Boundary integral equations of time harmonic wave scattering at complex structures

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Contents

Foreword

The present report provides an overview of my research activity since the end of my PhD thesis in 2008. Although this research is structured according to two distinct topics - boundary integral equations for electromagnetic scattering, and asymptotic modelling of singularly perturbed wave scattering - I deliberately chose to dedicate the most part of this document to boundary integral equations, because it clearly departs from my PhD thesis that concerned asymptotic modelling of electromagnetic scattering at thin wires, and also because it forms a more structured set of contributions than the remaining part of my work. One chapter will nevertheless present other projects related to asymptotic modelling. In this introduction, I would like to point a few contextual facts that played an important role in the work to be presented here.

My activity on boundary integral equations began with a postdoctoral stay at the Seminar of Applied Mathematics (SAM) at ETH Zürich where a collaboration had been initiated by Ralf Hiptmair with the radio frequency modelling department of Thales Airborne System in Élancourt (France). The initial goal was to devise and analyse boundary integral formulations fulfilling three requirements: it should

i) be adapted to multi-subdomain scattering,
ii) lend itself to Calderón preconditioning techniques,
iii) be a convenient starting point for domain decomposition.

Although my postdoctoral stay was one year long, collaboration continued since then. Trying to answer the points above first "accidentally" led to introduce the second kind formulation presented in Chapter 2. Work on this formulation fuelled further collaboration with Ralf Hiptmair at ETH Zürich, and led me to co-supervise the PhD thesis of Elke Spindler dedicated to this subject. This formulation was also the main topic of the internship of Hadrien Montanelli (student at ISAE-Supaero in Toulouse) that I supervised during Spring 2012.

Though appealing in other respects, this formulation was not a good candidate for domain decomposition. This motivated the search for an alternative approach, and led to the multi-trace formalism presented in Chapter 3. Although much analysis and preliminary numerical works have been now produced on multi-trace formalism, there is still a lot to be done for exploiting its potentialities for high performance computing, and in particular domain decomposition. Human resources (PhD students or postdoctoral fellows) would be desirable in this respect which led me to submit, as a principal investigator, a research project called NonLocalDD to the French National Research Agency (ANR). This project focuses on integral equation based domain decomposition. It has been accepted in July 2015 and will be 4 years

long. Besides myself it involves the INRIA teams Alpines (L.Grigori and F.Nataf), POems (P.Joly, S.Chaillat and F.Collino), and Magique3D (M.Durufle).

In connection with multi-trace formulations, I supervised Hassan Frissane (student at Univ. Versailles) and Alan Ayala (student at UPMC) for master internships during Spring 2015. Together with Laura Grigori from INRIA team Alpines, I presently co-supervise the PhD thesis of Alan on this same subject.

To conclude, since the end of 2012, together with the INRIA/ENSTA/CNRS team POems (S.Fliss, P.Joly, A.-S. Bonnet-Bendhia, P.Ciarlet, L.Chesnel), the INRIA team DEFI (H.Haddar), and the university of Toulon (G.Bouchitté), I have been a regular participant of another ANR project, entitled METAMATH, that focuses on the modelling of wave propagation in metamaterials. My contributions to this project will be reported in Chapter 5. These are new types of material consisting in periodic assemblies of small resonators, where the size of the periodicity cell is much smaller than the wavelength, and where the resonators are chosen so that the effective propagation characteristics of the medium admit negative real parts. In the context of the METAMATH project, I took part in the supervision of Valentin Vinoles that prepared his PhD thesis at UMA ENSTA and is now ATER at UPMC.

Outline of the thesis The first chapter will be a brief recapitulation of well known results concerning layer potentials in the context of wave propagation in harmonic regime. In Chapter 2, we give an overview of the Rumsey reaction principle that is the most popular boundary integral formulation for multi-subdomain scattering, and we present a new alternative integral formulation that seems to be the first boundary integral formulation of the second kind for multi-subdomain scattering in geometrical configurations involving junction points. Chapter 3 is dedicated to the multi-trace formalism which is a completely new approach to boundary integral formulation of multi-subdomain scattering: we briefly describe the local multi-trace formulation, and describe in detail the derivation of the global multi-trace formulation developed by us, as well as its sparsified counterpart that we dubbed quasi-local multi-trace formulation. In Chapter 4 we present a new functional framework adapted to well-posed boundary integral equations for scattering by particular types of object dubbed multi-screens as they take the form of arbitrary arrangements of thin panels of impenetrable material. In Chapter 5 we describe several works on asymptotic modelling in the context wave propagation in harmonic regime. The last chapter presents research perspectives.

Chapter 1

A brief review of integral equations for a single scatterer

The present chapter is a preparatory one. It aims at settling a few notations corresponding to classical concepts related to boundary integral equations. It will provide a common language useful for the subsequent chapters. We shall focus here on Helmholtz equation with wave number $\kappa \in \mathbb{C}$. Wave equations will be posed in a Lipschitz domain $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$. In addition, we will assume that that either Ω itself or $\mathbb{R}^d \setminus \overline{\Omega}$ is bounded and denote $\Gamma := \partial\Omega$.

1.1 Function spaces

We first introduce standard notations, setting $\mathrm{H}^1(\Omega) := \{v \in \mathrm{L}^2(\Omega) \mid \nabla v \in \mathrm{L}^2(\Omega)\}$ equipped with $\|v\|^2_{\mathrm{H}^1(\Omega)} := \|\nabla v\|^2_{\mathrm{L}^2(\Omega)} + \|v\|^2_{\mathrm{L}^2(\Omega)}$, and $\mathbf{H}(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in \mathrm{L}^2(\Omega)^d \mid \mathrm{div}(\boldsymbol{v}) \in \mathrm{L}^2(\Omega)\}$ equipped with $\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathrm{div},\Omega)} := \|\boldsymbol{v}\|^2_{\mathrm{L}^2(\Omega)} + \|\mathrm{div}(\boldsymbol{v})\|^2_{\mathrm{L}^2(\Omega)}$, as well as $\mathrm{H}^1(\Delta, \Omega) := \{v \in \mathrm{H}^1(\Omega) \mid \nabla v \in \mathbf{H}(\mathrm{div}, \Omega)\}$ equipped with $\|v\|^2_{\mathrm{H}^1(\Delta,\Omega)} := \|\Delta v\|^2_{\mathrm{L}^2(\Omega)} + \|v\|_{\mathrm{H}^1(\Omega)}$. If $\mathrm{H}(\Omega)$ refers to any one of the spaces mentioned above, we shall denote $\mathrm{H}_{\mathrm{loc}}(\overline{\Omega}) := \{v \in \mathrm{L}^2_{\mathrm{loc}}(\Omega) \mid \varphi v \in \mathrm{H}(\Omega)\ \forall \varphi \in \mathscr{C}^\infty_{\mathrm{K}}(\mathbb{R}^d)\}$ where $\mathscr{C}^\infty_{\mathrm{K}}(\mathbb{R}^d)$ refers to $\mathscr{C}^\infty$ functions with bounded support.

1.1.1 Trace spaces

We shall also make regular reference to functions defined on the boundary $\Gamma = \partial\Omega$ that are elements of the fractional order Sobolev space $\mathrm{H}^{1/2}(\Gamma)$ consisting in the closure of $\mathrm{H}^1(\Gamma)$ for the norm

$$\|v\|^2_{\mathrm{H}^{1/2}(\Gamma)} := \|v\|^2_{\mathrm{L}^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|v(\boldsymbol{x}) - v(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^d} d\sigma(\boldsymbol{x}) d\sigma(\boldsymbol{y})$$

where d is the dimension of the ambient space i.e. $\Omega \subset \mathbb{R}^d$, and $d\sigma(\boldsymbol{x})$ refers to the surface Lebesgue measure on Γ. The space $\mathrm{H}^{-1/2}(\Gamma)$ will refer to the topological dual to $\mathrm{H}^{1/2}(\Gamma)$ equipped with the norm

$$\|q\|_{\mathrm{H}^{-1/2}(\Gamma)} := \sup_{v \in \mathrm{H}^{1/2}(\Gamma)\setminus\{0\}} \frac{|\langle q, v\rangle_\Gamma|}{\|v\|_{\mathrm{H}^{1/2}(\Gamma)}}.$$

We shall repeatedly refer to the space of Dirichlet/Neumann pairs $\mathbb{H}(\Gamma) := \mathrm{H}^{1/2}(\Gamma) \times \mathrm{H}^{-1/2}(\Gamma)$ equipped with the norm $\|(v, q)\|^2_{\mathbb{H}(\Gamma)} := \|v\|^2_{\mathrm{H}^{1/2}(\Gamma)} + \|q\|^2_{\mathrm{H}^{-1/2}(\Gamma)}$. On this space we will consider the following pairing

$$
\begin{aligned}
&[\mathfrak{u},\mathfrak{v}]_\Gamma := \int_\Gamma u\,q - v\,p\;d\sigma \\
&\text{for} \quad \mathfrak{u}=(u,p),\ \mathfrak{v}=(v,q)\in\mathbb{H}(\Gamma).
\end{aligned}
\tag{1.1}
$$

This pairing satisfies an inf-sup condition so that it puts $\mathbb{H}(\Gamma)$ in duality with itself in the sense that for each continuous linear form $\varphi : \mathbb{H}(\Gamma) \to \mathbb{C}$ there exists a unique $\mathfrak{v}_\varphi \in \mathbb{H}(\Gamma)$ such that $\langle \varphi, \mathfrak{u}\rangle = [\mathfrak{v}_\varphi, \mathfrak{u}]_\Gamma$, $\forall \mathfrak{u} \in \mathbb{H}(\Gamma)$.

1.1.2 Trace operators

According to Rademacher's theorem [105], the boundary of a Lipschitz open set Ω admits a normal vector field $\boldsymbol{n} \in \mathrm{L}^\infty(\partial\Omega)^3$ that we shall systematically assume to be directed toward the exterior of Ω. Define the Dirichlet and Neumann trace operators $\gamma, \gamma_{\mathrm{D}}$ and γ_{N} by the formula

$$
\begin{aligned}
&\gamma(\varphi) := (\gamma_{\mathrm{D}}(\varphi), \gamma_{\mathrm{N}}(\varphi)) \quad \text{with} \\
&\gamma_{\mathrm{D}}(\varphi) := \varphi|_\Gamma \qquad \gamma_{\mathrm{N}}(\varphi) := \boldsymbol{n}\cdot\nabla\varphi|_\Gamma \qquad \forall \varphi \in \mathscr{C}^\infty(\overline{\Omega}).
\end{aligned}
\tag{1.2}
$$

According to [113, Thm.2.6.8 and Thm.2.7.7], the operator γ_{D} (resp. γ_{N}) continuously maps $\mathrm{H}^1_{\mathrm{loc}}(\Omega)$ to $\mathrm{H}^{1/2}(\partial\Omega)$ (resp. $\mathrm{H}^1_{\mathrm{loc}}(\Delta,\Omega)$ to $\mathrm{H}^{-1/2}(\partial\Omega)$). Moreover these maps are onto. From this we also deduce that the map $\gamma : \mathrm{H}^1_{\mathrm{loc}}(\Delta,\Omega) \to \mathbb{H}(\Gamma)$ is continuous (although its is not onto, see [49, Lemma 3.5] for more details).

It is important to emphasise that, for the definition of these operators, the trace is *taken from the interior* of Ω. We shall also consider trace operators $\gamma_c := (\gamma_{\mathrm{D},c}, \gamma_{\mathrm{N},c})$ defined in the same manner as in (1.2) except that the traces are taken from the exterior. Finally, we shall consider the jump and mean value of these traces as follows

$$
[\gamma] := \gamma - \gamma_c \qquad \{\gamma\} := (\gamma + \gamma_c)/2.
$$

Physical relevance of the notations We would like to underline the physical relevance of the notations we have just introduced for traces and pairings. Assume for example that $\Omega \subset \mathbb{R}^d$ is unbounded and that $u \in \mathrm{H}^1_{\mathrm{loc}}(\Omega)$ refers the complex amplitude of some harmonic scalar wave (acoustic for example), so that $\Delta u + \kappa^2 u = 0$ in Ω. Then, up to some physically relevant constant, the quantity

$$
\Im m\{\,[\gamma(u), \gamma(\overline{u})]_\Gamma\,\}
$$

is the power radiated toward infinity averaged over a period of time, see for example Chap.8 of [112] for further discussion on this point.

1.2 Integral representation theorem

All through this document, we will be considering time harmonic wave propagation problems in piece-wise homogeneous media. This is why we shall make repeated reference to the outgoing Green's kernel of Helmholtz equation

$$\mathscr{G}_\kappa(\boldsymbol{x}) := \begin{cases} \frac{\imath}{4} H_0^{(1)}(\kappa|\boldsymbol{x}|) & \text{if } \mathbb{R}^d = \mathbb{R}^2 \\ \dfrac{\exp(\imath\kappa|\boldsymbol{x}|)}{4\pi|\boldsymbol{x}|} & \text{if } \mathbb{R}^d = \mathbb{R}^3. \end{cases} \tag{1.3}$$

In the formula above $\imath := \sqrt{-1}$ is the imaginary unit and $H_0^{(1)}$ refers to the Hankel function of the first kind of order 0, see [85, Chap.5] for example. This function is exponentially decreasing at infinity in the case where $\Im m\{\kappa\} > 0$, and for $\kappa \in (0, +\infty)$ it satisfies Sommerfeld's radiation condition. In the sequel, we will say that a function v satisfies Sommerfeld's radiation condition at infinity if

$$\lim_{\rho\to\infty} \int_{\partial \mathrm{B}_\rho} |\partial_\rho v - \imath\kappa v|^2 d\sigma_\rho = 0 \tag{1.4}$$

where B_ρ is the ball of radius ρ centred at 0, ∂_ρ is the radial partial derivative of spherical coordinates, and $d\sigma_\rho$ is the surface Lebesgue measure on $\partial\mathrm{B}_\rho$. Note that, in this radiation condition, the wave-number κ comes into play as a parameter.

Convention on the radiation condition In the sequel, we shall systematically consider wave numbers $\kappa \neq 0$ satisfying $\Re e\{\kappa\} \geq 0$ and $\Im m\{\kappa\} \geq 0$. As a convention, we shall write that a function v is κ-outgoing whenever it satisfies

- Sommerfeld's condition (1.4) if $\kappa \in (0, +\infty)$
- $\lim_{|\boldsymbol{x}|\to\infty} |v(\boldsymbol{x})| = 0$ otherwise.

Potential operators The most elementary building blocks of boundary integral formulation to scattering problems are potential operators defined through the formulas

$$\begin{aligned} \mathrm{SL}_\kappa(q)(\boldsymbol{x}) &:= \int_\Gamma \mathscr{G}_\kappa(\boldsymbol{x}-\boldsymbol{y}) q(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) \\ \mathrm{DL}_\kappa(v)(\boldsymbol{x}) &:= \int_\Gamma \boldsymbol{n}(\boldsymbol{y}) \cdot (\nabla \mathscr{G}_\kappa)(\boldsymbol{x}-\boldsymbol{y}) v(\boldsymbol{y})\, d\sigma(\boldsymbol{y}) \end{aligned} \tag{1.5}$$

for all $v \in \mathrm{H}^{1/2}(\Gamma)$, $q \in \mathrm{H}^{-1/2}(\Gamma)$ and for any $\boldsymbol{x} \in \mathbb{R}^d \setminus \Gamma$. These operators are respectively known as single and double layer potentials, and they induce continuous maps $\mathrm{SL}_\kappa : \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}) \times \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \Omega)$, and $\mathrm{DL}_\kappa : \mathrm{H}^{+1/2}(\Gamma) \to \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}) \times \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \Omega)$ (see [49, Thm.1] for example), where we consider that $v \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}) \times \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \Omega)$ if and only if $v|_\Omega \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega})$ and $v|_{\mathbb{R}^d\setminus\overline{\Omega}} \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \Omega)$. In addition to potential operators (1.5), we use the following notation,

$$\mathrm{G}_\kappa(v, q)(\boldsymbol{x}) := \mathrm{DL}_\kappa(v)(\boldsymbol{x}) + \mathrm{SL}_\kappa(q)(\boldsymbol{x}) \tag{1.6}$$

for any $(v, q) \in \mathbb{H}(\Gamma)$ and any $\boldsymbol{x} \in \mathbb{R}^d \setminus \Gamma$. According to the remarks above, this induces a continuous operator $\mathrm{G}_\kappa : \mathbb{H}(\Gamma) \to \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}) \times \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \Omega)$. While operators (1.5) are systematically considered in the literature on boundary integral equations, such is not the case for (1.6).

Let us also point that, in formula (1.6), there is a "+" appearing in front of the double layer potential, while a "-" appears at this place when this formula is considered in most of the literature. Indeed, in Formula (1.5), we consider $(\nabla \mathscr{G}_\kappa)(\boldsymbol{x}-\boldsymbol{y})$ instead of $\nabla_{\boldsymbol{y}}(\mathscr{G}_\kappa(\boldsymbol{x}-\boldsymbol{y}))$.

We introduced the notation (1.6) because it is particularly convenient for integral representation of solution to homogeneous Helmholtz equations, as can be seen from the following classical result, see [113, Thm.3.1.6] for a detailed proof.

Theorem 1.2.1.
Let $u \in \mathrm{H}^1_{\mathrm{loc}}(\Omega)$ satisfy $-\Delta u - \kappa^2 u = 0$ in Ω (and u is κ-outgoing if Ω is unbounded). Then we have the representation formula

$$\mathrm{G}_\kappa(\gamma(u))(\boldsymbol{x}) = \begin{cases} u(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega \\ 0 & \text{for } \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Omega} \end{cases}$$

Similarly, if $v \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \Omega)$ satisfies $-\Delta v - \kappa^2 v = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$ (and v is κ-outgoing if Ω is bounded), then we have $\mathrm{G}_\kappa(\gamma_c(v))(\boldsymbol{x}) = -v(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Omega}$, and $\mathrm{G}_\kappa(\gamma(v)_c)(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega$.

This theorem immediately implies $(\gamma \cdot \mathrm{G}_\kappa)\gamma(u) = \gamma(u)$ for any solution u to an homogeneous Helmholtz equation. Taking $u = \mathrm{G}_\kappa(\mathfrak{u})$ for any $\mathfrak{u} \in \mathbb{H}(\Gamma)$, immediately yields $(\gamma \cdot \mathrm{G}_\kappa)^2\mathfrak{u} = (\gamma \cdot \mathrm{G}_\kappa)\mathfrak{u}$, which shows that $\gamma \cdot \mathrm{G}_\kappa : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ is a projector. More precisely, we have the following result, see [113, Prop.3.6.2] for a detailed proof.

Proposition 1.2.1.
Define the space $\mathcal{C}^+_\kappa(\Omega) := \{\gamma(u) \mid \Delta u + \kappa^2 u = 0$ in Ω, and u is κ-outgoing if Ω unbounded$\}$. Then $\gamma \cdot \mathrm{G}_\kappa : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ is a continuous projector, so called interior Calderón projector, whose range coincides with $\mathcal{C}^+_\kappa(\Omega)$ i.e. for any $\mathfrak{u} \in \mathbb{H}(\Gamma)$ we have

$$\mathfrak{u} \in \mathcal{C}^+_\kappa(\Omega) \quad \Longleftrightarrow \quad (\gamma \cdot \mathrm{G}_\kappa)(\mathfrak{u}) = \mathfrak{u}.$$

In the literature, the space $\mathcal{C}^+_\kappa(\Omega)$ is sometimes called the set of Cauchy data. When deriving boundary integral equations for a scattering problem, characterisation of this space by means of Calderón projectors is the main tool for reformulating homogeneous wave equations. To conclude this paragraph, we also want to point out a remarkable result describing the behaviour of the potential operators across Γ.

Theorem 1.2.2.
We have $[\gamma] \cdot \mathrm{G}_\kappa = \mathrm{Id}$.

A detailed proof of this result can be found for example in [113, Thm.3.3.1]. Note that $[\gamma] \cdot \mathrm{G}_\kappa$ is a continuous map sending $\mathbb{H}(\Gamma) = \mathrm{H}^{1/2}(\Gamma) \times \mathrm{H}^{-1/2}(\Gamma)$ onto $\mathbb{H}(\Gamma)$, so that this result is systematically written in the literature under the form of four identities. One of the motivations here for choosing the notations (1.2) and (1.6) is precisely that it makes the statement of Theorem 1.2.2 very concise. It is customary to introduce the operator

$$\mathrm{A}_\kappa := 2\,\{\gamma\} \cdot \mathrm{G}_\kappa \tag{1.7}$$

We will use this operator repeatedly. Its definition implies in particular that $\gamma \cdot \mathrm{G}_\kappa = \frac{1}{2}(\mathrm{Id} + \mathrm{A}_\kappa)$. A direct consequence of Proposition 1.2.1 is that $(\mathrm{A}_\kappa)^2 = \mathrm{Id}$, which is commonly known as Calderón's identity, see for example [96, Thm.3.1.3].

Chapter 2

Integral equations for multiple subdomain scattering

The main concern here and in the next chapter will be attached to boundary integral equations for *multi-subdomain* scattering, as opposed to most of the existing literature in integral equations that considers scattering by a *single* scatterer.

So far, this topic has been the subject of numerous contributions in the electrical engineering literature [38, 132, 78, 131, 88, 104, 130]. Most of these references propose various treatments of material junctions (i.e. points where three subdomains associated to different material characteristics abut) and provide numerical investigations on efficiency of their strategies. However, besides description of the integral formulations under consideration, these references do not provide mathematical analysis.

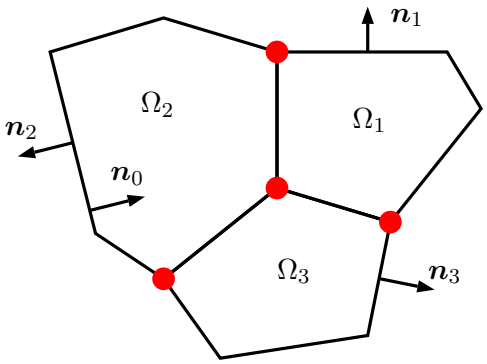
On this matter, fully mathematical (numerical) analysis had been provided in connection with mainly two approaches that will be presented in Section 2.2: the Rumsey reaction principle (also known as PMCHWT), and the Boundary Element Tearing and Interconnecting method. We believe that one of the main contribution of the present thesis is the development of at least two new and different boundary integral equation approaches related to multi-subdomain scattering. The first of these novel approaches will be presented in Section 2.3. The second one will be discussed in Chapter 3.

The first purpose of the present chapter is to introduce a class of boundary value problems corresponding to wave propagation in piece-wise constant media, and to define notations well adapted to this type of problem. Next we will briefly describe the state of the art prior to this thesis as regards boundary integral formulations for this type of problem, and finally present a first remarkable original contribution in this direction. The main focus will be on pure transmission problems i.e. every part of the propagation medium is penetrable. At the end of the present chapter and chapter 3, we shall comment on the situation where part of the medium is impenetrable.

Below is a picture giving an example of the kind of geometrical configuration under consideration here. We simply consider a partition of the whole space as $\mathbb{R}^d = \cup_{j=0}^{n} \overline{\Omega}_j$ where each Ω_j is a Lipschitz open set, bounded if $j \neq 0$. We do not formulate any further assumption concerning the geometry, which makes the work to be presented here very generic from a geometrical point of view. We shall regularly refer to the skeleton of this partition i.e. the

union of all interfaces, and denote it

$$\Sigma := \cup_{j=0}^{n} \Gamma_j \quad \text{with} \quad \Gamma_j := \partial\Omega_j.$$



Ω_0 = exterior domain

We do not make any particular assumption about the dimension, nor impose any further regularity restriction on the subdomains. In particular we allow the presence of so-called junction points i.e. points where at least three subdomains abut. In the three dimensional case $\mathbb{R}^d = \mathbb{R}^3$, the set of junction points is composed of Lipschitz curves and is sometimes called "wire basket" in the domain decomposition literature [123].

