Contributions au calcul variationnel géométrique et applications

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Habilitation à Diriger les Recherches en mathématiques

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[P16] J.F. BABADJIAN, V. MILLOT : Unilateral gradient flow of the Ambrosio-Tortorelli functional by minimizing movements, *Ann. Inst. H. Poincaré Analyse Non Linéaire* **31** (2014), 779–822.

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[P10] J.F. BABADJIAN, V. MILLOT : Homogenization of variational problems in manifold valued Sobolev spaces, *ESAIM : Control, Optim., Calc. Var.* **16** (2010), 833-855.

[P09] J.F. BABADJIAN, V. MILLOT : Homogenization of variational problems in manifold valued BV-spaces, *Calc. of Var. and Partial Differential Equations* **36** (2009), 7–47.

[P08] V. MILLOT, A. PISANTE : Relaxed energies for $H^{1/2}$–maps with values into the circle and measurable weights, *Indiana Univ. Math. J.* **58** (2009), 49–136.

[P07] V. MILLOT : The dipole problem for $H^{1/2}(S^2, S^1)$–maps and application, dans *Singularities in PDE and the Calculus of Variations*, CRM Proceedings and Lecture Notes **44** (2008), 165–178.

[P06] R. IGNAT, V. MILLOT : Energy expansion and vortex location for a two-dimensional rotating Bose-Einstein condensate, *Rev. Math. Phys.* **18** (2006), 119–162.

[P05] R. IGNAT, V. MILLOT : The critical velocity for vortex existence in a two-dimensional rotating Bose-Einstein condensate, *J. Funct. Anal.* **233** (2006), 260–306.

[P04] V. MILLOT : The relaxed energy for S^2-valued maps and measurable weights, *Ann. Inst. H. Poincaré Analyse Non Linéaire* **23** (2006), 135–157.

[P03] R. IGNAT, V. MILLOT : Vortices in a 2d rotating Bose-Einstein condensate, *C. R. Acad. Sci. Paris Sér.* I **340** (2005), 571–576.

[P02] V. MILLOT : Energy with weight for S^2-valued maps with prescribed singularities, *Calc. of Var. and Partial Differential Equations* **24** (2005), 83–109.

[P01] J.I. DÍAZ, V. MILLOT : Coulomb friction and oscillation : stabilization in finite time for a system of damped oscillators, *Actas XVIII CEDYA / VIII CMA, (editors) Public. Univ. of Tarragona, 2003.*

Les articles [P01, P02, P03, P04, P05, P06, P15] ne sont pas présentés dans ce mémoire.

Préambule

Les travaux présentés dans ce mémoire portent sur l'analyse de quelques problèmes mathématiques issus du *Calcul des Variations*. Loin de couvrir l'ensemble de ce vaste sujet, ils se concentrent essentiellement sur les aspects suivants : existence et relaxation, étude de minima ou de points stationnaires, théorie de la régularité, ou encore convergence variationnelle. A l'image du très célèbre problème de Plateau, les questions posées ont le plus souvent une nature géométrique, certaines d'entre elles ayant trait aux surfaces minimales, ou aux applications harmoniques. Toutefois, le cadre mathématique reste celui de l'*analyse*. Il fait appel à la théorie elliptique des équations aux dérivées partielles et à la théorie (géométrique) de la mesure. Les différentes études sont pour la plupart motivées par un souci de compréhension de phénomèmes issus de la physique de la matière condensée, ou de la mécanique des milieux continus.

Chaque section de ce mémoire constitue un résumé d'un travail spécifique ayant fait l'objet d'une publication (ou prépublication). Nous avons essayé de présenter chaque sujet d'une façon suffisamment précise pour mettre en lumière les mathématiques sous-jacentes. En outre, nous proposons un plan que nous espérons être le plus progressif et logique possible. Il ne correspond pas à la chronologie de nos travaux, mais plutôt à la vision d'ensemble que nous en avons aujourd'hui. Les résultats présentés ici sont le fruit de collaborations et de rencontres datant (déjà !) de plusieurs années.

Les différentes sections sont regroupées en deux chapitres. Le premier chapitre est consacré aux problèmes de nature "vectorielle" qui font intervenir des espaces de fonctions à valeurs dans une variété. Dans ce cadre, nous y parlerons d'applications harmoniques, d'équations de Ginzburg-Landau, ou encore d'homogénéisation. Le deuxième chapitre, quant à lui, s'attache aux questions "scalaires" où les objets géométriques inhérents sont de dimension ou de codimension 1. Il y sera fait mention d'hypersurfaces minimales, d'inégalités isopérimétriques, et de problèmes aux discontinuités libres.

Nous avons fait le choix de la langue anglaise pour la suite de la rédaction. Il n'y a de notre part aucun rejet de la langue de Molière, mais plutôt une habitude de travail.

Chapitre 1

Ginzburg-Landau systems and Sobolev maps into a manifold

1.1 The fractional Ginzburg-Landau equation & $1/2$-harmonic maps into spheres

The article [P17], in collaboration with Y. SIRE, is essentially devoted to the asymptotic analysis in a singular limit of a fractional version of the Ginzburg-Landau equation where the Laplacian is replaced by the square root Laplacian as defined in Fourier space. The classical complex Ginzburg-Landau equation has been widely studied because it shares many of the relevant features of more elaborate systems arising in the physics of superconductivity or superfluidity, see e.g. [2, 38, 173, 183]. In the spirit of the classical Landau's theory of phase transtions, fractional Ginzburg-Landau equations have been recently suggested in the physics literature in order to incorporate a long-range dependence posed by a nonlocal ordering, as it might appear in certain high temperature superconducting compounds, see [157, 202, 207]. In arbitrary dimensions, the Ginzburg-Landau equation has also a geometrical interest as it approximates in the singular limit "$\varepsilon \to 0$" the geometric equation of harmonic maps into a sphere (or into a more general manifold according to the potential well). Harmonic maps can be seen as higher dimensional generalizations of geodesics and are defined as critical points of the Dirichlet energy with respect to perturbations on the image. In our fractional setting, the singular limit still provides a geometric equation that we refer to as the $1/2$-*harmonic map system*. It is the fractional analogue of the classical harmonic maps and it corresponds to the Euler-Lagrange equation obtained from variations of a fractional Dirichlet energy associated to the square root Laplacian. In particular, if the domain dimension is one, $1/2$-harmonic lines are fractional versions of geodesics. The notion of $1/2$-harmonic maps from $\mathbb{R}$ into a manifold has been recently introduced by F. DA LIO & T. RIVIÈRE in [75, 76] where the regularity of weak solutions is established. A quite interesting fact about $1/2$-harmonic maps is that they naturally appear in the theory of *minimal surfaces with free boundary* [74, 199] and in some (related) problems of spectral geometry [109]. They are also intimately related to *harmonic maps with free boundary* [124, 127, 164], and to the so-called *semi-stiff boundary condition* arising in some Ginzburg-Landau theories, see [33] and the references therein.

The plan of this section is the following. The first part briefly introduces the appropriate functional framework. In the second part, we define the notion of $1/2$-harmonic map into a manifold in arbitrary dimensions, and provide a general partial regularity theory for such maps. Then we discuss, in the light of the theory for sphere valued harmonic maps, some specific situations where the target manifold is either $\mathbb{S}^1$ or $\mathbb{S}^2$. The last and third part is devoted to the fractional Ginzburg-Landau equation. We shall described a convergence result in the spirit of the blow-up analysis for stationary harmonic maps by F.H. LIN [145], or usual Ginzburg-Landau equations by F.H. LIN & C. WANG [68, 148, 149, 150].

1.1.1 The fractional Laplacian and its Dirichlet integral

For $s \in (0,1)$, the factional Laplacian $(-\Delta)^s$ on $\mathbb{R}^n$ is defined to be the operator whose symbol in Fourier space[1] is given by $(2\pi|\xi|)^{2s}$ (compare to the symbol $4\pi^2|\xi|^2$ of the Laplacian). Back to the physical space, $(-\Delta)^s$ is an integro-differential operator whose action on smooth bounded functions $v : \mathbb{R}^n \to \mathbb{R}^d$ is given by

$$(-\Delta)^s v(x) := \mathrm{p.v.}\left(\gamma_{n,s}\int_{\mathbb{R}^n}\frac{v(x)-v(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\right), \qquad \gamma_{n,s} := s2^s\pi^{-\frac{n}{2}}\frac{\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma(1-s)}, \tag{1.1.1}$$

where the notation $\mathrm{p.v.}$ means that the integral is taken in the *Cauchy principal value* sense.

If $\Omega \subseteq \mathbb{R}^n$ is a smooth bounded open set, the restriction to Ω of the distribution $(-\Delta)^s v$ can be equivalently written as

$$\begin{aligned}\big\langle(-\Delta)^s v,\varphi\big\rangle := \frac{\gamma_{n,s}}{2}\iint_{\Omega\times\Omega}&\frac{\big(v(x)-v(y)\big)\cdot\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\\ &+\gamma_{n,s}\iint_{\Omega\times(\mathbb{R}^n\setminus\Omega)}\frac{\big(v(x)-v(y)\big)\cdot\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\,,\end{aligned} \tag{1.1.2}$$

where $\varphi \in \mathscr{D}(\Omega;\mathbb{R}^d)$ is a test function (tacitly extended by zero outside Ω). This last formulation enlightens the variational structure of the operator $(-\Delta)^s$. More precisely, formula (1.1.2) turns out to define a distribution on Ω whenever the function $v \in L^2_{\rm loc}(\mathbb{R}^n;\mathbb{R}^d)$ satisfies

$$\mathcal{E}_s(v,\Omega) := \frac{\gamma_{n,s}}{4}\iint_{\Omega\times\Omega}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y + \frac{\gamma_{n,s}}{2}\iint_{\Omega\times(\mathbb{R}^n\setminus\Omega)}\frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y < \infty\,. \tag{1.1.3}$$

From now on we shall denote by $\widehat{H}^s(\Omega;\mathbb{R}^d)$ this class of functions, i.e.,

$$\widehat{H}^s(\Omega;\mathbb{R}^d) := \left\{v \in L^2_{\rm loc}(\mathbb{R}^n;\mathbb{R}^d) : \mathcal{E}_s(v,\Omega) < \infty\right\}.$$

In the case $v \in \widehat{H}^s(\Omega;\mathbb{R}^d)$, the distribution $(-\Delta)^s v$ belongs to $H^{-s}(\Omega;\mathbb{R}^d)$, the topological dual space of $H^s_{00}(\Omega;\mathbb{R}^d)$, i.e., the strong closure of $\mathscr{D}(\Omega;\mathbb{R}^d)$ in $H^s(\mathbb{R}^n;\mathbb{R}^d)$, and

$$\big\langle(-\Delta)^s v,\varphi\big\rangle = \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_s(v+t\varphi,\Omega)\right]_{t=0} \qquad \forall\varphi\in H^s_{00}(\Omega;\mathbb{R}^d)\,. \tag{1.1.4}$$

Hence, the energy $\mathcal{E}_s(\cdot,\Omega)$ can be interpreted as fractional s-Dirichlet energy associated to $(-\Delta)^s$ in the open set Ω.

Remark. The class $\widehat{H}^s(\Omega)$ is actually a (strongly) dense subset of $L^2(\mathbb{R}^n,\mathfrak{m}_s)$ for the finite measure $\mathfrak{m}_s := (1+|x|)^{-(n+2s)}\,\mathrm{d}x$, and the functional $\mathcal{E}_s(\cdot,\Omega)$ defines a *Dirichlet form* on $L^2(\mathbb{R}^n,\mathfrak{m}_s)$ in the sense of A. BEURLING & J. DENIS [40, 41].

1. We consider the ordinary frequency Fourier transform $v \mapsto \hat{v}$ given by $\hat{v}(\xi) := \int_{\mathbb{R}^n} v(x)e^{-2i\pi x\cdot\xi}\,\mathrm{d}\xi$.

1.1.2 $1/2$-harmonic maps into a manifold

Definition and regularity theory. Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain, and let $\mathcal{N} \subseteq \mathbb{R}^d$ be a smooth compact submanifold without boundary. As usual, the Sobolev space $\widehat{H}^{1/2}(\Omega;\mathcal{N})$ is defined by

$$\widehat{H}^{1/2}(\Omega;\mathcal{N}) := \left\{ v \in \widehat{H}^{1/2}(\Omega;\mathbb{R}^d) : v(x) \in \mathcal{N} \text{ for a.e. } x \in \mathbb{R}^n \right\}.$$

A map $v \in \widehat{H}^{1/2}(\Omega;\mathcal{N})$ is said to be a *weak* $1/2$*-harmonic map into* $\mathcal{N}$ in the open set Ω if

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{\frac{1}{2}}\big(\pi_{\mathcal{N}}(v+t\varphi),\Omega\big) \right]_{t=0} = 0 \qquad \forall \varphi \in H^{1/2}_{00} \cap L^\infty(\Omega;\mathbb{R}^d)\,, \tag{1.1.5}$$

where $\pi_{\mathcal{N}}$ denotes the nearest point retraction on $\mathcal{N}$ from some tubular neighborhood.

The Euler-Lagrange equation for weak $1/2$-harmonic maps reads

$$(-\Delta)^{\frac{1}{2}} v \perp T_v\mathcal{N} \quad \text{ in } H^{-1/2}(\Omega;\mathbb{R}^d)\,, \tag{1.1.6}$$

where $T\mathcal{N}$ denotes the tangent bundle to $\mathcal{N}$. More explicitly, equation (1.1.6) means that

$$\left\langle (-\Delta)^{\frac{1}{2}} v, \varphi \right\rangle = 0 \quad \forall \varphi \in H^{1/2}_{00}(\Omega;\mathbb{R}^d) \text{ such that } \varphi(x) \in T_{v(x)}\mathcal{N} \text{ for a.e. } x \in \Omega\,.$$

In case $\mathcal{N} = \mathbb{S}^{d-1}$, the Lagrange multiplier takes a quite simple form and yields the instructive equation

$$\left\langle (-\Delta)^{\frac{1}{2}} v, \varphi \right\rangle = \left(\frac{\gamma_{n,\frac{1}{2}}}{2} \int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}y \right) v(x) \quad \text{in } \mathscr{D}'(\Omega;\mathbb{R}^d)\,. \tag{1.1.7}$$

Equations (1.1.6) and (1.1.7) are in clear analogy with the equation $-\Delta u \perp T_u\mathcal{N}$ of harmonic maps into $\mathcal{N}$, and $-\Delta u = |\nabla u|^2 u$ for harmonic maps into spheres, respectively. In particular, they share the same critical structure concerning regularity. Indeed, the right hand side of equation (1.1.7) has *a priori* no better integrability than L^1, and this is precisely the borderline case where linear elliptic regularity does not apply. In the conformal dimension $n = 2$, weakly harmonic maps are smooth by the famous result of F. HÉLEIN [133]. For the square root Laplacian, the conformal dimension is $n = 1$, and entire $1/2$-harmonic maps from $\mathbb{R}$ into $\mathcal{N}$ are also smooth. This is the result of F. DA LIO & T. RIVIÈRE [75, 76], that can be localized to any domain of the real line. In higher dimensions $n \geqslant 3$, an arbitrary weak harmonic map can be highly discontinuous as shown by the counterexample of T. RIVIÈRE [172] (of a weakly harmonic map from the unit ball of $\mathbb{R}^3$ into $\mathbb{S}^2$ which is everywhere discontinuous). For $1/2$-harmonic maps we expect that in dimension $n \geqslant 2$ such a counterexample to regularity do exist (actually, one may try to follow the argument of [172] using the tools developed in [P08], see Section 1.2). In any case, one quickly realizes that "full" regularity can not hold in dimension $n \geqslant 2$ due (essentially) to topological constraints. For instance, if Ω is the unit disc of $\mathbb{R}^2$ and $\mathcal{N} = \mathbb{S}^1$, the map $v(x) = \frac{x}{|x|}$ is a weak $1/2$-harmonic map, and it exhibits an isolated singularity at the origin. To obtain partial regularity (i.e., regularity away from some "small" set), one has to require to a $1/2$-harmonic map to be either *stationary* or *minimizing*, exactly as for classical harmonic maps, see [36, 92, 188].

A weak $1/2$-harmonic map $v \in \widehat{H}^{1/2}(\Omega;\mathcal{N})$ is said to be :

(a) *stationary* in Ω if

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{\frac{1}{2}}\big(v\circ\phi_t,\Omega\big) \right]_{t=0} = 0$$

for any differentiable 1-parameter family of smooth diffeomorphisms $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi_0 = \mathrm{id}_{\mathbb{R}^n}$ and $\phi_t - \mathrm{id}_{\mathbb{R}^n}$ is compactly supported in Ω ;

(b) *minimizing* in Ω if

$$\mathcal{E}_{\frac{1}{2}}(v,\Omega)\leqslant\mathcal{E}_{\frac{1}{2}}(w,\Omega)$$

for all $w\in\widehat{H}^{1/2}(\Omega;\mathcal{N})$ such that $w-v$ is compactly supported in Ω.

Any smooth $1/2$-harmonic map, as well as any minimizing $1/2$-harmonic map, is stationary.

In the standard harmonic map theory, the stationary assumption is used when performing radial deformations around a point in the domain. This leads to the crucial *monotonicity formula* for the Dirichlet energy. In our fractional setting, the situation is slightly more subtle, and no reasonable monotonicity formula seems to come out directly from the stationary assumption. Before we enter into details on how to circumvent this problem we state our main regularity result on $1/2$-harmonic maps. In the following, $\mathscr{H}^k$ denotes the k-dimensional Hausdorff measure, and $\dim_{\mathscr{H}}$ stands for Hausdorff dimension.

Theorem 1.1.1. *Let $v\in\widehat{H}^{1/2}(\Omega;\mathcal{N})$ be a weak 1/2-harmonic map into $\mathcal{N}$ in Ω. Then $v\in C^\infty\big(\Omega\setminus\operatorname{sing}(v)\big)$ where $\operatorname{sing}(v)$ denotes the complement of the largest open set on which v is continuous, and*

(i) *if $n=1$, then $\operatorname{sing}(v)\cap\Omega=\emptyset$;*

(ii) *if $n\geqslant 2$ and v is stationary, then $\mathscr{H}^{n-1}\big(\operatorname{sing}(v)\cap\Omega\big)=0$;*

(iii) *if v is minimizing, then $\dim_{\mathscr{H}}\big(\operatorname{sing}(v)\cap\Omega\big)\leqslant n-2$ for $n\geqslant 3$, and $\operatorname{sing}(v)\cap\Omega$ is discrete for $n=2$.*

The proof of this theorem rests on the representation of $(-\Delta)^{\frac{1}{2}}$ as the *Dirichlet-to-Neumann operator* associated to the harmonic extension to the open half space $\mathbb{R}^{n+1}_+:=\mathbb{R}^n\times(0,\infty)$ given by the convolution product with the Poisson kernel. More precisely, denoting by $v\mapsto v^{\mathrm{e}}$ this harmonic extension, i.e.,

$$v^{\mathrm{e}}(x):=\gamma_{n,\frac{1}{2}}\int_{\mathbb{R}^n}\frac{x_{n+1}v(y)}{(|x'-y|^2+x_{n+1}^2)^{\frac{n+1}{2}}}\,\mathrm{d}y\,,\qquad x=(x',x_{n+1})\in\mathbb{R}^{n+1}_+\,,\tag{1.1.8}$$

we have proved that it is well defined on $\widehat{H}^{1/2}(\Omega;\mathbb{R}^d)$, and that $\partial_\nu v^{\mathrm{e}}=(-\Delta)^{\frac{1}{2}}v$ as distributions on the open set Ω. Here, ∂_ν denotes the exterior normal differentiation on $\partial\mathbb{R}^{n+1}_+\simeq\mathbb{R}^n$. When applying the extension procedure to a weak $1/2$ harmonic map v, we end up with the following system

$$\begin{cases}\Delta v^{\mathrm{e}}=0 & \text{in } \mathbb{R}^{n+1}_+\,,\\[1ex] \dfrac{\partial v^{\mathrm{e}}}{\partial\nu}\perp T_{v^{\mathrm{e}}}\mathcal{N} & \text{on } \Omega\,.\end{cases}\tag{1.1.9}$$

This system turns out to be (almost) included in the class of *harmonic maps with free boundary* for which a rather complete regularity theory do exist [32, 90, 91, 124, 127, 131, 187]. The theory of harmonic maps with free boundary essentially deals with mappings $u:G\subseteq\mathbb{R}^{n+1}\to\mathcal{M}\subseteq\mathbb{R}^d$ where G is an open set such that $\Omega\subseteq\partial G$, and $\mathcal{M}$ is a smooth compact manifold without boundary such that $\mathcal{N}\subseteq\mathcal{M}$. The boundary portion Ω is called the free boundary, and $\mathcal{N}$ its supporting manifold. Then $\mathcal{M}$-valued (weak) harmonic maps in G with the partially free boundary condition $u(\Omega)\subseteq\mathcal{N}$ are defined as critical points of the (classical) Dirichlet energy under the constraints $u(x)\in\mathcal{M}$ for $\mathscr{L}^{n+1}$-a.e. $x\in G$ and $u(x)\in\mathcal{N}$ for $\mathscr{H}^n$-a.e. $x\in\Omega$. In case G is the upper unit half ball of $\mathbb{R}^3$, Ω is the unit disc of $\mathbb{R}^2$, $\mathcal{M}=\mathbb{S}^2$, and $\mathcal{N}=\mathbb{S}^1$, the typical example of a (minimizing) harmonic map with free boundary is again $u(x)=\frac{x}{|x|}$. The theory of harmonic maps with free boundary provides partial regularity results, and for this, it requires stationarity or minimality *up to the free boundary*. Here, "up to the free boundary" means that deformations (of domain or image) are allowed to be compactly supported in $G\cup\Omega$. The key point in the proof of Theorem 1.1.1 is to observe that the stationary assumption, or the minimality assumption, on a weak $1/2$-harmonic map v leads to stationarity, or minimality (respectively), up to Ω of the extension v^{e}.

This is then enough to apply the partial regularity theory for harmonic maps with free boundary (and to obtain in addition a suitable monotonicity formula in terms of v^{e}).

At this stage, we would like to mention that many basic questions remain open concerning the partial regularity of $1/2$-harmonic maps. The first one is certainly the regularity at the boundary $\partial\Omega$ when prescribing some smooth $\mathcal{N}$-valued function outside Ω. In terms of the extended problem, it leads to a *nonlinear mixed boundary value problem Dirichlet/Neumann* for which no regularity results are known (to the best of our knowledge). Notice that even for linear mixed boundary value problems, the regularity issue at the interface between the Dirichlet and the Neumann region of the boundary is a very delicate task, and Hölder continuity is essentially the higher level of regularity accessible, see e.g. [184] and the references therein. Another interesting direction would be to give a better description of the singular set (i.e., to obtain some reduction or stratification results), and to extend to $1/2$-harmonic maps some well established results for standard harmonic maps (see [74] for a recent blow-up analysis in the one dimensional case). We now take a few steps in this sense, underlying the strong analogies between $1/2$-harmonic maps and classical harmonic maps.

$1/2$-harmonic circles and minimal surfaces. We shall now discuss some geometric properties of entire $1/2$-harmonic maps on $\mathbb{R}$ of finite energy. Note that in the case $\Omega = \mathbb{R}$, the space $\widehat{H}^{1/2}(\Omega)$ simply reduces to the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R})$, and it is well known that for every $v \in \dot{H}^{1/2}(\mathbb{R})$,

$$\mathcal{E}_{\frac{1}{2}}(v, \mathbb{R}) = \frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla v^{\mathrm{e}}|^2 \,\mathrm{d}x \,, \tag{1.1.10}$$

where v^{e} denotes the harmonic extension of v to $\mathbb{R}^2_+$ given by (1.1.8). We say that a map $v \in \dot{H}^{1/2}(\mathbb{R};\mathcal{N})$ is an entire $1/2$-harmonic line into $\mathcal{N}$ if v is a $1/2$-harmonic map in every bounded open subset $\Omega \subseteq \mathbb{R}$.

In dimension 2, harmonic maps are known to be closely linked with conformal mappings, and hence with minimal surfaces. For entire $1/2$-harmonic lines, we have the following lemma (which has also been discovered independently in [33] and [74]).

Lemma 1.1.2. *Let $v \in \dot{H}^{1/2}(\mathbb{R};\mathcal{N})$ be a nontrivial entire 1/2-harmonic line into $\mathcal{N}$. Its harmonic extension v^{e} to $\mathbb{R}^2_+$ is either a conformal or an anti-conformal transformation.*

If $\mathcal{N} = \mathbb{S}^1$ the unit circle of $\mathbb{R}^2$, then the harmonic extension of every $1/2$-harmonic line maps $\mathbb{R}^2_+$ into the unit disc $D_1 \subseteq \mathbb{R}^2$ by the maximum principle. On the other hand, it turns out that every conformal transformation with finite energy from $\mathbb{R}^2_+$ into D_1 and sending $\mathbb{R}$ into $\mathbb{S}^1 = \partial D_1$ has to be a finite Blaschke product (see [160]). In other words, Lemma 1.1.2 yields the following classification result. In its statement, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

Theorem 1.1.3. *Let $v \in \dot{H}^{1/2}(\mathbb{R};\mathbb{S}^1)$ be an entire 1/2-harmonic map into $\mathbb{S}^1$. There exist $d \in \mathbb{N}$, $\theta \in \mathbb{R}$, $\{\lambda_k\}_{k=1}^d \subseteq (0,\infty)$, and $\{a_k\}_{k=1}^d \subseteq \mathbb{R}$ such that $v^{\mathrm{e}}(z)$ or its complex conjugate equals*

$$e^{i\theta}\prod_{k=1}^{d}\frac{\lambda_k(z-a_k)-i}{\lambda_k(z-a_k)+i}\,. \tag{1.1.11}$$

In particular, $\mathcal{E}_{\frac{1}{2}}(v,\mathbb{R}) = \pi d$.

Remark (*$1/2$-harmonic circles*). By analogy with (1.1.3), we can consider on $H^{1/2}(\mathbb{S}^1;\mathcal{N})$ the $1/2$-Dirichlet energy

$$\mathcal{E}_{\frac{1}{2}}(g,\mathbb{S}^1) := \frac{\gamma_1}{4}\iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{|g(x)-g(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y\,.$$

Exactly as in (1.1.5), we define $1/2$-harmonic circles into $\mathcal{N}$ as critical points of $\mathcal{E}(\cdot,\mathbb{S}^1)$ with respect to perturbations on the image. It turns out that $1/2$-harmonic circles are in one-to-one correspondence with $1/2$-harmonic lines into $\mathcal{N}$ by conformal invariance. Indeed, first recall that for every $g \in H^{1/2}(\mathbb{S}^1;\mathbb{R}^d)$,

$$\mathcal{E}_{\frac{1}{2}}(g,\mathbb{S}^1) = \frac{1}{2}\int_{\mathbb{D}} |\nabla w_g|^2 \,\mathrm{d}x\,, \tag{1.1.12}$$

where $w_g \in H^1(D_1;\mathbb{C})$ denotes the harmonic extension of g to the whole disc. Using the (conformal) *Cayley transform* $\mathfrak{C} : \overline{\mathbb{R}}^2_+ \to \overline{D}_1$ defined by $\mathfrak{C}(z) := \frac{z-i}{z+i}$, one has $w_g \circ \mathfrak{C} = (g \circ \mathfrak{C}_{|\mathbb{R}})^{\mathrm{e}}$ for every $g \in \dot{H}^{1/2}(\mathbb{S}^1;\mathbb{R}^d)$. Then, by conformal invariance, (1.1.10) and (1.1.12),

$$\mathcal{E}_{\frac{1}{2}}(g,\mathbb{S}^1) = \mathcal{E}_{\frac{1}{2}}(g\circ\mathfrak{C}_{|\mathbb{R}},\mathbb{R}) \quad \text{for all } g \in H^{1/2}(\mathbb{S}^1;\mathbb{R}^d)\,.$$

As a consequence, $g \in H^{1/2}(\mathbb{S}^1;\mathcal{N})$ is a $1/2$-harmonic circle if and only if $g\circ\mathfrak{C}_{|\mathbb{R}}$ is a $1/2$-harmonic line into $\mathcal{N}$.

For $\mathcal{N} = \mathbb{S}^1$, Theorem 1.1.3 shows that the energy is quantized by the topological degree, i.e., $\mathcal{E}_{\frac{1}{2}}(g,\mathbb{S}^1) = \pi|\deg(g)|$ for any $1/2$-harmonic circle g into $\mathbb{S}^1$. By a result of P. Mironescu & A. Pisante [160], it shows in particular that every $1/2$-harmonic circle into $\mathbb{S}^1$ is minimizing in its own homotopy class. This property is in clear analogy with the theory of harmonic maps for which it is well known that *harmonic 2-spheres into* $\mathbb{S}^2$ are minimizing in their own homotopy class and have an energy quantized by the degree.

Other interesting geometric consequences of Lemma 1.1.2 together with equation (1.1.9) are the following. Assume that $d = 3$. First, if $\mathcal{N}$ is a smooth closed curve and g is a $1/2$-harmonic circle into $\mathcal{N}$, then the image of the unit disc by w_g is a disc-type minimal surface spanned by $\mathcal{N}$ (compare with [124, Section 5]). Second, if $\mathcal{N}$ is a surface, then the image of D_1 by w_g is a disc-type minimal surface whose boundary lies in $\mathcal{N}$, and meets $\mathcal{N}$ orthogonally (compare with [199]). In this last case, such surfaces are called *minimal surface with free boundary* (in $\mathcal{N}$), see e.g. [89]. Conversely, if $w : D_1 \to \mathcal{N}$ is a disc-type minimal surface with free boundary in $\mathcal{N}$, then $w_{|\mathbb{S}^1}$ is a $1/2$-harmonic circle into $\mathcal{N}$.

In the particular case $\mathcal{N} = \mathbb{S}^2$ the unit sphere of $\mathbb{R}^3$, for every $1/2$-harmonic circle g, the image of D_1 by w_g is a disc-type minimal surface whose free boundary lies in $\mathbb{S}^2$. For such minimal surfaces, it is known that the image is a plane disc (through the origin), see e.g. [89, Section 1.7 in Chapter 1]. By Theorem 1.1.3, all harmonic circles into $\mathbb{S}^2$ are thus classified, and we have the following corollary [2] which parallels a rigidity result about harmonic maps from $\mathbb{S}^2$ into $\mathbb{S}^3$, see [189, Lemma 1.1]. In the statement, we identify $\mathbb{S}^1 \subseteq \mathbb{R}^2$ with the "horizontal circle" $\mathbb{S}^1 \times \{0\} \subseteq \mathbb{R}^3$.

Corollary 1.1.4. *Let $h : \mathbb{S}^1 \to \mathbb{S}^2$ be a $1/2$-harmonic circle. Then h has an image lying in an equator. In particular, there exists a $1/2$-harmonic circle $g : \mathbb{S}^1 \to \mathbb{S}^1$ and an orthogonal transformation $R \in SO(3)$ such that $h = R\circ g$. As a consequence, $\mathcal{E}_{\frac{1}{2}}(h,\mathbb{S}^1)$ is an integral multiple of π.*

Remark (*Tangent maps*). Classically, one defines $1/2$-harmonic tangent maps as 0-homogeneous $1/2$-harmonic maps. In dimension 2, any $1/2$-harmonic tangent map is given by the 0-homogeneous extension of some $1/2$-harmonic circle. In other words, $v : \mathbb{R}^2 \to \mathcal{N}$ is a 0-homogeneous (weak) $1/2$-harmonic map if and only if $v(x) = g(\frac{x}{|x|})$ for some $1/2$-harmonic circle $g : \mathbb{S}^1 \to \mathcal{N}$.

Remark (*Tangent maps into $\mathbb{S}^1$ or $\mathbb{S}^2$*). By Theorem 1.1.3 and Corollary 1.1.4, tangent maps from $\mathbb{R}^2$ into $\mathbb{S}^1$ or $\mathbb{S}^2$ are completely classified. In particular, in the case of $\mathbb{S}^1$, it shows that $v(x) = \frac{x}{|x|}$ is a weak $1/2$-harmonic map. It would interesting to determine which tangent maps are minimizing or stationary.

2. This result has also been announced in [74].

As a matter of fact, we can prove that $\frac{x}{|x|}$ is a minimizing $1/2$-harmonic map from $\mathbb{R}^2$ into $\mathbb{S}^1$. It uses the material in Section 1.2 together arguments taken from H. BREZIS, J.M. CORON, & E.H. LIEB [57]. At this stage, we expect that $\frac{x}{|x|}$ is actually the unique non trivial minimizing tangent map (up to isometries) by analogy with [57].

In the case of $\mathbb{S}^2$, the situation is quite different. In fact, we can prove that all minimizing $1/2$-harmonic tangent maps from $\mathbb{R}^2$ into $\mathbb{S}^2$ are constant, similarly to the result of R. SCHOEN & K. UHLENBECK [189, Proposition 1.2] about minimizing tangent harmonic maps from $\mathbb{R}^3$ into $\mathbb{S}^3$. The argument is based on Corollary 1.1.4 together with the second variation of the Dirichlet energy of the extension (and using radial deformation orthogonal to the image very much like in [189]). By a standard blow-up analysis near an isolated singularity, this rigidity result implies in turn that a minimizing $1/2$-harmonic map in a planar domain into $\mathbb{S}^2$ is smooth [3].

1.1.3 Asymptotics for the fractional Ginzburg-Landau equation

Let $d \geqslant 2$ be a given integer, and let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded open set. We now describe our results on the asymptotic behavior, as $\varepsilon \downarrow 0$, of weak solutions $v_\varepsilon : \mathbb{R}^n \to \mathbb{R}^d$ to the fractional Ginzburg-Landau equation

$$(-\Delta)^{\frac{1}{2}} v_\varepsilon = \frac{1}{\varepsilon}(1 - |v_\varepsilon|^2)v_\varepsilon \qquad \text{in } \Omega\,, \tag{1.1.13}$$

subject to an exterior Dirichlet condition

$$v_\varepsilon = g \qquad \text{on } \mathbb{R}^n \setminus \Omega\,, \tag{1.1.14}$$

where $g : \mathbb{R}^n \to \mathbb{R}^d$ is a smooth function satisfying $g(x) \in \mathbb{S}^{d-1}$ for every $x \in \mathbb{R}^n \setminus \Omega$.

The weak sense for equation (1.1.13) is understood through the variational formulation (1.1.4) of the fractional Laplacian in the open set Ω, i.e.,

$$\left\langle (-\Delta)^{\frac{1}{2}} v_\varepsilon, \varphi \right\rangle = \frac{1}{\varepsilon}\int_\Omega (1 - |v_\varepsilon|^2) v \cdot \varphi \,\mathrm{d}x \qquad \forall \varphi \in \mathscr{D}(\Omega; \mathbb{R}^d)\,.$$

In this way, equation (1.1.13) corresponds to the Euler-Lagrange equation for critical points of the fractional $1/2$-*Ginzburg-Landau energy* $\mathcal{F}_\varepsilon(\cdot, \Omega)$ defined for $v \in \widehat{H}^{1/2} \cap L^4(\Omega; \mathbb{R}^d)$ by

$$\mathcal{F}_\varepsilon(v, \Omega) := \frac{1}{2}\int_\Omega \mathrm{e}_{\frac{1}{2}}(v, \Omega) + \frac{1}{2\varepsilon}(1 - |v|^2)^2 \,\mathrm{d}x\,,$$

where we have set $\mathrm{e}_{\frac{1}{2}}(v, \Omega)$ to be the *nonlocal energy density in* Ω given by

$$\mathrm{e}_{\frac{1}{2}}(v, \Omega) := \frac{\gamma_{n,\frac{1}{2}}}{2}\int_\Omega \frac{|v(x) - v(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}y + \gamma_{n,\frac{1}{2}}\int_{\mathbb{R}^n \setminus \Omega} \frac{|v(x) - v(y)|^2}{|x-y|^{n+1}}\,\mathrm{d}y\,. \tag{1.1.15}$$

The most standard way to obtain weak solutions to (1.1.13)-(1.1.14) is certainly to minimize $\mathcal{F}_\varepsilon(\cdot, \Omega)$ under the exterior Dirichlet condition (1.1.14), i.e., setting

$$H^{1/2}_g(\Omega; \mathbb{R}^d) := g + H^{1/2}_{00}(\Omega; \mathbb{R}^d) \subseteq \widehat{H}^{1/2}(\Omega; \mathbb{R}^d)\,,$$

one considers the minimization problem

$$\min\left\{ \mathcal{F}_\varepsilon(v, \Omega) : v \in H^{1/2}_g \cap L^4(\Omega; \mathbb{R}^d) \right\}\,, \tag{1.1.16}$$

3. The results announced here are part of an ongoing project.

whose resolution follows directly from the Direct Method of Calculus of Variations. However, we emphasize that the main object of our analysis is to provide a careful study of *arbitrary critical points* of the $1/2$-Ginzburg-Landau energy $\mathcal{E}_\varepsilon$ as $\varepsilon \to 0$. As we have already explained, $1/2$-harmonic maps behave very much like classical harmonic maps. This is also the case for the fractional Ginzburg-Landau equation. Before stating our result, let us briefly recall what is known for the classical Ginzburg-Landau equation. In a serie of articles, F.H. LIN & C. WANG [148, 149, 150] have shown that a sequence of arbitrary solutions to the Ginzburg-Landau equation with equibounded energy (in ε) converges weakly to a weak harmonic map into $\mathbb{S}^{d-1}$ as $\varepsilon \to 0$. Unfortunately, the convergence only holds *a priori* in the weak sense since the limiting system of harmonic maps is itself not strongly compact (due to the conformal invariance in dimension 2). Exactly as in the blow-up analysis for harmonic maps of F.H. LIN [145] (see also [147]), one can however provide a very good description of the defect measure arising in weak convergence process. For the fractional Ginzburg-Landau equation, we have obtained the following analogous result which corresponds in some sense to the first step in the full program of [145, 147, 148, 149, 150].

