Control, stabilization and classification, for various PDE

Camille Laurent

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Assia Benabdallah (Examinatrice)	-	Aix-Marseille Université
Jean-Michel Coron (Examinateur)	-	Sorbonne Université
Thomas Duyckaerts (Examinateur)	-	Université Sorbonne Paris Nord
Enrique Fernandez-Cara (Rapporteur)	-	Universidad de Sevilla
Patrick Gérard (Examinateur)	-	Université Paris-Saclay
Jérôme Le Rousseau (Examinateur)	-	Université Sorbonne Paris Nord
Luc Robbiano (Rapporteur)	-	Université Versailles Saint-Quentin-en-Yvelines

après les rapports de

Enrique Fernandez-Cara	-	Universidad de Sevilla
Luc Robbiano	-	Université Versailles Saint-Quentin-en-Yvelines
Enrique Zuazua	-	Friedrich-Alexander Universität Erlangen

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Liste des travaux de l'auteur

My publications and pre-publications are listed below and are not included in the general bibliography of this dissertation. To keep it concise and coherent, only a portion of these works is presented here. Thus, the publications listed below with a ⋆ will not be dealt with in this manuscript; I apologies in advance to the co-authors concerned. Also, most of the article presented are collaborations. In all the manuscript, the term "we" as "we proved" will always refer to the author of this memoir and the coauthors of the article.

All the articles are also available on my website https://www.ljll.math.upmc.fr/~laurent/, on Hal and on Arxiv.

Preprint :

[P2] *A scattering operator for some nonlinear elliptic equations* (with Raphaël Côte) (80p) https://arxiv.org/abs/2312.17514

[P1] *Lectures on unique continuation for waves*, (with Matthieu Léautaud), (survey) https://arxiv.org/abs/2307.02155

Publications :

[A21] *On uniform observability of gradient flows in the vanishing viscosity limit* (with Matthieu Léautaud), Comptes Rendus Mathématique 2023, Vol. 361, p. 265-312 (ce n'est pas une note)

[A20] *Uniform observation of semiclassical Schrödinger eigenfunctions on an interval* (with Matthieu Léautaud), Tunisian Journal of Mathematics, Vol. 5 (2023), No. 1, pp. 125-170

[A19] C. Laurent et M. Léautaud, *Tunneling estimates and approximate controllability for hypoelliptic equations* (with Matthieu Léautaud), Mem. Amer. Math. Soc. 276 (2022), no. 1357, vi+95 pp.

[A18] *Concentration close to the cone for linear waves* (with Raphaël Côte), Rev. Mat. Iberoam. 40 (2024), no. 1, pp. 201-250

[A17] *Logarithmic decay for damped hypoelliptic wave and Schrödinger equations* (with Matthieu Léautaud), SIAM J. Control Optim. 59 (2021), no. 3, 1881-1902.

[A16] *On uniform observability of gradient flows in the vanishing viscosity limit* (with Matthieu Léautaud), Journal de l'École polytechnique - Mathématiques, Tome 8 (2021), pp. 439-506.

[A15] *Quantitative unique continuation for hyperbolic and hypoelliptic equations* (with Matthieu Léautaud), Séminaire X-EDP (2020) (survey)

[A14] *Decay of semilinear damped wave equations: cases without geometric control condition* (with Romain Joly), Annales Henri Lebesgue 3 (2020) 1241-1289

⋆ **[A13]** *Exact Controllability of Nonlinear Heat Equations in Spaces of Analytic Functions* (with Lionel Rosier), Ann. Inst. H. Poincaré Anal. Non Linéaire 37 (2020), no. 4, 104–1073.

- ⋆ **[A12]** *On the Observability Inequality of Coupled Wave Equations: the Case without Boundary* (with Yan Cui and Zhiqiang Wang), ESAIM: COCV 26 (2020) 14
- **[A11]** *Observability of the heat equation, geometric constants in control theory, and a conjecture of Luc Miller* (with Matthieu Léautaud), Analysis & PDE Vol. 14 (2021), No. 2, 355-423
- **[A10]** *Quantitative unique continuation for operators with partially analytic coefficients. Application to approximate control for waves.* (with Matthieu Léautaud) J. Eur. Math. Soc. 21 (2019), 957-1069.
- ⋆ **[A9]** *Bilinear control of high frequencies for a 1D Schrödinger equation* (with Karine Beauchard), Math. Control Signals Systems 29 (2017), no. 2, Art. 11, 14 pp.
- ⋆ **[A8]** *Local exact controllability of the 2D-Schrödinger-Poisson system* (with Karine Beauchard), J. Éc. polytech. Math. 4 (2017), 287-336.
- **[A7]** *Uniform observability estimates for linear waves.* (with Matthieu Léautaud) ESAIM Control Optim. Calc. Var. 22 special issue in honor of J.-M. Coron (2016), no. 4, 1097-1136.
- ⋆ **[A6]** *Control and stabilization of the Benjamin-Ono in* $L^2(\mathbb{T})$ (with Felipe Linares and Lionel Rosier), Arch. Ration. Mech. Anal. 218 (2015), no. 3, 1531-1575.
- ⋆ **[A5]** *Internal control of the Schrödinger equation* (survey), Mathematical Control and Related Fields 4(2) 2014.
- **[A4]** *A note on the global controllability of the semilinear wave equation* (with Romain Joly), SIAM J. Control Optim. 52 (2014), no. 1, 439-450.
- **[A3]** *Stabilization for the semilinear wave equation with geometric control condition* (with Romain Joly), Analysis & PDE Vol. 6 (2013), No. 5, 1089-1119 , 2012.
- ⋆ **[A2]** *On stabilization and control for the critical Klein-Gordon equation on a 3-D compact manifold*, Proceedings of the conference Journées Équations aux Dérivées Partielles, Biarritz, 2011.
- ⋆ **[A1]** *The Laplace-Beltrami operator in almost-Riemannian Geometry* (with Ugo Boscain), Annales de l'Institut Fourier, 63 no. 5 (2013), p. 1739-1770 .

The following works are the one written during my PhD thesis. They will not be described precisely, but sometimes mentioned since some of the works I did later are closely related to these subjects.

- ⋆ **[T]** *Contrôle d'équations aux dérivées partielles non linéaires dispersives*, **Thèse de doctorat**, Université Paris-Sud, sous la direction de Patrick Gérard, 2010.
- ⋆ **[T5]** *On stabilization and control for the critical Klein Gordon equation on 3-D compact manifolds*, Journal of Functional Analysis, 260(5):1304-1368, 2011.
- ⋆ **[T4]** *Local controllability of 1D linear and nonlinear Schrödinger equations* (with Karine Beauchard), Journal de Mathématiques Pures et Appliquées, 94(5):520-554, 2010.
- ⋆ **[T3]** *Control and Stabilization of the Korteweg-de Vries Equation on a Periodic Domain* (with Lionel Rosier and Bingyu Zhang, Communications in PDE 35(4):707-744, 2010.
- ⋆ **[T2]** *Global controllability and stabilization for the nonlinear Schrödinger equation on some compact manifolds of dimension 3*, SIAM Journal on Mathematical Analysis 42(2):785-832, 2010.
- ⋆ **[T1]** *Global controllability and stabilization for the nonlinear Schrödinger equation on an interval*, ESAIM-COCV, 16(2): 356-379, 2010.

With Matthieu Léautaud, we also wrote some Lecture Notes concerning Carleman estimates and their applications. They are not published for the moment but available online.

[L] *Unique continuation and applications* (with Matthieu Léautaud) https://www.ljll.math.upmc.fr/~laurent/papiers/UCPApplications.pdf .

General presentation of the manuscript

This memoir is divided into 1+4 chapters:

0) Chapter 0 is the Introduction where we try to put in a broader context the results and methods we presented in this manuscript. It is also meant as a toolbox that we will refer to all along the manuscript.

1) Chapter 1, certainly the longest, concerns the quantitative unique continuation and its applications to approximate and exact observability for wave equations, to the approximate controllability and tunneling estimates for hypoelliptic operator.

2) Chapter 2 describes how some methods coming from dynamical systems can be useful to the stabilization and controllability of semilinear wave equations.

3) Chapter 3 concerns the cost of the control in some asymptotic regimes: the heat equation in small time and the transport equation in the vanishing viscosity limit.

4) Chapter 4 describes some situations where we can classify the solutions of nonlinear elliptic and linear wave equation that have a prescribed behavior at infinity.

Let us detail a bit more the content of each chapter.

Chapter 1 contains all our results about quantitative unique continuation in the case of partial (or complete) analyticity and several of the applications we obtained from it.

- We first describe our general result in **[A10]** where we give the optimal stability estimates associated to the general unique continuation theorem of Tataru-Robbiano-Zuily-Hörmander. For this result, we develop a general strategy
  - to quantify locally and in an optimal way the unique continuation,
  - to ensure that these estimates can be propagated in order to obtain *in fine* some global estimates.