In this geometrical setting, Ω_0 represents a background medium and $\cup_{j=1}^{n} \overline{\Omega}_j$ a composite scatterer. The material characteristics in each subdomain will be characterised by two constants $\kappa_j \in \mathbb{C}$ and $\mu_j \in (0, +\infty)$. The constants κ_j refer to the effective wave number in each subdomain. The proper physical meaning of the constants μ_j depends on the context.

As an example of a source term, we consider an incident field u_{inc} supposed to be a priori *not an outgoing radiating* solution to the wave equation $\Delta u_{\text{inc}} + \kappa_0^2 u_{\text{inc}} = 0$ in $\mathbb{R}^d$. For example a plane wave $u_{\text{inc}}(\boldsymbol{x}) = \exp(\imath\kappa_0 \mathbf{d} \cdot \boldsymbol{x})$ for some vector $\mathbf{d} \in \mathbb{C}^3, |\mathbf{d}| = 1$. The pure transmission problem we wish to consider writes: find $u \in \mathrm{H}^1_{\text{loc}}(\mathbb{R}^d)$ such that

$$\begin{cases} -\Delta u - \kappa_j^2 u = 0 \quad \text{in } \Omega_j, \quad j = 0 \dots n \\ u - u_{\text{inc}} \text{ is } \kappa_0\text{-outgoing.} \end{cases} \tag{2.1a}$$

$$\begin{cases} \gamma_{\mathrm{D}}^j(u) - \gamma_{\mathrm{D}}^k(u) = 0 \quad \text{and} \\ \mu_j^{-1}\gamma_{\mathrm{N}}^j(u) + \mu_k^{-1}\gamma_{\mathrm{N}}^k(u) = 0 \quad \text{on } \Gamma_j \cap \Gamma_k, \quad \forall j, k. \end{cases} \tag{2.1b}$$

This problem was proved to be systematically well posed in [127, Sect.1] under certain assumptions on the wave numbers. For the sake of simplicity and conciseness, we shall assume here that, for all $j = 0 \dots n$ we have

$$\Re e\{\kappa_j\} \geq 0, \quad \Im m\{\kappa_j\} \geq 0 \quad \text{and} \quad \kappa_j \neq 0. \tag{2.2}$$

Recall that the radiation condition either refers to Sommerfeld's condition if $\kappa_0 > 0$, or imposes decay at infinity otherwise. For the sake of clarity, all through this chapter and chapter 3, we shall assume that $\mu_0 = \mu_1 = \cdots = \mu_n$. However, we emphasise that this assumption is not mandatory for most of the results we are going to present.

2.1 Functional setting adapted to multi-domain geometries

To write properly boundary integral formulations of the problem above, we will first need counterparts of the operators introduced in Chapter 1. We shall denote $\gamma_{\mathrm{D}}^j, \gamma_{\mathrm{N}}^j, \gamma^j$ (resp. $\gamma_{\mathrm{D},c}^j, \gamma_{\mathrm{N},c}^j, \gamma_c^j$) interior (resp. exterior) traces on the boundary of Ω_j. We shall also denote $\mathrm{SL}_\kappa^j, \mathrm{DL}_\kappa^j, \mathrm{G}_\kappa^j, \mathrm{A}_\kappa^j$ the operators defined in (1.5)-(1.6)-(1.7) associated to the choice $\Omega = \Omega_j$ and $\Gamma = \Gamma_j$ (with a normal vector $\boldsymbol{n}_j$ directed toward the exterior of Ω_j).

2.1.1 Multi-trace space

In addition to notations adapted to operators, we need to introduce trace spaces adapted to the study of Problem (2.1a)-(2.1b). The most natural space simply consists in a cartesian product of traces on the boundary of each subdomain

$$\begin{aligned}&\mathbb{H}(\Sigma) := \mathbb{H}(\Gamma_0) \times \cdots \times \mathbb{H}(\Gamma_n)\\ &\text{where} \quad \mathbb{H}(\Gamma_j) := \mathrm{H}^{+\frac{1}{2}}(\Gamma_j) \times \mathrm{H}^{-\frac{1}{2}}(\Gamma_j)\end{aligned}$$

This so-called multi-trace space will be equipped with its natural cartesian product norm $\|\mathfrak{u}\|_{\mathbb{H}(\Sigma)}^2 := \|\mathfrak{u}_0\|_{\mathbb{H}(\Gamma_0)}^2 + \cdots + \|\mathfrak{u}_n\|_{\mathbb{H}(\Gamma_n)}^2$ for all $\mathfrak{u} = (\mathfrak{u}_j)_{j=0}^n$, where the norm $\|\cdot\|_{\mathbb{H}(\Gamma_j)}$ was defined in §1.1.1. The multi-trace space will be equipped with the following duality pairing

$$[\![\mathfrak{u}, \mathfrak{v}]\!] := \sum_{j=0}^{n} [\mathfrak{u}_j, \mathfrak{v}_j]_{\Gamma_j} \tag{2.3}$$

for any $\mathfrak{u} = (\mathfrak{u}_j)_{j=0}^n$ and $\mathfrak{v} = (\mathfrak{v}_j)_{j=0}^n$ belonging to $\mathbb{H}(\Sigma)$. Note that this pairing puts $\mathbb{H}(\Sigma)$ in duality with itself, so that we shall identify $\mathbb{H}(\Sigma)$ to its own dual in the rest of this thesis. More precisely for any bounded linear form $\varphi : \mathbb{H}(\Sigma) \to \mathbb{C}$ there exists a unique $\mathfrak{v}_\varphi \in \mathbb{H}(\Sigma)$ such that $\langle \varphi, \mathfrak{u} \rangle = [\![\mathfrak{v}_\varphi, \mathfrak{u}]\!]$. This is guaranteed by the following inf-sup condition

$$\inf_{\mathfrak{u} \in \mathbb{H}(\Sigma)\setminus\{0\}} \sup_{\mathfrak{v} \in \mathbb{H}(\Sigma)\setminus\{0\}} \frac{|[\![\mathfrak{u}, \mathfrak{v}]\!]|}{\|\mathfrak{u}\|_{\mathbb{H}(\Sigma)} \|\mathfrak{v}\|_{\mathbb{H}(\Sigma)}} = 1.$$

2.1.2 Single-trace space

Next we introduce a subspace of $\mathbb{H}(\Sigma)$ that takes account of transmission conditions as a characterising constraint. Although the idea of considering this space came from personal discussions with A.Bendali (presently professor at INSA in Toulouse), to our knowledge explicit definition and systematic study of this space was achieved for the first time in [4, 10]. This space is pivotal in the forthcoming analysis. Below is its precise definition

$$\begin{aligned}\mathbb{X}(\Sigma) := \{&\mathfrak{u} = (v_j, q_j)_{j=0}^n \in \mathbb{H}(\Sigma) \mid \\ &\exists v \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d) \text{ such that } v|_{\Gamma_j} = v_j \\ &\exists \boldsymbol{q} \in \mathrm{H}_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^d) \text{ such that } \boldsymbol{n}_j \cdot \boldsymbol{q}|_{\Gamma_j} = q_j \quad \forall j = 0 \dots n\}\end{aligned} \tag{2.4}$$

We called it single-trace space. Indeed, as will become clear from subsequent results, for tuples of traces chosen in this space, at each point of each interface only one pair of Dirichlet-Neumann traces come into play, not two like in the multi-trace space. A careful inspection

of this definition shows that an element $\mathfrak{u} = (v_j, q_j)_{j=0}^n \in \mathbb{H}(\Sigma)$ belongs to $\mathbb{X}(\Sigma)$ if and only if we have

$$v_j = v_k \quad \text{and} \quad q_j = -q_k \qquad \text{on } \Gamma_j \cap \Gamma_k, j < k. \tag{2.5}$$

which mimics the transmission conditions (2.1b) without taking account of the μ_j's.

This space admits at least two remarkable characterisations. In the first one, it is characterised as its own polar set under the duality pairing $[\![\cdot, \cdot]\!]$. This was established in [4, Prop.2.1]

Proposition 2.1.1.
For any $\mathfrak{u} \in \mathbb{H}(\Sigma)$ *we have* $\mathfrak{u} \in \mathbb{X}(\Sigma) \iff [\![\mathfrak{u}, \mathfrak{v}]\!] = 0 \ \forall \mathfrak{v} \in \mathbb{X}(\Sigma)$.

The proof given in [4, Prop.2.1] or [10, Prop.2.1] only relies on an elementary play with Green's formula. A second result characterises the single-trace space as the closure of traces of functions that are smooth across the interfaces, see [9, Lemma 7.4].

Proposition 2.1.2.
The space $\mathbb{X}(\Sigma)$ *is the closure of the set* $\{(\gamma^j(u))_{j=0}^n \mid u \in \mathscr{C}^\infty(\mathbb{R}^d)\}$ *for the norm of* $\| \ \|_{\mathbb{H}(\Sigma)}$.

The proof can be found in [9, Lemma 7.4]. Note in particular that this implies that $\mathbb{X}(\Sigma)$ is a closed subspace of $\mathbb{H}(\Sigma)$.

2.1.3 Cauchy data on the skeleton

Another subspace that plays an important role in boundary integral equations consists in traces induced by those functions that are piece-wise solutions to homogeneous Helmholtz equations. The present multi-subdomain context thus leads to considering

$$\begin{aligned} \mathcal{C}^+_{(\kappa)}(\Sigma) := \{\mathfrak{u} = (\gamma^j(u_j))_{j=0}^n \mid u_j \in \mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega}) \\ \text{and } \Delta u_j + \kappa_j^2 u_j = 0 \ \text{ in } \Omega_j \\ \text{and } u_j \text{ is } \kappa_j\text{-outgoing if } \Omega_j \text{ unbounded } \}. \end{aligned}$$

Subsequently, we shall use the notation (κ) to refer to the tuple $(\kappa_0, \kappa_1, \ldots, \kappa_n)$. Similarly to Proposition 2.1.1, this space can be characterised as its own polar set under the duality pairing we have chosen for $\mathbb{H}(\Sigma)$. The proof of the next result can be found in [10, Lemma 6.2].

Lemma 2.1.1.
For any $\mathfrak{u} \in \mathbb{H}(\Sigma)$ *we have* $\mathfrak{u} \in \mathcal{C}^+(\Sigma) \iff [\![\mathfrak{u}, \mathfrak{v}]\!] = 0 \ \forall \mathfrak{v} \in \mathcal{C}^+(\Sigma)$.

This result holds true only for the choice of duality pairing (1.1)-(2.3). Another remarkable feature of this space is that it complements the single-trace space. The following result was first pointed out and proved in [10, Prop.6.1].

Lemma 2.1.2.
$\mathbb{H}(\Sigma) = \mathbb{X}(\Sigma) \oplus \mathcal{C}^+(\Sigma)$.

2.2 State of the art prior to the thesis

Before completion of the present work, there already existed several boundary integral approaches of Problem (2.1a)-(2.1b). We will present here what seems to be the two main existing approaches.

2.2.1 Rumsey's reaction principle (PMCHWT)

The first pre-existing approach is called Rumsey's reaction principle, also known in the literature as PMCHWT, which stands for Poggio-Miller-Chang-Harrington-Wu-Tsai from the various authors that independently introduced this formulation see [109, 66, 92, 41]. In its scalar version, this formulation was analysed by T.Von Petersdorff [127], and this was later generalised for Maxwell's equations by A.Buffa [34]. In many industrial contexts, it is considered as *the* method of choice for solving (2.1) due to its versatility and its level of accuracy. It is probably the oldest boundary integral formulation of this problem. To describe this formulation, we will assume in this paragraph that $\mu_0 = \mu_1 = \cdots = \mu_n$ for the sake of simplicity. All the result that we will mention, though, can be generalised for the case where this assumption is not satisfied. Denote $\mathfrak{u}^{\rm inc} := (\gamma^0(u_{\rm inc}), 0, \ldots, 0)$, and define $\mathbb{A}_{(\kappa)} : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ by

$$[\![\mathbb{A}_{(\kappa)}\mathfrak{v}, \mathfrak{v}']\!] := \sum_{j=0}^{n} [\mathrm{A}^j_{\kappa_j}\mathfrak{v}_j, \mathfrak{v}'_j]_{\Gamma_j} \tag{2.6}$$

for all $\mathfrak{v} = (\mathfrak{v}_j), \mathfrak{v}' = (\mathfrak{v}'_j) \in \mathbb{H}(\Sigma)$. If u refers to the unique solution to Problem (2.1), considering $\mathfrak{u} = (\gamma^j(u))_{j=0}^n$, then (2.1b) can be rewritten $\mathfrak{u} \in \mathbb{X}(\Sigma)$, and (2.1a) can be rewritten $(\mathbb{A}_{(\kappa)} - \mathrm{Id})(\mathfrak{u} - \mathfrak{u}^{\rm inc}) = 0$. Testing the latter equation with an arbitrary $\mathfrak{v} \in \mathbb{X}(\Sigma)$ finally yields

$$\begin{cases} \mathfrak{u} \in \mathbb{X}(\Sigma) \quad \text{and} \\ [\![\mathbb{A}_{(\kappa)}\mathfrak{u}, \mathfrak{v}]\!] = 2\,[\![\mathfrak{u}^{\rm inc}, \mathfrak{v}]\!] \quad \forall \mathfrak{v} \in \mathbb{X}(\Sigma). \end{cases} \tag{2.7}$$

In the equation above, we used the homogeneous wave equation satisfied by $u_{\rm inc}$ in Ω_j for $j \neq 0$ in order to simplify the expression of the right-hand side. The following result was proved in [127].

Theorem 2.2.1.
Set $\theta(v, q) = (-v, q)$ and $\Theta(\mathfrak{v}) = (\theta(\mathfrak{v}_j))_{j=0}^n$. There exists a constant $C > 0$ and a compact operator $\mathrm{R} : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ such that

$$\Re e\{[\![(\mathbb{A}_{(\kappa)} + \mathrm{R})\mathfrak{v}, \Theta(\overline{\mathfrak{v}})]\!]\} \geq C\,\|\mathfrak{v}\|^2_{\mathbb{H}(\Sigma)} \qquad \forall \mathfrak{v} \in \mathbb{H}(\Sigma).$$

This implies that $\mathbb{A}_{(\kappa)}$ is a Fredholm operator with index 0. Hence it is an isomorphism if and only if it is one-to-one, which is confirmed by the following result also contained in [127]. Note that Rumsey's formulation does not suffer from the so-called spurious resonance phenomenon (i.e. the integral formulation is well-posed if and only if the volumic formulation is).

Theorem 2.2.2.
Under Assumption (2.2) the PMCWHT formulation (2.7) admits a unique solution and the operator $\mathbb{A}_{(\kappa)}$ induces a continuous isomorphism mapping $\mathbb{X}(\Sigma)$ onto its dual $\mathbb{X}(\Sigma)^*$.

The Garding inequality established in Theorem 2.2.1 guarantees a uniform discrete inf-sup condition, and thus quasi-optimal consistency of conformal Galerkin based discretisations of the PMCHWT formulation. We should mention in addition that PMCHWT is commonly accepted as one of the most accurate formulations for problems of the same type as (2.1).

From a numerical point of view though, a remarkable drawback of this formulation is that it does not lend itself to any known efficient preconditioning strategy when the geometrical configurations (decomposition in subdomain) involves junction points. In particular, it does not fit the operator preconditioning framework described in [43, 67, 117].

2.2.2 Boundary Element Tearing and Interconnecting

The second pre-existing approach already well developed in the mathematical literature for dealing with multi-subdomain scattering stems from a domain decomposition perspective of the problem. It is called Boundary Element Tearing and Interconnecting method (BETI), because it is a boundary element counterpart of the Finite Element Tearing and Interconnecting method (FETI) where equations local to each subdomain are expressed by means of boundary integral operators, instead of being formulated in a volumic form prone to solvers of finite element type. This approach was introduced and extended in many directions in a series of papers including [83, 100, 82, 98, 118, 84].

In its most primary version, the idea of this method is the following. In each subdomain, Neumann traces are related to the corresponding Dirichlet traces by means of a Steklov-Poincare map expressed in terms of boundary integral operators, which takes account of the PDE in each subdomain. Then Dirichlet transmission conditions appear as constraints characterising the variational space, while Neumann transmission conditions are imposed weakly.

This approach is well adapted to parallel computations. Provided that boundary integral operators can be inverted in each subdomain, a robust preconditioner can be constructed. Hence preconditioning is easily parallelisable but is computationally costly, which suggests that, in a domain decomposition context, this approach may be more relevant to situations where parallelisation is appropriate.

2.3 Integral equation of the second kind

Now, we present a first alternative to the PMCHWT and BETI formulations. The formulation of the present section, originally introduced in [4], was dubbed Single-Trace Formulation of the second kind (2nd kind STF). Further investigation of this formulation led to the co-supervision together with Ralf Hiptmair (Seminar of Applied Mathematics, ETH Zürich) of the PhD thesis of Elke Spindler. This led to the publications [16, 15]. We should also mention that, parallel to the present work and independently, a similar formulation was introduced by L.Greengard and his collaborators [61].

Prior to the work presented in this section, there already existed an integral formulation of the second kind valid in the case of geometrical configurations with no junction point. This formulation was proposed by Müller [93] and Rokhlin [108], and could be written for any values of κ_j and μ_j. However, with Lipschitz interfaces, the operator associated to the latter formulation is truly a compact perturbation of the identity only if $\mu_0 = \cdots = \mu_1$.

The formulation presented below agrees with Rokhlin-Müller's formulation in the case of two domains $\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_1$ and one interface. Both formulations are strictly different, though, for more subdomains i.e. $n \geq 2$ even if there is no junction point. This was commented upon in detail and explicitly in [4].

Although the formulation we shall present in this section may be considered for the general problem (2.1), it is particularly relevant for the case where all μ_j's take the same value. Hence, in the remaining of this section we will make the following assumption

$$\mu_0 = \mu_1 = \cdots = \mu_n = 1. \tag{2.8}$$

2.3.1 A new characterisation of Cauchy data

The main ingredient in the derivation of the 2nd kind STF is a new characterisation of the space $\mathcal{C}^+_{(\kappa)}(\Sigma)$. This characterisation directly relies on an operator obtained as the sum of all layer potentials on all subdomains, a so-called "multi-potential" defined as $\mathrm{G}^0_{\kappa_0}(\mathfrak{u}_0)(\boldsymbol{x})+\cdots+\mathrm{G}^n_{\kappa_n}(\mathfrak{u}_n)(\boldsymbol{x})$ for any $\boldsymbol{x}\in\mathbb{R}^d\setminus\Sigma$ and any $\mathfrak{u}=(\mathfrak{u}_j)_{j=0}^n\in\mathbb{H}(\Sigma)$. It is a continuous operator as each of the potentials $\mathrm{G}^j_{\kappa_j}$ is continuous. We have the following remarkable property, see [4, Lemma 5.1].

Lemma 2.3.1.
If $\kappa_0=\kappa_1=\cdots=\kappa_n$ *then* $\mathrm{G}^0_{\kappa_0}(\mathfrak{u}_0)+\cdots+\mathrm{G}^n_{\kappa_n}(\mathfrak{u}_n)=0\ \forall\mathfrak{u}\in\mathbb{X}(\Sigma)$.

Note that $\mathrm{G}^0_{\kappa_0}(\mathfrak{u}_0)+\cdots+\mathrm{G}^n_{\kappa_n}(\mathfrak{u}_n)$ is not the solution to any specific wave equation, except if all wave numbers equal. We shall also refer to the operator $\mathbb{P}_{(\kappa)}:\mathbb{H}(\Sigma)\to\mathbb{H}(\Sigma)$ obtained by taking the traces of this multi-potential on the boundary of each subdomain

$$[\![\mathbb{P}_{(\kappa)}\mathfrak{u},\mathfrak{v}]\!]:=\sum_{j=0}^n\sum_{q=0}^n[\gamma^j\cdot\mathrm{G}^q_{\kappa_q}(\mathfrak{u}_q),\mathfrak{v}_j]_{\Gamma_j}$$

This last operator satisfies several remarkable properties. The first one generalises, for the multi-subdomain geometric configurations, the jump formulas of Theorem 1.2.2. The next result was established in [4, Prop.5.1].

Theorem 2.3.1.
We have $[\![\mathbb{P}_{(\kappa)}\mathfrak{u},\mathfrak{v}]\!]=[\![\mathfrak{u},\mathfrak{v}]\!]$ *for all* $\mathfrak{u}\in\mathbb{H}(\Sigma),\mathfrak{v}\in\mathbb{X}(\Sigma)$.

According to Proposition 2.1.1, this implies in particular that $(\mathrm{Id}-\mathbb{P}_{(\kappa)})\mathfrak{u}\in\mathbb{X}(\Sigma)$ for all $\mathfrak{u}\in\mathbb{H}(\Sigma)$. The next theorem shows that this operator can be used to express homogeneous wave equations. It is a reformulation of [4, Lemma 5.1 & 5.2] combined with [4, Thm. 5.1].

Theorem 2.3.2.
In the case where $\kappa_0=\kappa_1=\cdots=\kappa_n$ *the operator* $\mathbb{P}_{(\kappa)}$ *is a projector whose kernel is* $\mathbb{X}(\Sigma)$ *and range is* $\mathcal{C}^+_{(\kappa)}(\Sigma)$. *For the general case where the* κ_j*'s do not coincide, the operator* $\mathbb{P}_{(\kappa)}$ *is (a priori) not a projector anymore, but there exists a constant* $\delta>0$ *such that*

$$\ker(\mathbb{P}_{(\kappa)}-\mathrm{Id})=\mathcal{C}^+_{(\kappa)}(\Sigma)\qquad\text{if}\ \max_{j,q=0\ldots n}|\kappa_j-\kappa_q|<\delta.$$

This is a direct consequence of the representation Theorem 1.2.1 that, for *any* choice of $\kappa_0,\ldots,\kappa_n$ satisfying (2.2) we have $\mathfrak{u}\in\mathcal{C}^+_{(\kappa)}(\Sigma)\Rightarrow\mathbb{P}_{(\kappa)}\mathfrak{u}=\mathfrak{u}$. The previous theorem shows that the reciprocal is true under the assumption that the wave numbers are close enough to each others. It is legitimate to ask whether this reciprocal remains true without any other assumption than (2.2) on the wave numbers.

Conjecture 2.3.1.
For any tuple $\kappa_0,\ldots,\kappa_n\in\mathbb{C}$ *satisfying (2.2) we have* $\ker(\mathbb{P}_{(\kappa)}-\mathrm{Id})=\mathcal{C}^+_{(\kappa)}(\Sigma)$.

Whether this conjecture is actually a valid result remains an open question. We tried to prove it without success. All numerical investigations we tried systematically suggested that this result holds true.

