Integrated Information Theory (IIT) with Simple Maths (slides)

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ASSC 27
Tokyo, 4 July, 2024

For the 'preprint' go here: here

- What we do:
  - $\rightarrow$ **Reinforcement learning** + **phenomenological** aspects of consciousness

Reproduce the experience of *space* in 'robots'

Consciousness involves a subjective perspective, characterized by viewpoint-structured organization, a sense of unity (holistic world), embodiment, and an internal representation of the world in perspective from a specific standpoint.

$\hookrightarrow$ the **Projective Consciousness Model**

- Initiated by D. Rudrauf, K. Williford, D. Bennequin, K. Friston [RBG+17, WBFR18, RBW20]

$\longrightarrow$See K. Williford's MoC4 presentation of phenomenological motivation
`https://www.youtube.com/watch?v=eHyVZWZMqzg&t=853s`

- What do I do?
  - $\rightarrow$ Background: mathematical physics (PhD)
  - $\rightarrow$ Machine learning $\cap$ geometry $\rightarrow$ computational biology
  - $\rightsquigarrow$ Embarked on the *consciousness* adventure
    - $\hookrightarrow$ Geometric structure of the world model [RSPB$^+$20, SPRR$^+$23]
- Why this work?
  - $\rightarrow$ Disseminate formal and computational models of consciousness
  - $\rightarrow$ Simplify entering into the mathematical details of the models: IIT, active inference, etc.

Based on a seminar given at:

- PMMC2: Paris Mathematical Models of Cognition and Consciousness (PMMC2, link to the seminar page)
  - → On mathematical and computational models of consciousness and cognition
  - → Longer version on YouTube: link to the talk
  - ↪ Come and give a talk

Today's presentation is a simplification of the excellent presentation:

→ *The mathematical structure of integrated information theory*, Johannes Kleiner and Sean Tull. In Frontiers in Applied Mathematics and Statistics, 2020.

Definition (Probability measures)

Let E be a finite set, denote by $\mathbb{P}(E)$ the set of probability measures on E,

$$p \in \mathbb{P}(E) \iff \forall x \in E, p(x) \geq 0 \text{ and } \sum_{x \in E} p(x) = 1 \tag{0.1}$$

Definition (Markov kernels (stochastic maps))

A Markov kernel or stochastic map T from E to F, denoted as $T : E \to \mathbb{P}(F)$, sends any point $x \in E$ to a probability measure $T_x \in \mathbb{P}(F)$. For $x \in E$ and $y \in F$, we will denote $T_x(y)$ as $T(y|x)$.

- We will be considering multiple variables $(X_i, i \in S)$ denotes X_S
- When S is a finite set:
  - $\rightarrow$ enumerate S, $S \simeq [1, ..., N]$
- We will consider sub-collections of variable:
  - $\rightarrow$ $a \subseteq S$
  - $\rightarrow$ $\overline{a}$ the complement of a in S,

$$a \cup \overline{a} = S \qquad\qquad a \cap \overline{a} = \emptyset$$

- a subset $a \subseteq S$ is associates to a random variable $(X_i, i \in a)$
  - $\rightarrow$ X_i takes values in E_{X_i}
  - $\rightarrow$ Denote $(X_i, i \in a)$ as X_a
  - $\rightarrow$ X_a takes values in $\prod_{i \in a} E_{X_i}$ denoted as E_{X_a}

Definition (Conditional expectation)

Let $(Y_i, i \in S_1)$ be a collection of variables taking values in $F = \prod_{i \in S_1} F_{Y_i}$. Each F_{Y_i} is a finite set. Let $p \in \mathbb{P}(F)$. For any $a \subseteq S_1$, and any function $f : F \to \mathbb{R}$, one defines the conditional expectation with respect to Y_a as;

$$\forall y_a \in F_a, \quad \mathbb{E}[f|Y_a](y_a) = \sum_{y_{\overline{a}} \in Y_{\overline{a}}} \frac{f(y_{\overline{a}}, y_a) p(y_{\overline{a}}, y_a)}{\sum_{y_{\overline{a}} \in Y_{\overline{a}}} p(y_{\overline{a}}, y_a)}$$

Example with two random variables X, Y, assume:

- $\rightarrow$ $X \in E$, $Y \in F$, where E, F are finite sets
- $\rightarrow$ $x \in E$, $y \in F$

then,

- $\rightarrow$ $\mathbb{E}[Y = y|X = x] := P(y|x)$
- $\rightarrow$ and

$$P(y|x) = \frac{P(X = x, Y = y)}{P(X = x)} \qquad \text{(Bayes' Rule)}$$

We want to see the effect of the stochastic dynamic T on sub-collections of variables.