Theorem 1.1.5. *Let $\varepsilon_k \downarrow 0$ be an arbitrary sequence, and let $\{v_k\}_{k\in\mathbb{N}} \subseteq H^{1/2}_g \cap L^4(\Omega;\mathbb{R}^d)$ be such that*

$$\begin{cases} (-\Delta)^{\frac12} v_k = \dfrac{1}{\varepsilon_k}(1-|v_k|^2)v_k & \text{in } \Omega\,, \\[2ex] v_k = g & \text{in } \mathbb{R}^n\setminus\Omega\,. \end{cases}$$

If $\sup_k \mathcal{F}_{\varepsilon_k}(v_k,\Omega) < \infty$, then there exist a (not relabeled) subsequence and $v_* \in H^{1/2}_g(\Omega;\mathbb{S}^{d-1})$ a weak 1/2-harmonic map in Ω such that $v_k - v_* \rightharpoonup 0$ weakly in $H^{1/2}_{00}(\Omega)$. In addition, there exist a finite nonnegative Radon measure $\mu_{\rm sing}$ on Ω, a countably $\mathscr{H}^{n-1}$-rectifiable relatively closed set $\Sigma \subseteq \Omega$ of finite $(n-1)$-dimensional Hausdorff measure, and a Borel function $\Theta : \Sigma \to (0,\infty)$ such that

(i) $\mathrm{e}_{\frac12}(v_k,\Omega)\,\mathscr{L}^n \llcorner \Omega \overset{*}{\rightharpoonup} \mathrm{e}_{\frac12}(v_*,\Omega)\,\mathscr{L}^n \llcorner \Omega + \mu_{\rm sing}$ *weakly* as Radon measures on Ω;*

(ii) $\dfrac{(1-|v_k|^2)^2}{\varepsilon_k} \to 0$ *in $L^1_{\rm loc}(\Omega)$;*

(iii) $\dfrac{1-|v_k(x)|^2}{\varepsilon_k} \rightharpoonup \dfrac{\gamma_{n,\frac12}}{2}\displaystyle\int_{\mathbb{R}^n} \frac{|v_*(x)-v_*(y)|^2}{|x-y|^{n+1}}\,{\rm d}y + \mu_{\rm sing}$ *in $\mathscr{D}'(\Omega)$;*

(iv) $\mu_{\rm sing} = \Theta\mathscr{H}^{n-1}\llcorner\Sigma$*;*

(v) $v_* \in C^\infty(\Omega\setminus\Sigma)$ *and $v_n \to v_*$ in $C^\ell_{\rm loc}(\Omega\setminus\Sigma)$ for every $\ell\in\mathbb{N}$;*

(vi) *if $n \geqslant 2$, the limiting 1/2-harmonic map v_* and the defect measure $\mu_{\rm sing}$ satisfy the stationary relation*

$$\left[\frac{\rm d}{{\rm d}t}\,\mathcal{E}_{\frac12}\big(v_*\circ\phi_t,\Omega\big)\right]_{t=0} = \frac12\int_\Sigma {\rm div}_\Sigma X\,{\rm d}\mu_{\rm sing} \tag{1.1.17}$$

for all vector fields $X \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in Ω, where $\{\phi_t\}_{t\in\mathbb{R}}$ denotes the flow on $\mathbb{R}^n$ generated by X;

(vii) *if $n=1$, the set Σ is finite and $v_*\in C^\infty(\Omega)$.*

Remark. In the case of minimizing solutions, i.e., assuming that v_k solves (1.1.16), the defect measure $\mu_{\rm sing}$ vanishes, and v_k converges strongly towards v_* which is then minimizing in Ω. In addition, the smooth convergence in (v) holds locally away from ${\rm sing}(v_*)$ if $n\geqslant 2$, and locally in Ω for $n=1$.

As in Theorem 1.1.1, the proof of Theorem 1.1.5 rests on the harmonic extension to $\mathbb{R}^{n+1}_+$ given by (1.1.8), and on the the representation of $(-\Delta)^{\frac12}$ as associated Dirichlet-to-Neumann operator. This extension procedure leads the following *system of Ginzburg-Landau boundary reactions*

$$\begin{cases} \Delta v_\varepsilon^{\mathrm{e}} = 0 & \text{in } \mathbb{R}^{n+1}_+ \,, \\ \dfrac{\partial v_\varepsilon^{\mathrm{e}}}{\partial \nu} = \dfrac{1}{\varepsilon}(1 - |v_\varepsilon^{\mathrm{e}}|^2) v_\varepsilon^{\mathrm{e}} & \text{on } \Omega \,. \end{cases} \tag{1.1.18}$$

The asymptotic analysis of this system as $\varepsilon \to 0$ gives the main conclusions. To perform such an analysis, we have first established an *epsilon-regularity* type of estimate for (1.1.18) in the spirit of the regularity theory for harmonic maps [36, 188] or usual Ginzburg-Landau equations [68]. Together with a fundamental *monotonicty formula*, this estimate is the key to derive the convergence result and the rectifiability of the defect measure. Note that Theorem 1.1.5 actually says that μ_{sing} is a $(n-1)$-rectifiable varifold in the sense of F.J. ALMGREN, see e.g. [193]. We emphasize that identity (1.1.17) is precisely the coupling equation between the limiting $1/2$-harmonic map v_* and the defect measure μ_{sing}. It states that the first inner variation of the $1/2$-Dirichlet energy of v_* is equal to $-\frac{1}{2}$ times the first inner variation of the varifold μ_{sing}, see [193, formulas 15.7 and 16.2]. We have achieved (1.1.17) in two independent steps. The first step consists in proving an analogous identity when passing to the limit $\varepsilon \to 0$ in system (1.1.18). In the spirit of [150], the convenient way to let $\varepsilon \to 0$ in the first inner variation of the Dirichlet energy of $v_\varepsilon^{\mathrm{e}}$ is to use the notion of *generalized varifold* of L. AMBROSIO & M.H. SONER [29], once adapted to the boundary setting. In turn, the second step allows us to return to the original formulation on $\mathbb{R}^n$. It shows that the first inner variation of the $1/2$-Dirichlet energy of an arbitrary map v is equal to the first inner variation up to Ω of the Dirichlet energy of its harmonic extension v^{e}. This is precisely the observation who led us to conclude that a stationary $1/2$-harmonic map yields a stationary map when applying the extension.

Before concluding this section, let us briefly comment on some possible extension of the present results and an open question. In the fractional Ginzburg-Landau energy one could replace the potential $(1 - |u|^2)^2$ by a more general nonnegative potential $W(u)$ having a zero set $\{W = 0\}$ given by a smooth compact submanifold $\mathcal{N}$ of $\mathbb{R}^d$ without boundary, and then consider the corresponding fractional Ginzbug-Landau equation. In this context, the singular limit $\varepsilon \to 0$ leads to the $1/2$-harmonic map system into $\mathcal{N}$. If the codimension of $\mathcal{N}$ is equal to 1 (plus some non degeneracy assumptions on W), the proof of Theorem 1.1.5 can certainly be reproduced with minor modifications. However, the higher codimension case seems to require additional analysis since our *espilon-regularity estimate* strongly uses the codimension 1 structure. It would be interesting to have a proof handling both cases.

1.2 Geometric analysis of $H^{1/2}$-maps from $\mathbb{R}^2$ into $\mathbb{S}^1$

Within the theory of harmonic maps, the article of H. BREZIS, J.M. CORON, & E.H. LIEB [57] certainely represents a fundamental contribution to the understanding of singularities in dimensions greater than 3. Together with the subsequent article [37] by F. BETHUEL, H. BREZIS, & J.M. CORON, it gave appropriate mathematical tools for the construction of highly singular (weak) harmonic maps from three dimensional domains into $\mathbb{S}^2$ (see [132, 171, 172]). In [57] the main problem consists in minimizing the Dirichlet energy over a class of $\mathbb{S}^2$-valued maps which are smooth away from finitely many prescribed singular points with prescribed topological degree. It is proved that minimizing sequences concentrate on a set made by finitely many (oriented) segments connecting the singularities according to the degree, and that the energy density converges to a multiple of the one dimensional Hausdorff measure restricted to this union of segments. In addition, this connection has to be minimal in a sense of minimal length. For instance, if the prescribed singular set consists of two isolated points with opposite degree, then the previous infimum is simply a multiple of the *distance* between these two points. For this reason, the name of *minimal connection* has been given to the value of the infimum. It has then been noticed by F. BETHUEL, H. BREZIS, & J.M. CORON [37] that the notion of minimal connection makes sense for an arbitrary $\mathbb{S}^2$-valued map with finite energy, and provide in some sense the "total weight" of the topological singularities present in a given map. By computing the so-called *relaxed energy* (name taken from the relaxation theory in the Calculus of Variations, see e.g. [72]), it is shown in [37] that the minimal connection quantify exactly the lack of strong approximation in the Sobolev space H^1 by smooth maps into $\mathbb{S}^2$ (see also [34]).

The purpose of the article [P08], in collaboration with A. PISANTE, was to investigate the analogues of these results about H^1-maps with values in $\mathbb{S}^2$ in the case of $H^{1/2}$-maps from the plane into $\mathbb{S}^1$. In this manuscript we are of course trying to present the problem in the perspective of the $1/2$-harmonic maps discussed in the previous section, but our original motivation comes from the three dimensional (complex) Ginzburg-Landau equation and questions left open by J. BOURGAIN, H. BREZIS, & P. MIRONESCU in [50]. $H^{1/2}$-maps into $\mathbb{S}^1$ with singularities turn out to be essentially the appropriate boundary conditions in the 3D Ginzburg-Landau theory for the emergence of *vortex lines* in the singular limit $\varepsilon \to 0$, see [39, 50, 146, 173]. Historically, the analogy between the H^1 and the $H^{1/2}$ case has been first discovered by T. RIVIÈRE [174] showing that a map in $H^{1/2}(\mathbb{S}^2;\mathbb{S}^1)$ can be strongly approximated by smooth $\mathbb{S}^1$-valued maps if and only if its *distributional Jacobian* vanishes, i.e., the exact analogue of a result by F. BETHUEL [34] for H^1-maps into $\mathbb{S}^2$. In both cases, the distributional Jacobian is the right quantity to consider in order to detect the topological part of the singular set of a given map. In [50], J. BOURGAIN, H. BREZIS, & P. MIRONESCU have performed a rather complete analysis of the topological *and* analytical part of the singular set of a $H^{1/2}$-function into $\mathbb{S}^1$, and our contribution departs from there. We finally mention that our study recovers some result proved in [50], and in [117, 118] the setting of Cartesian currents. In contrast with these articles, our analysis is performed in the entire space and is not restricted to the Euclidean metric.

1.2.1 The distributional Jacobian and the dipole problem

Let us consider $X := \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{S}^1)$, and denote by $[\cdot]_{1/2}$ the standard (Gagliardo) $H^{1/2}$–seminorm

$$[v]^2_{1/2} = \frac{1}{4\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|v(x)-v(y)|^2}{|x-y|^3}\,\mathrm{d}x\mathrm{d}y \quad \big(= 2\mathcal{E}_{\frac{1}{2}}(v,\mathbb{R}^2)\big) \tag{1.2.1}$$

which makes X modulo constants a complete metric space. In this way, X naturally appears as a closed subset of the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$. By our normalization choice,

$$[v]_{1/2} = \|\nabla v^{\mathrm{e}}\|_{L^2(\mathbb{R}^3_+)}\,, \tag{1.2.2}$$

where v^{e} is the harmonic extension of v given by (1.1.8), i.e., the unique finite energy harmonic extension of v to the half space $\mathbb{R}^3_+ := \mathbb{R}^2\times(0,\infty)$ (whose boundary is identify with $\mathbb{R}^2$).

We now present some properties of maps in X related to the nontrivial topology of the target. These properties were known in the bounded domain case (see [50, 174], and [118] for a different approach). In particular, the strong density of the subspace of smooth maps $X\cap C^\infty(\mathbb{R}^2)$ was known to fail and the sequential weak density to hold. However, strong density holds for maps with finitely many singularities, see [174]. For any $v\in X$, a characterization of the topological singularities can be obtained in terms of a distribution $T(v)$, as in [50, 129, 174]. In a few words, this distribution measures how much v fails to preserve closed forms under pull-back.

Given $v \in X$ and $\varphi \in \mathrm{Lip}(\mathbb{R}^2;\mathbb{R})$, we consider $u \in \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ and $\Phi \in \mathrm{Lip}(\mathbb{R}^3_+;\mathbb{R})$ with $u_{|\mathbb{R}^2} = v$ and $\Phi_{|\mathbb{R}^2} = \varphi$. Setting

$$H(u) := -2\big(\partial_2 u \wedge \partial_3 u, \partial_3 u \wedge \partial_1 u, \partial_1 u \wedge \partial_2 u\big)\,,$$

the distribution $T(v)$ is defined through its action on φ by

$$\langle T(v), \varphi\rangle := \int_{\mathbb{R}^3_+} H(u)\cdot \nabla\Phi \,\mathrm{d}x\,. \tag{1.2.3}$$

Noticing that $\operatorname{div} H(u) = 0$ in $\mathscr{D}'(\mathbb{R}^3_+)$, it is routine to check that such a definition makes sense, i.e., it is independent of the extensions u and Φ, and $T(v) \in \big(\mathrm{Lip}(\mathbb{R}^2)\big)'$. As shown in [50, 174], $T(v) = 0$ if and only if v can be approximated strongly by smooth functions. For maps which are slightly more regular, namely if $v \in X \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$, an integration by parts in (1.2.3) yields

$$\langle T(v), \varphi\rangle = -\int_{\mathbb{R}^2} (v \wedge \nabla v)\cdot \nabla^\perp \varphi \,\mathrm{d}x \qquad \forall \varphi \in C^1_c(\mathbb{R}^2;\mathbb{R})\,. \tag{1.2.4}$$

In other words,

$$T(v) = \mathrm{curl}(v \wedge \nabla v) = 2\mathrm{Det}(\nabla v)\,,$$

where $\mathrm{Det}(\nabla v)$ is the *distributional Jacobian* of v. Formula (1.2.4) actually holds for an arbitrary $v \in X$ whenever $\varphi \in C^2_c(\mathbb{R}^2;\mathbb{R})$, if we interpret (1.2.4) in terms of $\dot{H}^{1/2} - \dot{H}^{-1/2}$ duality. In addition, if v is smooth except at finitely many points $\{a_j\}_{j=1}^k$ and u is taken to be smooth in the open half space, then

$$H(u)\cdot \nabla\Phi \, dx_1 \wedge dx_2 \wedge dx_3 = -2u^\# d\omega \wedge d\Phi = -d(u^\#\omega \wedge d\Phi)\,,$$

where $\omega(y_1, y_2) := y_1 dy_2 - y_2 dy_1$ induces the standard volume form on $\mathbb{S}^1 = \big\{y_1^2 + y_2^2 = 1\big\}$. In this way,

$$\langle T(v), \varphi\rangle = -2\int_{\mathbb{R}^3_+} u^\# d\omega \wedge d\Phi = \int_{\mathbb{R}^2} v^\#\omega \wedge d\varphi = 2\pi \sum_{j=1}^k d_j\, \varphi(a_j)\,,$$

where $d_j := \deg(v, a_j) \in \mathbb{Z}$ is the topological degree of v restricted to any small circle around a_j. In addition, $\sum_j d_j = 0$ because $v \in \dot{H}^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$. The same finite sum representation holds if $T(v)$ is a finite measure, see [50], this result being the $H^{1/2}$-counterpart of the same statement for $W^{1,1}$-maps proved in [58, 117, 139].

As a consequence of the strong $\dot{H}^{1/2}$-continuity of $T(v)$, we easily see that, no matter which semi-norm $\langle\cdot\rangle$ equivalent to $[\cdot]_{1/2}$ is used, given $v_0 \in X$ such that $T_0 := T(v_0) \neq 0$, we have

$$\mathbf{m}_{\langle\cdot\rangle}(T_0) := \inf\big\{\langle v\rangle^2 : v \in X\,,\ T(v) = T_0\,\big\} > 0\,. \tag{1.2.5}$$

A slightly different quantity actually plays the decisive role. It can be introduced as follows :

$$\widetilde{\mathbf{m}}_{\langle\cdot\rangle}(T_0) := \inf\bigg\{\liminf_{n\to+\infty} \langle v_n\rangle^2 : \{v_n\}_{n\in\mathbb{N}} \subseteq X\,,\ T(v_n) = T_0\,,$$
$$v_n \rightharpoonup \alpha \text{ weakly in } \dot{H}^{1/2} \text{ for some constant } \alpha \in \mathbb{S}^1\bigg\}\,. \tag{1.2.6}$$

It is a nontrivial fact that $\widetilde{\mathbf{m}}_{\langle\cdot\rangle}(T_0)$ is well defined, but in any case, we obviously have $\widetilde{\mathbf{m}}_{\langle\cdot\rangle}(T_0) \geqslant \mathbf{m}_{\langle\cdot\rangle}(T_0)$ since sequences weakly converging to a constant are the only competitors allowed in $\widetilde{\mathbf{m}}_{\langle\cdot\rangle}(T_0)$.

In the situation where $T_0 = 2\pi(\delta_P - \delta_Q)$ with $P, Q \in \mathbb{R}^2$, it is tempting to show that the numbers

$$\boldsymbol{\rho}(P,Q) := \mathbf{m}_{\langle\cdot\rangle}\big(2\pi(\delta_P - \delta_Q)\big) \quad \text{and} \quad \widetilde{\boldsymbol{\rho}}(P,Q) := \widetilde{\mathbf{m}}_{\langle\cdot\rangle}\big(2\pi(\delta_P - \delta_Q)\big)\,, \tag{1.2.7}$$

as functions of P and Q are distances on the plane. At least for suitable seminorms, it is the case, these functions giving heuristically the minimal $H^{1/2}$-energy necessary to move the singularity P up to the singularity Q.

We address two natural questions concerning (1.2.7), namely

(Q1) Can we compute (1.2.7) in terms of $\langle\cdot\rangle$?

(Q2) What is the behavior of a minimizing sequence in (1.2.7) ?

Both questions are very delicate in nature and intimately related to the specific choice of the seminorm. Since smooth maps are dense in the weak topology and $T(v) = 0$ for any such map, it is obvious that the constraint $T(v) = T_0$ is not sequentially weakly closed. Hence, each of the minimization problems above is highly nontrivial.

We have restricted ourselves to a class of seminorms which come from second order linear elliptic operators in the half space. We have shown that, no matter which regularity we assume on the coefficients of the operators, concentration occurs near the boundary of the half space. These phenomena can be regarded as the boundary analogues of the concentration phenomena in the Ginzburg-Landau theories, and we have explained them in terms of concentration and quantization effects of Jacobians.

The class of seminorms we are interested in is defined as follows. Let $\mathscr{S}^+$ be the set of all positive definite symmetric 3×3 matrices and consider $A : \overline{\mathbb{R}^3_+} \to \mathscr{S}^+$ satisfying an ellipticity assumption

$$\lambda|\xi|^2 \leqslant A(x)\xi\cdot\xi \leqslant \Lambda|\xi|^2\,, \quad \forall x \in \overline{\mathbb{R}^3_+}\,,\ \forall \xi \in \mathbb{R}^3\,, \tag{1.2.8}$$

for some constants $\lambda = \lambda(A) > 0$ and $\Lambda = \Lambda(A) > 0$. We denote by $\boldsymbol{\mathcal{S}}^+$ the set of all *continuous* matrix fields satisfying (1.2.8). Thus, $\boldsymbol{\mathcal{S}}^+ \subseteq C^0(\overline{\mathbb{R}^3_+};\mathscr{S}^+)$. We shall also consider $\boldsymbol{\mathcal{S}}^+_\times \subseteq \mathcal{A}$ the subset of those $A \in \boldsymbol{\mathcal{S}}^+$ of *product-type* at the boundary, i.e., such that

$$A_{|\mathbb{R}^2} = \begin{pmatrix} B & 0 \\ 0 & b \end{pmatrix}\,, \tag{1.2.9}$$

for some 2×2 matrix field $B = B(x_1,x_2)$ and scalar function $b = b(x_1,x_2)$.

Given $A \in \boldsymbol{\mathcal{S}}^+$, we introduce the "$A$-energy functional" on $\dot H^1(\mathbb{R}^3_+;\mathbb{R}^2)$ as follows

$$E_A(u) := \frac{1}{2}\int_{\mathbb{R}^3_+} \operatorname{tr}\big(\nabla u\, A\, {}^{\mathrm t}\nabla u\big)\, \mathrm{d}x\,.$$

Then we define the "fractional A-energy" $\mathcal{E}_A$ and a seminorm $\langle\cdot\rangle_A$ on $\dot H^{1/2}(\mathbb{R}^2;\mathbb{R}^2)$ by setting

$$\mathcal{E}_A(v) := \langle v\rangle_A^2 := \inf\Big\{E_A(u) : u \in \dot H^1_v(\mathbb{R}^3_+;\mathbb{R}^2)\Big\}\,, \tag{1.2.10}$$

where $\dot H^1_v(\mathbb{R}^3_+;\mathbb{R}^2) := \big\{u \in \dot H^1(\mathbb{R}^3_+;\mathbb{R}^2) : u_{|\mathbb{R}^2} = v\big\}$. Due to the uniform ellipticity assumption (1.2.8), this seminorm is equivalent to $[\cdot]_{1/2}$. Moreover, the infimum in (1.2.10) is attained by a unique map $v^A \in \dot H^1_v(\mathbb{R}^3_+;\mathbb{R}^2)$ satisfying

$$\operatorname{div}(A\nabla v^A) = 0 \quad \text{in } \dot H^{-1}(\mathbb{R}^3_+ : \mathbb{R}^2)\,.$$

In the sequel, we will refer to v^A as the A-harmonic extension of v.

It is instructive to observe that the A-energy has a natural geometric interpretation when considering on $\overline{\mathbb{R}^3_+}$ the Riemannian metric $g = (g_{ij})$ given by $g = \operatorname{Cof} A$. Indeed, for maps $u = (u_1,u_2) : (\mathbb{R}^3_+, g) \to (\mathbb{R}^2, \mathrm{id})$ the squared length of the differential $du = \partial_1 u\, dx_1 + \partial_2 u\, dx_2 + \partial_3 u\, dx_3$ at a point x is precisely given by $|du|^2_{\hat g_x} = \hat g_x(du_1,du_1) + \hat g_x(du_2,du_2)$, where $\hat g = (g^{ij})$ denotes the dual metric. Hence,

$$E_A(u) = \frac{1}{2}\int_{\mathbb{R}^3_+} |du|^2_{\hat g}\, \mathrm{dvol}_g\,.$$

For curves $\gamma : [0,1] \to \overline{\mathbb{R}^3_+}$, the squared length of the tangent vector $\dot\gamma(t)$ at the point $\gamma(t)$ is given by $|\dot\gamma(t)|^2_{g_{\gamma(t)}} = g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))$. In this way, the matrix field A induces a canonical Riemannian length $\mathfrak{L}_A : \mathrm{Lip}\big([0,1];\overline{\mathbb{R}^3_+}\big) \to [0,\infty)$ given by

$$\mathfrak{L}_A(\gamma) := \int_0^1 \ell_A\big(\gamma(t),\dot\gamma(t)\big)\,\mathrm{d}t\,, \tag{1.2.11}$$

where we have set

$$\ell_A(x,\tau) := \sqrt{\mathrm{Cof}\,A(x)\tau\cdot\tau}\,.$$

To the functional $\mathfrak{L}_A$, we associate the geodesic distance $\mathbf{d}_A$ on $\overline{\mathbb{R}^3_+}$ defined by

$$\mathbf{d}_A(P,Q) := \inf\big\{\mathfrak{L}_A(\gamma) : \gamma\in\mathrm{Lip}\big([0,1];\overline{\mathbb{R}^3_+}\big)\,,\ \gamma(0)=P\,,\ \gamma(1)=Q\,\big\}\,.$$

In the same way, $\mathfrak{L}_A$ induces a distance $\widetilde{\mathbf{d}}_A$ on $\mathbb{R}^2$ by taking the previous infimum over curves lying on the boundary $\partial\mathbb{R}^3_+ \simeq \mathbb{R}^2$, i.e., for $P,Q\in\mathbb{R}^2$,

$$\widetilde{\mathbf{d}}_A(P,Q) := \inf\big\{\mathfrak{L}_A(\gamma) : \gamma\in\mathrm{Lip}([0,1];\partial\mathbb{R}^3_+)\,,\ \gamma(0)=P\,,\ \gamma(1)=Q\,\big\}\,.$$

Both distances are of course equivalent to the Euclidean distance.

Given $A\in\boldsymbol{\mathcal{S}}^+$ and assuming the choice $\langle\cdot\rangle = \langle\cdot\rangle_A$ in (1.2.5) and (1.2.6), we denote by $\mathbf{m}_A(T_0)$ and $\widetilde{\mathbf{m}}_A(T_0)$ the corresponding respective quantities, and $\boldsymbol{\rho}_A(P,Q)$ and $\widetilde{\boldsymbol{\rho}}_A(P,Q)$ the functions defined in (1.2.7). Our first result concerns question (**Q1**). It compares the functions $\boldsymbol{\rho}_A$, $\widetilde{\boldsymbol{\rho}}_A$, $\mathbf{d}_A$ and $\widetilde{\mathbf{d}}_A$.

Theorem 1.2.1. *Let $A\in\boldsymbol{\mathcal{S}}^+$. Then,*

(i) $\widetilde{\boldsymbol{\rho}}_A = \pi\widetilde{\mathbf{d}}_A$ *;*

(ii) *we have*

$$\boldsymbol{\rho}_A(P,Q) \geqslant \pi\mathbf{d}_A(P,Q) \qquad \forall P,Q\in\mathbb{R}^2\,; \tag{1.2.12}$$

(iii) *if $A\in\boldsymbol{\mathcal{S}}^+_\times$ and A does not depend on x_3, then $\boldsymbol{\rho}_A = \widetilde{\boldsymbol{\rho}}_A = \pi\widetilde{\mathbf{d}}_A = \pi\mathbf{d}_A$;*

(iv) *if $\boldsymbol{\rho}_A(P,Q) = \pi\mathbf{d}_A(P,Q)$ for some distinct points $P,Q\in\mathbb{R}^2$, then any minimizing sequence $\{v_n\}_{n\in\mathbb{N}}$ for $\boldsymbol{\rho}_A(P,Q)$ tends weakly (up to subsequences) to some constant $\alpha\in\mathbb{S}^1$. As a consequence, $\pi\mathbf{d}_A(P,Q) = \boldsymbol{\rho}_A(P,Q) = \widetilde{\boldsymbol{\rho}}_A(P,Q) = \pi\widetilde{\mathbf{d}}_A(P,Q)$ and $\boldsymbol{\rho}_A(P,Q)$ is not attained.*

Remark. Inequality (1.2.12) can be strict. In fact, we can construct a matrix field $A\in\boldsymbol{\mathcal{S}}^+$ of the form $A(x) = a(x_3)\mathrm{Id}$ such that $\pi\mathbf{d}_A(P,Q) < \boldsymbol{\rho}_A(P,Q) < \pi\widetilde{\mathbf{d}}_A(P,Q)$ whenever $P\neq Q$. The same phenomena appears when considering $H^{1/2}(\mathbb{S}^2;\mathbb{S}^1)$ endowed with its standard Gagliardo seminorm, see [P07].

Remark. In [P08] we actually consider matrix fields with *measurable coefficients*. In that case, the formula (1.2.11) for the length of a curve is of course meaningless. We have succeeded however to show that such a matrix field induces canonical distances on $\overline{\mathbb{R}^3_+}$ and its boundary associated to some generalized *Finsler metric*. For a matrix field with measurable coefficients, Theorem 1.2.1 holds in a slightly weaker form where statement (i) is replaced by : *$\widetilde{\boldsymbol{\rho}}_A$ is a distance which is greater than or equal to π times the "A-geodesic distance" on $\partial\mathbb{R}^3_+$*. All other items remain unchanged.

In the proof of Theorem 1.2.1, the lower bounds in (i) and (ii) come from a duality argument involving the vector field $H(u)$ and the characterization of 1-Lipschitz functions with respect to a geodesic distance as subsolutions of a suitable eikonal equation. In the Euclidean setting, the argument was originally introduced in [57]. Another basic ingredient providing the upper bound in (i), is the construction of an explicit *optimal dipole* $\{v_n\}_{n\in\mathbb{N}}$ with respect to a constant matrix. As first noticed by P. MIRONESCU &

A. PISANTE in [160], the crucial role is played by *Möbius transformations*, see also Theorem 1.1.3. Under the structure assumption in (iii), $\mathbf{d}_A$ and $\widetilde{\mathbf{d}}_A$ coincide as distances on the plane and this fact leads to the full equality. About claim (iv), we show that the energy has to stay in a bounded set, therefore concentration follows from the strong maximum principle.

1.2.2 Graph currents and bubbling off of circles

Now we would like to answer question (**Q2**), i.e., to describe the behavior of an optimal sequence for $\widetilde{\boldsymbol{\rho}}_A(P,Q)$. From the analytical point of view, such an optimal sequence has an energy density concentrating on a minimizing geodesic connecting the point P to the point Q, very much like in [57]. On the other hand, to interpret geometrically the lack of compactness of an optimal sequence $\{v_n\}_{n\in\mathbb{N}}$, it is very convenient to consider the graphs of the v_n's as two dimensional currents in the product space $\mathbb{R}^2\times\mathbb{S}^1$ in the spirit of the theory of Cartesian currents [116]. Our approach to graph currents for maps in X extends the construction of [160] in the one dimensional case.

Given $v \in X\cap C^\infty(\mathbb{R}^2)$, the graph of v is a 2-dimensional smooth submanifold without boundary $\mathsf{G}_v \subseteq \mathbb{R}^2\times\mathbb{S}^1$, endowed with the natural orientation induced by the parametrization $x\mapsto \big(x,v(x)\big)$. The *graph current* G_v associated to v is defined by its action on smooth compactly supported 2-forms $\beta\in\mathscr{D}^2(\mathbb{R}^2\times\mathbb{S}^1)$ through the formula

$$\langle \mathsf{G}_v,\beta\rangle := \int_{\mathsf{G}_v}\beta\,. \tag{1.2.13}$$

If we denote by ω the standard volume form on $\mathbb{S}^1$, then every 2-form $\beta\in\mathscr{D}^2(\mathbb{R}^2\times\mathbb{S}^1)$ can be uniquely and globally written as

$$\beta(x,y) = f_0(x,y)dx_1\wedge dx_2 + \big(f_1(x,y)dx_1 + f_2(x,y)dx_2\,\big)\wedge\omega(y)\,, \tag{1.2.14}$$

for some smooth functions $f_0,f_1,f_2 \in C_c^\infty(\mathbb{R}^2\times\mathbb{S}^1;\mathbb{R})$. Using decomposition (1.2.14), we can rewrite (1.2.13) as

$$\langle \mathsf{G}_v,\beta\rangle = \int_{\mathbb{R}^2} f_0(x,v)\,\mathrm{d}x + \int_{\mathbb{R}^2}\Big(f_1(x,v)\,v\wedge\partial_2 v - f_2(x,v)\,v\wedge\partial_1 v\Big)\,\mathrm{d}x\,. \tag{1.2.15}$$

Clearly, if v is smooth, then the right hand side of (1.2.15) defines a current, i.e.,

$$\mathsf{G}_v \in \mathscr{D}_2(\mathbb{R}^2\times\mathbb{S}^1) := \big(\mathscr{D}^2(\mathbb{R}^2\times\mathbb{S}^1)\big)'\,,$$

and by construction, it coincides with the integration over the graph of v.

Since the f_j's in (1.2.15) are compactly supported smooth functions, formula (1.2.15) can be interpreted as an $\dot{H}^{1/2}-\dot{H}^{-1/2}$ duality for an arbitrary $v\in X$, and it still defines an element of $\mathscr{D}_2(\mathbb{R}^2\times\mathbb{S}^1)$. In other words, we use (1.2.15) as *definition of graph current* associated to $v\in X$.

Remark. By Stokes Theorem, if $v\in X\cap C^\infty(\mathbb{R}^2)$, then $\int_{\mathsf{G}_v} d\beta = \int_{\partial\mathsf{G}_v}\beta = 0$ for any $\beta\in\mathscr{D}^1(\mathbb{R}^2\times\mathbb{S}^1)$, since the graph G_v has no boundary in $\mathbb{R}^2\times\mathbb{S}^1$. On the contrary, for an arbitrary map $v\in X$, the graph current G_v can have a boundary. More precisely, for $\beta\in\mathscr{D}^1(\mathbb{R}^2\times\mathbb{S}^1)$,

$$\langle\partial\mathsf{G}_v,\beta\rangle := \langle\mathsf{G}_v,d\beta\rangle = \langle T(v),\beta_0\rangle\,,$$

where $\beta_0(x) := \fint_{\mathbb{S}^1}\beta(x,\cdot)\in C_c^\infty(\mathbb{R}^2;\mathbb{R})$.

Remark. If a sequence $\{v_n\}_{n\in\mathbb{N}}\subseteq X$ converges *strongly* to v in $\dot{H}^{1/2}(\mathbb{R}^2)$, then $\mathsf{G}_{v_n}\rightharpoonup\mathsf{G}_v$ weakly as currents.

In view of the two preceding remarks and the discussion above, if $\{v_n\}_{n\in\mathbb{N}}$ is an optimal sequence $\widetilde{\boldsymbol{\rho}}_A(P,Q)$, then the graphs $\{\mathsf{G}_{v_n}\}$ must undergo a change of topology along a geodesic between P and Q. Geometrically, what happens is that a vertical circle is formed above each point of this geodesic in order to compensate the loss of boundary in the limit. The precise statement is given in Theorem 1.2.2 below, and it requires the matrix field to satisfy the structure assumption (1.2.9).

Theorem 1.2.2. *Let $A \in \boldsymbol{\mathcal{S}}^+_\times$, and let $P,Q \in \mathbb{R}^2$ be two distinct points, $\{v_n\}_{n\in\mathbb{N}} \subseteq X$ an optimal sequence for $\widetilde{\boldsymbol{\rho}}_A(P,Q)$, and $\{v_n^A\}_{n\in\mathbb{N}} \subseteq \dot{H}^1(\mathbb{R}^3_+;\mathbb{R}^2)$ the corresponding A-harmonic extensions. Then, up to subsequences,*

(i) *there exists an injective curve $\gamma \in \mathrm{Lip}([0,1];\partial\mathbb{R}^3_+)$ satisfying $\gamma(0) = P$, $\gamma(1) = Q$, and $\mathfrak{L}_A(\gamma) = \widetilde{\mathbf{d}}_A(P,Q)$ such that*

$$\frac{1}{2}\,\mathrm{tr}\big(\nabla v_n^A\, A\, {}^{\mathrm{t}}\nabla v_n^A\big)\, \mathscr{L}^{n+1} \llcorner \mathbb{R}^3_+ \overset{*}{\rightharpoonup} \pi \ell_A(x,\tau_x)\mathscr{H}^1 \llcorner \Gamma\,, \tag{1.2.16}$$

weakly as Radon measures, where $\Gamma = \gamma([0,1])$ and τ_x denotes a unit tangent vector to Γ at x ;*

(ii) *the sequence of graph currents $\{\mathsf{G}_n\}_{n\in\mathbb{N}}$ associated to $\{v_n\}_{n\in\mathbb{N}}$ satisfies*

$$\langle \mathsf{G}_n, \beta\rangle \to \langle \mathsf{G}_\alpha, \beta\rangle + \big\langle \vec{\Gamma} \times [\![\mathbb{S}^1]\!], \beta\big\rangle \qquad \forall \beta \in \mathscr{D}^2(\mathbb{R}^2 \times \mathbb{S}^1)\,, \tag{1.2.17}$$

where $\vec{\Gamma}$ is the 1–rectifiable current relative to the oriented curve γ ;

(iii) *the energy is carried by the vorticity sets, i.e.,*

$$\frac{1}{2}\int_{\{|v_n^A| \leqslant \delta\}} \mathrm{tr}\big(\nabla v_n^A\, A\, {}^{\mathrm{t}}\nabla v_n^A\big)\,\mathrm{d}x \to \pi\delta^2\, \widetilde{\mathbf{d}}_A(P,Q) \qquad \forall \delta \in (0,1)\,. \tag{1.2.18}$$

In this theorem, (i) describes lack of compactness of optimal sequences and the structure of the limiting defect measure. The analysis of this quantization phenomena is based on a study of the pre-Jacobians $j(v) := v \wedge \nabla v$ for $v \in X$, and their weak limits. Claim (ii) is the announced topological counterpart of energy concentration interpreted in terms of bubbling-off of a vertical current as already pursued in the $H^{1/2}$-setting in [117, 118, 160]. In contrast with [117, 118], our approach to graph currents is direct and does not rely too heavily on Geometric Measure Theory. Instead, it essentially relies on a representation formula for the pre-Jacobian 1-current $j(v)$ in terms of a suitable lifting of v. Our lifting construction is based on a deep result of J. BOURGAIN, H. BREZIS, & P. MIRONESCU [50]. Finally, (iii) asserts that the energy is carried by the vorticity sets of the extensions, much in the spirit of Ginzburg-Landau theories. This statement is the higher dimensional analogue of [160, Remark 7, formula (3.54)], and it is proved using the oriented coarea formula of G. ALBERTI, S. BALDO, & G. ORLANDI [7].

