_0, V_1, V_2, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

Accordingly, $\mathcal{L}_{\mathcal{W}}$ and $\mathcal{L}^{\#}_{\mathcal{W}}$ play the role, in our fractal context, of $\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial \bar{z}}$, respectively.

Note that, in terms of the generators *of H^m (see Proposition 4.17, on page 52), namely,*

$$\varepsilon_k^{k\left((2-D_{\mathcal{W}})+i\,k\,\ell_{k,j,m}\,\mathbf{p}\right)}\,,\quad \text{for } 0 \leqslant k \leqslant m\,,$$

and because

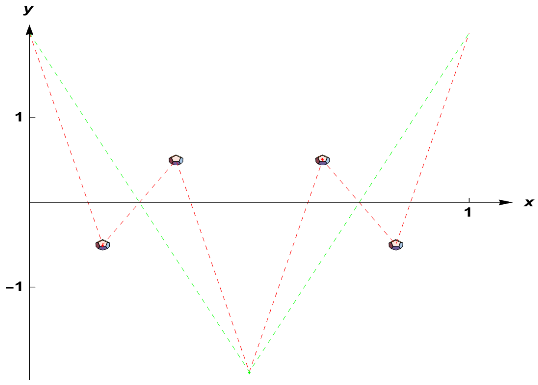
$$\varepsilon_{k+1}^{k+1} = \frac{1}{N_b}\,\varepsilon_k^k = \varepsilon\,\varepsilon_k^k\,,$$

along with the fact that, for $\ell_{k',j',m+1} \in \mathbb{Z}$,

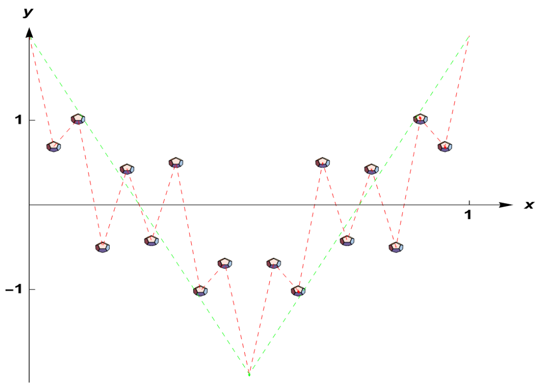
$$\varepsilon_{k+1}^{i\,(k+1)\,\ell_{k',j',m+1}\,\mathbf{p}} = \varepsilon^{i\,(k+1)\,\ell_{k',j',m+1}\,\mathbf{p}}\,\varepsilon_k^{i\,(k+1)\,\ell_{k',j',m+1}\,\mathbf{p}} = \varepsilon_k^{i\,\ell_{k',j',m+1}\,\mathbf{p}}\,\varepsilon_k^{i\,k\,\ell_{k',j',m+1}\,\mathbf{p}}\,,$$

since

$$\varepsilon_{k+1}^{i\,(k+1)\,\ell_{k',j',m+1}\,\mathbf{p}} = e^{-2\,i\,\pi\,(k+1)\,\ell_{k',j',m+1}\,\frac{\ln\varepsilon}{\ln\varepsilon}} = 1\,,$$



(a) **The set of vertices $V_1 \setminus V_0$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**



(b) **The set of vertices $V_2 \setminus V_0$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

Figure 7: **The set of vertices $V_1 \setminus V_0$ and $V_2 \setminus V_0$, in the case when $\lambda = \frac{1}{2}$ and $N_b = 3$.**

the action of $\mathcal{L}_{\mathcal{W}}$ on $H^m \otimes \mathbb{C}$ amounts to the multiplication by $\varepsilon^{(2-D_{\mathcal{W}})}\,\varepsilon_k^{i\,\ell\,\mathbf{p}}$, for a suitable $\ell \in \mathbb{Z}$.

And similarly, the action of $\mathcal{L}_{\mathcal{W}}^{\#}$ on $H^m \otimes \mathbb{C}$ amounts to the multiplication by $\varepsilon^{(2-D_{\mathcal{W}})}\,\varepsilon_k^{-i\,\ell\,\mathbf{p}}$, for a suitable $\ell \in \mathbb{Z}$.

Also, for $0 \leqslant k' \leqslant N_b^m - 1$, $0 \leqslant k" \leqslant N_b - 1$ and $0 \leqslant i \leqslant N_b - 1$ arbitrary, thanks to the correspondence between

$$\left(H^{m-1} \otimes \mathbb{C}\right)^{(N_b-1)\left(k'-i\,(N_b-1)\,N_b^{m-1}\right),k"} \quad \text{and} \quad \left(H^{m-1} \otimes \mathbb{C}\right)^{k',k"}$$

(see Theorem 4.24 above, on page 59, along with Notation 13, on page 62), we have that (see Figure 8, on page 67),

$$\mathcal{L}_{\mathcal{W}}\left(\left(H^{m-1} \otimes \mathbb{C}\right)^{N_b-1)\left(k'-i\,(N_b-1)\,N_b^{m-1}\right),k"}\right) = \left(H^{m-1} \otimes \mathbb{C}\right)^{k',k"} . \qquad (\mathcal{R}\,62)$$

And similarly for $\mathcal{L}_{\mathcal{W}}^{\#}$.

Proof. This simply follows from the fact that $V_{m+1} = \bigcup_{i=0}^{N_b-1} T_i\left(V_m\right)$ (see Definition 2.3, on page 10).

$$
\begin{array}{c}
H^m \\
\Big\downarrow \delta_{m,m+1} \\
H_{m+1}
\end{array}
\qquad\qquad
\begin{array}{c}
H^m \otimes \mathbb{C} \\
\Big\downarrow \mathcal{L}_{\mathcal{W}} \\
H_{m+1} \otimes \mathbb{C}
\end{array}
$$

Figure 8: **Respective (and corresponding) actions of the differential $\boldsymbol{\delta_{m,m+1}}$ and the natural transfer operator $\boldsymbol{\mathcal{L}_{\mathcal{W}}}$.**

Indeed, since the m^{th} cohomology group H^m is comprised of the set of the restrictions to V_m of $(m, N_b^m + 1)$-fermions, i.e., the restrictions to $V_m^{N_b^m+1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of continuous functions f on $\Gamma_{\mathcal{W}}$, such that, for any $j \in \{0, \ldots, \#V_m - 1\}$, we have that

$$f\left(M_{j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \varepsilon_k^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,j,m}\,\mathbf{p}} \qquad (\mathcal{R}\,63)$$

(with $\mathbf{p} = \dfrac{2\,\pi}{\ln N_b}$ denoting the oscillatory period introduced in [DL22a]), the action of $\mathcal{L}_{\mathcal{W}}$ on $H^m \otimes \mathbb{C}$ gives the set of the restrictions to V_{m+1} of $(m+1, N_b^{m+1} + 1)$-fermions, i.e., the restrictions to $V_{m+1}^{N_b^m+1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_{m+1}), involving the restrictions to V_{m+1} of continuous functions f on $\Gamma_{\mathcal{W}}$, such that, for any $j \in \{0, \ldots, \#V_{m+1} - 1\}$, we have that

$$f\left(M_{j,m+1}\right) = \sum_{k=0}^{m+1} c_k\left(f, M_{j,m+1}\right) \varepsilon_k^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,j,m+1}\,\mathbf{p}} \;;$$

i.e., it yields $H^{m+1} \otimes \mathbb{C}$.

□

Theorem 4.29 (**A Second Hodge Decomposition of the Prefractal Cohomology Groups (Fractal Counterpart of Theorem 4.3, on page 42)**).

For all $m \in \mathbb{N}$, and any $j' \in \left\{0, \ldots, \dfrac{\#V_m - 1}{2}\right\}$, we introduce the sets

$$H^{m,j'} \quad \textit{and} \quad H^{m,\#V_m - j'}$$

as the set of the restrictions to V_m of $(m, N_b^m + 1)$-fermions, i.e., the restrictions to $V_m^{N_b^m+1}$ of antisymmetric maps on $\Gamma_{\mathcal{W}}$, with $N_b^m + 1$ variables (corresponding to the vertices of V_m), involving the restrictions to V_m of continuous functions f on $\Gamma_{\mathcal{W}}$, such that, for any $j \in \left\{0, \ldots, j'\right\}$, we have that

$$f\left(M_{j,m}\right) = \sum_{k=0}^{m} c_k\left(f, M_{j,m}\right) \varepsilon_k^{k\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,j,m}\,\mathbf{p}} \,, \qquad (\mathcal{R}\,64)$$

where $\mathbf{p}$ denotes the oscillatory period introduced in [DL22a].

We then have that

$$H^m = \bigoplus_{j'=0}^{\frac{\#V_m-1}{2}} \left(H^{m,j'} \oplus H^{m,\#V_m-j'}\right) . \qquad (\mathcal{R}\,65)$$

Upon tensoring by $\mathbb{C}$, we deduce at once from relation ($\mathcal{R}$65) the following orthogonal decomposition of the (finite-dimensional) complex Hilbert space $H^m \otimes \mathbb{C}$:

$$H^m \otimes \mathbb{C} = \bigoplus_{j'=0}^{\frac{\#V_m-1}{2}} \left(H^{m,j'} \otimes \mathbb{C} \oplus H^{m,\#V_m-j'} \otimes \mathbb{C} \right) . \qquad (\mathcal{R}\,66)$$

Proof. Relation ($\mathcal{R}$65) directly follows from the fact that

$$V_m = \left\{ M_{j,m} \, , \, 0 \leqslant j \leqslant \#V_m - 1 \right\} = \left\{ M_{j',m} \, , \, 0 \leqslant j' \leqslant \frac{\#V_m - 1}{2} \right\} \bigcup \left\{ M_{j',m} \, , \, \frac{\#V_m - 1}{2} \leqslant j' \leqslant \#V_m - 1 \right\} ,$$

where the subsets involved in the second equality are pairwise disjoint.

As was mentioned above, relation ($\mathcal{R}$66) then follows at once from relation ($\mathcal{R}$65) upon tensoring by $\mathbb{C}$.

□

Theorem 4.30 (**Orthogonal Hodge Decomposition of the Total Cohomology**).