  The most notable application of this result, described in the first part of Section 1.3, is a frequency-dependent observability estimate for solutions of the wave equation on a bounded domain that takes the form

  $$\|(u_0, u_1)\|_{H^1\times L^2} \leq Ce^{C\Lambda} \|u\|_{L^2((0,T);H^1(\omega))}, \qquad \text{(appobs)}$$

  where $\Lambda = \frac{\|(u_0,u_1)\|_{H^1\times L^2}}{\|(u_0,u_1)\|_{L^2\times H^{-1}}}$ represents the typical frequency of the initial datum (u_0, u_1). $\omega \subset \Omega$ can be any non empty open set and $T > 2\sup_{x\in\mathcal{M}} \operatorname{dist}(x, \omega)$. The result is optimal with respect to the cost and the minimal time. It provides some estimates of the cost of approximate controllability.

- Then, we give some applications to the exact controllability for waves which correspond to **[A7]** and are described in the second part of Section 1.3. It provides a constructive proof of the classical result of Bardos-Lebeau-Rauch [16] of the controllability of the wave equation under the geometric control condition. By constructive, we mean that there is no argument by compactness or contradiction. It has the advantage of allowing some estimates of the cost of the control, at least in some asymptotic regimes, as when the time T gets close to the geometric control condition or when the equation is perturbed by a large potential.

- Finally, we develop some applications to hypoelliptic operators that were obtained in [A19] and [A17] . They are presented in Section 1.4. It concerns evolution equations associated to some operator $\mathcal{L}$ (replacing the Laplacian) that is hypoelliptic. It has the form "sum of the squares" of vector fields that satisfy the Chow-Rashevski-Hörmander hypothesis: the Lie algebra of these vector fields spans the full tangent plane, ensuring hypoellipticity. Assuming the analyticity of the coefficients, we obtain an analog of (appobs) for the associated wave equation. The minimal time has to be modified with the associated sub-Riemannian distance adapted to the vector fields while $e^{C\Lambda}$ has to be replaced by $e^{C\Lambda^k}$, k is the *hypoellipticity index*, that is the number of iterated Lie brackets necessary to span the tangent plane. We prove that this result is optimal in general with respect to the time and form of the cost. We then deduce some tunneling estimates for the eigenfunctions of $\mathcal{L}$ and several results of approximate controllability for the associated heat-like operator.

Chapter 2, concerning dynamical system methods, contains three parts corresponding respectively to [A3] , [A4] and [A14] . More precisely, the results we obtain are as follows:

- Section 2.2 presents the results we obtained in [A3] concerning the nonlinear damped wave equation $\Box u + \gamma_\omega(x)\partial_t u + f(u) = 0$ where γ_ω is a damping active on a zone ω satisfying the geometric control condition and f is a defocusing subcritical analytic nonlinearity. Many results were available but with stronger geometrical assumptions. The main missing argument was the unique continuation that we obtained thanks to some ideas of dynamical systems proving that some compact global attractor is actually analytic in time. We also simplify the proof of propagation of compactness and regularity that already existed. We obtain exponential decay when zero is the only equilibrium and the existence of a compact global attractor otherwise.

- In Section 2.3 corresponding to [A14] , we present some results of stabilization of the nonlinear damped wave equation under some weaker geometrical assumptions. They typically happen in some cases where the trapped set is small. We prove the decay to zero of the nonlinear damped equation in several examples. The proof is actually done in an abstract way under the assumptions that the linear damped wave equation decays sufficiently fast. We present a new proof of "asymptotic smoothness" in this context and of unique continuation.

- In Section 2.4 corresponding to [A4] , we present applications to global controllability. We prove some global controllability results for some equation of the form $\Box u + f(x,u) = \mathbb{1}_\omega g$ where g is the control and ω satisfies the geometric control condition. The nonlinearity $f(x,u)$ is "asymptotically defocusing", but allows some non trivial equilibria for the associated damped equation. The idea for the control is to use the damped equation and to travel through the attractor to get to the expected target.

Chapter 3, about the cost of the control, is divided in two parts:

- Cost of the control of the heat equation in small time in [A11] : there was a conjecture in the field about the asymptotics of the cost of small-time control. This conjecture was based on the assumption that the heat kernel consisted of the worst case. We show, with counterexamples, that the conjecture is false in general, but remains true if we restrict ourselves to the observability of positive solutions, which also has implications for control. We also give some upper bounds of the cost of fast control when the zone of internal control is a small ball around one point.

- Uniform controllability of the transport equation in the vanishing viscosity limit: in this part, we consider transport equations with an evanescent viscosity term $\varepsilon\Delta$. It was hoped that an observability estimate uniform in ε would be possible if the limiting transport equation was controllable. We show in [A16] that this is not true in the general case, but it is true for positive solutions. Then, in [A21] and [A20] , we give upper and lower bounds of the time of uniform controllability in the one-dimensional case with varying coefficients. It seems to give some hints on what could be the correct time.

Chapter 4 has two parts corresponding to some classification result for nonlinear elliptic equation and non-radiative solutions of the linear wave equation

- in **[P2]** , we prove the existence, in a suitable space, of a scattering operator for some class of nonlinear elliptic equations on $\mathbb{R}^d$ as $r \to +\infty$. We show that there is a *one-to-one correspondence* between the nonlinear solution u defined there, and the linear solution u_L to the Laplace equation, such that, in an adequate space, $u - u_L \to 0$ as $|x| \to +\infty$. It is a way of classifying all solutions of nonlinear elliptic equation close to infinity by the identification with linear solutions. The full classification is made in the following examples: semilinear energy critical equation, Harmonic maps in dimension 2, H-system.

  More precisely, we define a functional space Z_r on $\mathbb{S}^{d-1}$, depending on the radius r and fitted to the elliptic equation so that for any finite energy solution of $\Delta u = f(u, \nabla u)$ (for f in the class of the previous examples) on $\mathbb{R}^d \setminus B(0,1)$, we can find a unique linear solution of $\Delta u_L = 0$ on $\mathbb{R}^d \setminus B(0, r_0)$, $r_0 \geq 1$ so that

  $$\|(u - u_L)(r\cdot)\|_{Z_{r/r_0}} \underset{r\to+\infty}{\longrightarrow} 0.$$

  Reciprocally, for u_L linear solution, we prove that there exists $r_0 \geq 1$ and u nonlinear solution so that the previous asymptotic holds.

  The second part actually holds in a much general class of nonlinearity. The first part also holds for several other examples, assuming some additional decay on the solutions, adapted to each equation.

  We also obtain a similar classification considering the asymptotic close to a point.

- in the second result **[A18]** , we classify linear solutions of the wave equations on $\mathbb{R}^d$ that, asymptotically as $t \to \pm\infty$, have no energy outside of a truncated cone. More precisely, we consider the asymptotic energy outside the light cone

  $$E_{ext,R}(v) := \frac{1}{2}\Bigg(\lim_{t\to+\infty}(\|\nabla v\|^2_{L^2(|x|\geq t+R)} + \|\partial_t v\|^2_{L^2(|x|\geq t+R)}) \\ + \lim_{t\to-\infty}(\|\nabla v\|^2_{L^2(|x|\geq |t|+R)} + \|\partial_t v\|^2_{L^2(|x|\geq |t|+R)})\Bigg).$$

  In odd dimension, for $R > 0$, we fully describe the space $P(R)$ of initial data giving rise to solutions asymptotically zero outside the light cone, that is so that $E_{ext,R}(v) = 0$. The description involves the decomposition in spherical harmonics. We also derive a Pythagorean formula of the form

  $$\|(u_0, u_1)\|^2_{\dot{H}^1\times L^2(\mathbb{R}^d)} = 2E_{ext,R}(u) + \|\pi_R(u_0, u_1)\|^2_{\dot{H}^1\times L^2(\mathbb{R}^d)}.$$

  where π_R is the orthogonal projection (in $\dot{H}^1 \times L^2$) onto the space $P(R)$. Also, along the proof, we derive further expressions of the exterior energy (outside a shifted light cone), valid in all dimensions and for nonradial data. This, in particular, generalizes previous results obtained in the radial setting.

CHAPITRE 0

Introduction

This memoir presents several works that I completed since I finished my PhD thesis [T]. The redaction of such a memoir can be the occasion to try to find some unity in the works that we completed. The main topic of my research is control and stabilization, but thinking a little more, it seemed to me that the common point of all the works I did is the propagation of the information for Partial Differential Equations: if you know something about a solution of a PDE (which means in the application that you know your system only partly), what can you say about the full solution?

To be more precise, we need to say what kind of information we want to propagate. Mathematically speaking, the following kind of "information" [1] is studied in this manuscript at several points and for several applications:

Assume for instance that u is solution of a certain Partial Differential Equation on a big set Ω, we can study the following concepts :

- propagation of nullity (or uniqueness): if I know that u is zero somewhere (say u is zero on a subset $\omega \subset \Omega$), can I say that it is zero in a bigger set or even in Ω? This will sometimes be called the **unique continuation** property.
- propagation of smallness: if I know that u is small somewhere on a subset $\omega \subset \Omega$, can I say that it is small in a bigger set or even in Ω? This is obviously stronger and will sometimes be called the **quantification of the unique continuation** property.
- propagation of compactness: consider instead a sequence of solutions such that a certain energy is bounded and so that u_n restricted to ω is strongly convergent to a function u. Can we extract a subsequence that is strongly convergent to a function on Ω? We will talk about **propagation of compactness**.
- propagation of regularity: if I know that u is regular somewhere on a subset $\omega \subset \Omega$, can I say that it is regular in a bigger set? This will be called the **propagation of regularity**.