Remark. By analogy with [74, 160] and in view of Theorem 1.1.3, we expect that, for almost every point $x \in \Gamma$, the sequence $\{v_n^A\}_{n\in\mathbb{N}}$ behaves like a rescaled Möbius transformation on the half plane passing through x and orthogonal to Γ.

1.2.3 The relaxed energy

As we have already mentioned, smooth maps are dense in X only for the $\dot{H}^{1/2}$-weak topology. In analogy to [37], the last question we address is : for a given $v \in X$, how far from v remains a smooth approximating sequence ? Given the energy functional $\mathcal{E}_A$ on X, we can evaluate the "smooth approximation defect" via the so-called *relaxed functional* $\overline{\mathcal{E}}_A : X \to [0,\infty)$ defined by

$$\overline{\mathcal{E}}_A(v) := \inf\left\{\liminf_{n\to+\infty}\, \mathcal{E}_A(v_n) : \{v_n\}_{n\in\mathbb{N}} \subseteq X \cap C^\infty(\mathbb{R}^2)\,,\; v_n \rightharpoonup v \text{ weakly in } \dot{H}^{1/2}(\mathbb{R}^2)\right\}.$$

Obviously $\overline{\mathcal{E}}_A \geqslant \mathcal{E}_A$ and the gap between $\overline{\mathcal{E}}_A(v)$ and $\mathcal{E}_A(v)$ is the quantity we want to determine. In the context of $\mathbb{S}^2$-valued maps from three dimensional domains, it has been proved in [37] that the gap occurring in the approximation process is proportional to the *length of a minimal connection* between the topological singularities of positive and negative degree. This notion is genuinely related to the metric under consideration, and in our setting, Theorem 1.2.1 already suggests that $\widetilde{\mathbf{d}}_A$ is the appropriate distance. The length of a minimal connection relative to the distance $\widetilde{\mathbf{d}}_A$ corresponds to the functional $L_A : X \to [0,\infty)$ defined by

$$L_A(v) := \frac{1}{2\pi} \sup \Big\{ \langle T(v), \varphi \rangle : \varphi \in \mathrm{Lip}(\mathbb{R}^2,\mathbb{R}) \,,\; |\varphi(P) - \varphi(Q)| \leqslant \widetilde{\mathbf{d}}_A(P,Q) \;\; \forall P,Q \in \mathbb{R}^2 \Big\} \,.$$

In other words, $L_A(v)$ is (up to a multiplicative factor) the dual norm of $T(v) \in \big(\mathrm{Lip}(\mathbb{R}^2;\mathbb{R})\big)'$ with $\mathbb{R}^2$ endowed with the metric $\widetilde{\mathbf{d}}_A$. To picture analytically the value of $L_A(v)$, it is instructive to consider the case where v has only finitely many singularities $\{a_1,\dots,a_k\}$. We have seen that $\sum_i d_i = 0$ where d_i is degree of v around a_i. Hence, we can relabel the a_i's taking into account their multiplicity $|d_i|$, as two lists of positive and negative points say $(P_1,\dots,P_m)$ and $(Q_1,\dots,Q_m)$. In that case, we have $T(v) = 2\pi \sum_j (\delta_{P_j} - \delta_{Q_j})$, and

$$L_A(v) = \min_{\sigma \in \mathfrak{S}_m} \sum_{j=1}^{m} \widetilde{\mathbf{d}}_A(P_j, Q_{\sigma(j)}) \,,$$

where $\mathfrak{S}_m$ denotes the set of all permutations of m indices.

We have the following representation result for $\overline{\mathcal{E}}_A$, and eventually for $\widetilde{\mathbf{m}}_A$.

Theorem 1.2.3. *Let $A \in \boldsymbol{\mathcal{S}}^+$. For every $v \in X$,*

$$\overline{\mathcal{E}}_A(v) = \mathcal{E}_A(v) + \pi L_A(v) \,, \tag{1.2.19}$$

and

$$\widetilde{\mathbf{m}}_A\big(T(v)\big) = \pi L_A(v) \,.$$

For the upper bounds, the heart of the matter is a combination of the density of maps with finitely many singularities with Theorem 1.2.1 through a *dipole removing* technique in the spirit of [34]. The lower bounds are obtained again by duality arguments. As already mentioned, when $A = \mathrm{Id}$, formula (1.2.19) could be proved using the theory of Cartesian currents, adapted to the case of the entire space, combining the lower semicontinuity of the energy functional and the approximation in energy, see [118, Proposition 2.11 and Theorem 6.1]. Our proof is elementary and does not make any use of currents.

Remark. As for Theorem 1.2.1, we have also considered in [P08] the case of a matrix field with measurable coefficients. If the matrix field satisfies (1.2.9) and does not depend on x_3, then a similar representation for the relaxed energy holds. In the general case, we only have upper and lower bounds involving the length of a minimal connection relative to the distance $\frac{1}{\pi}\widetilde{\boldsymbol{\rho}}_A$, and the one relative to $\widetilde{\mathbf{d}}_A$, respectively.

1.3 Equivariant symmetry for the 3D Ginzburg-Landau equation

Symmetry results for nonlinear elliptic PDE's are difficult and usually rely on a clever use of the maximum principle as in the celebrate Aleksandrov's moving planes method, or the use of rearrengement techniques as the Schwarz symmetrization (see e.g. [140] for a survey). In case of systems the situation is much more involved since there are no general tools for proving this kind of results.

In the article [P12], in collaboration with A. PISANTE, we have investigated symmetry properties of maps $u : \mathbb{R}^3 \to \mathbb{R}^3$ which are entire (smooth) solutions of the Ginzburg-Landau system

$$\Delta u + (1 - |u|^2)u = 0 \tag{1.3.1}$$

possibly subject to the condition at infinity

$$|u(x)| \to 1 \quad \text{as} \quad |x| \to \infty \,. \tag{1.3.2}$$

This system is naturally associated to the localized energy functional

$$E(v, \Omega) := \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2}(1 - |v|^2)^2 \, \mathrm{d}x \tag{1.3.3}$$

defined for $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ and a bounded open set $\Omega \subseteq \mathbb{R}^3$. Indeed, if $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ is a critical point of $E(\cdot, \Omega)$ for every Ω then u is a weak solution of (1.3.1) and thus a classical solution according to the standard regularity theory for elliptic equations. In addition, any weak solution u of (1.3.1) satisfies the natural bound $|u| \leqslant 1$ in the entire space by a result of A. FARINA [94, Proposition 1.9].

Here the "boundary condition" (1.3.2) is added to rule out solutions with values in a lower dimensional Euclidean space like the scalar valued solutions relevant for the De Giorgi conjecture (see e.g. [17]), or the explicit vortex solutions of [128] arising in the 2D Ginzburg-Landau model (see also [38]). More precisely, under assumption (1.3.2) the map u has a well defined topological degree at infinity given by

$$\deg_\infty u := \deg\left(\frac{u}{|u|}, \partial B_R\right)$$

whenever R is large enough, and we are interested in solutions satisfying $\deg_\infty u \neq 0$.

A special symmetric solution U to (1.3.1)-(1.3.2) with $\deg_\infty U = 1$ has been constructed by V. AKOPIAN & A. FARINA [4] and S. GUSTAFSON [126] in the form

$$U(x) = f(|x|) \frac{x}{|x|} \,, \tag{1.3.4}$$

for a unique function f vanishing at zero and increasing to one at infinity. Taking into account the obvious invariance properties of (1.3.1) and (1.3.3), infinitely many solutions can be obtained from (1.3.4) by translations on the domain and orthogonal transformations on the image. In addition, these solutions satisfy $r^{-1}E(u, B_r) \to 4\pi$ as $r \to +\infty$. It is easy to check that U as in (1.3.4) is the unique solution u of (1.3.1)-(1.3.2) such that $u^{-1}(\{0\}) = \{0\}$, $\deg_\infty u = 1$ and u is $O(3)$-equivariant, i.e.,

$$u(Tx) = Tu(x) \qquad \forall x \in \mathbb{R}^3 \,, \; \forall T \in O(3) \,.$$

In addition $u = U$ satisfies $|u(x)| = 1 + O(|x|^{-2})$ as $|x| \to \infty$.

In [56] H. BREZIS has formulated the following problem :

> (**Q**) *Is any solution to* (1.3.1) *satisfying* (1.3.2) *(possibly with a "good rate" of convergence) and* $\deg_\infty u = \pm 1$*, of the form* (1.3.4) *(up to isometries)?*

We have investigated this problem focusing on *local minimizers* of the energy in the following sense. We say that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ is local minimizer of $E(\cdot)$ if

$$E(u, \Omega) \leqslant E(v, \Omega)$$

for any bounded open set Ω and $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ such that $u - v \in H^1_0(\Omega; \mathbb{R}^3)$. In other words, we have considered the following alternative question :

$(\mathbf{Q}')$ *Is any non-trivial local minimizers of $E(\cdot)$ of the form (1.3.4) (up to isometries) ?*

Our main result gives a positive answer to this question under a natural condition on the growth of the energy on balls of increasing radius, i.e.,

$$\sup_{r>0} r^{-1} E(u, B_r) < \infty \,. \tag{1.3.5}$$

An entire solution u to (1.3.1) satisfying this condition can be studied near infinity through a "blow-down" analysis. More precisely, for each $r > 0$ we introduce the scaled map u_r defined by

$$u_r(x) := u(rx)\,, \tag{1.3.6}$$

which is a smooth entire solution of

$$\Delta u_r + r^2(1 - |u_r|^2)u_r = 0\,. \tag{1.3.7}$$

Whenever $E(u, B_r)$ grows at most linearly with r,

$$E_r(u_r, \Omega) := \frac{1}{2}\int_\Omega |\nabla u_r|^2 + \frac{r^2}{2}(1 - |u_r|^2)^2 \,\mathrm{d}x$$

is equibounded for every bounded open set Ω, and thus $\{u_r\}_{r>0}$ is bounded in $H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$. Any weak limit $u_\infty : \mathbb{R}^3 \to \mathbb{R}^3$ of $\{u_r\}_{r>0}$ as $r \to \infty$ is called a *tangent map* to u at infinity. By a result due to F.H. LIN & C. WANG [149], any tangent map u_∞ is a 0-homogeneous entire harmonic map into $\mathbb{S}^2$. Assuming in addition that u is a local minimizer, we have shown that any tangent map u_∞ is a locally minimizing harmonic map, i.e.,

$$\int_\Omega |\nabla u_\infty|^2 \,\mathrm{d}x \leqslant \int_\Omega |\nabla v|^2 \,\mathrm{d}x$$

for any bounded open set Ω and $v \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{S}^2)$ such that $u_\infty - v \in H^1_0(\Omega; \mathbb{R}^3)$. On the other hand, it was known from F.J. ALMGREN & E.H. LIEB [12] that any locally minimizing harmonic map from $\mathbb{R}^3$ into $\mathbb{S}^2$ is either trivial or of in form $\frac{x}{|x|}$ up to translations and orthogonal transformations. In case the harmonic map is assumed to be 0-homogeneous, the same result was already contained in H. BREZIS, J.M. CORON, & E.H. LIEB [57]. Our formulation of question $(\mathbf{Q}')$ was clearly motivated by those rigidity results for entire harmonic maps from $\mathbb{R}^3$ into $\mathbb{S}^2$.

1.3.1 Existence of local minimizers

Obviously local minimizers are smooth entire solutions of (1.3.1) but it is not clear that non-trivial local minimizers do exist or if the solutions obtained from (1.3.4) are locally minimizing. In case of maps from the plane into itself, these questions were essentially solved affirmatively in [158, 159, 180] (see also [175] for the more difficult gauge-dependent problem, i.e., in presence of a magnetic field). As a preliminary step of our analysis, we have shown the existence of non-constant local minimizers for the 3D Ginzburg-Landau equation.

Theorem 1.3.1. *There exists a smooth non-constant solution $u : \mathbb{R}^3 \to \mathbb{R}^3$ of (1.3.1)-(1.3.2) which is a local minimizer of $E(\cdot)$. In addition, $u(0) = 0$, $\deg_\infty u = 1$ and $r^{-1}E(u, B_r) \to 4\pi$ as $r \to +\infty$.*

Our construction relies on a careful analysis of the vorticity set for solutions u_λ to

$$\begin{cases} \Delta u + \lambda^2 u(1 - |u|^2) = 0 & \text{in } B_1\,, \\ u = \mathrm{id} & \text{on } \partial B_1\,, \end{cases} \qquad \lambda > 0\,, \tag{1.3.8}$$

which are absolute minimizers of the Ginzburg-Landau functional $E_\lambda(u, B_1)$ over $H^1_{\rm id}(B_1;\mathbb{R}^3)$. Up to a translation, we have obtained a locally minimizing solution to (1.3.1) as a limit of $u_{\lambda_n}(x/\lambda_n)$ for some sequence $\lambda_n \to +\infty$. The heart of the matter was to prove the following

Proposition 1.3.2. *Let $\lambda \geqslant 1$ and u_λ be a global minimizer of $E_\lambda(\cdot, B_1)$ over $H^1_{\rm id}(B_1;\mathbb{R}^3)$. For any $\delta \in (0,1)$, there exists a constant $C_\delta > 0$ independent of λ such that*

$$\mathrm{diam}\big(\{|u_\lambda| \leqslant \delta\}\big) \leqslant C_\delta \lambda^{-1} \quad \textit{and} \quad \mathrm{dist}_{\mathscr{H}}\big(\{|u_\lambda| \leqslant \delta\}, \{0\}\big) = o(1) \quad \textit{as } \lambda \to \infty\,,$$

where $\mathrm{dist}_{\mathscr{H}}$ denotes the Haussdorff distance.

Part of the proof rests on classical estimates for Ginzburg-Landau equations [66, 67, 68], and the result of H. BREZIS, J.M. CORON, & E.H. LIEB [57] telling us that the map $\frac{x}{|x|}$ is the unique asymptotic limit as $\lambda \to \infty$. In turn, the estimate on the size of the vorticity set $\{|u_\lambda| \leqslant \delta\}$ relies the asymptotic analysis of F.H. LIN & C. WANG [149] and on a quantization result for stationary harmonic maps into $\mathbb{S}^2$ due to F.H. LIN & T. RIVIÈRE [147]. This estimate is proved by contradiction using the easy upper bound $E_\lambda(u_\lambda, B_1) \leqslant 4\pi$ (choosing $\frac{x}{|x|}$ as competitor for minimality).

1.3.2 Tangent maps and asymptotic symmetry

In order to prove full symmetry of a non-constant local minimizer, a natural approach is to prove uniqueness and symmetry of the tangent map at infinity, and then try to propagate the symmetry from infinity to the entire space. To be able to follow this path, one has to determine first the nature of the convergence toward tangent maps. The possible lack of compactness of the scaled maps $\{u_r\}_{r>0}$ in (1.3.6)-(1.3.7) has been carefully analyzed by F.H. LIN & C. WANG [148, 149]. They have obtained a complete description of a defect measure, and as a byproduct, proved a quantization effect for the normalized energy in the spirit of [147], namely $r^{-1}E(u, B_r) \to 4\pi d$ as $r \to \infty$ for some $d \in \mathbb{N}$. Note that the case $d = 1$ is already valid both for the solution (1.3.4) and the local minimizer constructed in Theorem 1.3.1. As a matter of fact, we have proved that the same property holds for *any* non-constant local minimizer satisfying (1.3.5), and that the induced scaled maps $\{u_r\}_{r>0}$ are strongly relatively compact in $H^1_{\rm loc}(\mathbb{R}^3;\mathbb{R}^3)$. In proving this result, the first step is to apply the blow-down analysis from infinity of [149]. Then concentration is excluded by a comparison argument involving the "dipole removing technique" of [34]. This yields the strong compactness of the scaled maps. A further comparison argument based on [57] gives the desired value for the limit of the normalized energy.

Once the compactness of the scaled maps is obtained, one can address a finer convergence analysis using elliptic theory. In this direction, we have the following result inspired by the asymptotic analysis for minimizing harmonic maps at isolated singularities of L. SIMON [194].

Theorem 1.3.3. *Let u be an entire solution of (1.3.1) satisfying (1.3.5) and such that the scaled maps $\{u_r\}_{r>0}$ are strongly relatively compact in $H^1_{\rm loc}(\mathbb{R}^3;\mathbb{R}^3)$. Then there exist a constant $C > 0$ such that*

$$|x|^2(1-|u(x)|^2) + |x||\nabla u(x)| + |x|^3|\nabla(1-|u(x)|^2)| + |x|^2|\nabla^2 u(x)| \leqslant C \qquad \forall x \in \mathbb{R}^3\,, \tag{1.3.9}$$

and a unique harmonic map $\phi : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\deg\phi = \deg_\infty u$ and setting $u_\infty(x) = \phi(x/|x|)$,

(i) $\|u_{r|\mathbb{S}^2} - \phi\|_{C^2(\mathbb{S}^2;\mathbb{R}^3)} \to 0$ *as $r \to \infty$;*

(ii) $E_r(u_r, \Omega) \to \frac{1}{2}\int_\Omega |\nabla u_\infty|^2\,\mathrm{d}x$ *for every bounded open set Ω .*

If in addition $\deg_\infty u = \pm 1$, then $\phi(x) = Tx$ for some $T \in O(3)$.

This result strongly relies on the a priori bound (1.3.9) which is proved by contradiction. Whenever (1.3.9) holds, we can write for $|x|$ sufficiently large the polar decomposition of the solution u as $u(x) = \rho(x)w(x)$ for some positive function ρ and some $\mathbb{S}^2$-valued map w which have to solve the system

$$\begin{cases} \operatorname{div}(\rho^2(x)\nabla w(x)) + w(x)\rho^2(x)|\nabla w(x)|^2 = 0\,, \\ \Delta\rho(x) + \rho(x)(1-\rho^2(x)) = \rho(x)|\nabla w(x)|^2\,, \end{cases} \tag{1.3.10}$$

for $|x|$ large. It is clear from (1.3.9) that ρ smoothly tends to 1 at infinity. Hence the unit map w tends to be harmonic as $|x| \to \infty$, and system (1.3.10) can be considered as a perturbation of the harmonic map system. Using a suitable "Pohozaev identity", we were able to derive an elementary (but tricky) estimate on the radial derivative of w leading to the uniqueness of the asymptotic limit. In particular, we avoid the use of the Simon-Lojasievicz inequality [194].

1.3.3 Full symmetry

Concerning the original symmetry problem, we finally state our main result answering both questions $(\mathbf{Q})$ and $(\mathbf{Q}')$.

Theorem 1.3.4. *Let u be an entire solution of* (1.3.1)*. The following conditions are equivalent :*

(i) *u is a non-constant local minimizer of $E(\cdot)$ satisfying* (1.3.5)*;*

(ii) *$E(u, B_r) = 4\pi r + o(r)$ as $r \to \infty$;*

(iii) *u satisfies $|u(x)| = 1 + O(|x|^{-2})$ as $|x| \to \infty$ and $\deg_\infty u = \pm 1$;*

(iv) *up to a translation and an orthogonal transformation on the image, u is $O(3)$-equivariant, i.e., $u = U$ as given by* (1.3.4)*.*

The first chains of implications come from the results and techniques we have described above. In turn, (iii) $\Rightarrow$ (iv) rests on the asymptotic symmetry in Theorem 1.3.3, and it is proved considering the map u/f (where f is given in (1.3.4)) in the spirit of the division method of [159, 175]. We emphasize that Theorem 1.3.4 requires the energy bound (1.3.5) in (i). It would be interesting to determine whether or not any local minimizer satisfies this bound. For the 2D Ginzburg-Landau equation, the analogous bound is know to be true [180].

1.4 Vortex curves in some 2D Ginzburg-Landau systems

The Ginzburg-Landau theories have had an enormous influence on both physics and mathematics. Physicists employ Ginzburg-Landau models in superconductivity, superfluidity, or Bose-Einstein condensates (BECs), all systems which present quantized defects commonly known as vortices. Starting with the work by F. Bethuel, H. Brezis, & F. Hélein [38], many powerful methods have been developed to study the physical London limit, i.e., as the characteristic length scale ε tends to 0. This limit corresponds to the Thomas-Fermi regime in BECs, and to an analogous regime in superfluids. In a two-dimensional setting, vortices are essentially characterized as isolated zeroes of the order parameter carrying a winding number, and in the London limit as point defects where energy concentration occurs. The question of whether energy minimizers develop vortices, where they appear in the domain, and how many there should be (for given boundary conditions, constant applied fields or angular velocities) has been analyzed in many contexts and parameter regimes.

In the article [P13], in collaboration with S. Alama & L. Bronsard, we have considered a certain type of Ginzburg-Landau energies, arising for instance in the physical context of a rotating superfluid.

Given a smooth bounded domain $\mathcal{D} \subseteq \mathbb{R}^2$, a smooth vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$, an angular speed $\mathbf{\Omega} > 0$, and a parameter $\varepsilon > 0$, the general form of the energy is

$$u \in H^1(\mathcal{D};\mathbb{C}) \mapsto \mathscr{F}_\varepsilon(u) := \frac{1}{2}\int_{\mathcal{D}} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \,\mathrm{d}x - \mathbf{\Omega}\int_{\mathcal{D}} V(x)\cdot j(u)\,\mathrm{d}x\,.$$

Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, we denote by

$$j(u) := u \wedge \nabla u \in L^1(\mathcal{D};\mathbb{R}^2)\,,$$

the *pre-Jacobian* of u. The L^1-vector field $j(u)$ is often written as $j(u) = (iu, \nabla u)$, where $(\cdot,\cdot)$ is the standard inner product of two complex numbers, viewed as vectors in $\mathbb{R}^2$. We have chosen this particular model because it is the simplest setting to analyze concentration of vortices on arbitrary sets. Nevertheless, similar results can be extended to other physical models, see [P13].

In the case of uniform rotation, that is $V(x) = x^\perp$ and with $\mathcal{D}$ a disc, S. SERFATY [190] studied minimizers of a very closely related functional to determine the critical value $\mathbf{\Omega}_1 = \mathbf{\Omega}_1(\varepsilon)$ of the angular speed $\mathbf{\Omega}$ at which vortices first appear (see also [P05, P06] for BECs). She finds that minimizers acquire vorticity at $\mathbf{\Omega}_1 = \boldsymbol{k}(D)|\ln\varepsilon| + O(\ln|\ln\varepsilon|)$ for an explicitly determined constant $\boldsymbol{k}(\mathcal{D})$. In a series of papers, culminating with the research monograph [183], E. SANDIER & S. SERFATY developed powerful tools to study vortices in Ginzburg-Landau models. Although they primarily work with the full Ginzburg-Landau model with magnetic field, the methods apply as well to the functional $\mathscr{F}_\varepsilon$ above. In particular, their results apply to the near-critical regime in *simply connected domains*. In our setting, their results show that for any simply connected domain $\mathcal{D}$, the first order expansion of the critical value $\mathbf{\Omega}_1$ for vortex existence in minimizing configurations is also of the form $\boldsymbol{k}(\mathcal{D})|\ln\varepsilon|$ for some constant $\boldsymbol{k}(\mathcal{D})$. Moreover the locus of concentration of vortices for $\mathbf{\Omega} = \mathbf{\Omega}_1 + o(|\ln\varepsilon|)$ is given by the set of maxima of $|\zeta|$, with ζ the solution of the following boundary-value problem :

$$\begin{cases} -\Delta\zeta = \operatorname{curl} V & \text{in } \mathcal{D}\,,\\ \zeta = 0 & \text{on } \partial\mathcal{D}\,. \end{cases} \tag{1.4.1}$$

The constant $\boldsymbol{k}(\mathcal{D})$ is then determined by

$$\boldsymbol{k}(\mathcal{D}) = \frac{1}{2|\zeta|_{\max}}\,,$$

where $|\zeta|_{\max}$ denotes the maximum value of $|\zeta|$. If, for instance, V is real-analytic and $\operatorname{curl} V$ is nonnegative, then so is the solution ζ, and the maximum is generically attained at a finite number of points in the domain. In this situation, if $\mathbf{\Omega} = \mathbf{\Omega}_1 + o(|\ln\varepsilon|)$, minimizers exhibit concentration of vortices *at isolated points*, and the number of vortices remains uniformly bounded whenever $\mathbf{\Omega} - \mathbf{\Omega}_1$ is of order $O(\ln|\ln\varepsilon|)$, see [183, 190].

The case of a multiply connected domain provides a slightly different qualitative picture. In a work on rotating BECs, A. AFTALION, S. ALAMA & L. BRONSARD [3] considered a similar functional in a domain given by a circular annulus $\mathcal{A}$ (centered at the origin) and again with uniform rotation $V(x) = x^\perp$. Unlike the simply connected case, minimizers in the annulus may have vorticity without vortices, as the hole acquires positive winding at bounded rotation $\mathbf{\Omega}$. Then point vortices are nucleated inside the interior of $\mathcal{A}$ at a critical value $\mathbf{\Omega}_1$, again of leading order $|\ln\varepsilon|$. Solving equation (1.4.1) in the annulus $\mathcal{A}$, one finds out that the set of maxima of the function ζ is given by a circle inside $\mathcal{A}$. Hence one can expect that, rather than accumulating at isolated points, vortices concentrate along this circle in the limit $\varepsilon \to 0$. The main feature proved in [3] is that if $\mathbf{\Omega} \sim \mathbf{\Omega}_1 + O(\ln|\ln\varepsilon|)$, then vortices are indeed essentially supported by a circle Σ and that the total degree of these vortices is of order $\ln|\ln\varepsilon|$. In other words, in the limit $\varepsilon\to 0$, infinitely many vortices accumulate on Σ. However the determination of the limiting vorticity measure on the circle was left open.

Our primary objective was to answer this question on the nature of the limiting vorticity measure in the case where infinitely many vortices accumulates on a curve. We underline that this situation was not covered by the results in [183], and it can be seen as intermediate between the regime $\mathbf{\Omega}\approx\mathbf{\Omega}_1$ with finitely many vortices, and the "free boundary regime" $\mathbf{\Omega}\gg\mathbf{\Omega}_1$ where the limiting vorticity measure is supported on a set of positive area (see [183, 190]).

1.4.1 Prescribed concentration set in simply connected domains

To effectively separate the question of the nature of the concentration set from the question of localizing vortices, we instead start with a *simply connected domain* $\mathcal{D}$, and we prescribe the function ζ with $\zeta\geq 0$ in $\mathcal{D}$ and $\zeta|_{\partial D}=0$, in such a way that ζ is maximized on *a prescribed closed curve* $\Sigma\subseteq\mathcal{D}$. Then, we choose as our vector field

$$V(x)=-\nabla^{\perp}\zeta(x)=\left(\frac{\partial\zeta}{\partial x_2},-\frac{\partial\zeta}{\partial x_1}\right)$$

(so that (1.4.1) is trivially satisfied). With this choice, we have shown that vortices are forced to accumulate on Σ as ε tends to 0. The curve Σ can be either a smooth Jordan curve or a smooth embedded simple arc, compactly contained in $\mathcal{D}$. In this setting, we have resolved the problem of distribution of vortices along Σ, both for minimizers and in the more general setting of Γ-convergence. While this arbitrary choice of concentration set Σ may seem unphysical, in fact Ginzburg-Landau functionals of this form appear naturally in thin shell limits for superconductors in strong constant magnetic fields, see [5]. In the next subsection we will consider a general vector field V in a multiply connected domain, and we will see that in this case, the two problems do not differ too much in nature.

To state our first result we need some specific hypotheses on ζ and the angular speed $\mathbf{\Omega}$. We assume that ζ satisfies the following assumptions [4] :

$(\mathbf{H_1})$ $\zeta\in\mathrm{Lip}_0(\mathcal{D})$, $\zeta\geq 0$ in $\overline{\mathcal{D}}$, and $\zeta_{\max}:=\max_{x\in\overline{\mathcal{D}}}\zeta(x)>0$;

$(\mathbf{H_2})$ $\Sigma:=\{x\in\mathcal{D}:\ \zeta(x)=\zeta_{\max}\}\subseteq\mathcal{D}$ is a Jordan curve or a simple embedded arc of class C^2.

We further assume that $\mathbf{\Omega}=\mathbf{\Omega}(\varepsilon)$ is near to the critical value needed for the presence of vortices. More precisely,

$$\mathbf{\Omega}_\varepsilon:=\frac{|\ln\varepsilon|}{2\zeta_{\max}}+\boldsymbol{\omega}(\varepsilon)\,, \tag{1.4.2}$$

for some function $\boldsymbol{\omega}:(0,+\infty)\to(0,+\infty)$ satisfiying $\omega(\varepsilon)\to+\infty$ with $|\ln\varepsilon|^{-1}\boldsymbol{\omega}(\varepsilon)\to 0$ as $\varepsilon\to 0$.

For $u\in H^1(\mathcal{D};\mathbb{C})$ we consider the normalized functional

$$F_\varepsilon(u):=\frac{1}{\boldsymbol{\omega}^2(\varepsilon)}\int_{\mathcal{D}}\left\{\frac{1}{2}|\nabla u|^2+\frac{1}{4\varepsilon^2}(1-|u|^2)^2+\mathbf{\Omega}_\varepsilon\nabla^{\perp}\zeta\cdot j(u)\right\}\mathrm{d}x\,,$$

and for a nonnegative Radon measure μ on $\mathcal{D}$, we define

$$I(\mu):=\frac{1}{2}\iint_{\mathcal{D}\times\mathcal{D}}G(x,y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y)\,,$$

4. We denote by $\mathrm{Lip}_0(\mathcal{D})$ the space of Lipschitz functions on $\overline{\mathcal{D}}$ vanishing on $\partial\mathcal{D}$.

where the function G denotes the Dirichlet Green's function of the domain $\mathcal{D}$, i.e., for every $y \in \mathcal{D}$, $x \mapsto G(x, y)$ is the solution of

$$\begin{cases} -\Delta G(\cdot, y) = \delta_y & \text{in } \mathscr{D}'(\mathcal{D})\,, \\ G(\cdot, y) = 0 & \text{on } \partial\mathcal{D}\,. \end{cases} \tag{1.4.3}$$

Our result describes the asymptotic behavior of the family of functionals $\{F_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0$, and it is stated in terms of the vorticity distribution given by (twice) the weak Jacobian, i.e., the distributional curl of the pre-Jacobian.

Theorem 1.4.1. *Assume that* $(\mathbf{H_1})$, $(\mathbf{H_2})$, *and* (1.4.2) *hold. Let* $\varepsilon_n \to 0^+$ *be an arbitrary sequence. Then,*

(i) *for any* $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\mathcal{D};\mathbb{C})$ *satisfying* $\sup_n F_{\varepsilon_n}(u_n) < \infty$, *there exist a (not relabeled) subsequence and a nonnegative Radon measure* μ *in* $H^{-1}(\mathcal{D})$ *supported by* Σ *such that*

$$\frac{1}{\boldsymbol{\omega}(\varepsilon_n)} \operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \textit{strongly in } (\mathrm{Lip}_0(\mathcal{D}))'\,; \tag{1.4.4}$$

(ii) *for any* $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\mathcal{D};\mathbb{C})$ *such that* (1.4.4) *holds for some nonnegative Radon measure* μ *in* $H^{-1}(\mathcal{D})$ *supported by* Σ, *we have*

$$\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) \geqslant I(\mu) - \zeta_{\max}\,\mu(\mathcal{D})\,;$$

(iii) *for any nonnegative Radon measure* μ *in* $H^{-1}(\mathcal{D})$ *supported by* Σ, *there exists a sequence* $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\mathcal{D};\mathbb{C})$ *such that* (1.4.4) *holds and*

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_n) = I(\mu) - \zeta_{\max}\,\mu(\mathcal{D})\,.$$

The proof of Theorem 1.4.1 makes an essential use of the general estimates of E. SANDIER & S. SERFATY [183], specially for the lower bound and the compactness of normalized weak Jacobians which rest on the so-called "vortex-ball construction". The upper bound is obtained by constructing trial functions in two steps. First, we consider measures which are absolutely continuous with respect to $\mathscr{H}^1 \llcorner \Sigma$ and have a smooth density. In the second step, we prove that an arbitrary measure in H^{-1} supported by Σ can be approximated by measures of the previous kind.

The conclusions of Theorem 1.4.1 are reminiscent of Γ-convergence theory, see [78]. In this context, it yields the following convergence result for the vorticity of global minimizers, and hence solving the problem on the limiting distribution of vortices along Σ.

Corollary 1.4.2. *Assume that* $(\mathbf{H_1})$, $(\mathbf{H_2})$, *and* (1.4.2) *hold. Let* $\varepsilon_n \to 0^+$ *be an arbitrary sequence. For every integer* $n \in \mathbb{N}$, *let* $u_n \in H^1(\mathcal{D};\mathbb{C})$ *be a global minimizer of* F_{ε_n}. *Then,*

$$\frac{1}{\boldsymbol{\omega}(\varepsilon_n)} \operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \frac{\zeta_{\max}}{2I_*}\,\mu_* \quad \textit{strongly in } (\mathrm{Lip}_0(\mathcal{D}))'\,,$$

where μ_* *is the unique minimizer of* I *over all probability measures supported on* Σ, *and* $I_* := I(\mu_*)$.

Remark (Renormalized energy). By the results in [138, 181, 183], the vorticity distribution $\operatorname{curl} j(u_\varepsilon)$ can be approximated by a measure of the form $2\pi \sum_{i\in I_\varepsilon} d_i \delta_{a_i}$ for some finite set of points $\{a_i\}_{i\in I_\varepsilon} \subseteq \mathcal{D}$ and integers $\{d_i\}_{i\in I_\varepsilon} \subseteq \mathbb{Z}$. In other words, each point a_i can be viewed as an "approximate vortex" with winding number d_i. Then the integer $D_\varepsilon = \sum_{i\in I_\varepsilon} |d_i|$ represents the total vorticity of u_ε. It is commonly known that those vortices carry a kinetic energy of leading order at least $\pi D_\varepsilon |\ln\varepsilon|$. In view of such estimate, we actually have a more refined lower bound for the energy than the one given by Theorem 1.4.1. More precisely, if $\{u_\varepsilon\}$ satisfies the uniform energy bound of Theorem 1.4.1, then

$$\liminf_{\varepsilon\to 0} \frac{1}{\boldsymbol{\omega}^2(\varepsilon)} \left(\int_{\mathcal{D}} \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \, \mathrm{d}x - \pi D_\varepsilon |\ln \varepsilon| \right) \geqslant I(\mu)\,,$$

and

$$\liminf_{\varepsilon\to 0} \frac{1}{\boldsymbol{\omega}^2(\varepsilon)} \left(\boldsymbol{\Omega}_\varepsilon \int_{\mathcal{D}} \nabla^\perp \zeta \cdot j(u_\varepsilon) \, \mathrm{d}x + \pi D_\varepsilon |\ln \varepsilon| \right) \geqslant -\zeta_{\max}\, \mu(\mathcal{D})\,.$$

In particular, if $\{u_\varepsilon\}$ is any recovery sequence (in the sense of (iii) in Theorem 1.4.1), the $\liminf$'s above become limits, and equality holds in each case. In analogy with [38], we may then say that $I(\mu)$ plays the role of "renormalized energy".

In particular, we deduce from Corollary 1.4.2 that if u_ε is energy minimizing, then

$$D_\varepsilon = \frac{\zeta_{\max}}{4\pi I_*}\, \boldsymbol{\omega}(\varepsilon) + o(\boldsymbol{\omega}(\varepsilon)) \quad \text{as } \varepsilon \to 0\,,$$

and from Theorem 1.4.1, the minimal value of the energy expands as

$$\min_{H^1(\mathcal{D};\mathbb{C})} \boldsymbol{\omega}^2(\varepsilon) F_\varepsilon = -\frac{\zeta_{\max}^2}{4I_*}\, \boldsymbol{\omega}^2(\varepsilon) + o(\boldsymbol{\omega}^2(\varepsilon))\,.$$

Remark (*Equilibrium measures).* The value $I(\mu)$ gives the electrostatic energy of a positive charge distribution μ on the set $\Sigma \subseteq \mathcal{D}$. The minimizer μ_* of I over all probability measures on Σ is called the *Green equilibrium measure* in $\mathcal{D}$ associated to the set Σ, and gives the equilibrium charge distribution of a charged conductor inside of a neutral conducting shell, represented by $\partial\mathcal{D}$. The value $1/I_*$ is refered to as to the *capacity of the condenser* $(\Sigma, \partial\mathcal{D})$. The interested reader can find in [178] many results on the existence and general (regularity) properties of the equilibrium measures as well as some examples. For instance, if $\mathcal{D}$ is a disc and Σ is a concentric circle, then the equilibrium measure μ_* is the normalized arclength measure on Σ, see [178, Example II.5.13], and thus vortices are asymptotically equidistributed along Σ as $\varepsilon \to 0$. However for an arbitrary curve Σ, the distribution is of course non-uniform in general. In the case where Σ is an embedded arc, it is even singular at the endpoints, see [178, Example II.5.14].

