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Sergeant-Perthuis

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Integrated Information Theory (IIT) with Simple Maths

David Rudrauf, **Tonglin Yan**¹, Nils Ruet, Kenneth Williford,
Grégoire Sergeant-Perthuis²

[1] CIAMS, Université Paris-Saclay, [2] LCQB, Sorbonne Université

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For the 'preprint' go here: [here](#)

- What we do:
 - **Reinforcement learning** + **phenomenological** aspects of consciousness

Reproduce the experience of *space* in 'robots'

Consciousness involves a subjective perspective, characterized by viewpoint-structured organization, a sense of unity (holistic world), embodiment, and an internal representation of the world in perspective from a specific standpoint.

↔ the **Projective Consciousness Model**

- Initiated by D. Rudrauf, K. Williford, D. Bennequin, K. Friston [RBG⁺17, WBFR18, RBW20]

→ See K. Williford's MoC4 presentation of phenomenological motivation
<https://www.youtube.com/watch?v=eHyVZWZMqzg&t=853s>

- What do I do?
 - Background: mathematical physics (PhD)
 - Machine learning \cap geometry → computational biology
 - ↪ Embarked on the *consciousness* adventure
 - ↔ Geometric structure of the world model [RSPB⁺20, SPRR⁺23]
- Why this work?
 - Disseminate formal and computational models of consciousness
 - Simplify entering into the mathematical details of the models: IIT, active inference, etc.

Based on a seminar given at:

- PMMC2: Paris Mathematical Models of Cognition and Consciousness (PMMC2, [link to the seminar page](#))
 - On mathematical and computational models of consciousness and cognition
 - Longer version on YouTube: [link to the talk](#)
 - ↪ Come and give a talk

Today's presentation is a simplification of the excellent presentation:

- *The mathematical structure of integrated information theory*, Johannes Kleiner and Sean Tull. In *Frontiers in Applied Mathematics and Statistics*, 2020.

Definition (Probability measures)

Let E be a finite set, denote by $\mathbb{P}(E)$ the set of probability measures on E ,

$$p \in \mathbb{P}(E) \iff \forall x \in E, p(x) \geq 0 \text{ and } \sum_{x \in E} p(x) = 1 \quad (0.1)$$

Definition (Markov kernels (stochastic maps))

A Markov kernel or stochastic map T from E to F , denoted as $T : E \rightarrow \mathbb{P}(F)$, sends any point $x \in E$ to a probability measure $T_x \in \mathbb{P}(F)$. For $x \in E$ and $y \in F$, we will denote $T_x(y)$ as $T(y|x)$.

- We will be considering multiple variables $(X_i, i \in S)$ denotes X_S
- When S is a finite set:
 - enumerate S , $S \simeq [1, \dots, N]$
- We will consider sub-collections of variable:
 - $a \subseteq S$
 - \bar{a} the complement of a in S ,

$$a \cup \bar{a} = S \qquad a \cap \bar{a} = \emptyset$$

- a subset $a \subseteq S$ is associates to a random variable $(X_i, i \in a)$
 - X_i takes values in E_{X_i}
 - Denote $(X_i, i \in a)$ as X_a
 - X_a takes values in $\prod_{i \in a} E_{X_i}$ denoted as E_{X_a}

Definition (Conditional expectation)

Let $(Y_i, i \in S_1)$ be a collection of variables taking values in $F = \prod_{i \in S_1} F_{Y_i}$. Each F_{Y_i} is a finite set. Let $p \in \mathbb{P}(F)$. For any $a \subseteq S_1$, and any function $f : F \rightarrow \mathbb{R}$, one defines the conditional expectation with respect to Y_a as;

$$\forall y_a \in F_a, \quad \mathbb{E}[f | Y_a](y_a) = \sum_{y_{\bar{a}} \in Y_{\bar{a}}} \frac{f(y_{\bar{a}}, y_a) p(y_{\bar{a}}, y_a)}{\sum_{y_{\bar{a}} \in Y_{\bar{a}}} p(y_{\bar{a}}, y_a)}$$

Example with two random variables X, Y , assume:

→ $X \in E, Y \in F$, where E, F are finite sets

→ $x \in E, y \in F$

then,

→ $\mathbb{E}[Y = y | X = x] := P(y|x)$

→ and

$$P(y|x) = \frac{P(X = x, Y = y)}{P(X = x)} \quad (\text{Bayes' Rule})$$

We want to see the effect of the stochastic dynamic T on sub-collections of variables.

- We start with $T : E_{X_1} \times \dots \times E_{X_N} \rightarrow \mathbb{P}(F_{Y_1} \times \dots \times F_{Y_M})$
- Choose $a = (1, \dots, n_1)$ and $b = (1, \dots, m_1)$ with $n_1 \leq N$ and $m_1 \leq M$
- How to deduce a transition

$$T^{a,b} : E_{X_1} \times \dots \times E_{X_{n_1}} \rightarrow \mathbb{P}(F_{Y_1} \times \dots \times F_{Y_{m_1}})$$

Definition (Building a Markov kernel $E_{X_a} \rightarrow \mathbb{P}(F_{Y_b})$ from a prior Q)

Let $X_S := (X_i, i \in S)$, $Y_{S_1} := (Y_i, i \in S_1)$ and $E = \prod_{i \in S} E_{X_i}$,
 $F = \prod_{i \in S_1} F_{Y_i}$.

