

# Integrated Information Theory (IIT) with Simple Maths (slides)

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# Integrated Information Theory (IIT) with Simple Maths

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For the 'preprint' go here: here

- What we do:
  - → Reinforcement learning + phenomenological aspects of consciousness

#### Reproduce the experience of space in 'robots'

Consciousness involves a subjective perspective, characterized by viewpoint-structured organization, a sense of unity (holistic world), embodiment, and an internal representation of the world in perspective from a specific standpoint.

- → the Projective Consciousness Model
  - Initiated by D. Rudrauf, K. Williford, D. Bennequin, K. Friston [RBG<sup>+</sup>17, WBFR18, RBW20]
- $\longrightarrow$  See K. Williford's MoC4 presentation of phenomenological motivation https://www.youtube.com/watch?v=eHyVZWZMqzg&t=853s

- What do I do?
  - → Background: mathematical physics (PhD)
  - → Machine learning ∩ geometry → computational biology
  - --- Embarked on the consciousness adventure
    - → Geometric structure of the world model [RSPB+20, SPRR+23]
- Why this work?
  - → Disseminate formal and computational models of consciousness
  - $\rightarrow$  Simplify entering into the mathematical details of the models: IIT, active inference, etc.

#### Based on a seminar given at:

- PMMC2: Paris Mathematical Models of Cognition and Consciousness (PMMC2, link to the seminar page)
  - $\,\rightarrow\,$  On mathematical and computational models of consciousness and cognition
  - → Longer version on YouTube: link to the talk
  - $\hookrightarrow$  Come and give a talk

Today's presentation is a simplification of the excellent presentation:

→ The mathematical structure of integrated information theory, Johannes Kleiner and Sean Tull. In Frontiers in Applied Mathematics and Statistics, 2020.

#### Definition (Probability measures)

Let E be a finite set, denote by  $\mathbb{P}(E)$  the set of probability measures on E,

$$p \in \mathbb{P}(E) \iff \forall x \in E, p(x) \ge 0 \text{ and } \sum_{x \in E} p(x) = 1$$
 (0.1)

#### Definition (Markov kernels (stochastic maps))

A Markov kernel or stochastic map T from E to F, denoted as  $T: E \to \mathbb{P}(F)$ , sends any point  $x \in E$  to a probability measure  $T_x \in \mathbb{P}(F)$ . For  $x \in E$  and  $y \in F$ , we will denote  $T_x(y)$  as T(y|x).

- We will be considering multiple variables  $(X_i, i \in S)$  denotes  $X_S$
- When S is a finite set:
  - $\rightarrow$  enumerate  $S, S \simeq [1, ..., N]$
- We will consider sub-collections of variable:
  - $\rightarrow a \subseteq S$
  - $\rightarrow \overline{a}$  the complement of a in S,

$$a \cup \overline{a} = S$$
  $a \cap \overline{a} = \emptyset$ 

- a subset  $a \subseteq S$  is associates to a random variable  $(X_i, i \in a)$ 
  - $\rightarrow X_i$  takes values in  $E_{X_i}$
  - $\rightarrow$  Denote  $(X_i, i \in a)$  as  $X_a$
  - $\rightarrow X_a$  takes values in  $\prod_{i \in a} E_{X_i}$  denoted as  $E_{X_a}$

#### Definition (Conditional expectation)

Let  $(Y_i, i \in S_1)$  be a collection of variables taking values in  $F = \prod_{i \in S_1} F_{Y_i}$ . Each  $F_{Y_i}$  is a finite set. Let  $p \in \mathbb{P}(F)$ . For any  $a \subseteq S_1$ , and any function  $f : F \to \mathbb{R}$ , one defines the conditional expectation with respect to  $Y_a$  as;

$$\forall y_a \in F_a, \quad \mathbb{E}[f|Y_a](y_a) = \sum_{y_{\overline{a}} \in Y_{\overline{a}}} \frac{f(y_{\overline{a}}, y_a) p(y_{\overline{a}}, y_a)}{\sum_{y_{\overline{a}} \in Y_{\overline{a}}} p(y_{\overline{a}}, y_a)}$$

Example with two random variables X, Y, assume:

- $\rightarrow X \in E, Y \in F$ , where E, F are finite sets
- $\rightarrow x \in E, y \in F$

then,

$$\rightarrow \mathbb{E}[Y = y | X = x] := P(y | x)$$

 $\rightarrow$  and

$$P(y|x) = \frac{P(X = x, Y = y)}{P(X = x)}$$
 (Bayes' Rule)

We want to see the effect of the stochastic dynamic T on sub-collections of variables.

- We start with  $T: E_{X_1} \times \ldots \times E_{X_N} \to \mathbb{P}(F_{Y_1} \times \ldots \times F_{Y_M})$
- Choose  $a = (1, ..., n_1)$  and  $b = (1, ..., m_1)$  with  $n_1 \le N$  and  $m_1 \le M$
- How to deduce a transition

$$T^{a,b}: E_{X_1} \times \ldots \times E_{X_{n_1}} \rightarrow \mathbb{P}(F_{Y_1} \times \ldots \times F_{Y_{m_1}})$$

### Definition (Building a Markov kernel $E_{X_a} \to \mathbb{P}(F_{Y_b})$ from a prior Q)

Let  $X_S := (X_i, i \in S)$ ,  $Y_{S_1} := (Y_i, i \in S_1)$  and  $E = \prod_{i \in S} E_{X_i}$ ,  $F = \prod_{i \in S_1} F_{Y_i}$ .