The Fractal Complex of $\Gamma_{\mathcal{W}}$, $H^\star = H^\bullet \left(\mathcal{F}^\bullet \left(\Gamma_{\mathcal{W}}, \mathbb{C} \right) , \delta^\bullet \right)$ (see Theorem 4.19 above, on page 55), can be decomposed as follows,

$$H^\star = \bigoplus_{m=0}^{\infty} \bigoplus_{j'=0}^{\frac{\#V_m-1}{2}} \left(H^{m,j'} \oplus H^{m,\#V_m-j'} \right) , \qquad (\mathcal{R}\,67)$$

or, else,

$$H^\star = \bigoplus_{m=0}^{\infty} \bigoplus_{m'=0}^{\infty} \bigoplus_{j'=0}^{\frac{\#V_m-1}{2}} \left(H^{m,j'} \oplus H^{m+m',\#V_{m+m'}-(\#V_{m+m'}-j)\,(N_b-2)^{m'}-j'} \right) . \qquad (\mathcal{R}\,68)$$

Those two decompositions (which should be interpreted as inductive limits, much as towards the end of Theorem 4.19, on page 55) also ensure the resulting ($\mathbb{C}$-tensored) orthogonal decompositions of the (infinite dimensional and separable) complex Hilbert space $H^\star \otimes \mathbb{C}$, with respect to the inner product defined by $\mathbf{Q}_m$ (see Theorem 4.25, on page 60),

$$H^\star \otimes \mathbb{C} = \bigoplus_{m=0}^{\infty} \bigoplus_{j'=0}^{\frac{\#V_m-1}{2}} \left(\left(H^{m,j'} \otimes \mathbb{C} \right) \oplus \left(H^{m,\#V_m-j'} \otimes \mathbb{C} \right) \right) , \qquad (\mathcal{R}\,69)$$

or, else,

$$\begin{aligned} H^\star \otimes \mathbb{C} \quad &= \quad \bigoplus_{m=0}^{\infty} \bigoplus_{m'=0}^{\infty} \bigoplus_{j'=0}^{\frac{\#V_m-1}{2}} \left(\left(H^{m,j'} \otimes \mathbb{C} \right) \oplus \left(H^{m+m',\#V_{m+m'}-(\#V_{m+m'}-j)\,(N_b-2)^{m'}-j'} \right) \otimes \mathbb{C} \right) \\ &= \bigoplus_{m=0}^{\infty} \bigoplus_{m'=0}^{\infty} \bigoplus_{j'=0}^{\frac{\#V_m-1}{2}} \left(\left(H^{m,j'} \otimes \mathbb{C} \right) \oplus \left(\left(\mathcal{L}_{\mathcal{W}}^{m'} \left(H^m \otimes \mathbb{C} \right) \right)^{\#V_{m+m'}-(\#V_{m+m'}-j)\,(N_b-2)^{m'}-j'} \right) \otimes \mathbb{C} \right) , \end{aligned}$$

($\mathcal{R}\,70$)

where, for any $m' \in N$, $\mathcal{L}_{\mathcal{W}}^{m'}$ denotes the $(m')^{th}$ iterate of the transfer operator $\mathcal{L}$.

Proof. Since, by definition, $H^\star = \bigoplus_{m=0}^{\infty} H^m$, the first decomposition, given in relation ($\mathcal{R}$67), directly follows from Theorem 4.29, on page 67.

As for the second decomposition, given in relation ($\mathcal{R}$68), we obtain it thanks to the symmetry, with respect to the vertical line $x = \frac{1}{2}$, of the vertices

$$M_{j',m} \quad \text{and} \quad M_{\#V_{m+m'}-(\#V_{m+m'}-j')\,(N_b-2)^{m'}-j',m+m'}\,,$$

for $0 \leqslant j' \leqslant \dfrac{(N_b-1)\,N_b^m}{2}$ (see also the equivalent property in part *ii.* of Theorem 4.9, on page 45).

Relations ($\mathcal{R}$69) and ($\mathcal{R}$70) then follow at once from relation ($\mathcal{R}$67) and ($\mathcal{R}$68), respectively, upon tensoring by $\mathbb{C}$.

□

Proposition 4.31 (**Operators Induced by the Natural Transfer Operator and its Dual**).

The natural transfer operator $\mathcal{L}_{\mathcal{W}}$ and its dual $\mathcal{L}_{\mathcal{W}}^{\#}$, introduced in Definition 4.14, on page 46, induce operators – still denoted by $\mathcal{L}_{\mathcal{W}}$ and $\mathcal{L}_{\mathcal{W}}^{\#}$, respectively, for the sake of simplicity – acting on the (total) $\mathbb{C}$-tensored cohomology $H^\star \otimes \mathbb{C}$.

Proposition 4.32 (**(Fractal) Lefschetz Operator (Fractal Counterpart of the Classical Lefschefz Operator, Given in Theorem 4.5, on page 42)**).

We introduce the (fractal) Lefschetz Operator $\mathcal{L}ef_{\mathcal{W}}$ *via*

$$\mathcal{L}ef_{\mathcal{W}} = i\,\mathcal{L}_{\mathcal{W}}\,\mathcal{L}_{\mathcal{W}}^{\#} = i\,\mathcal{L}_{\mathcal{W}}^{\#}\,\mathcal{L}_{\mathcal{W}}\,,$$

where $\mathcal{L}_{\mathcal{W}}$ is the natural transfer operator introduced in Definition 4.14, on page 46, while $\mathcal{L}_{\mathcal{W}}^{\#}$ is its dual (see Proposition 4.10, on page 46).

The Lefschetz Operator $\mathcal{L}ef_{\mathcal{W}}$ is a bigraded operator, of bigrading $(1,1)$, which, given $m \in \mathbb{N}^\star$, acts on each tensor product $H^m \otimes \mathbb{C}$ (see Theorem 4.24 above, on page 59, along with Theorem 4.28 above, on page 64), in the following way,

$$\mathcal{L}ef_{\mathcal{W}}\left(H^m \otimes \mathbb{C}\right) = H^{m+2} \otimes \mathbb{C}\,. \qquad (\mathcal{R}\,71)$$

This yields a fractal counterpart of the classic (primitive) Lefschetz decomposition, *given (in the case of Kähler manifolds) in Corollary 4.7, on page 43, since, given $m \in \mathbb{N}^\star$ and $0 \leqslant k \leqslant m$, with*

$$P^k\left(H^m \otimes \mathbb{C}\right) = \ker\left(\mathcal{L}ef_{\mathcal{W}}^{m-k+1}{}_{|H^m \otimes \mathbb{C}}\right)\,,$$

we obviously have the following orthogonal decomposition of the (finite-dimensional) complex Hilbert space $H^m \otimes \mathbb{C}$:

$$H^m \otimes \mathbb{C} = P^m \left(H^m \otimes \mathbb{C}\right) \oplus \mathcal{L}ef_{\mathcal{W}} P^{m-2} \left(H^m \otimes \mathbb{C}\right) \oplus \mathcal{L}ef_{\mathcal{W}} P^{m-4} \left(H^m \otimes \mathbb{C}\right) \oplus \ldots .$$

Remark 4.11 (**Hodge Star Relation Induced by the Lefschetz Operator**).

Obviously, the Lefschetz operator $\mathcal{L}ef_{\mathcal{W}}$ introduced in Proposition 4.32 above, on page 69, induces a Hodge Star relation on the functions defined on the Weierstrass Curve $\Gamma_{\mathcal{W}}$ and on all the higher differential forms. Naturally, an entirely similar comment applies to the transfer operator $\mathcal{L}_{\mathcal{W}}$ and its adjoint $\mathcal{L}_{\mathcal{W}}^{\#}$, as well as to the fractal Hodge Laplacian, $\Box_{\mathcal{W}}$, which we next introduce.

Definition 4.22 (**Fractal Hodge Laplacian (Fractal Counterpart of the Classical Hodge Lapacian, Given in Definition 4.11, on page 41)**).

The *fractal Hodge Laplacian*is given by

$$\Box_{\mathcal{W}} = \mathcal{L}ef_{\mathcal{W}} \, \mathcal{L}ef_{\mathcal{W}}^{\#} + \mathcal{L}ef_{\mathcal{W}}^{\#} \, \mathcal{L}ef_{\mathcal{W}} \, .$$

Along the lines of Remark 4.11 just above, and as alluded to in the introduction, we can now wonder about the possible connections between the local and global polyhedral effective zeta functions, and the cohomology. This issue is addressed in Proposition 4.33 just below.

Proposition 4.33 (**Operators Induced on the Fractal Cohomology by the Local and Global Polyhedral Effective Zeta Functions**).

Given any $m \in \mathbb{N}^\star$, the m^{th} local polyhedral effective zeta function $\tilde{\zeta}_m^e$ introduced in Theorem 2.20, on page 26, induces a (necessarily bounded, since the Hilbert space $H^m \otimes \mathbb{C}$ is finite-dimensional) linear operator – denoted by $\tilde{\zeta}_m^{e,op} = \tilde{\zeta}_m^{e,op}(s)$, for the sake of simplicity – which, for each $s \in \mathbb{C}$, acts on $H^m \otimes \mathbb{C}$.

In the same way, the global polyhedral effective zeta function $\tilde{\zeta}_{\mathcal{W}}^e = \tilde{\zeta}_{\mathcal{W}}^e(s)$ introduced in Theorem 2.20, on page 26, induces a (possibly unbounded) linear operator – denoted similarly by $\tilde{\zeta}_{\mathcal{W}}^{e,op} = \tilde{\zeta}_{\mathcal{W}}^{e,op}(s)$, for the sake of simplicity – which, again, for each fixed $s \in \mathbb{C}$, acts on $H^\star \otimes \mathbb{C}$. (See Remark 4.12 below, on page 71.)

Proof. This simply comes from the respective expressions of the local and global polyhedral effective zeta functions according to the generators of the $\mathbb{C}$-tensored cohomology groups $H^m \otimes \mathbb{C}$; see Theorem 2.20, on page 26. Indeed, given $m \in \mathbb{N}$, and $0 \leqslant k \leqslant m$,

$$\varepsilon^{k\,(2-D_{\mathcal{W}})} = (N_b - 1)\, \varepsilon_k^{k\,(2-D_{\mathcal{W}})}$$

and we can write each complex number $\varepsilon^{i\,\ell_{k,j,m}\,\mathbf{p}}$ in the following form,

$$\varepsilon^{i\,\ell_{k,j,m}\,\mathbf{p}} = c_{k,j,m}\, \varepsilon_k^{i\,k\,\ell'_{k,j,m}\,\mathbf{p}} ,$$

with $c_{k,j,m} \in \mathbb{C}$ and $\ell'_{k,j,m} \in \mathbb{Z}$ (where ε_k^k is the k^{th} cohomology infinitesimal, introduced in Definition 2.8, on page 21, with $\mathbf{p} = \dfrac{2\pi}{\ln N_b}$ denoting *the oscillatory period* of the Weierstrass Curve).