The understanding of all these properties will be crucial for all the applications we have in mind. Most of them will concern controllability and stabilization, but the last part will also concern some classification.

Before presenting more precisely our results, we found it useful to give a short introduction to the goals and tools that are common to several of the results presented in this memoir. We will try to give a partial answer to the following questions:

Why to understand the propagation of the information for PDE?

How to understand the propagation of the information for PDE?

[1] we use the word information in the common meaning, not for the mathematical theory of information. We could also talk about the propagation of knowledge.

Of course, the answer will not be complete and it will be rather an excuse to introduce the reader to the methods that will be discussed later.

This very general presentation is an attempt to give the broad context of our works and to present it in a unified way. The methods presented in this introduction are mostly very general and presented without precision in order to avoid complicated hypotheses. Moreover, it was therefore sometimes very hard to know precisely where the general idea comes from. As a consequence, the bibliography is very scarce. More precise theorems and a more complete bibliography will be presented in the following chapters where the results will be presented with more precision.

All the ideas and methods apply to several equations, but in order to make things more concrete, we will illustrate them many times by the linear or nonlinear wave equation.

0.1 Why to understand the propagation of the information?

Control and observability

The link between observability and controllability is very classical. We give (without rigorous statements) the ideas of this implication and discuss some links with the cost. Some more precise statements can be found in [41, Section 2.3.].

Most of the problems of control of linear PDE can be put under the following abstract form

$$\left\{ \begin{array}{rcll} \dot{y} & = & \mathcal{A}y + Bg & t \in [0,T] \\ y(0) & = & y_0. \end{array} \right. \tag{0.1.1}$$

where $\mathcal{A}$ is an unbounded operator on a Hilbert space H that generates a strongly continuous semigroup, B is the operator of control (assumed bounded for simplicity) and g is the control to be chosen. The internal control corresponds to the case where $B = 1_\omega$ is the multiplication by the indicatrix function of an open set ω (for the waves, the equations can be put under the form of a system and then $B = (0, 1_\omega)$. Moreover, by commodity, we are often led to replace the function 1_ω by a regular function $a(x) \approx 1_\omega$, so that the operator B is bounded on H.

Now, two important questions are the following

- The question of exact **controllability** is to know if, given an initial state y_0, and a final y_1 fixed in advance, can we find a control $g \in L^2([0,T], H)$ so that the solution of (0.1.1) satisfies $y(T) = y_1$?
- The question of **approximate controllability** is that for $\varepsilon > 0$, y_0, and a final y_1 fixed in advance, can we find a control $g \in L^2([0,T], H)$ so that the solution of (0.1.1) satisfies $\|y(T) - y_1\| \leq \varepsilon$? Here, the norm can be another one than H.

Considering the surjectivity of an operator is usually a complicated task while, in analysis, it is often easier to prove injectivity or inequalities. The HUM method of Russel and Jacques-Louis Lions [129] translates the controllability to zero (or to the trajectories), that is $y_1 = 0$, into the obtention of an observability inequality for the adjoint problem. More precisely, we consider the dual system

$$\left\{ \begin{array}{rcll} \dot{z} & = & -\mathcal{A}^* z & t \in [0,T] \\ z(T) & = & z_T \end{array} \right. \tag{0.1.2}$$

Note that this equation is backward and we usually consider the unknown $z(T-t)$ solution of a similar that exchange the initial and final data and $-\mathcal{A}^*$ to $\mathcal{A}^*$.

The idea is to observe that by multiplying (actually applying the duality) and integrating in time, we get (at least formally)

$$\int_0^T \langle z(t), \dot{y}(t) \rangle_{H',H} \, dt = \int_0^T \langle z(t), \mathcal{A}y(t) + Bg(t) \rangle_{H',H} = \int_0^T \langle \mathcal{A}^* z(t), y(t) \rangle_{H',H} \, dt + \int_0^T \langle B^* z(t), g(t) \rangle_{H',H} \, dt$$

Integration by parts in time gives $\int_0^T \langle z(t), \dot{y}(t) \rangle_{H',H} \, dt = -\int_0^T \langle \dot{z}(t), y(t) \rangle_{H',H} \, dt + \langle z_T, y(T) \rangle_{H',H} - \langle z(0), y(0) \rangle_{H',H}$, which thanks to (0.1.2) gives

$$\langle z_T, y(T) \rangle_{H',H} - \langle z(0), y(0) \rangle_{H',H} = \int_0^T \langle B^* z(t), g(t) \rangle_{H',H} \, dt.$$

If we consider the control from $y_0 = 0$, we can define the application "endpoint map" $g \mapsto y(T) := \mathcal{F}_T(g)$ and have the identity

$$\langle z_T, y(T)\rangle_{H',H} = \langle z(0), \mathcal{F}_T(g)\rangle_{H',H} = \int_0^T \langle B^* z(t), g(t)\rangle_{H',H}\, dt.$$

That means that we have found $\mathcal{F}_T^*$: this is the application $z_T \mapsto B^* z(\cdot)$ from H' to $L^2([0,T], H')$, which is the observation of the solution z by the operator B^*. This justifies the rough statement "the observability is the dual to the controllability".

- The question of exact controllability can be interpreted as the surjectivity of the operator $\mathcal{F}_T$. It is standard in analysis that this is often equivalent to an inequality of the form $\|y_T\|_H \leq \|\mathcal{F}_T^* y_T\|_H$ which in our context takes the form of an observability inequality

$$\|z_T\|_{H'}^2 \leq C_{obs} \int_0^T \|B^* z(t)\|_{H'}^2\, dt. \qquad (0.1.3)$$

- The question of approximate controllability can be interpreted as the density of the image of $\mathcal{F}_T$. It is often equivalent to the injectivity of $\mathcal{F}_T^*$, which takes the form of the unique continuation

$$\left(B^* z = 0 \text{ in } L^2(0,T), H')\right) \Rightarrow z_T = 0.$$

A very important question is then the **cost of the control**. It is answering: what is the size of the control g with respect to the parameters of the system?

- In the case of exact controllability, it is easy to see that the cost of the control is directly linked to the constant C_{obs} in (0.1.3). So, computing the cost of the control is to give estimates on C_{obs}. In practice, it is often very complicated to compute C_{obs}. Yet, we can expect to get some dependence on some parameters, like the time of observation T. Also, in the case of an internal observation when $B = 1_\omega$, we can expect to have bounds when ω becomes small for instance.
- For the approximate controllability, the first natural question is the dependence of the cost of the control g with respect to the parameter ε. How expensive is it to get very close to the expected target? By a similar duality that we presented for the exact controllability, it is equivalent to proving a quantification of the unique continuation property.

A typical example is the wave equation

$$\begin{cases} \partial_t^2 u - \Delta_g u = 0 & \text{in } (0,T)\times\Omega, \\ u_{|\partial\mathcal{M}} = 0 & \text{in } (0,T)\times\partial\Omega, \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) & \text{in } \Omega. \end{cases} \qquad (0.1.4)$$

In this case, the internal observability inequality on an open subset ω

$$\|(u_0,u_1)\|_{H^1\times L^2} \leq C\, \|u\|_{L^2((0,T);H^1(\omega))}\,. \qquad (0.1.5)$$

is equivalent to the controllability of the control problem

$$\begin{cases} \partial_t^2 v - \Delta_g v = \mathbb{1}_\omega h & \text{in } (0,T)\times\Omega, \\ v_{|\partial\mathcal{M}} = 0 & \text{in } (0,T)\times\partial\Omega, \\ (v, \partial_t v)_{|t=0} = (v_0, v_1) & \text{in } \Omega. \end{cases} \qquad (0.1.6)$$

As we said, the unique continuation implies an approximate controllability result, and it is sometimes possible to quantify this. For instance, we proved a quantification of the unique continuation of the form

$$\|(u_0,u_1)\|_{L^2\times H^{-1}} \leq Ce^{\kappa\mu}\, \|u\|_{L^2((0,T);H^1(\omega))} + \frac{1}{\mu}\, \|(u_0,u_1)\|_{H^1\times L^2} \qquad (0.1.7)$$

uniform in $\mu \geq \mu_0$. It implies some result of approximate controllability at cost $e^{C/\varepsilon}$: if we want to obtain the final state with precision ε, the control will be of size of the order $e^{C/\varepsilon}$. We refer to Theorem 1.3.3 for a more precise statement.

Where it appears in the manuscript?

- the whole Chapter 1 concerns the quantification of the unique continuation and some applications. We provide estimates for the approximate controllability of the type (0.1.5) for various models: wave, Schrödinger equations, Hypoelliptic equations.

A large part of the manuscript is about the exact observability and controllability

- Chapter 3 concerns the cost of observability/controllability for some parabolic equations in some singular limit: small time or the vanishing viscosity limit. The question is to obtain some estimates of the constant of observability with some precise dependence on the various parameters when they converge to some singular limit: the limit $T \to 0$, that is small time controllability, the size of the observation set, a uniform dependence on $\varepsilon > 0$ in the context of a vanishing viscosity $\varepsilon\Delta$.