2.3.2 Derivation of the formulation

For the derivation of the formulation we assume that we are in a situation where the result of the conjecture above is valid. Consider $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ as the solution to Problem (2.1), then $\mathfrak{u} = (\gamma^j(u))_{j=0}^n$ satisfies $\mathfrak{u} \in \mathbb{X}(\Sigma)$ and $\mathfrak{u} - \mathfrak{u}^{\mathrm{inc}} \in \mathcal{C}^+_{(\kappa)}(\Sigma)$, where $\mathfrak{u}^{\mathrm{inc}}$ was defined in §2.2.1. These equations can be variationally recast as

$$\mathfrak{u} \in \mathbb{X}(\Sigma) \quad \text{and} \quad [\![(\mathrm{Id} - \mathbb{P}_{(\kappa)})(\mathfrak{u} - \mathfrak{u}^{\mathrm{inc}}), \mathfrak{v}]\!] = 0 \quad \forall \mathfrak{v} \in \mathbb{H}(\Sigma). \tag{2.9}$$

According to Theorem 2.3.1, the equation above is trivial for $\mathfrak{v} \in \mathbb{X}(\Sigma)$ as in this case it writes $0 = 0$. This means that, in this formulation, the test traces $\mathfrak{v}$ may be chosen in any closed subspace $\mathbb{Y}(\Sigma) \subset \mathbb{H}(\Sigma)$ that is complementary to $\mathbb{X}(\Sigma)$,

$$\mathbb{H}(\Sigma) = \mathbb{X}(\Sigma) \oplus \mathbb{Y}(\Sigma).$$

Besides let $\mathbb{P}_{\kappa_0}$ refer to the same operator as $\mathbb{P}_{(\kappa)}$ except that all wave numbers are chosen equal to κ_0. Since in (2.9), the unknown $\mathfrak{u}$ is sought in $\mathbb{X}(\Sigma)$, we have $\mathbb{P}_{\kappa_0}(\mathfrak{u}) = 0$, so that we may add this term to (2.9), which finally yields

$$\begin{cases} \mathfrak{u} \in \mathbb{X}(\Sigma) \quad \text{and} \\ [\![(\mathrm{Id} + \mathbb{P}_{\kappa_0} - \mathbb{P}_{(\kappa)})\mathfrak{u}, \mathfrak{v}]\!] = [\![\mathfrak{f}, \mathfrak{v}]\!] \quad \forall \mathfrak{v} \in \mathbb{Y}(\Sigma). \end{cases} \tag{2.10}$$

where $\mathfrak{f} = (\gamma^j(u_{\mathrm{inc}}))_{j=0}^n$. The attractive feature of this formulation is that the difference $\mathbb{P}_{\kappa_0} - \mathbb{P}_{(\kappa)}$ is compact, as it only involves differences of integral operators whose kernels only differ through the wave number. The next result was established in [4, Prop.5.2].

Lemma 2.3.2.
The operator $\mathbb{P}_{\kappa_0} - \mathbb{P}_{(\kappa)} : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ *is compact.*

As a consequence the operator in (2.10) is a compact perturbation of the identity, which justifies calling it a formulation of the second kind. Provided that a discretisation scheme is chosen so as to guarantee a stable discrete inf-sup condition for the identity operator, the matrix associated to this formulation will be systematically well conditioned, see [23, §3.6].

In addition, Fredholm alternative implies that the operator in (2.10) is an isomorphism if and only if it is one-to-one i.e. if and only if there is no spurious resonance. Establishing the existence or non-existence of spurious resonance turned out to be a tricky question directly related to the conjecture stated above. The following lemma was established by [4, Cor.5.2]

Lemma 2.3.3.
The operator $\mathrm{Id} + (\mathbb{P}_{\kappa_0} - \mathbb{P}_{(\kappa)})$ *maps isomorphically* $\mathbb{X}(\Sigma)$ *onto its dual* $\mathbb{X}(\Sigma)^*$ *if and only if* $\ker(\mathbb{P}_{(\kappa)} - \mathrm{Id}) = \mathcal{C}^+_{(\kappa)}(\Sigma)$.

2.3.3 Lifting and explicit expression

Many remarkable boundary integral equations involve a smooth kernel which usually allows for choosing square integrable functions rather that half-integer Sobolev spaces as functional framework for traces. This is also an advantage from a numerical point of view as it widens

the possibilities regarding discretisation schemes. These remarks also apply in the present case, and this was studied in detail in [16]. Define the spaces

$$\mathbb{L}^2(\Sigma) := \prod_{j=0}^{n} \mathrm{L}^2(\Gamma_j) \times \mathrm{L}^2(\Gamma_j),$$
$$\mathcal{L}^2(\Sigma) := \{\ (v_j, q_j)_{j=0}^n \in \mathbb{L}^2(\Sigma) \mid v_j - v_k = 0,\ q_j + q_k = 0 \text{ on } \Gamma_j \cap \Gamma_k\ \}.$$

Observe that $\mathcal{L}^2(\Sigma)$ is a closed subspace of $\mathbb{L}^2(\Sigma)$. Examining the explicit expression of $[\![\ ,\ ]\!]$ given by (1.1)-(2.3), it is clear that it also puts $\mathbb{L}^2(\Sigma)$ in duality with itself. First of all, the operator coming into play in Formulation (2.10) makes sense in this functional framework.

Proposition 2.3.1.
The operator $\mathbb{P}_{\kappa_0} - \mathbb{P}_{(\kappa)}$ induces a continuous map that sends $\mathbb{L}^2(\Sigma)$ compactly into itself, and continuity holds with respect to the natural norm of $\mathbb{L}^2(\Sigma)$.

Also note that, since we have assumed that $\mu_j = 1\ \forall j = 0 \dots n$, classical elliptic regularity results imply that $u \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^d)$, so that $\mathfrak{u} = (\gamma^j(u))_{j=0}^n \in \mathcal{L}^2(\Sigma)$. Formulation (2.10) can actually be recast in a new and somewhat simpler functional setting. The next result summarises [16, Cor.4.2] when combined with [16, Thm.4.1].

Theorem 2.3.3.
The operator $\mathrm{Id} + (\mathbb{P}_{\kappa_0} - \mathbb{P}_{(\kappa)})$ isomorphically maps $\mathbb{X}(\Sigma)$ onto $\mathbb{X}(\Sigma)^*$ if and only if it isomorphically maps $\mathcal{L}^2(\Sigma)$ onto $\mathcal{L}^2(\Sigma)^\perp$. Moreover the tuple $\mathfrak{u} = (\gamma^j(u))_{j=0}^n$ (where u solves the scattering problem) is the unique solution to

$$\begin{cases} \mathfrak{u} \in \mathcal{L}^2(\Sigma) \quad \textit{and} \\ [\![(\mathrm{Id} + \mathbb{P}_{\kappa_0} - \mathbb{P}_{(\kappa)})\mathfrak{u}, \mathfrak{v}]\!] = [\![\mathfrak{f}, \mathfrak{v}]\!] \quad \forall \mathfrak{v} \in \mathcal{L}^2(\Sigma)^\perp. \end{cases} \tag{2.11}$$

In this theorem $\mathcal{L}^2(\Sigma)^\perp$ refers to the orthogonal complement to $\mathcal{L}^2(\Sigma)$ with respect to the natural scalar product of $\mathbb{L}^2(\Sigma)$ given by

$$(\mathfrak{u}, \mathfrak{v})_{\mathbb{L}^2(\Sigma)} := \sum_{j=0}^{n} \int_{\Gamma_j} u_j \overline{v}_j + p_j \overline{q}_j d\sigma$$

for $\mathfrak{u} = (u_j, p_j)_{j=0}^n$ and $\mathfrak{v} = (v_j, q_j)_{j=0}^n$. An element $(v_j, q_j)_{j=0}^n \in \mathbb{L}^2(\Sigma)$ belongs to $\mathcal{L}^2(\Sigma)$ if and only if it satisfies (2.5). Similarly, such an element belongs to $\mathcal{L}^2(\Sigma)^\perp$ if and only if $v_j = -v_k$ and $q_j = q_k$ on $\Gamma_j \cap \Gamma_k$ for $j < k$. We will use this to develop and re-arrange the expression of the bilinear form (2.11) so as to obtain a more explicit form for it, which is particularly helpful as regards effective implementation of this formulation. We first introduce a concise notation for indexing the interfaces of the skeleton

$$\mathfrak{I} := \{(j,k) \in \{0, \dots n\}^2 \mid \Gamma_j \cap \Gamma_k \neq \emptyset, j < k\}$$
$$\Gamma_{\mathrm{J}} := \Gamma_j \cap \Gamma_k$$
$$\mathfrak{u}_{\mathrm{J}} := \mathfrak{u}_j|_{\Gamma_j \cap \Gamma_k} \quad \text{for } \mathrm{J} = (j,k) \in \mathfrak{I}$$

For $\mathrm{J} = (j,k) \in \mathfrak{I}$, we shall also denote $\mathrm{J}_- := j$ and $\mathrm{J}_+ := k$. Observe that the application $(\mathfrak{u}_j)_{j=0}^n \mapsto (\mathfrak{u}_{\mathrm{J}})_{\mathrm{J} \in \mathfrak{I}}$ isomorphically maps $\mathcal{L}^2(\Sigma)$ (resp. $\mathcal{L}^2(\Sigma)^\perp$) onto the space

$\mathrm{L}^2(\Sigma)^2 := \mathrm{L}^2(\Sigma) \times \mathrm{L}^2(\Sigma)$ where $\mathrm{L}^2(\Sigma) := \Pi_{\mathrm{J}\in\mathfrak{J}}\,\mathrm{L}^2(\Gamma_{\mathrm{J}})$. These new notations allow substantial simplification of the bilinear form of (2.11). Setting $\gamma^{\mathrm{I}} := \gamma^j$ for $\mathrm{I} = (j,k) \in \mathfrak{J}$, direct calculus yields that, for all $\mathfrak{u} \in \mathcal{L}^2(\Sigma), \mathfrak{v} \in \mathcal{L}^2(\Sigma)^\perp$, we have

$$[\![\mathfrak{u},\mathfrak{v}]\!] = 2\sum_{\mathrm{J}\in\mathfrak{J}}[\mathfrak{u}_{\mathrm{J}},\mathfrak{v}_{\mathrm{J}}]_{\Gamma_{\mathrm{J}}}$$
$$[\![(\mathbb{P}_{\kappa_0}-\mathbb{P}_{(\kappa)})\mathfrak{u},\mathfrak{v}]\!] = 2\sum_{\mathrm{I}\in\mathfrak{J}}\sum_{\mathrm{J}\in\mathfrak{J}}[\gamma^{\mathrm{I}}\cdot(\mathrm{G}^{\mathrm{J}}_{\kappa_{\mathrm{J}_+}}-\mathrm{G}^{\mathrm{J}}_{\kappa_{\mathrm{J}_-}})\mathfrak{u}_{\mathrm{J}},\mathfrak{v}_{\mathrm{I}}]_{\Gamma_{\mathrm{I}}}$$

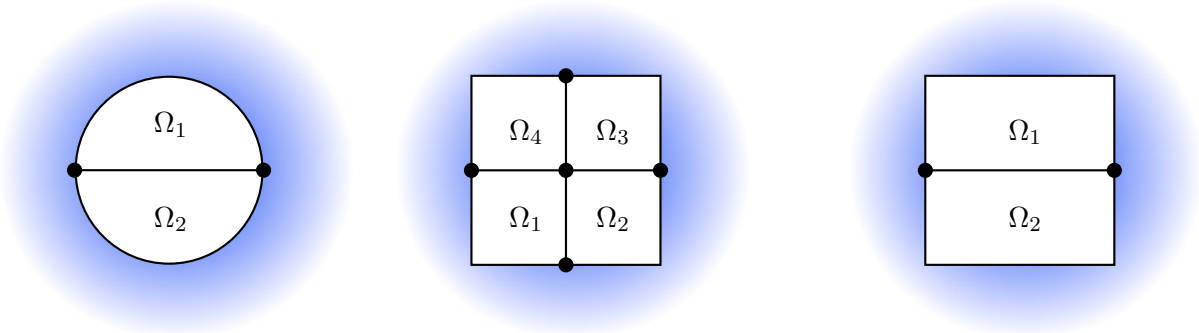
where, for $v, q \in \mathrm{L}^2(\Gamma_{\mathrm{J}})$,

$$(\mathrm{G}^{\mathrm{J}}_{\kappa_{\mathrm{J}_+}}-\mathrm{G}^{\mathrm{J}}_{\kappa_{\mathrm{J}_-}})(v,q)(\boldsymbol{x}) := \int_{\Gamma_{\mathrm{J}}} v(\boldsymbol{y})\boldsymbol{n}_{\mathrm{J}_-}(\boldsymbol{y})\cdot\big(\nabla(\mathscr{G}_{\kappa_{\mathrm{J}_+}}-\mathscr{G}_{\kappa_{\mathrm{J}_-}})\big)(\boldsymbol{x}-\boldsymbol{y}) + q(\boldsymbol{y})(\mathscr{G}_{\kappa_{\mathrm{J}_+}}-\mathscr{G}_{\kappa_{\mathrm{J}_-}})(\boldsymbol{x}-\boldsymbol{y})\,d\sigma(\boldsymbol{y}).$$

2.3.4 Numerical experiments

In the present section, we show numerical results concerning the integral formulation of the second kind (2.11) that confirm its relevance. These numerical results have been obtained by Elke Spindler during her PhD thesis co-supervised by R.Hiptmair and myself. These numerical experiments were achieved using a 2D C++ code launched on a single desktop workstation. Numerical results were also presented in [4], but they were less detailed.

In these experiments, we shall consider Problem (2.1) in 2D with $\mu_0 = \cdots = \mu_n = 1$, and with various values for the κ_j's. We will consider three different geometrical configurations represented in the figure below (Ω_0 is systematically the exterior domain).



Case (a)
$\kappa_0 = 3,\ \kappa_1 = 1$
$\kappa_2 = 5$

Case (b)
$\kappa_0 = 3,\ \kappa_1 = 1$
$\kappa_2 = 5,\ \kappa_3 = 2,\ \kappa_4 = 3$

Case (c)
$\kappa_0 = 1,\ \kappa_1 = 10$
$\kappa_2 = 2$

Figure 2.1: Geometrical configurations under consideration.

We take $u_{\mathrm{inc}}(\boldsymbol{x}) = \exp(\imath\kappa_0\mathbf{d}\cdot\boldsymbol{x})$ as incident field with $\mathbf{d} := (1,0,0)$ i.e. a plane wave directed toward growing x-axis. The geometries were approximated by straight segment panels. Formulation (2.11) has been discretised by a Galerkin approximation process, using piece-wise

constant shape functions for both Dirichlet and Neumann traces. This in particular guarantees a uniform discrete inf-sup condition for the term Id coming into play in the formulation. As regards Dirichlet traces, such a low order discretisation is only possible because only weakly singular kernels come into play in the formulation we are considering.

Consistency We first show a series of results focused on the consistency of the method. For the sake of comparison, consistency results were also obtained for the PMCHWT formulation (2.7) discretised in the same manner, except that continuous piece-wise linear shape functions were used for Dirichlet traces. A reference solution was obtained by means of the hp-FEM code CONCEPTS [128].

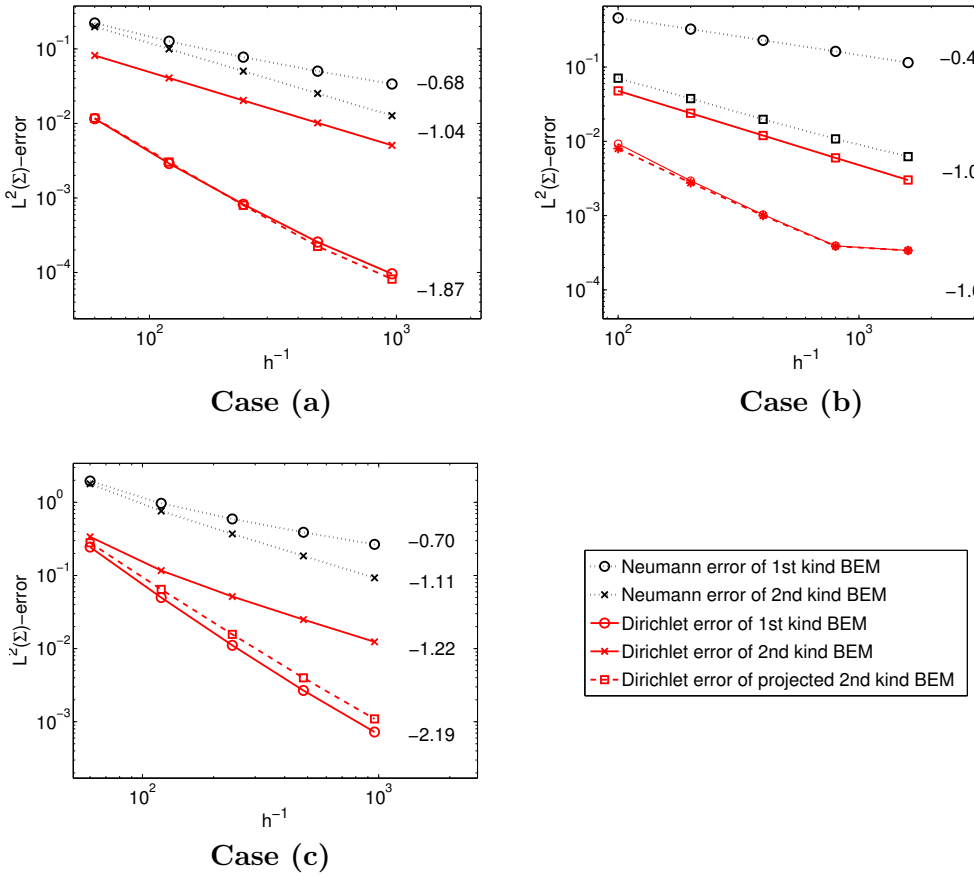
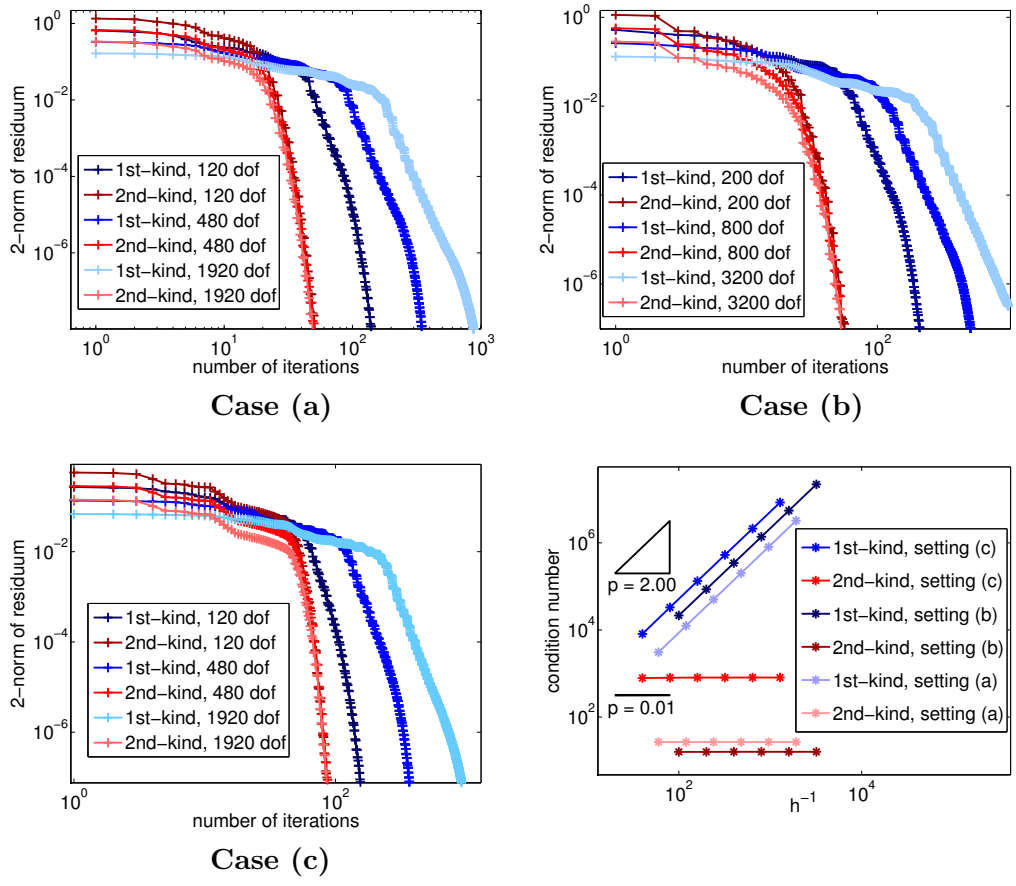


Figure 2.2: Mesh width versus error in L^2-norm: consistency of the second kind STF compared to that of the PMCHWT formulation.

When computing the solution to Formulation (2.11) a post-process was applied to the Dirichlet traces, consisting in an L^2-projection into the space of piece-wise linear continuous functions. In the pictures above, the fifth graphs labeled "Dirichlet error of projected 2nd kind BEM" refer to the error obtained when taking account of this post-process. One can see this error is remarkably lower than the Dirichlet error of the second kind BEM *without* this post-process, but theoretical explanation of this is still missing.

Stability The next series of result shows history of the quadratic norm of the residue when applying GMRes solver with the same restart parameter to the matrix obtained from both the first kind PMCHWT formulation (without any preconditioning) and the Formulation (2.11), for each of the geometrical configurations i.e. cases a), b) and c). We see that the total number of iterations before convergence is stable in the case of the second kind formulation, whereas it deteriorates as $h \to 0$ for PMCHWT, which agrees with theoretical predictions. The fourth picture in the figure below (lower right corner) presents the behaviour of the condition numbers of the matrices as $h \to 0$, which confirms the stabilisation of the condition number for the matrices obtained from the 2nd kind STF (2.11).



2.4 Extensions

We dedicate this paragraph to extensions that we have already conducted concerning Formulation (2.9). Let us first mention that further work on this topic has been achieved in collaboration with R.Hiptmair and E.Spindler.

Numerical tests in 3D A first direction of work has consisted in implementing and testing this formulation not only in 2D but also in 3D. Elke Spindler has worked on such an implementation based on the Boundary Element Template Library (BETL) previously developed

at ETH Zürich by Lars Kielhorn. As regards programming, a remarkable difficulty in the present situations is that assembly of matrices can only be achieved using a numbering of unknowns that keeps track of interfaces between subdomains. Core assembly routines here remain the same as for a classical BEM code though. This work has been the subject of a presentation given by Elke at the Waves 2015 conference in Karlsruhe.

Scattering by a partially impenetrable scatterer Another direction of work has focused on extending this formulation to the case where one or more subdomains Ω_j in Problem (2.1) is impenetrable, which corresponds to imposing a Dirichlet or Neumann condition at the boundary of it. This work has been the subject of a research report [15] submitted for publication.

Pure transmission problems As was already mentioned in paragraph 2.3.1, we unfortunately still not have found how to prove Conjecture 2.3.1, so that absence of spurious resonance for Formulation (2.10) could not be established in the general case.

However, there exists a boundary value problem that is relevant from applicative point of view, and for which this uniqueness issue does not arise. Assume that $\kappa_0 = \kappa_1 = \cdots = \kappa_n = 0$, and the μ_j's take arbitrary non-negative values. Let f refer to a square integrable function with bounded support, and define $\mu(\boldsymbol{x}) := \mu_j$ for $\boldsymbol{x} \in \Omega_j$. This new problem consists in looking for

$$\begin{cases} u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d) \quad \text{such that} \\ -\mathrm{div}(\mu \nabla u) = f \quad \text{in } \mathbb{R}^d \\ \limsup_{|\boldsymbol{x}|\to\infty} |u(\boldsymbol{x})| < +\infty \end{cases}$$

In an ongoing work we have derived for the above problem a formulation analogous to (2.10). This new formulation once again involves $\mathbb{X}(\Sigma)$ but, this time, well-posedness can be fully established in this functional setting, because the same Green's kernel comes into play in all subdomains.

This formulation can also be reformulated in the framework of square integrable traces, and we are now investigating well posedness in this modified setting. This requires harmonic analysis techniques, see [125, 116]. In particular, it seems that we need to establish a well posedness result similar to the one obtained in [57] but for a multi-subdomain transmission problem (with junctions...).

2.5 Open questions

We finally also point two open questions that seem to us relevant.

No spurious resonance The proof that the conjecture of §2.3.1 either holds or is not true is an open question. Consider the case of Problem (2.1) with $\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_1 \cup \overline{\Omega}_2$, with $\mathrm{dist}(\Omega_1, \Omega_2) > 0$ (two well separated scatterer and no junction point) and $\kappa_0 \neq \kappa_1$, $\kappa_1 \neq \kappa_2$ and $\kappa_2 \neq \kappa_0$, and such that $\kappa_0, \kappa_1, \kappa_2 \in \imath\mathbb{R}$ (purely imaginary). Even for this apparently simple case we did not manage to prove this conjecture.