Remark (*Regularity of* Σ*).* In the present results the structure and regularity assumptions on the set Σ given in $(\mathbf{H_2})$ are mainly motivated by the physical context of [3]. However $(\mathbf{H_2})$ can be relaxed into weaker statments. More precisely, our proof of Theorem 1.4.1 relies on $(\mathbf{H_2})$ only for conclusion (iii). Our construction of the recovery sequence could be applied with minor modifications if the set Σ is for instance a finite union of piecewise C^2 arcs/Jordan curves. Actually Σ could even have a more general structure such as a non-empty interior.

1.4.2 Asymptotics in domains with a single hole

In this subsection we discuss the case of a general vector field V in a multiply connected domain. Our method can be applied for an arbitrary genus, but for simplicity we have restricted ourselves to domains which are topological annuli. $\mathcal{D}$ still denotes a simply connected domain in $\mathbb{R}^2$ with smooth boundary, and we consider $\mathcal{B} \subseteq \mathcal{D}$ a smooth, simply connected domain compactly contained inside $\mathcal{D}$. Then we set $\mathcal{A} := \mathcal{D} \setminus \mathcal{B}$.

For $u \in H^1(\mathcal{A};\mathbb{C})$ we define the functional

$$\mathcal{F}_\varepsilon(u) := \int_{\mathcal{A}} \left\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 - \boldsymbol{\Omega}_\varepsilon V(x) \cdot j(u) \right\} \mathrm{d}x\,.$$

Here the given vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$ is assumed to be locally Lipschitz continuous. We are interested in the asymptotic behavior of $\min \mathcal{F}_\varepsilon$ as $\varepsilon \to 0$, with an angular speed $\mathbf{\Omega}_\varepsilon$ as in (1.4.2).

The hole as a "giant vortex". For multiply connected domains, the highest order term in an expansion of the minimal energy is partially due to the turning of the phase of a minimizer around the holes. The first step in studying vortices in the interior is to identify the *vorticity of the hole*, and then split the energy into contributions from the hole and from the interior. To this purpose we had to study first the minimization of the functional $\mathcal{F}_\varepsilon$ over $\mathbb{S}^1$-valued maps. Observe that for $\mathbb{S}^1$-valued maps, the functional $\mathcal{F}_\varepsilon$ only depends on the angular speed $\mathbf{\Omega} = \mathbf{\Omega}(\varepsilon)$, and not ε itself, i.e., for every $u \in H^1(\mathcal{A}; \mathbb{S}^1)$,

$$\mathcal{F}_\varepsilon(u) = \mathcal{G}_{\mathbf{\Omega}}(u) := \int_{\mathcal{A}} \left\{ \frac{1}{2}|\nabla u|^2 - \mathbf{\Omega}\, V(x) \cdot j(u) \right\} \mathrm{d}x\,.$$

To effectively minimize $\mathcal{G}_{\mathbf{\Omega}}$, we notice that the connected components of $H^1(\mathcal{A}; \mathbb{S}^1)$ are classified by the topological degree (or winding number) around the hole $\mathcal{B}$. Hence, minimizing first in each homotopy class and then choosing the lowest energy level, one reaches the minimum of $\mathcal{G}_{\mathbf{\Omega}}$, i.e.,

$$\min_{H^1(\mathcal{A};\mathbb{S}^1)} \mathcal{G}_{\mathbf{\Omega}} = \min_{d\in\mathbb{Z}} g(d, \mathbf{\Omega})\,, \tag{1.4.5}$$

where

$$g(d, \mathbf{\Omega}) := \min \left\{ \mathcal{G}_{\mathbf{\Omega}}(u) \ :\ u \in H^1(\mathcal{A}; \mathbb{S}^1)\,,\ \deg u = d \right\}.$$

For each integer d, the minimum value can be computed explicitly noticing that $\mathcal{G}_{\mathbf{\Omega}}(u)$ depends only on the divergence free vector field $j(u)$ (since u is $\mathbb{S}^1$-valued). Using Hodge decompositions as in [38, Chapter 1], we have found that

$$g(d, \mathbf{\Omega}) = \frac{1}{2} \int_{\mathcal{A}} \left\{ |\nabla \Phi_d|^2 - \mathbf{\Omega}^2 |V|^2 \right\} \mathrm{d}x\,,$$

where Φ_d is given by

$$\Phi_d = \mathbf{\Omega}\,\zeta + \left(\frac{\gamma_V \mathbf{\Omega} - 2\pi d}{\mathrm{cap}(\mathcal{B})} \right) \xi\,, \tag{1.4.6}$$

and the functions ζ and ξ are determined by

$$\begin{cases} -\Delta\zeta = \mathrm{curl}\, V & \text{in } \mathcal{A} \\ \zeta = 0 & \text{on } \partial\mathcal{A} \end{cases} \quad \text{and} \quad \begin{cases} \Delta\xi = 0 & \text{in } \mathcal{A} \\ \xi = 0 & \text{on } \partial\mathcal{D} \\ \xi = 1 & \text{on } \partial\mathcal{B} \end{cases}, \tag{1.4.7}$$

with

$$\gamma_V := \int_{\partial\mathcal{D}} \left\{ \frac{\partial\zeta}{\partial\nu} + V \cdot \tau \right\} \quad \text{and} \quad \mathrm{cap}(\mathcal{B}) := -\int_{\partial\mathcal{B}} \frac{\partial\xi}{\partial\nu}\,.$$

In particular,

$$g(d, \mathbf{\Omega}) = \frac{|\gamma_V \mathbf{\Omega} - 2\pi d|^2}{2\mathrm{cap}(\mathcal{B})} - \frac{\mathbf{\Omega}^2}{2} \int_{\mathcal{A}} \left\{ |V|^2 - |\nabla\zeta|^2 \right\} \mathrm{d}x\,.$$

As a consequence, the optimal $d_{\mathbf{\Omega}} \in \mathbb{Z}$ in (1.4.5) is unique, except of course for half-integer values of $\frac{\gamma_V \mathbf{\Omega}}{2\pi}$. In any case,

$$d_{\mathbf{\Omega}} \in \left\{ \left\lfloor \frac{\gamma_V \mathbf{\Omega}}{2\pi} \right\rfloor, \left\lfloor \frac{\gamma_V \mathbf{\Omega}}{2\pi} \right\rfloor + 1 \right\},$$

where $\lfloor \cdot \rfloor$ denotes the floor function, and

$$\min_{H^1(\mathcal{A};\mathbb{S}^1)} \mathcal{G}_{\mathbf{\Omega}} = -\frac{\mathbf{\Omega}^2}{2} \int_{\mathcal{A}} \left\{ |V|^2 - |\nabla\zeta|^2 \right\} \mathrm{d}x + O(1) \quad \text{as } \mathbf{\Omega} \to \infty\,.$$

Asymptotics for the reduced energy. To state our parallel result for the annular domain case we must now give more specific hypotheses on the potential V and the angular speed $\boldsymbol{\Omega}_\varepsilon$. We assume in the sequel that V satisfies the following assumptions [5] :

($\mathbf{H_1}'$) the function ζ in (1.4.7) is such that $\zeta_{\max} := \max_{x\in\overline{\mathcal{A}}} \zeta(x) = \max_{x\in\overline{\mathcal{A}}} |\zeta(x)| > 0$;

($\mathbf{H_2}'$) the set $\Sigma := \{x \in \mathcal{A} :\ \zeta(x) = \zeta_{\max}\} \subseteq \mathcal{A}$ is a Jordan curve or a simple embedded arc of class C^2.

As for the simply connected case, we assume that $\boldsymbol{\Omega}_\varepsilon$ is near the critical value needed for the presence of vortices which again reads

$$\boldsymbol{\Omega}_\varepsilon = \frac{|\ln\varepsilon|}{2\zeta_{\max}} + \boldsymbol{\omega}(\varepsilon)\,, \tag{1.4.8}$$

for some positive function $\boldsymbol{\omega}$ satisfying $\boldsymbol{\omega}(\varepsilon) \to +\infty$ with $\boldsymbol{\omega}(\varepsilon) \leqslant o(|\ln\varepsilon|)$ as $\varepsilon \to 0^+$, exactly as in (1.4.2).

For $u \in H^1(\mathcal{A};\mathbb{C})$ we consider the functional

$$\overline{F}_\varepsilon(u) := \frac{\mathcal{F}_\varepsilon(u) - \min \mathcal{G}_{\boldsymbol{\Omega}_\varepsilon}}{\boldsymbol{\omega}^2(\varepsilon)}\,,$$

and for a nonnegative Radon measure μ on $\mathcal{A}$, we define

$$\bar{I}(\mu) := \frac{1}{2}\iint_{\mathcal{A}\times\mathcal{A}} \overline{G}(x,y)\,\mathrm{d}\mu(x)\mathrm{d}\mu(y)\,,$$

where $\overline{G}$ is the Dirichlet Green's function of the annulus $\mathcal{A}$, i.e.,

$$\begin{cases} -\Delta\overline{G}(\cdot,y) = \delta_y & \text{in } \mathscr{D}'(\mathcal{A})\,,\\ \overline{G}(\cdot,y) = 0 & \text{on } \partial\mathcal{A}\,. \end{cases} \tag{1.4.9}$$

Our second result addresses the Γ-convergence of $\overline{F}_\varepsilon$ as $\varepsilon \to 0$. It shows that the second order Γ-development of $\mathcal{F}_\varepsilon$ is completely similar to the limit obtained in the simply connected case.

Theorem 1.4.3. *Assume that* ($\mathbf{H_1}'$), ($\mathbf{H_2}'$), *and* (1.4.8) *hold. Let* $\varepsilon_n \to 0^+$ *be an arbitrary sequence. Then,*

(i) *for any* $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\mathcal{A};\mathbb{C})$ *satisfying* $\sup_n \overline{F}_{\varepsilon_n}(u_n) < +\infty$, *there exist a (not relabelled) subsequence and a nonnegative Radon measure* μ *in* $H^{-1}(\mathcal{A})$ *supported by* Σ *such that*

$$\frac{1}{\boldsymbol{\omega}(\varepsilon_n)}\operatorname{curl} j(u_n) \underset{n\to+\infty}{\longrightarrow} \mu \quad \textit{strongly in } (\mathrm{Lip}_0(\mathcal{A}))'\,; \tag{1.4.10}$$

(ii) *for any* $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\mathcal{A};\mathbb{C})$ *such that* (1.4.10) *holds for some nonnegative Radon measure* μ *in* $H^{-1}(\mathcal{A})$ *supported by* Σ, *we have*

$$\liminf_{n\to+\infty} \overline{F}_{\varepsilon_n}(u_n) \geqslant \bar{I}(\mu) - \zeta_{\max}\,\mu(\mathcal{A})\,;$$

(iii) *for any nonnegative Radon measure* μ *in* $H^{-1}(\mathcal{A})$ *supported by* Σ, *there exists a sequence* $\{u_n\}_{n\in\mathbb{N}} \subseteq H^1(\mathcal{A};\mathbb{C})$ *such that* (1.4.10) *holds and*

$$\lim_{n\to+\infty} \overline{F}_{\varepsilon_n}(u_n) = \bar{I}(\mu) - \zeta_{\max}\,\mu(\mathcal{A})\,.$$

5. Note that in ($\mathbf{H_1}'$), the assumption that $\zeta_{\max}$ is achieved at positive values of ζ is not restrictive. Indeed, considering the complex conjugate of an admissible function replaces V by $-V$ in the energy and hence ζ by $-\zeta$.

The proof Theorem 1.4.3 rests on the fact that the energy due the presence of the hole decouples almost exactly. More precisely, under the assumption in (i), if we denote by $u_\varepsilon^\star$ a minimizer of $\mathcal{G}_{\boldsymbol{\Omega}_\varepsilon}$ in $H^1(\mathcal{A};\mathbb{S}^1)$, then

$$\mathcal{F}_\varepsilon(u) = \min \mathcal{G}_{\boldsymbol{\Omega}_\varepsilon} + \boldsymbol{\omega}^2(\varepsilon)\widetilde{F}_\varepsilon\big(\overline{u_\varepsilon^\star}\, u\big) + o(1)\,,$$

where

$$\widetilde{F}_\varepsilon(v) := \frac{1}{\boldsymbol{\omega}^2(\varepsilon)}\int_{\mathcal{A}} \left\{\frac{1}{2}|\nabla v|^2 + \frac{1}{4\varepsilon^2}(1-|v|^2)^2 + \nabla^\perp \Phi_{d_{\boldsymbol{\Omega}_\varepsilon}} \cdot j(v)\right\} \mathrm{d}x\,,$$

the function $\Phi_{d_{\boldsymbol{\Omega}_\varepsilon}}$ being given by (1.4.6) and $d_{\boldsymbol{\Omega}_\varepsilon}$ is an optimal integer for (1.4.5). The main conclusion in Theorem 1.4.3 comes from the Γ-convergence analysis of $\widetilde{F}_\varepsilon$, very much like in the previous subsection.

As in the simply connected case, this Γ-convergence result leads to the asymptotic description of the internal vorticity in $\mathcal{F}_\varepsilon$-global minimizers.

Corollary 1.4.4. *Assume that* $(\mathbf{H_1}')$, $(\mathbf{H_2}')$, *and* (1.4.8) *hold. Let* $\varepsilon_n \to 0^+$ *be an arbitrary sequence. For every integer* $n \in \mathbb{N}$, *let* $u_n \in H^1(\mathcal{A};\mathbb{C})$ *be a global minimizer of* $\mathcal{F}_{\varepsilon_n}$. *Then,*

$$\frac{1}{\boldsymbol{\omega}(\varepsilon_n)}\,\mathrm{curl}\, j(u_n) \underset{n\to+\infty}{\longrightarrow} \frac{\zeta_{\max}}{2\bar{I}_*}\,\bar{\mu}_* \quad \textit{strongly in } (\mathrm{Lip}_0(\mathcal{A}))'\,,$$

where $\bar{\mu}_*$ *is the unique minimizer of* $\bar{I}(\cdot)$ *over all probability measures supported by* Σ, *and* $\bar{I}_* := \bar{I}(\bar{\mu}_*)$. *In addition,*

$$\min \mathcal{F}_\varepsilon = -\frac{\boldsymbol{\Omega}_\varepsilon^2}{2}\int_{\mathcal{A}} \{|V|^2 - |\nabla\zeta|^2\}\,\mathrm{d}x - \frac{\zeta_{\max}^2}{4\bar{I}_*}\boldsymbol{\omega}^2(\varepsilon) + o(\boldsymbol{\omega}^2(\varepsilon))\,. \tag{1.4.11}$$

We conclude this section with an elementary example motivated by [3].

Example. Assume that $\mathcal{D} = B_R(0)$, $\mathcal{B} = B_\rho(0)$ for some $0<\rho<R$ and $V(x) = x^\perp$. Then the function ζ in (1.4.7) is given by

$$\zeta(x) = -\frac{|x|^2}{2} + \frac{R^2-\rho^2}{2\ln(R/\rho)}\ln|x| + \frac{\rho^2\ln R - R^2\ln\rho}{2\ln(R/\rho)}\,.$$

In particular, the set Σ is the concentric circle $\partial B_{r_*}(0)$ of radius

$$r_* = \sqrt{\frac{R^2-\rho^2}{2\ln(R/\rho)}} \in (\rho, R)\,.$$

Here again, the uniform measure $\bar{\mu}_* = (2\pi r_*)^{-1} d\mathscr{H}^1 \llcorner \Sigma$ turns out to be the Green equilibrium measure for Σ in $\mathcal{A}$, i.e., $\bar{I}(\bar{\mu}_*) = \bar{I}_*$ (see [178, Theorem II.5.12]).

1.5 Homogenization of multiple integrals for manifold valued maps

The homogenization theory aims to find an effective description of materials whose heterogeneities scale is much smaller than the size of the body. The simplest example is periodic homogenization for which the microstructure is assumed to be periodically distributed within the material. In the framework of the Calculus of Variations, periodic homogenization problems rest on the study of equilibrium states, or minimizers, of integral functionals of the form

$$\int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u\right)\,\mathrm{d}x\,, \quad u:\Omega\to\mathbb{R}^d\,, \tag{1.5.1}$$

under suitable boundary conditions, where $\Omega \subseteq \mathbb{R}^n$ is a bounded open set and $f:\mathbb{R}^n\times\mathbb{R}^{d\times n}\to[0,+\infty)$ is some oscillating integrand with respect to the first variable. To understand the asymptotic behavior as $\varepsilon \downarrow 0$ of (almost) minimizers of such energies, it is convenient to perform a Γ-convergence analysis, see e.g. [78], which is an adequate theory to study such variational problems. It is usual to assume that the integrand f satisfies uniform p-growth and p-coercivity conditions (with $1 \leqslant p < +\infty$) so that one should require to admissible fields to belong to the Sobolev space $W^{1,p}$. For energies with superlinear growth, i.e., $p > 1$, this problem has a quite long history, and we refer to [154] in the convex case. Then it has received the most general answer in the independent works of [51] and [167], showing that such materials asymptotically behave like homogeneous ones. These results have been subsequently generalized into a lot of different manners. Let us mention [53] where the authors add a surface energy term allowing for fractured media. In that case, Sobolev spaces are not adapted to take into account eventual discontinuities of the deformation field across the cracks. For energies growing linearly, the situation is somehow intermediate, and the pathological nature of $W^{1,1}$ leads to relaxation in the space of functions of Bounded Variation. The problem of finding *integral representations* of relaxed functionals (i.e., lower semicontinuous envelopes) in BV took many years and it is has been widely investigated, see e.g. [18, 23, 25, 46, 77, 101, 102, 104, 115, 122]. The corresponding homogenization problems in BV have been successively studied in [45, 85], and in [46] with an extra surface energy term.

In many applications admissible fields have to satisfy additional constraints. This is for example the case in some study of equilibria for liquid crystals, in ferromagnetism or for magnetostrictive materials where order parameters take their values in a given manifold (e.g. the sphere $\mathbb{S}^2$, the circle $\mathbb{S}^1$, or the real projective plane $\mathbb{RP}^2$). It then becomes necessary to understand the behaviour of integral functionals of the type (1.5.1) under this additional constraint. At $\varepsilon > 0$ fixed, the possible lack of lower semicontinuity of the energy may prevent the existence of minimizers (with eventual boundary conditions). It leads to compute its relaxation under the pointwise constraint to take values in the manifold. In the framework of Sobolev spaces, it has been first studied by B. DACOROGNA, I. FONSECA, J. MALÝ, & K. TRIVISA [73] for $p > 1$, and the relaxed energy is obtained by replacing the integrand by its *tangential quasiconvexification*, i.e., the analogue of the quasiconvex envelope in the non constrained case. The case of an integrand with linear growth has been addressed in [10] for sphere valued maps, and then in [165] for a more general manifold but with a strong isotropy assumption on the integrand.

In the articles [P09, P10], in collaboration with J.F. BABADJIAN, we tackle the general homogenization problem for integral functionals of the form (1.5.1) and defined for manifold valued Sobolev mappings, in the superlinear and linear growth case. We have shown that the constraint leads to a quantitatively different limit compare to unconstrained homogenization, even in the simpler quadratic case where the integrand f comes from a second order linear elliptic operator with periodic coefficients. As a byproduct of our analysis in the linear growth case, we have also obtained a relaxation result in BV for a general target manifold under standard assumptions on the integrand.

1.5.1 A brief review of Γ-convergence

For completeness and in order to appreciate our results, we recall in this subsection the basic notions of Γ-convergence. We refer to the monographs by G. DAL MASO [78] and A. BRAIDES & A. DEFRANCESCHI [54] for a detailed description of the subject and the applications to homogenization.

Consider a metric space X and functions $F_\varepsilon : X \to [0, +\infty]$, where $\varepsilon > 0$. The family $\{F_\varepsilon\}_{\varepsilon>0}$ is said to Γ-converge as $\varepsilon \to 0$ to $F_0 : X \to [0, +\infty]$ if for every sequence $\varepsilon_j \to 0$ and every $u \in X$ the following conditions are satisfied :

(a) $F_0(u) \leqslant \liminf_j F_{\varepsilon_j}(u_j)$ for every sequence $u_j \to u$;

(b) there exists a sequence $u_j \to u$ such that $F_0(u) = \lim_j F_{\varepsilon_j}(u)$.

In the particular case of a constant family $\{F_\varepsilon\}$, i.e., $F_\varepsilon = F$, the function F_0 is simply the (sequential) lower semicontinuous envelope of F or "relaxed function" usually denoted by $\overline{F}$:

$$\overline{F}(u) := \inf\left\{\liminf_j F(u_j) : u_j \to u\right\} .$$

In the general case, it follows from the very definition of Γ-convergence that F_0 is sequentially lower semicontinuous.

The most fundamental result of the theory shows that Γ-convergence is the appropriate notion of variational convergence when dealing with a family of minimization problems. Under a compactness assumption, it implies that sequences of minimizers converge to minimizers of the Γ-limit. More precisely, if the family $\{F_\varepsilon\}$ is equi-mildly coercive (i.e., there exists a compact set K such that $\inf_X F_\varepsilon = \inf_K F_\varepsilon$ for every $\varepsilon > 0$), then

$$\exists \min_X F_0 = \lim_{\varepsilon\to 0} \inf_X F_\varepsilon .$$

In addition, if (u_j) is a converging sequence such that $\lim_j F_{\varepsilon_j}(u_j) = \lim_\varepsilon \inf_X F_\varepsilon$, then its limit is a minimum point of F_0. To conclude, we mention that Γ-convergence is stable under continuous perturbations, a property which turns out to be very useful in applications. It means that $\Gamma - \lim_\varepsilon(F_\varepsilon + G) = F_0 + G$ whenever $G : X \to [0,\infty]$ is continuous.

1.5.2 Homogenization in Sobolev spaces

Let Ω be a bounded open subset of $\mathbb{R}^n$. We consider throughout this section a connected smooth submanifold $\mathcal{M}$ of $\mathbb{R}^d$ without boundary. The tangent bundle to $\mathcal{M}$ is denoted by $T\mathcal{M}$. The class of admissible maps we are interested in is defined for $p \in [1,\infty)$ as

$$W^{1,p}(\Omega;\mathcal{M}) := \left\{u \in W^{1,p}(\Omega;\mathbb{R}^d) : u(x) \in \mathcal{M} \text{ for a.e. } x \in \Omega\right\} .$$

For a smooth $\mathcal{M}$-valued map, it is well known that first order derivatives belong to $T\mathcal{M}$. For $u \in W^{1,p}(\Omega;\mathcal{M})$, this property still holds in the sense that $\nabla u(x) \in [T_{u(x)}\mathcal{M}]^n$ for a.e. $x \in \Omega$.

The function $f : \mathbb{R}^n \times \mathbb{R}^{d\times n} \to [0,+\infty)$ is assumed to be a Carathéodory integrand [6] satisfying :

($\mathbf{H_1}$) for every $\xi \in \mathbb{R}^{d\times n}$ the function $f(\cdot,\xi)$ is 1-periodic, i.e., if $\{e_1,\dots,e_n\}$ denotes the canonical basis of $\mathbb{R}^n$, one has $f(y+e_i,\xi) = f(y,\xi)$ for every $i = 1,\dots,n$ and $y \in \mathbb{R}^n$;

($\mathbf{H_2}$) there exist $0 < \alpha \leqslant \beta < +\infty$ and $1 \leqslant p < +\infty$ such that

$$\alpha|\xi|^p \leqslant f(y,\xi) \leqslant \beta(1+|\xi|^p) \quad \text{for a.e. } y \in \mathbb{R}^n \text{ and all } \xi \in \mathbb{R}^{d\times n} .$$

For $\varepsilon > 0$, we define the functional $\mathcal{F}_\varepsilon : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$ by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega f\left(\frac{x}{\varepsilon},\nabla u\right)\,\mathrm{d}x & \text{if } u \in W^{1,p}(\Omega;\mathcal{M}), \\ +\infty & \text{otherwise}. \end{cases} \tag{1.5.2}$$

For energies with superlinear growth, we have the following result.

Theorem 1.5.1. *Let $f : \mathbb{R}^n \times \mathbb{R}^{d\times n} \to [0,+\infty)$ be a Carathéodory function satisfying assumptions ($\mathbf{H_1}$) and ($\mathbf{H_2}$) with $1 < p < +\infty$. As $\varepsilon \to 0$, the family $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ Γ-converges for the strong $L^p(\Omega)$-topology to the*

6. f is a function such that $f(x,\cdot)$ is continuous for a.e. $x \in \Omega$, and $f(\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^{d\times n}$.

functional $\mathcal{F}_0 : L^p(\Omega;\mathbb{R}^d) \to [0,+\infty]$ *defined by*

$$\mathcal{F}_0(u) := \begin{cases} \displaystyle\int_\Omega Tf_{\rm hom}(u,\nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(\Omega;\mathcal{M})\,,\\ +\infty & \text{otherwise}\,,\end{cases}$$

where for every $s \in \mathcal{M}$ *and* $\xi \in [T_s\mathcal{M}]^n$*,*

$$Tf_{\rm hom}(s,\xi) := \lim_{t\to+\infty} \inf_{\varphi} \left\{ \fint_{(0,t)^n} f\big(y,\xi+\nabla\varphi(y)\big)\,\mathrm{d}y : \varphi \in W^{1,\infty}_0\big((0,t)^n;T_s\mathcal{M}\big)\right\} \tag{1.5.3}$$

is the "tangentially homogenized energy density".

If the integrand f has a linear growth in the ξ-variable, i.e., if f satisfies $(\mathbf{H_2})$ with $p=1$, we assume in addition that $\mathcal{M}$ is compact, and that

$(\mathbf{H_3})$ there exists $L>0$ such that

$$|f(y,\xi)-f(y,\xi')| \leqslant L|\xi-\xi'| \quad \text{for a.e. } y\in\mathbb{R}^n \text{ and all } \xi,\, \xi' \in \mathbb{R}^{d\times n}\,.$$

Then the following representation result on $W^{1,1}(\Omega;\mathcal{M})$ holds :

Theorem 1.5.2. *Assume that* $\mathcal{M}$ *is compact, and let* $f:\mathbb{R}^n\times\mathbb{R}^{d\times n}\to[0,+\infty)$ *be a Carathéodory function satisfying assumptions* $(\mathbf{H_1})$ *to* $(\mathbf{H_3})$ *with* $p=1$*. As* $\varepsilon\to 0$*, the family* $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ Γ*-converges for the strong* $L^1(\Omega)$*-topology at every* $u\in W^{1,1}(\Omega;\mathcal{M})$ *to* $\mathcal{F}_0 : W^{1,1}(\Omega;\mathcal{M})\to[0,+\infty)$*, where*

$$\mathcal{F}_0(u) := \int_\Omega Tf_{\rm hom}(u,\nabla u)\,\mathrm{d}x\,,$$

and $Tf_{\rm hom}$ *is given by* (1.5.3)*.*

Remark. The use of assumption $(\mathbf{H_3})$ is not too restrictive. Indeed, the Γ-limit remains unchanged upon first relaxing the functional $\mathcal{F}_\varepsilon$ (at fixed $\varepsilon>0$) in $W^{1,1}(\Omega;\mathbb{R}^d)$. It would lead to replace the integrand f by its tangential quasiconvexification which, by virtue of the growth condition $(\mathbf{H_1})$, does satisfy such a Lipschitz continuity assumption, see [73]. However, Theorem 1.5.2 is not really satisfactory in its present form. In the case of an integrand with linear growth, the domain of the Γ-limit is obviously larger than the Sobolev space $W^{1,1}(\Omega;\mathcal{M})$ and the analysis has to be performed in the space of functions of bounded variation. In fact Theorem 1.5.2 is a first step in this direction and the complete result in BV-spaces is the object of the following subsection.

We underline that a main novelty of our result compare to standard homogenization is the emergence of a dependence on the u-variable in the expression of the homogenized energy density $Tf_{\rm hom}$. To show that it is not an artifact of the abstract formula defining $Tf_{\rm hom}$, we now present two simple examples based on a "rank-one laminate" where it can be computed explicitly. In the first example, the key point is to assume that the manifold is one dimensional so that (1.5.3) reduces to scalar problem which can be solved explicitly in case of separation of variables, see [78, Example 25.6]. In the second example, we assume that $n=1$ so that (1.5.3) can computed explicitly following e.g. [54, Chapter 13].

Example. Let $a_1,\dots,a_d\in L^\infty(\mathbb{R})$ be 1-periodic functions bounded from below by a positive constant. Assume that the manifold $\mathcal{M}$ is one dimensional, and that

$$f(x,\xi) = \sum_{i=1}^d \sum_{j=1}^n a_i(x_1)|\xi_{ij}|^2\,,$$

where we write $\xi = (\xi_{ij}) \in \mathbb{R}^{d\times n}$. Then, for every $s = (s_i) \in \mathcal{M}$ and $\xi \in [T_s\mathcal{M}]^n$,

$$T f_{\text{hom}}(s,\xi) = \sum_{i=1}^{d}\sum_{j=1}^{n} \boldsymbol{\alpha}_j(s)|\xi_{ij}|^2 ,$$

where

$$\boldsymbol{\alpha}_1(s) := \left(\int_0^1 \frac{1}{\sum_i a_i(t)\tau_i^2(s)}\,\mathrm{d}t\right)^{-1} \quad \text{and} \quad \boldsymbol{\alpha}_j(s) := \int_0^1 \sum_i a_i(t)\tau_i^2(s)\,\mathrm{d}t \quad \text{for } j = 2,\dots,n ,$$

and $\tau(s) = (\tau_i(s))$ denotes a unit tangent vector to $\mathcal{M}$ at the point s.

Example. Assume that $n = 1$. Let $A \in L^\infty(\mathbb{R};\mathbb{R}^{d\times d})$ be a 1-periodic field of symmetric matrices such that $\langle A(x)\xi,\xi\rangle \geqslant \alpha|\xi|^2$ for every $\xi \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}$, for some constant $\alpha > 0$. Consider $f : \mathbb{R}\times\mathbb{R}^d \to [0,\infty)$ given by

$$f(x,\xi) = \langle A(x)\xi,\xi\rangle .$$

For $s \in \mathcal{M}$, let $P(s) \in \mathbb{R}^{d\times d}$ be the orthogonal projection on $T_s\mathcal{M}$. Noticing that the matrix ${}^{\mathrm{t}}P(s)A(x)P(s)$ induces a self adjoint isomorphism $B(x,s) \in \mathrm{GL}(T_s\mathcal{M})$, we denote by $B^{-1}(x,s)$ its inverse. By the uniform ellpticity assumption, we also have $\int_0^1 B^{-1}(t,s)\,\mathrm{d}t \in \mathrm{GL}(T_s\mathcal{M})$.

Then, for every $s \in \mathcal{M}$ and $\xi \in T_s\mathcal{M}$,

$$T f_{\text{hom}}(s,\xi) = \langle A_{\text{hom}}(s)\xi,\xi\rangle ,$$

where $A_{\text{hom}}(s) \in \mathrm{GL}(T_s\mathcal{M})$ is given by

$$A_{\text{hom}}(s) := \left(\int_0^1 B^{-1}(t,s)\,\mathrm{d}t\right)^{-1} .$$

Remark. The two examples above can be interpreted as follows. In both cases, we take a measurement for length of a differential $du(x)$ with respect to a Riemannian metric on $\mathbb{R}^d$ which depends on the point x. In both cases, we implicitly endow $\mathcal{M}$ with the induced "x-dependent" Riemannian metric. Therefore, when letting $\varepsilon \to 0$, homogenization not only takes place on Ω, but also on $T\mathcal{M}$.

The proof of Theorems 1.5.1 and 1.5.2 consists in proving sharp upper and lower bounds for the upper and lower Γ-limits, respectively. The upper bound is obtained by the "localization method". In the spirit of [73], we introduce a modified "Γ-$\limsup$" to handle the manifold constraint. When localized to open subsets of Ω, it is the restriction to open sets of some Radon measure. We then estimate from above the Radon-Nikodým derivative of this measure with respect to $\mathscr{L}^n$ by means of the blow-up method of I. FONSECA & S. MÜLLER [101, 102]. For the lower bound, the estimate is more classical and the constraint do not induces too much difficulties. The analysis relies again on the blow-up method, and makes use of the classical "Decomposition Lemma" of [103] in the superlinear case. The linear case is treated in a way similar to [101].

1.5.3 Homogenization in BV-spaces

In case the integrand in (1.5.2) has linear growth with respect to ∇u, the domain of Γ-limit should include functions of bounded variations. In view of the results in [119], the domain is precisely given by

$$BV(\Omega;\mathcal{M}) := \left\{u \in BV(\Omega;\mathbb{R}^d) : u(x) \in \mathcal{M} \text{ for a.e. } x \in \Omega\right\} .$$

To describe the structure of maps in this space, we first recall that for $u \in BV(\Omega;\mathbb{R}^d)$, the $\mathbb{R}^{d\times n}$-valued Radon measure Du can be decomposed into an absolutely continuous part ∇u and a singular part $D^s u$ with respect to $\mathscr{L}^n$, i.e.,

$$Du = \nabla u\, \mathscr{L}^n \llcorner \Omega + D^s u\,. \tag{1.5.4}$$

In turn the singular measure $D^s u$ can be decomposed into two mutually singular measures

$$D^s u = (u^+ - u^-) \otimes \nu_u\, \mathscr{H}^{n-1} \llcorner J_u + D^c u\,, \tag{1.5.5}$$

where J_u is the *approximate jump set* of u. It is a Borel subset of the *approximate discontinuity set* S_u. In addition, J_u is countably $\mathscr{H}^{n-1}$-rectifiable and it can be oriented by a (normal) direction of jump $\nu_u : J_u \to \mathbb{S}^{n-1}$. Then, $u^\pm$ are the one-sided approximate limits of u on J_u according to ν_u. Finally, $D^c u$ is the so-called "Cantor part" of Du defined by $D^c u := D^s u \llcorner (\Omega \setminus S_u)$.

For a map $u \in BV(\Omega;\mathcal{M})$, the following properties hold :

(i) the approximate limit $\tilde{u}(x)$ belongs to $\mathcal{M}$ at every point $x \in \Omega \setminus S_u$;

(ii) $u^\pm(x) \in \mathcal{M}$ for every $x \in J_u$;

(iii) $\nabla u(x) \in [T_{u(x)}\mathcal{M}]^n$ for $\mathscr{L}^n$-a.e. $x \in \Omega$;

(iv) $\dfrac{dD^c u}{d|D^c u|}(x) \in [T_{\tilde{u}(x)}\mathcal{M}]^n$ for $|D^c u|$-a.e. $x \in \Omega$.

We are now ready to state our result extending Theorem 1.5.2 to BV-maps. It only requires the following additional (standard) assumption,

$(\mathbf{H_4})$ there exist $C > 0$ and $0 < q < 1$ such that

$$|f(y,\xi) - f^\infty(y,\xi)| \leqslant C(1 + |\xi|^{1-q}) \qquad \text{for a.e. } y \in \mathbb{R}^n \text{ and all } \xi \in \mathbb{R}^{d\times n}\,,$$

where $f^\infty : \mathbb{R}^n \times \mathbb{R}^{d\times n} \to [0,+\infty)$ is the recession function of f defined by

$$f^\infty(y,\xi) := \limsup_{t\to+\infty} \frac{f(y,t\xi)}{t}\,.$$

Theorem 1.5.3. *Assume that $\mathcal{M}$ is compact, and let $f : \mathbb{R}^n \times \mathbb{R}^{d\times n} \to [0,+\infty)$ be a Carathéodory function satisfying assumptions $(\mathbf{H_1})$ to $(\mathbf{H_4})$. As $\varepsilon \to 0$, the family $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ Γ-converges for the strong $L^1(\Omega)$-topology to the functional $\overline{\mathcal{F}}_0 : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined by*

$$\overline{\mathcal{F}}_0(u) := \begin{cases} \displaystyle\int_\Omega Tf_{\mathrm{hom}}(u,\nabla u)\,\mathrm{d}x + \int_{\Omega\cap J_u} K_{\mathrm{hom}}(u^+,u^-,\nu_u)\,\mathrm{d}\mathscr{H}^{n-1} + \\ \qquad\qquad \displaystyle + \int_\Omega Tf^\infty_{\mathrm{hom}}\left(\tilde{u}, \frac{dD^c u}{d|D^c u|}\right) \mathrm{d}|D^c u| & \text{if } u \in BV(\Omega;\mathcal{M})\,, \\ +\infty & \text{otherwise}\,, \end{cases}$$

where Tf_{hom} is given in (1.5.3), Tf^∞_{hom} is the recession function of Tf_{hom} defined for every $s \in \mathcal{M}$ and every $\xi \in [T_s\mathcal{M}]^n$ by

$$Tf^\infty_{\mathrm{hom}}(s,\xi) := \limsup_{t\to+\infty} \frac{Tf_{\mathrm{hom}}(s,t\xi)}{t}\,,$$

and for all $(a,b,\nu) \in \mathcal{M} \times \mathcal{M} \times \mathbb{S}^{n-1}$,

$$K_{\mathrm{hom}}(a,b,\nu) := \lim_{t\to+\infty} \inf_\varphi \left\{ \frac{1}{t^{n-1}} \int_{tQ_\nu} f^\infty(y,\nabla\varphi(y))\,\mathrm{d}y : \varphi \in W^{1,1}(tQ_\nu;\mathcal{M})\,, \right.$$
$$\left. \varphi = a \text{ on } \partial(tQ_\nu) \cap \{x\cdot\nu > 0\} \text{ and } \varphi = b \text{ on } \partial(tQ_\nu) \cap \{x\cdot\nu < 0\} \right\}, \tag{1.5.6}$$

Q_ν being any open unit cube in $\mathbb{R}^n$ centered at the origin with two of its faces orthogonal to ν.