- in **[A7]** , described in Section 1.3, we give a constructive proof of the classical result of Bardos-Lebeau-Rauch proving the controllability of the wave equation under the geometric control condition. This allows to give several estimates of the constant of observability.

Stabilization

The stabilization problem is quite close to the problem of controllability. We are asking if we can choose the control g of our control problem (0.1.1) (or some nonlinear version) under the form $g = K(u)$, that is in a closed loop form. The question is then if our system will converge to some equilibrium, very often zero. Note that in the applications, the term $K(u)$ might be chosen by the user or inherent to the system as a damping.

Instead of being too abstract, let us present the example of the nonlinear wave equation. We consider the following system, where for instance, Ω is a bounded smooth domain of $\mathbb{R}^3$.

$$\begin{cases} \Box u + \gamma_\omega(x)\partial_t u + f(u) = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u_{|\partial\mathcal{M}} = 0 & \text{in } \mathbb{R}_+ \times \partial\Omega, \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) & \text{in } \Omega. \end{cases} \tag{0.1.8}$$

We assume that $\gamma_\omega \geq 0$ is smooth and $\gamma_\omega \geq \varepsilon > 0$ on an open set $\omega \subset \Omega$. So, we have to think as γ_ω as a smooth version of the indicatrix function $\mathbb{1}_\omega$ of ω. The associated energy is given by

$$E(u) := E(u, \partial_t u) = \frac{1}{2}\int_\Omega (|\partial_t u|^2 + |\nabla u|^2) + \int_\Omega V(u) \ , \tag{0.1.9}$$

where $V(u) = \int_0^u f(s)ds$. Under natural assumption on f (see for instance (2.1.2) below), this energy is well-defined and positive in suitable spaces. Moreover, if u solves (0.1.8), multiplying the equation by $\partial_t u$ and integrating by parts, we get, at least formally,

$$E(u(T)) = E(u(0)) - \int_0^T \int_\Omega \gamma_\omega(x)|\partial_t u(x,t)|^2 \, dx \, dt \leq \ 0 \ . \tag{0.1.10}$$

The system is therefore dissipative. We are interested in the exponential decay of the energy of the nonlinear damped wave equation (2.1.1), that is the following property:

- (ED) For any $E_0 \geq 0$, there exist $K > 0$ and $\lambda > 0$ such that, for all solutions u of (0.1.8) with $E(u(0)) \leq E_0$,
$$\forall t \geq 0 \ , \quad E(u(t)) \leq K e^{-\lambda t} E(u(0)).$$

Property (ED) means that the damping term $\gamma_\omega \partial_t u$ stabilizes any solution of (0.1.8) to zero, which is an important property from the dynamical and control points of view. It is then easy to see that (ED) is implied by the existence of one $T > 0$ so that we have the following observability estimate

$$E(u(0)) \leq C \int_0^T \int_\Omega \gamma_\omega(x) |\partial_t u(x,t)|^2 \, dx \, dt, \tag{0.1.11}$$

for any $u(0)$ with $E(u(0)) \leq E_0$. Since $\gamma_\omega \geq \varepsilon > 0$ on ω, (0.1.11) translates some propagation of information from $[0,T] \times \omega$ to the full solution. It is exactly the same situation as in the observability estimates described in the previous section, but in a nonlinear context. If we know that a certain quantity of energy has passed through ω during the interval $[0,T]$, what can we say about the initial condition, that is the full solution?

Note that we have asked here the stronger property, that is the exponential decay, and the required condition was an exact observability inequality. Yet, in a parallel way as in the previous section where a (quantitative) unique continuation had some consequences for approximate controllability, some weaker observation can lead to some different results of stabilization.

Where it appears in the manuscript?

- Chapter 2 concerns the proof of the stabilization for nonlinear waves. Section 2.2 concerns the case where we can prove exponential decay under the geometric control condition while Section 2.3 concerns the case where we obtain some weaker decay, but under weaker geometric assumptions.
- in Section 1.4, we prove some logarithmic decay for the hypoelliptic damped wave equation as a consequence of quantitative unique continuation for the undamped equation.

Note also that the stabilization problem is also one of the main purposes of the papers **[T5]** , **[A6]** , **[T3]** , **[T2]** , **[T1]** , that we have chosen not to describe precisely. They concern critical nonlinear waves, Benjamin-Ono, Korteweg-de Vries and Non Linear Schrödinger equations.

Classifying solutions with specific behavior

In the unique continuation problem, we want to prove a result of the form: assume that u, solution of a certain PDE is zero somewhere, then $u = 0$. This rigidity property is sometimes too demanding. In some situations, the set of solutions satisfying such property is not trivial but can be precisely described. A basic example where the information comes from infinity is the Liouville theorem: a holomorphic function with polynomial growth on $\mathbb{C}$ is a polynomial.

This description of the rigidity problem might seem quite general. Let us be more specific to the context we have in mind that concerns cases where the final goal would be to prove that the only solutions having some particular behavior are solitons, that is solutions of some nonlinear elliptic equations.

In this manuscript, we give two examples of classification, that, we expect could end up later to such classification. They are both partially motivated by the general program of Duyckaerts-Kenig-Merle (see for instance [108] for a survey) for the soliton resolution conjecture in the case of the energy-critical wave equation (for example, in $\mathbb{R}^3$, the energy-critical, focusing $\Box u = u^5$). The purpose is to prove that bounded solutions decouple in a sum of rescaled solitons and a radiative term solution of the free wave equation. That is, for global bounded solutions, do we have?

$$u(t) \underset{t\to+\infty}{\approx} u_L(t) + \sum_{j=1}^{N} \frac{1}{\lambda_j(t)} Q_{l_j}^j \left(\frac{\cdot - c_j(t)}{\lambda_j(t)} \right)$$

where $u_L(t)$ is linear solution and $Q_{l_j}^j$ are traveling waves, that is Lorentz transform of a soliton.

They develop in particular the channel of energy method: in the 3D radial setting [60], the authors manage to conclude that some initial datum giving rise to "nonlinear non radiative solutions" (solutions that are asymptotically zero outside of a truncated cone) should behave at infinity as the Newtonian potential $\frac{1}{r}$ and next, should actually *be* the ground state $W : x \mapsto (1 + |x|^2/3)^{-1/2}$ (up to scaling). This idea to "catch the ground-state by the tail" has been extended with many more subtleties to other dimensions and other equations [43, 44, 46, 61].

The proof are performed in the radial case. They require (at least) to

- have a good understanding of all the linear and nonlinear solutions that are asymptotically zero outside of the truncated cone $|x| \geq |t| + R$ for a certain R and as $|t| \to +\infty$.
- make a link between (some of) these pathological solutions and solitons.

In both of these problems, it is necessary to make some classifications of solutions from their behavior at infinity.

Where it appears in the manuscript?

In Chapter 4, we give two classification results that intend to generalize the previous tools in the non radial setting.

- the first part deals with the link between elliptic equation (solitons) and linear solutions (the Newtonian potential in the radial setting) from their asymptotic behavior,
- in the second part, we give precisions for the general global question for the linear wave equation: does the information outside the (translated) light cone gives you informations about the full solution? In particular, in dimension 3, we classify all the linear solutions that have zero asymptotic energy outside of the translated light cone.

0.2 How to understand the propagation of the information?

In this Section, we intend to present several tools and methods that allow to get more information on the propagation of some information. The purpose is not to be exhaustive, but only to present techniques that will be used all along the manuscript, in different places. Some of them are very classical in control theory, but others are a little less. The reader familiar with these notions could jump this part without affecting the reading of the results obtained below.

The purpose will also be to give the flavor of more complicated results or ideas presented after. For instance, we hope that some of the easy computations performed in Section 0.2 for the flat Laplacian can help to get the intuition of some more technical spaces used in Section 4.1 for the resolution of nonlinear elliptic equations.

Carleman estimates

The theory of Carleman estimates is very rich and it would be very optimistic to give a fair survey in a memoir. With Matthieu Léautaud, we wrote some unpublished (for the moment) lecture Notes about Carleman estimates [L] . We also wrote a survey [P1] where we focus more on the wave equation. There are of course much more other references that are listed therein. The introduction of Chapter 1 below will also give more precisions about the whole theory of Carleman estimates. Here, we just want to describe very shortly what are Carleman estimates and why they are useful before getting to more precisions.

Carleman estimates were first designed for local unique continuation problems but were later seen to be useful in many other contexts as controllability, inverse problems, spectral theory... Let us first present them for unique continuation.

We consider Ω a bounded open subset of $\mathbb{R}^n$, P a differential operator on Ω, $x_0 \in \Omega$ a point, and a hypersurface $S = \{\Psi = 0\}$ containing x_0. We aim at proving local unique continuation for an operator P across the hypersurface $S = \{\Psi = 0\}$, say, a statement like

$$Pu = 0 \text{ in } \Omega, \quad u = 0 \text{ in } \Omega \cap \{\Psi \leq 0\} \Longrightarrow x_0 \notin \operatorname{supp}(u), \tag{0.2.1}$$

In particular, we want to prevent the situation in which a smooth function w both solves $Pw = 0$ and vanishes (possibly "flately", in the sense that all its derivatives vanish) on S. We thus need to "emphasize" the local behavior of functions close to the hypersurface S.