Location of eigenvalues When computing numerically the spectrum of the operator associated to Formulation (2.10), we have observed that it is systematically located in the right half complex plane $\{\lambda \in \mathbb{C},\ \Re e\{\lambda\} > 0\ \}$. We have no explanation for this, but it would be interesting to prove this result for the continuous operator of (2.10). Maybe one possible approach for investigating this question would consist in establishing a result stating that (part of) the operator involved in Formulation (2.10) is a contraction, trying to adapt the proof presented in [113, Thm.3.8.7] (that relies on a variational technique), or adapting the approach of [57].

Chapter 3

Multi-trace formalism

In the context of multi-subdomain scattering, the boundary integral formulation of the second kind presented in Section 2.3 provided a new answer to the issue of preconditioning as it is intrinsically well conditioned. In addition, it offers the possibility of discretising by means of collocation or Nyström approaches, see [65, 23, 32, 81, 54].

Yet it does not seem to lend itself to an efficient domain decomposition strategy, as its natural functional setting is the single-trace space $\mathbb{X}(\Sigma)$ that encompasses constraints coupling adjacent subdomains. Besides, in this formulation, each subdomain is coupled with any other subdomain, not just its neighbours, which induces fully populated (not even block sparse) matrices. Last but not least, the operator associated to this formulation is not a compact perturbation of the identity anymore in the case where Hypothesis (2.8) on the coefficients μ_j does not hold. Hence the performances of this formulation are not optimal for many situations of interest in physics.

This was a motivation for looking at yet other formulations still adapted to multi-subdomain scattering and at the same time prone to some efficient preconditioning strategy. The work presented in this chapter was initiated by a collaboration between the Seminar of Applied Mathematics at ETH Zürich, and the radio frequency modelling department of Thales Airbone System that, for practical reasons related to code development, was specifically interested in an integral formulation of the first kind which may lend itself to Calderón preconditioning. The outcome of this collaboration was a new theoretical framework developed in a series of articles (see [11, 8, 5, 14, 13, 10] as well as [68, 69]) that we are going to describe here.

This framework was dubbed multi-trace because it covers several new formulations sharing the common feature that, at each point of each interface, two pairs of Dirichlet/Neumann traces come into play. It does not involve just one pair of traces at interfaces like in the 2nd kind STF (2.10) or PMCHWT (2.7). In these formulations, the functional setting simply consists in a cartesian product of traces on each subdomain, so that the coupling between subdomains is enforced by the equation itself. We speak of a formalism because the formulations we are going to describe seem to be adaptable to any physical situation (with piece-wise constant propagation medium) and any geometrical configuration.

There exists two types of multi-trace formulations that differ in the manner they take account of transmission conditions. In the first type, called local Multi-Trace Formulations (later referred to as "local MTF"), transmission conditions are imposed by means of a local

operator so that only adjacent subdomains are coupled to each other. Hence local MTF induces block-sparse matrices (each block being associated with a subdomain). In the second type, called global Multi-Trace Formulations (later referred to as "global MTF"), transmission conditions are imposed by means of a fully non-local integral operator so that all subdomains are coupled to all other subdomains. As a consequence, global MTF induces fully populated matrices, non even block-sparse. We will conclude by presenting a third type of formulation, so called quasi-local multi-trace formulation (quasi-local MTF) that stands as intermediate between local and global MTF.

In this chapter we will present the derivation of these two types of formulation by considering the pure transmission problem (2.1), although one could consider more complicated mixed Dirichlet-Neumann-transmission boundary value problems instead, or even Maxwell's equations. We shall comment on such extensions at the end of the chapter.

3.1 Local multi-trace

We first present the derivation of the local MTF for Problem (2.1). This formulation was introduced in [68], and further investigated in [13, 14]. It has then also been studied from the point of view of domain decomposition in [102, 69]. The key ingredient in this formulation is what we shall call in the following a transmission operator Π defined by the formula

$$\Pi(\mathfrak{u}) = \mathfrak{v} \quad \Longleftrightarrow \quad \left\{ \begin{array}{ll} v_j = +u_k \\ q_j = -p_k \end{array} \right. \quad \text{on } \Gamma_{j,k} := \Gamma_j \cap \Gamma_k$$

where $\mathfrak{u} = (u_j, p_j)_{j=0}^n$ and $\mathfrak{v} = (v_k, q_k)_{k=0}^n$. Clearly, for any function $u \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^d)$ we have $\gamma(u) := (\gamma^j(u))_{j=0}^n \in \mathbb{L}^2(\Sigma)$ and $\gamma(u) = \Pi(\gamma(u))$. Conversely, considering any function $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ such that $u|_{\Omega_j} \in \mathrm{H}^2_{\mathrm{loc}}(\overline{\Omega}_j)$ for all $j = 0 \dots n$, then $\gamma(u) \in \mathbb{L}^2(\Sigma)$ is well defined, and if $\gamma(u) = \Pi(\gamma(u))$ then we have $u \in \mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^d)$. Routine calculus shows that the transmission operator satisfies the following remarkable identities

$$\Pi^2 = \mathrm{Id}\,, \quad \text{and} \quad [\![\Pi(\mathfrak{u}), \mathfrak{v}]\!] = [\![\Pi(\mathfrak{v}), \mathfrak{u}]\!] \quad \forall \mathfrak{u}, \mathfrak{v} \in \mathbb{L}^2(\Sigma). \tag{3.1}$$

As is readily checked, the operator Π maps continuously $\mathbb{H}(\Sigma)$ onto $\mathbb{H}(\Sigma)$ *under the assumption that* $\Gamma_{j,k} := \Gamma_j \cap \Gamma_k$ *is a Lipschitz manifold without boundary for any* j, k, which corresponds to situations where there is no junction point i.e. points where at least three subdomains abut. Without this assumption, Π does not map $\mathbb{H}(\Sigma)$ into itself. In the general case with junctions, we only have a continuity property $\Pi : \mathbb{H}(\Sigma) \to \widetilde{\mathbb{H}}(\Sigma)'$ where $\widetilde{\mathbb{H}}(\Sigma)'$ is the topological dual to the space

$$\begin{aligned} \widetilde{\mathbb{H}}(\Sigma) := \{\, \mathfrak{v} = (v_j, q_j)_{j=0}^n \in \mathbb{H}(\Sigma) \mid\ & v_j|_{\Gamma_{j,k}} \in \widetilde{\mathrm{H}}^{+\frac{1}{2}}(\Gamma_{j,k})\,, \\ & q_j|_{\Gamma_{j,k}} \in \widetilde{\mathrm{H}}^{-\frac{1}{2}}(\Gamma_{j,k}) \quad \forall j, k = 0 \dots n \,\} \end{aligned}$$

equipped with

$$\|\mathfrak{v}\|_{\widetilde{\mathbb{H}}(\Sigma)} := \Big(\sum_{j=0}^{n} \sum_{\substack{k=0 \\ k \neq j}}^{n} \|v_j\|^2_{\widetilde{\mathrm{H}}^{1/2}(\Gamma_{j,k})} + \|q_j\|^2_{\widetilde{\mathrm{H}}^{-1/2}(\Gamma_{j,k})} \Big)^{1/2}$$

where, following [91, Chap.3], for $\sigma = \pm 1/2$ we adopt the standard definitions $\mathrm{H}^\sigma(\Gamma_{j,k}) = \{\varphi|_{\Gamma_{j,k}} \ \varphi \in \mathrm{H}^\sigma(\Gamma_j)\}$ and $\widetilde{\mathrm{H}}^\sigma(\Gamma_{j,k}) := \mathrm{H}^{-\sigma}(\Gamma_{j,k})'$.

Let us come back to the pure transmission problem (2.1). Assuming that $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ is the unique solution to this problem, and taking $\mathfrak{u} = (\gamma^j(u))_{j=0}^n$ as unknown, we start by reformulating the wave equations (2.1a) like for PMCHWT using Calderón's projector $(\mathbb{A}_{(\kappa)} - \mathrm{Id})\mathfrak{u} = 2\,\mathfrak{u}^{\mathrm{inc}}$. Plugging the transmission conditions $\mathfrak{u} = \Pi(\mathfrak{u})$ into this equation finally yields the formulation

$$\begin{cases} \mathfrak{u} \in \mathbb{H}(\Sigma) \quad \text{and} \\ [\![(\mathbb{A}_{(\kappa)} - \Pi)\mathfrak{u}, \mathfrak{v}]\!] = 2[\![\mathfrak{u}^{\mathrm{inc}}, \mathfrak{v}]\!] \quad \forall \mathfrak{v} \in \widetilde{\mathbb{H}}(\Sigma) \end{cases} \tag{3.2}$$

This variational formulation admits a unique solution, see [68, Thm. 9 & 11].

Lemma 3.1.1.
For any $\mathfrak{f} \in \mathbb{H}(\Sigma)$*, the formulation (3.2) admits a unique solution* $\mathfrak{u} \in \mathbb{H}(\Sigma)$ *that depends continuously on* $\mathfrak{f}$*. In particular* $\ker(\mathbb{A}_{(\kappa)} - \Pi) = \{0\}$.

Whether this formulation satisfies a Garding inequality remains an open question though. For this reason, so far, uniform discrete inf-sup conditions (and hence quasi-uniform approximation property) for Galerkin discretisations of (3.2) could only be established in the particular case of 2-D scalar problems.

3.2 Global multi-trace

We shall now present the derivation of the global MTF introduced for the first time in [10]. It was developed in parallel to the local MTF, and completely independently. This is a type of formulation strictly different from local MTF and, there is no *a priori* direct relationship between the two formulations. However there are similarities which raises the natural question of trying to find a general framework encompassing both local and global MTF. We will comment on this in the next section.

3.2.1 The gap idea

We first provide an intuition of the derivation of the global MTF. Consider Problem (2.1), but in a slightly modified geometrical configuration. In the modified configuration, the subdomains $\Omega_j, j = 1 \dots n$ are teared apart, which introduces a thin gap of exterior material associated to Ω_0, see Figure 3.1.

In the new geometry, there is no junction anymore and we have $\Gamma_0 = \cup_{j=1}^n \Gamma_j$ and $\Gamma_j \cap \Gamma_k = \emptyset$ for $j, k \neq 0$ and $j \neq k$. With this modified geometrical configuration we can write the PMCHWT formulation (2.7), and eliminate the contributions attached to Γ_0 by means of the relations

$$\mathfrak{v}_0|_{\Gamma_j} + \theta \cdot \mathfrak{v}_j = 0 \qquad \forall j = 1 \dots n, \ \ \forall \mathfrak{v} = (\mathfrak{v}_j)_{j=0}^n \in \mathbb{X}(\Sigma)$$

$$\text{where} \quad \theta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

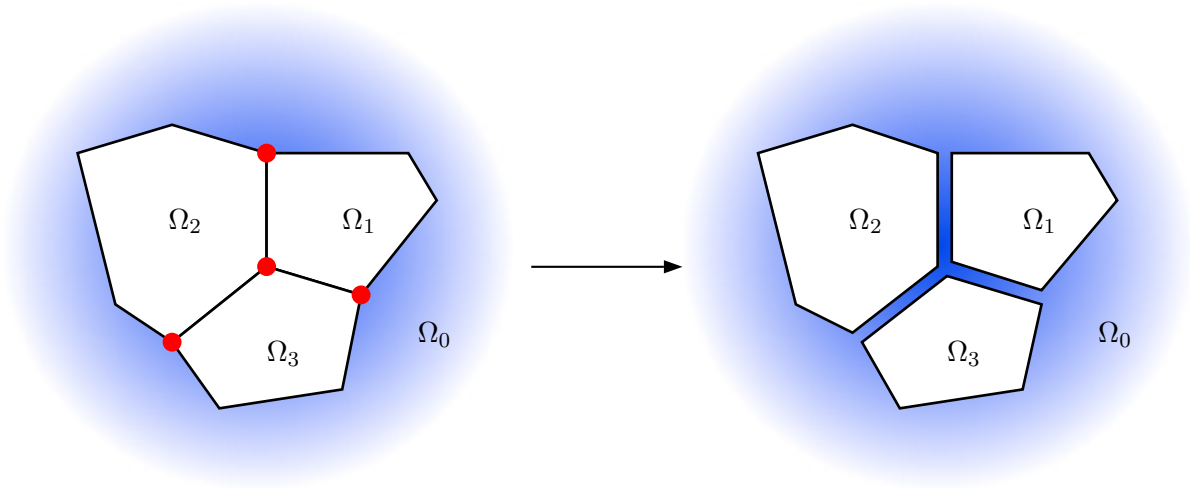


Figure 3.1: Geometrical configuration with junction points (in red): initial geometry (left) and modified geometry with gaps (right).

This leads to a natural rearrangement of PMCHWT not containing any contribution attached to Γ_0 anymore, so that the corresponding functional framework is now a reduced multi-trace space

$$\widehat{\mathbb{H}}(\Sigma) := \Pi_{j=1}^{n} \mathbb{H}(\Gamma_j).$$

We also equip this space with the duality pairing $[\![\mathfrak{u}, \mathfrak{v}]\!] = \sum_{j=1}^{n} [\mathfrak{u}_j, \mathfrak{v}_j]_{\Gamma_j}$ (we take the same notation as for the pairing of $\mathbb{H}(\Sigma)$, which should not bring confusion). The transformed bilinear form contains contributions inherited from Γ_0 that couple each Γ_j with any other Γ_k, for $j, k \neq 0$. It is given by the formula

$$[\![\hat{\mathbb{A}}_{(\kappa)}\hat{\mathfrak{u}}, \hat{\mathfrak{v}}]\!] = \sum_{j=1}^{n} [(\mathrm{A}^j_{\kappa_j} + \mathrm{A}^j_{\kappa_0})\mathfrak{u}_j, \mathfrak{v}_j]_{\Gamma_j} + \sum_{j=1}^{n} \sum_{\substack{k=1 \\ k\neq j}}^{n} [\gamma^j \cdot \mathrm{G}^k_{\kappa_0}(\mathfrak{u}_k), \mathfrak{v}_j]_{\Gamma_j} \tag{3.3}$$

defined for all $\hat{\mathfrak{u}}, \hat{\mathfrak{v}} \in \widehat{\mathbb{H}}(\Sigma)$. PMCHWT re-arranged in this manner now takes the following form, where $\mathfrak{f} \in \widehat{\mathbb{H}}(\Sigma)$ is some right-hand side related to $\mathfrak{u}^{\mathrm{inc}}$,

$$\begin{cases} \text{Find } \hat{\mathfrak{u}} \in \widehat{\mathbb{H}}(\Sigma) \text{ such that} \\ [\![\hat{\mathbb{A}}_{(\kappa)}\hat{\mathfrak{u}}, \hat{\mathfrak{v}}]\!] = [\![\mathfrak{f}, \hat{\mathfrak{v}}]\!] \quad \forall \mathfrak{v} \in \widehat{\mathbb{H}}(\Sigma). \end{cases} \tag{3.4}$$

Observe that all the terms coming into play in the bilinear form (3.3), as well as the space $\widehat{\mathbb{H}}(\Sigma)$, still make sense when considering the initial geometry (see picture on the left in Figure 3.1). The global multi-trace formulation simply consists in writing (3.4) for this initial geometry, no matter whether there are junction points or not.

The gap idea described above is a convenient insight for describing the global multi-trace formulation. It was initially suggested by Jean-Claude Nédélec in personnal discussions. It is not a technical ingredient of the actual rigorous derivation of the global MTF though. Complete rigorous derivation of this formulation was provided in [10] based on the results of Section 2.1, and in particular thorough analysis of PMCHWT.

3.2.2 Well posedness and Garding inequality

The operator coming into play in the global MTF enjoys several interesting properties. First, this formulation is free of spurious resonance. The following result was established in [10, Thm.10.5].

Theorem 3.2.1.
In the general case of wave numbers satisfying (2.2) the operator $\hat{\mathbb{A}}_{(\kappa)} : \widehat{\mathbb{H}}(\Sigma) \to \widehat{\mathbb{H}}(\Sigma)$ *is an isomorphism. Moreover the unique solution to (3.4) is* $\hat{\mathfrak{u}} = (\gamma^j(u))_{j=1}^n$ *where* u *is the unique solution to the scattering problem (2.1).*

In addition, a modified Garding inequality can be proved for this operator, which guarantees quasi-optimal convergence of Galerkin discretisations, see [10, Thm.10.4] for a proof of the next theorem.

Theorem 3.2.2.
Set $\theta(v,q) = (-v,q)$ *and* $\Theta(\hat{\mathfrak{v}}) = (\theta(\mathfrak{v}_j))_{j=1}^n$. *The exists a constant* $C > 0$ *and a compact operator* $\mathrm{R} : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ *such that*

$$\Re e\{[\![(\hat{\mathbb{A}}_{(\kappa)} + \mathrm{R})\hat{\mathfrak{v}}, \overline{\Theta(\hat{\mathfrak{v}})}]\!]\} \geq C \left\|\hat{\mathfrak{v}}\right\|^2_{\widehat{\mathbb{H}}(\Sigma)} \qquad \forall \hat{\mathfrak{v}} \in \widehat{\mathbb{H}}(\Sigma).$$

Note in addition that there is no a priori requirement that the discretisation chosen for one subdomain Γ_j be related to the one chosen for another subdomain Γ_k even if they share a common interface $\Gamma_j \cap \Gamma_k \neq \emptyset$. This paves the way to domain-decomposition based on non-conforming grids, in spite of potential mild complications related to quadrature of BEM integrals.

We conclude this section by mentioning an algebraic identity satisfied by the operator of the global multi-trace formulation that generalizes Calderón's identity mentioned at the end of §1.2. The following result was established in [10, Thm.11.1].

Theorem 3.2.3.
Assume that all wave numbers equal $\kappa_0 = \cdots = \kappa_n$. *Then we have* $(\hat{\mathbb{A}}_{(\kappa)})^2 = \mathrm{Id}$.

Finally we point out that, since $\hat{\mathbb{A}}_{(\kappa)}$ maps $\widehat{\mathbb{H}}(\Sigma)$ onto $\widehat{\mathbb{H}}(\Sigma)$, it naturally lends itself to the operator preconditioning strategy presented in [67]. Considering a finite dimensional subspace $\widehat{\mathbb{H}}_h \subset \widehat{\mathbb{H}}(\Sigma)$, let us denote A_h (resp. M_h) the Galerkin matrices obtained from the bilinear forms $[\![\hat{\mathbb{A}}_{(\kappa)}\cdot, \cdot]\!]$ (resp. $[\![\cdot, \cdot]\!]$). Consider also, for example, the matrix D_h obtained from a Galerkin discretisation of the block diagonal bilinear form $[\![\mathrm{D}_{(\kappa)}\mathfrak{u}, \mathfrak{v}]\!] = \sum_{j=1}^n [\mathrm{A}^j_{\kappa_j}\mathfrak{u}_j, \mathfrak{v}_j]_{\Gamma_j}$. Finally assume that the discretisation is chosen so as to guarantee that the pairing $[\![\cdot, \cdot]\!]$ satisfies a uniform discrete inf-sup property. Then, according to [67, Thm.2.1], the spectral condition number of the matrix $(\mathrm{M}_h^{-1} \cdot \mathrm{D}_h \cdot \mathrm{M}_h^{-T}) \cdot \mathrm{A}_h$ remains bounded independently of h.

3.2.3 Numerical results

We now present numerical results confirming the quasi-optimal approximation property of the global MTF together with the possibility of an efficient preconditioning strategy for it.

Beforehand, we would like to underline how tricky is the implementation of multi-trace boundary integral equations. This is related to the necessity of carefully handling adjacency

between subdomains. This prevents the use of black-box softwares that take surface boundary meshes as input, and give boundary integral operator matrices as output. This does not prevent from using classical elementary assembly routine that would take a pair of elements as input, and would return elementary interactions as output. Since there only exists very few freely available boundary elements libraries (essentially BEM++ [114] and BETL [71]), it has appeared more profitable to redevelop a "home-made" code that would be better adapted to handling multi-domain geometries (with junction points) and multi-trace formulations.

The outcome of this programming effort was a boundary element code in 2D based on continuous piece-wise linear or piece-wise constant shape functions for Laplace and Helmholtz problems. This code is based on data structures specially adapted to multi-subdomain geometrical configurations involving junction points.

The numerical results presented here have been obtained with this code. All computations have been achieved on a single laptop equipped with a dual core Intel i7-3520M processor at 2.9GHz with 4GB of RAM. Meshes have been generated using Gmsh [60] (see also `http://geuz.org/gmsh/`), and, for linear algebra including GMRes routines, we relied on the Gmm++ library (see `http://download.gna.org/getfem/html/homepage/gmm/index.html`). We considered Problem (2.1) with the geometrical configuration pictured below.

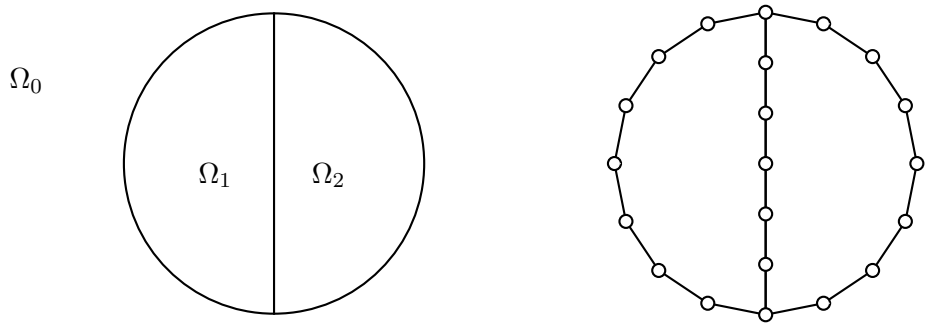
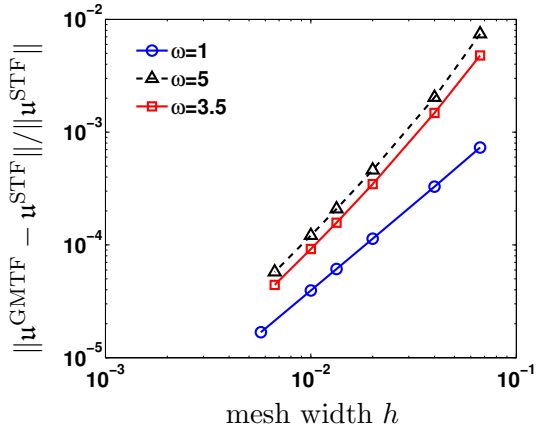


Figure 3.2: Geometrical configuration of numerical experiments on global MTF

We take $u_{\text{inc}}(\boldsymbol{x}) = \exp(\imath\kappa_0 \mathbf{d} \cdot \boldsymbol{x})$ with $\mathbf{d} = (1, 0, 0)$ as incident field. Based on a Galerkin discretisation, we approximate both Dirichlet and Neumann traces by means of continuous $\mathbb{P}_1$ shape functions. We take the discrete solution to PMCHWT formulation (2.7) as reference. The first series of numerical results presented below focuses on consistency. It shows the relative error between the solution $\mathfrak{u}^{\text{GMTF}}$ of the global MTF, and the solution $\mathfrak{u}^{\text{STF}}$ of PMCHWT versus h = max. of the length of the panels of the mesh.

In Figure 3.4, we examine the convergence history of GMRes applied to the global MTF (denoted "multi-trace"), to PMCHWT (denoted "single-trace"), and also to a preconditioned version of the global MTF where we use the operator preconditioning strategy mentioned at the end of §3.2.2 above. The values for the material characteristics (i.e. the κ_j's and μ_j's) are indicated in Fig.3.3.