□

Remark 4.12. Since $H^\star \otimes \mathbb{C} = \varinjlim H^m \otimes \mathbb{C}$ is the inductive limit of the finite-dimensional complex Hilbert spaces $H^m \otimes \mathbb{C}$ – a fact wich follows (upon tensoring by $\mathbb{C}$) from the latter part of the statement of Theorem 4.19, on page 55 (see also Theorem 4.24, on page 59) – with associated orthogonal projections $\pi_{m+1}^{\mathbb{C}} : H^\star \otimes \mathbb{C} \to H^m \otimes \mathbb{C}$, such that – if we write (much as at the end of the statement of Theorem 4.19, on page 55) every $\varphi \in H^\star \otimes \mathbb{C}$ in the form $\varphi = (\varphi_m)_{m\in\mathbb{N}}$, with $\varphi_m = \pi_{m+1}^{\mathbb{C}}(\varphi)$, for all $m \in \mathbb{N}$ – then, with this notation, we expect that the (possibly) unbounded operor $\tilde{\zeta}_{\mathcal{W}}^{e,op} = \tilde{\zeta}_{\mathcal{W}}^{e,op}(s)$ can be viewed as a kind of inductive limit of the bounded operors $\tilde{\zeta}_m^{e,op} = \tilde{\zeta}_m^{e,op}(s)$; at least symbolically, $\tilde{\zeta}_{\mathcal{W}}^{e,op} = \varinjlim \tilde{\zeta}_m^{e,op}$, which is an inductive limit counterpart of the usual notion of the strong operator convergence in operator theory.

More specifically, we expect that

$$\tilde{\zeta}_{\mathcal{W}}^{e,op}(s)(\varphi) = \lim_{m\to\infty} \tilde{\zeta}_{\mathcal{W}}^{e,op}(s)(\varphi_m) \,, \qquad (\mathcal{R}\,72)$$

for all $\varphi = (\varphi_m)_{m\in\mathbb{N}}$ in $D\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}(s)\right)$, where the convergence holds in the norm topology of the Hilbert space $H^\star \otimes \mathbb{C}$ and is locally uniform in $s \in \mathbb{C}$. Here, for each $s \in \mathbb{C}$, $D\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}(s)\right) \subseteq H^\star \otimes \mathbb{C}$ is the domain of the operator $\tilde{\zeta}_{\mathcal{W}}^{e,op}(s)$, which should precisely coincide with the set of $\varphi \in H^\star \otimes \mathbb{C}$ such that the limit in relation ($\mathcal{R}$72) exists in $H^\star \otimes \mathbb{C}$, with the obvious identification of $H^m \otimes \mathbb{C} = \pi_{m+1}^{\mathbb{C}}\left(H^\star \otimes \mathbb{C}\right)$ with a closed subspace of $H^\star \otimes \mathbb{C}$.

We note that the kind of *quantization* of fractal zeta functions introduced in the present work is analogous – but not at all identical – to the one introduced and studied in the book on *Quantized Number Thery*, by Hafedh Herichi and the second author in [HL21].

Corollary 4.34 (**A First Step Towards A Functional Equation: A Hodge Star Relatio**n**).**

By applying the result mentioned in Remark 4.11, on page 70, we immediately obtain that the Lefschetz Operator $\mathcal{L}ef_{\mathcal{W}}$ introduced in Proposition 4.32 above, on page 69, induces a Hodge Star relation on the (differential) operator $\tilde{\zeta}_{\mathcal{W}}^{e,op}$ induced by the global polyhedral effective zeta function $\tilde{\zeta}_{\mathcal{W}}^{e}$, in the following form,

$$\tilde{\zeta}_{\mathcal{W}}^{e,op} = \overline{\tilde{\zeta}_{\mathcal{W}}^{e,op}} \,,$$

from which we deduce that, for all $s \in \mathbb{C}$,

$$\tilde{\zeta}_{\mathcal{W}}^{e}(\bar{s}) = \overline{\tilde{\zeta}_{\mathcal{W}}^{e}(s)} \,,$$

an identity that can also be verified directly from the definition and the expression of $\tilde{\zeta}_{\mathcal{W}}^{e}$.

The forthcoming consequence of Theorem 4.35 below, on page 72, about the resolvent of the (differential) operator $\tilde{\zeta}_{\mathcal{W}}^{e,op} = \tilde{\zeta}_{\mathcal{W}}^{e,op}(s)$ induced by the global zeta function, is also new. It is based upon its decomposition with respect to the total cohomology, given in Theorem 4.19, on page 55. The proof of this result, where the difficulty is to go to the limit when the integer m tends to infinity, can be obtained via two completely different methods: first, by using the explicit expression for the global effective polyhedral zeta function; second, by relying on the (natural) correspondence between the Alexander-Kolmogorov Complex, and the de Rham complex, by means of a small scale parameter; see [DL23d]. In short, this latter result will enable us to go to the limit when the integer m tends to infinity, and obtain the result we expect in the case of the global zeta function. This passage to the limit is done in a very elegant way, by means of canonical projectors between the de Rham Complex and a h-scale cohomology, where h is a very small parameter (in our present setting, we will take $h = \varepsilon^m$, the intrinsinc m^{th} cohomology infinitesimal, introduced in Definition 2.8, on page 21).

Theorem 4.35 (**Resolvent of the (Differential) Operator Induced by the Global Zeta Function).**

Given $m \in \mathbb{N}$ sufficiently large (i.e., for all $m \geqslant m_0$, for some optimal $m_0 \in \mathbb{N}$*), the operator $\tilde{\zeta}_m^{e,op}$ induced by the m^{th} local polyhedral effective zeta function $\tilde{\zeta}_m^{e}$ (see Proposition 4.33, on page 70) introduced in Theorem 2.20 above, on page 26, can be decomposed with respect to the $\mathbb{C}$-tensored cohomology groups $H^{m_0} \otimes \mathbb{C}$, ..., $H^{m-m_0+1} \otimes \mathbb{C}$ in the following two ways:*

$$\begin{aligned}\tilde{\zeta}_m^{e,op} &= \left(\tilde{\zeta}_m^{e,op}\right)_{|H^{m_0}\otimes\mathbb{C}} + \left(\tilde{\zeta}_m^{e,op}\right)_{|(H^{m_0+1}\otimes\mathbb{C})\setminus(H^{m_0}\otimes\mathbb{C})} + \ldots + \left(\tilde{\zeta}_m^{e,op}\right)_{|(\otimes\mathbb{C})H^{m}\setminus(H^{m-1}\otimes\mathbb{C})} \\ &= \left(\tilde{\zeta}_m^{e,op}\right)_{|H^{m_0}\otimes\mathbb{C}} + \sum_{k=m_0+1}^{m} \left(\tilde{\zeta}_m^{e,op}\right)_{|(H^{k}\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})},\end{aligned}$$

which corresponds to a graded decomposition of the cohomology according to the real parts of the generators (see Proposition 4.17, on page 52, along with Theorem 4.24, on page 59), or, going further, i.e., taking into account a graded decomposition of the cohomology according to the real and imaginary parts of the generators (see relation ($\mathcal{R}$58), on page 63),

$$\tilde{\zeta}_m^{e,op} = \sum_{k'=0}^{N_b^{m_0}-1}\sum_{k''=0}^{N_b-1} \left(\tilde{\zeta}_m^{e,op}\right)_{|(H^{m_0}\otimes\mathbb{C})^{k',k''}} + \sum_{k'=0}^{N_b^{k}-1}\sum_{k''=0}^{N_b-1}\sum_{k=m_0+1}^{m} \left(\tilde{\zeta}_m^{e,op}\right)_{|(H^{k}\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})^{k',k''}}.$$

In the same way, the operator $\tilde{\zeta}_{\mathcal{W}}^{e,op}$ induced by the global effective zeta function $\tilde{\zeta}_{\mathcal{W}}^{e}$ introduced in Theorem 2.20 above, on page 26, can be decomposed with respect to the $\mathbb{C}$-tensored cohomology groups $H^{m_0} \otimes \mathbb{C}$, $H^{m_0+1} \otimes \mathbb{C}$, ..., in the following two ways:

$$\begin{aligned}\tilde{\zeta}_{\mathcal{W}}^{e,op} &= \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|H^{m_0}\otimes\mathbb{C}} + \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|(H^{m_0+1}\otimes\mathbb{C})\setminus(H^{m_0}\otimes\mathbb{C})} + \ldots + \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|(\otimes\mathbb{C})H^{m}\setminus(H^{m-1}\otimes\mathbb{C})} \\ &= \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|H^{m_0}\otimes\mathbb{C}} + \sum_{k=m_0+1}^{\infty} \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|(H^{k}\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})},\end{aligned}$$

and

$$\tilde{\zeta}_{\mathcal{W}}^{e,op} = \sum_{k'=0}^{N_b^{m_0}-1}\sum_{k''=0}^{N_b-1} \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|(H^{m_0}\otimes\mathbb{C})^{k',k''}} + \sum_{k'=0}^{N_b^{k}-1}\sum_{k''=0}^{N_b-1}\sum_{k=m_0+1}^{\infty} \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|(H^{k}\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})^{k',k''}}.$$

This ensures, for all $z \in \mathbb{C} \setminus Sp\left(\tilde{\zeta}_m^{e,op}\right)$, where $Sp\left(\tilde{\zeta}_m^{e,op}\right)$ denotes the spectrum of $\tilde{\zeta}_m^{e,op}$ and $Id_{|H^m\otimes\mathbb{C}}$ the identity map on $H^m \otimes \mathbb{C}$, that for all $m \geqslant m_0$, with $m_0 \in \mathbb{N}$ as above, the determinant of the *resolvent* $\left(z\,Id_{|H^m\otimes\mathbb{C}}-\tilde{\zeta}_m^{e,op}\right)^{-1}$ *of the* m^{th} *local operator* $\tilde{\zeta}_m^{e,op}$ *can be expressed in the following two forms,*

$$\det\left(z\,Id_{|Hm\otimes\mathbb{C}}-\tilde{\zeta}_m^{e,op}\right)^{-1}=$$

$$=\det\left(z\,Id_{|H^{m_0}\otimes\mathbb{C}}-\left(\tilde{\zeta}_m^{e,op}\right)_{|H^{m_0}\otimes\mathbb{C}}\right)^{-1}\prod_{k=m_0+1}^{m}\det\left(z\,Id_{|(H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})}-\left(\tilde{\zeta}_m^{e,op}\right)_{|(H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})}\right)^{-1}, \qquad (\mathcal{R}\,73)$$