The general idea of Carleman to do so, and thus prove unique continuation, is to consider weighted estimates of the form

$$\left\|e^{\tau\Phi}w\right\|_{L^2(\Omega)} \leq C\left\|e^{\tau\Phi}Pw\right\|_{L^2(\Omega)}, \tag{0.2.2}$$

which hold:

- for some well-chosen weight function $\Phi : \overline{\Omega} \to \mathbb{R}$ (related to Ψ as discussed below);
- for all $w \in C_c^\infty(\Omega)$ (related to u as discussed below);
- and *uniformly* for τ sufficiently large, i.e. $\tau \geq \tau_0$.

To prove the relevance/efficiency of this approach, two different things need to be explained:

1) how to exploit Carleman estimates to get unique continuation properties like (1.1.1) and its quantification?
2) how to prove such Carleman estimates?

Let us first discuss point 1. Note first that (0.2.2) says directly that if $w \in C_c^\infty(\Omega)$ is solution of $Pw = 0$ on $\{\Phi \geq 0\}$, then the right hand side will tend to zero as τ tends to infinity. Therefore, the left hand side will converge to zero, which implies that w is supported in $\{\Phi \leq 0\}$.

However, statements like (1.1.3) that are useful in applications are not concerned with functions w having compact support. Moreover, in general, as we shall see, usual differential operators P do not admit solutions w to $Pw = 0$ having compact support!

The heart of the Carleman method to pass from the estimate (0.2.2) to the unique continuation statement (1.1.3) resides in applying (0.2.2) to $w = \chi u$, where u is the function for which unique continuation has to be proved (hence solving $Pu = 0$ in Ω and $u = 0$ on $\Psi \geq 0$), and $\chi \in C_c^\infty(\Omega)$ is a cut-off function (to be chosen) allowing to apply (0.2.2).

Using that $P\chi u = \chi Pu + [P,\chi]u = [P,\chi]u$ (where $[P,\chi]$ denotes the commutator of P and the multiplication operator by χ), this then yields

$$\left\|e^{\tau\Phi}\chi u\right\|_{L^2(\Omega)} \leq C\left\|e^{\tau\Phi}[P,\chi]u\right\|_{L^2(\Omega)}.$$

We then notice that $\operatorname{supp}[P,\chi] \subset \operatorname{supp}\nabla\chi$. If we now *assume* (this can be achieved if Φ is a slight convexification of Ψ), that the functions Ψ, Φ, χ are chosen such that $\operatorname{supp}(\nabla\chi) \cap \{\Psi \leq 0\} \subset \{\Phi \leq -\eta\}$, for some $\eta > 0$ (small!), then the support property of u (namely $u = 0$ on $\Psi \geq 0$) implies that $\operatorname{supp}([P,\chi]u) \subset \{\Phi \leq -\eta\}$, and we thus obtain

$$\left\|e^{\tau\Phi}\chi u\right\|_{L^2(\Omega)} \leq C_u e^{-\eta\tau}, \qquad \text{for all } \tau \geq \tau_0.$$

It is easy to see by making τ converge to $+\infty$ that it implies that χu vanishes identically in $\{\Phi \geq -\eta\}$ which contains a neighborhood of the point x_0. This sketch of proof could also yield a quantitative unique continuation result.

To conclude, this brief discussion of point 1 suggests that unique continuation (1.1.3) will hold (across $\{\Psi = 0\}$) provided the Carleman estimate (0.2.2) is true for some weight function Φ satisfying an appropriate *geometric convexity condition*.

But when we turn to the second point 2 of how to prove such estimates, it turns out that the right condition for classical Carleman estimates to hold is some pseudoconvexity assumption (see Definition 1.2.1 below with $n_a = 0$). We will see later that in many examples, this gives some geometric conditions that are too demanding. To solve this problem, several authors (Tataru first, then Robbiano-Zuily, Hörmander) developed some more complicated Carleman estimates of the form

$$\|e^{-\varepsilon\frac{|D_t|^2}{2\tau}}e^{\tau\Phi}u\|_{H^s}^2 \leq C\left\|e^{-\varepsilon\frac{|D_t|^2}{2\tau}}e^{\tau\Phi}Pu\right\|_{L^2}^2 + Ce^{-\mathsf{d}\tau}\left\|e^{\tau\Phi}u\right\|_{H^s}^2 \tag{0.2.3}$$

where $e^{-\varepsilon\frac{|D_t|^2}{2\tau}}$ is the Fourier multiplier. This allowed to decrease drastically the geometric conditions involved but at the cost of asking for some analyticity of the coefficients of P in some variable. Also, because of the nonlocal term $e^{-\varepsilon\frac{|D_t|^2}{2\tau}}$, the proof that Carleman estimates like (0.2.3) imply unique continuation is more involved. An optimal quantitative version of this was also to be done, and it is the object of Chapter 1 where we obtain such quantitative unique continuation in this context.

Where it appears in the manuscript?

- the main theme in Chapter 1 is to quantify the general Theorem of Tataru-Zuily-Robbiano-Hörmander from Carleman estimates and give several applications of this. We refer to this Chapter and its introduction for more precisions and as an introduction to the domain.
- Carleman estimates for elliptic operators are used (and proved) in Section 3.1 where we want to have estimates of the cost of the control of the heat equations in different regimes: small time and small zone of observation. Note also that in the same chapter, we also use Agmon estimates which are quite closely related to Carleman estimates. They are actually also weighted type estimates but are more suited to obtain upper bounds on the solutions. Carleman will be used to get lower bounds of the solutions while Agmon estimates will give upper bounds.

The resolution of ill-posed problems

The definition of ill-posed and well-posed problem in the sense of Hadamard is somewhat informal unless when we make some specific choices of spaces and what are the data.

Very often, there is an evolution variable, often called the time t, and the problem is posed under the form: Let P an operator. Given a function u_0 in a space X, can I find a function $t \mapsto u(t)$, solution of my problem $Pu = 0$ for t in a small interval so that $u(t) \in X$. We ask moreover that the solution is unique and the application $u_0 \mapsto u(t)$ is continuous from X to itself. In that case, we say that the problem is well-posed. This is for example the case for the evolution problem (3.1.23) for the wave equation, which is well posed for initial datum $(u_0, u_1) \in H_0^1 \times L^2(\Omega)$. Therefore, the most favorable case when we want to prove some propagation of information is when a Cauchy problem is well-posed. Assume that we know our solution (locally) on a hypersurface S. If the Cauchy problem $Pu = 0$ is well posed with respect to $S = \{t = 0\}$, we obtain some information on a neighborhood of S from the knowledge of u on S.

Yet, in most of the applications in control theory, as the local unique continuation of the type (0.2.1), the problem is ill-posed. A typical example where we can solve the Cauchy problem but it is ill-posed is the Laplace equation.

Consider the elliptic equation

$$\partial_t^2 u + \partial_\theta^2 u = 0$$

that we want to solve in $\mathbb{R} \times \mathbb{T}^1$ where $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the unit circle. Consider the initial value problem $(u, \partial_t u)(0) = (u_0, u_1)$. If we write formally the Fourier series $u_0 = \sum_{k\in\mathbb{Z}} u_{0,k}e^{ik\theta}$ and $u_1 = \sum_{k\in\mathbb{Z}} u_{1,k}e^{ik\theta}$, the formal solution would be

$$u(t, \theta) = \sum_{k\in\mathbb{Z}} \left(\cosh(kt)u_{0,k} + \frac{\sinh(kt)}{k}u_{1,k}\right) e^{ik\theta}. \tag{0.2.4}$$

This is not well defined for a general (u_0, u_1) in some Sobolev space for instance. But if we assume some decay of the form $|u_{0,k}| + |u_{1,k}| \leq e^{-r_0 k}$, we can give some meaning to (0.2.4) for $|t| < r_0$. Yet, the assumption $|u_{0,k}| + |u_{1,k}| \leq e^{-r_0 k}$ is severely downgraded for positive time. Indeed, if we denote $u_k(t)$ the Fourier coefficients of $u(t)$, we get the bound $|u_k(t)| \leq 2e^{-(r_0 - |t|)k}$. This is the typical situation of an ill-posed problem. Even if we can define a solution, the solution will not remain in the same space. In our example, if the initial datum is in a space X_0 of function $|u_{0,k}| + |u_{1,k}| \leq Ce^{-r_0 k}$, the solution will be in some space X_t of functions with

$$|u_k(t)| \leq Ce^{-r(t)k}. \tag{0.2.5}$$

Remark here that for small t, X_t is a space of analytic functions, and $r(t)$ can be seen as the radius of analyticity that decreases. This is typical of how we can solve some ill-posed problems. We have to allow the space where we solve our equation to depend on the "time" t.

This is the typical application of some theorem of the type Cauchy-Kowalesky where we see some loss of derivatives as follows.