As in the previous theorems, we prove this integral representation by matching suitable upper and lower bounds. The upper bound is again derived through the localization method. Here, we make use of two essential ingredients : a suitable projection on $\mathcal{M}$ taken from [130] to construct admissible maps, and the density result of [35]. In an intermediate step we obtain an abstract integral representation for the surface energy as in [15, 53]. Then the upper bound is obtained by blow-up treating the diffuse and the concentrated part separately. The lower bound also rests on the blow-up method of [102], proving lower bounds for the absolutely continuous part, the Cantor part, and the jump part, also separately. As usual, the estimate for the Cantor part is essentially based on Alberti's rank-one Theorem [6].

1.5.4 A relaxation result in BV

We finally present for completeness a relaxation result in BV for Sobolev maps taking values in the manifold $\mathcal{M}$. It extends the result of [10] which is restricted to $\mathcal{M} = \mathbb{S}^{d-1}$, and the result of [165] dealing with a general manifold but assuming a strong isotropy condition on the integrand. As a matter of fact, we have simply realized that the approach of [10] can be reproduced using the appropriate geometric tools. In particular, a key point is again the projection on $\mathcal{M}$ of [130] when constructing competitors at various stages of the analysis. Let us now give the precise setting of the result.

Let $h : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to [0, +\infty)$ be a continous function satisfying :

$(\mathbf{H_1'})$ h is tangentially quasiconvex, i.e., for all $x \in \Omega$, all $s \in \mathcal{M}$ and all $\xi \in [T_s\mathcal{M}]^n$,

$$h(x, s, \xi) \leqslant \int_Q h\big(x, s, \xi + \nabla\varphi(y)\big)\, dy \quad \text{for every } \varphi \in W_0^{1,\infty}(Q; T_s\mathcal{M}) \;;$$

$(\mathbf{H_2'})$ there exist $\alpha > 0$ and $\beta > 0$ such that

$$\alpha|\xi| \leqslant h(x, s, \xi) \leqslant \beta(1 + |\xi|) \quad \text{for every } (x, s, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times n}\,;$$

$(\mathbf{H_3'})$ for every compact subset $K \subseteq \Omega$, there exists a continuous function $\omega : [0, +\infty) \to [0, +\infty)$ satisfying $\omega(0) = 0$ and

$$|h(x, s, \xi) - h(x', s', \xi)| \leqslant \omega(|x - x'| + |s - s'|)(1 + |\xi|)$$

for every x, $x' \in \Omega$, every s, $s' \in \mathbb{R}^d$, and every $\xi \in \mathbb{R}^{d\times n}$;

$(\mathbf{H_4'})$ there exist $C > 0$ and $q \in (0, 1)$ such that

$$|h(x, s, \xi) - h^\infty(x, s, \xi)| \leqslant C(1 + |\xi|^{1-q}), \quad \text{for every } (x, s, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times n}\,,$$

where $h^\infty : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to [0, +\infty)$ is the recession function of h defined by

$$h^\infty(x, s, \xi) := \limsup_{t\to+\infty} \frac{h(x, s, t\xi)}{t}\,.$$

We consider the functional $F : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty]$ given by

$$F(u) := \begin{cases} \displaystyle\int_\Omega h(x, u, \nabla u)\, dx & \text{if } u \in W^{1,1}(\Omega; \mathcal{M}), \\ +\infty & \text{otherwise,} \end{cases}$$

and its relaxation for the strong $L^1(\Omega)$-topology $\overline{F} : L^1(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined by

$$\overline{F}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} F(u_n) : u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d) \right\} .$$

The following integral representation of result holds :

Theorem 1.5.4. *Assume that $\mathcal{M}$ is compact, and let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times n} \to [0,+\infty)$ be a continuous function satisfying $(\mathbf{H_1'})$ to $(\mathbf{H_4'})$. Then for every $u \in L^1(\Omega;\mathbb{R}^d)$,*

$$\overline{F}(u) = \begin{cases} \displaystyle\int_\Omega h(x,u,\nabla u)\,\mathrm{d}x + \int_{\Omega\cap S_u} H(x,u^+,u^-,\nu_u)\,\mathrm{d}\mathscr{H}^{n-1} + \\ \qquad\qquad + \displaystyle\int_\Omega h^\infty\left(x,\tilde{u},\frac{dD^c u}{d|D^c u|}\right)\,\mathrm{d}|D^c u| & \textit{if } u \in BV(\Omega;\mathcal{M})\,, \\ +\infty & \textit{otherwise}\,, \end{cases}$$

where for every $(x,a,b,\nu) \in \Omega \times \mathcal{M} \times \mathcal{M} \times \mathbb{S}^{n-1}$,

$$H(x,a,b,\nu) := \inf_{\varphi} \left\{ \int_{Q_\nu} h^\infty(x,\varphi(y),\nabla\varphi(y))\,dy : \varphi \in W^{1,1}(Q_\nu;\mathcal{M}),\ \varphi = a \textit{ on } \{x\cdot\nu = 1/2\}, \right.$$
$$\left. \varphi = b \textit{ on } \{x\cdot\nu = -1/2\} \textit{ and } \varphi \textit{ is } 1\textit{-periodic in the } \nu_2,\dots,\nu_n \textit{ directions} \right\},$$

$\{\nu,\nu_2,\dots,\nu_n\}$ forms any orthonormal basis of $\mathbb{R}^n$, and Q_ν stands for the open unit cube in $\mathbb{R}^n$ centered at the origin associated to this basis.

Remark. In the simple case where $h(x,u,\nabla u) = |\nabla u|$, the surface energy reduces to $H(x,u^+,u^-,\nu_u) = d_{\mathcal{M}}(u^+,u^-)$ where $d_{\mathcal{M}}$ is the geodesic distance on $\mathcal{M}$.

Chapitre 2

Isoperimetry, phase transitions, and free discontinuity problems

2.1 Quantitative isoperimetry for fractional perimeters

Isoperimetric inequalities play a crucial role in many areas of mathematics such as geometry, linear and nonlinear PDE's, or probability theory. In the Euclidean setting, it states that among all sets of prescribed measure, balls have the least perimeter. More precisely, for any Borel set $E \subseteq \mathbb{R}^n$ of finite Lebesgue measure,

$$P(E) \geqslant \frac{P(B)}{|B|^{\frac{n-1}{n}}} \, |E|^{\frac{n-1}{n}} \,, \tag{2.1.1}$$

where B denotes the unit ball of $\mathbb{R}^n$ centered at the origin. Here $|E|$ is the Lebesgue measure of E, and $P(E)$ denotes the distributional perimeter of E, i.e.,

$$P(E) := \sup\left\{ \int_E \operatorname{div} X \, \mathrm{d}x : X \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)\,,\ |X| \leqslant 1 \right\} , \tag{2.1.2}$$

which coincides with the $(n-1)$-dimensional Hausdorff measure of ∂E when E has a (piecewise) smooth boundary (see e.g. [20]). Note that the right hand side of (2.1.1) is equal to $P(B_{r_E})$, the perimeter of a ball of radius $r_E := (|E|/|B|)^{1/n}$ – so that $|E| = |B_{r_E}|$. It is a well known fact that inequality (2.1.1) is strict unless E is a ball up to a set of vanishing Lebesgue measure.

The *quantitative isoperimetric inequality* aims to provide a quantitative stability estimate in L^1 for the validity of (2.1.1). In other words, it is a second order lower expansion of the perimeter with respect to the L^1-distance to the collections of all balls with prescribed volume. In this context, the relevant quantity is the so-called *Fraenkel asymmetry* defined for a set E of finite and positive measure by

$$A(E) := \inf\left\{ \frac{|E \triangle (x + B_{r_E})|}{|E|} : x \in \mathbb{R}^n \right\} \in [0,2)\,,$$

where $\triangle$ is the symmetric difference between sets. Then the quantitative isoperimetric inequality reads

$$P(E) \geqslant \frac{P(B)}{|B|^{\frac{n-1}{n}}} \, |E|^{\frac{n-1}{n}} \left(1 + \frac{A(E)^2}{C(n)}\right) , \tag{2.1.3}$$

for a constant $C(n)$ which only depends on the dimension. We shall not attempt here to sketch the history of this inequality, but simply refer to the article by N. Fusco, F. Maggi, & A. Pratelli [112] where this inequality has been first proved in its sharp form, and to A. FIGALLI, F. MAGGI, & A. PRATELLI [96] where (2.1.3) is extended to anisotropic perimeter functionals by mass transportation (see also [153] for a survey).

In the article [P18], in collaboration with A. FIGALLI, N. FUSCO, F. MAGGI, & M. MORINI, we have investigated isoperimetric inequalities for fractional perimeters functionals arising from Sobolev semi-norm of fractional order. For $s \in (0,1)$ and a Borel set $E \subseteq \mathbb{R}^n$, $n \geqslant 2$, the fractional s-perimeter of E is defined by

$$P_s(E) := \iint_{E\times(\mathbb{R}^n\setminus E)} \frac{1}{|x-y|^{n+s}}\,\mathrm{d}x\mathrm{d}y\,.$$

If $P_s(E) < \infty$, we have

$$P_s(E) = \frac{1}{2}[\chi_E]^p_{W^{\sigma,p}(\mathbb{R}^n)}\,,$$

for $p \geqslant 1$ and $\sigma p = s$, where $[\cdot]_{W^{\sigma,p}(\mathbb{R}^n)}$ denotes the Gagliardo $W^{\sigma,p}$-seminorm and χ_E the characteristic function of E. The functional $P_s(E)$ can be thought as a $(n-s)$-dimensional perimeter in the sense that $P_s(\lambda E) = \lambda^{n-s}P_s(E)$ for any $\lambda > 0$ (compare to the $(n-1)$-homogeneity of the standard perimeter), and $P_s(E)$ can be finite even if the Hausdorff dimension of ∂E is strictly greater than $n-1$ (see e.g. [177]). It is also immediately checked that $P_s(E) < \infty$ for any set $E \subseteq \mathbb{R}^n$ of finite perimeter and finite measure since $BV(\mathbb{R}^n) \hookrightarrow W^{s,1}(\mathbb{R}^n)$.

The fractional s-perimeter has been first considered by L. CAFFARELLI, J.M. ROQUEJOFFRE, & O. SAVIN [60] who have initiated the study of Plateau type problems in this context (see Section 2.2.1). Such perimeter functional arises naturally in some phase transitions models with fractional diffusion as we shall see in Section 2.2. Besides this fact, a further motivation for studying s-perimeters appears when we look at the asymptotic $s \uparrow 1$. It turns out that s-perimeters give an approximation of the standard perimeter. More precisely, it follows from [83] and [19] that for any (bounded) set E of finite perimeter,

$$\lim_{s\uparrow 1}(1-s)P_s(E) = \omega_{n-1}P(E)\,, \tag{2.1.4}$$

both in the pointwise and Γ-convergence sense. Here ω_{n-1} denotes the volume of an $(n-1)$-dimensional ball of radius 1. Note that (2.1.4) is reminiscent of the results of J. BOURGAIN, H. BREZIS, & P. MIRONESCU [49] on the behavior of the $W^{\sigma,p}$-seminorm as $\sigma \uparrow 1$. Concerning the behavior of $P_s(E)$ as $s \downarrow 0$, we finally mention that

$$\lim_{s\downarrow 0} sP_s(E) = n\omega_n|E|\,, \tag{2.1.5}$$

for any set E of finite measure and finite s_0-perimeter for some $s_0 \in (0,1)$, as shown in [156].

An isoperimetric inequality for s-perimeters has been recently obtained by R.L. FRANK & R. SEIRINGER [108] as a consequence of more general functional inequalities. It states that for any Borel set $E \subseteq \mathbb{R}^n$ of finite measure,

$$P_s(E) \geqslant \frac{P_s(B)}{|B|^{\frac{n-s}{n}}}\,|E|^{\frac{n-s}{n}}\,, \tag{2.1.6}$$

with equality holding if and only if E is a ball (up to a null set). Actually, inequality (2.1.6) can be deduced from a symmetrization result due to F.J. ALMGREN & E.H. LIEB [13], and the cases of equality have been determined in [108]. Note that, in view of (2.1.4) and (2.1.5), one recovers the classical isoperimetric inequality letting $s \uparrow 1$, and (2.1.6) degenerates as $s \downarrow 0$.

We have obtained in [P18] a sharp quantitative version of (2.1.6), uniform with respect to s bounded away from 0. It allowed us to address the minimization of a free energy consisting of a fractional s-perimeter plus a nonlocal repulsive interaction term. Such a free energy comes from a generalization of the Gamow model for the nucleus. In the following subsections, we are going to present first the quantitative fractional isoperimetric inequality, and then its application to the aforementioned minimization problem. We shall conclude with first and second variations formulae for general nonlocal perimeters. Those formula come into play in our analysis but are also of independent interest.

2.1.1 Stability of the fractional isoperimetric inequality

We start with the sharp quantitative version of (2.1.6). Here, sharpness means that the exponent on the Fraenkel asymmetry can not be lowered. Indeed, for the classical perimeter, the use of ellipsoids asymptotically close to the unit ball shows that the decay rate is sharp.

Theorem 2.1.1. *For every $n \geqslant 2$ and $s_0 \in (0,1)$, there exists a positive constant $C(n,s_0)$ such that*

$$P_s(E) \geqslant \frac{P_s(B)}{|B|^{\frac{n-s}{n}}}\,|E|^{\frac{n-s}{n}}\left(1+\frac{A(E)^2}{C(n,s_0)}\right), \tag{2.1.7}$$

whenever $0<|E|<\infty$ and $s \in [s_0,1)$.

Remark. The constant $C(n,s_0)$ appearing in (2.1.7) is not explicit. We conjecture that $C(n,s_0) \simeq 1/s_0$ as $s_0 \downarrow 0$, see (2.1.8) below. Letting $s \uparrow 1$ we recover the quantitative isoperimetric inequality. The constant $C(n)$ appearing in (2.1.3) is known to grow polynomially in the dimension [96].

Remark. In a previous article [P12], in collatoration with N. Fusco & M. Morini, we have obtained a weaker version of (2.1.7) with exponent $4/s$ instead of 2 on the asymmetry. The proof is based on symmetrization arguments in the spirit of [112] which can nevertheless be useful in other contexts.

Theorem 1.1 is obtained by means of a Taylor expansion of the s-perimeter near balls together with a uniform version of the regularity theory developed by L. Caffarelli, J.M. Roquejoffre, & O. Savin [60] and M.C. Caputo & N. Guillen [62] for sets minimizing or almost minimizing the s-perimeter. These two tools are combined through a suitable version of Ekeland's variational principle. This approach has been introduced in the case $s=1$ by M. Cicalese & G.P. Leonardi [69] to provide an alternative proof the quantitative isoperimetric inequality. In our case, we have implemented this method through a penalization argument closer to the one adopted in [1]. Due to the nonlocality of the s-perimeter, the implementation itself is far from being straightforward, and it requires to develop some specific arguments of independent interest.

We now describe in more details the two main steps leading to Theorem 2.1.1.

Stability for nearly spherical sets – The Fuglede estimate. The first result needed to establish (2.1.1) is a stability estimate for *nearly spherical sets*. For the standard perimeter, such estimate is due B. Fuglede [111]. According to [111], we say that a bounded open set E is nearly spherical if $|E|=|B|$, $\int_E x\,\mathrm{d}x = 0$, and

$$\partial E = \left\{(1+u_E(x))x : x \in \partial B\right\}, \qquad \text{where } u_E \in C^1(\partial B),$$

for some function u_E with $\|u_E\|_{C^1(\partial B)}$ small. Our estimate provides a control of the fractional Sobbolev seminorm

$$[u_E]^2_{\frac{1+s}{2}} := \iint_{\partial B\times\partial B} \frac{|u_E(x)-u_E(y)|^2}{|x-y|^{n+s}}\,\mathrm{d}x\mathrm{d}y$$

in terms of the difference $P_s(E)-P_s(B)$.

Theorem 2.1.2. *There exist constants $\boldsymbol{\varepsilon}_\star \in (0,1/2)$ and $\mathbf{c}_\star > 0$, depending only on n, with the following property : if E is a nearly spherical set with $\|u_E\|_{C^1(\partial B)} < \boldsymbol{\varepsilon}_\star$, then*

$$P_s(E)-P_s(B) \geqslant \mathbf{c}_\star\left([u_E]^2_{\frac{1+s}{2}} + sP_s(B)\|u_E\|^2_{L^2(\partial B)}\right) \qquad \forall s \in (0,1). \tag{2.1.8}$$

Remark. If we multiply the inequality above by $(1-s)$, and then take the limit $s \uparrow 1$, we recover the original Fuglede's inequality $P(E) - P(B) \geqslant c(n)\|u_E\|^2_{H^1(\partial B)}$.

To obtain Theorem 2.1.2, we have performed an asymptotic expansion of $P_s(E)$ with respect to $\|u_E\|_{C^1(\partial B)} \to 0$, along nearly spherical sets. The zero order term is of course $P_s(B)$, and the first order term vanishes since the ball is minimizing. In turn the second order term is given by

$$\iint_{\partial B \times \partial B} \frac{|u_E(x) - u_E(y)|^2}{|x-y|^{n+s}}\,\mathrm{d}x\mathrm{d}y - s(n-s)\frac{P_s(B)}{P(B)}\int_{\partial B} |u_E|^2\,\mathrm{d}x\,, \tag{2.1.9}$$

which reveals that the nonlocal "Jacobi operator" $\mathcal{L}_s$ of the sphere is

$$\mathcal{L}_s u = \mathcal{I}_s u - s(n-s)\frac{P_s(B)}{P(B)}\,u\,,$$

where $\mathcal{I}_s$ is (up to a multiplicative constant) the *hypersingular spherical Riesz operator* of order $(1+s)$[1], i.e.,

$$\mathcal{I}_s u(x) := 2\,\mathrm{p.v.}\left(\int_{\partial B} \frac{u(x)-u(y)}{|x-y|^{n+s}}\,\mathrm{d}y\right). \tag{2.1.10}$$

By means of the classical *Funk-Hecke formula* (see e.g. [166]), one can show that $\mathcal{I}_s$ is diagonalized on the basis of spherical harmonics. Moreover, the eigenvalues $\{\lambda_k^s\}$ can be explicitly computed using integral identities for ultra spherical polynomials. In particular, the sequence of eigenvalues is strictly increasing and λ_2^s is quantitatively larger that λ_1^s. Evaluating $\mathcal{I}_s$ at coordinate functions, we have discovered that

$$\lambda_1^s = s(n-s)\frac{P_s(B)}{P(B)}\,.$$

It is now clear that expanding (2.1.9) in sherical harmonics led us to the result.

The penalization method. We shall now briefly explain how to prove (2.1.7), at least for a constant $C(n,s)$ in the right hand side which may depend on $s \in (0,1)$. Since the asymmetry is always smaller than 2, it is enough to show the existence of $\delta_s > 0$ such that for $M > 0$ sufficiently large,

$$A(E)^2 \leqslant M D_s(E) \quad \text{whenever} \quad D_s(E) := \frac{P_s(E) - P_s(B)}{P_s(B)} \leqslant \delta_s\,. \tag{2.1.11}$$

To prove (2.1.11), one argues by contradiction assuming the existence of a sequence of Borel sets $\{E_k\}$ satisfying $|E_k| = |B|$, $D_s(E_k) \to 0$, and $A(E_k)^2 > M D_s(E_k)$ for each $k \in \mathbb{N}$. By a preliminary continuity lemma (see [P12, Lemma 3.1] for its original version), we deduce that $A(E_k) \to 0$ as $k \to 0$.

At this stage it is not clear how to proceed without any further properties on the sequence $\{E_k\}$. The purpose of the *penalization method* is to select from $\{E_k\}$ a "better" sequence having at least the same properties. The selection we made is based on the minimization of the free energy

$$\mathcal{E}_{\Lambda,k}(F) := P_s(F) + \Lambda\big||F| - |B|\big| + |\boldsymbol{\alpha}(F) - \alpha_k|\,,$$

where $\Lambda > 0$ is a large constant, $\boldsymbol{\alpha}(F) := \inf_x |F \triangle (x+B)|$, and $\alpha_k := \boldsymbol{\alpha}(E_k) \to 0$ as $k \to \infty$. By a suitable truncation argument, we have proved that for Λ large enough and α_k small enough (i.e., k sufficiently large), the energy $\mathcal{E}_{\Lambda,k}$ admits a minimizer F_k contained in a ball of radius uniformly bounded in k. Comparing the energy of F_k and E_k shows that (up to a translation) $|F_k \triangle B| \to 0$ as $k \to \infty$. On the

1. The symbol p.v. means that the integral is taken in the Cauchy principal value sense

other hand, the nature of $\mathcal{E}_{\Lambda,k}$ implies that each F_k enjoys an almost minimality property with respect to the fractional perimeter, i.e.,

$$P_s(F_k) \leqslant P_s(G) + (\Lambda+1)|F_k \triangle G| \qquad \forall G \subseteq \mathbb{R}^n\,.$$

This is then enough to apply the regularity theory of [62], and deduce that

$$\partial F_k = \big\{(1+v_k(x))x : x \in \partial B\big\} \quad \text{with } \|v_k\|_{C^1(\partial B)} \to 0\,.$$

Dilating and translating the F_k's, one obtains a further sequence of nearly spherical sets which still satisfies the contradiction hypothesis. Applying Theorem 2.1.2 to this new sequence leads immediatetly to a contradiction for M sufficiently large.

2.1.2 Application to a nonlocal liquid drop model

We now present an application of our previous results and methods to a nonlocal isoperimetric problem in presence of a repulsive interaction term. The starting point is provided by the Gamow model for the nucleus, which consists in the volume constraint minimization of the energy $P(E) + V_\alpha(E)$, where, given $\alpha \in (0,n)$, $V_\alpha(E)$ is the Riesz potential of E

$$V_\alpha(E) := \iint_{E\times E} \frac{1}{|x-y|^{n-\alpha}}\,\mathrm{d}x\mathrm{d}y\,.$$

By minimizing $P(E)+V_\alpha(E)$ with $|E| = \mathbf{m}$ fixed, we observe a competition between the surface energy, which tries to round-up competitors into ball, and the Riesz potential which, on the contrary, prefers to smear them around. This last effect is in fact due to the Riesz rearrangement inequality implying that balls are actually volume constrained *maximizers* of V_α.

Recently, H. Knüpfer & C.B. Muratov [142, 143] have shown the existence of a mass $\mathbf{m}_\star = \mathbf{m}_\star(n,\alpha) > 0$ such that :

- **(a)** if $n = 2$ and $\alpha \in (0,2)$, then balls of volume $\mathbf{m} \leqslant \mathbf{m}_\star$ are the unique minimizers of $P + V_\alpha$ under the volume constraint $|E| = \mathbf{m}$;
- **(b)** if $n = 2$ and α is sufficiently close to 2, then there are no minimizers for $\mathbf{m} > \mathbf{m}_\star$;
- **(c)** if $3 \leqslant n \leqslant 7$ and $\alpha \in (1,n)$, then **(a)** holds.

In [43], M. Bonacini & R. Cristoferi have extend **(b)** and **(c)** to any dimension $n \geqslant 3$, and have also shown that balls of volume $\mathbf{m}$ are volume constrained L^1-local minimizers if $\mathbf{m} < \mathbf{m}_0(n,\alpha)$, while they are unstable if $\mathbf{m} > \mathbf{m}_0(n,\alpha)$. The critical mass $\mathbf{m}_0(n,\alpha)$ is characterized in terms of a spectral minimization problem that is explicitly solved for $n = 3$. In particular, in the physically relevant case $\alpha = 2$ one obtains $\mathbf{m}_0(3,2) = 5$, a result that was already known in the physics literature since the 30's [42, 95, 110]. Let us also mention that, in addition to **(b)**, further nonexistence results are contained in [143, 152].

We stress that, apart from the special case $n = 2$, all the results above are limited to the case $\alpha \in (1,n)$. We have extended **(a)** and **(c)** in two directions : first, by covering the full range $\alpha \in (0,n)$ for all $n \geqslant 2$, and second, by including the possibility for the surface energy to be a nonlocal s-perimeter. The global minimality threshold $\mathbf{m}_\star(n,\alpha,s)$ is shown to be uniformly positive with respect to s and α provided they both stay away from zero. To state our result in a unified way including the classical perimeter, it is convenient to define

$$\mathrm{Per}_s(E) := \begin{cases} \dfrac{1-s}{\omega_{n-1}} P_s(E) & \text{for } s \in (0,1)\,, \\ P(E) & \text{for } s = 1\,. \end{cases}$$

Theorem 2.1.3. *For every $n \geqslant 2$, $s_0 \in (0,1)$, and $\alpha_0 \in (0,n)$, there exists $\mathbf{m}_\star = \mathbf{m}_\star(n,\alpha_0,s_0) > 0$ such that, if $\mathbf{m} \in (0,\mathbf{m}_\star)$, $s \in (s_0,1]$, and $\alpha \in (\alpha_0,n)$, then the variationl problem*

$$\inf\left\{\mathrm{Per}_s(E) + V_\alpha(E) : |E| = \mathbf{m}\right\}$$

admits balls of volume $\mathbf{m}$ as unique minimizers.

We have also recovered the L^1-local minimality threshold $\mathbf{m}_0 = \mathbf{m}_0(n,\alpha,s)$ through a suitable spectral minimization problem. Compare to [43], a different approach allowed us to compute its explicit value, which is given by

$$\mathbf{m}_0(n,\alpha,s) = \begin{cases} \omega_n \left(\dfrac{n+s}{n-\alpha}\,\dfrac{s(1-s)P_s(B)}{\omega_{n-1}\alpha V_\alpha(B)}\right)^{\frac{n}{\alpha+s}} & \text{if } s \in (0,1)\,, \\[2ex] \omega_n \left(\dfrac{n+1}{n-\alpha}\,\dfrac{P(B)}{\alpha V_\alpha(B)}\right)^{\frac{n}{\alpha+1}} & \text{if } s = 1\,. \end{cases}$$

Theorem 2.1.4. *For every $n \geqslant 2$, $s \in (0,1]$, $\alpha \in (0,n)$, and $\mathbf{m} \in (0,\mathbf{m}_0)$, there exists $\boldsymbol{\kappa}_0 = \boldsymbol{\kappa}_0(n,\alpha,s,\mathbf{m}) > 0$ such that, if $B[\mathbf{m}]$ denotes a ball of volume $\mathbf{m}$, then*

$$\mathrm{Per}_s\big(B[\mathbf{m}]\big) + V_\alpha\big(B[\mathbf{m}]\big) \leqslant \mathrm{Per}_s(E) + V_\alpha(E)\,, \tag{2.1.12}$$

whenever $|E| = \mathbf{m}$ and $\big|E \triangle B[\mathbf{m}]\big| \leqslant \boldsymbol{\kappa}_0$. Moreover, if $\mathbf{m} > \mathbf{m}_0$, then $B[\mathbf{m}]$ is unstable, i.e., there exists a sequence of sets $\{E_k\}_{k\in\mathbb{N}}$ with $|E_k| = \mathbf{m}$ and $\big|E_k \triangle B[\mathbf{m}]\big| \to 0$ as $k \to \infty$ such that (2.1.12) fails with $E = E_k$ for every $k \in \mathbb{N}$.

Remark. Note that the ball $B[\mathbf{m}]$ is always a volume constrained critical point of $\mathrm{Per}_s + V_\alpha$. From the proof of Theorem 2.1.4, we may expect that there exists an increasing sequence of critical masses $\{\mathbf{m}_k\}_{k\in\mathbb{N}}$ (depending on n, α, and s) such that $\mathbf{m}_k \to \infty$, and that if $\mathbf{m} \in (\mathbf{m}_k, \mathbf{m}_{k+1})$, then $B[\mathbf{m}]$ has a finite Morse index $d(k)$ with $d(k) < d(k+1)$.

In proving both Theorem 2.1.3 and Theorem 2.1.4, it is convenient to rescale the problem in such a way that admissible sets E satisfies the volume constraint $|E| = |B|$. The corresponding rescaled (and normalized) energy reduces to

$$\mathcal{F}^\beta_{s,\alpha}(E) := \mathrm{Per}_s(E) + \beta V_\alpha(E) \qquad \text{with } \beta = \left(\frac{\mathbf{m}}{|B|}\right)^{\frac{\alpha+s}{n}}\,.$$

In this way, $\mathcal{F}^\beta_{s,\alpha}$ clearly appears as a lower order perturbation of the fractional perimeter, and it is not surprising, in view of the quantitative isoperimetric inequality, that B is the unique (up to translation) minimizer of $\mathcal{F}_{s,\alpha}\beta$ for β small enough.

The proof of Theorem 2.1.4 is divided in two steps. The first step consists in finding the stability threshold β_0 of the ball B. Here stability means non-negativety of the *second variation* of the energy $\mathcal{F}^\beta_{s,\alpha}$ at B along volume preserving deformations. The computation of the second variation, that we present in the following subsection, leads to the nonlocal operator (for $s \in (0,1)$)

$$\mathcal{J}^\beta_{s,\alpha}u := \frac{(1-s)}{\omega_{n-1}}\big(\mathcal{I}_s u - \lambda^s_1 u\big) - \beta\big(\mathcal{R}_\alpha u - \mu^\alpha_1 u\big)$$

acting on (smooth) functions $u : \partial B \to \mathbb{R}$ with zero average (which come from the linearization of the volume constraint). In the expression above, $\mathcal{I}_s$ is given by (2.1.10), $\mathcal{R}_\alpha$ is the spherical operator

$$\mathcal{R}_\alpha u(x) := 2\int_{\partial B} \frac{u(x)-u(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y\,,$$

and λ_1^s, μ_1^α are the principal eigenvalues of $\mathcal{I}_s$ and $\mathcal{R}_\alpha$, respectively. Here again, spherical harmonics diagonalize $\mathcal{R}_\alpha$ and the eigenvalues $\{\mu_k^\alpha\}$ can be computed explicitly. Once $\mathcal{J}^\beta_{s,\alpha}$ is diagonalized, one easily discovers that $\mathcal{J}^\beta_{s,\alpha} \geqslant 0$ if and only if $\beta \leqslant \beta_0$ where

$$\beta_0 := \frac{(1-s)}{\omega_{n-1}} \inf_{k\geqslant 2} \frac{\lambda_k^s - \lambda_1^s}{\mu_k^\alpha - \mu_1^\alpha}\,.$$

Taking advantage of the explicit expressions for the eigenvalues, we have proved that the infimum above is achieved at $k=2$, and, as a byproduct, we have found the explicit value of β_0.

The second part of the proof aims to show that the stability of the ball for $\beta < \beta_0$ implies its L^1-local minimality. It is achieved by a contradiction very much like in Section 2.1.1. By a similar penalization method, it reduces to prove a stability estimate for nearly spherical sets.

Theorem 2.1.5. *There exist constants $\varepsilon_0 \in (0,1/2)$ and $\mathbf{c}_0 > 0$, depending only on n, with the following property : if $\beta \in (0,\beta_0)$ and E is a nearly spherical set with $\|u_E\|_{C^1(\partial B)} < (1-\beta/\beta_0)\varepsilon_0$, then*

$$\mathcal{F}^\beta_{s,\alpha}(E) - \mathcal{F}^\beta_{s,\alpha}(B) \geqslant \mathbf{c}_0\left(1-\frac{\beta}{\beta_0}\right)\left((1-s)[u_E]^2_{\frac{1+s}{2}} + \|u_E\|^2_{L^2(\partial B)}\right) \qquad \forall s\in(0,1)\,,\ \forall \alpha \in (0,n)\,.$$

Remark. Taking the limit $s\uparrow 1$ in the inequality above, we obtain

$$\big(P(E)+\beta V_\alpha(E)\big) - \big(P(B)+\beta V_\alpha(B)\big) \geqslant c(n)\left(1-\frac{\beta}{\beta_0}\right)\|u_E\|^2_{H^1(\partial B)}\,.$$

2.1.3 First and second variations formulae for nonlocal perimeters

In this section we present the first and second variation formulae for the functionals P_s and V_α, and actually for more general nonlocal functionals behaving like P_s and V_α. We mention that the second variation of P_s has also been found very recently (and independently) by J. DÁVILA, M. DEL PINO, & J. WEI [84], where it is used to discuss the stability of certain nonlocal minimal surfaces such as Lawson cones (see Section 2.2.1 for a discussion on *nonlocal minimal surfaces*). In the case of the perimeter functional, the well known first and second variation formulae can be found for instance in the classical monograph by L. SIMON [193, Section 9].

Given $s\in(0,1)$ and $\alpha\in(0,n)$, we consider two convolution kernels $K,G \in C^1(\mathbb{R}^n\setminus\{0\};[0,\infty))$ which are symmetric by the origin (i.e., $K(-z)=K(z)$ and $G(-z)=G(z)$ for every $z\in\mathbb{R}^n\setminus\{0\}$) and satisfying the pointwise bounds

$$\sup_{z\neq 0}|z|^{n+s}K(z) < \infty \quad \text{and} \quad \sup_{z\neq 0}|z|^{n-\alpha}G(z)<\infty\,. \tag{2.1.13}$$

Correspondingly, given an open set $\Omega\subseteq\mathbb{R}^n$ and $E\subseteq\mathbb{R}^n$, we consider the nonlocal functionals localized to the open set Ω,

$$P_K(E,\Omega) := \int_{E\cap\Omega}\int_{E^c\cap\Omega} K(x-y)\,\mathrm{d}x\mathrm{d}y + \int_{E\cap\Omega}\int_{E^c\setminus\Omega} K(x-y)\,\mathrm{d}x\mathrm{d}y + \int_{E\setminus\Omega}\int_{E^c\cap\Omega} K(x-y)\,\mathrm{d}x\mathrm{d}y\,, \tag{2.1.14}$$

and

$$V_G(E,\Omega) := \int_{E\cap\Omega}\int_{E\cap\Omega} G(x-y)\,\mathrm{d}x\mathrm{d}y + 2\int_{E\cap\Omega}\int_{E\setminus\Omega} G(x-y)\,\mathrm{d}x\mathrm{d}y\,.$$

We have computed the first and second variations of these functionals along *flows*. Given a vector field $X \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in Ω, the flow $\{\phi_t\}_{t\in\mathbb{R}}$ induced by X is the smooth map $(t,x) \in \mathbb{R}\times\mathbb{R}^n \mapsto \phi_t(x) \in \mathbb{R}^n$ defined by solving the system of ODE's parametrized by x,

$$\begin{cases} \partial_t \phi_t(x) = X\big(\phi_t(x)\big)\,, & t \in \mathbb{R}\,, \\ \phi_0(x) = x\,. \end{cases}$$

It is well know $\{\phi_t\}_{t\in\mathbb{R}}$ is a 1-parameter group of smooth diffeomorphisms on $\mathbb{R}^n$, and $\phi_t - \mathrm{id}_{\mathbb{R}^n}$ is compactly supported in Ω. If $|E| < \infty$, we say that X induces a *volume preserving flow* on E whenever $|\phi_t(E)| = |E|$ for t sufficiently small.

If $P_K(E,\Omega) < \infty$ and $E_t := \phi_t(E)$, one can deduce from the area formula that $t \mapsto P_K(E_t,\Omega)$ is smooth (at least for t small). Accordingly, the first and second variations of $P_K(\cdot,\Omega)$ at E along X can be defined as

$$\delta P_K(E,\Omega)[X] := \left[\frac{\mathrm{d}}{\mathrm{d}t}\, P_K(E_t;\Omega)\right]_{t=0}\,, \qquad \delta^2 P_K(E;\Omega)[X] := \left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, P_K(E_t;\Omega)\right]_{t=0}\,. \tag{2.1.15}$$

Identical definitions are adopted when V_G is considered in place of P_K and E is such that $V_G(E;\Omega) < \infty$.