Material parameters
$\kappa_0 = \omega,\ \mu_0 = 1$
$\kappa_1 = 2\omega,\ \mu_1 = 1/2$
$\kappa_2 = 3\omega,\ \mu_2 = 2$

Figure 3.3: Relative error versus mesh width: consistency of global MTF for three different frequency values

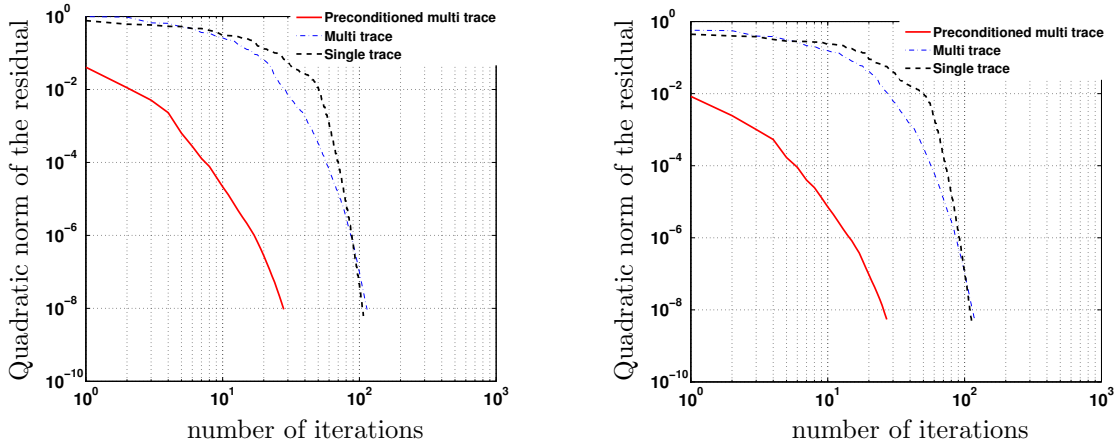


Figure 3.4: Quadratic norm of GMRes residue versus number of iterations for mesh width $h = 0.02$ (left) and $h = 0.0066$ (right).

3.3 Quasi-local multi-trace

The local MTF looks more appealing than the global MTF because it is block sparse. At the same time, much more theoretical results can be established about the global MTF, whereas local MTF is more difficult to analyze. Besides, the relationship between these two formulations is not so clear at first sight. It was thus natural to try to clarify this relationship so that theoretical results already available for the global MTF could be transferred to the local MTF.

This was the purpose of the work presented in [5] that led to the derivation of quasi-local MTF which are family of formulations that appear as intermediate between local and global MTF. Quasi-local MTF can be seen as a sparsified version of global MTF. It can also be seen as a regularized version of local MTF. The idea is to derive a formulation similar to the local MTF where the transmission operator Π is replaced by another operator $\widetilde{\Pi}$ that

coincides with Π outside some neighbourhood of junctions, that is fully non-local only inside neighbourhoods of junctions, and is continuous as an operator mapping $\mathbb{H}(\Sigma)$ to $\mathbb{H}(\Sigma)$.

3.3.1 Non-local transmission operator

The idea underlying this alternative transmission operator is based on the simple observation that the operator $\mathbb{P}_{\kappa_0}$ introduced in §2.3.2 is a projector. Actually, one can propose a whole class of quasi-local transmission operators. To describe it, let us consider a kernel $\mathscr{K}(\boldsymbol{x})$ that coincides, in a neighbourhood of $\boldsymbol{x}=0$, with kernel $\mathscr{K}_0(\boldsymbol{x})$ satisfying $-\Delta\mathscr{K}_0+\mu\mathscr{K}_0=\delta_0$ for some $\mu\in\mathbb{C}$. For any $\mathfrak{v}=(v,q)\in\Gamma_j$, denote

$$\mathrm{K}^j(\mathfrak{v})(\boldsymbol{x})=\int_{\Gamma_j} q(\boldsymbol{y})\mathscr{K}(\boldsymbol{x}-\boldsymbol{y})+v(\boldsymbol{y})\boldsymbol{n}_j(\boldsymbol{y})\cdot(\nabla\mathscr{K})(\boldsymbol{x}-\boldsymbol{y})\,d\sigma(\boldsymbol{y}) \tag{3.5}$$

This operator is defined similarly to G^j_κ, except that here we do not impose that the kernel $\mathscr{K}$ be a Green kernel. The kernel $\mathscr{K}$ actually does not need to satisfy any particular equation. In practice, one can simply choose $\mathscr{K}(\boldsymbol{x})=\psi(\boldsymbol{x})\mathscr{K}_0(\boldsymbol{x})$ where ψ is a $\mathscr{C}^\infty$ cut-off function with bounded support satisfying $\psi(\boldsymbol{x})=1$ for $|\boldsymbol{x}|\leq\rho$ (for some fixed $\rho>0$), and $\mathscr{K}_0(\boldsymbol{x})$ is the Green kernel of the Laplace equation. We only need it to have the appropriate singularity at $\boldsymbol{x}=0$. Similarly to $\mathbb{P}_{(\kappa)}$, define $\mathbb{K}:\mathbb{H}(\Sigma)\to\mathbb{H}(\Sigma)$ by

$$[\![\mathbb{K}(\mathfrak{u}),\mathfrak{v}]\!]:=\sum_{j=0}^n\sum_{q=0}^n[\gamma^j\cdot\mathrm{K}^q(\mathfrak{u}_q),\mathfrak{v}_j]_{\Gamma_j}\qquad\forall\mathfrak{u},\mathfrak{v}\in\mathbb{H}(\Sigma).$$

Due to its structure, the operator $\mathbb{K}$ inherits many properties from classical potential operators. It satisfies jump formulas, as well as symmetry properties. The following proposition was established with [5, Lemma 6.2 & Cor.6.1].

Proposition 3.3.1.

i) $[\![\mathbb{K}(\mathfrak{u}),\mathfrak{v}]\!]=[\![\mathbb{K}(\mathfrak{v}),\mathfrak{u}]\!]+[\![\mathfrak{u},\mathfrak{v}]\!]$

ii) The operator $\mathbb{K}$ *is a continuous projector and* $\ker(\mathbb{K})=\mathbb{X}(\Sigma)$.

These remarkable properties allow to devise a continuous transmission operator, well defined over $\mathbb{H}(\Sigma)$ unlike Π, and adapted to a characterization of the single trace space $\mathbb{X}(\Sigma)$, see [5, Prop.6.1]

Proposition 3.3.2.
The map $\mathbb{Q}:=\mathrm{Id}-2\,\mathbb{K}$ *is a continuous operator satisfying:*

i) $\mathbb{Q}^2=\mathrm{Id}$

ii) $[\![\mathbb{Q}(\mathfrak{u}),\mathfrak{v}]\!]=[\![\mathbb{Q}(\mathfrak{v}),\mathfrak{u}]\!]\ \ \forall\mathfrak{u},\mathfrak{v}\in\mathbb{H}(\Sigma)$

iii) $\mathfrak{v}\in\mathbb{X}(\Sigma)\iff\mathbb{Q}(\mathfrak{v})=\mathfrak{v}$

In addition, since the kernel $\mathscr{K}$ differs from a standard Green kernel outside a neighbourhood of 0, the operator $\mathbb{K}$ is a compact perturbation of an operator of the form $\mathbb{P}_\alpha$. This allows to prove a Garding inequality for $\mathbb{Q}$. The next result was established in [5, Prop.6.2].

Proposition 3.3.3.
Define $\theta(v,q):=(-v,q)$ *and* $\Theta(\mathfrak{v})=(\theta(\mathfrak{v}_j))_{j=0}^n$ *for* $\mathfrak{v}=(\mathfrak{v}_j)_{j=0}^n$. *Then there exists a compact operator* $\mathrm{R}:\mathbb{H}(\Sigma)\to\mathbb{H}(\Sigma)$ *and a constant* $C>0$ *such that*

$$\Re e\{[\![(-\mathbb{Q}+\mathrm{R})\mathfrak{v},\Theta(\overline{\mathfrak{v}})]\!]\}\geq C\|\mathfrak{v}\|^2_{\mathbb{H}(\Sigma)}\qquad\forall\mathfrak{v}\in\mathbb{H}(\Sigma).$$

3.3.2 Quasi-local transmission operator

Although the transmission operator we have just constructed has many interesting properties it has one important drawback from the point of view of numerics: it is a fully non-local operator which, under Galerkin discretisation, will induce densely populated matrices. This is why we introduce a process to "sparsify" it. Let

$$\mathscr{J} = \bigcup_{0\leq j<k<l\leq n} \Gamma_j \cap \Gamma_k \cap \Gamma_l$$

refer to the set of junction points. Let $\mathscr{V}, \mathscr{V}'$ be two open neighbourhoods of this set satisfying $\mathscr{J} \subset \mathscr{V} \subset \overline{\mathscr{V}} \subset \mathscr{V}'$. The neighbourhood $\mathscr{V}'$ may be chosen as small and narrow as desired.

Next consider a $\mathscr{C}^\infty$ cut-off function ψ such that $\psi = 1$ in $\mathscr{V}$, and $\psi = 0$ in $\mathbb{R}^d \setminus \mathscr{V}'$. We also set $\chi = 1 - \psi^2$. Finally we define a continuous operator $\widetilde{\Pi} : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ by the formula

$$[\![\widetilde{\Pi}(\mathfrak{u}), \mathfrak{v}]\!] := [\![\Pi(\chi\mathfrak{u}), \mathfrak{v}]\!] + [\![\mathbb{Q}(\psi\mathfrak{u}), \psi\mathfrak{v}]\!] \qquad \forall \mathfrak{u}, \mathfrak{v} \in \mathbb{H}(\Sigma). \tag{3.6}$$

This new operator coincides with Π outside $\mathscr{V}'$, and is thus no-local only inside this neighbourhood that can be chosen as small and narrow as desired. It satisfies the same remarkable properties as $\mathbb{Q}$, see [5, Prop.6.4].

Proposition 3.3.4.
There exists a compact operator $\mathrm{R} : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ *and a constant* $C > 0$ *such that, for any* $\mathfrak{v} \in \mathbb{H}(\Sigma)$ *the following inequality holds* $\Re e\{[\![(-\widetilde{\Pi} + \mathrm{R})\mathfrak{v}, \Theta(\overline{\mathfrak{v}})]\!]\} \geq C\|\psi\mathfrak{v}\|^2_{\mathbb{H}(\Sigma)}$. *In addition we have:*

i) $(\widetilde{\Pi})^2 = \mathrm{Id}$

ii) $[\![\widetilde{\Pi}(\mathfrak{u}), \mathfrak{v}]\!] = [\![\widetilde{\Pi}(\mathfrak{v}), \mathfrak{u}]\!] \ \ \forall \mathfrak{u}, \mathfrak{v} \in \mathbb{H}(\Sigma)$

iii) $\mathfrak{v} \in \mathbb{X}(\Sigma) \iff \widetilde{\Pi}(\mathfrak{v}) = \mathfrak{v}$

3.3.3 Quasi-local multi-trace formulation

Now let us see how to use this operator to reformulate the scattering problem (2.1). We follow the same path as for the local multi-trace formulation: we rewrite wave equations as $(-\mathrm{Id} + \mathbb{A}_{(\kappa)})\mathfrak{u} = 2\mathfrak{u}^{\mathrm{inc}}$, and then plug the transmission equation into it $\mathfrak{u} = \widetilde{\Pi}(\mathfrak{u})$. Writing this in a weak form finally yields

$$\begin{cases} \mathfrak{u} \in \mathbb{H}(\Sigma) \quad \text{and} \\ [\![(\mathbb{A}_{(\kappa)} - \widetilde{\Pi})\mathfrak{u}, \mathfrak{v}]\!] = 2[\![\mathfrak{u}^{\mathrm{inc}}, \mathfrak{v}]\!] \quad \forall \mathfrak{v} \in \widetilde{\mathbb{H}}(\Sigma) \end{cases} \tag{3.7}$$

This time, thanks to the first property mentioned in Proposition 3.3.4, we can prove that the operator satisfies a modified Garding inequality so that it is of Fredholm type with index 0. The next result was established in [5, Thm.7.1].

Theorem 3.3.1.
Assume that the wave numbers $\kappa_0, \ldots, \kappa_n$ *satisfy (2.2). Then there exists a compact operator* $\mathrm{R} : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ *and a constant* $C > 0$ *such that*

$$\Re e\{[\![(\mathbb{A}_{(\kappa)} - \widetilde{\Pi} + \mathrm{R})\mathfrak{v}, \Theta(\overline{\mathfrak{v}})]\!]\} \geq C\|\mathfrak{v}\|^2_{\mathbb{H}(\Sigma)} \qquad \forall \mathfrak{v} \in \mathbb{H}(\Sigma).$$

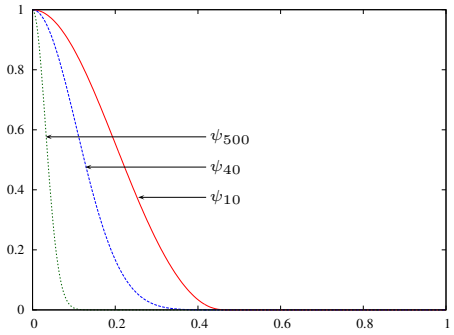
Well posedness can be guaranteed under an additional assumption on the transmission operator: it is sufficient that the kernel be real valued, see [5, Thm.7.2].

Proposition 3.3.5.
Assume that $\mathscr{K}$ is real valued, and the wave numbers satisfy (2.2). Then $\mathbb{A}_{(\kappa)} - \widetilde{\Pi} : \mathbb{H}(\Sigma) \to \mathbb{H}(\Sigma)$ is one-to-one, and hence is a continuous isomorphism.

3.3.4 Numerical experiments

In this paragraph we present numerical results that aim at studying the effective computational properties of the quasi-local formulation. We consider once again Problem (2.1) in the geometrical configuration of §3.2.3, and take $u_{\text{inc}}(\boldsymbol{x}) = \exp(\imath\kappa_0 \mathbf{e}_1 \cdot \boldsymbol{x})$ as incident field, and the material parameters $\kappa_0 = 1$, $\kappa_1 = 5$, $\kappa_2 = 0.5$, and $\mu_0 = \mu_1 = \mu_2 = 1$. The results presented here involve the discrete solutions to three different boundary integral formulations

- $\mathfrak{u}_h^{\mathrm{L}}$ solves local MTF (3.2)
- $\mathfrak{u}_h^{\mathrm{QL}}$ solves quasi-local MTF (3.7)
- $\mathfrak{u}_h^{\mathrm{STF}}$ solves PMCHWT (2.7)



Once again we used a Galerkin discretisation by means of continuous piece-wise linear shape functions for both Dirichlet and Neumann traces. When assembling the matrix associated to the operator $\widetilde{\Pi}$ we considered the cut-off function $\psi(\boldsymbol{x}) := \psi_\alpha(|\boldsymbol{x} - \boldsymbol{x}_+|) + \psi_\alpha(|\boldsymbol{x} - \boldsymbol{x}_-|)$ with

$$\psi_\alpha(t) := \exp(-\alpha t^2) \exp\left(4t^2/(4t^2-1)\right) 1_{4t^2<1}$$

where $\boldsymbol{x}_\pm := (0, \pm 1)$ are the junction points (see Figure 3.2), and $\alpha > 0$ is some fitting parameter allowing to control how much the function ψ is concentrated around junction points. For the first results we consider the value $\alpha = 40$. As regards the kernel coming into play in (3.5), we chose the Laplace kernel (which guarantees that Proposition 3.3.5 holds)

$$\mathscr{K}(\boldsymbol{x}) = -(2\pi)^{-1} \ln |\boldsymbol{x}|$$

The results of Figure 3.5 focus on consistency. They show that quasi-local MTF provides slightly better approximation than the local MTF (about twice more accurate for the same mesh width).

Let $\mathrm{M}_h, \mathrm{B}_h^{\mathrm{L}}, \mathrm{B}_h^{\mathrm{QL}}$ be the matrices obtained by Galerkin discretisation of the bilinear forms $[\![\cdot,\cdot]\!]$, $[\![(\mathrm{A}-\Pi)\cdot,\cdot]\!]$, $[\![(\mathrm{A}-\widetilde{\Pi})\cdot,\cdot]\!]$. Figure 3.6 below represents the spectrum of the matrices $\mathrm{M}_h^{-1}\mathrm{B}_h^{\mathrm{L}}/2$ and $\mathrm{M}_h^{-1}\mathrm{B}_h^{\mathrm{QL}}/2$ that approximate the spectrum of $(\mathrm{A}-\Pi)/2$ and $(\mathrm{A}-\widetilde{\Pi})/2$.The eigenvalues have been computed by means of the Arpack++ library (see `http://www.caam.rice.edu/software/ARPACK/arpack++.html`), using a mesh with 167 nodes corresponding to 672 unknowns for both local and quasi-local multi-trace discrete formulations. In both cases, eigenvalues clearly cluster in two packets symmetric with respect to the origin.

In Figure 3.7 we have represented the convergence history of GMRes iterations when applied to the matrices $\mathrm{M}_h^{-1}\mathrm{B}_h^{\mathrm{QL}}$ and $\mathrm{M}_h^{-1}\mathrm{B}_h^{\mathrm{L}}$ corresponding respectively to quasi-local and local multi-trace. We used a restart of 20 iterations and a threshold of 10^{-8}. Convergence of GMRes is stable with respect to the mesh width in both cases, and both formulations converge at a comparable speed.

h	$\|\mathfrak{u}_h^{QL} - \mathfrak{u}_h^{STF}\|_{\mathbb{H}}$	$\|\mathfrak{u}_h^{L} - \mathfrak{u}_h^{STF}\|_{\mathbb{H}}$
0.0981353	0.00563797	0.010623
0.0490825	0.00127879	0.00237613
0.0253347	0.000310327	0.000574992
0.00994171	4.77015e-05	8.70664e-05
0.00500253	1.18548e-05	2.15064e-05

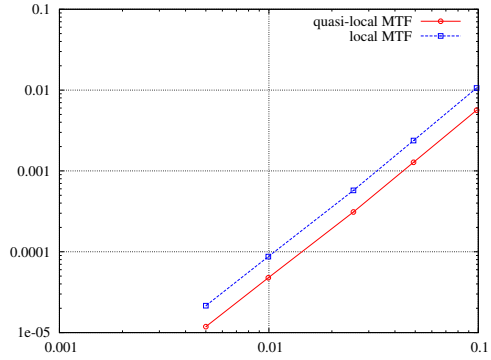


Figure 3.5: Error versus mesh width: consistency of local and quasi-local MTF

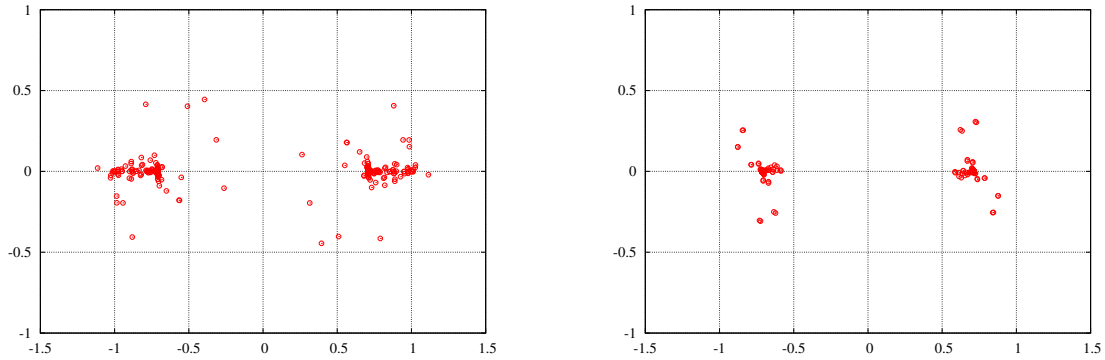


Figure 3.6: Approximate spectrum of the operator associated to the quasi-local (left) and local (right) multi-trace formulation

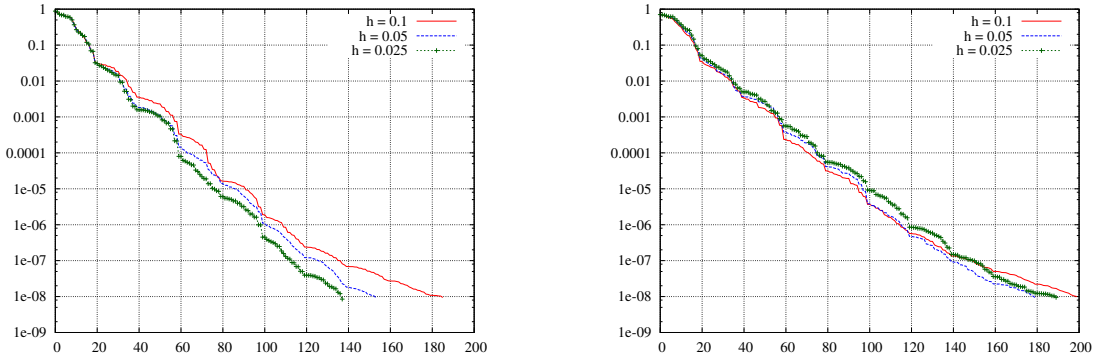


Figure 3.7: Behaviour of the quadratic norm of the GMRes residual for both quasi-local (left) and local (right) multi-trace formulation

In the table below, we compare the errors $\|\mathfrak{u}_h^{\mathrm{QL}} - \mathfrak{u}_h^{\mathrm{STF}}\|_{\mathbb{H}}$ associated to different values of the mesh width and for $\alpha = 10, 40, 100, 1000$. The value of this parameter seems to have very little impact on the accuracy.

	Error $\|\mathfrak{u}_h^{\mathrm{QL}} - \mathfrak{u}_h^{\mathrm{STF}}\|_{\mathbb{H}}$ with			
h	$\alpha = 10$	$\alpha = 40$	$\alpha = 100$	$\alpha = 1000$
0.0981353	0.00572272	0.00563797	0.0054629	0.00505463
0.0490825	0.00128214	0.00127879	0.00127138	0.00115966
0.0253347	0.000310538	0.000310327	0.000309988	0.000303461
0.00994171	4.77078e-05	4.77015e-05	4.7693e-05	4.75972e-05
0.00500253	1.18553e-05	1.18548e-05	1.18541e-05	1.18484e-05

Finally, in Figure 3.8 below we examine the impact of a variation of the parameter α on the convergence of GMRes solver (with a restart of 20 iterations) applied to quasi-local multi-trace formulation at a fixed mesh width $h = 0.05$. We can see that convergence deteriorates as $\alpha \to \infty$: for a fixed mesh, the steeper the function ψ, the slower the convergence.

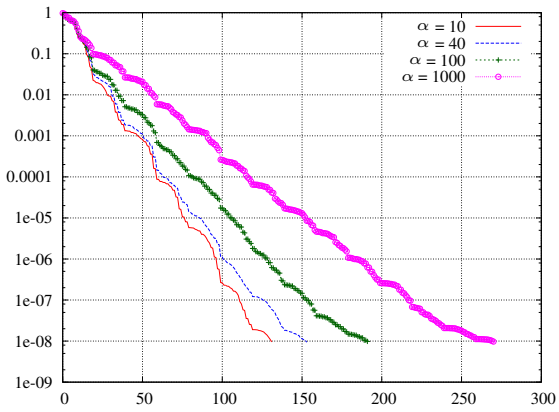


Figure 3.8: Behaviour of the quadratic norm of the GMRes residual for the quasi-local multi-trace formulation with more or less concentrated cut-off functions

In conclusion, the results above suggest to choose a cut-off function ψ supported on a limited number of mesh cells around the junctions. The quasi-local approach then appears as a post-treatment applied to the local mutli-trace formulation at junction points. An improvement in consistency appears as a first numerical advantage. Taking a cut-off function with a larger support will induce, in addition, gains in the convergence of iterative solvers. This can be obtained at the price of implementing the operator $\widetilde{\Pi}$ which only requires calling standard boundary element assembly routines.