Theorem 0.2.1 (Cauchy-Kowaleski Theorem). *Assume F is real analytic in its arguments and g_j are real analytic on Ω. Then, for $x_0 \in \Omega \subset \mathbb{R}^n$, there exists an open set $\Omega_1 \Subset \Omega$ containing x_0, $\varepsilon > 0$ and a unique $u(t,x)$ that is real analytic for $x \in \Omega_1$, $t \in (-\varepsilon, \varepsilon)$ and satisfies*

$$\begin{cases} \partial_t^m u &= F(t,x,u,\partial_t^j \partial_x^\alpha u, \dots), \\ \partial_t^j u(0,x) &= g_j(x), \quad \text{for all } 0 \le j \le m-1, \end{cases} \tag{0.2.6}$$

where the parameter j, α inside of F runs over all the terms with $j < m$ and $j + |\alpha| \le m$.

There are many different proof of this theorem, but we would like to stress the one of Nirenberg [144] that works with scale some spaces of analytic functions X_t with variable radius as before. We will use this idea in a slightly different context in Section 4.1.

The most common application of the Cauchy-Kowaleski Theorem to control theory is by its implication to unique continuation of the form (1.1.3). More precisely, by some duality argument and change of variable, it gives the classical Holmgren-John Theorem, stating that, for operators with analytic coefficients, unique continuation holds across any noncharacteristic hypersurface S.

Let P be a differential operator of order m on Ω, $x_0 \in \Omega$ and S a local hypersurface passing through x_0, that is $S = \{\Psi = 0\}$, $\Psi(x_0) = 0$ and $d\Psi(x_0) \neq 0$ with $\Psi \in C^1(\Omega)$. We say that S is characteristic (resp. non-characteristic) for P at x_0 if $p_m(x_0, d\Psi(x_0)) = 0$ (resp. $p_m(x_0, d\Psi(x_0)) \neq 0$).

Also, given a local hypersurface $S = \{\Psi = 0\}$, it has locally two sides which we write

$$S^{\pm} = \{x \in \Omega; \pm\Psi(x) > 0\}.$$

Theorem 0.2.2 (Holmgren-John Theorem). *Let P be a differential operator of order m on Ω, having all coefficients* real analytic *in a neighborhood of $x_0 \in \Omega$ and $S \ni x_0$ being a local hypersurface. Assume that S is noncharacteristic for P at x_0. Then, there exists a neighborhood V of x_0 so that every $u \in \mathcal{D}'(\Omega)$ satisfying $Pu = 0$ on Ω and $u = 0$ in the set S^+ vanishes identically in V.*

The non-characteristicity condition is very weak, and in some sense optimal.

Before ending this section, we would like to get back to a variant of the explicit computations by changing the Laplace equation to the heat equation (with the space variable being named t and the time variable being θ, that is

$$\partial_t^2 u - \partial_\theta u = 0.$$

A similar formal analysis yields the formula

$$u(t,\theta) = \sum_{k \in \mathbb{Z}} \left(\cos(\sqrt{ik}\, t) u_{0,k} + \frac{\sin(\sqrt{ik}\, t)}{\sqrt{ik}} u_{1,k} \right) e^{ik\theta}. \tag{0.2.7}$$

In that case, a similar analysis would yield that it is enough that the Fourier coefficients decay as $e^{-r_0\sqrt{k}}$, which is the typical behavior of functions in the class Gevrey 2. There are also abstract results of the type of Cauchy-Kowaleski in the class of Gevrey functions but for anisotropic operators.

Where it appears in the manuscript?

- in Section 4.1, we construct some solution of $\Delta u = f(u, \nabla u)$ with some prescribed behavior at infinity. After some appropriate change of coordinates, we are led to solve some problem $\partial_t^2 v + \Delta_{\mathbb{S}^{d-1}} v = g(t, v, \nabla v)$ that we solve in some spaces adapted to the Laplace equation that look like the spaces X_t depending on t with bounds of the type (0.2.5).

Note also that in Chapter 1, we give a general theorem of quantification of the unique continuation that, as a particular case, gives some quantification of the Holmgren theorem. Yet, the proof does not use the methods of this section, but Carleman estimates instead.

Also, in the article [A13] (that we have chosen not to present in detail), we solve the ill-posed problem $\partial_x^2 u - \partial_t u = f(u, \partial_x u)$ with x as an evolution variable. The resolution is in functions Gevrey in time and allows us to prove that some sets of analytic data are controllable.

High-frequency estimates and microlocal analysis

The theory of microlocal analysis is the world of high-frequency results which are always stated ”up to more regular terms” or compact term or when a parameter (often related to the frequency) is small. It has proved to be very useful for understanding propagation that involves high-frequency regimes as the propagation of regularity or compactness. It has a huge number of applications in control and stabilization theory, particularly for the wave and Schrödinger equations.

Let us say a few words about the wave equation, which is more present in this manuscript for which the optimal result was obtained in [16] using deeply microlocal analysis.

The natural condition is the following.

We say that (ω, T) satisfies the Geometric Control Condition if any generalized geodesic of (assuming that it is uniquely defined, see [135]) Ω of length T meets the set ω.

It led to the following classical theorem in [16].

Theorem 0.2.3 (Bardos-Lebeau-Rauch)**.** *Assume that (ω, T) satisfies the Geometric Control Condition, then the observability* (0.1.5) *holds.*

All the known proofs of this result use microlocal analysis. The proof was originally performed relying on the propagation of regularity. Later, alternative proofs emerged using microlocal defect and Egorov Theorem.

Let us give an idea of the proof using Egorov Theorem when we are on a compact Riemannian manifold $\mathcal{M}$ without boundary. The idea emerged in [53] for getting more information on the HUM operator and was used later to give a constructive proof in our paper **[A7]** . We give the idea for the simpler half-wave equation.

Denoting $\Lambda = \sqrt{-\Delta_g}$, the square root of the Laplace Beltrami operator, $e^{it\Lambda}$ is the semigroup resolving the half wave equation

$$\begin{cases} \partial_t u(t) - i\Lambda u(t) &= 0 \\ u(0) &= u_0, \end{cases}$$

If consider the observation on L^2 for a smooth observation $\chi_\omega \in C^\infty(\mathcal{M})$ that is positive on ω, the observation we are looking for is

$$\|u_0\|^2_{L^2(\mathcal{M})} \leq C \int_0^T \|\chi_\omega u(t)\|^2_{L^2(\mathcal{M})}.$$

The Egorov theorem states if P_m is a pseudo differential operator of order m, then $e^{it\Lambda}P_m e^{-it\Lambda}$ is (modulo a 1-smoothing operator) a pseudodifferential operator of order m with principal symbol q_t, with $q_t(\rho) = p_m(\phi_t(\rho))$ for $\rho \in S^*\mathcal{M}$ and where ϕ_t is the geodesic flow on $S^*\mathcal{M}$.Denoting $\langle \cdot, \cdot \rangle_{L^2}$ the Hermitian product on $L^2(\mathcal{M})$, we notice that for any $t \in [0, T]$,

$$\|\chi_\omega u(t)\|^2_{L^2(\mathcal{M})} = \left\|\chi_\omega e^{it\Lambda} u_0\right\|^2_{L^2(\mathcal{M})} = \left\langle e^{-it\Lambda}\chi_\omega^2 e^{it\Lambda} u_0, u_0\right\rangle_{L^2}.$$

χ_ω can be considered as a pseudodifferential operator and the Egorov Theorem states that $e^{-it\Lambda}\chi_\omega^2 e^{it\Lambda}$ is (modulo smoothing operator), a pseudodifferential operator of order zero and principal symbol $q_t(\rho) = \chi_\omega^2(\phi_{-t}(\rho))$. In particular, we obtained (modulo regularizing operators), the following description for the HUM operator

$$\int_0^T \|\chi_\omega u(t)\|^2_{L^2(\mathcal{M})} = \langle H u_0, u_0 \rangle_{L^2}$$

where H is (modulo smoothing operator) a pseudodifferential operator of order zero and symbol $h(\rho) = \int_0^T \chi_\omega^2(\phi_{-t}(\rho))dt$. It is not so hard to notice that the geometric control condition and compactness imply $h(\rho) > C_0 > 0$ for any $\rho \in S^*\mathcal{M}$. A Gårding inequality implies then

$$\int_0^T \|\chi_\omega u(t)\|^2_{L^2(\mathcal{M})} \geq C_0 \|u_0\|^2_{L^2(\mathcal{M})} - C\|u_0\|_{H^{-1/2}(\mathcal{M})}.$$

This is what we call a weak observability or high-frequency inequality.

Another alternative proof, that will not really appear in this manuscript is the propagation of microlocal defect measure. We refer to [32] for more precisions.

Where it appears in the manuscript?

- in **[A7]** , described in Section 1.3, we give a constructive proof of the classical result of Bardos-Lebeau-Rauch proving the controllability of the wave equation under the geometric control condition. This proof contains a high-frequency part and a low-frequency part. The high-frequency part uses the Egorov theorem which allows to quantify the Geometric Control Condition by a certain quantity. We have therefore some estimates uniform when the time converges to the time of Geometric Control Condition, as we described in this Section.
- in **[A18]** , we use some elementary stationnary phase Lemma to obtain precise asymptotics of the linear wave equation in $\mathbb{R}^d$ in large time.

Note also, that the Egorov theorem is a crucial tool of the article **[A12]** (that we have chosen not to present) that is concerned with the control of some system of waves coupled by some lower order terms. The Egorov theorem allows to make the link between the observability of the wave equation and the observability of some ODE along the bicharacteristics. It turns out to be a necessary and sufficient condition when we are interested in only weak observability, that is up to a lower-order term.