Having set our terminology, we now turn to the question of expressing first and second variations along X in terms of boundary integrals involving X and its derivatives. These formulas involve some "nonlocal" variants of the *mean curvature* and the *length squared of the second fundamental form*. Given a point $x \in \mathbb{R}^n$, we define (as elements of $[-\infty,\infty]$)

$$\mathrm{H}_{K,\partial E}(x) := \mathrm{p.v.}\left(\int_{\mathbb{R}^n} \big(\chi_{E^c}(y) - \chi_E(y)\big)\, K(x-y)\,\mathrm{d}y\right)\,, \qquad \mathrm{H}^*_{G,\partial E}(x) := 2\int_E G(x-y)\,\mathrm{d}y\,.$$

Assuming that E is smooth enough and denoting by ν_E the outer unit normal to ∂E, we also define for $x \in \partial E$ and $J \in \{K,G\}$,

$$\mathrm{c}^2_{J,M}(x) := \int_{\partial E} J(x-y)|\nu_E(x) - \nu_E(y)|^2\,\mathrm{d}\mathscr{H}^{n-1}_y\,.$$

In the following statement, $X_\tau := X - (X\cdot\nu_E)\nu_E$ is the projection of X on $T\partial E$ and div_τ is the tangential divergence operator on ∂E.

Theorem 2.1.6. *Let $K, G \in C^1(\mathbb{R}^n\setminus\{0\};[0,\infty))$ be even functions satisfying (2.1.13) for some $s \in (0,1)$ and $\alpha \in (0,n)$. Let Ω be an open set in $\mathbb{R}^n$, and let $E \subseteq \mathbb{R}^n$ be an open set with $C^{1,1}$-boundary such that $\partial E \cap \Omega$ is a C^2-hypersurface. Given $X \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ compactly supported in Ω, set $\zeta = X\cdot\nu_E$.*

(a) *If $P_K(E,\Omega) < \infty$ and $\int_{\partial E}(1+|z|)^{-n-s}\,\mathrm{d}\mathscr{H}^{n-1} < \infty$, then*

$$\delta P_K(E,\Omega)[X] = \int_{\partial E} \mathrm{H}_{K,\partial E}\,\zeta\,\mathrm{d}\mathscr{H}^{n-1}\,,$$

and

$$\delta^2 P_K(E,\Omega)[X] = \iint_{\partial E\times\partial E} K(x-y)|\zeta(x)-\zeta(y)|^2\,\mathrm{d}\mathscr{H}^{n-1}_x \mathrm{d}\mathscr{H}^{n-1}_y - \int_{\partial E} \mathrm{c}^2_{K,\partial E}\,\zeta^2\,\mathrm{d}\mathscr{H}^{n-1} + \int_{\partial E} \mathrm{H}_{K,\partial E}\Big((\mathrm{div}\,X)\,\zeta - \mathrm{div}_\tau\big(\zeta\,X_\tau\big)\Big)\,\mathrm{d}\mathscr{H}^{n-1}\,.$$

(b) *If $V_G(E,\Omega) < \infty$, $\int_E |z|^{-n+\alpha}\,\mathrm{d}z < \infty$, and $\int_{\partial E}(1+|z|)^{-n+\alpha}\,\mathrm{d}\mathscr{H}^{n-1} < \infty$, then*

$$\delta V_G(E,\Omega)[X] = \int_{\partial E} \mathrm{H}^*_{G,\partial E}\,\zeta\,\mathrm{d}\mathscr{H}^{n-1}\,,$$

and

$$\delta^2 V_G(E,\Omega)[X] = -\iint_{\partial E\times\partial E} G(x-y)|\zeta(x)-\zeta(y)|^2\,\mathrm{d}\mathscr{H}^{n-1}_x \mathrm{d}\mathscr{H}^{n-1}_y + \int_{\partial E} \mathrm{c}^2_{G,\partial E}\,\zeta^2\,\mathrm{d}\mathscr{H}^{n-1} + \int_{\partial E} \mathrm{H}^*_{G,\partial E}\Big((\operatorname{div} X)\,\zeta - \operatorname{div}_\tau\big(\zeta\, X_\tau\big)\Big)\,\mathrm{d}\mathscr{H}^{n-1}\,.$$

Remark. Under the assumptions above, $\mathrm{H}_{K,\partial E}$ and $\mathrm{H}^*_{G,\partial E}$ are real-valued and continuous on ∂E.

Remark. The assumptions of Theorem 2.1.6 are satisfied if E is bounded and ∂E is a C^2-hypersurface.

Remark. Under the assumptions of Theorem 2.1.6, if E is a volume constrained stationary set of P_K in Ω, i.e., $\delta P_K(E,\Omega)[X]=0$ for every vector field X inducing a volume preserving flow on E, then $\mathrm{H}_{K,\partial E}$ is constant on $\partial E\cap\Omega$ and

$$\delta^2 P_K(E,\Omega)[X] = \iint_{\partial E\times\partial E} K(x-y)|\zeta(x)-\zeta(y)|^2\,\mathrm{d}\mathscr{H}^{n-1}_x \mathrm{d}\mathscr{H}^{n-1}_y - \int_{\partial E} \mathrm{c}^2_{K,\partial E}\,\zeta^2\,\mathrm{d}\mathscr{H}^{n-1}\,.$$

Similarly, if E is a volume constrained stationary set of V_G in Ω, then $\mathrm{H}^*_{G,\partial E}$ is constant on $\partial E\cap\Omega$ and the third integral in the expression of $\delta^2 V_G(E,\Omega)[X]$ vanishes.

2.2 Asymptotics for a fractional Allen-Cahn equation

In the classical van der Waals-Cahn-Hilliard theory of phase transitions (see e.g. M.E. GURTIN [125]), two-phase systems are driven by energy functionals of the form

$$F_\varepsilon(u) = \int_\Omega \varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,\mathrm{d}x\,, \qquad \varepsilon\in(0,1)\,,$$

where $u:\Omega\subseteq\mathbb{R}^n\to\mathbb{R}$ is a normalized density distribution of the two phases, and $W:\mathbb{R}\to[0,\infty)$ is a potential with exactly two global minimal at ±1, and $W(\pm1)=0$. Critical points of F_ε satisfies the so-called (elliptic) Allen-Cahn equation

$$-\Delta u_\varepsilon + \varepsilon^{-2}W'(u_\varepsilon) = 0 \quad \text{in}\,\Omega\,.$$

When ε is small, a control on the potential term in F_ε implies that $u_\varepsilon\simeq\pm1$ away from a region whose area is of order ε. Formally, the transition layer from the phase -1 to the phase $+1$ has a characteristic width of order ε. It should take place along an hypersurface which is expected to be a critical point of the area functional, i.e., a minimal surface. More precisely, the region delimited by this hypersurface and the container Ω should be a stationary set in Ω of the (distributional) perimeter.

For minimizing solutions of the Allen-Cahn equation (under their own boundary condition), this picture has been rigorously proved by L. MODICA & S. MORTOLA [162] through one of the first examples of Γ-convergence. Their result shows that if the energy is equibounded, then $u_\varepsilon\to u_*$ in $L^1(\Omega)$ as $\varepsilon\to0$ for some function $u_*\in BV(\Omega;\{\pm1\})$, and the set $\{u_*=+1\}$ is (locally) perimeter minimizing in Ω (see also L. MODICA [161] and P. STERNBERG [198] for the volume constrained problem).

The more difficult case of general critical points (which may not be energy minimizing) has been addressed by J.E. HUTCHINSON & Y. TONEGAWA [135], and it presents an additional qualitative feature. Namely, if the energy is equibounded, then the energy density converges as $\varepsilon\to0$ to a stationary integral $(n-1)$-varifold, i.e., a generalized minimal surface with integer multiplicity. The multiplicity of the limiting hypersurface comes from an eventual "folding" of the interface as $\varepsilon\to0$. In particular, the limiting interface between the two regions $\{u_*=+1\}$ and $\{u_*=-1\}$ can be strictly smaller than the support of the varifold, and it may not be stationary. In fact, the boundary of the region $\{u_* = +1\}$ corresponds to the set of points where the limiting hypersurface has odd multiplicity.

This effect of energy loss is in complete analogy with the lack of strong compactness for solutions of the (vectorial) Ginzburg-Landau equation, or fractional Ginzburg-Landau equation. In Section 1.1.3, we have seen that a weak limit of solutions of the fractional Ginzburg-Landau equation may not be a stationary fractional harmonic map. However, stationarity is restored when adding the defect measure in the first variation of energy, see (vi) in Theorem 1.1.5.

In the ongoing article [P20], in collaboration with Y. SIRE & K. WANG, we perform the asymptotic analysis as $\varepsilon \to 0$ of a fractional version of the Allen-Cahn equation where the diffusion operator is replaced by a (small) power of the Laplacian. In the spirit of [135], our study focuses on general critical points and reveals some unexpected effects compare to the classical case. Our work is motivated by the recent theory of nonlocal minimal surfaces of L. CAFFARELLI, J.M. ROQUEJOFFRE, & O. SAVIN [60] and some nonlocal phase transitions problems arising in Peirls-Nabarro models for dislocations in crystals, or in stochastic Ising models from statistical mechanics, see e.g. C. IMBERT [136] and C. IMBERT & P.E. SOUGANIDIS [137]. To tackle efficiently the problem, we took advantage of the techniques and methods developed for the fractional Ginzburg-Landau equation presented in Section 1.1.

2.2.1 The fractional Allen-Cahn equation and nonlocal minimal surfaces

Let $n \geqslant 2$, $s \in (0,1)$, and let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded open set. We are now addressing the asymptotic behavior, as $\varepsilon \downarrow 0$, of weak solutions $v_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ to the fractional Allen-Cahn equation

$$(-\Delta)^s v_\varepsilon + \varepsilon^{-2s} W'(v_\varepsilon) = 0 \qquad \text{in } \Omega\,, \tag{2.2.1}$$

subject to an exterior Dirichlet condition

$$v_\varepsilon = g_\varepsilon \qquad \text{on } \mathbb{R}^n \setminus \Omega\,, \tag{2.2.2}$$

where $g_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ is a smooth bounded function. The potential $W : \mathbb{R} \to [0,\infty)$ is assumed to be of double-well type. More precisely, we assume that

(H1) $W \in C^2(\mathbb{R})$;

(H2) $\{W = 0\} = \{\pm 1\}$ and $W''(\pm 1) > 0$;

(H3) $tW'(t) \geqslant 0$ for $|t| \geqslant 1$;

(H4) there exit an exponent $p \in [1,\infty)$ and a constant $c > 0$ such that

$$\frac{1}{c}\big(|t|^{p-1} - 1\big) \leqslant |W'(t)| \leqslant c\big(|t|^{p-1} + 1\big) \qquad \forall t \in \mathbb{R}\,.$$

Those assumption are of course satisfied by the prototypical potential $W(t) = (1-t^2)^2$.

The fractional Laplace operator $(-\Delta)^s$ is given in (1.1.1), and the weak sense for equation (2.2.1) is understood through the variational formulation (1.1.4) in the open set Ω, i.e.,

$$\big\langle (-\Delta)^s v_\varepsilon, \varphi \big\rangle + \frac{1}{\varepsilon^{2s}} \int_\Omega W'(v_\varepsilon)\varphi \,\mathrm{d}x = 0 \qquad \forall \varphi \in \mathscr{D}(\Omega)\,.$$

In this way, equation (2.2.1) corresponds to the Euler-Lagrange equation for critical points of the fractional *s-Allen-Cahn energy* $\mathcal{F}_\varepsilon(\cdot,\Omega)$ defined for $v \in \widehat{H}^s \cap L^p(\Omega)$ by

$$\mathcal{F}_\varepsilon(v,\Omega) := \mathcal{E}_s(v,\Omega) + \frac{1}{\varepsilon^{2s}} \int_\Omega W(v)\,\mathrm{d}x\,,$$

where $\mathcal{E}_s(\cdot,\Omega)$ is the fractional s-Dirichlet energy given by (1.1.3).

To construct weak solutions to (2.2.1)-(2.2.2), the simplest way is to minimize $\mathcal{F}_\varepsilon(\cdot,\Omega)$ under the exterior Dirichlet condition (2.2.2), i.e., setting

$$H^s_{g_\varepsilon}(\Omega) := g_\varepsilon + H^s_{00}(\Omega) \subseteq \widehat{H}^s(\Omega)\,,$$

one considers the minimization problem

$$\min\left\{\mathcal{F}_\varepsilon(v,\Omega) : v \in H^s_{g_\varepsilon} \cap L^p(\Omega)\right\}\,, \tag{2.2.3}$$

whose resolution follows from the Direct Method of Calculus of Variations.

For what concerns *minimizers*, i.e., solutions of (2.2.3), the asymptotic behavior as $\varepsilon \downarrow 0$ has been investigated recently by O. SAVIN & E. VALDINOCI [185] through a Γ-convergence analysis of the functionals $\mathcal{F}_\varepsilon(\cdot,\Omega)$. Their result reveals a dichotomy between the two cases $s \geqslant 1/2$ and $s < 1/2$. In the case $s \geqslant 1/2$, the normalized energies

$$\widetilde{\mathcal{F}}_\varepsilon(\cdot,\Omega) := \begin{cases} \varepsilon^{2s-1}\mathcal{F}_\varepsilon(\cdot,\Omega) & \text{if } s \in (1/2,1)\,,\\ |\ln\varepsilon|^{-1}\mathcal{F}_\varepsilon(\cdot,\Omega) & \text{if } s = 1/2\,, \end{cases}$$

Γ-converge as $\varepsilon \to 0$ to the functional $\widetilde{\mathcal{F}}_0(\cdot,\Omega)$ defined on $BV(\Omega;\{\pm1\})$ by

$$\widetilde{\mathcal{F}}_0(v,\Omega) := \sigma_{n,s}(W)P\big(\{v=+1\},\Omega\big)\,,$$

where $\sigma_{n,s}(W)$ is a constant, and $P(E,\Omega)$ denotes the distributional perimeter of a set E in Ω [2]. In other words, for $s \geqslant 1/2$, Allen-Cahn energies (and thus minimizers) behave exactly as in the classical case, and area-minimizing hypersurfaces arise in the limit $\varepsilon \to 0$.

Remark. The fractional Allen-Cahn energy has been originally introduced for $s = 1/2$ by G. ALBERTI, G. BOUCHITTÉ, & P. SEPPECHER in [8, 9], where (essentially) the same Γ-convergence result is proved.

On the contrary, the variational convergence of $\mathcal{F}_\varepsilon(\cdot,\Omega)$ in the case $s \in (0,1/2)$ appears to be almost trivial. Indeed, the H^s-regularity does not exclude characteristic functions and the class

$$\widehat{H}^s(\Omega;\{\pm1\}) := \{v \in \widehat{H}^s(\Omega) : |v| = 1 \text{ a.e. in } \mathbb{R}^n\}$$

is not reduced to constants. In particular, there is no need to normalize $\mathcal{F}_\varepsilon(\cdot,\Omega)$, and $\mathcal{F}_\varepsilon(\cdot,\Omega)$ converges as $\varepsilon \to 0$ both in the variational and pointwise sense to

$$\mathcal{F}_0(v,\Omega) := \begin{cases} \mathcal{E}_s(v,\Omega) & \text{if } v \in \widehat{H}^s(\Omega;\{\pm1\})\,,\\ +\infty & \text{otherwise}\,. \end{cases}$$

Now it is worth noting that

$$\mathcal{E}_s(v,\Omega) = 2\gamma_{n,s}P_{2s}\big(\{v=+1\},\Omega\big) \qquad \forall v \in \widehat{H}^s(\Omega;\{\pm1\})\,,$$

where the constant $\gamma_{n,s}$ is given in (1.1.1), and $P_{2s}(E,\Omega)$ is the fractional $2s$-perimeter in Ω of a set $E \subseteq \mathbb{R}^n$ as defined in (2.1.14) (with $K(x-y) = |x-y|^{-(n+2s)}$).

2. The perimeter $P(E,\Omega)$ is defined as in (2.1.2) with the additional restriction that the supremum is taken over vector fields with compact support in Ω.

As a consequence, under suitable assumptions on the exterior Dirichlet condition (see Theorem 2.2.1), solutions $\{v_\varepsilon\}$ of (2.2.3) converge (up to subsequences) in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to some function $v_* \in \widehat{H}^s(\Omega)$ of the form $v_* = \chi_E - \chi_{\mathbb{R}^n\setminus E}$, and the set $E \subseteq \mathbb{R}^n$ is minimizing the $2s$-perimeter in Ω, i.e.,

$$P_{2s}(E,\Omega) \leqslant P_{2s}(F,\Omega) \qquad \forall F \subseteq \mathbb{R}^n\,,\ F\setminus\Omega = E\setminus\Omega\,. \tag{2.2.4}$$

Sets E satisfying the minimality condition (2.2.4) have been introduced and studied by L. Caffarelli, J.M. Roquejoffre, & O. Savin [60]. Their boundary ∂E are referred to as (minimizing) *nonlocal minimal surfaces*. The minimality condition implies that the first variation of the $2s$-perimeter (as defined in (2.1.15)) vanishes at E, i.e.,

$$\delta P_{2s}(E,\Omega)[X] = 0 \qquad \forall X \in C^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ compactly supported in } \Omega\,. \tag{2.2.5}$$

If ∂E is smooth enough (see e.g. Theorem 2.1.6), (2.2.5) is equivalent to

$$\mathrm{H}_{2s,\partial E}(x) := \mathrm{p.v.}\left(\int_{\mathbb{R}^n} \frac{\chi_{\mathbb{R}^n\setminus E}(y) - \chi_E(y)}{|x-y|^{n+2s}}\,\mathrm{d}y\right) = 0 \qquad \forall x \in \partial E\cap\Omega\,,$$

where $\mathrm{H}_{2s,\partial E}$ is the so-called *fractional mean curvature*. The boundary of a set $E \subseteq \mathbb{R}^n$ satisfying (2.2.5) shall be refered to as *stationary nonlocal minimal surface* in Ω.

The main purpose in [60] was to determine the regularity of a minimizing nonlocal minimal surface ∂E. It is proved that $\partial E\cap\Omega$ is a $C^{1,\alpha}$-hypersurface away from a (relative) closed set $\Sigma \subseteq \partial E\cap\Omega$ of Hausdorff dimension less than or equal to $(n-2)$. This clearly parallels the classical theory of minimizing hypersurfaces, except that, in the classical theory, the dimension of the singular set is less than or equal to $(n-8)$ (see e.g. [120]). More recently, O. Savin & E. Valdonici [186] improved the dimension estimate to $(n-3)$, and B. Barrios Barrera, A. Figalli, & E. Valdinoci [31] have shown the C^∞-regularity of the regular part of the boundary.

We finally recall that all these results are valid for *minimizers and only minimizers*. In particular, nothing was known about the asymptotic $\varepsilon \to 0$ of general critical points for the fractional Allen-Cahn energy. We shall answer this question in the case $s \in (0,1/2)$, i.e., in the regime of nonlocal minimal surfaces.

2.2.2 Strong convergence of the fractional Allen-Cahn equation

We now come to our main result on general (weak) solutions to equation (2.2.1). We recall our assumptions (H1)–(H4) on the double-well potential W.

Theorem 2.2.1. *Assume that $s \in (0,1/2)$. Let $\varepsilon_k \downarrow 0$ be an arbitrary sequence, and let $\{g_k\}_{k\in\mathbb{N}} \subseteq C^1(\mathbb{R}^n)$ be such that $|g_k| \leqslant 1$ and $g_k \to g$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n\setminus\Omega)$ for a function g satisfying $|g| = 1$ a.e. in $\mathbb{R}^n\setminus\Omega$. Let $\{v_k\}_{k\in\mathbb{N}} \subseteq H^s_{g_k}\cap L^p(\Omega)$ be such that*

$$\begin{cases} (-\Delta)^s v_k + \dfrac{1}{\varepsilon_k^{2s}} W'(v_k) = 0 & \textit{in } \Omega\,,\\ v_k = g_k & \textit{in } \mathbb{R}^n\setminus\Omega\,. \end{cases}$$

If $\sup_k \mathcal{F}_{\varepsilon_k}(v_k,\Omega) < \infty$, then there exist a (not relabeled) subsequence and a Borel set $E \subseteq \mathbb{R}^n$ of finite $2s$-perimeter in Ω such that $v_k \to \chi_E - \chi_{\mathbb{R}^n\setminus E}$ strongly in $H^s_{\mathrm{loc}}(\Omega)$ and $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, $E\cap\Omega$ is an open set, ∂E is a stationary nonlocal $2s$-minimal surface *in Ω (i.e., (2.2.5) holds), and*

(i) *$\mathcal{E}_s(v_k,\Omega') \to 2\gamma_{n,s}P_{2s}(E,\Omega')$ for every smooth open set $\Omega' \subseteq \Omega$ such that $\overline{\Omega'} \subseteq \Omega$;*

(ii) *$\varepsilon_k^{-2s}W(v_k) \to 0$ in $L^1_{\mathrm{loc}}(\Omega)$;*

(iii) $v_k \to 1$ *locally uniformly in* $E \cap \Omega$, *and* $v_k \to -1$ *locally uniformly in* $\Omega \setminus \overline{E}$;

(iv) *for each* $\delta \in (-1,1)$, *the level set* $\{v_k = \delta\}$ *converges locally uniformly in* Ω *to* $\partial E \cap \Omega$, *i.e., for every compact set* $K \subseteq \Omega$ *and every* $r > 0$,

$$\{v_k = \delta\} \cap K \subseteq \big\{x : \mathrm{dist}(x, \partial E \cap \Omega) < r\big\} \quad \textit{and} \quad \partial E \cap K \subseteq \big\{x : \mathrm{dist}(x, \{v_k = \delta\} \cap \Omega) < r\big\}$$

whenever k *is large enough.*

Remark. In the particular case where $\{v_k\}$ is assumed to be minimizing, Theorem 2.2.1 recovers at least the results in [185] for $s \in (0, 1/2)$. Note that for minimizers, the uniform energy bound assumption is always satisfied since the function $\chi_\Omega + (1 - \chi_\Omega) g_k$ is an admissible competitor with uniformly bounded energy (depending on $P_{2s}(\Omega)$).

Remark. Theorem 2.2.1 parallels J.E. HUTCHINSON & Y. TONEGAWA convergence result [135]. In contrast with [135], there is no loss of energy, and solutions of the fractional Allen-Cahn equation are strongly compact. In particular, "even folding" of interfaces is excluded. However "odd folding" might occur but the H^s-regularity is not fine enough to capture multiplicity as $\varepsilon \to 0$.

The proof of Theorem 2.2.1 is based on the strategy we have developed for the fractional Ginzburg-Landau equation, see Section 1.1.3. It rests on the result of L. CAFFARELLI & L. SILVESTRE [61] for the representation of $(-\Delta)^s$ as the generalized *Dirichlet-to-Neumann operator* associated to an extension to the open half space $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$ given by the convolution product with a "fractional" Poisson kernel. More precisely, the $2s$-Poisson kernel is the function $\mathbf{K}_{2s} : \mathbb{R}^{n+1}_+ \to [0, \infty)$ defined by

$$\mathbf{K}_{2s}(\mathbf{x}) := \sigma_{n,s} \frac{z^{2s}}{|\mathbf{x}|^{n+2s}}\,, \qquad \sigma_{n,s} := \pi^{-\frac{n}{2}} \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(s)}\,,$$

where we write $\mathbf{x} = (x, z) \in \mathbb{R}^n \times (0, \infty)$. Setting $a := 1 - 2s$, the kernel $\mathbf{K}_{2s}$ solves the equation

$$\begin{cases} \mathrm{div}\big(z^a \nabla \mathbf{K}_{2s}\big) = 0 & \text{in } \mathbb{R}^{n+1}_+\,, \\ \mathbf{K}_{2s} = \delta_0 & \text{on } \partial\mathbb{R}^{n+1}_+ \simeq \mathbb{R}^n\,, \end{cases}$$

where δ_0 is the Dirac distribution at the origin.

Denoting by $v \mapsto v^{\mathrm{e}}$ the convolution in x of v with $\mathbf{K}_{2s}$, i.e.,

$$v^{\mathrm{e}}(x) := \sigma_{n,s} \int_{\mathbb{R}^n} \frac{z^{2s} v(y)}{(|x - y|^2 + z^2)^{\frac{n+1}{2}}}\, \mathrm{d}y\,, \tag{2.2.6}$$

we have proved that it is well defined on $\widehat{H}^s(\Omega)$, and that $(-\Delta)^s v = \mathbf{\Lambda}^{2s} v$ as distributions on the open set Ω, where $\mathbf{\Lambda}^{2s}$ is the Dirichlet-to-Neumann operator

$$\mathbf{\Lambda}^{2s} v(x) := -d_s \lim_{z \downarrow 0} z^a \partial_z v^{\mathrm{e}}(x, z)\,, \qquad d_s := 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)}\,.$$

When applying the extension procedure to a solution v of the fractional Allen-Cahn equation, we end up with the following degenerate equation

$$\begin{cases} \mathrm{div}\big(z^a \nabla v^{\mathrm{e}}\big) = 0 & \text{in } \mathbb{R}^{n+1}_+\,, \\ \displaystyle\lim_{z \downarrow 0} z^a \partial_z v^{\mathrm{e}} = \frac{1}{d_s \varepsilon^{2s}} W'(v^{\mathrm{e}}) & \text{on } \Omega\,. \end{cases} \tag{2.2.7}$$

The asymptotic analysis of this system as $\varepsilon \to 0$ led us to the main conclusions. To perform such an analysis, we had first to establish a suitable (boundary) regularity theory for (2.2.7) relying on the classical results of E.B. FABES, C.E. KENIG, & R.P. SERAPIONI [93] and a recent work of X. CABRÉ & Y. SIRE [59]. Then, the key ingredient was to prove an appropriate *monotonicity formula* for solutions of (2.2.7). This monotonicity formula is completely analogous to the one we found for the fractional Ginzburg-Landau equation, and, in sharp contrast with the classical Allen-Cahn equation, no "discrepancy term" appears. In the spirit of Ginzburg-Landau theories, the monotonicity formula implies a "clearing-out lemma" : if the energy in a half ball (centered at a point of Ω) is small enough compare to the radius raised to the power $(n-2s)$, then the solution is uniformly close to the wells of W in the ball of half radius. By convexity of the potential near the wells, it implies that the solution is actually minimizing in this smaller ball. This is then enough to derive strong convergence in H^s and in L^∞ to one of the wells whenever the energy is sufficiently small. At this stage, the asymptotic analysis as $\varepsilon \to 0$ follows closely [P17], and energy concentration is excluded by Marstrand's theorem, see e.g. [155]. In turn, the strong local convergence in H^s implies the stationarity of the limiting function.

Remark. At the present time, there is no regularity theory for *stationary* nonlocal minimal surfaces. In an ongoing project, we are addressing this question. Many of the arguments we have developed for the fractional Allen-Cahn equation turn out to be useful for this problem. In fact, the general philosophy is that nonlocal minimal surfaces are better understood when interpreted as fractional harmonic maps with values in $\mathbb{S}^0 \simeq \{\pm 1\}$.

2.3 Unilateral gradient flow of an approximate Mumford-Shah functional

Many free discontinuity problems are variational in nature and involve two unknowns, a function u and a discontinuity set Γ across which u may jump. The most famous example is certainly the minimization of the MUMFORD-SHAH (MS) functional introduced in [168] to approach image segmentation. It is defined by

$$\mathcal{E}_*(u,\Gamma) := \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2\,\mathrm{d}x + \mathscr{H}^{n-1}(\Gamma) + \frac{\beta}{2}\int_\Omega (u-g)^2\,\mathrm{d}x\,,$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded Lipschitz open set, $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, $\beta > 0$ is a fidelity (constant) factor, and $g \in L^\infty(\Omega)$ stands for the grey level of the original image. In the resulting minimization process, we end up with a segmented image $u : \Omega \setminus \Gamma \to \mathbb{R}$ and a set of contours $\Gamma \subseteq \Omega$. To efficiently tackle this problem, a weak formulation in the space of *Special functions of Bounded Variation* has been suggested and solved in [87], where the set Γ is replaced by the jump set J_u of u. The new energy is defined for $u \in SBV^2(\Omega)$ by

$$\frac{1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \mathscr{H}^{n-1}(J_u) + \frac{\beta}{2}\int_\Omega (u-g)^2\,\mathrm{d}x\,, \tag{2.3.1}$$

where ∇u is now intended to be the measure theoretic gradient of u.

A related model based on the Mumford-Shah functional has been introduced by G. FRANCFORT & J.J. MARIGO in [48, 107] to describe quasi-static crack propagation inside elastic bodies. It is a variational model relying on three fundamental principles : *(i)* the fractured body must stay in elastic equilibrium at each time (*quasi-static hypothesis*) ; *(ii)* the crack can only grow (*irreversibility constraint*) ; *(iii)* an energy balance holds. In the anti-plane setting, the equilibrium and irreversibility principles lead to a constrained local minimization of MS at each time, where u stands now for the scalar displacement while Γ is the crack. Unfortunately, there is no canonical notion of local minimality and most of the study consider global minimizers instead, see [81, 79, 105]. In the discrete setting, one looks at each time step for a pair (u_i, Γ_i) minimizing

$$(u, \Gamma) \mapsto \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx + \mathscr{H}^{n-1}(\Gamma) \,,$$

among all cracks $\Gamma \supseteq \Gamma_{i-1}$ and all displacements $u : \Omega \setminus \Gamma \to \mathbb{R}$ satisfying an updated boundary condition, where Γ_{i-1} is the crack found at the previous time step.

While such static free discontinuity problems start to be well understood, many questions remain open concerning their evolutionary version. Apart from the quasi-static case, the closest evolution problem to statics consists in finding a steepest gradient descent of the energy, and thus in solving a gradient flow type equation. A major difficulty in this setting is to define a suitable notion of gradient since the functional is neither regular nor convex, and standard theories such as maximal monotone operators [55] do not apply. However, using a time discretization, an implicit Euler scheme can always be defined. Letting the time step tend to zero, the possible limits are refered to as DE GIORGI's *minimizing movements* [14, 86, 88]. Minimizing movements for the Mumford-Shah functional have been first considered by L. AMBROSIO & A. BRAIDES [16], and further developed by A. CHAMBOLLE & F. DOVERI [63]. Motivated by the crack growth model, the authors apply an implicit iterative scheme with respect to the variable u while minimizing the energy with respect to Γ under the constraint of irreversibility. More precisely, denoting by u_{i-1} the displacement at the previous time step, one looks for minimizers of

$$(u, \Gamma) \mapsto \frac{1}{2} \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx + \mathscr{H}^{n-1}(\Gamma) + \frac{\beta}{2} \int_{\Omega} (u - g)^2 \, dx + \frac{1}{2\delta} \|u - u_{i-1}\|^2_{L^2(\Omega)} \,,$$

again among all $\Gamma \supseteq \Gamma_{i-1}$ and $u : \Omega \setminus \Gamma \to \mathbb{R}$, where $\delta > 0$ is the time step. In any space dimension, the limiting displacement $t \mapsto u(t)$ satisfies some degenerate heat equation, and an energy inequality with respect to the initial time holds.

On another hand, the Mumford-Shah functional enjoys good variational approximation properties by means of regular energies. Constructing $L^2(\Omega)$-gradient flows for those regularized energies and taking the limit in the approximation parameter could be an alternative way to derive a generalized gradient flow for MS. It was actually the path followed in [121] where a gradient flow equation for the one-dimensional Mumford-Shah functional is obtained as a limit of ordinary differential equations derived from a non-local approximation. Many other approximations are available, and the most famous one is certainly the AMBROSIO-TORTORELLI functional defined for $(u, \rho) \in [H^1(\Omega)]^2$ by

$$AT_\varepsilon(u, \rho) := \frac{1}{2} \int_{\Omega} (\eta_\varepsilon + \rho^2) |\nabla u|^2 \, dx + \frac{\beta}{2} \int_{\Omega} (u - g)^2 \, dx + \frac{1}{2} \int_{\Omega} \left(\varepsilon |\nabla \rho|^2 + \frac{1}{\varepsilon}(1 - \rho)^2 \right) dx \,,$$

where $\eta_\varepsilon > 0$ is a parameter satisfying $\eta_\varepsilon = o(\varepsilon)$ as $\varepsilon \to 0$. The idea is to replace the discontinuity set Γ by a (diffuse) phase field variable, denoted by $\rho : \Omega \to [0, 1]$, which is "smooth" and essentially 0 in some ε-neighborhood of Γ. Such energies are of great importance for numerical simulations in imaging or brittle fracture, see [47, 48]. From the mechanical point of view, it is interpreted as a non-local damage approximation of fracture models, where ρ represents a damage density. The approximation result of L. AMBROSIO & V.M. TORTORELLI [27, 28] states that AT_ε Γ-converges as $\varepsilon \to 0$ to MS (in the form (2.3.1)) with respect to a suitable topology. For the static problem, it implies the convergence of AT_ε-minimizers towards MS-minimizers, see Section 1.5.1. However, the convergence of general critical points is *a priori* not guaranteed. The only positive results in this direction have been obtained in [106, 144] but are restricted to the one-dimensional case. The Ambrosio-Tortorelli approximation of quasi-static crack evolution has been considered in [114], where the irreversibility constraint translates into the decrease of the phase field $t \mapsto \rho(t)$. The main result of [114] concerns the convergence of this regularized model towards the original one in [105], and it relies strongly on global minimality. About parabolic evolutions, not much is known, specially for what concerns the singular limit $\varepsilon \to 0$.

The object of the article [P16], in collaboration with J.F. BABADJIAN, was to study a *unilateral gradient flow* for the Ambrosio-Tortorelli functional taking into account the irreversibility constraint on the phase field variable. The idea was to construct minimizing movements starting from a discrete Euler scheme which is precisely an Ambrosio-Tortorelli regularization of the one studied in [15, 63]. As in the quasi-static case [114], the irreversibility of the process has to be encoded into the decrease of the phase field variable, and leads at each time step to a constrained minimization problem. More precisely, given an initial data (u_0, ρ_0), one may recursively define pairs (u_i, ρ_i) by minimizing at each time $t_i \sim i\delta$,

$$(u, \rho) \mapsto AT_\varepsilon(u, \rho) + \frac{1}{2\delta}\|u - u_{i-1}\|^2_{L^2(\Omega)} ,$$

among all u and $\rho \leqslant \rho_{i-1}$, where (u_{i-1}, ρ_{i-1}) is a pair found at the previous time step. The objective is then to pass to the limit as the time step δ tends to 0. In this formulation, an important issue is to deal with the asymptotics of the obstacle problems induced by the irreversibility constraint. It is known that such problems are not stable with respect to weak H^1-convergence, and that undesirable "strange terms" of capacitary type may appear [70, 80]. However, ensuring uniform convergence of obstacles would be enough to rule out this situation. For that reason, instead of AT_ε, we have considered a modified Ambrosio-Tortorelli functional with p-growth in $\nabla\rho$, $p > n$. By the Sobolev imbedding theorem, with such a functional in hand, uniform convergence on the ρ variable is now ensured. The choice we made is certainly the closest analogue of AT_ε, and it is defined for $(u, \rho) \in H^1(\Omega) \times W^{1,p}(\Omega)$ by

$$\mathcal{E}_\varepsilon(u, \rho) := \frac{1}{2}\int_\Omega (\eta_\varepsilon + \rho^2)|\nabla u|^2 \,\mathrm{d}x + \frac{\beta}{2}\int_\Omega (u - g)^2 \,\mathrm{d}x + \int_\Omega \left(\frac{\varepsilon^{p-1}}{p}|\nabla\rho|^p + \frac{\alpha_p}{\varepsilon}|1 - \rho|^p\right) \mathrm{d}x\,, \quad p > n\,,$$

where $\alpha_p := \frac{p-1}{p}\left(\frac{p}{2}\right)^{\frac{p}{p-1}}$ is a normalizing factor. Here, an immediate adaptation of [97] shows that $\mathcal{E}_\varepsilon$ is still an approximation of the Mumford-Shah functional in the sense of Γ-convergence.

2.3.1 Unilateral minimizing movements

Given a time step $\delta > 0$ and an initial condition $u_0 \in H^1(\Omega)$, we consider the implicit scheme [3] [4] :

- set $u^0_\delta = u_0$ and $\rho^0_\delta = \operatorname{argmin}\{\mathcal{E}_\varepsilon(u_0, \rho) : \rho \in W^{1,p}(\Omega)\}$;
- select recursively for $i \geqslant 1$,

$$(u^i_\delta, \rho^i_\delta) \in \operatorname{argmin}\left\{\mathcal{E}_\varepsilon(u, \rho) + \frac{1}{2\delta}\|u - u^{i-1}_\delta\|^2_{L^2(\Omega)} : (u, \rho) \in H^1(\Omega) \times W^{1,p}(\Omega)\,,\ \rho \leqslant \rho_{i-1}\right\} .$$

From iterates $\{(u^i_\delta, \rho^i_\delta)\}_{i\in\mathbb{N}}$ we define the *discrete trajectory* $(u_\delta, \rho_\delta) : [0, +\infty) \to L^2(\Omega) \times L^p(\Omega)$ to be the piecewise constant interpolation given by $\big(u_\delta(t), \rho_\delta(t)\big) = (u^i_\delta, \rho^i_\delta)$ for $t \in (t_{i-1}, t_i]$.