3.4 Extensions

Here we describe remarkable extensions to the work presented previously in this chapter. They concern the global multi-trace formulation (3.4).

3.4.1 Arbitrary material parameters

For the sake of clarity, all through this chapter, we have considered that $\mu_0 = \mu_1 = \cdots = \mu_n$. This indeed significantly simplifies the notations. However, as regards global multi-trace formulation (3.4), in [10] we considered the general case where the μ_j's could take any arbitrarily real non-negative values. The approach adopted in this article can be used without any particular difficulty to extend the analysis of the quasi-local multi-trace formulation (3.7) to the case of arbitrary μ_j's. To sum up, *all the results presented in this chapter can be extended to the case where the μ_j's take arbitrary real non-negative values.*

3.4.2 Composite scatterer with impenetrable parts

Another extension concerns the case where some of the subdomains are non-penetrable. In the context of scalar scattering (acoustic for example) a Dirichlet or Neumann boundary condition is imposed at the boundary of such subdomains. This situation was addressed in [11] that proposed several extensions of the global multi-trace formulation in this direction.

When considering a single scatterer, a phenomenon of so-called spurious resonance may arise at certain critical values of the wave numbers. The possibility of spurious resonance for the PMCHWT formulation (2.7) had already been mentioned in [127], but this reference did not give a clear criterion for deciding when do such resonance appear (or do not appear).

In [11] we proved that PMCHWT only suffers from spurious resonance when the non-penetrable parts are completely surrounded by another homogeneous penetrable subdomain and, in this case, spurious resonance arises at the same critical values of the wave number as in the case of a single scatterer.

In this contribution, we also showed that the gap idea can be applied to this case also for the derivation of a global multi-trace formulation. Finally, we showed that both PMCHWT and global MTF can be modified so as to treat the non-penetrable parts by means of a Generalized Combined Field Integral Equation approach (so called GCFIE, see [18, 22, 31]), and we proved that this eliminates spurious resonances.

3.4.3 Electromagnetics

In the contribution [8], we adapted the results of [10] (that focuses on the global MTF) to the case of Maxwell's equations with arbitrary coefficients. We proved completely analog well-posedness results for electromagnetics.

We can only prove a modified Garding's inequality, but this is rather classical a situation (in the context of electromagnetics) and still yields Fredholmness (with index 0) for the global multi-trace boundary integral operators. In addition, the analysis presented in [10] fits the framework of [35] which allows to use our modified Garding's inequality for establishing a uniform discrete inf-sup condition for Galerkin discretisations of the global multi-trace formulation.

In conclusion, although in [8] we only considered pure transmission problems, there seems to be no particular theoretical obstruction to adapting this analysis for the case of partially non-penetrable composite scatterers.

Chapter 4

Scattering at multi-screens

In contrast to preceding chapters, we will not consider here transmission problems with piecewise homogeneous media. We will rather be interested in wave propagation in an homogeneous medium perturbed by the presence of a multi-screen scatterer. This type of scatterer consists in complex arrangements of thin (i.e. with thickness much smaller than the wavelength) impenetrable panels. From a geometrical point of view, multi-screens are non-trivial instances of rectifiable varifolds. We considered this scattering problem in [9, 12], with a particular focus on boundary integral formulations.

From a geometrical point of view, screens are infinitely thin impenetrable scatterers taking the form of (at least) Lipschitz manifolds with boundaries. These are the commonly adopted mathematical objects for modelling metallic substrates in electromagnetic contexts. In the mathematical literature, analysis of wave scattering by screens had already received much attention in the past. Early works [119, 17, 50, 120, 74, 75] focused on well-posedness, analytic properties and numerical analysis of boundary integral operators, in the case of Maxwell's equations or elliptic problems. In the case of electromagnetics, lowering the geometrical regularity assumptions on the screen was a challenge due to the cumbersome functional analytic natural framework of Maxwell's equations. This challenge was overcome in [36] by Buffa and Christiansen who proved well-posedness and uniform discrete inf-sup condition for the EFIE operator in the case where the surface bearing the screen is a Lipschitz manifold with (Lipschitz) boundary. Recently Chandler-Wilde and Hewett [39, 40] have studied boundary integral formulation of acoustic scattering by flat thin screens in the case of particularly poor geometrical regularity of the scatterer (fractal screens for example). The edges of screens are geometrical singularities that dramatically deteriorate the performance of preconditioning strategies available for EFIE, so several other contributions [33, 73, 70] have focused on bypassing this issue, with preconditioning strategies that incorporate a weight centred at the edges.

While screens are usually supposed to be at least locally orientable which discards the presence of branches, multi-screens are generalisations that allow such features. This is relevant in applications as there is no reason that metallic substrates do not admit any branch. This type of geometrical configuration has already caught attention in the electrical engineering literature [124, 87, 46]. But, to our knowledge, the articles [9, 12] have been the first to address scattering by such objects from a fully rigorous mathematical view point.

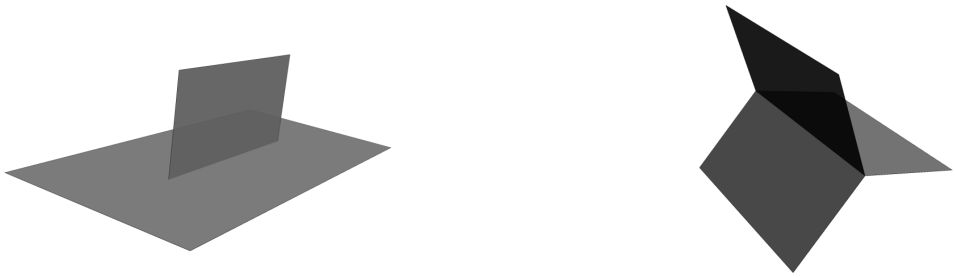


Figure 4.1: Two examples of multi-screen geometries

In the context of scattering at multi-screens, we studied both acoustics and electromagnetics. In the present chapter, for the sake of simplicity, we will mainly focus on acoustic scattering. Only at the end of the chapter, we will briefly mention similar results obtained with Maxwell's equations.

4.1 Geometry

Before providing a detailed definition for the geometries that we consider, let us recall the definition of a Lipschitz screen as proposed by Buffa and Christiansen [36].

Definition 4.1.1.
A Lipschitz screen *(in the sense of Buffa-Christiansen) is a subset $\Sigma \subset \mathbb{R}^3$ that satisfies the following properties:*

- *the set $\overline{\Sigma}$ is a compact Lipschitz two-dimensional sub-manifold with boundary,*
- *denoting $\partial\Sigma$ the boundary of $\overline{\Sigma}$, we have $\Sigma = \overline{\Sigma} \setminus \partial\Sigma$,*
- *there exists a finite covering of $\overline{\Sigma}$ with cubes such that, for each such cube C, denoting by a the length of its sides, we have*

  * *if C contains a point of $\partial\Sigma$, there exists an orthonormal basis of $\mathbb{R}^3$ in which C can be identified with $(0,a)^3$ and there are Lipschitz continuous functions $\psi : \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ with values in $(0,a)$ such that*

$$\begin{aligned} \Sigma \cap C &= \{\ (x,y,z) \in C \ \mid\ y < \psi(x),\ z = \phi(x,y)\ \}\ , \\ \partial\Sigma \cap C &= \{\ (x,y,z) \in C \ \mid\ y = \psi(x),\ z = \phi(x,y)\ \}\ , \end{aligned} \tag{4.1}$$

  * *if C contains no boundary point, there exists a Lipschitz open set $\Omega \subset \mathbb{R}^3$ such that we have $\Sigma \cap C = \partial\Omega \cap C$.*

The definition of a Lipschitz screen in $\mathbb{R}^2$ is very similar, but simpler. The only difference compared to Definition 4.1.1 is that Condition (4.1) should be replaced by: *there is a Lipschitz continuous function* $\phi : \mathbb{R} \to \mathbb{R}$ *with values in* $(0, a)$ *and a constant* $a_0 \in (0, a)$ *such that*

$$\Sigma \cap C = \{\ (x, y) \in C \ \mid\ x < a_0,\ y = \phi(x)\ \}$$

$$\text{and} \quad \partial\Sigma \cap C = \{\ (x, y) \in C \ \mid\ x = a_0,\ y = \phi(x)\ \}.$$

Now let us focus on potentially more complicated surfaces. In order to propose a convenient definition for surfaces shaped like screens with several branches, we first introduce an intermediary definition.

Definition 4.1.2.
A Lipschitz partition *of* $\mathbb{R}^d$ *is a finite collection of Lipschitz open sets* $(\Omega_j)_{j=0\ldots n}$ *such that* $\mathbb{R}^d = \cup_{j=0}^n \overline{\Omega}_j$ *and* $\Omega_j \cap \Omega_k = \emptyset$*, if* $j \neq k$*.*

Definition 4.1.3.
A multi-screen *is a subset* $\Sigma \subset \mathbb{R}^d$ *such that there exists a Lipschitz partition of* $\mathbb{R}^d$ *denoted* $(\Omega_j)_{j=0\ldots n}$ *satisfying* $\Sigma \subset \cup_{j=0}^n \partial\Omega_j$ *and such that, for each* $j = 0 \ldots n$*, we have* $\overline{\Sigma} \cap \partial\Omega_j = \overline{\Gamma}_j$ *where* $\Gamma_j \subset \partial\Omega_j$ *is some Lipschitz screen (in the sense of Buffa-Christiansen).*

Note that a Lipschitz screen, in the sense of Definition 4.1.1, is a multi-screen. The surfaces represented in Figure 4.1 are multi-screens, but are not Lipschitz screens. Besides, the skeleton $\cup_{j=0\ldots n}\partial\Omega_j$ of a Lipschitz partition $(\Omega_j)_{j=0\ldots n}$ of $\mathbb{R}^d$ is a multi-screen.

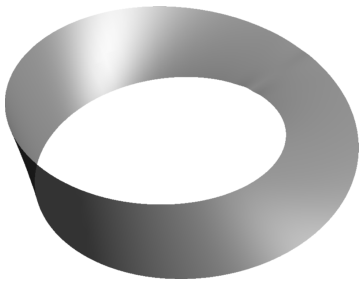


Figure 4.2: Möbius strip

A multi-screen is not a priori orientable which makes it more delicate to analyse compared to a more standard surface such as the boundary of a $C^\infty-$domain. For example, a Möbius strip is a Lipschitz screen in the sense of Buffa and Christiansen, as was pointed out in [36], although it is not globally orientable.

Of course, the Möbius strip fits the definition of a multi-screen: as is shown in Figure 4.3, one can find a Lipshitz partition that contains the Möbius strip in its skeleton.

Remark 4.1.1. *A multi-screen according to Definition 4.1.3 may contain points where three or more "branches" meet so that, at these points, the multi-screen is not two-sided. This situation compounds difficulties and forces us to adopt an abstract point of view for concepts such as trace operators and trace spaces that are more straightforward in other contexts. Hence part of the present work has focused on properly defining objects and results that are already very well known in other classical situations.*

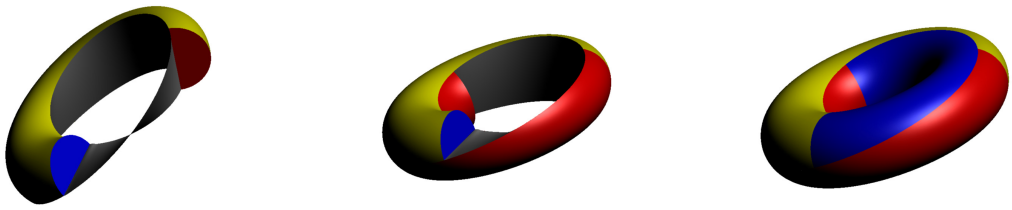


Figure 4.3: Lipschitz partition with Möbius strip in its skeleton

4.2 Function spaces

The trace spaces adapted to multi-screens that we introduced in [9] are built upon two domain based functional spaces. The first one, denoted $\mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Sigma})$, is defined as the space of functions $u\in\mathrm{L}^2(\mathbb{R}^3)$ such that there exists $\boldsymbol{p}\in\mathrm{L}^2(\mathbb{R}^3)$ satisfying[1]

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus\overline{\Sigma}} u\,\mathrm{div}(\boldsymbol{q})\,d\boldsymbol{x}=-\int_{\mathbb{R}^3\setminus\overline{\Sigma}}\boldsymbol{p}\cdot\boldsymbol{q}\,d\boldsymbol{x}\qquad\forall\boldsymbol{q}\in(\mathscr{D}(\mathbb{R}^3\setminus\overline{\Sigma}))^3\\&\text{with}\quad\|u\|^2_{\mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Sigma})}:=\|u\|^2_{\mathrm{L}^2(\mathbb{R}^3)}+\|\boldsymbol{p}\|^2_{\mathrm{L}^2(\mathbb{R}^3)}\,.\end{aligned}\tag{4.2}$$

Naturally, this norm is well defined since, if such a $\boldsymbol{p}$ as above exists, it is unique. The Sobolev space $\mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Sigma})$ equipped with the norm defined in (4.2) is a Hilbert space. We also define $\mathrm{H}^1_{0,\Sigma}(\mathbb{R}^3)$ the closure of $\mathscr{D}(\mathbb{R}^3\setminus\overline{\Sigma})$ in $\mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Sigma})$ with respect to this norm. The second domain based space that we introduced in [9, Section 4], denoted by $\mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Sigma})$, is the space of fields $\boldsymbol{p}\in\mathrm{L}^2(\mathbb{R}^3)^3$ such that there exists $u\in\mathrm{L}^2(\mathbb{R}^3)$ satisfying

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus\overline{\Sigma}}\boldsymbol{p}\cdot\nabla v\,d\boldsymbol{x}=-\int_{\mathbb{R}^3\setminus\overline{\Sigma}}u\,v\,d\boldsymbol{x}\qquad\forall v\in\mathscr{D}(\mathbb{R}^3\setminus\overline{\Sigma})\,,\\&\text{with}\quad\|\boldsymbol{p}\|^2_{\mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Sigma})}:=\|u\|^2_{\mathrm{L}^2(\mathbb{R}^3)}+\|\boldsymbol{p}\|^2_{\mathrm{L}^2(\mathbb{R}^3)}\,.\end{aligned}\tag{4.3}$$

Once again, if such a u as above exists, it is unique, so that the norm $\|\ \|_{\mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Sigma})}$ is well defined. The space $\mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Sigma})$ equipped with this norm is a Hilbert space. We also define $\mathbf{H}_{0,\Sigma}(\mathrm{div},\mathbb{R}^3)$ as the closure of $\mathscr{D}(\mathbb{R}^3\setminus\overline{\Sigma})^3$ with respect to this norm.

4.2.1 Multi-trace spaces

We introduce a generalisation of the multi-trace concept considered in §2.1.1 adapted to the geometrical context of multi-screens. These trace spaces are defined in an abstract manner as factor spaces, see [9, Section 5].

Definition 4.2.1.
Scalar valued Dirichlet and Neumann multi-trace spaces, equipped with their respective canon-

[1] Given any open subset $\omega\subset\mathbb{R}^3$, $\mathscr{D}(\omega)$ denotes the set of elements of $\mathcal{C}^\infty(\mathbb{R}^3)$ that vanish in $\mathbb{R}^3\setminus\overline{\omega}$, and $\mathscr{D}'(\omega)$ designates its dual i.e., the space of distributions in ω.

ical quotient norms $\|\ \|_{\mathbb{H}^{\pm 1/2}(\Sigma)}$*, are defined as*

$$\mathbb{H}^{+\frac{1}{2}}(\Sigma) \ := \ \mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Sigma})/\mathrm{H}^1_{0,\Sigma}(\mathbb{R}^3)$$

$$\mathbb{H}^{-\frac{1}{2}}(\Sigma) \ := \ \mathbf{H}(\mathrm{div}, \mathbb{R}^3 \setminus \overline{\Sigma})/\mathbf{H}_{0,\Sigma}(\mathrm{div}, \mathbb{R}^3).$$

We also consider trace operators $\pi_{\mathrm{D}} : \mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Sigma}) \to \mathbb{H}^{1/2}(\Sigma)$ and $\pi_{\mathrm{N}} : \mathbf{H}(\mathrm{div}, \mathbb{R}^3 \setminus \overline{\Sigma}) \to \mathbb{H}^{-1/2}(\Sigma)$ simply as the canonical projections for these quotient spaces. The multi-trace spaces $\mathbb{H}^{\pm 1/2}(\Sigma)$ are dual to each other via the bilinear pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ defined by the formula

$$\langle\!\langle \pi_{\mathrm{D}}(u), \pi_{\mathrm{N}}(\boldsymbol{p}) \rangle\!\rangle := \int_{\mathbb{R}^3 \setminus \overline{\Sigma}} \boldsymbol{p} \cdot \nabla u + \mathrm{div}(\boldsymbol{p})\, u \, d\boldsymbol{x} \ , \tag{4.4}$$

for all $u \in \mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Sigma})$ and all $\boldsymbol{p} \in \mathbf{H}(\mathrm{div}, \mathbb{R}^3 \setminus \overline{\Sigma})$. Identity (4.4) should be understood as a generalised Green formula where Σ plays the role of the "boundary" of $\mathbb{R}^3 \setminus \overline{\Sigma}$.

Interpretation Let us describe in more detail how the Dirichlet/Neumann multi-trace spaces can be embedded in a cartesian product of classical trace spaces. This should help make Definition 4.2.1 more explicit. Consider a decomposition $\overline{\Sigma} = \cup_{j=0}^n \overline{\Gamma}_j$ of the multi-screen like in Definition 4.1.3, where $\Gamma_j \subset \partial\Omega_j$. For any $\dot{u} \in \mathbb{H}^{1/2}(\Sigma)$ denote $\dot{u}|_{\Gamma_j} = (u|_{\Omega_j})|_{\Gamma_j}$ where $u \in \mathrm{H}^1(\mathbb{R}^d \setminus \overline{\Sigma})$ is any function admitting $\dot{u}$ as Dirichlet multi-trace on Σ i.e. $\pi_{\mathrm{D}}(u) = \dot{u}$. One can easily check that $\dot{u}|_{\Gamma_j}$ does not depend on the choice of the representative u. Then the application

$$\dot{u} \mapsto (\, \dot{u}|_{\Gamma_j}\,)_{j=0}^n$$

maps continuously $\mathbb{H}^{1/2}(\Sigma)$ into $\mathrm{H}^{1/2}(\Gamma_0) \times \cdots \times \mathrm{H}^{1/2}(\Gamma_n)$, and it is injective. As a consequence one may identify $\mathbb{H}^{1/2}(\Sigma)$ with a subspace of $\mathrm{H}^{1/2}(\Gamma_0) \times \cdots \times \mathrm{H}^{1/2}(\Gamma_n)$, and simple density result show that this subspace is dense. Similarly $\mathbb{H}^{-1/2}(\Sigma)$ can be identified with a dense subspace of $\mathrm{H}^{-1/2}(\Gamma_0) \times \cdots \times \mathrm{H}^{-1/2}(\Gamma_n)$.

4.2.2 Single-trace spaces and jumps

The elements of $\mathbb{H}^{\pm 1/2}(\Sigma)$ may be regarded as double-valued functions defined on Σ (each value being associated to a different face of Σ). We also consider subspaces of the multi-trace spaces that correspond to single valued functions.

Definition 4.2.2.
Scalar valued single trace spaces for Dirichlet and Neumann data, respectively, are closed subspaces of $\mathbb{H}^{\pm 1/2}(\Sigma)$ *defined as*

$$\mathrm{H}^{+\frac{1}{2}}([\Sigma]) \ := \ \mathrm{H}^1(\mathbb{R}^3)/\mathrm{H}^1_{0,\Sigma}(\mathbb{R}^3) \ = \ \pi_{\mathrm{D}}\big(\,\mathrm{H}^1(\mathbb{R}^3)\,\big)$$

$$\mathrm{H}^{-\frac{1}{2}}([\Sigma]) \ := \ \mathbf{H}(\mathrm{div}, \mathbb{R}^3)/\mathbf{H}^1_{0,\Sigma}(\mathrm{div}, \mathbb{R}^3) \ = \ \pi_{\mathrm{N}}\big(\,\mathbf{H}(\mathrm{div}, \mathbb{R}^3)\,\big).$$

The single trace spaces $\mathrm{H}^{\pm 1/2}([\Sigma])$ are polar to each other under the duality pairing (4.4). In particular we have $\langle\!\langle \dot{u}, \dot{p} \rangle\!\rangle = 0$ for every $\dot{u} \in \mathrm{H}^{1/2}([\Sigma])$, $\dot{p} \in \mathrm{H}^{-1/2}([\Sigma])$. We also define jump spaces as duals of the single trace spaces

$$\widetilde{\mathrm{H}}^{+\frac{1}{2}}([\Sigma]) \ = \ \mathrm{H}^{-\frac{1}{2}}([\Sigma])' \qquad \text{and} \qquad \widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Sigma]) \ = \ \mathrm{H}^{+\frac{1}{2}}([\Sigma])' \tag{4.5}$$

We equip the jump spaces (4.5) with the dual norms. Note that any element of $\mathbb{H}^{\pm 1/2}(\Sigma)$ naturally induces an element of $\widetilde{\mathrm{H}}^{\pm 1/2}([\Sigma])$ via the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. This allows to consider a continuous and surjective "jump" operator $[\,\cdot\,] : \mathbb{H}^{+1/2}(\Sigma) \to \widetilde{\mathrm{H}}^{+1/2}([\Sigma])$ defined by the formula

$$\langle [\dot{u}], \dot{q} \rangle \;:=\; \langle\!\langle \dot{u}, \dot{q} \rangle\!\rangle \qquad \forall \dot{q} \in \mathrm{H}^{-\frac{1}{2}}([\Sigma]) \tag{4.6}$$

where this holds for any $\dot{u} \in \mathbb{H}^{+1/2}(\Sigma)$. In a completely analogous manner we can define a jump operator $[\,\cdot\,] : \mathbb{H}^{-1/2}(\Sigma) \to \widetilde{\mathrm{H}}^{-1/2}([\Sigma])$ that is continuous and surjective as well.

4.3 Layer potentials and representation theorem

In the sequel we shall denote $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Sigma}) := \{u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \overline{\Sigma}) \mid \nabla u \in \mathrm{H}_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^d \setminus \overline{\Sigma})\}$. We shall also re-note the Dirichlet trace $\gamma_{\mathrm{D}}(u) := \pi_{\mathrm{D}}(u)$, and take the following definition for the Neumann trace $\gamma_{\mathrm{N}}(u) := \pi_{\mathrm{N}}(\nabla u)$. The following result shows that these operators can be used to impose boundary conditions at multi-screens, see [9, Prop.7.2].

Proposition 4.3.1.
Assume that Σ is a multi-screen in the sense of Definition 4.1.3, and that $\mathbb{R}^d \setminus \overline{\Sigma}$ is connected. Take $f \in \mathrm{L}^2(\mathbb{R}^d)$ with bounded support, and $g \in \mathbb{H}^{1/2}(\Sigma)$, and $\kappa \in \mathbb{R}_+$. Then there exists a unique $u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Sigma})$ satisfying the following equations

$$\begin{cases} -\Delta u - \kappa^2 u = f \quad \textit{in } \mathbb{R}^d \setminus \overline{\Sigma}, \\ u \;\; \kappa\textit{-outgoing radiating}, \\ \gamma_{\mathrm{D}}(u) = g \textit{ on } \Sigma \end{cases}$$

and u continuously depends on f and g with respect to the topology of in $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Sigma})$. The same result holds if the Dirichlet condition on Σ is replaced by Neumann condition of the form $\gamma_{\mathrm{N}}(u) = h$ for some $h \in \mathbb{H}^{-1/2}(\Sigma)$.