Moreover, the propagation of microlocal defect measure is a key ingredient of the articles of my thesis **[T2]** about the control of the Non Linear Schrödinger equation and **[T5]** for the critical nonlinear wave equation.

The compactness-uniqueness method

The compactness uniqueness method is a standard method to prove some inequality by contradiction. The idea is the following. Assuming the inequality is false, then, there is a sequence of element u_n (solutions in the case of PDE) contradicting the estimates. Then, there are 2 steps:

- Compactness step: prove that the sequence u_n converges to some element u
- Uniqueness: prove that the element u satisfies estimates for some constant.

In order to not make it too abstract, let us take the example of the stabilization of the nonlinear wave equation as described in Section 0.1. The following scheme of proof in this context originates from [54]. Assume that we want to prove (0.1.11) for every u_0 so that $E(u_0) \leq E_0$. If we argue by contradiction, we obtain a sequence of solutions u_n, with bounded energy, so that

$$\int_0^T \int_\Omega \gamma_\omega(x)|\partial_t u_n(x,t)|^2\, dx\, dt \leq \frac{1}{n} E(u_n(0)). \tag{0.2.8}$$

By assumption, $E(u_n(0))$ is bounded, so, up to a subsequence, it converges to some $\alpha \geq 0$. The case $\alpha = 0$ is easier and can be treated apart since the solution is very small and therefore almost linear. Note that for a linear equation, we can impose $\alpha = 1$ by linearity. Here, we assume up to now that $\alpha > 0$ and want to get a contradiction.

Up to taking a subsequence, we can assume that u_n weakly converges (in an appropriate topology) to some solutions u. Also, recalling the geometric assumption on γ_ω, we see that (0.2.8) expresses that

$$\partial_t u_n \underset{L^2([0,T]\times\omega)}{\longrightarrow} 0,$$

that is a strong converges to zero in $[0,T]\times\omega$. Note that it implies that $\partial_t u = 0$ on $[0,T]\times\omega$. Next, the first step consists in using some ideas of propagation of compactness, that is to prove that

$$u_n \underset{H^1([0,T]\times\Omega)}{\longrightarrow} u,$$

that is a strong converges to zero in energy in the full space $[0,T]\times\Omega$. In the nonlinear case, it was done in [54] using microlocal defect measures as described in the previous Section 0.2 under some condition of Geometric Control Condition. In our work **[A3]** , we provide an easier proof using the exponential decay of the damped equation, but when we allow $T = T_n \to +\infty$, which is sufficient for applications. In any case, the idea of this compactness step is to obtain the compactness of the sequence in the energy space.

Next, comes the step of uniqueness. It relies on proving some rigidity of the limit u. In most of the cases, this step relies on proving that the limit u is actually zero, but sometimes, it is sufficient to obtain some known solutions, like static solutions. In this example of the defocusing nonlinear wave equation, we see that the limit u satisfies

$$\begin{cases} \Box u + f(u) = 0 & \text{on } [0,T] \times \Omega \ , \\ \partial_t u = 0 & \text{on } [0,T] \times \omega \end{cases} \tag{0.2.9}$$

A unique continuation argument is exactly to prove that $u = 0$. Yet, it is quite difficult to obtain. The usual argument is by Carleman estimates, as described in Section 0.2, but it requires some strong geometric assumptions. We develop a general method in Chapter 2 to relax this geometric assumptions. We refer to this chapter for more precisions.

When the compactness can not be obtained, like in critical cases, the method can be refined to the so-called concentration-compactness-rigidity method. In that case, the compactness is replaced by the concentration-compactness, that is u_n converges to some element u modulo some terms of the form $\varphi_n(t,x) = h_n^{\alpha} \varphi\left(\frac{t-t_n}{h_n^{\beta}}, \frac{x-x_n}{h_n^{\gamma}}\right)$ where α, β, γ are dictated by the scaling of the problem. For nonlinear wave equations, this was developed by Kenig-Merle [109] following the earlier works on the concentration-compactness by Bahouri-Gérard [15].

Where it appears in the manuscript?

- Chapter 2 concerns the stabilization of nonlinear waves as we just presented. We use the strategy of compactness-uniqueness as described just above.

Note also that in the papers **[A6]** , **[T3]** , **[T2]** , **[T1]** , that we have chosen not to describe precisely, the compactness uniqueness method is one key method but needs to be adapted to the specific propagation of each dispersive model: Benjamin-Ono, Korteweg-de Vries and Non Linear Schrödinger equations. **[T5]** uses the more refined concentration-compactness-rigidity methods with profile decomposition.

Now, we present some other tools that we used in this memoir that seem less usual in the context of control and stabilization theory.

Dynamical system methods

In this section, we present a few definitions of dynamical systems that will be useful later and how they relate to the properties we are interested in. We refer for example [79, 80] and [151] for a review on this concept.

Let X be a Banach space. We consider S a continuous group on X, that is a one-parameter family of mappings $S(t)$, $t \in \mathbb{R}$ from X into X such that

- $S(0) = Id$
- for any $t, s \in \mathbb{R}$, $S(t+s) = S(t) \circ S(s)$
- for any $t \in \mathbb{R}$, $S(t) \in C(X,X)$
- for any $x_0 \in X$, the application $t \mapsto S(t)(x_0)$ is continuous from $\mathbb{R}$ to X.

The main application we will have in mind will be the group of the solution of the nonlinear wave equation (0.1.8), eventually with a more general $f = f(x,u)$. Note that most of the theory works for semigroups, but we only wrote the definitions in the case of groups, which allows a few simplifications.

For $x_0 \in X$, we define the positive trajectory $\gamma_+(x_0) = \{S(t)x_0, t \geq 0\}$ and the negative trajectory $\gamma_-(x_0) = \{S(t)x_0, t \leq 0\}$.

The ω-limit set is $\omega(x_0) := \bigcap_{t\geq 0} \overline{\{S(s)(x_0), s \geq t\}}$. This is the set of points $x \in X$ so that there exists a sequence $(t_n)_{n\in\mathbb{N}}$ increasing to $+\infty$ so that $S(t_n)x_0$ converges to x.

The α-limit set is $\alpha(x_0) := \bigcap_{t\leq 0} \overline{\{S(s)(x_0), s \leq t\}}$. This is the set of points $x \in X$ so that there exists a sequence $(t_n)_{n\in\mathbb{N}}$ decreasing to $-\infty$ so that $S(t_n)x_0$ converges to x.

An equilibrium point is a point $e \in X$ so that $S(t)e = e$ for any $t \in \mathbb{R}$. We denote $\mathcal{E}$ the set of equilibrium points.

Definition 0.2.4. A nonempty subset $\mathcal{A}$ of X is called a global attractor of the group S if:

- $\mathcal{A}$ is a closed, bounded subset of X,
- $\mathcal{A}$ is invariant under the group S (i.e. $S(t)\mathcal{A} = \mathcal{A}$ for every $t \in \mathbb{R}$)
- $\mathcal{A}$ attracts every bounded subset B of X under the group S, that is for every $\mathcal{V}$ neighborhood of $\mathcal{A}$, there exists $t > 0$ so that $S(t)B \subset \mathcal{V}$.

If it exists, it is given by $\mathcal{A} = \{\text{bounded complete orbits of } S\}$.

The case where there is a compact global attractor is of particular interest. The existence of such an attractor is an important dynamical property because it roughly says that the dynamics of the PDE for large times may be reduced to dynamics on a compact set, which is often finite-dimensional.

The following properties will be typical of systems that dissipate energy

Definition 0.2.5. A group S is called

- **point dissipative** if there exist a bounded set $\mathcal{B}$ so that for any $x_0 \in X$, there exists $t_0(x_0) \geq 0$ so that $S(t)x_0 \subset \mathcal{B}$ for all $t \geq t_0(x_0)$
- **asymptotically compact** if, for any bounded subset $B \subset X$ such that $\bigcup_{t\geq t_0} S(t)B$ is bounded for one $t_0 \geq 0$, for any sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in B$ and $t_n \to +\infty$, there is a subsequence so that $S(t_{\varphi(n)})x_{\varphi(n)}$ is convergent.

Theorem 0.2.6. *[80, Theorem 9.3.] We have the equivalence*

- *S admit a global compact attractor $\mathcal{A}$,*
- *S is point dissipative, asymptotically compact, and for any bounded set $B \subset X$, there exists $t_0 \geq 0$ such that $\bigcup_{t\geq t_0} S(t)B$ is bounded.*

Moreover, $\mathcal{A}$ is connected.

In practical situations, the previous configuration happens in the presence of some decreasing "energy".

Definition 0.2.7. A function $\Phi \in C^0(X, \mathbb{R})$ is a **Lyapunov** functional for S if

- $\Phi(S(t)x_0) \leq \Phi(x_0)$, for all $t \geq 0$ and $x_0 \in X$,
- Φ is bounded by below and $\Phi(x) \underset{|x|\to+\infty}{\longrightarrow} +\infty$,
- $\Phi(S(t)x_0) = \Phi(x_0)$ for all $t \in \mathbb{R}$ implies that x_0 is an equilibrium point.