or,

$$\det\left(z\,Id_{|H^m\otimes\mathbb{C}}-\tilde{\zeta}_m^{e,op}\right)^{-1}=$$

$$\begin{aligned}
= &\det\left(z\,Id_{|H^{m_0}\otimes\mathbb{C}}-\left(\tilde{\zeta}_m^{e,op}\right)_{|H^{m_0}\otimes\mathbb{C}}\right)^{-1}\\
&\times\prod_{k=m_0+1}^{m}\prod_{k'=0}^{N_b^k-1}\prod_{k''=0}^{N_b-1}\det\left(z\,Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k''}}-\left(\tilde{\zeta}_m^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k''}}\right)^{-1}\\
= &\det\left(z\,Id_{|(H^{m_0}\otimes\mathbb{C})}-\left(\tilde{\zeta}_m^{e,op}\right)_{|H_{m_0}\otimes\mathbb{C}}\right)^{-1}\\
&\times\prod_{k=m_0+1}^{m}\prod_{k'=0}^{N_b^k-1}\prod_{k''=0}^{N_b-1}\det\left(z\,Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k''}}-\left(\tilde{\zeta}_m^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k''}}\right)^{-1}\\
= &\det\left(z\,Id_{|(H^{m_0}\otimes\mathbb{C})}-\left(\tilde{\zeta}_m^{e,op}\right)_{|H_{m_0}\otimes\mathbb{C}}\right)^{-1}\\
&\times\prod_{k=m_0+1}^{m}\prod_{k'=0}^{\left[\frac{N_b^k}{2}\right]}\prod_{k''=0}^{N_b-1}\det\left(z\,Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k''}}-\left(\tilde{\zeta}_m^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k''}}\right)^{-1}\\
&\times\det\left(z\,Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{(N_b-1)\,N_b^k-k',-k''}}-\left(\tilde{\zeta}_m^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{(N_b-1)\,N_b^k-k',-k''}}\right)^{-1}\\
&\times\prod_{k''=0}^{N_b-1}\left(\frac{1+(-1)^{k+1}}{2}\right)\det\left(z\,Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{[\frac{N_b^k}{2}]+1,k''}}-\left(\tilde{\zeta}_m^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{[\frac{N_b^k}{2}]+1,k''}}\right)^{-1};
\end{aligned} \qquad (\mathcal{R}\,74)$$

i.e.,

$$\det\left(z\,Id_{|H^m\otimes\mathbb{C}} - \tilde{\zeta}_m^{e,op}\right)^{-1} =$$

$$
\begin{aligned}
= \quad & \det\left(z\,Id_{|(H_{m_0}\otimes\mathbb{C})} - \left(\tilde{\zeta}_m^{e,op}\right)_{|H_{m_0}\otimes\mathbb{C}}\right)^{-1} \\
& \times \prod_{k=m_0+1}^{m}\ \prod_{k'=0}^{\left[\frac{N_b^k}{2}\right]}\ \prod_{k''=0}^{N_b-1} \det\left(z\,Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k"}} - \left(\tilde{\zeta}_m^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k''}}\right)^{-1} \\
& \overline{\times \det\left(z\,Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k''}} - \left(\tilde{\zeta}_m^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k"}}\right)^{-1}} \\
& \times \prod_{k''=0}^{N_b-1}\left(\frac{1+(-1)^{k+1}}{2}\right)\det\left(z\,Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{[\frac{N_b^k}{2}]+1,k"}} - \left(\tilde{\zeta}_m^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{[\frac{N_b^k}{2}]+1,k''}}\right)^{-1} \\
= \quad & \det\left(z\,Id_{|(H^{m_0}\otimes\mathbb{C})} - \left(\tilde{\zeta}_m^{e,op}\right)_{|H^{m_0}\otimes\mathbb{C}}\right)^{-1} \\
& \times \prod_{k=m_0+1}^{m}\ \prod_{k'=0}^{\left[\frac{N_b^k}{2}\right]}\ \prod_{k''=0}^{N_b-1}\left|\det\left(z\,Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k"}} - \left(\tilde{\zeta}_m^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k''}}\right)^{-1}\right|^2 \\
& \times \prod_{k''=0}^{N_b-1}\left(\frac{1+(-1)^{k+1}}{2}\right)\det\left(z\,Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{[\frac{N_b^k}{2}]+1,k"}} - \left(\tilde{\zeta}_m^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{[\frac{N_b^k}{2}]+1,k''}}\right)^{-1}
\end{aligned}
\tag{$\mathcal{R}$ 75}
$$

with, of course,

$$\det\left(z\,Id_{|H^{m_0}\otimes\mathbb{C}} - \left(\tilde{\zeta}_m^{e,op}\right)_{|H^{m_0}\otimes\mathbb{C}}\right)^{-1} = \prod_{k'=0}^{N_b^{m_0}-1}\ \prod_{k''=0}^{N_b-1}\det\left(z\,Id_{|H^{m_0}\otimes\mathbb{C}} - \left(\tilde{\zeta}_m^{e,op}\right)_{|(H^{m_0}\otimes\mathbb{C})^{k',k''}}\right)^{-1} . \tag{$\mathcal{R}$ 76}$$

We also have that, for all $z \in \mathbb{C} \setminus Sp\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)$, where $Sp\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)$ denotes the spectrum of the (differential) operator $\tilde{\zeta}_{\mathcal{W}}^{e,op}$ (where $\tilde{\zeta}_{\mathcal{W}}^{e}$ is given in Theorem 2.20, on page 26), the resolvent $\det\left(z\, Id_{|H^\star\otimes\mathbb{C}} - \tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1}$ of the global operator $\tilde{\zeta}_{\mathcal{W}}^{e,op}$ can be expressed in the following form:

$$\begin{aligned}
&\det\left(z\, Id_{|H^\star\otimes\mathbb{C}} - \tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1} = \\
&= \det\left(z\, Id_{|H^{m_0}\otimes\mathbb{C}} - \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|H^{m_0}\otimes\mathbb{C}}\right)^{-1} \prod_{k=m_0+1}^{\infty} \det\left(z\, Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k"}} - \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k"}}\right)^{-1} \\
&= \det\left(z\, Id_{|H^{m_0}\otimes\mathbb{C}} - \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|H^{m_0}\otimes\mathbb{C}}\right)^{-1} \\
&\quad \times \prod_{k=m_0+1}^{\infty} \prod_{k'=0}^{N_b^k-1} \prod_{k''=0}^{N_b-1} \prod_{k=m_0+1}^{m} \det\left(z\, Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k"}} - \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k"}}\right)^{-1} \\
&= \det\left(z\, Id_{|H^{m_0}\otimes\mathbb{C}} - \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|H^{m_0}\otimes\mathbb{C}}\right)^{-1} \\
&\quad \times \lim_{m\to\infty} \prod_{k'=0}^{N_b^k-1} \prod_{k''=0}^{N_b-1} \det\left(z\, Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k"}} - \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k"}}\right)^{-1}
\end{aligned}$$

which, of course, can also be written as

$$\begin{aligned}
&\det\left(z\, Id_{|H^\star\otimes\mathbb{C}} - \tilde{\zeta}_{\mathcal{W}}^{e,op}\right) = \\
&= \det\left(z\, Id_{|H^{m_0}\otimes\mathbb{C}} - \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|H_{m_0}\otimes\mathbb{C}}\right) \\
&\quad \times \lim_{m\to\infty} \prod_{k'=0}^{N_b^k-1} \prod_{k''=0}^{N_b-1} \det\left(z\, Id_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k"}} - \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)_{|((H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C}))^{k',k"}}\right) .
\end{aligned}$$

Since, for $m \geqslant m_0$, $0 \leqslant k \leqslant m$, $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k" \leqslant N_b - 1$, the expression (for each fixed $s \in \mathbb{C}$) of the operator $\tilde{\zeta}_{\mathcal{W}}^{e,op}$ on each $\left(H^k \otimes \mathbb{C}\right) \setminus \left(H^{k-1} \otimes \mathbb{C}\right)^{k',k''}$ only involves terms in $\varepsilon^{k\,(2-D_{\mathcal{W}})\,s + i\,k\,\ell_{k',k",j,m}\,\mathbf{p}}$, we note that the eigenvalues of $\tilde{\zeta}_{\mathcal{W}}^{e,op} = \tilde{\zeta}_{\mathcal{W}}^{e,op}(s)$ are precisely the complex numbers

$$\begin{aligned}
\lambda(s) \quad = \quad & \frac{1}{2}\,\varepsilon^{m+1}\,\alpha_q(N_b)\,\frac{c_{k,(N_b-1)\,j+q,m+1}\,\varepsilon^{s}}{s - D_{\mathcal{W}} + (k-1)\,(2-D_{\mathcal{W}}) + i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}} \\
& + \frac{1}{2}\,\varepsilon^{m+1}\,\alpha_q(N_b)\,\frac{\overline{c_{k,(N_b-1)\,j+q,m+1}}\,\varepsilon^{s}}{s - D_{\mathcal{W}} + (k-1)\,(2-D_{\mathcal{W}}) - i\,\ell_{k,(N_b-1)\,j+q,m+1}\,\mathbf{p}} \,,
\end{aligned}$$

for $0 \leqslant q \leqslant N_b$, and where the coefficients $\alpha_q(N_b)$ have been introduced in Theorem 2.20, on page 26, while the coefficients $c_{k,(N_b-1)\,j+q,m+1}$ are given in Theorem 2.18, on page 23.

The proof of Theorem 4.35 is given on page 76, just after Remark 4.13.

Remark 4.13 (**On the Optimality of the Integer m_0 Involved in the Expression of the Global Polyhedral Effective Zeta Function**).

Note that the integer m_0, involved in the expression of the global polyhedral effective zeta function $\tilde{\zeta}^e_{\mathcal{W}}$ given in Theorem 2.20, on page 26, has to be chosen as the optimal admissible integer, since this integer plays a key role. More precisely, it is the value of m_0 which fixes the global polyhedral effective zeta function $\tilde{\zeta}^e_{\mathcal{W}}$ and hence helps determine not only its poles (i.e., the intrinsinc Complex Dimensions) but also its zeros.

Proof. (Of Theorem 4.35, on page 72)

The first part of the theorem, i.e., the result in relation ($\mathcal{R}73$), comes from Proposition 4.17, on page 52, which provides the *generators* of the cohomology groups H^m, as the following generalized *(fractional)* polynomials

$$\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,\varepsilon_k^{i\,k\,\ell_{j,m}\,\mathbf{p}} = \varepsilon_k^{k\,((2-D_{\mathcal{W}})+i\,k\,\ell_{j,m}\,\mathbf{p})}$$

and

$$\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,\varepsilon_k^{-i\,k\,\ell_{j,m}\,\mathbf{p}} = \varepsilon_k^{k\,((2-D_{\mathcal{W}})-i\,k\,\ell_{j,m}\,\mathbf{p})}\,,$$

with $0 \leqslant k \leqslant m$, $0 \leqslant \ell_{j,m} \leqslant \# V_m - 1$, along with Proposition 4.33, on page 70, and its proof, on page 70.