Recall that $\mathscr{G}_\kappa(\boldsymbol{x})$ refers to the outgoing Green kernel of Helmholtz equation, see (1.3). By analogy with the single and double layer potential operator, for any tuple $\varphi_j \in \mathrm{H}^1(\Gamma_j), j = 0 \dots n$, denoting $\dot{\varphi} = (\varphi_0, \dots, \varphi_n)$ we define layer potentials based on the multi-screen

$$\begin{aligned} \mathrm{SL}_\kappa(\dot{\varphi})(\boldsymbol{x}) &:= \sum_{j=0}^{n} \int_{\Gamma_j} \mathscr{G}_\kappa(\boldsymbol{x} - \boldsymbol{y}) \varphi_j(\boldsymbol{y}) d\sigma(\boldsymbol{y}) \\ \mathrm{DL}_\kappa(\dot{\varphi})(\boldsymbol{x}) &:= \sum_{j=0}^{n} \int_{\Gamma_j} \boldsymbol{n}_j(\boldsymbol{y}) \cdot (\nabla \mathscr{G}_\kappa)(\boldsymbol{x} - \boldsymbol{y}) \varphi_j(\boldsymbol{y}) d\sigma(\boldsymbol{y}) \qquad \boldsymbol{x} \in \mathbb{R}^d \setminus \overline{\Sigma}. \end{aligned} \tag{4.7}$$

In this expression, the vector fields $\boldsymbol{n}_j$ refers to the normal to $\partial\Omega_j$ (that contains Γ_j) directed toward the exterior of Ω_j. First of all these operators naturally give rise to potential operators continuously acting on multi-trace spaces, see Proposition 8.3 and 8.4 in [9].

Proposition 4.3.2.
The operator SL_κ (resp. DL_κ) defined by (4.7) induces a continuous map from $\mathbb{H}^{-1/2}(\Sigma)$ (resp. $\mathbb{H}^{+1/2}(\Sigma)$) into $\mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Sigma})$, and we have $\ker(\mathrm{SL}_\kappa) = \mathrm{H}^{-1/2}([\Sigma])$ and $\ker(\mathrm{DL}_\kappa) = \mathrm{H}^{+1/2}([\Sigma])$.

As a direct byproduct of the preceding result, we see that the layer potentials act naturally on jump spaces $\widetilde{\mathrm{H}}^{\pm 1/2}([\Sigma]) = \mathbb{H}^{\pm 1/2}(\Sigma)/\mathrm{H}^{\pm 1/2}([\Sigma])$. These potential operators also satisfy jump relations similar to the result of Theorem 1.2.2, but they actually satisfy a modified version of it, see [9, Prop.8.5].

Proposition 4.3.3.

$$[\gamma_{\mathrm{D}}] \cdot \mathrm{DL}_\kappa(\dot{u}) = [\dot{u}] \quad [\gamma_{\mathrm{N}}] \cdot \mathrm{DL}_\kappa(\dot{u}) = 0 \quad \forall \dot{u} \in \mathbb{H}^{+\frac{1}{2}}(\Sigma),$$

$$[\gamma_{\mathrm{D}}] \cdot \mathrm{SL}_\kappa(\dot{p}) = 0 \qquad [\gamma_{\mathrm{N}}] \cdot \mathrm{SL}_\kappa(\dot{p}) = [\dot{p}] \quad \forall \dot{p} \in \mathbb{H}^{-\frac{1}{2}}(\Sigma).$$

These layer potentials provide particular functions that are solutions to homogeneous Helmholtz equations in $\mathbb{R}^d \setminus \overline{\Sigma}$. Actually, one can prove an integral representation formula completely analogous to Theorem 1.2.1.

Lemma 4.3.1.
Assume that $u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d \setminus \overline{\Sigma})$ satisfies Sommerfeld's radiation condition. Assume that there is a $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ such that $f = -\Delta u - \kappa^2 u$ in the sense of distributions in $\mathbb{R}^d \setminus \overline{\Sigma}$, and suppose in addition that f has bounded support. Then we have

$$u = \mathscr{G}_\kappa * f + \mathrm{SL}_\kappa \cdot \gamma_{\mathrm{N}}(u) + \mathrm{DL}_\kappa \cdot \gamma_{\mathrm{D}}(u) \qquad \text{in } \mathbb{R}^d \setminus \overline{\Sigma}.$$

This result can be used to reformulate a boundary value problem involving a multi-screen as a boundary integral equation. The following well-posedness result was proved in Proposition 8.8 and 8.9 in [9].

Proposition 4.3.4.
For any wave number $\kappa \in \mathbb{C} \setminus \{0\}$ such that $\Re e\{\kappa\} \geq 0, \Im m\{\kappa\} \geq 0$, define the operators $\mathrm{V} : \widetilde{\mathrm{H}}^{-1/2}([\Sigma]) \to \mathrm{H}^{+1/2}([\Sigma])$ and $\mathrm{W} : \widetilde{\mathrm{H}}^{+1/2}([\Sigma]) \to \mathrm{H}^{-1/2}([\Sigma])$ by

$$\mathrm{V} := \gamma_{\mathrm{D}} \cdot \mathrm{SL}_\kappa \qquad \text{and} \qquad \mathrm{W} := \gamma_{\mathrm{N}} \cdot \mathrm{DL}_\kappa. \tag{4.8}$$

Then these two operators are isomorphisms and there exist compact operators $\mathrm{K_V} : \widetilde{\mathrm{H}}^{-1/2}([\Sigma]) \to \mathrm{H}^{+1/2}([\Sigma])$ and $\mathrm{K_W} : \widetilde{\mathrm{H}}^{+1/2}([\Sigma]) \to \mathrm{H}^{-1/2}([\Sigma])$ such that the following Garding inequalities are satisfied

$$\Re e\{\langle\!\langle (\mathrm{V} + \mathrm{K_V})q, \overline{q} \rangle\!\rangle\} \geq C \|q\|^2_{\widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Sigma])} \qquad \forall q \in \widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Sigma])$$

$$\Re e\{\langle\!\langle (\mathrm{W} + \mathrm{K_W})v, \overline{v} \rangle\!\rangle\} \geq C \|v\|^2_{\widetilde{\mathrm{H}}^{+\frac{1}{2}}([\Sigma])} \qquad \forall v \in \widetilde{\mathrm{H}}^{+\frac{1}{2}}([\Sigma])$$

The Garding inequalities above straightforwardly lead to discrete inf-sup conditions, which provides quasi-optimal convergence of conforming Galerkin discretisations of boundary integral equations at screens associated to operators (4.8).

4.4 Extension to electromagnetics

The content of this chapter was adapted to the case of electromagnetic scattering in [12]. We introduced multi-trace/single-trace and jump spaces for tangential traces of fields $\mathbf{H}(\mathbf{curl}, \mathbb{R}^d \setminus \overline{\Sigma})$, as well as a counterpart of the EFIE (i.e. single-trace operator) for multi-screens. We

also introduced tangential differential operators and proved surface integration by parts for traces at multi-screens.

We also established a characterisation similar to the one discussed in [37]: the traces of elements in $\mathbf{H}(\mathbf{curl}, \mathbb{R}^d \setminus \overline{\Sigma})$ are those tangential fields that have $\mathrm{H}^{-1/2}$ regularity with divergence also in $\mathrm{H}^{-1/2}$.

We proved that the vector potential operators attached to a multi-screen satisfy jump formulas, and we proved that the "multi-screen EFIE operator" satisfies a modified Garding inequality, which guarantees well-posedness of boundary integral formulations of electromagnetic scattering at multi-screens. Let us underline that, to establish this result, we had to resort on Hodge decompositions, which was not trivial due to the domain $\mathbb{R}^d \setminus \overline{\Sigma}$ being not even Lipschitz.

Note that, in the present case, the modified Garding inequality satisfied by the continuous formulation does not straightforwardly lead to a uniform discrete inf-sup condition. The theoretical numerical analysis of the EFIE for scattering at multi-screen remains an open issue that we shall address in a forthcoming work.

Chapter 5

Asympotic models for wave propagation in perturbed media

This chapter gives a brief overview of contributions to the modelling of wave propagation in media submitted to multi-scale perturbations giving rise to boundary layer phenomena. The derivation of these models strongly relied on the method of matched asymptotic expansions on the one hand (see [90] for an overview on these techniques), and Kondratiev's theory for the analysis of elliptic PDE in singular domains on the other hand (see [79, 80, 95] for reference books on this topic).

5.1 Wave propagation through thin oscillating interfaces

The present contribution is a joint work with B.Delourme published in [7]. For some $f \in \mathrm{L}^2(\mathbb{R}^2)$ with bounded support and a frequency $\omega > 0$, we were interested in studying the asymptotic behaviour of the solution u_δ of a 2-D acoustic wave propagation problem

$$\begin{cases} -\mathrm{div}(\epsilon_\delta^{-1}\nabla u_\delta) - \omega^2 \mu_\delta u_\delta = f \quad \text{in } \mathbb{R}^2 \\ \\ u_\delta \ \text{ outgoing radiating} \end{cases} \tag{5.1}$$

where $\epsilon_\delta, \mu_\delta$ are piece-wise constant functions. These functions are actually assumed constant equal to background values ϵ_0, μ_0 everywhere except in the neighbourhood of a line interface Γ where they fastly oscillate, adopting a periodic behaviour with a period proportional to $1/\delta$ with $\delta \to 0$, see Figure 5.1.

For this problem, we have derived and justified, the matched asymptotic expansion up to any order. This expansion follows two different ansatz, either close or far from the interface Γ. In the far field zone the ansatz is $u_\delta(\boldsymbol{x}) = u_0(\boldsymbol{x}) + \delta u_1(\boldsymbol{x}) + \delta^2 u_2(\boldsymbol{x}) + \dots$. In the near field zone i.e. close to Γ the ansatz takes the more sophisticated form

$$u_\delta(\boldsymbol{x}) = \sum_{n=0}^{+\infty} \delta^n U_n(s/\delta, t/\delta, t)$$

where t (resp. s) refers to the curvilinear abscissa along Γ (resp. the distance from Γ), and the functions $U_n(\xi, \eta, t)$ are supposed to be periodic with respect to η. For this matched

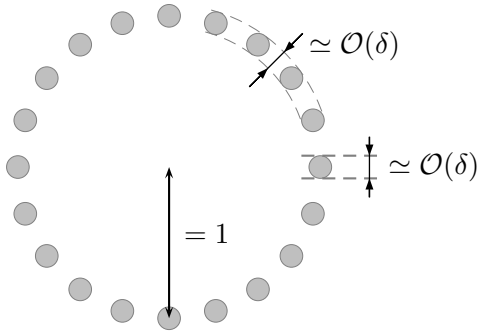


Figure 5.1: Thin fastly oscillating interface geometry. Here the interface Γ is the circle with unit radius, $\delta = 2\pi/N$ for some integer $N \to \infty$, and s, t are coordinates related to $\boldsymbol{x} = (x, y)$ by $x = s \cdot \cos(t)$ and $y = s \cdot \sin(t)$.

expansion we proved an error estimate in H^1-norm. Let us provide a simplified version of this estimate. For any bounded open subdomain $\Omega \subset \mathbb{R}^2$ such that $\Omega \cap \Gamma = \emptyset$, and any $N > 0$, there exists a constant $C_{\Omega,N} > 0$ such that

$$\|u_\delta - \sum_{n=0}^{N} \delta^n u_n\|_{\mathrm{H}^1(\Omega)} \leq C_{\Omega,N}\ \delta^{N+1} \quad \forall \delta > 0.$$

In addition, we exhibited modified transmission conditions to be imposed at the interface Γ such that the truncated sums $\tilde{u}_{\delta,N} := \sum_{n=0}^{N} \delta^n u_n$ are solutions to the problem $-\mathrm{div}(\epsilon_0^{-1}\nabla \tilde{u}_{\delta,N}) - \omega^2 \mu_0 \tilde{u}_{\delta,N} = f$ supplemented with these transmission conditions. We could provide such conditions for any N which seemed to be new in the existing literature on oscillatory interfaces.

There already existed numerous references about elliptic problems posed in a domains with fastly oscillating boundaries. In particular the problem described by (5.1) and Fig.5.1 had already been considered in [29, 30, 44, 52]. Yet the contributions already avaiable only investigated the first few terms of the expansion of u_δ. The main originality of [7] consisted in deriving asymptotic expansions up to any order, and to provide the corresponding approximate transmission condition and error estimate regardless of the order of approximation.

5.2 Transmission problem with a fastly oscillating medium

Further work has been achieved during the PhD thesis of Valentin Vinoles in the context of the ANR project METAMATH. Together with Valentin and Sonia Fliss (ENSTA POems) we considered the same problem as (5.1) except that, this time, the material characteristics $\epsilon_\delta, \mu_\delta$ are assumed to oscillate in the whole domain (not just along a line), see Figure 5.2 below.

We studied the asymptotic behaviour of the solution u_δ to a scattering problem in this geometrical configuration as $\delta \to 0$. We applied matched asymptotic techniques for deriving the expansion of u_δ in the homogeneous part of the propagation medium, and close to the interface. In these two regions, we adopted the same ansatz as in the oscillating interface of the previous section. In the fastly oscillating part of the medium we used homogenisation,

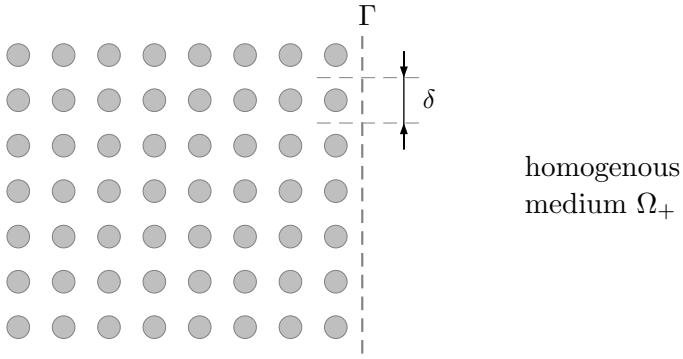


Figure 5.2: Interface between an homogeneous and a fastly oscillating medium

adopting a two-scale ansatz

$$u_\delta(\boldsymbol{x}) = \sum_{n=0}^{+\infty} \delta^n u_n(\boldsymbol{x}/\delta, \boldsymbol{x})$$

where the functions $u_n(\boldsymbol{y}, \boldsymbol{x})$ are supposed to be periodic in the $\boldsymbol{y}$ variable in all the directions of a fixed cartesian coordinate frame. We derived and justified by means of error estimates, such an expansion for u_δ up to terms of orders $\mathcal{O}(\delta^3)$. This work is still in progress.

5.3 Non-smooth transmission problem with negative material

The great potential of new artificial materials, called metamaterials, is motivating an intense research in electromagnetics. These metamaterials, which have a complex periodic structure, can be modelized in some frequency range by homogeneous isotropic materials, whose effective dielectric permittivity and magnetic permeability have a negative real part and a small imaginary part, see e.g. [126, 115]. Neglecting losses leads to represent a metamaterial at a given frequency by constants ϵ and μ which are negative real numbers.

This simple model allows to explain the main phenomena which make the interest of metamaterials. For instance, the unusual negative refraction effect at the interface between a dielectric and a metamaterial is due to the change of sign of ϵ and μ. On the other hand, this sign change raises original questions for both the mathematical analysis and the numerical simulation [99, 106, 58]. Quite general answers have been recently obtained by a variational approach [26, 27, 28, 97, 133]. Concerning the time harmonic transmission problem between non-dissipative dielectric and meta- material, set in a bounded 2D/3D domain, the problem is proved to be of Fredholm type in the classical functional framework if the contrasts (ratios of the values of ϵ and μ across the interface) are outside some interval, which always contain the value -1. These intervals reduce to $\{-1\}$ if (and only if) the interface is a regular curve/surface. Before the contribution presented in this paragraph, the effect of corners on an interface was not very well understood from a mathematical point of view.

Clarifying this question was precisely the goal of the work described in this section. This work was published in [1] and was achieved in collaboration with Lucas Chesnel and Anne-Sophie Bonnet-BenDhia (CNRS POems) in the context of the ANR project METAMATH. We focused on a simplified model, considering a 2-D diffusion problem with a sign changing diffusion coefficient. Such a sign change in the principal part of the PDE spoils the coercivity of the appropriate bilinear form, and the problem is not elliptic anymore, not even up to a

compact perturbation. Let $\Omega, \Omega_\pm$ refer to bounded open domains with $\overline{\Omega} = \overline{\Omega}_- \cup \overline{\Omega}_+ \subset \mathbb{R}^2$ and $\Omega_- \cap \Omega_+ = \emptyset$.

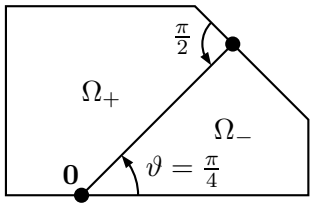


Figure 5.3: Geometry of the problem.

The boundary Ω is assumed to be polygonal and the interface $\Gamma = \partial\Omega_+ \cap \partial\Omega_-$ be such that Γ meets the boundary $\partial\Omega$ at two points, with an incidence angle of $\pi/2$ and $\pi/4$, see Figure 5.3. Define the diffusion coefficient by $\sigma = \sigma_\pm$ in $\Omega_\pm$ with $\sigma_+ > 0$ and $\sigma_- < 0$, and $\sigma_\pm$ constants. For some $f \in \mathrm{L}^2(\Omega)$, we considered the problem

$$-\mathrm{div}(\sigma\nabla u) = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega. \tag{5.2}$$

The appropriate function space where the solution u to Problem (5.2) should be sought strongly depends on the contrast parameter $\sigma_-/\sigma_+ < 0$. Define $\mathrm{A} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega) := \mathrm{H}^1_0(\Omega)^*$ by $\langle \mathrm{A}(u), v\rangle = \int_\Omega \sigma\nabla u\nabla v d\boldsymbol{x}$, $\forall u, v \in \mathrm{H}^1_0(\Omega)$. It was established in [42] that this operator is of Fredholm type if and only if $\sigma_-/\sigma_+ \in \mathbb{C} \setminus [-1, -1/3]$.

The article [1] focused on this problem for the particular case where $\sigma_-/\sigma_+ \in (-1, -1/3)$, so that A is not Fredholm and the classical Sobolev space $\mathrm{H}^1_0(\Omega)$ does not appear appropriate anymore. The main result of this contribution was to provide a modified functional setting where Fredholmness with index zero is restored for the operator related to (5.2). The definition of this functional setting relies on the following lemma.

Lemma 5.3.1.
Let r, θ refer to the polar coordinates centred at $\mathbf{0}$ (= the point where Γ meets $\partial\Omega$ with an angle of $\pi/4$). For $\sigma_-/\sigma_+ \in (-1, -1/3)$ there exists a unique eigenpair $(\eta, \varphi) \in \mathbb{R} \times \mathrm{H}^1_0(0, \pi)$ such that $\mathrm{div}(\sigma\nabla r^{\pm i\eta}\varphi(\theta)) = 0$ and $\eta > 0$.

Let $\chi : \mathbb{R} \to [0, 1]$ refer to a $\mathscr{C}^\infty$ cut-off function such that $\chi(r) := 0$ for $|\boldsymbol{x}| > 2r_0$ and $\chi(r) := 1$ for $|\boldsymbol{x}| < r_0$ for some sufficiently small $r_0 > 0$, and set $\chi(\boldsymbol{x}) := \chi(|\boldsymbol{x}|)$. Observe that the function

$$\mathbf{s}^\pm(\boldsymbol{x}) := \chi(r) r^{\pm i\eta}\varphi(\theta) \tag{5.3}$$

vanishes on $\partial\Omega \setminus \{\mathbf{0}\}$ since $\varphi = \varphi(\theta) \in \mathrm{H}^1_0(0, \pi)$ for r_0 small enough. Next, for any $\beta \in \mathbb{R}$, define the space $\mathrm{V}^1_\beta(\Omega)$ as the completion of $\mathscr{C}^\infty_0(\Omega) := \{\, \psi \in \mathscr{C}^\infty(\mathbb{R}^2) \mid \psi = 0 \text{ in } \mathbb{R}^2 \setminus \Omega \,\}$ for the following weighted norm

$$\|v\|^2_{\mathrm{V}^1_\beta(\Omega)} := \int_\Omega |\boldsymbol{x}|^{2\beta}|\nabla v|^2 + |\boldsymbol{x}|^{2\beta-2}|v|^2 d\boldsymbol{x}.$$

This type of space is classical in Kondratiev's theory where one studies the properties of the solutions to elliptic problems posed in domains with singularities at the boundary, see [80].

The functional setting that was proposed in [1] is built upon $\mathrm{V}^1_\beta(\Omega)$ for some well chosen β, imposing in addition some kind a radiation condition at the corner at $\mathbf{0}$ whose expression relies on Lemma 5.3.1.

Theorem 5.3.1.
Assume that $\sigma_-/\sigma_+ \in (-1,-1/3)$, and take $\beta \in (0,2)$. Define the space $\mathscr{V}^+_\beta(\Omega) := \mathrm{span}\{\mathbf{s}^+\} \oplus \mathrm{V}^1_{-\beta}(\Omega)$. The bilinear form $u,v \mapsto \int_\Omega \sigma \nabla u \nabla v d\boldsymbol{x}$ induces a unique continuous operator A_β mapping $\mathscr{V}^+_\beta(\Omega)$ into $\mathrm{V}^1_\beta(\Omega)^*$. Moreover this operator is of Fredholm type with index equal to 0.

In the result above, the radiation condition comes into play through the presence of $\mathbf{s}^+$ in the definition of $\mathscr{V}^+_\beta(\Omega)$.

5.4 Rounded corner asymptotics with negative material

The work presented here was published in [3]. It deals with a singular perturbation problem related to Equation (5.2) of the previous section. In [42] it was proved that, in the case where the interface $\Gamma = \partial\Omega_+ \cap \partial\Omega_-$ intersects $\partial\Omega$ in such a way that it forms right angles at intersection points, then the operator $\mathrm{A} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ defined by $\langle \mathrm{A}(u), v\rangle = \int_\Omega \sigma \nabla u \nabla v d\boldsymbol{x}$ is of Fredholm type of index 0 provided that $\sigma_-/\sigma_+ \neq -1$.

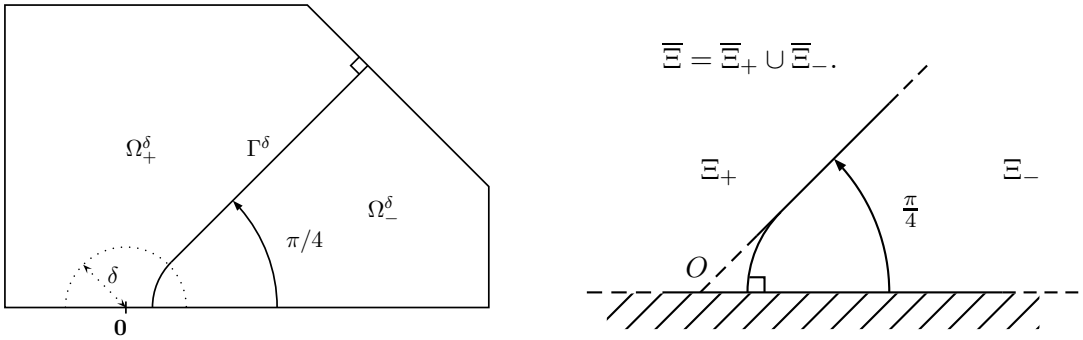


Figure 5.4: Geometry of the problem.

Admittedly, in the previous section $\mathrm{A} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ was not Fredholm for $\sigma_-/\sigma_+ \in (-1,-1/3)$, but that was due to the angle $\pi/4$ that Γ formed with $\partial\Omega$ at $\mathbf{0}$. A natural regularisation process would thus consists in slightly rounding this corner so as to force Γ to meet $\partial\Omega$ according to right angles, see the left picture in Figure 5.4.