In analogy with [14, 21], we say that $(u_\varepsilon, \rho_\varepsilon) : [0, +\infty) \to L^2(\Omega) \times L^p(\Omega)$ is a (generalized) *unilateral minimizing movement* starting from u_0 if

$$\big(u_{\delta_k}(t), \rho_{\delta_k}(t)\big) \to \big(u_\varepsilon(t), \rho_\varepsilon(t)\big) \quad \text{strongly in } L^2(\Omega) \times L^p(\Omega) \text{ for every } t \geqslant 0\,,$$

for some sequence of discrete trajectories $\{(u_{\delta_k}, \rho_{\delta_k})\}_{k\in\mathbb{N}}$ such that $\delta_k \to 0$. Setting $GUMM(\varepsilon, u_0)$ to be the collection of all unilateral minimizing movements starting from u_0, our first main result provides their existence and main qualitative properties.

3. To simplify the presentation we only consider here uniform time discretizations, see [P16] for the general case.
4. The resolution of this scheme follows from the Direct Method, noticing that $\mathcal{E}_\varepsilon$ is separately strictly convex in (u, ρ).

Theorem 2.3.1. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with $C^{1,1}$ boundary. Given $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, the collection $GUMM(\varepsilon, u_0)$ is not empty, and any $(u_\varepsilon, \rho_\varepsilon) \in GUMM(\varepsilon, u_0)$ satisfies*

$$u_\varepsilon \in H^1_{\mathrm{loc}}([0,+\infty); L^2(\Omega)) \cap L^\infty(0,+\infty; H^1(\Omega)) \cap L^2_{\mathrm{loc}}(0,+\infty; H^2(\Omega)) ,$$

$$\rho_\varepsilon \in L^\infty(0,+\infty; W^{1,p}(\Omega)) , \quad 0 \leqslant \rho_\varepsilon(t) \leqslant \rho_\varepsilon(s) \leqslant 1 \ \textit{for every } t \geqslant s \geqslant 0 .$$

In addition,

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div}\big((\eta_\varepsilon + \rho_\varepsilon^2)\nabla u_\varepsilon\big) + \beta(u_\varepsilon - g) = 0 & \textit{in } L^2(0,+\infty; L^2(\Omega)) , \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \textit{in } L^2(0,+\infty; H^{1/2}(\partial\Omega)) , \\ u_\varepsilon(0) = u_0 , \end{cases} \tag{2.3.2}$$

and

$$\begin{cases} \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) \leqslant \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho) & \textit{for every } t > 0 \textit{ and } \rho \in W^{1,p}(\Omega) \textit{ such that } \rho \leqslant \rho_\varepsilon(t) \textit{ in } \Omega , \\ \rho_\varepsilon(0) = \operatorname{argmin}\big\{\mathcal{E}_\varepsilon(u_0, \rho) : \rho \in W^{1,p}(\Omega)\big\} . \end{cases} \tag{2.3.3}$$

Moreover, $t \mapsto \mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t))$ has finite pointwise variation in $[0,+\infty)$, and there exists an (at most) countable set $\mathcal{N}_\varepsilon \subseteq (0,+\infty)$ such that

(i) $(u_\varepsilon, \rho_\varepsilon) : [0,+\infty) \setminus \mathcal{N}_\varepsilon \to H^1(\Omega) \times W^{1,p}(\Omega)$ *is strongly continuous ;*

(ii) *for every $s \in [0,+\infty) \setminus \mathcal{N}_\varepsilon$, and every $t \geqslant s$,*

$$\mathcal{E}_\varepsilon(u_\varepsilon(t), \rho_\varepsilon(t)) + \int_s^t \|\partial_t u_\varepsilon(r)\|^2_{L^2(\Omega)} \, \mathrm{d}r \leqslant \mathcal{E}_\varepsilon(u_\varepsilon(s), \rho_\varepsilon(s)) . \tag{2.3.4}$$

In this theorem, the fact that $GUMM(\varepsilon, u_0) \neq \emptyset$ follows from the strong compactness implied by a priori estimates on discrete trajectories. In particular, we have proved a global H^2-estimate for the elliptic operator in divergence form appearing in (2.3.2) (see Section 2.3.3). The same a priori estimates yield the announced regularity on the limiting trajectory $(u_\varepsilon, \rho_\varepsilon)$. In a quite classical way, the heat type equation on u_ε is established by passing to the limit in the Euler-Lagrange equations satisfied by the iterates u^i_δ. Taking advantage of a semi-group property in equation (2.3.2), we have shown that the "bulk energy"

$$t \mapsto \frac{1}{2} \int_\Omega (\eta_\varepsilon + \rho_\varepsilon(t)^2)|\nabla u_\varepsilon(t)|^2 \, \mathrm{d}x + \frac{\beta}{2} \int_\Omega (u_\varepsilon(t) - g)^2 \, \mathrm{d}x \tag{2.3.5}$$

is non-increasing, and thus continuous away from a countable set of times. In turn, the minimality property in (2.3.3) holds at the discrete level and passes to the limit by comparison arguments. It implies that the "diffuse surface energy"

$$t \mapsto \int_\Omega \left(\frac{\varepsilon^{p-1}}{p} |\nabla \rho_\varepsilon(t)|^p + \frac{\alpha_p}{\varepsilon} |1 - \rho_\varepsilon(t)|^p \right) \mathrm{d}x \tag{2.3.6}$$

is non-decreasing, hence continuous away from a countable set of times. From the monotonicity in (2.3.5) and (2.3.6) we derive the strong continuity of $(u_\varepsilon, \rho_\varepsilon)$ stated in (i). Once strong continuity is obtained, we can prove that the discrete trajectories converge strongly pointwise in time away from the exceptional set $\mathcal{N}_\varepsilon$. The energy inequality (2.3.4) is then deduced from this convergence and from the analogous inequalities for discrete trajectories.

We point out that the Lyapunov inequality (2.3.4) is reminiscent of gradient flow type equations, and that it usually reduces to equality whenever the flow is regular enough. In any case, an energy equality would be equivalent to the absolute continuity in time of the total energy, see Section 2.3.3. The reverse inequality might be obtained through an abstract infinite-dimensional chain-rule formula in the spirit of [176]. In our case, if we formally differentiate in time the total energy, we obtain

$$\frac{\mathrm{d}d}{\mathrm{d}t}\,\mathcal{E}_\varepsilon(u_\varepsilon,\rho_\varepsilon) = \left\langle \partial_u\mathcal{E}_\varepsilon(u_\varepsilon,\rho_\varepsilon), \partial_t u_\varepsilon \right\rangle + \left\langle \partial_\rho\mathcal{E}_\varepsilon(u_\varepsilon,\rho_\varepsilon), \partial_t \rho_\varepsilon \right\rangle . \tag{2.3.7}$$

From (2.3.3) we may expect that

$$\langle \partial_\rho\mathcal{E}_\varepsilon(u_\varepsilon,\rho_\varepsilon), \partial_t\rho_\varepsilon\rangle = 0\,, \tag{2.3.8}$$

which would lead, together with (2.3.2), to energy equality. Now observe that (2.3.8) is precisely the regularized version of Griffith's criterion stating that a crack evolves if and only if the release of bulk energy is compensated by the increase of surface energy (see e.g. [48, Section 2.1]). Unfortunately, such a chain-rule is not available since we do not have enough control on the time regularity of ρ_ε. In the quasi-static case, one observes discontinuous time evolutions for the surface energy. Since the evolution law for ρ_ε is quite similar to the quasi-static case (see [114]), it is reasonable to expect time discontinuities for the diffuse surface energy.

2.3.2 Asymptotics in the Mumford-Shah limit

As we already mentioned, the main motivation for considering the gradient flow of the Ambrosio-Tortorelli functional is to understand the asymptotic $\varepsilon \to 0$ and to derive a limiting evolution rule for what could be a "generalized gradient flow" of the Mumford-Shah functional. We stress that in this context the general theory on Γ-convergence of gradient flows of E. SANDIER & S. SERFATY [182, 191] does not apply since it requires a well defined gradient structure for the Γ-limit. In this direction, we believe that our results, although not completely satisfactory, will shed a new light on the problem. For completeness, let us recall that a function u belongs to $SBV^2(\Omega)$ if $u \in BV(\Omega)$ and, in the decomposition (1.5.4)-(1.5.5) of the measure Du, the absolutely continuous part $\nabla u \in L^2(\Omega)$, the Cantor part $D^c u$ vanishes, and the "jump set" J_u has a finite $\mathscr{H}^{n-1}$-measure, see [20]. Our second main result is the following [5]

Theorem 2.3.2. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with $C^{1,1}$ boundary. Let $\varepsilon_k \downarrow 0$ be an arbitrary sequence, $u_0 \in H^1(\Omega)\cap L^\infty(\Omega)$ and for each $k\in\mathbb{N}$, $(u_{\varepsilon_k},\rho_{\varepsilon_k}) \in GUMM(\varepsilon_k,u_0)$. There exist a (not relabeled) subsequence and $u \in H^1_{\mathrm{loc}}([0,+\infty);L^2(\Omega))$ such that*

$$\begin{cases} \rho_{\varepsilon_k}(t) \to 1 \;\; \textit{strongly in } L^p(\Omega) \textit{ for every } t \geqslant 0\,, \\ u_{\varepsilon_k}(t) \to u(t) \;\; \textit{strongly in } L^2(\Omega) \;\; \textit{for every } t\geqslant 0\,, \\ \partial_t u_{\varepsilon_k} \rightharpoonup \partial_t u \;\; \textit{weakly in } L^2(0,+\infty;L^2(\Omega))\,. \end{cases}$$

For every $t \geqslant 0$ the function $u(t)$ belongs to $SBV^2(\Omega)\cap L^\infty(\Omega)$,

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant \max\{\|u_0\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}\}\,,$$

and $\nabla u \in L^\infty(0,+\infty;L^2(\Omega;\mathbb{R}^n))$. Moreover, u solves

$$\begin{cases} \partial_t u - \operatorname{div}(\nabla u) + \beta(u-g) = 0 & \textit{in } L^2(0,+\infty;L^2(\Omega))\,, \\ \nabla u\cdot\nu = 0 & \textit{in } L^2(0,+\infty;H^{-1/2}(\partial\Omega))\,, \\ u(0) = u_0\,, \end{cases} \tag{2.3.9}$$

5. The symbol $\widetilde{\subseteq}$ is used for inclusion up to $\mathscr{H}^{n-1}$-null sets.

and there exists a family of $\mathscr{H}^{n-1}$-rectifiable subsets $\{\Gamma(t)\}_{t\geqslant 0}$ of Ω such that

(i) $\Gamma(s) \subseteq \Gamma(t)$ *for every* $0 \leqslant s \leqslant t$*;*

(ii) $J_{u(t)} \,\widetilde{\subseteq}\, \Gamma(t)$ *for every* $t \geqslant 0$*;*

(iii) *for every* $t \geqslant 0$,

$$\frac{1}{2}\int_\Omega |\nabla u(t)|^2 \,\mathrm{d}x + \mathscr{H}^{n-1}(\Gamma(t)) + \frac{\beta}{2}\int_\Omega (u(t)-g)^2 \,\mathrm{d}x + \int_0^t \|\partial_t u(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s \leqslant \frac{1}{2}\int_\Omega |\nabla u_0|^2 \,\mathrm{d}x + \frac{\beta}{2}\int_\Omega (u_0-g)^2 \,\mathrm{d}x\,.$$

The first part of this theorem is obtained by means of the Arzelà-Ascoli Theorem together with a general compactness result of A. Braides, A. Chambolle, & M. Solci [52]. Then the limiting equation (2.3.9) follows from (2.3.2). The family $\{\Gamma(t)\}_{t\geqslant 0}$ is essentially constructed according to the following idea. Passing to the weak* limit in the family of time-dependent measures

$$\mu_\varepsilon(t) := \left(\frac{\varepsilon^{p-1}}{p}|\nabla\rho_\varepsilon(t)|^p + \frac{\alpha_p}{\varepsilon}|1-\rho_\varepsilon(t)|^p\right)\mathscr{L}^n \llcorner \Omega\,,$$

yields a non-decreasing (in time) family of measures $\mu(t)$, thanks to the monotonicity of the diffuse surface energy (2.3.6). Each crack $\Gamma(t)$ is then obtained by taking the $\mathscr{H}^{n-1}$-rectifiable part of the set of points x where the $(n-1)$-upper density $\Theta^*_{n-1}(\mu(t),x)$ is at least 1. The key point was then to prove that $\mathscr{H}^{n-1}\llcorner J_{u(t)} \ll \mu(t)$ in order to deduce (ii). We have obtained this estimate and the energy inequality using some asymptotic lower bounds from [52] for both surface and bulk energies.

Comparing our result with the one of A. Chambolle & F. Doveri [63], we find that u solves the same generalized heat equation with an improvement in the energy inequality where an increasing family of cracks appears. The optimality of this inequality and the convergence of energies remain open questions. The (pointwise in time) convergence of the bulk energy usually follows by taking the solution itself as a test function in the equation. In our case it asks the question wether or not $SBV^2(\Omega)$ functions whose jump set is contained in $\Gamma(t)$ can be used in the variational formulation of (2.3.9). It would yield a weak form of the relation

$$\big(u^+(t)-u^-(t)\big)\frac{\partial u(t)}{\partial\nu} = 0 \quad \text{on } \Gamma(t)\,,$$

where $u^\pm(t)$ are the one-sided traces of $u(t)$ on $\Gamma(t)$. This is indeed the missing equation to complement (2.3.9), and it is intimately related to the finiteness of the unilateral slope of the Mumford-Shah functional (evaluated at $(u(t),\Gamma(t))$) defined by G. Dal Maso & R. Toader [82], see the following subsection.

2.3.3 Curves of maximal unilateral slope

An alternative approach to minimizing movements (but actually related), is to make use of the general theory of gradient flows in metric spaces introduced by E. De Giorgi, A. Marino, & M. Tosques [88]. In this setting the notion of gradient is replaced by the concept of slope, and the standard gradient flow equation is recast in terms of *curves of maximal slope*, see the recent monograph of L. Ambrosio, N. Gigli, & G. Savaré [21] for a detailed description of this subject. This point of view has been investigated by G. Dal Maso & R. Toader in [82], introducing the *unilateral slope* $|\partial\mathcal{E}_*|$ of the Mumford-Shah functional

$$|\partial\mathcal{E}_*|(u,\Gamma) := \limsup_{v\to u \text{ in } L^2(\Omega)} \frac{(\mathcal{E}_*(u,\Gamma)-\mathcal{E}_*(v,\Gamma\cup J_v))^+}{\|v-u\|_{L^2(\Omega)}}\,,$$

where $u \in SBV^2(\Omega)$ and Γ is a subset of Ω such that $\mathscr{H}^{n-1}(\Gamma) < \infty$ and $J_u \,\widetilde{\subseteq}\, \Gamma$. The main results of [82] concern explicit representations of $|\partial\mathcal{E}_*|$ and its relaxation, but a complete description is still missing.

In analogy with [82], we have introduced the unilateral slope of the Ambrosio-Tortorelli functional

$$|\partial\mathcal{E}_\varepsilon|(u,\rho) := \limsup_{v\to u \text{ in } L^2(\Omega)} \sup_{\hat\rho \leqslant \rho} \frac{\big(\mathcal{E}_\varepsilon(u,\rho) - \mathcal{E}_\varepsilon(v,\hat\rho)\big)^+}{\|v-u\|_{L^2(\Omega)}}\,,$$

where $(u,\rho) \in H^1(\Omega) \times W^{1,p}(\Omega)$. We have proved that $|\partial\mathcal{E}_\varepsilon|$ is lower semicontinuous with respect to the strong $L^2 \times L^p$-topology, but our main result provide an explicit representation formula. It rests on an aforementioned regularity estimate for elliptic operators in divergence form with $W^{1,p}$-coefficients.

Proposition 2.3.3. *Assume that Ω is a bounded open set with $C^{1,1}$ boundary. Let $D(|\partial\mathcal{E}_\varepsilon|)$ be the proper domain of $|\partial\mathcal{E}_\varepsilon|$. Then,*

$$D(|\partial\mathcal{E}_\varepsilon|) = \Big\{(u,\rho) \in H^2(\Omega)\times W^{1,p}(\Omega) : \frac{\partial u}{\partial\nu} = 0 \text{ in } H^{1/2}(\partial\Omega)\,, \text{ and } \\ \mathcal{E}_\varepsilon(u,\rho) \leqslant \mathcal{E}_\varepsilon(u,\hat\rho) \text{ for all } \hat\rho \in W^{1,p}(\Omega) \text{ such that } \hat\rho \leqslant \rho \text{ in } \Omega\Big\}\,. \tag{2.3.10}$$

In addition, if $(u,\rho) \in D(|\partial\mathcal{E}_\varepsilon|)$, then

$$|\partial\mathcal{E}_\varepsilon|(u,\rho) = \big\|\mathrm{div}((\eta_\varepsilon + \rho^2)\nabla u) - \beta(u-g)\big\|_{L^2(\Omega)}\,.$$

Together with the notion unilateral slope comes the definition of *curves of maximal unilateral slope* : a pair $(u,\rho) : [0,+\infty) \to L^2(\Omega)\times L^p(\Omega)$ is a curve of maximal unilateral slope if $u \in H^1([0,+\infty);L^2(\Omega))$, $t \mapsto \rho(t) \in L^p(\Omega)$ is non-increasing, and if there exists a non-increasing function $\lambda : [0,+\infty) \to [0,+\infty)$ satisfying

$$\lambda(t) = \mathcal{E}_\varepsilon(u(t),\rho(t)) \quad \text{and} \quad \lambda'(t) \leqslant -\frac{1}{2}\|\partial_t u(t)\|^2_{L^2(\Omega)} - \frac{1}{2}|\partial\mathcal{E}_\varepsilon|(u(t),\rho(t)) \quad \text{for a.e. } t \in (0,+\infty)\,. \tag{2.3.11}$$

This definition is motivated by the fact that any curve of maximal unilateral slope satisfies

$$(u(t),\rho(t)) \in D(|\partial\mathcal{E}_\varepsilon|) \quad \text{and} \quad \|\partial_t u(t)\|^2_{L^2(\Omega)} = |\partial\mathcal{E}_\varepsilon|(u(t),\rho(t)) \qquad \text{for a.e. } t \in (0,+\infty)\,,$$

and equality holds in (2.3.11) whenever $t \mapsto \mathcal{E}_\varepsilon(u(t),\rho(t))$ is absolutely continuous.

As a consequence of the definition, curves of maximal unilateral slope are curves of steepest $L^2(\Omega)$-descent of $\mathcal{E}_\varepsilon$ with respect to u in the direction of non-increasing ρ's. We have established that any unilateral minimizing movement is actually a curve of maximal unilateral slope. As a matter of fact, any curve satisfying (2.3.2)-(2.3.3)-(2.3.4) has maximal unilateral slope. If one drops the energy inequality (2.3.4), system (2.3.2)-(2.3.3) admits infinitely many solutions which are not in general curves of maximal unilateral slope. The other way around, the question wether or not curves of maximal unilateral slope provide solutions of (2.3.2) is actually connected to the validity of the generalized chain-rule formula (2.3.7). To conclude this section, we finally mention that we have obtained some partial results in the spirit of [82] for the limit of $|\partial\mathcal{E}_\varepsilon|$ as $\varepsilon \to 0$. However, a complete asymptotic analysis of $|\partial\mathcal{E}_\varepsilon|$ remains an open problem.

2.4 On a free boundary problem for material voids

Understanding surface roughening of materials plays a central role in many fields of physics, chemistry, and metallurgy. Since the pioneer work of R.J. Asaro & W.A. Tiller [30] (see also [192, 209] and the references therein), it has been recognized that in continuous models of crystals surface instability is driven by the competition between elastic energy and surface energy. The stress, acting parallel to a flat surface of an elastic solid, causes atoms to diffuse on the surface and the surface to undulate. In turn such a migration of atoms has an energetic prize in terms of surface tension. This phenomenon may lead to the formation of isolated islands on the substrate surface, see e.g. [195, 196, 197], or of cracks running into the bulk of the solid. Island formation in systems such as In-GaAs/GaAs or SiGe/Si turns out to be useful in the fabrication of modern semiconductor electronic and optoelectronic devices such as quantum dots laser.

Similarly, a void in a grain can collapse into a crack by surface diffusion when the applied stress exceeds a critical value [71, 113, 200, 201, 206]. Note that, since the lattice diffusion is much slower as compared to the surface diffusion, the evolving void in a grain can be assumed to conserve its volume, only changes its shape. In [201], Z. SUO & W. WANG have conducted numerical experiments on the shape change of a pore in an infinite solid. Assuming that the surface tension is isotropic and that the solid is under a uniaxial stress σ_1, they observed that the pore changes shape as the atoms diffuse on the surface driven by surface and elastic energy variation, expressed in term of the dimensionless number $\Lambda = \sigma_1^2 R_0/(Y\gamma)$, where Y is the Young's modulus, R_0 the initial circular pre radius, and γ the surface tension. Their experiments showed that under no stress, the pore has a rounded shape maintained by surface tension. On the other hand, if the applied stress is small (Λ small), the pore reaches an equilibrium shape close to an ellipse (thus compromising the stress and the surface tension), while if the applied stress Λ is large, the pore does not reach equilibrium and noses emerge, which sharpen into crack tips. Similar results were also obtained for anisotropic surface tension.

The purpose of the article [P14], in collaboration with I. FONSECA, N. FUSCO, & G. LEONI, was to formulate a simple variational model describing the competition between elastic energy and highly anisotropic surface energy for problems involving a material void in a linearly elastic solid. Following the fundamental work of C. HERRING [134], we take the surface free energy of a body to be an integral of the form $\int \varphi(\nu)\, dS$ extended over the surface of the body, where the surface energy density φ is, for anisotropic bodies, a function of the orientation of the outer unit normal ν at each surface point. The unique shape that minimizes surface energy for fixed volume is known as the *Wulff shape* [208]. The existence and uniqueness proof is originally due to J. TAYLOR [203, 204, 205] (see also [98, 100]). Under no stress, C. HERRING [134] argued that if a given macroscopic surface of a crystal does not coincide in orientation with some portion of the boundary of the Wulff shape, then there exists a hill-and-valley structure that has a lower free energy than a flat surface. On the other hand, the minimum energy configuration of the bulk material occurs at the stress-free state for each solid. Thus, at the interface between the void and the elastic solid these two opposing mechanisms compete to determine the resulting structure.

2.4.1 Variational formulation and relaxation

We now describe the model considered in [P14]. Our formulation follows M. SIEGEL, M.J. MIKSIS, & P.W. VOORHEES [192]. We consider *starshaped* cavities, which occupy closed regions $F \subseteq \mathbb{R}^2$ (with Lipschitz boundary), embedded in an elastic solid. The solid region is assumed to obey the usual laws of linear elasticity, and we consider a bulk energy of the form

$$\frac{1}{2}\int_{B_0\setminus F} \mathcal{W}\big(\mathbf{E}(u)\big)\,\mathrm{d}z\,, \qquad \mathcal{W}(\mathbf{E}) := \mathbb{T}(\mathbf{E})\cdot\mathbf{E}\,,$$

where B_0 is a fixed large ball centered at the origin, $\mathbb{T}$ is a constant positive definite fourth order tensor, and $\mathbf{E}(u)$ is the symmetrized gradient

$$\mathbf{E}(u) = \frac{1}{2}\big(\nabla u + {}^{\mathrm{t}}(\nabla u)\big)\,.$$

We assume that far from the cavity a Dirichlet condition is prescribed, i.e., $u = u_0$ a.e. in $\mathbb{R}^2 \setminus B_0$ for some given Lipschitz map $u_0 : \mathbb{R}^2 \to \mathbb{R}^2$. The surface energy is taken of the form

$$\int_{\partial F} \varphi(\nu^i_F)\,\mathrm{d}\mathscr{H}^1$$

where the anisotropy function $\varphi : \mathbb{R}^2 \to [0,\infty)$ is assumed to be convex [6], positively 1-homogeneous, and positive on $\{|z| = 1\}$. Here, ν^i_F denotes the inner normal on the surface of the cavity, that is the outer normal to the elastic body. Thus, the total energy is

$$\mathcal{E}(F,u) := \frac{1}{2}\int_{B_0\setminus F} \mathcal{W}\big(\mathbf{E}(u)\big)\,\mathrm{d}z + \int_{\partial F} \varphi(\nu^i_F)\,\mathrm{d}\mathscr{H}^1\,.$$

Given a volume $d \in (0, |B_0|)$, we have addressed the minimization problem

$$\inf\Big\{\mathcal{E}(F,u) : (F,u) \in X_{\mathrm{Lip}}\,,\ |F| = d\Big\}\,, \tag{2.4.1}$$

where the class of competitors $\mathbf{X}_{\mathrm{Lip}}$ is defined by

$$\mathbf{X}_{\mathrm{Lip}} := \Big\{(F,u) : F \in \mathcal{V}_{\mathrm{Lip}}\,,\ u \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus F;\mathbb{R}^2)\,,\ u = u_0 \text{ a.e. in } \mathbb{R}^2 \setminus F\Big\}\,,$$

and

$$\mathcal{V}_{\mathrm{Lip}} := \Big\{F \subseteq \overline{B}_0 \text{ closed, starshaped with respect to the origin, and } \partial F \text{ Lipschitz}\Big\}\,.$$

In general problem (2.4.1) is ill-posed since the limit of an arbitrary sequence of closed sets in $\mathcal{V}_{\mathrm{Lip}}$ might not belong to $\mathcal{V}_{\mathrm{Lip}}$. To efficiently tackle this minimization problem, one has to determine precisely the closure of $\mathcal{V}_{\mathrm{Lip}}$, and then relax the energy to the enlarged class of competitors.

The natural topology on $\mathcal{V}_{\mathrm{Lip}}$ is provided by the Hausdorff distance $\mathrm{dist}_{\mathscr{H}}$. In this way, $\mathcal{V}_{\mathrm{Lip}}$ appears to be a subset of the metric space made of all closed subsets of $\overline{B}_0$. This space is compact by Blaschke's theorem, and by our assumption on the surface energy, we are interested in all possible limits of sequences in $\mathcal{V}_{\mathrm{Lip}}$ with equibounded perimeter, i.e.,

$$\overline{\mathcal{V}} := \Big\{F \subseteq \overline{B}_0 \text{ such that } \mathrm{dist}_{\mathscr{H}}(F,F_k) \to 0 \text{ for some } \{F_k\}_{k\in\mathbb{N}} \subseteq \mathcal{V}_{\mathrm{Lip}} \text{ with } \sup_k \mathscr{H}^1(\partial F_k) < \infty\Big\}\,.$$

Clearly, the starshapedness property is closed under Hausdorff convergence, but the Lipschitz regularity of the boundary is in general lost in the limit. However a certain regularity is preserved, and one can prove that

$$\overline{\mathcal{V}} = \Big\{F \subseteq \overline{B}_0 \text{ closed, starshaped with respect to the origin, and } \mathscr{H}^1(\partial F) < \infty\Big\}\,.$$

The inclusion $\subseteq$ is actually a consequence of the Golab lower semicontinuity theorem, while the reverse inclusion follows from explicit constructions. In fact, for any $F \in \overline{\mathcal{V}}$, we can construct a sequence

6. In [P14] we actually do not assume that φ is convex. It leads to an intricate relaxation analysis that we prefer to avoid for clarity reasons. We only mention that the relaxed surface energy density is affected by the starshpeness condition, and it does not reduces to the convex envelope of φ.

$\{F_k\}_{k\in\mathbb{N}} \subseteq \mathcal{V}_{\rm Lip}$ with equibounded perimeter such that $F \subseteq F_k$ for every $k \in \mathbb{N}$, and $\mathrm{dist}_{\mathscr{H}}(F, F_k) \to 0$. In particular, it suggests that the class of admissible competing pairs relaxes to

$$\overline{\mathbf{X}} := \Big\{(F,u) : F \in \overline{\mathcal{V}}\,,\ u \in H^1_{\rm loc}(\mathbb{R}^2 \setminus F;\mathbb{R}^2)\,,\ u = u_0 \text{ a.e. in } \mathbb{R}^2 \setminus F\Big\}\,.$$

Accordingly, we have introduced the relaxed energy $\overline{\mathcal{E}} : \overline{\mathbf{X}} \to [0,\infty)$ defined by

$$\overline{\mathcal{E}}(F,u) := \inf\Big\{\liminf_{k\to\infty} \mathcal{E}(F_k,u_k) : \{(F_k,u_k)\}_{n\in\mathbb{N}} \subseteq \mathbf{X}_{\rm Lip}\,,\ (F_k,u_k) \xrightarrow{\mathbf{X}} (F,u)\Big\}\,,$$

where our notation $(F_k,u_k) \xrightarrow{\mathbf{X}} (F,u)$ means that

$$\begin{aligned}
&(\mathbf{a})\ \ \sup_k \mathscr{H}^1(\partial F_k) < \infty\,;\\
&(\mathbf{b})\ \ \mathrm{dist}_{\mathscr{H}}(F,F_k) \to 0\,;\\
&(\mathbf{c})\ \ u_k \rightharpoonup u \text{ weakly in } H^1(\Omega;\mathbb{R}^2) \text{ for any bounded open set } \Omega \text{ such that } \overline{\Omega} \subseteq \mathbb{R}^2 \setminus F\,.
\end{aligned} \tag{2.4.2}$$

Our first main result deals with an integral representation of the relaxed energy $\overline{\mathcal{E}}$. In order to state it properly, we first need to describe the geometry of sets in $\overline{\mathcal{V}}$. We have shown that a closed set $F \subseteq \overline{B}_0$, starshaped with respect to the origin, belongs to $\overline{\mathcal{V}}$ if and only if its *radial function* $\rho_F : \mathbb{S}^1 \to [0,\infty)$ defined by

$$\rho_F(\sigma) := \sup\{r \geqslant 0 : r\sigma \in F\}\,,$$

has finite pointwise variation on $\mathbb{S}^1$. Obviously, $F = \{r\sigma : \sigma \in \mathbb{S}^1\,, 0 \leqslant r \leqslant \rho_F(\sigma)\}$ and ρ_F is an upper semicontinuous function.

To identify the boundary of a given set $F \in \overline{\mathcal{V}}$, it is useful to introduce the functions

$$\rho_F^+(\sigma) = \max\big\{\rho_F(\sigma+), \rho_F(\sigma-)\big\} \quad \text{and} \quad \rho_F^-(\sigma) = \min\big\{\rho_F(\sigma+), \rho_F(\sigma-)\big\}\,,$$

where $\rho_F(\sigma\pm)$ denote the right and left limits of ρ_F at $\sigma \in \mathbb{S}^1$. Then,

$$\partial F = \big\{r\sigma : \sigma \in \mathbb{S}^1\,,\ \rho_F^-(\sigma) \leqslant r \leqslant \rho_F(\sigma)\big\}\,,$$

and the set

$$\Gamma_F := \big\{r\sigma : \sigma \in \mathbb{S}^1\,,\ \rho_F^+(\sigma) < r \leqslant \rho_F(\sigma)\big\} \subseteq \partial F\,,$$

is made of countably many segments.

On the other hand, it is well known that the subgraph of a BV-function has (locally) finite perimeter, see e.g. [77]. For radial functions we have proved that the same property holds, i.e., $\chi_F \in BV(\mathbb{R}^2)$ for every $F \in \overline{\mathcal{V}}$, where χ_F denotes the characteristic function of F. For such a set F, the reduced boundary $\partial^* F$ is defined as the set of points $z \in \mathrm{spt}\,|D\chi_F|$ such that the limit

$$\nu_F^i(z) := \lim_{r\downarrow 0} \frac{D\chi_F(B_r(z))}{|D\chi_F|(B_r(z))}$$

exists and satisfies $|\nu_F^i(z)| = 1$. Then $\partial^* F$ is a countably 1-rectifiable set, and $|D\chi_F| = \mathscr{H}^1 \llcorner \partial^* F$. In addition, we have proved that for any $F \in \overline{\mathcal{V}}$,

$$\partial F \simeq \partial^* F \cup \Gamma_F\,,$$

where $\simeq$ means that equality holds up to a set of vanishing $\mathscr{H}^1$-measure.

Theorem 2.4.1. *For every* $(F,u) \in \overline{\mathbf{X}}$,

$$\overline{\mathcal{E}}(F,u) = \frac{1}{2}\int_{B_0\setminus F} \mathcal{W}(\mathbf{E}(u))\,\mathrm{d}z + \int_{\partial^* F} \varphi(\nu_F^i)\,\mathrm{d}\mathscr{H}^1 + \int_{\Gamma_F} \Big(\varphi(\nu_F) + \varphi(-\nu_F)\Big)\,\mathrm{d}\mathscr{H}^1\,, \tag{2.4.3}$$

where ν_F *denotes a unit normal vector on* Γ_F*.*

Remark. The emergence of the integral over Γ_F is due to the "folding effect" necessary to create Γ_F from a sequence in $\mathcal{V}_{\mathrm{Lip}}$. If the elastic energy were not present, this term could be neglected since erasing Γ_F would yield a better competitor. However, in the present case it can be advantageous to keep it since $H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus F)$ is clearly larger than $H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \widetilde{F})$ with $\widetilde{F} = F \setminus \Gamma_F$.

Remark. In the isotropic case (i.e., $\varphi = 1$), a similar relaxation result has been first obtained in 2D for subgraphs by E. Bonnetier & A. Chambolle [44] (see also [99]). Closely related results in arbitrary dimension have also been obtained by A. Chambolle & M. Solci [65] for subgraphs in the isotropic case, and by A. Braides, A. Chambolle, & M. Solci [52] for more general sets with a convex anisotropy function.

The proof Theorem 2.4.1 is obtained in two independent steps, the upper and the lower inequalities. The upper bound is obtained by means of a suitable Moreau-Yosida regularization of the radial function as in [44], and Reshetnyak continuity theorem [7]. In turn, the lower bound relies on the "blow-up method". More precisely, given a recovery sequence $\{(F_n,u_n)\}_{n\in\mathbb{N}} \subseteq \mathbf{X}_{\mathrm{Lip}}$, we consider the weak* limit μ of the measures $\mu_n := \varphi(\nu^i_{F_n})\mathscr{H}^1 \llcorner \partial F_n$. From Reshetnyak lower semicontinuity theorem it follows that $\mu \geqslant \varphi(\nu_F^i)\mathscr{H}^1 \llcorner \partial^* F$ [8]. Then we identify the remaining part of the surface energy computing the Radon-Nikodým derivative of μ with respect to $\mathscr{H}^1 \llcorner \Gamma_F$.

As a corollary to Theorem 2.4.1, we have obtained the following existence result.

Theorem 2.4.2. *The relaxed energy* $\overline{\mathcal{E}} : \overline{\mathbf{X}} \to [0,\infty)$ *is sequentially lower semicontinuous with respect to the convergence in* (2.4.2)*, and, given* $d \in (0, |B_0|)$*, the constrained minimization problem*

$$\min\Big\{\overline{\mathcal{E}}(F,u) : (F,u) \in \overline{\mathbf{X}}\,,\ |F| = d\Big\} \tag{2.4.4}$$

admits at least one solution.

The lower semicontinuity of $\overline{\mathcal{E}}$ is essentially a straightforward consequence of the representation (2.4.3). In turn, the required compactness of minimizing sequences is a classical application of Blaschke's theorem for the sets and Korn's inequality for the functions.

2.4.2 Regularity for crystalline and strictly convex surface energies

In a second part of [P14] we have studied the regularity of the free boundary ∂F for minimizers (F,u) of the relaxed energy $\overline{\mathcal{E}}$ under the volume constraint. We have adopted a strategy first implemented by I. Fonseca, N. Fusco, G. Leoni, & M. Morini in [99], where the sets F are assumed to be subgraphs of the plane and the surface energy is isotropic. In the spirit of that paper we were able to show that volume constrained minimizers of $\overline{\mathcal{E}}$ are also unconstrained minimizers if we add to $\overline{\mathcal{E}}$ a suitable volume penalization. This allows to consider a larger class of variations and to prove an "exterior Wulff shape condition". This condition, originally introduced by A. Chambolle & C.J. Larsen [64] in the isotropic

7. If φ is not convex, one needs to perform a preliminary relaxation on the class of Lipschitz radial functions.
8. If φ is not convex, the analysis is much more involved and relies on the measure theoretic description of graphs of BV-functions in the spirit of [77, 116].

case, provides a weak estimate on the (anisotropic) curvature of the boundary. At this point that our analysis focuses on two distinct cases : the *crystalline* case and the *strictly convex* case.

We recall that the Wulff shape $\mathbf{W}$ of the anisotropy function φ is the open and convex bounded set determined by

$$\mathbf{W} := \left\{ z \in \mathbb{R}^2 : \max_{\varphi(y) \leqslant 1} y \cdot z < 1 \right\}.$$

The other way around, φ is the support function of its (closed) Wulff shape, i.e.,

$$\varphi(z) = \sup \left\{ y \cdot z : y \in \mathbf{W} \right\}.$$

The anisotropy φ is said to be crystalline if its Wulff shape is a *polygon*. The typical examples of crystalline anisotropy are given by the ℓ_1-norm or the ℓ_∞-norm, while the ℓ_p-norm with $1 < p < \infty$ has a strictly convex unit ball.