Let $T = E \rightarrow \mathbb{P}(F)$ be a Markov kernel. For any $a \subseteq S$ and $b \subseteq S_1$, a choice of $Q \in \mathbb{P}(E)$ allows us to derive from T the kernel denoted $T^{Q,a,b} : E_{X_a} \rightarrow \mathbb{P}(F_{Y_b})$, which encodes the effect of the variables X_a on Y_b . It is defined as,

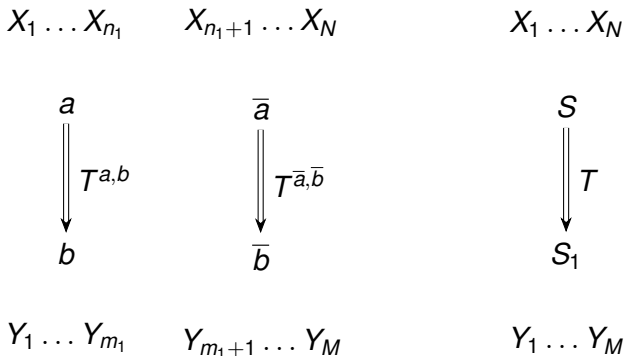
$$\forall y_b \in F_{Y_b}, \forall x_a \in E_{X_a} \quad T^{Q,a,b}(y_b|x_a) := \mathbb{E}[Y_b = y_b | X_a = x_a]$$

– How? \leftrightarrow Joint distribution,

$$P(Y_{S_1} = y, X_S = x) := T(y|x) \times Q(x)$$

\leftrightarrow Sum out (*marginalize*) $X_{\bar{a}}$, $Y_{\bar{b}}$

'Cutting' interactions: the central operation



Left: Cutting the interactions. Right: Overall interaction.

- We quantify the effect of ‘cutting’ interactions between variables
- $(T^{a,b}, T^{\bar{a},\bar{b}})$ should be in the same space as T

Definition (Product of local kernels)

For any two probability kernels, $T^{a,b} : E_{X_a} \rightarrow \mathbb{P}(F_{Y_b})$ and $T^{\bar{a},\bar{b}} : E_{X_{\bar{a}}} \rightarrow \mathbb{P}(F_{Y_{\bar{b}}})$ posit,

$$(T^{a,b} \otimes T^{\bar{a},\bar{b}})(y|x) := T^{a,b}(y_b|x_a) \cdot T^{\bar{a},\bar{b}}(y_{\bar{b}}|x_{\bar{a}}) \quad (0.2)$$

→ We want to quantify how far $T^{a,b} \otimes T^{\bar{a},\bar{b}}$ is from T

Definition (Informal definition of divergence)

For a finite space Y , we define a divergence D on $\mathbb{P}(Y)$ as a function $D : \mathbb{P}(Y) \times \mathbb{P}(Y) \rightarrow \mathbb{R}_{\geq 0}$ such that, for any two probability distributions P and P_1 in $\mathbb{P}(Y)$, $D(P, P_1)$ decreases as the two distributions P and P_1 get 'closer'; and it reaches its minimum value of 0 when and only when $P = P_1$.

- The dissimilarity is on *probability distributions*.
- Fix $x \in E$ then, $T^{a,b} \otimes T^{\bar{a},\bar{b}}(\cdot|x)$ is a probability distribution.
- Similarly $T(\cdot|x) \in \mathbb{P}(F)$

→ For a fixed x

↪ denote $T^{a,b} \otimes T^{\bar{a},\bar{b}}(\cdot|x)$ as $T_{x_a}^{a,b} \otimes T_{x_{\bar{a}}}^{\bar{a},\bar{b}}$

↪ denote $T(\cdot|x)$ as T_x

- Little φ_e focusing on effects.
 - $S \simeq [1, \dots, N]$ and $X_1 \dots X_N$
 - $M \subseteq S$, 'old' X_S is now X_M
 - $P \subseteq S$ 'old' Y_{S_1} is now X_P
 - To remember that we start with X_M and we go to X_P denote T_M^P the associated kernel

Definition

For any $M, P \subseteq S$ and $x_M \in X_M$,

$$\varphi_{M, x_M}^P := \inf_{\substack{a \subseteq M \\ b \subseteq P}} D(T_{M, x_M}^P | T_{M, x_a}^{P, (a, b)} \otimes T_{M, x_{\bar{a}}}^{P, (\bar{a}, \bar{b})}) \quad (0.3)$$

And

$$\varphi_{M, x_M}^* := \max_{P \subseteq S} \varphi_{M, x_M}^P \quad (0.4)$$

→ One more step to compute *big* Φ , focusing on effects (see [here](#))

$$\Phi_{M,x,b} := \sum_{a \subseteq M} \varphi_{a,x_a}^* D(T_{a,x_a}^{\psi(a,x_a)} | T_{a,x_a}^{b_a} \otimes T_{a,x_a}^{\bar{b}_a})$$

with $\psi(M, x) := \mathbf{argmax} \varphi_M^P$

$$\Phi_{M,x} = \mathbf{argmin}_{b \subseteq \psi(M, x_M)} \Phi_{M,x,b}$$




→ IIT 4.0 introduces an important difference in how information is quantified \rightsquigarrow dissimilarity function.

→ Go check their paper [ABF⁺23].

Thank you very much for your attention

Thank you very much for your attention!

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