Let  $T = E \to \mathbb{P}(F)$  be a Markov kernel. For any  $a \subseteq S$  and  $b \subseteq S_1$ , a choice of  $Q \in \mathbb{P}(E)$  allows us to derive from T the kernel denoted  $T^{Q,a,b}: E_{X_a} \to \mathbb{P}(F_{Y_b})$ , which encodes the effect of the variables  $X_a$  on  $Y_b$ . It is defined as,

$$\forall y_b \in F_{Y_b}, \forall x_a \in E_{X_a} \quad T^{Q,a,b}(y_b|x_a) := \mathbb{E}[Y_b = y_b|X_a = x_a]$$

$$P(Y_{S_1} = y, X_S = x) := T(y|x) \times Q(x)$$

 $\hookrightarrow$  Sum out (*marginalize*)  $X_{\overline{a}}$ ,  $Y_{\overline{b}}$ 

#### 'Cutting' interactions: the central operation

Left: Cutting the interactions. Right: Overall interaction.

- We quantify the effect of 'cutting' interactions between variables
- $(T^{a,b}, T^{\overline{a},\overline{b}})$  should be in the same space as T

#### Definition (Product of local kernels)

For any two probability kernels,  $T^{a,b}: E_{X_a} \to \mathbb{P}(F_{Y_b})$  and

$$T^{\overline{a},\overline{b}}: E_{X_{\overline{a}}} o \mathbb{P}(F_{Y_{\overline{b}}})$$
 posit,

$$(T^{a,b}\otimes T^{\overline{a},\overline{b}})(y|x):=T^{a,b}(y_b|x_a).T^{\overline{a},\overline{b}}(y_{\overline{b}}|x_{\overline{a}})$$
(0.2)

ightarrow We want to quantify how far  $T^{a,b}\otimes T^{\overline{a},\overline{b}}$  is from T

#### Definition (Informal definition of divergence)

For a finite space Y, we define a divergence D on  $\mathbb{P}(Y)$  as a function  $D: \mathbb{P}(Y) \times \mathbb{P}(Y) \to \mathbb{R}_{\geq 0}$  such that, for any two probability distributions P and  $P_1$  in  $\mathbb{P}(Y)$ ,  $D(P, P_1)$  decreases as the two distributions P and  $P_1$  get 'closer'; and it reaches its minimum value of 0 when and only when  $P = P_1$ .

- The dissimilarity is on *probability distributions*.
- Fix  $x \in E$  then,  $T^{a,b} \otimes T^{\overline{a},\overline{b}}(.|x)$  is a probability distribution.
- Similarly  $T(.|x) \in \mathbb{P}(F)$
- $\rightarrow$  For a fixed x
  - $\rightsquigarrow$  denote  $T^{a,b} \otimes T^{\overline{a},\overline{b}}(.|x)$  as  $T^{a,b}_{x_a} \otimes T^{\overline{a},\overline{b}}_{x_{\overline{a}}}$
  - $\rightsquigarrow$  denote T(.|x) as  $T_x$

- Little  $\varphi_e$  focusing on effects.
  - $ightarrow \mathcal{S} \simeq [1,...,N] \text{ and } X_1...X_N$
  - $\rightarrow M \subseteq S$ , 'old'  $X_S$  is now  $X_M$
  - ightarrow  $P\subseteq \mathcal{S}$  'old'  $Y_{\mathcal{S}_1}$  is now  $X_P$
  - $\rightarrow$  To remember that we start with  $X_M$  and we go to  $X_P$  denote  $T_M^P$  the associated kernel

#### **Definition**

For any  $M, P \subseteq S$  and  $x_M \in X_M$ ,

$$\varphi_{M,x_M}^P := \inf_{\substack{a \subseteq M \\ b \subseteq P}} D(T_{M,x_M}^P | T_{M,x_a}^{P,(a,b)} \otimes T_{M,x_{\overline{a}}}^{P,(\overline{a},\overline{b})})$$
(0.3)

And

$$\varphi_{M,x_M}^* := \max_{P \subset S} \varphi_{M,x_M}^P \tag{0.4}$$

 $\rightarrow$  One more step to compute *big*  $\Phi$ , focusing on effects (see here)

$$\Phi_{M,x,b} := \sum_{a \subseteq M} \varphi_{a,x_a}^* D(T_{a,x_a}^{\psi(a,x_a)} | T_{a,x_a}^{b_a} \otimes T_{a,x_a}^{\overline{b}_a})$$

with  $\psi(M, x) := \operatorname{argmax} \varphi_M^P$ 

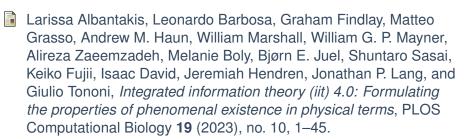
$$\Phi_{M,x} = \operatorname{argmin}_{b \subseteq \psi(M,x_M)} \Phi_{M,x,b}$$

- ightarrow IIT 4.0 introduces an important difference in how information is quantified  $\leadsto$  dissimilarity function.
  - $\rightarrow$  Go check their paper [ABF+23].

### Thank you very much for your attention

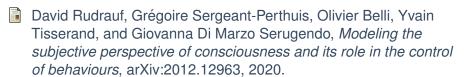
Thank you very much for your attention!

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