The exterior Wulff condition. As we have just mentioned, the way to tackle the regularity of optimal configurations in (2.4.4) is to show that they coincide with the free minima of a suitable energy with volume penalization. For $d \in (0, |B_0|)$ and $\ell \geqslant 0$, we have considered the modified energy

$$\overline{\mathcal{E}}_\ell(F, u) := \overline{\mathcal{E}}(F, u) + \ell \big| d - |F| \big| ,$$

and proved the following

Proposition 2.4.3. *There exists $\ell_0 \geqslant 0$ such that for all $\ell \geqslant \ell_0$, $(F_\sharp, u_\sharp) \in \overline{\mathbf{X}}$ is a minimizer of the constrained problem (2.4.4) if and only if $(F_\sharp, u_\sharp)$ is a minimizer in $\overline{\mathbf{X}}$ of $\overline{\mathcal{E}}_\ell$.*

The proof of this proposition is essentially based on comparison arguments. It resembles [99, Proposition 3.1] but its proof is much more involved due to the different geometric context.

Once Proposition 2.4.3 is obtained, the regularity problem for constrained minimizers reduces to the regularity of a free minimizer $(F_\sharp, u_\sharp)$ of $\overline{\mathcal{E}}_{\ell_0}$. The first construction of competitors consists in choosing sets F containing $F_\sharp$ and leaving $u_\sharp$ unchanged. This obviously reduces the elastic energy and we are left with the balance between the amount of surface energy we won and the amount of volume energy we lost. It is then clear that the *isoperimetric inequality* comes into play in the determination of optimal shapes. In the anisotropic setting, this inequality states that the minimum

$$\min \left\{ \int_{\partial^* E} \varphi(-\nu_E^i) \, \mathrm{d}\mathscr{H}^1 : E \subseteq \mathbb{R}^2 \text{ of finite perimeter, } |E| = |\mathbf{W}| \right\} =: c_{\mathbf{W}} |\mathbf{W}|^{1/2}$$

is uniquely achieved by $\mathbf{W}$ (up to translations and sets of vanishing measure).

In view of this isoperimetric inequality, the natural competing sets are the ones obtained by replacing a piece $\partial F_\sharp$ by a piece of boundary of (a translated and dilated) $\mathbf{W}$, provided that the resulting set is larger than $F_\sharp$. To efficiently compare surface energies, we make use of the following key estimate based on the isoperimetric inequality. In the statement we denote by $A(\sigma_1, \sigma_2)$ the open angular sector of the plane (centered at the origine) delimited by $\sigma_1 \in \mathbb{S}^1$ and $\sigma_2 \in \mathbb{S}^1$ according to the counterclockwise orientation.

Proposition 2.4.4. *There exists a constant $c_0 > 0$ such that the following holds. For $F \in \overline{\mathcal{V}}$, let $C := z_0 + \varrho_0 \mathbf{W}$ with $z_0 \in \mathbb{R}^2$ and $\varrho_0 > 0$, be such that $0 \notin \partial C$, $C \subseteq \mathbb{R}^2 \setminus F$, and $\partial C \cap \partial F$ contains at least two distinct points p_1 and p_2 with $\sigma_1 := \frac{p_1}{|p_1|} \neq \frac{p_2}{|p_2|} =: \sigma_2$. Let G be the bounded component of $A(\sigma_1, \sigma_2) \cap (\mathbb{R}^2 \setminus \overline{C})$ and let $D := G \setminus F$. Then,*

$$\int_{\partial^* D \setminus \partial C} \varphi\left(\nu_D\right) \, \mathrm{d}\mathscr{H}^1 - \int_{\partial C \cap \partial^* D} \varphi\left(\nu_C\right) \, \mathrm{d}\mathscr{H}^1 \geqslant \frac{c_0}{\varrho_0} |D| \, ,$$

where ν_D and ν_C denote the exterior normals to D and C, respectively.

By the comparison argument explained above, this proposition allowed us to show that the "contact set" between $\partial F_\sharp$ and the boundary of an exterior Wulff shape is a connected arc. More precisely, if $\varrho_0 \in (0, c_0/\ell_0)$, then for any $C = z_0 + \varrho_0 \mathbf{W} \subseteq \mathbb{R}^2 \setminus F_\sharp$, the set $\partial C \cap \partial F_\sharp$ is either empty or it is a connected arc. From this fact, we have deduced by elementary (but intricate) geometric arguments the following "pre-regularity" theorem.

Theorem 2.4.5 (Uniform exterior Wulff condition)**.** *Let $\varrho_0 \in (0, c_0/\ell_0)$, and let $(F_\sharp, u_\sharp) \in \overline{\mathbf{X}}$ be a minimizer of the penalized energy $\overline{\mathcal{E}}_{\ell_0}$. Then for all $z \in \partial F_\sharp$ there exists $w \in \mathbb{R}^2$ such that $w + \varrho_0 \mathbf{W} \subseteq \mathbb{R}^2 \setminus F_\sharp$ and $z \in \partial\,(w + \varrho_0 \mathbf{W})$.*

Regularity in the crystalline case. By Theorem 2.4.5, at each point $z_0 \in \partial F_\sharp$ there is an exterior Wulff shape of size ϱ_0 "touching" $F_\sharp$ at z_0. If $\mathbf{W}$ is assumed to be polygonal, then we can find a (maximal) solid cone of finite height with vertex at z_0 which is included in the touching Wulff shape. The opening angle α of such cone can obviously take only finitely many values $\alpha_0 < \alpha_1 < \ldots < \alpha_K = \pi$ according to the "corners" of $\mathbf{W}$. Taking advantage of this uniform exterior cones condition and assuming that $\alpha_0 > \pi/2$, we have obtained the following resulting using arguments reminiscent of the well-known "bow tie" lemma for rectifiability.

Theorem 2.4.6. *Assume that $\mathbf{W}$ is a polygon with internal angles greater than $\pi/2$. If $(F_\sharp, u_\sharp) \in \overline{\mathbf{X}}$ is a minimizer of the penalized functional $\overline{\mathcal{E}}_{\ell_0}$, then $\partial F_\sharp$ is the union of finitely many Lipschitz graphs. More precisely, $\Gamma_{F_\sharp}$ contains at most finitely many segments, and there exists a finite set $\Sigma_{\rm sing} \subseteq \partial F_\sharp \setminus \Gamma_{F_\sharp}$ such that :*

(i) *if $z \in \partial F_\sharp \setminus (\Sigma_{\rm sing} \cup \Gamma_{F_\sharp})$, then there exists a neighborhood $\mathcal{N}(z)$ of z such that $\partial F \cap \mathcal{N}(z)$ is the graph of a Lipschitz function ;*

(ii) *if $z \in \Sigma_{\rm sing} \setminus \{0\}$, then there exists a neighborhood $\mathcal{N}(z)$ of z such that $\big(\partial F_\sharp \cap \mathcal{N}(z)\big) \setminus \Gamma_{F_\sharp}$ is the union of two graphs of Lipschitz functions intersecting only at z ;*

(iii) *if $0 \in \Sigma_{\rm sing}$, then there exists a neighborhood $\mathcal{N}_0$ of 0 such that $\partial F_\sharp \cap \mathcal{N}_0$ is the union of at most six graphs of Lipschitz functions intersecting only at 0.*

Remark. The proof of this theorem only relies on elementary geometric considerations and do not make use of the elastic part of the energy. It is not clear whether of not the elastic energy could help in improving the qualitative properties of $\partial F_\sharp$. In any case, Lipschitz regularity seems to be the higher level of regularity accessible. To illustrate this, recall that, for a crystalline norm φ, there exist infinitely many geodesic curves connecting two given points, and those geodesics are in general not better than Lipschitz regular, see e.g. [163, Section 2].

Remark. Related results and questions appear in studies by L. Ambrosio, M. Novaga, & E. Paolini [24] and M. Novaga & E. Paolini [169] on "almost minimizers" of the anisotropic perimeter functional induced by φ. Here again, Lipschitz regularity of almost minimal boundaries is proved, and it is in general optimal.

Regularity in the strictly convex case. We now consider the case where the set $\{\varphi \leqslant 1\}$ is strictly convex. Under this assumption, the boundary of the Wulff shape $\mathbf{W}$ is known to be of class C^1. This situation includes the particular case where the anisotropy φ is *elliptic*, i.e., $z \mapsto \varphi(z) - \varepsilon|z|$ is convex for some $\varepsilon > 0$. If φ is elliptic, then the regularity of $\mathbf{W}$ improves up to $C^{1,1}$, see [163]. In view of Theorem 2.4.5, we may expect that a higher order regularity for $\partial F_\sharp$ holds compare to the crystalline case. For this, we needed to prove that $\partial F_\sharp$ is (away from singular points) almost minimizing the surface energy (in the additive sense). It requires good decay estimates of the elastic energy near $\partial F_\sharp$ in balls of small radius.

Such decay estimates have been proved by I. FONSECA, N. FUSCO, G. LEONI, & M. MORINI [99] for linearly isotropic materials, i.e.,

$$\mathcal{W}(\mathbf{E}) = \frac{1}{2}\lambda\left[\operatorname{tr}(\mathbf{E})\right]^2 + \mu \operatorname{tr}\left(\mathbf{E}^2\right) , \tag{2.4.5}$$

where λ and μ are the (constant) Lamé moduli with $\mu > 0$, $\mu + \lambda > 0$.

In a first step, we have obtained a result similar to Theorem 2.4.6 by means of the exterior Wulff condition. In this preliminary result, the additional C^1-regularity of $\mathbf{W}$ implies the existence of right and left tangents at every point of $\partial F_\sharp \setminus \Gamma_{F_\sharp}$. It also allows to define a set $\Sigma_{\text{cusp}} \subseteq \partial F_\sharp \setminus \Gamma_{F_\sharp}$ of "cusp points" : a point $z \in \partial F_\sharp \setminus \Gamma_{F_\sharp}$ is a cusp point if there exist $w_1, w_2 \in \mathbb{R}^2$ such that $(w_1 + \varrho_0 \mathbf{W}) \cap (w_2 + \varrho_0 \mathbf{W}) = \emptyset$ and $z \in \partial(w_1 + \varrho_0 \mathbf{W}) \cap \partial(w_2 + \varrho_0 \mathbf{W})$. Note that Σ_{cusp} must contain the countable set $\overline{\Gamma}_{F_\sharp} \setminus \Gamma_{F_\sharp}$.

Using the aforementioned decay estimates on the elastic part of the energy, we have then proved by comparison arguments that it always favorable to "cut corners". It shows that away from cusp points, left and right tangents to $\partial F_\sharp$ have to agree. The C^1-regularity of $\mathbf{W}$ then yields the continuity of these tangents.

Theorem 2.4.7. *Assume that $\{\varphi \leqslant 1\}$ is strictly convex and that (2.4.5) holds. Let $(F_\sharp, u_\sharp) \in \overline{\mathbf{X}}$ be a minimizer of the penalized functional $\overline{\mathcal{E}}_{\ell_0}$. There exists a finite set $\Sigma_{\text{cusp}} \subseteq \partial F_\sharp \setminus (\Gamma_{F_\sharp} \cup \{0\})$ such that*

(i) *if $z \notin \Gamma_{F_\sharp} \cup \Sigma_{\text{cusp}}$ and $z \in B_0 \setminus \{0\}$, then there exists a neighborhood $\mathcal{N}(z)$ of z such that $\partial F_\sharp \cap \mathcal{N}(z)$ coincides with the graph of a C^1-function ;*

(ii) *if $0 \in \partial F_\sharp$, then there exists a neighborhood $\mathcal{N}_0$ of 0 such that $\partial F_\sharp \cap \mathcal{N}_0$ is the union of at most two graphs of Lipschitz functions intersecting only at 0, and $\partial F_\sharp$ admits at most two tangents at 0 forming an angle of at least π ;*

(iii) *if $z \in \Sigma_{\text{cusp}}$, then there exist $\delta > 0$ and two Lipschitz functions $h, g : (|z| - \delta, |z|\,] \to \mathbb{R}$, left-differentiable at $|z|$, satisfying $g \leqslant 0 \leqslant h$, $h(|z|) = g(|z|) = 0$, $h(t) > g(t)$ for $t \in (|z| - \delta, |z|)$ and $h'_-(|z|) = g'_-(|z|) = 0$, and such that*

$$\left\{ t\frac{z}{|z|} + g(t)\frac{z^\perp}{|z|} : t \in (|z| - \delta, |z|\,] \right\} \cup \left\{ t\frac{z}{|z|} + h(t)\frac{z^\perp}{|z|} : t \in (|z| - \delta, |z|\,] \right\}$$

coincides with $\partial F_\sharp \setminus \Gamma_{F_\sharp}$ in an open neighborhood of z.

Near a point $z \in \partial F_\sharp$ where $\partial F_\sharp$ is C^1, the decay estimates on the elastic energy of [99] improves. If the anisotropy φ is assumed to be elliptic, those estimates imply that the amount of length of $\partial F_\sharp$ in small balls $B_r(z)$ is order $2r + O(r^\beta)$ for every $\beta > 1$. This classically yields the $C^{1,\alpha}$-regularity of $\partial F_\sharp$ for every $\alpha \in (0, 1/2)$ in a neighborhood of z.

Theorem 2.4.8. *Assume that φ is elliptic and that (2.4.5) holds. Let $(F_\sharp, u_\sharp) \in \overline{\mathbf{X}}$ be a minimizer for the penalized functional $\overline{\mathcal{E}}_{\ell_0}$. If $z \in \partial F_\sharp \cap B_0 \setminus \left(\Sigma_{\text{cusp}} \cup \Gamma_{F_\sharp}\right)$ and $z \neq 0$, then $\partial F_\sharp$ coincides in a neighborhood of z with the graph of a function of class $C^{1,\alpha}$ for every $0 < \alpha < 1/2$.*

Remark. In Theorem 2.4.8, the ellipticity assumption on φ can be strongly weakened. In fact, the comparison arguments used to prove this theorem shows that near regular points, $\partial F_\sharp$ is an almost-minimal set of the (1-dimensional) surface energy with a gauge of almost-minimality of power type. From this property, we can obtained a quantitative C^1-regularity of $\partial F_\sharp$ in terms of the modulus of convexity of the set $\{\varphi \leqslant 1\}$, provided that this modulus of convexity satisfies a suitable Dini condition (see Section 2.5.3). This kind of issue is actually the object of the next and last section.

2.5 Almost minimal 1-sets in anisotropic spaces

The article [P19], in collaboration with T. De Pauw & A. Lemenant, contributes to the study of one dimensional geometric variational problems in an ambient Banach space X. We have addressed both existence and partial regularity issues. The paradigmatic *weighted Steiner problem* is

$$(\mathscr{P}) \quad \begin{cases} \text{minimize } \int_\Gamma w \, \mathrm{d}\mathscr{H}^1 \\ \text{among compact connected sets } \Gamma \subseteq X \text{ containing } F \,. \end{cases}$$

Here $\mathscr{H}^1$ denotes the one dimensional Hausdorff measure (relative to the metric of X), $w : X \to (0, +\infty]$ is a weight, and F is a finite set implementing the boundary condition.

Assuming that problem $(\mathscr{P})$ admits finite energy competing sets, we have proved existence of a minimizer in case X is the dual of a separable Banach space, and w is weakly* lower semicontinuous and bounded away from zero. Ideas on how to circumvent the lack of compactness that ensues from X being possibly infinite dimensional go back to M. Gromov [123], and have been implemented by L. Ambrosio & B. Kirchheim [22] in the context of metric currents, as well as by L. Ambrosio & P. Tilli [26] in the context of the Steiner problem (with $w \equiv 1$). The novelty here is to allow for a varying weight w through a relevant lower semicontinuity result for the weighted length.

In studying the regularity of a minimizer Γ of problem $(\mathscr{P})$, we have regarded Γ as a member of the larger class of *almost minimizing* sets. Our definition is less restrictive than that of F.J. Almgren [11] who first introduced the concept. A gauge is a nondecreasing function $\xi : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+$ such that $\xi(0+) = 0$. We say a compact connected set $\Gamma \subseteq X$ of finite length is (ξ, r_0)-almost minimizing in an open set $\Omega \subseteq X$ whenever the following holds : for every $x \in \Gamma \cap \Omega$, every $0 < r \leqslant r_0$ such that $\overline{B}(x, r) \subseteq \Omega$, and every compact connected $\Gamma' \subseteq X$ with

$$\Gamma \setminus \overline{B}(x,r) = \Gamma' \setminus \overline{B}(x,r)$$

one has

$$\mathscr{H}^1\big(\Gamma \cap \overline{B}(x,r)\big) \leqslant \big(1 + \xi(r)\big)\mathscr{H}^1\big(\Gamma' \cap \overline{B}(x,r)\big) \,.$$

One easily checks that if Γ is a solution of $(\mathscr{P})$ then it is (ξ, ∞)-almost minimizing in $\Omega = X \setminus F$, where ξ is (related to) the oscillation of the weight w. For instance if w is Hölder continuous of exponent α then $\xi(r)$ behaves asymptotically like r^α near $r = 0$.

In order to appreciate the hypotheses of our regularity results, we now make elementary observations. In case $\operatorname{card} F = 2$ and w is bounded from above and from below by positive constants, each minimizer Γ of $(\mathscr{P})$ is actually a minimizing geodesic curve with respect to the conformal metric induced by w, with endpoints those of F. Since $\mathscr{H}^1(\Gamma) < \infty$ we infer that Γ is a Lipschitz curve. In general not much more regularity seems to ensue from the minimizing property of Γ. Indeed in the plane $X = \ell^2_\infty$ with $w \equiv 1$, every 1-Lipschitz graph over one of the coordinate axes is length minimizing, as the reader will happily check. However if X is a rotund [9] Banach space, then Γ must be a straight line segment. Finally, in case w is merely Hölder continuous the Euler-Lagrange equation for geodesics cannot be written in the classical or even weak sense, and our regularity results, providing quantitative C^1-regularity (see Section 2.5.3), do not seem to entail from ODE or PDE arguments, even when the ambient space $X = \ell^2_2$ is the Euclidean plane.

9. or strictly convex

2.5.1 Existence for the weighted Steiner problem in Banach spaces

One of our aim was to solve the minimization problem $(\mathscr{P})$ following the Direct Method of Calculus of Variations. To do so, we had to investigate the compactness properties of minimizing sequences, and accordingly, to determine the lower semicontinuity of the length energy. A main assumption for existence is that X is the dual of a separable Banach space, with norm $\|\cdot\|$. Its closed unit ball $\overline{B}_X$ equipped with the restriction of the weak* topology of X is a compact separated topological space. It is metrizable as well, owing to the separability of a predual of X, and we let d^* denote any metric on $\overline{B}_X$ compatible with its weak* topology [10].

It is clear that a first compactness property for problem $(\mathscr{P})$ comes from the Blaschke selection principle applied to the compact metric space $(\overline{B}_X, d^*)$. In particular, if we denote by $\operatorname{dist}^*_{\mathscr{H}}$ the corresponding Hausdorff distance, every sequence $\{\Gamma_k\}_{k\in\mathbb{N}}$ of compact subsets $\overline{B}_X$ admits a subsequence $\{\Gamma_{k(j)}\}_{j\in\mathbb{N}}$ such that $\operatorname{dist}^*_{\mathscr{H}}(\Gamma_{k(j)}, \Gamma) \to 0$ for some closed set $\Gamma \subseteq \overline{B}_X$.

Theorem 2.5.1. *Assume that*

(a) $\{\Gamma_k\}_{k\in\mathbb{N}}$ *is a sequence of nonempty compact connected subsets of* $\overline{B}_X$ *;*

(b) $\operatorname{dist}^*_{\mathscr{H}}(\Gamma_k, \Gamma) \to 0$ *for some nonempty closed subset* Γ *of* $\overline{B}_X$ *;*

(c) $w : \overline{B}_X \to (0, +\infty]$ *is weakly* lower semicontinuous and* $\sup_k \int_{\Gamma_k} w \, \mathrm{d}\mathscr{H}^1 < \infty$.

It follows that

(i) Γ *is compact and connected ;*

(ii) $\int_\Gamma w \, \mathrm{d}\mathscr{H}^1 \leqslant \liminf_k \int_{\Gamma_k} w \, \mathrm{d}\mathscr{H}^1$ *;*

(iii) $F \subseteq \Gamma$ *whenever* $F \subseteq \Gamma_k$ *for every* $k = 1, 2, \ldots$.

Remark. If the function w fails to be weakly* lower semicontinuous, conclusion (ii) does not need to hold, as the following counterexample shows. Denote by $\{e_k\}_{k=1}^\infty$ the canonical orthonormal basis of $X = \ell_2$, and define $w : X \to [1, 2]$ by $w(x) := \max\{1, 2 - 8 \operatorname{dist}(x, \operatorname{span}\{e_1\})\}$. Then consider the sequence $\{\Gamma_k\} \subseteq \overline{B}_X$ of compact connected sets $\Gamma_k := \gamma_k([0,1])$ where

$$\gamma_k(t) := \begin{cases} te_k & \text{for } 0 \leqslant t \leqslant 1/8\,, \\ \frac{1}{8}e_k + (t - \frac{1}{8})e_1 & \text{for } 1/8 < t \leqslant 7/8\,, \\ (1-t)e_k + \frac{3}{4}e_1 & \text{for } 7/8 < t \leqslant 1\,. \end{cases}$$

One easily checks that (b) holds with $\Gamma = [0, \frac{3}{4}e_1]$. On the other hand we have $\int_{\Gamma_k} w \, \mathrm{d}\mathscr{H}^1 = 9/8$ for every $k = 1, 2, \ldots$, while $\int_\Gamma w \, \mathrm{d}\mathscr{H}^1 = 3/2 > 9/8$.

Part of the proof of Theorem 2.5.1 follows the argument of L. Ambrosio & P. Tilli [26] in the case $w = 1$. Noticing the uniform bound on the weighted length implies that the family $\{\Gamma_k\}$ is equicompact, we can apply the Gromov compactness theorem. Up to a subsequence, it provides a compact metric space (Z, d_Z) and isometric embeddings $i_k : (\Gamma_k, \|\cdot\|) \to (Z, d_Z)$ such that the compact and connected sets $G_k := i_k(\Gamma_k)$ converge for the Hausdorff metric in Z to some compact and connected set $G \subseteq Z$. Passing to the Hausdorff limit in the graphs of the (1-Lipschitz) mappings $j_k := i_k^{-1} \llcorner G_k$, one can show that $\Gamma = j(G)$ for some 1-Lipschitz map $j : G \to \overline{B}_X$, and both compactness and connectedness of Γ follow.

10. We consider two metrizable topologies on $\overline{B}_X$: that induced by the norm of X, and that induced by the weak* topology of X. When we refer to *closed* (respectively *compact*) subsets $\Gamma \subseteq \overline{B}_X$ we always mean strongly closed (respectively compact), i.e., with respect to the norm topology of X.

To establish the lower semicontinuity of the weighted length, we have applied the blow-up method to the sequence of Borel measures on Z,

$$\mu_k(B) := \int_{B \cap G_k} w\big(j_k(z)\big)\, \mathrm{d}\mathscr{H}^1_Z(z)\,.$$

More precisely, assuming in addition that w is Lipschitz continuous, we have (essentially) proved that the one dimensional density of any weak* limit is greater than $w\big(j(z)\big)$ at every loint $z \in G$. Since j is a 1-Lipschitz map, this yields the announced inequality. Finally, the case of a general weakly* continuous weight reduces to the Lipschitz case by means of a Moreau-Yosida regularization.

From Theorem 2.5.1 we have classically deduced

Corollary 2.5.2. *Assume that $w : X \to (0, +\infty]$ is weakly* lower semicontinuous, and that $\inf_X w > 0$. Let $F \subseteq X$ be a nonempty finite set, and let $\mathscr{C}_F$ denote the collection of all compact connected sets $\Gamma \subseteq X$ such that $F \subseteq \Gamma$. If $\inf_{\mathscr{C}_F} \int_\Gamma w\, \mathrm{d}\mathscr{H}^1 < \infty$, then the variational problem $(\mathscr{P})$ admits at least one solution.*

Remark. If $w : X \to [a, b]$ for some $0 < a, b < \infty$, then the finiteness assumption above holds. Indeed if $F = \{x_0, x_1, \ldots, x_J\}$ we let $\Gamma_0 = \cup_{j=1}^J [x_0, x_j]$, so that $\Gamma_0 \in \mathscr{C}_F$ and $\int_{\Gamma_0} w\, \mathrm{d}\mathscr{H}^1 \leqslant b \sum_{j=1}^J \|x_j - x_0\|$. In addition, any solution Γ of problem $(\mathscr{P})$ is (ξ, ∞)-almost minimizing in $X \setminus F$, relative to the gauge

$$\xi(r) = \operatorname{osc}(w, r) \left(\frac{a+b}{a^2} \right)\,,$$

where $\operatorname{osc}(w, r)$ is the *oscillation* of w at scale $r > 0$ defined by

$$\operatorname{osc}(w, r) := \sup \big\{ |w(x_1) - w(x_2)| : x_1, x_2 \in X \text{ and } \|x_1 - x_2\| \leqslant r \big\}\,.$$

Note that $\lim_{r \to 0^+} \operatorname{osc}(w, r) = 0$ if and only if w is uniformly continuous.

2.5.2 Almost minimal 1-sets in arbitrary Banach spaces

We now report on some properties of sets Γ which are (ξ, r_0)-almost minimizing in some open set Ω of a general Banach space X. It is convenient – but not always necessary – to assume that the gauge ξ verifies a Dini growth condition, specifically that

$$\zeta(r) := \int_0^r \frac{\xi(t)}{t}\, \mathrm{d}t < \infty\,, \tag{2.5.1}$$

for each $r > 0$. We have shown that for each $x \in \Gamma \cap \Omega$ the weighted density ratio

$$\exp[\zeta(r)] \frac{\mathscr{H}^1\big(\Gamma \cap \overline{B}(x, r)\big)}{2r}$$

is a nondecreasing function of $0 < r \leqslant \min\{r_0, \operatorname{dist}(x, X \setminus \Omega)\}$. Its limit as $r \downarrow 0$, denoted $\Theta^1(\mathscr{H}^1 \llcorner \Gamma, x)$, verifies the following dichotomy :

$$\textbf{either } \Theta^1(\mathscr{H}^1 \llcorner \Gamma, x) = 1 \quad \textbf{or } \Theta^1(\mathscr{H}^1 \llcorner \Gamma, x) \geqslant 3/2\,. \tag{2.5.2}$$

We have established that the set of points $x \in \Gamma$ where this density equals 1 characterizes the "regular part" of Γ, i.e., where Γ is locally a Lipschitz curve. To be more precise, we say that :

(a) x is a *regular point* of Γ if for each $\delta > 0$, there exists $0 < r < \delta$ such that $\Gamma \cap \overline{B}(x,r)$ is a Lipschitz curve $\boldsymbol{\gamma}$ and $\Gamma \cap \partial B(x,r)$ consists of the two endpoints of $\boldsymbol{\gamma}$ [11] ;

(b) x is a *singular point* of Γ if it is not a regular point of Γ.

The *set of regular points* of Γ is denoted $\operatorname{reg}(\Gamma)$, and the *set of singular points* is $\operatorname{sing}(\Gamma) := \Gamma \setminus \operatorname{reg}(\Gamma)$.

Theorem 2.5.3. *Assume that :*

(a) $\Gamma \subseteq X$ *is compact and connected,* $\Omega \subseteq X$ *is open,* $r_0 > 0$ *;*

(b) ξ *is a Dini gauge, i.e.,* (2.5.1) *holds ;*

(c) Γ *is* (ξ, r_0) *almost minimizing in* Ω.

It follows that $\operatorname{reg}(\Gamma) \cap \Omega = \Omega \cap \{x : \Theta^1(\mathscr{H}^1 \llcorner \Gamma, x) = 1\}$, *that* $\operatorname{sing}(\Gamma) \cap \Omega$ *is relatively closed in* $\Gamma \cap \Omega$, *and that* $\mathscr{H}^1(\operatorname{sing}(\Gamma) \cap \Omega) = 0$.

In this theorem, the closedness of $\operatorname{sing}(\Gamma) \cap \Omega$ comes from the upper semicontinuity of the density function $x \mapsto \Theta^1(\mathscr{H}^1 \llcorner \Gamma, x)$ together with (2.5.2). In turn the fact that $\operatorname{sing}(\Gamma) \cap \Omega$ has a vanishing $\mathscr{H}^1$-measure is a consequence of a result of B. KIRCHHEIM [141] noticing that Γ is rectifiable as a compact and connected set of finite length, see [26, Theorem 4.4.8].

2.5.3 Partial regularity in uniformly rotund spaces

We now present an improvement on the regularity of $\operatorname{reg}(\Gamma)$ in case the ambient Banach space X is uniformly rotund [12]. We recall that the *modulus of rotundity* of X is the gauge function $\varepsilon \in (0,2] \to \delta_X(\varepsilon)$ defined by

$$\delta_X(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in X, \max\{\|x\|, \|y\|\} \leqslant 1 \text{ and } \|x-y\| \geqslant \varepsilon \right\} ,$$

and X is called uniformly rotund if $\delta_X(\varepsilon) > 0$ for every $0 < \varepsilon \leqslant 2$.

We define for $t > 0$,

$$\delta_X^{-1}(t) := \sup \left\{ \varepsilon > 0 : \delta_X(\varepsilon) \leqslant t \right\} .$$

The gauge δ_X^{-1}, particularly its growth, is the relevant quantity for our regularity theory.

Theorem 2.5.4. *Assume that :*

(a) $\Gamma \subseteq X$ *is compact and connected,* $\Omega \subseteq X$ *is open,* $r_0 > 0$, $x_0 \in \Gamma$, *and* $\overline{B}(x_0, r_0) \subseteq \Omega$ *;*

(b) ξ *is a gauge and the gauge* $\delta_X^{-1} \circ \xi$ *is Dini ;*

(c) Γ *is* (ξ, r_0) *almost minimizing in* Ω *;*

(d) $\Theta^1(\mathscr{H}^1 \llcorner \Gamma, x_0) = 1$.

It follows that there exists $r > 0$ *such that* $\Gamma \cap \overline{B}(x_0, r)$ *is a* C^1 *curve* $\boldsymbol{\gamma}$. *Furthermore if* g *is an arclength parametrization of* $\boldsymbol{\gamma}$ *then*

$$\operatorname{osc}(g', t) \leqslant C \int_0^{Ct} \frac{\delta_X^{-1} \circ \xi(t)}{t} \, dt ,$$

for some constant $C > 0$.

---

11. A *curve* is a topological line segment, i.e., a set $\boldsymbol{\gamma} \subseteq X$ of the type $\boldsymbol{\gamma} = g([a,b])$ where $a < b$ and $g : [a,b] \to X$ is an injective continuous map. We call $g(a)$ and $g(b)$ the *endpoints* of $\boldsymbol{\gamma}$.

12. or uniformly convex

Remark. Since $\delta_X(\varepsilon) \leqslant O(\varepsilon^2)$ (see e.g. [151, Chapter E]), we have $t \leqslant \sqrt{t} \leqslant C\delta_X^{-1}(t)$ for every $t \leqslant 1$. Therefore, if $\xi(t) \leqslant 1$, then $\xi(t) \leqslant \delta_X^{-1} \circ \xi(t)$. In particular, assumption (b) implies that ξ is a Dini gauge, and Theorem 2.5.3 applies.

Remark. If $\xi(r) \simeq r^\alpha$ for some $\alpha > 0$, then assumption (b) reduces to δ_X^{-1} being Dini.

Remark. If X is a Hilbert space, then $\delta_X(\varepsilon) = \varepsilon^2/8 + o(\varepsilon^2)$ as $\varepsilon \to 0$. Hence Theorem 2.5.4 requires $\sqrt{\xi}$ to be Dini. If $\xi(r) \simeq r^\alpha$ for some $\alpha > 0$, it implies that $\boldsymbol{\gamma}$ is a $C^{1,\alpha/2}$ curve. More generally, if X is an L_p space, $1 < p < \infty$, relative to any measure space, then $\delta_X(\varepsilon) = C_p\varepsilon^{\max\{2,p\}} + o(\varepsilon^{\max\{2,p\}})$, see [151, Chapter E], in which case $\boldsymbol{\gamma}$ is a $C^{1,\alpha/\max\{2,p\}}$ curve.

The proof of Theorem 2.5.4 is technically quite involved, but the geometric idea behind it is rather simple. From Theorem 2.5.3 we can find sequences $r_j \downarrow 0$ such that $\Gamma \cap \overline{B}(x_0, r_j)$ is a Lipschitz curve γ_j with endpoints x_j^- and x_j^+ on $\partial B(x_0, r_j)$. Considering the affine line L_j containing x_j^- and x_j^+, one has to show that γ_j does not wander too far away from L_j. Suppose $\max_{z\in\Gamma_j} \operatorname{dist}(z, L_j) = h_j r_j$ and the maximum is achieved at $z \in \gamma_j$. The triangle inequality implies $\mathscr{H}^1(\gamma_j) \geqslant \|z - x_j^-\| + \|x_j^+ - z\|$. As X is uniformly rotund, the latter is quantitatively larger than the length of the straight line segment joining x_j^- and x_j^+. Specifically, we have proved that

$$\|z - x_j^-\| + \|x_j^+ - z\| \geqslant \|x_j^+ - x_j^-\| \left(1 + \delta_X(Ch_j)\right).$$

On the other hand, the almost minimizing property of Γ says that $\mathscr{H}^1(\gamma_j) \leqslant (1 + \xi(r_j))\|x_j^+ - x_j^-\|$. It now becomes clear that h_j cannot be too large, in fact $h_j \leqslant C(\delta_X^{-1} \circ \xi)(r_j)$, which in turns yields the Hausdorff distance estimate

$$\operatorname{dist}_{\mathscr{H}}\left(\gamma_j, L_j \cap \overline{B}(x_0, r_j)\right) \leqslant C(\delta_X^{-1} \circ \xi)(r_j).$$

Upon noticing that the radii r_j can be chosen in near geometric progression, we infer that the sequence of affine secant lines $\{L_j\}$ is Cauchy provided $\sum_j (\delta_X^{-1} \circ \xi)(2^{-j}) < \infty$. The fact that the relevant inequalities are also locally uniform in x then yields C^1 regularity under the assumption that δ_X and ξ verify the Dini growth condition.

2.5.4 Differentiability in 2-dimensional rotund spaces

To enlighten again the reader on the regularity result in Theorem 2.5.4, it is perhaps worth noting that even in the finite dimensional setting $X = \ell_p^n$, $2 < p < \infty$, the problem is not "elliptic", or rather the metric is not Finslerian, as the smooth unit sphere $\mathbb{S}_{\ell_p^n}$ has vanishing curvature at $\pm e_1, \dots, \pm e_n$. In fact, in case X is finite dimensional and the unit sphere $\mathbb{S}_X$ is C^∞ smooth, the Dini condition on δ_X^{-1} may be understood as a condition on the order of vanishing of

$$f_v : T_v\mathbb{S}_X \to \mathbb{R} : h \mapsto \|v + h\| - 1\,, \qquad v \in \mathbb{S}_X\,.$$

With this in mind, we have shown how to completely dispense with the Dini condition on δ_X^{-1} in case $\dim X = 2$, and the norm of X is rotund and smooth. The relevant regularity states that $\operatorname{reg}(\Gamma)$ is made of *differentiable* curves. The question wether or not this regularity is optimal is a very intriguing (and difficult) question.

Theorem 2.5.5. *Let X be a 2 dimensional Banach space whose norm is rotund and of class C^2 on $X \setminus \{0\}$. Assume that*

(a) *$\Gamma \subseteq X$ is compact and connected, $\Omega \subseteq X$ is open, $x_0 \in \Gamma$, $r_0 > 0$, and $\overline{B}(x_0, r_0) \subseteq \Omega$;*

(b) ξ *is a gauge and* $\sqrt{\xi}$ *is Dini ;*

(c) Γ *is* (ξ, r_0) *almost minimizing in* Ω *;*

(d) $\Theta^1(\mathscr{H}^1 \llcorner \Gamma, x_0) = 1$.

It follows that there exists $r > 0$ *such that* $\Gamma \cap \overline{B}(x_0, r)$ *is a differentiable curve.*

In order to prove this theorem, we have localized the modulus of continuity $\delta_X(v; \varepsilon)$ relative to each direction $v \in \mathbb{S}_X$. The key point is to observe that the subset $G = \mathbb{S}_X \cap \{v : \partial^2_{h,h} f_v(0) > 0\}$ is relatively open in $\mathbb{S}_X$, and that its complement $\mathbb{S}_X \setminus G$ is nowhere dense because the norm is rotund, i.e., $\mathbb{S}_X$ contains no line segment. Furthermore, if $v \in G$ then $\delta_X(v; \varepsilon) \geqslant c(v)\varepsilon^2$, the best case scenario for regularity. To prove the differentiability at $x_0 \in \operatorname{reg}(\Gamma)$ we need only to establish that the set of tangent lines $\operatorname{Tan}(\Gamma, x_0)$ is a singleton. This set is connected, according to D. PREISS [170]. Thus either $L \in \operatorname{Tan}(\Gamma, x_0) \cap G \neq \emptyset$ and we can run the regularity proof of Theorem 2.5.4 "in a cone about L", or $\operatorname{Tan}(\Gamma, x_0) \subseteq \mathbb{S}_X \setminus G$ and therefore $\operatorname{Tan}(\Gamma, x_0)$ is a singleton.

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