This immediately ensures, for the (differential) operator $\tilde{\zeta}^{e,op}_m$ induced by the m^{th} local polyhedral effective zeta function $\tilde{\zeta}^e_m$, the expression of which with respect to the aforementioned generators, is given by relation ($\mathcal{R}21$), on page 26, that $\tilde{\zeta}^{e,op}_m$ can be decomposed as

$$\tilde{\zeta}^{e,op}_m = \left(\tilde{\zeta}^{e,op}_m\right)_{|H^{m_0}\otimes\mathbb{C}} + \ldots + \left(\tilde{\zeta}^{e,op}_m\right)_{|H^{m}\otimes\mathbb{C}}\,,$$

with

$$\begin{aligned}\left(\tilde{\zeta}^{e,op}_m\right)_{|H^{m_0}\otimes\mathbb{C}} = \; & \frac{1}{2}\,\varepsilon^m \sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b}\alpha_q(N_b)\sum_{k=1}^{m_0}\frac{c_{k,(N_b-1)\,j+q,m}\,\varepsilon_k^{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}\\ & +\frac{1}{2}\,\varepsilon^m \sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b}\alpha_q(N_b)\sum_{k=1}^{m_0}\frac{\overline{c_{k,(N_b-1)\,j+q,m}\,\varepsilon_k}^{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})-i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})-i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}} \in H^{m_0}\otimes\mathbb{C}\end{aligned}$$

where the coefficients $c_{k,(N_b-1)\,j+q,m+1}$ are given in Theorem 2.18, on page 23, and where, for all $m_0+1 \leqslant k \leqslant m$,

$$\begin{aligned}\left(\tilde{\zeta}^{e,op}_m\right)_{|(H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})} = \; & \frac{1}{2}\,\varepsilon^m \sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b}\alpha_q(N_b)\,\frac{c_{k,(N_b-1)\,j+q,m}\,\varepsilon^{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})+i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}\\ & +\frac{1}{2}\,\varepsilon^m \sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b}\alpha_q(N_b)\,\frac{\overline{c_{k,(N_b-1)\,j+q,m}}\,\varepsilon^{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})-i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-D_{\mathcal{W}}+(k-1)\,(2-D_{\mathcal{W}})-i\,k\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}\end{aligned}$$

and acts on $\left(H^k\otimes\mathbb{C}\right)\setminus\left(H^{k-1}\otimes\mathbb{C}\right)$, as desired.

The second part of the theorem, i.e., the result in relation ($\mathcal{R}74$), also comes from Proposition 4.17, on page 52, which also provides the *generators* of the cohomology groups H^m, as the following generalized *(fractional)* polynomials

$$\varepsilon_k^{k\,(2-D_{\mathcal{W}})}\,\varepsilon_k^{i\,k\,\ell_{k',k'',m}\,\mathbf{p}} = \varepsilon_k^{k\left((2-D_{\mathcal{W}})+i\,k\,\ell_{k',k'',m}\,\mathbf{p}\right)},$$

with $0 \leqslant k \leqslant m$, $0 \leqslant k' \leqslant N_b^m - 1$, $0 \leqslant k'' \leqslant N_b - 1$. This immediately ensures, for the m^{th} local polyhedral effective zeta function $\tilde{\zeta}_m^e$, the expression of which with respect to the aforementioned generators, is given by relation ($\mathcal{R}21$), on page 26, that the (differental) operator $\tilde{\zeta}_m^{e,op}$ can be decomposed as

$$\tilde{\zeta}_m^{e,op} = \sum_{k'=0}^{N_b^{m0}-1}\sum_{k''=0}^{N_b-1}\left(\tilde{\zeta}_m^{e,op}\right)_{|(H^{m_0}\otimes\mathbb{C})^{k',k''}} + \sum_{k'=0}^{N_b^k-1}\sum_{k''=0}^{N_b^k-1}\sum_{k=m_0+1}^{m}\left(\tilde{\zeta}_m^{e,op}\right)_{|(H^k\otimes\mathbb{C})\setminus(H_{k-1}\otimes\mathbb{C})^{k',k''}}.$$

The third step is to go to the limit when the integer m tends to ∞. We can, indeed, use the explicit expression for the global polyhedral effective zeta function $\tilde{\zeta}_{\mathcal{W}}^e$ given in relation ($\mathcal{R}24$), on page 27, which directly yields the expected result.

We can also apply the same method as in [DL23d], based on the (natural) correspondence between the Alexander–Kolmogorov Complex, and the de Rham Complex, by means of a small scale parameter. To this purpose, we require the following scaled cohomology, obtained by letting, for all $m \in \mathbb{N}^\star$,

$$\delta_{m-1,m}^{\varepsilon} = \varepsilon^{-m}\,\delta_{m-1,m}.$$

where the $(m-1,m)$-*differential* $\delta_{m-1,m}$ from $\mathcal{F}^0(V_m,\mathbb{C})$ to $\mathcal{F}^{N_b+1}(V_m,\mathbb{C})$ has been introduced in Definition 4.17, on page 49. This provides the scaled cohomology

$$\bigoplus_{m=0}^{m}\left(H^m\right)^{\varepsilon}, \qquad (\mathcal{R}\,77)$$

where, for all $m \in \mathbb{N}^\star$,

$$\left(H^m\right)^{\varepsilon} = \ker\delta_{m-1,m}^{\varepsilon}/Im\,\delta_{m-2,m-1}^{\varepsilon}.$$

The method given in [DL23d], also applies in our present context. For the sake of concision, we do not rewrite here all the details. The important point is that it relies on canonical projections between the scaled complex and the de Rham Complex. It enables us to go to the limit when $m \to \infty$ (or, equivalently, when $\varepsilon^m \to 0$), thus also providing the result obtained by using the recurrence relation ($\mathcal{R}23$), on page 27.

□

Remark 4.14 (**About the Natural Correspondence Between the Alexander-Kolmogorov Complex, and the de Rham Complex – Remarkable Consequences on Local Zeta Functions**).

We stress that the method given in [DL23d] enables us to go to the limit in the case of local zeta functions which do not satisfy a recurrence relation. It thus provides a very powerful and natural tool which, thus far, was missing from the theory.

Theorem 4.36 (**The (Complex Dimensions) Frobenius Operator**).

Given $m \in \mathbb{N}$ sufficiently large (i.e., for all $m \geqslant m_0$, for some optimal $m_0 \in \mathbb{N}$*), the operator $\left(\tilde{\zeta}_m^{e,op}\right)^{-1}$ – the inverse operator of $\tilde{\zeta}_m^{e,op}$ (see Proposition 4.33, on page 70), can also be decomposed with respect to the $\mathbb{C}$-tensored cohomology groups $H^{m_0} \otimes \mathbb{C}$, ..., $H^{m-m_0+1} \otimes \mathbb{C}$ in the following two ways:*

$$\begin{aligned}\left(\tilde{\zeta}_m^{e,op}\right)^{-1} &= \left(\left(\tilde{\zeta}_m^{e,op}\right)^{-1}\right)_{|H^{m_0}\otimes\mathbb{C}} + \left(\left(\tilde{\zeta}_m^{e,op}\right)^{-1}\right)_{|(H^{m_0+1}\otimes\mathbb{C})\setminus(H^{m_0}\otimes\mathbb{C})} + \ldots + \left(\left(\tilde{\zeta}_m^{e,op}\right)^{-1}\right)_{|(\otimes\mathbb{C})H^{m}\setminus(H^{m-1}\otimes\mathbb{C})} \\ &= \left(\left(\tilde{\zeta}_m^{e,op}\right)^{-1}\right)_{|H^{m_0}\otimes\mathbb{C}} + \sum_{k=m_0+1}^{m} \left(\left(\tilde{\zeta}_m^{e,op}\right)^{-1}\right)_{|(H^{k}\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})},\end{aligned}$$

where the last sum is equal to zero if $m = m_0$, or, going further, i.e., taking into account a graded decomposition of the cohomology according to the real and imaginary parts of the generators (see relation ($\mathcal{R}58$), on page 63),

$$\left(\tilde{\zeta}_m^{e,op}\right)^{-1} = \sum_{k'=0}^{N_b^{m_0}-1}\sum_{k''=0}^{N_b-1} \left(\left(\tilde{\zeta}_m^{e,op}\right)^{-1}\right)_{|(H^{m_0}\otimes\mathbb{C})^{k',k''}} + \sum_{k'=0}^{N_b^{k}-1}\sum_{k''=0}^{N_b-1}\sum_{k=m_0+1}^{m} \left(\left(\tilde{\zeta}_m^{e,op}\right)^{-1}\right)_{|(H^{k}\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})^{k',k''}}.$$

Similarly, the operator $\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1}$ – the inverse operator of $\tilde{\zeta}_{\mathcal{W}}^{e,op}$, can be decomposed with respect to the $\mathbb{C}$-tensored cohomology groups $H^{m_0} \otimes \mathbb{C}$, $H^{m_0+1} \otimes \mathbb{C}$, ..., in the following two ways:

$$\begin{aligned}\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1} &= \left(\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1}\right)_{|H^{m_0}\otimes\mathbb{C}} + \left(\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1}\right)_{|(H^{m_0+1}\otimes\mathbb{C})\setminus(H^{m_0}\otimes\mathbb{C})} + \ldots + \left(\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1}\right)_{|(\otimes\mathbb{C})H^{m}\setminus(H^{m-1}\otimes\mathbb{C})} \\ &= \left(\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1}\right)_{|H^{m_0}\otimes\mathbb{C}} + \sum_{k=m_0+1}^{\infty} \left(\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1}\right)_{|(H^{k}\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})},\end{aligned}$$

and

$$\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1} = \sum_{k'=0}^{N_b^{m_0}-1}\sum_{k''=0}^{N_b-1} \left(\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1}\right)_{|(H^{m_0}\otimes\mathbb{C})^{k',k''}} + \sum_{k'=0}^{N_b^{k}-1}\sum_{k''=0}^{N_b-1}\sum_{k=m_0+1}^{\infty} \left(\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1}\right)_{|(H^{k}\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})^{k',k''}}.$$

Since, for $m \geqslant m_0$, $0 \leqslant k \leqslant m$, $0 \leqslant k' \leqslant N_b^m - 1$ and $0 \leqslant k" \leqslant N_b - 1$, the expression of the operator $\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1}$ on each $\left(H^k \otimes \mathbb{C}\right) \setminus \left(H^{k-1} \otimes \mathbb{C}\right)^{k',k''}$ only involves terms in $\varepsilon_k^{k\,(2-D_{\mathcal{W}})\,s+i\,k\,\ell_{k',k",j,m}\,\mathbf{p}}$, we note that the eigenvalues of $\left(\tilde{\zeta}_{\mathcal{W}}^{e,op}\right)^{-1} = \left(\tilde{\zeta}_{\mathcal{W}}^{e,op}(s)\right)^{-1}$, for each fixed $s \in \mathbb{C}$, are precisely the complex numbers

$$\lambda^{+}_{k''',(N_b-1)\,j+q,k+1}(s)^{-1} = \frac{2}{\alpha_q(N_b)\,\varepsilon^{k+1}}\,\frac{s+2\,(k'''-1)-k\,D_{\mathcal{W}}+i\,\ell_{k''',(N_b-1)\,j+q,k+1}\,\mathbf{p}}{c_{k''',(N_b-1)\,j+q,k+1}\,\varepsilon^{s}}$$

and

$$\lambda^{-}_{k''',(N_b-1)\,j+q,k+1}(s)^{-1} = \frac{2}{\alpha_q(N_b)\,\varepsilon^{k+1}}\,\frac{s+2\,(k'''-1)-k\,D_{\mathcal{W}}-i\,\ell_{k''',(N_b-1)\,j+q,k+1}\,\mathbf{p}}{c_{k''',(N_b-1)\,j+q,k+1}\,\varepsilon^{s}},$$

where $0 \leqslant k''' \leqslant k+1$, along with

$$\lambda^+_{k+1,(N_b-1)\,j+q,k+1}(s)^{-1} = \frac{2}{\alpha_q(N_b)\,\varepsilon^{k+1}}\,\frac{s+2\,k-(k+1)\,D_{\mathcal{W}}+i\,\ell_{k+1,(N_b-1)\,j+q,k+1}\,\mathbf{p}}{c_{k+1,(N_b-1)\,j+q,k+1}\,\varepsilon^s}$$

and

$$\lambda^-_{k+1,(N_b-1)\,j+q,k+1}(s)^{-1} = \frac{2}{\alpha_q(N_b)\,\varepsilon^{k+1}}\,\frac{s+2\,k-(k+1)\,D_{\mathcal{W}}-\,\ell_{k+1,(N_b-1)\,j+q,k+1}\,\mathbf{p}}{c_{k+1,(N_b-1)\,j+q,k+1}\,\varepsilon^s}\,,$$

with $0 \leqslant q \leqslant N_b$*, where the coefficients* $\alpha_q(N_b)$ *have been introduced in Theorem 2.20, on page 26, while the coefficients* $c_{k,(N_b-1)\,j+q,k+1}$ *are given in Theorem 2.18, on page 23.*