- We start with $T : E_{X_1} \times \ldots \times E_{X_N} \to \mathbb{P}(F_{Y_1} \times \ldots \times F_{Y_M})$
- Choose $a = (1, \ldots, n_1)$ and $b = (1, \ldots, m_1)$ with $n_1 \leq N$ and $m_1 \leq M$
- How to deduce a transition

$$T^{a,b} : E_{X_1} \times \ldots \times E_{X_{n_1}} \to \mathbb{P}(F_{Y_1} \times \ldots \times F_{Y_{m_1}})$$

Definition (Building a Markov kernel $E_{X_a} \to \mathbb{P}(F_{Y_b})$ from a prior Q)

Let $X_S := (X_i, i \in S)$, $Y_{S_1} := (Y_i, i \in S_1)$ and $E = \prod_{i\in S} E_{X_i}$, $F = \prod_{i\in S_1} F_{Y_i}$.
Let $T = E \to \mathbb{P}(F)$ be a Markov kernel. For any $a \subseteq S$ and $b \subseteq S_1$, a choice of $Q \in \mathbb{P}(E)$ allows us to derive from T the kernel denoted $T^{Q,a,b} : E_{X_a} \to \mathbb{P}(F_{Y_b})$, which encodes the effect of the variables X_a on Y_b. It is defined as,

$$\forall y_b \in F_{Y_b}, \forall x_a \in E_{X_a} \quad T^{Q,a,b}(y_b|x_a) := \mathbb{E}[Y_b = y_b | X_a = x_a]$$

– How? $\hookrightarrow$ Joint distribution,

$$P(Y_{S_1} = y, X_S = x) := T(y|x) \times Q(x)$$

$\hookrightarrow$ Sum out (*marginalize*) $X_{\overline{a}}$, $Y_{\overline{b}}$

‘Cutting’ interactions: the central operation

$$\begin{array}{ccccc} X_1 \dots X_{n_1} & & X_{n_1+1} \dots X_N & & X_1 \dots X_N \\ a & & \overline{a} & & S \\ \Big\Downarrow T^{a,b} & & \Big\Downarrow T^{\overline{a},\overline{b}} & & \Big\Downarrow T \\ b & & \overline{b} & & S_1 \\ Y_1 \dots Y_{m_1} & & Y_{m_1+1} \dots Y_M & & Y_1 \dots Y_M \end{array}$$

Left: Cutting the interactions. Right: Overall interaction.

- We quantify the effect of 'cutting' interactions between variables
- $(T^{a,b}, T^{\overline{a},\overline{b}})$ should be in the same space as T

Definition (Product of local kernels)

For any two probability kernels, $T^{a,b} : E_{X_a} \to \mathbb{P}(F_{Y_b})$ and $T^{\overline{a},\overline{b}} : E_{X_{\overline{a}}} \to \mathbb{P}(F_{Y_{\overline{b}}})$ posit,

$$(T^{a,b} \otimes T^{\overline{a},\overline{b}})(y|x) := T^{a,b}(y_b|x_a).T^{\overline{a},\overline{b}}(y_{\overline{b}}|x_{\overline{a}}) \tag{0.2}$$

$\rightarrow$ We want to quantify how far $T^{a,b} \otimes T^{\bar{a},\bar{b}}$ is from T

Definition (Informal definition of divergence)

For a finite space Y, we define a divergence D on $\mathbb{P}(Y)$ as a function $D : \mathbb{P}(Y) \times \mathbb{P}(Y) \rightarrow \mathbb{R}_{\geq 0}$ such that, for any two probability distributions P and P_1 in $\mathbb{P}(Y)$, $D(P, P_1)$ decreases as the two distributions P and P_1 get 'closer'; and it reaches its minimum value of 0 when and only when $P = P_1$.

- The dissimilarity is on *probability distributions*.
- Fix $x \in E$ then, $T^{a,b} \otimes T^{\bar{a},\bar{b}}(.|x)$ is a probability distribution.
- Similarly $T(.|x) \in \mathbb{P}(F)$

$\rightarrow$ For a fixed x

- $\rightsquigarrow$ denote $T^{a,b} \otimes T^{\bar{a},\bar{b}}(.|x)$ as $T^{a,b}_{x_a} \otimes T^{\bar{a},\bar{b}}_{x_{\bar{a}}}$
- $\rightsquigarrow$ denote $T(.|x)$ as T_x

- Little φ_e focusing on effects.
  - $\rightarrow$ $S \simeq [1, ..., N]$ and $X_1 ... X_N$
  - $\rightarrow$ $M \subseteq S$, 'old' X_S is now X_M
  - $\rightarrow$ $P \subseteq S$ 'old' Y_{S_1} is now X_P
  - $\rightarrow$ To remember that we start with X_M and we go to X_P denote T_M^P the associated kernel

Definition

For any $M, P \subseteq S$ and $x_M \in X_M$,

$$\varphi^P_{M,x_M} := \inf_{\substack{a \subseteq M \\ b \subseteq P}} D(T^P_{M,x_M} | T^{P,(a,b)}_{M,x_a} \otimes T^{P,(\overline{a},\overline{b})}_{M,x_{\overline{a}}}) \tag{0.3}$$

And

$$\varphi^*_{M,x_M} := \max_{P \subseteq S} \varphi^P_{M,x_M} \tag{0.4}$$

→ One more step to compute *big* Φ, focusing on effects (see here)

$$\Phi_{M,x,b} := \sum_{a \subseteq M} \varphi^{*}_{a,x_a} D(T^{\psi(a,x_a)}_{a,x_a} | T^{b_a}_{a,x_a} \otimes T^{\bar{b}_a}_{a,x_a})$$

with $\psi(M, x) := \textbf{argmax}\, \varphi^{P}_{M}$

$$\Phi_{M,x} = \text{argmin}_{b \subseteq \psi(M,x_M)} \Phi_{M,x,b}$$

→ IIT 4.0 introduces an important difference in how information is quantified ⇝ dissimilarity function.
  - → Go check their paper [ABF+23].

Thank you very much for your attention

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References I

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