One may expect that the effect of this regularisation vanishes for $\delta \to 0$. In [3] we proved that this does not occur. With such a singular perturbation, an oscillatory phenomenon with non-decreasing amplitude takes place as $\delta \to 0$. To prove this, we studied the behaviour of the solution $u_\delta \in \mathrm{H}^1_0(\Omega)$ to the problem

$$u_\delta \in \mathrm{H}^1_0(\Omega) \quad \text{such that} \quad -\operatorname{div}(\sigma_\delta \nabla u_\delta) = f \ \text{ in } \Omega,$$

where $\sigma_\delta = \sigma_\pm$ in $\Omega^\delta_\pm$ and $\sigma_-/\sigma_+ \in (-1,-1/3)$. We know from [1] and Theorem 5.3.1 above that Problem (5.2) posed in the "limit geometry" is of Fredholm type with index 0, so we

took as an hypothesis that $\ker(\mathrm{A}_\beta) = \{0\}$ for some $\beta \in (0,2)$ (which actually implies that, for *any* $\beta \in (0,2)$, we have $\ker(\mathrm{A}_\beta) = \{0\}$).

When $\delta \to 0$, a boundary layer arises at the corner in the neighbourhood of $\mathbf{0}$. Deriving the matched asymptotic expansion of u_δ, we proved that

$$\lim_{\delta\to 0} \|u_\delta\|_{\mathrm{H}^1(\Omega)} = +\infty,$$

and also that there exists two functions $u_0, u_1 \in \mathrm{V}^1_\beta(\Omega)$ with $u_1 \notin \mathrm{H}^1(\Omega)$ and $\delta_\star > 0$ such that, for any disc D centred at $\mathbf{0}$ and any $\epsilon, \gamma > 0$, there exists constants $C_{\epsilon,\gamma} > 0$ independent of δ satisfying the error estimate

$$\left\| u_\delta - u_0 - \frac{(\delta/\delta_\star)^{2i\eta}}{1-(\delta/\delta_\star)^{2i\eta}} u_1 \right\|_{\mathrm{H}^1(\Omega\setminus\overline{\mathrm{D}})} \leq C_{\epsilon,\gamma}\, \delta^{2-\epsilon}, \qquad \forall \delta \in \mathrm{I}(\gamma)$$

$$\text{where} \quad \mathrm{I}(\gamma) = \bigcup_{k=-\infty}^{+\infty} [\delta_\star e^{(k+\gamma)\pi/\eta}, \delta_\star e^{(k+1-\gamma)\pi/\eta}].$$

Note that, although $\mathrm{I}(\gamma)$ does not contain any neighbourhood of 0, it does admit 0 as accumulation point. This set has been chosen so as to guarantee that the gauge function $(\delta/\delta_\star)^{2i\eta}/(1-(\delta/\delta_\star)^{2i\eta})$ remains bounded as $\delta \to 0$. Observe also that this gauge function is oscillating, and does not tend to 0 as $\delta \to 0$, which proves that u_δ does not converge toward u_0: the solution of the perturbed problem does *not* converge at all!

5.5 Spectrum for a small inclusion of negative material

We also investigated in [2] the asymptotics of eigenvalues of the diffusion operator in the case where the computational domain contains a small inclusion of negative material. For this problem, we considered two bounded open sets $\Omega, \Xi \subset \mathbb{R}^3$ with $\mathscr{C}^\infty$ boundary, such that $\mathbf{0} \in \Omega$, and denoted $\Omega^\delta_- := \delta\Xi$, $\Omega^\delta_+ := \Omega \setminus \overline{\Omega}^\delta_-$. In the sequel we shall also implicitely assume that δ is small enough to guarantee that $\overline{\Omega}^\delta_- \subset \Omega$. In this case we consider the problem

$$\left\{ \begin{array}{l} \text{Find } (\lambda, u) \in \mathbb{C} \times \mathrm{H}^1_0(\Omega) \setminus \{0\} \text{ such that} \\ \displaystyle\int_\Omega \sigma_\delta \nabla u \nabla v \, d\boldsymbol{x} = \lambda \int_\Omega uv \, d\boldsymbol{x} \quad \forall v \in \mathrm{H}^1_0(\Omega). \end{array} \right. \tag{5.4}$$

with $\sigma_\delta = \sigma_\pm \in \Omega^\delta_\pm$ such that $\sigma_-/\sigma_+ \neq -1$. Since $\partial\Omega^\delta_-$ is smooth, the unbounded operator $\mathcal{A}_\delta : D(\mathcal{A}_\delta) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ defined by

$$\begin{array}{l} \mathcal{A}_\delta(v) := -\mathrm{div}(\sigma_\delta \nabla v) \quad \text{with} \\ D(\mathcal{A}_\delta) := \{v \in \mathrm{H}^1_0(\Omega) \mid \mathrm{div}(\sigma_\delta \nabla v) \in \mathrm{L}^2(\Omega)\}, \end{array} \tag{5.5}$$

is a self-adjoint Fredholm operator with index 0 and compact resolvent. Hence its spectrum $\mathfrak{S}(\mathcal{A}_\delta)$ consists in an infinite sequence of real eigenvalues. We could show that $\sup \mathfrak{S}(\mathcal{A}_\delta) = +\infty$ and, due to the presence of the negative part Ω^δ_-, we also have $\inf \mathfrak{S}(\mathcal{A}_\delta) = -\infty$. So this

spectrum takes the form

$$\mathfrak{S}(\mathcal{A}_\delta) = \{\lambda_n^\delta\}_{n\in\mathbb{Z}} \text{ where}$$
$$\cdots \leq \lambda_{-n}^\delta \leq \cdots \leq \lambda_{-1}^\delta < 0 \leq \lambda_1^\delta \leq \cdots \leq \lambda_n^\delta \leq \ldots$$
$$\text{with } \lim_{n\to\pm\infty} \lambda_n^\delta = \pm\infty.$$

We studied the asymptotic behaviour of the λ_n^δ as $\delta \to 0$. To derive the predominant behaviour of these eigenvalues, we had to introduce two operators. The first one is related to the far field behaviour (i.e. away from the small negative inclusion). It is an unbounded self-adjoint operator $\mathcal{A}_0 : D(\mathcal{A}_0) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ defined by $\mathcal{A}_0(v) := -\sigma_+\Delta v$ with $D(\mathcal{A}_0) := \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$. We first established the following approximation result.

Proposition 5.5.1.
Define $\sigma_{\mathrm{N}}(\boldsymbol{\xi}) := \sigma_+$ for $\boldsymbol{\xi} \in \mathbb{R}^2 \setminus \overline{\Xi}$, and $\sigma_{\mathrm{N}}(\boldsymbol{\xi}) := \sigma_-$ for $\boldsymbol{\xi} \in \Xi$. Assume that $\sigma_- \neq -\sigma_+$ and that any $v \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ satisfying both $-\mathrm{div}(\sigma_{\mathrm{N}}\nabla v) = 0$ in $\mathbb{R}^3$ and $\int_{\mathbb{R}^2} |\nabla v|^2 + |\boldsymbol{\xi}|^{-2}|v(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} < +\infty$ is actually $v = 0$. Then, for δ small enough the operator $\mathcal{A}_\delta$ is invertible, and for any $\epsilon \in (0,1)$ there exists constants $\delta_0, C_\epsilon > 0$ independent of δ such that

$$\sup_{f\in\mathrm{L}^2(\Omega)} \frac{\|(\mathcal{A}_\delta^{-1} - \mathcal{A}_0^{-1})f\|_{\mathrm{L}^2(\Omega)}}{\|f\|_{\mathrm{L}^2(\Omega)}} \leq C_\epsilon\, \delta^{3/2-\epsilon} \qquad \forall \delta \in (0, \delta_0).$$

Note that $\mathcal{A}_0$ is an unbounded positive self-adjoint operator with compact resolvent and hence admits a countable spectrum of isolated eigenvalues located in $\mathbb{R}_+$. We shall denote $0 < \lambda_1^0 \leq \lambda_2^0 \leq \cdots \leq \lambda_n^0 \leq \ldots$ these (ordered) eigenvalues. From the previous proposition can be deduced a first result stating that the positive eigenvalues of the operator $\mathcal{A}_\delta$ are close to the eigenvalues of $\mathcal{A}_0$.

Proposition 5.5.2.
Under the hypothesis of Proposition 5.5.1, for any $n \in \mathbb{N}^*$ and any $\epsilon \in (0,1)$, there exists constants $C, \delta_0 > 0$ independent of δ (but dependent on ϵ and n) such that

$$|\lambda_n^\delta - \lambda_n^0| \leq C\, \delta^{3/2-\epsilon} \quad \forall \delta \in (0, \delta_0).$$

The remarkable feature of the spectral problem under consideration here is that, although $\mathcal{A}_0$ may at first glance be regarded as the limit of $\mathcal{A}_\delta$, the operator $\mathcal{A}_\delta$ admits *infinitely many* negative eigenvalues whereas $\mathcal{A}_0$ does *not* admit any negative eigenvalue. So the main challenge in [2] consisted in describing the behaviour of the negative part of $\mathfrak{S}(\mathcal{A}_\delta)$. We proved the following result.

Theorem 5.5.1.
Let $\mathcal{B}_0 : D(\mathcal{B}_0) \subset \mathrm{L}^2(\mathbb{R}^3) \to \mathrm{L}^2(\mathbb{R}^3)$ refer to the unbounded operator given by $\mathcal{B}_0(v) := -\mathrm{div}(\sigma_{\mathrm{N}}\nabla v)$ with $D(\mathcal{B}_0) = \{v \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3) \mid \mathrm{div}(\sigma_{\mathrm{N}}\nabla v) \in \mathrm{L}^2(\mathbb{R}^3)\}$. Then $\mathcal{B}_0$ is a self-adjoint operator whose spectrum lies in the real axis and satisfies $\inf \mathfrak{S}(\mathcal{B}_0) = -\infty$. Moreover the negative part of the spectrum is punctual: there exists an infinite ordered sequence $\cdots \leq \mu_{-n} \leq \cdots \leq \mu_{-2} \leq \mu_{-1} < 0$ such that $\mathfrak{S}(\mathcal{B}_0) \cap (-\infty, 0) = \{\mu_{-n}\}_{n\in\mathbb{N}^*}$. In addition, under the assumptions of Proposition 5.5.1, for any $n \in \mathbb{N}^*$, there exists constants $C, \gamma, \delta_0 > 0$ independent of δ such that

$$|\lambda_{-n}^\delta - \delta^{-2}\mu_{-n}| \leq C \exp(-\gamma/\delta) \qquad \forall \delta \in (0, \delta_0).$$

5.6 Small metallic inclusion in an electromagnetic cavity

The last contribution in asymptotics that we wish to mention was published in [6] and concerns the stability of the solution to Maxwell's equations in a cavity containing a small metallic inclusion. Although this kind of asymptotic stability result was well known for elliptic problems, Maxwell's equations had received much less attention than elliptic problems. Admittedly a series of works [20, 21, 51, 64] already dealt with electromagnetic scattering in homogeneous media, but the propagation medium containing the small penetrable heterogeneities was fixed which makes it possible to directly rely on compactness arguments. Other contributions [62, 63] dealt with the asymptotics of perfectly conducting (impenetrable) inclusions but, in these references, the analysis relies on boundary layer operators which is only possible with an homogeneous background medium.

In the case of a small metallic inclusion, the propagation medium changes as $\delta \to 0$, which discards the use of classical compactness arguments. The work in [6] circumvented this difficulty by means of an asymptotic version of Hardy's inequality.

Here is the result proved in [6]. Let $\Omega, \mathrm{D} \subset \mathbb{R}^3$ refer to bounded Lipschitz open sets with $0 \in \Omega$. Set $\Omega_\delta := \{\boldsymbol{x} \in \Omega \ , \ \boldsymbol{x}/\delta \notin \mathrm{D}\}$. Let $\epsilon, \mu : \Omega \mapsto \mathbb{C}^{3\times 3}$ refer to bounded matrix valued functions such that there exists constants $\epsilon_*, \mu_* > 0$ with $\epsilon_*|\boldsymbol{y}|^2 < \Re e\{\boldsymbol{y}^T \epsilon(\boldsymbol{x})\boldsymbol{y}\}$ and $\mu_*|\boldsymbol{y}|^2 < \Re e\{\boldsymbol{y}^T \mu(\boldsymbol{x})\boldsymbol{y}\}$ for all $\boldsymbol{x} \in \Omega, \boldsymbol{y} \in \mathbb{R}^3$. Choose $\omega > 0$ that is not an eigenfrequency of the Maxwell problem in Ω i.e. so that the only $\boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega) := \{\, \boldsymbol{u} \in \mathbf{L}^2(\Omega)\,,\ \mathbf{curl}(\boldsymbol{u}) \in \mathbf{L}^2(\Omega)\}$ satisfying $\mathbf{curl}(\mu^{-1}\mathbf{curl}\,\boldsymbol{v}) - \omega^2\epsilon\boldsymbol{v} = 0$ in Ω and $\boldsymbol{v} \times \boldsymbol{n} = 0$ on $\partial\Omega$ is $\boldsymbol{v} = \mathbf{0}$. Then we proved the following inf-sup condition that is uniform with respect to δ.

Theorem 5.6.1.
If ω is not an eigenfrequency of the Maxwell problem associated to ϵ, μ in Ω, there exist constants $C, \delta_0 > 0$ independent of δ such that, for any $\boldsymbol{u} \in \mathbf{H}_0(\mathbf{curl}, \Omega_\delta) := \{\, \boldsymbol{u} \in \mathbf{L}^2(\Omega_\delta)\,,\ \mathbf{curl}(\boldsymbol{u}) \in \mathbf{L}^2(\Omega_\delta),\ \boldsymbol{u} \times \boldsymbol{n} = 0$ on $\partial\Omega_\delta\,\}$ we have

$$\|\boldsymbol{u}\|_{\mathbf{H}(\mathbf{curl},\Omega_\delta)} \leq C \sup_{\boldsymbol{v}\in\mathbf{H}_0(\mathbf{curl},\Omega_\delta)\setminus\{0\}} \frac{\int_{\Omega_\delta} \mu^{-1}\mathbf{curl}(\boldsymbol{u})\mathbf{curl}(\boldsymbol{v}) - \omega^2(\epsilon\boldsymbol{u})\boldsymbol{v}\, d\boldsymbol{x}}{\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_\delta)}} \qquad \forall \delta \in [0, \delta_0].$$

Chapter 6

Research perspectives

In this final chapter we describe perspectives and possible future directions of research that seem both relevant, and directly connected to the work presented in this thesis. Some of these topics are current works in progress.

6.1 More numerics on MTF

As a preliminary remark, we should underline that there is still too few numerical results concerning multi-trace formulations, especially in 3-D geometries, and in an HPC context. In this respect, at least two points should be raised in the near future. First, it appears mandatory to conduct relevant numerical experiments comparing multi-trace formulations and the Boundary Element Tearing and Interconnecting Approach. Second, it would be interesting to examine how to combine multi-trace formulations with a Fast Multipole Method (FMM) or Adaptative Cross Approximation (ACA). Admittedly, combining MTF with these techniques does not seem to raise any conceptual difficulty. From a purely algorithmic/combinatoric/implementation point of view though, this point may be tricky due to the peculiarities of multi-trace formulations that must, at some point, keep track of adjacency relations between subdomains.

6.2 Quasi-local MTF for electromagnetics

Another research direction concerns extensions and further studies of the quasi-local multi-trace formulations described in §3.3. The numerical results that we have presented suggest that taking a cut-off function ψ in (3.6) with a very concentrated support does not deteriorate much the performances of the quasi-local MTF (3.7). It is thus natural to think of taking a cut-off function ψ whose support is only a few mesh cells wide which, de facto, localises the effect of the operator $\mathbb{Q}$. With this approach, it would be interesting to examine whether a uniform discrete inf-sup condition remains valid. This would yield an alternative to the local MTF (3.2) i.e. a "regularised local MTF" that would lend itself to much finer theoretical numerical analysis.

Many theoretical questions remain open as regards the analysis of the local MTF. One may consider the regularised local MTF described above as an intermediary step toward a more deep theoretical study of the local MTF, in the context of electromagnetics for example.

6.3 Multi-trace FEM-BEM coupling

Boundary integral approaches and, a fortiori, multi-trace formulations, are no longer applicable when considering fully heterogeneous media. Even with multi-trace formulations, only piece-wise constant media can be considered. To deal with heterogeneous media, a possible strategy may rely on finite element/boundary element coupling i.e. so-called FEM-BEM coupling. This kind of strategy has been widely studied in the case of two subdomains (one that is homogeneous, and the other one heterogeneous) separated by one interface. Let us mention three of the most widespread coupling schemes: the Bielak-MacCamy scheme [25], the Johnson-Nédélec scheme [76] and the Costabel symmetric coupling method [48]. When considering multi-subdomain problems with junction points, most of the contributions known to us are related to the BETI strategy and are domain decomposition oriented.

One interesting research direction would consist in examining whether the multi-trace (either local, global or quasi-local) formulations of Chapter 3 may be coupled with finite element strategies via some of the three coupling schemes mentioned above. The same question for the integral formulation of the second kind presented in §2.3 is also of interest.

6.4 BEM-based domain decomposition

Multi-trace formulations were devised to treat PDEs posed in piece-wise homogeneous media. On the other hand, this kind of problem lends itself to non-overlapping domain decomposition (DDM). Since, in addition, MTF allows a clean handling of junctions, it would thus appear natural to combine a multi-trace boundary integral approach with domain decomposition. This question has already been the subject of a few contributions of the literature [102, 69, 101]. These results are mostly of a purely numerical nature.

A direction we are pursuing at present consists in studying domain decomposition in conjunction with multi-trace formulations. A first simple idea consists in studying the performance of standard DDM strategies, such as block-Jacobi preconditioning, applied to MTF. It would also be interesting to devise global DDM solvers well adapted to multi-trace formulations.

A first remark is that spectral analysis of the multi-trace operators should play a pivotal role in answers to such questions. We have initiated a work in this direction in collaboration with V.Dolean (Université de Nice) and M.Gander (Université de Genève). One of the first important results we have obtained is a clear relation between Optimal Schwarz DDM strategies on the one hand (see [94] for example), and block-Jacobi preconditioning of local multi-trace formulations. We have also put into evidence a strong relationship between the adjacency graph of the sub-domain partition and the spectral structure of local multi-trace operators.

Research contract with ANR This topic and the next section are the main focus of the project NonLocalDD funded by the French National Research Agency (ANR), in which I am the principal coordinator. This project is 4 years long and has been accepted in July 2015.

6.5 Non-local Schwarz transmission conditions

In the context of non-overlapping domain decomposition for wave equations, the pioneering work of Després [53] then Collino, Ghanemi and Joly [45] and Gander, Magoules and Nataf [59] have shown that it is mandatory to use impedance type transmission conditions in the coupling of subdomains in order to obtain convergence of the DDM. In the approaches considered so far in the literature, the impedance operator involved in the transmission conditions was always local (a scalar in the most simplest cases). These methods lead to algebraic convergence of the DDM in the best cases.

In a recent work, F. Collino, P. Joly and M. Lecouvez [86] have observed that using non-local impedances such as integral operators could lead to an exponential convergence of the DDM. One of the strengths of this approach is to rely on a solid theoretical basis that systematically guarantees geometrical convergence, provided that certain properties of injectivity, surjectivity and positivity (in suitable trace spaces) are satisfied by the impedance.

We propose to study possible construction of this impendance operator by means of a (possibly modified) EFIE-type integral operator associated with a Yukawa potential (so as to guarantee positivity and self-adjointness of the operator). This approach would be easily adapted to Maxwell's equation with interfaces admitting a possibly low level of regularity (Lipschitz for example).

Besides, let us point out that the analysis of Collino, Joly and Lecouvez is valid only in the case where the interfaces between subdomains are closed i.e. there are no junctions. We propose to adapt their analysis, relying on the use of quasi-local transmission operators (3.6).

6.6 MTF-based finite element method

Recently Copeland, Langer and Pusch [47] have introduced a new class of non-standard finite element schemes. Considering for example a second order elliptic problem, their idea consists in considering polyhedral meshes with cells of small characteristic size (like in standard finite element methods) as a subdomain decomposition of the problem, and then applying a boundary element tearing and interconnecting method.

Since then, this approach has given rise to a growing number of contributions, see for example [72, 107, 129]. One appealing feature of this method is that it can deal with mesh cells with general polyhedral form. Since it can be seen as some sort of Trefftz method, in the context of wave propagation problems, there is possibility that this approach be less prone to dispersion phenomena like ultra weak formulations.

We plan to explore this direction of research basing our approach on the local multi-trace formulation. In addition to the advantage of the above mentioned BEM-based method, this may allow to take profit in a volumic finite-element-like context of preconditioning techniques already available for multi-trace formulations. In addition, due to the particular geometric context (i.e. the subdomains are mesh-cells) there may be a chance for exploiting already available techniques for discontinuous Galerkin methods to better analyse local multi-trace formulations in this context.

6.7 Integral equation for low frequency electromagnetics

All classical boundary integral equations used in electromagnetics such as, for example, the electric field integral equation (EFIE), are known to become singular at low frequency. This is related, at least at first sight, to a factor of the form $1/\kappa^2$ (where κ refers to the wave number) coming into play in the expression of the (vector) single layer potential. This issue makes the use of integral equations rather problematic in low frequency physical models such as eddy currents.

Certain formulations though, only recently introduced, have managed to bypass this issue. In this direction, one could mention the current and charge formulation [122, 121], or a new formulation introduced by Greengard and Epstein [55, 56]. All these new approaches seem to be related to an extended version of the Maxwell system that was considered by Picard [103] for the study of the low frequency asymptotics of Maxwell's equations. Analysis of boundary integral equations in the low frequency limit, and their precise relationship to the so-called Picard system seem ill understood up to now.

In this research project, we propose to develop an approach fully based on the Picard system, that would consist in adapting the analysis of the classical paper of Costabel [49] to this new context: proper definition of trace operators and spaces, proof of integral representation formulas, jumps and continuity properties of appropriate potential operators,... We shall adopt the methodology that we developed for the study of scattering by multi-screen that provides some kind of systematic theoretical machinery for deriving this type of result.

Industrial collaboration with CEA This work is currently the subject of a joint research contract between Laboratoire Jacques-Louis Lions and CEA LIST, involving Edouard Demaldent and myself as principal investigators. It is also the subject of a present active collaboration with Ralf Hiptmair (SAM ETH Zürich).

6.8 Time domain scattering by small objects

Although the asymptotics of wave scattering by small inclusions has been intensively studied for harmonic regime propagation, current literature still offers only few results concerning time dependent scattering problems. As far as we know, the most remarkable contributions in this direction are [19, 89]. In harmonic regime, asymptotic analysis becomes particularly challenging when considering small inclusions with Dirichlet boundary conditions in a 2-D propagation medium which, in a nutshell, is partly caused by the the special logarithmic behaviour of the Laplace Green kernel in 2-D.

This particular situation is not covered by the present contributions of the literature when considering time domain. In this case, matched asymptotics lead to an ill-posed scalar integro-differential equation. We plan to examine this problem in a forthcoming article.

6.9 Asymptotics for Pocklington's equation

Participating in new projects related to asymptotic analysis, and in particular collaborating with Sergei Nazarov for the articles [2, 3], led me to understand and learn new theoretical techniques. This raised new ideas and perspectives for revisiting the analysis of Pocklington's

equation that is a one-dimensional counterpart of the Electric Field Integral Equation (EFIE) adapted to the scattering by thin wires.

The well-posedness of Pocklington's equation is now well established since the work of Jones [77] and Rynne [110, 111]. However, this equation involves a small parameter, namely the ratio $\delta = (\text{thickness of the wire})/(\text{wave length})$ and there still does not exist any theoretical result about the stability of this equation as $\delta \to 0$. We propose to establish such a stability result by means of a $\delta-$dependent norm. We also intend to examine the robustness as $\delta \to 0$ of the results of [24] concerning the asymptotics of the eigenvalues of the Pocklington's operator.

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