Going further, by introducing the operator

$$\begin{aligned}
\mathcal{Z}^{e,op}_{\mathcal{W}} = & \sum_{k'=0}^{N_b^{m_0}-1}\sum_{k''=0}^{N_b-1}\left(\left(\tilde{\zeta}^{e,op}_{\mathcal{W}}\right)^{-1}\right)_{|(H^{m_0}\otimes\mathbb{C})^{k',k''}} \\
& +2\sum_{k'=0}^{N_b^k-1}\sum_{k''=0}^{N_b-1}\sum_{k=m_0+1}^{\infty}\sum_{k'''=0}^{k+1}\sum_{q=0}^{N_b}\frac{s+2\,(k'''-1)-k'''\,D_{\mathcal{W}}+i\,\ell_{k''',(N_b-1)\,j+q,k+1}\,\mathbf{p}}{\alpha_q(N_b)\,\varepsilon^{k+1}\,c_{k''',(N_b-1)\,j+q,k+1}\,\varepsilon^s}\left(\left(\tilde{\zeta}^{e,op}_{\mathcal{W}}\right)^{-1}\right)_{|(H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})^{k',k''}} \\
& +2\sum_{k'=0}^{N_b^k-1}\sum_{k''=0}^{N_b-1}\sum_{k=m_0+1}^{\infty}\sum_{k'''=0}^{k+1}\sum_{q=0}^{N_b}\frac{s+2\,(k'''-1)-k'''\,D_{\mathcal{W}}-i\,\ell_{k''',(N_b-1)\,j+q,k+1}\,\mathbf{p}}{\alpha_q(N_b)\,\varepsilon^{k+1}\,c_{k''',(N_b-1)\,j+q,k+1}\,\varepsilon^s}\left(\left(\tilde{\zeta}^{e,op}_{\mathcal{W}}\right)^{-1}\right)_{|(H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})^{k',k''}} \\
& +2\sum_{k'=0}^{N_b^k-1}\sum_{k''=0}^{N_b-1}\sum_{k=m_0+1}^{\infty}\sum_{q=0}^{N_b}\frac{s+2\,(k-1)-k\,D_{\mathcal{W}}+i\,\ell_{k,(N_b-1)\,j+q,k+1}\,\mathbf{p}}{\alpha_q(N_b)\,\varepsilon^{k+1}\,c_{k,(N_b-1)\,j+q,k+1}\,\varepsilon^s}\left(\left(\tilde{\zeta}^{e,op}_{\mathcal{W}}\right)^{-1}\right)_{|(H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})^{k',k''}} \\
& +2\sum_{k'=0}^{N_b^k-1}\sum_{k''=0}^{N_b-1}\sum_{k=m_0+1}^{\infty}\sum_{q=0}^{N_b}\frac{s+2\,(k-1)-k\,D_{\mathcal{W}}-i\,\ell_{k,(N_b-1)\,j+q,k+1}\,\mathbf{p}}{\alpha_q(N_b)\,\varepsilon^{k+1}\,c_{k,(N_b-1)\,j+q,k+1}\,\varepsilon^s}\left(\left(\tilde{\zeta}^{e,op}_{\mathcal{W}}\right)^{-1}\right)_{|(H^k\otimes\mathbb{C})\setminus(H^{k-1}\otimes\mathbb{C})^{k',k''}}\,.
\end{aligned}$$

we note that, for $s=0$*, the eigenvalues of* ${\mathcal{Z}^{e,op}_{\mathcal{W}}}_{|s=0} = \mathcal{Z}^{e,op}_{\mathcal{W}}(0)$ *are exactly the intrinsic Complex Dimensions of the Weierstrass Curve* $\Gamma_{\mathcal{W}}$*, as given by relation* ($\mathcal{R}$*25) of Corollary 2.21, on page 28.*

Remark 4.15 (**About the Frobenius Operator**).

Our Frobenius operator, as introduced in Theorem 4.36, on page 78, is simply a normalized version of the inverse of the (differential) operator $\tilde{\zeta}^{e,op}_{\mathcal{W}}$ induced by the global zeta function $\tilde{\zeta}^{e}_{\mathcal{W}}$. This can be understod intuitively, insofar as the global zeta function $\tilde{\zeta}^{e}_{\mathcal{W}}$ is, also, the global zeta function associated with the Weierstrass function $\mathcal{W}$, when an infinity of other functions enable us to obtain the cohomology groups; see Theorem 4.19 above, on page 55.

Theorem 4.37 (**Functional Equation for the Global (Polyhedral) Zeta Function** $\tilde{\zeta}^{e}_{\mathcal{W}}$).

We set

$$\lambda^\star = \frac{1}{\lambda\,N_b^2}\,. \qquad (\mathcal{R}\,78)$$

Since $\lambda N_b > 1$ and $0 < \lambda < 1$, we then have that

$$\lambda N_b^2 > N_b > 1 \quad \textit{and} \quad \lambda^\star < \varepsilon = \frac{1}{N_b} < 1 \, ,$$

while

$$\lambda^\star N_b = \frac{1}{\lambda N_b} < 1 \, ,$$

along with

$$D_{\mathcal{W}}^\star = 2 + \frac{\ln \lambda^\star}{\ln N_b} = 2 + \frac{-\ln \lambda - 2 \ln N_b}{\ln N_b} = -\frac{\ln \lambda}{\ln N_b} = 2 - D_{\mathcal{W}} \in \,]0,1[\; . \tag{$\mathcal{R}$ 79}$$

We then consider the Weierstrass star function $\mathcal{W}^\star$ (also called, in short, the $\mathcal{W}^\star$ function*, or the* dual Weierstrass function*) defined, for any real number x, by*

$$\mathcal{W}^\star(x) = \sum_{n=0}^{\infty} \left(\lambda^\star\right)^n \cos\left(2\,\pi\, N_b^n\, x\right) \, , \tag{$\mathcal{R}$ 80}$$

with associated fractal curve $\Gamma_{\mathcal{W}}^\star = \Gamma_{\mathcal{W}^\star}$, along with its complex counterpart, defined, for any real number x, by

$$\mathcal{W}_{comp}^\star(x) = \sum_{n=0}^{\infty} \left(\lambda^\star\right)^n e^{2\,i\,\pi\, N_b^n\, x} \, . \tag{$\mathcal{R}$ 81}$$

The global fractal effective zeta function $\tilde{\zeta}_{\mathcal{W}}^{e,\star}$, associated with the Weierstrass star function $\mathcal{W}^\star$, satisfies the following functional equation*, for all $s \in \mathbb{C}$,*

$$\tilde{\zeta}_{\mathcal{W}}^{e,\star}(2-s) = \tilde{\zeta}_{\mathcal{W}}^{e}(s) \, , \tag{$\mathcal{R}$ 82}$$

where, by definition, $\tilde{\zeta}_{\mathcal{W}}^{e,\star} = \tilde{\zeta}_{\mathcal{W}^\star}^{e}$.

Since $0 < \lambda^\star < 1$, the function $\mathcal{W}^\star$ is of class (at least) C^1 (i.e., continuously differentiable). Therefore, the corresponding Weierstrass Curve $\Gamma_{\mathcal{W}}^\star = \Gamma_{\mathcal{W}^\star}$ is also of class C^1 (in particular, it has a tangent line at every point), and yet, it is fractal, in the sense of the theory of fractal Complex Dimensions [LvF13], [LRŽ17b], [Lap19], [DL23b].

Indeed – according to the functional equation ($\mathcal{R}$82), and Corollary 2.21, on page 28 – it must have infinitely Complex Dimensions (all of which, but $2 - D_{\mathcal{W}} \in \,]0,1[$) are nonreal; namely, the Complex Dimensions of $\Gamma_{\mathcal{W}}^\star = \Gamma_{\mathcal{W}^\star}$ are all simple and exact, as well as given by

$$m\,(2 - D_{\mathcal{W}}) \pm i\,\ell_{j_{k,m},k}\,\mathbf{p} \quad , \quad \textit{with } m \in \mathbb{N} \textit{ arbitrary and } 0 \leqslant k \leqslant m \, , \tag{$\mathcal{R}$ 83}$$

where the integers $\ell_{j_{k,m},k} \in \mathbb{N}$ (which depend on k and m) are given in Theorem 2.18, on page 23) and where $\mathbf{p} = \dfrac{2\,\pi}{\ln N_b}$ is the oscillatory period of the dual Weierstrass Curve $\Gamma_{\mathcal{W}}^\star$ (as well as of the Weierstrass Curve $\Gamma_{\mathcal{W}}$).

Remark 4.16. Naturally, if, in Theorem 4.37, we assume, instead, that $\lambda N_b < 1$, with $0 < \lambda < 1$ – so that the Weierstrass function $\mathcal{W}$ and the Weierstrass Curve $\Gamma_{\mathcal{W}}$ are of class C^1 – the dual Weierstrass function $\mathcal{W}^\star$ and the dual Weierstrass Curve $\Gamma_{\mathcal{W}}^\star$ are nowhere differentiable, while the global (polyhedral) zeta functions of $\Gamma_{\mathcal{W}}$ and $\Gamma_{\mathcal{W}}^\star$, namely, $\tilde{\zeta}_{\mathcal{W}}^{e}$ and $\tilde{\zeta}_{\mathcal{W}}^{e,\star}$ are still related by the same functional equation ($\mathcal{R}$82), on page 80. Also, the (intrinsic) Complex Dimensions of $\Gamma_{\mathcal{W}}^\star$ (i.e., the poles of $\tilde{\zeta}_{\mathcal{W}}^{e,\star}$) are given by relation ($\mathcal{R}$25), in Corollary 2.21, on page 28.

Proof. (Of Theorem 4.37)

The functional equation given in relation ($\mathcal{R}$82) above can be obtained in the three following equivalent manners, where $m \in \mathbb{N}^\star$ is sufficiently large (i.e., for all $m \geqslant m_0$, where $m_0 \in \mathbb{N}^\star$ is optimal):

i. First, by using the Hodge star relation induced by the Lefschetz Operator (see Remark 4.11, on page 70).

ii. Second, by noting that the complex coefficients involved in the explicit expression for the global polyhedral effective zeta function $\tilde{\zeta}^e_{\mathcal{W}}$ given in relation ($\mathcal{R}$24), on page 27, both satisfy Hodge Diamond Star relations (for the coefficients $c_{k,j,m}$, Proposition 2.19, on page 25, and Theorem 4.20, on page 56, for the terms $\varepsilon^{i\,k\,\ell_{j,m}\,\mathbf{p}}$).

iii. The m^{th} local polyhedral effective zeta function $\tilde{\zeta}^{e,\star}_m$ is obtained by applying Theorem 2.20, on page 26, where λ is replaced by $\lambda^\star$, N_b by N_b^2, and $D_{\mathcal{W}}$ by $D^\star_{\mathcal{W}} - 2$,

$$\begin{aligned}\tilde{\zeta}^{e,\star}_m(s) \quad = \quad & \frac{1}{2}\,\varepsilon^m \sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b}\alpha_{j,q}(N_b)\sum_{k=1}^{m}\frac{c_{k,(N_b-1)\,j+q,m}\,\varepsilon^{s-D^\star_{\mathcal{W}}+(k-1)\,(2-D^\star_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-D^\star_{\mathcal{W}}+(k-1)\,(2-D^\star_{\mathcal{W}})+i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}\\ & +\frac{1}{2}\,\varepsilon^m \sum_{j=0}^{N_b^m-1}\sum_{q=0}^{N_b}\alpha_{j,q}(N_b)\sum_{k=1}^{m}\frac{\overline{c_{k,(N_b-1)\,j+q,m}}\,\varepsilon^{s-D^\star_{\mathcal{W}}+(k-1)\,(2-D^\star_{\mathcal{W}})-i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}}{s-D^\star_{\mathcal{W}}+(k-1)\,(2-D^\star_{\mathcal{W}})-i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}}\,.\end{aligned} \qquad (\mathcal{R}\,84)$$

The poles of $\tilde{\zeta}^{e,\star}_m(s)$ are then given by:

$$s \quad = \quad D^\star_{\mathcal{W}} - (k-1)\,(2-D^\star_{\mathcal{W}}) \pm i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}$$

while the poles of $\tilde{\zeta}^e_m(s)$ are given by:

$$s \quad = \quad D_{\mathcal{W}} - (k-1)\,(2-D_{\mathcal{W}}) \pm i\,\ell_{k,(N_b-1)\,j+q,m}\,\mathbf{p}\,.$$

The symmetry $s \mapsto 2-s$ thus interchanges $D_{\mathcal{W}}$ and $D^\star_{\mathcal{W}} = 2 - D_{\mathcal{W}}$.

Since the residues of $\tilde{\zeta}^{e,\star}_m(s)$ and $\tilde{\zeta}^e_m(s)$ are the same (they are given by Theorem 2.18, on page 2.18, where, again, λ is replaced by $\lambda^\star$, N_b by N_b^2, and $D_{\mathcal{W}}$ by $D^\star_{\mathcal{W}} - 2$, with $\varepsilon = \dfrac{1}{N_b^2}$); namely, $\tilde{\zeta}^e_m(s)$ satisfies, for all integers $m \geqslant m_0$, and for all $s \in \mathbb{C}$, the functional equation

$$\tilde{\zeta}^{e,\star}_m(s) = \tilde{\zeta}^e_m(2-s)\,.$$

This concludes the proof, since, according to Theorem 2.20, on page 26, the corresponding global zeta functions $\tilde{\zeta}^{e,\star}_{\mathcal{W}}(s)$ and $\tilde{\zeta}^e_{\mathcal{W}}(s)$ are obtained by taking the limits of the local zeta functions $\tilde{\zeta}^{e,\star}_m(s)$ and $\tilde{\zeta}^e_m(s)$, respectively.

(Note that again, for notational simplicity, in the statement of Theorem 4.37, on page 79, we use the notation $\ell_{j_{k,m},k}$ of Theorem 2.18, on page 23 – associated to the integer $0 \leqslant k \leqslant m$ – instead of the notation $\ell_{k,(N_b-1)\,j+q,m}$ – associated to the integer $k-1 \geqslant 0$ of Theorem 2.20, on page 26. Of course, both notations are equivalent.)

□

The following result is an immediate consquence of the proof of Theorem 4.37, on page 81,

Corollary 4.38. *For all $m \in \mathbb{N}^\star$ sufficiently large (i.e., for all $m \geqslant m_0$, where $m_0 \in \mathbb{N}^\star$ is optimal), $\tilde{\zeta}_m^e$, the m^{th} local (polyhedral) zeta function of the Weierstrass Curve $\Gamma_{\mathcal{W}}$, satisfies the following functional equation, for all $s \in \mathbb{C}$,*

$$\tilde{\zeta}_m^{e,\star}(2-s) = \tilde{\zeta}_m^e(s)\,, \qquad (\mathcal{R}\,85)$$

where $\tilde{\zeta}_m^{e,\star}$ denotes the m^{th} local (polyhedral, effective) zeta function of$\Gamma_{\mathcal{W}}^\star = \Gamma_{\mathcal{W}^\star}$, the dual Weierstrass Curve of $\Gamma_{\mathcal{W}}$.

Note that it is the exact local counterpart of the functional equation satisfied by $\tilde{\zeta}_{\mathcal{W}}^e$ in relation ($\mathcal{R}$82), on page 80.

Definition 4.23 (Supercritical, Subcritical and Critical Cases).

Let $\widetilde{\mathcal{W}}$ be a Weierstrass function, with associated parameters $\tilde{\lambda}$ and $\widetilde{N_b}$, where $0 < \tilde{\lambda} < 1$ and $\widetilde{N_b} \in \mathbb{N}^\star$. Then, $\widetilde{\mathcal{W}}$ (or, equivalently, the associated Weierstrass curve $\tilde{\Gamma}_{\mathcal{W}} = \Gamma_{\widetilde{\mathcal{W}}}$) is said to be *supercritical*, *subcritical* or *critical*, respectively, if $\tilde{\lambda}\,\widetilde{N_b} > 1$, $\tilde{\lambda}\,\widetilde{N_b} < 1$, or $\tilde{\lambda}\,\widetilde{N_b} = 1$. Note that – according to the definition of the dual function $\mathcal{W}^\star$ and of $\Gamma_{\mathcal{W}}^\star = \Gamma_{\mathcal{W}^\star}$ given in Theorem 4.37, on page 79, then, $\mathcal{W}^\star$ (or, equivalently, $\Gamma_{\mathcal{W}}^\star = \Gamma_{\mathcal{W}^\star}$) is subcritical (respectively, supercritical) if and only if $\mathcal{W}$ (or, equivalently, $\Gamma_{\mathcal{W}}$) is supercritical (respectively, subcritical).

Also, it is critical if and only if $\mathcal{W}^\star = \mathcal{W}$.

Finally, note that $\mathcal{W}^{\star\star} = \mathcal{W}$ (i.e., $\Gamma_{\mathcal{W}}^{\star\star} = \Gamma_{\mathcal{W}^{\star\star}}$). Indeed, since $\lambda^\star = \dfrac{1}{\lambda\,N_b^2}$, we have that

$$\lambda^{\star\star} = \frac{1}{\lambda^\star\,N_b^2} = \lambda\,.$$

Remark 4.17 (**Subcritical Case**).

Assume that $\mathcal{W}$ is supercritical (i.e., $\lambda\,N_b > 1$), as usual. Then, observe that $D_{\mathcal{W}}^\star = 2 - D_{\mathcal{W}} \in]0,1[$ (see relation ($\mathcal{R}$79), in Theorem 4.37, on page 79) cannot be equal to the Minkowski dimension of the dual Weierstrass Curve $\Gamma_{\mathcal{W}^\star}$. Indeed, the latter dimension must be equal to 1 ($> D_{\mathcal{W}^\star}$) because $\Gamma_{\mathcal{W}^\star}$ is of class C^1 and hence, $\Gamma_{\mathcal{W}}$ is rectifiable. Instead, $D_{\mathcal{W}}^\star$ should be equal to the anti-abscissa of convergence of $\tilde{\zeta}_{\mathcal{W}}^{e,\star}$ – or the anti-abscissa of holomorphic continuation of $\tilde{\zeta}_{\mathcal{W}}^{e,\star}$, in a sense analogous to that of [LRŽ17b], but with the implied convergence or holomorphic continuation holding in a left rather than in a right half-plane.

We expect that the methods and results of [DL23b] can be naturally extended and adapted in order to show that the poles of $\tilde{\zeta}_{\mathcal{W}}^{e,\star}$ – which, in light of Corollary 2.21, on page 28, and the functional equation ($\mathcal{R}$82) in Theorem 4.37, on page 79, must all be simple and precisely given by

$$\omega_{k,m}^\star = 2 - \omega_{k,m} = m\,(2 - D_{\mathcal{W}}) \pm i\,\ell_{j_{k,m},k}\,\mathbf{p} \quad , \quad \text{with } m \in \mathbb{N}\,\text{arbitrary and } 0 \leqslant k \leqslant m\,,$$

where the integers $\ell_{j_{k,m},k} \in \mathbb{N}$ (which depend on k and m) are given in Theorem 2.18, on page 23) and

$$\omega_{k,m} = D_{\mathcal{W}} - m\,(2 - D_{\mathcal{W}}) \pm i\,\ell_{j_{k,m},k}\,\mathbf{p}\,,$$

are precisely the exact (intrinsic) Complex Dimensions of the subcritical Weierstrass Curve $\Gamma_{\mathcal{W}^\star}$. Accordingly, it would follow that in addition to being (at least C^1) smooth, $\Gamma_{\mathcal{W}^\star}$ is fractal, in the sense of the extended theory of Complex Dimensions.

Finally, let

$$n = \max\left\{j \in \mathbb{N}^\star \,:\, \lambda^\star\, N_b^j < 1\right\}.$$

Then, we conjecture that the subcritical Weierstrass function $\mathcal{W}^\star$ is smooth of class C^n, with n being *optimal* (i.e., $\mathcal{W}^\star$ is not of class C^{n+1} and $\left(\mathcal{W}^\star\right)^{(n)}$, the n^{th} derivative of $\mathcal{W}^\star$, is Hölder continuous of *optimal* Hölder exponent $\alpha^\star$, given by

$$\alpha^\star = \left\{2 - D_{\mathcal{W}}^\star\right\} = \left\{2 - \frac{\ln \frac{1}{\lambda^\star}}{\ln N_b}\right\} = \left\{D_{\mathcal{W}}\right\} = \left\{2 - \frac{\ln \frac{1}{\lambda}}{\ln N_b}\right\} \in [0,1)\,, \qquad (\mathcal{R}\,86)$$

where $\{y\}$ is the fractional part of $y \in \mathbb{R}$, and just below, $[y]$ stands for the integer part of $y \in \mathbb{R}$.

In other words, conjecturally, $n = \left[\frac{\ln \frac{1}{\lambda^\star}}{\ln N_b}\right] \in \mathbb{N}^\star$ is the maximal order of smoothness of $\mathcal{W}^\star$, while $\alpha^\star = \left\{\frac{\ln \frac{1}{\lambda^\star}}{\ln N_b}\right\}$ is given by relation ($\mathcal{R}$86). Rephrased: Conjecturally, the subcritical Weierstrass funcion $\mathcal{W}^\star$ is Hölder continuous of class $C^{n+\alpha^\star}$ (e.g., in the sense of [JM96]), where $n + \alpha^\star = n + 2 - \frac{\ln \frac{1}{\lambda^\star}}{\ln N_b}$, with $n \in \mathbb{N}^\star$ and $\alpha^\star \in [0,1)$ being optimal and given as above.

Remark 4.18 (**Critical Weierstrass Curve and Self-Duality**).

Note that since $\lambda\, N_b > 1$, the Weierstrass function $\mathcal{W}$ (or, equivalently, the Weierstrass Curve $\Gamma_{\mathcal{W}}$) is nowhere differentiable. However, since $\lambda^\star\, N_b < 1$, its dual Weierstrass function $\mathcal{W}^\star$ (or, equivalently, the Weierstrass Curve $\Gamma_{\mathcal{W}^\star}$) is at least C^1-smooth – but is still fractal, in the sense of the theory of Complex Dimensions, and in light of the functional equation ($\mathcal{R}$82), and of [DL23b] (Theorem 2. 19 and Corollary 2. 20), as is discussed in more detail in the previous remark.

Observe that in *the critical case* when $\lambda\, N_b = 1$, then $\mathcal{W} = \mathcal{W}^\star$ and so, $\Gamma_{\mathcal{W}} = \Gamma_{\mathcal{W}^\star}$ and $\tilde{\zeta}_{\mathcal{W}}^e = \tilde{\zeta}_{\mathcal{W}}^{e,\star}$. Consequently, the critical Weierstrass function $\mathcal{W}$, the critical Weierstrass Curve $\Gamma_{\mathcal{W}}$ and the global effective zeta function $\tilde{\zeta}_{\mathcal{W}}^e$ are all *self-dual.* In particular, $\tilde{\zeta}_{\mathcal{W}}^e$ satisfies the following *self-dual functional equation*, for all $s \in \mathbb{C}$,

$$\tilde{\zeta}_{\mathcal{W}}^e(s) = \tilde{\zeta}_{\mathcal{W}}^e(2-s)\,. \qquad (\mathcal{R}\,87)$$

Since $D_{\mathcal{W}} = D_{\mathcal{W}}^\star = 2 - D_{\mathcal{W}} = 1$ in this case, it follows that, conjecturally (see the corresponding discussion in Remark 4.17, on page 82), the (exact) Complex Dimensions of the critical Weierstrass Curve $\Gamma_{\mathcal{W}}$ should be simple and given by

$$m \pm i\,\ell_{j_{k,m},k}\,\mathbf{p} \quad , \quad \text{with } m \in \mathbb{N} \text{ arbitrary and } 0 \leqslant k \leqslant m\,,$$

where the integers $\ell_{j_{k,m},k} \in \mathbb{N}$ (which depend on k and m) are given in Theorem 2.18, on page 23); i.e., the set of Complex Dimensions is included in the rank 2 lattice $\mathbb{Z} \oplus i\,\mathbf{p}\,\mathbb{Z}$, where $\mathbf{p} = \frac{2\,\pi}{\ln N_b}$ is the oscillatory period of $\Gamma_{\mathcal{W}}$.

Also, in light of the self-dual functional equation ($\mathcal{R}87$), it is natural to call the vertical line $\mathcal{R}e(s) = 1$ *the critical line* of the fractal zeta function $\widetilde{\zeta}^{e}_{\mathcal{W}}$. Furthermore, it follows from ($\mathcal{R}87$) that, in general, the poles ω (i.e., the Complex Dimensions of $\Gamma_{\mathcal{W}}$), as well as the zeros of $\widetilde{\zeta}^{e}_{\mathcal{W}}$, come in quadruplets $(\omega, \bar{\omega}, 2-\omega, 2-\bar{\omega})$, unless they are located on the real axis (i.e., $\omega = \bar{\omega}$, or, equivalently, $\omega \in \mathbb{R}$) or on the critical line $\mathcal{R}e(s) = 1$ (i.e., $\omega = 2-\bar{\omega}$, or, equivalently, $\mathcal{R}e(\omega) = 1$).

Remark 4.19 (**Possible Analogy with the Conjectural Fractal Flow in [Lap08]**).

The above comments in Remark 4.17, on page 82, and in Remark 4.18, on page 83, along with Definition 4.23, on page 82, suggest that there might be interesting connections between our present setting and the (noncommutative) fractal flow (called *the modular flow*) conjectured to exist in [Lap08] and assumed to play a key role in interpreting dynamically and geometrically, as well as possibly establishing, the (generalized) Riemann Hypothesis.

This fractal flow acts on the moduli space of fractal membranes and to it, are naturally associated a flow of zeta functions (or *partition functions*), along with a flow of zeros and poles.

Furthermore, the (noncommutative) fixed points of the flow are the self-dual geometries and self-dual zeta functions – naturally corresponding, in our present setting, to the critical (and hence also self-dual) Weierstrass Curves and their (global, effective) fractal zeta functions. Note that, here, since N_b is fixed, the parameter playing the role of *time* is the parameter λ (or some suitable function of λ), with the $\pm$ infinite time limit (or the $\pm$ zero temperature limit) corresponding to λ tending to $\dfrac{1}{N_b}$ from above or from below, respectively. In a later work, we hope to pursue this analogy with [Lap08].

4.2.6 A Summary of the Analogies Between the Classical Theory and our Fractal Theory

The following table may be helpful to the reader as a quick guide through the new fractal counterpart of classical Hodge theory developed in this paper.

Classical theory	**Fractal theory**
Exterior derivative, d, on page 37	Natural transfer operator $\mathcal{L}_{\mathcal{W}}$, on page 64
Adjoint of the exterior differential, $d^\star$	Dual natural transfer operator $\mathcal{L}^{\#}_{\mathcal{W}}$, on pages 46 and 64
Hodge Laplacian, $\square = \left(d + d^\star\right)^2 = d\,d^\star + d^\star\,d$, on page 41	Fractal Hodge Laplacian, $\mathcal{L}_{\mathcal{W}}\,\mathcal{L}^{\#}_{\mathcal{W}} + \mathcal{L}^{\#}_{\mathcal{W}}\,\mathcal{L}_{\mathcal{W}}$, on page 70
Lefschetz operator, $\mathcal{L}ef_{\mathcal{W}}$, on page 42	Fractal Lefschetz operator, $\mathcal{L}ef_{\mathcal{W}} = i\,\mathcal{L}_{\mathcal{W}}\,\mathcal{L}^{\#}_{\mathcal{W}} = i\,\mathcal{L}^{\#}_{\mathcal{W}}\,\mathcal{L}_{\mathcal{W}}$, on page 69
Polarization operator, $\mathbf{Q}$, on page 44	Prefractal polarization operator, $\mathbf{Q}_m$, on page 60
Poincaré Duality, on page 42	Fractal Poincaré Duality, on page 63
Hard Lefschetz Theorem, on page 43	Fractal Hard Lefschetz Theorem, on page 69
Hodge–Riemann Relations, on page 44	Fractal Hodge–Riemann Relations, on page 60
Hodge decomposition, on page 41	Fractal Hodge decomposition, on page 63

5 Concluding Comments

Thus far, the (explicit) determination of the connections between fractal zeta functions and differential operators remained an open problem, that we have partially addressed. In doing so, our results shed new lights on the theory of Complex Dimensions which, at the same time, enables us to extend to the fractal realm the classical Hodge theory in terms of an orthogonal decomposition of the fractal cohomology. Note that those results rely on our previous works [DL23b] and [DL22b] where, for the first time, in the case of a (non self-affine) fractal curve, we respectively obtain the exact expressions of the local and global polyhedral fractal zeta functions, as well as an explicit determination of the cohomology groups associated with the (fractal) curve.

A key remaining issue is to obtain a clear cohomological (and spectral) interpretation of the zeros of the global polyhedral zeta function $\tilde{\zeta}^e_{\mathcal{W}}$ – or rather, for example, of the poles and zeros of a zeta function whose logarithmic derivative coincides with $\tilde{\zeta}^e_{\mathcal{W}}$. Accordingly, the zeros and the poles would be placed on a truly level-playing field.

By necessity of concision, we postpone to later work a careful consideration of several of the functional analytic and operator-theoretic issues involved in this paper, especially once we consider (which is only partially done in this article; see Remark 4.12, on page 71) the action of the Lefschetz operator on the total (infinite dimensional) fractal cohomology $H^\star \otimes \mathbb{C}$, as well as of the various operators involved (transfer operator and its dual, Lefschetz operator, fractal Hodge Laplacian and Frobenius operator).

In the near future, we intend to keep on exploring the fields of fractal algebraic topology. In particular, our current quest is to find an appropriate fractal homology theory, dual of the existing fractal cohomology theory in [DL22b]. Homology, which enables us to algebraically represent and model topological spaces, appears as a compulsory step when studying fractals and their applications to real life, for instance, morphogenesis, where the occurrence of similar or quasi-similar polygonal patterns in fractal shaped living forms suggests that their abstract representation (for computational purposes again), can be obtained by means of fractal homology.

Clearly, our results and methods in this paper and in our previous work [DL22a], [DL22b], [DL24a], [DL24b], [DL23b], could be extended to a variety of fractal geometries; see, e.g., [DL23c], an extension of [DL23b] to the case of the classic Koch Curve. We expect (as in [LvF13], [Lap08], [Lap19], [Lap24]) that some of these fractal-like geometries may provide suitable geometric models for the classic number theories (e.g., number fields and function fields) arising in mathematics and associated with arithmetic L-functions, including the Riemann zeta function and its various generalizations.

If this is correct, then the theory presented in this paper may provide a very useful and powerful tool to understand arithmetic geometries, and to eventually resolve some of its most unattainable conjectures, including the Riemann Hypothesis and its natural generalizations.

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