A semigroup $S(t)$ is a **gradient system** if it has a Lyapunov function and each bounded positive orbit is precompact.

Note that the definition of a gradient system is not totally uniform in the literature (we took the one from [79]) but all of them has the important following property as a consequence.

Theorem 0.2.8 (Invariance principle of LaSalle)**.** *Let $S(t)$ be a gradient system on X. Then, for any $x_0 \in X$, $\omega(x_0)$ is a set of equilibrium points.*

If x_0 is so that the negative trajectory $\gamma_-(x_0)$ is relatively compact, then $\alpha(x_0)$ is a set of equilibrium points.

Concerning compact attractors, Theorem 0.2.6 simplifies to the following in the context of gradient system.

Theorem 0.2.9. *[151, Theorem 4.6.] Let $S(t)$, $t \in \mathbb{R}$, be a gradient system, so that*

- *S is asymptotically smooth,*
- *for any bounded set $B \subset X$, there exists $t_0 \geq 0$ such that $\bigcup_{t \geq t_0} S(t)B$ is bounded,*
- *the set of equilibrium points $\mathcal{E}$ is bounded.*

Then $S(t)$ has a compact global attractor $\mathcal{A}$.

Moreover, $\mathcal{A}$ is connected.

Note that when Theorem 0.2.9 applies, $\mathcal{A}$ is invariant by S and any trajectory is therefore compact in positive and negative time. In particular, Theorem 0.2.8 applies in positive and negative time: any trajectory inside $\mathcal{A}$ satisfies $d(S(t), \mathcal{E}) \underset{t \to \pm\infty}{\longrightarrow} 0$.

After having introduced all the notions of dynamical systems that will be used in this manuscript, we would like to make a small parallel between the different notions that are more common in control or propagation and what we believe to be their counterpart in dynamical systems. In what follows, the first notion is related to control and propagation and the second is the parallel one in dynamical systems:

- **Stabilisation/existence of a compact global attractor**: the notion of stabilization is the one of convergence to zero (or an equilibrium) of the system while the global compact attractor is the "convergence of bounded sets to a compact set".

- **Propagation of compactness / asymptotic compactness**: in problems of stabilization for nonlinear waves for instance, the propagation of compactness from the small set ω to the full set Ω described in the Section 0.2 about compactness-uniqueness is quite similar to the one that allows to prove the asymptotic compactness of the dynamical system.

- **Propagation of space regularity/regularity of the attractor**: The propagation of the regularity from ω to Ω actually corresponds in our context to the regularity of the attractor. This is the same for analytic regularity.

- **Unique continuation properties/gradient structure**: the unique continuation property of the form (0.2.9), but in infinite time is mostly equivalent to the gradient property, that is the equilibria are the only trajectories which do not dissipate the energy.

One could conclude from this discussion that the problems of stabilization of semilinear equations and the existence of a global compact attractor are closely related. Besides the result we present, one of the main interests of the articles presented in Chapter 2 is the use of arguments coming from both the dynamical study and the control theory of the damped wave equations. People familiar with the control theory could find interesting the use of the asymptotic smoothing effect to get a unique continuation property with smooth solutions. The one familiar with the dynamical study of PDEs could be interested in the use of Strichartz estimates to deal with more general nonlinearities and the tools of propagation.

Where it appears in the manuscript?

The full Chapter 2 is mainly focused on the application of dynamical system methods to control and stabilization of the nonlinear wave equation.

In Section 2.2 and 2.3, concerning stabilization, we will use some ideas of Hale-Raugel [81] that prove that for some for some dynamical systems coming from damped PDE, the elements of the compact global attractor are actually smoother than was is expected for usual solutions. When the nonlinearity is analytic, that will be crucial to prove some unique continuation in infinite time.

In Section 2.4, we will also use the compact attractor $\mathcal{A}$ to prove some global control property. Since the asymptotic dynamic of a damped equation is dictated by the global compact attractor, we will reduce a global control problem to the subproblem of control inside of the attractor (travelling inside the attractor). Indeed, inside of $\mathcal{A}$, the dynamic is simpler thanks to the LaSalle principle and the connectedness of $\mathcal{A}$.

Using positivity

In this section, we present a few tools that were used throughout the manuscript to propagate some information for **positive** solutions of partial differential equations. This will be restricted to elliptic or parabolic equations where the maximum principle holds. This is of course an extremely vast topic. The results we use in this manuscript are mainly Li-Yau estimates as in Theorem 0.2.11 below, but we present fastly Harnack estimates as an introduction.

The most basic estimates taking advantage of the positivity of a solution are Harnack type inequalities for elliptic equations. For $\Omega' \Subset \Omega \subset \mathbb{R}^d$ connected, it writes

$$\sup_{x\in\Omega'} u(x) \leq C \inf_{x\in\Omega'} u(x)$$

for any nonnegative function u solution of $\Delta u = 0$ on Ω.

We actually know very few applications of these estimates in the context of propagation of smallness or control, except in the recent work of Logunov-Malinnikova [132].

The parabolic equivalent of this is more subtle. It writes for u solution of $\partial_t u - \Delta u = 0$ on $[0,T] \times \Omega$, and $0 < t_1 < t_2 \leq T$,

$$\sup_{x\in\Omega'} u(t_1, x) \leq C \inf_{x\in\Omega'} u(t_2, x).$$

It is actually possible to get much more special information on the relative constants thanks to the following special version (see the survey [105]).

Theorem 0.2.10 (Harnack inequality for the heat equation)**.** *Let $u \in C^\infty((0,+\infty) \times \mathbb{R}^d$ be a nonnegative solution of the heat equation, that is, $\partial_t u - \Delta u = 0$. Then*

$$u(t_1, x) \leq \left(\frac{t_2}{t_1}\right)^{d/2} e^{\frac{|x-y|^2}{4(t_2-t_1)}} u(t_2, y), \quad \forall x, y \in \mathbb{R}^d, \quad \forall t_2 > t_1 > 0$$

Note that these inequalities can be read as upper or lower bound depending if we are interested in t_1 or t_2.

In both cases, we can interpret that as a propagation of the information. If I know for instance that the solution is very small for y in some opens set ω and one time, it gives me some small upper bound for x at smaller times. We will use it in this direction later.

This has an equivalent for varying coefficients in the celebrated Li-Yau estimates

Theorem 0.2.11 (Theorem 2.3 of Li-Yau [126])**.** *Let $\mathcal{M}$ be a compact manifold. Let*

$$-K = \min(0, \min_{x\in\mathcal{M}} Ricc(x)) \leq 0,$$

where $Ricc(x)$ is the Ricci curvature at x. We assume that the boundary of $\mathcal{M}$ is convex. Let $u(t,x)$ be a positive solution on $(0,+\infty)$ of the heat equation with Neumann boundary condition. Then for any $\alpha > 1$, $x, y \in \mathcal{M}$, and $0 < t_1 < t_2$, we have

$$u(t_1, x) \leq \left(\frac{t_2}{t_1}\right)^{n\alpha/2} e^{\frac{n\alpha K(t_2-t_1)}{\sqrt{2}(\alpha-1)}} e^{\alpha\frac{d(x,y)^2}{4(t_2-t_1)}} u(t_2, y).$$

Here, the assumption of convex boundary means $II > 0$ where $II(x)$ the second fundamental form of ∂M with respect to outward pointing normal at the point x and could be avoided, up to a loss in the exponent (see [173, Theorem 3.1]).

After all this section about the expectation of possible improvement of observability inequalities for positive solutions, we could wonder what is the consequence of the observability inequalities only valid for positive solutions. It was noticed by Le Balc'h [114] that it implies that we can impose the final state to be positive. We refer to Lemma 3.1.9 below for a more precise statement.

Where it appears in the manuscript?

In Chapter 3, we study the constants appearing in the observability of heat equations or transport equations in the vanishing viscosity limit. While a precise asymptotic of the constant of the control for a small time of observability T or small ε (the small constant in front of the vanishing viscosity) seem very hard to obtain, we make some very precise estimates of the observability constants when restricted to positive solutions. The proof relies crucially on Li-Yau estimates.

CHAPITRE 1

Quantitative unique continuation and applications

This chapter is about the work [A10] and some of its later applications, all obtained in collaboration with Matthieu Léautaud. It gives a general result of quantification of unique continuation under partial analyticity, which is described more precisely in Section 1.2. The applications to the exact controllability for waves corresponds to [A7] and are described in Section 1.3. The applications to hypoelliptic operators were obtained in [A19] and [A17] . They are presented in Section 1.4.

The text of this chapter is largely based on the conference proceedings [A15] .

1.1 Introduction

In this chapter, we are interested in the quantification of *global unique continuation* results of the following form: given a differential operator P on an open set $\Omega \subset \mathbb{R}^n$, and given a small subset U of Ω, having

$$Pu = 0 \text{ in } \Omega, \quad u|_U = 0 \Longrightarrow u = 0 \text{ on } \Omega. \tag{1.1.1}$$

More generally, in cases where (1.1.1) is known to hold, we are interested in proving a quantitative version of

$$Pu \text{ small in } \Omega, \quad u \text{ small in } U \Longrightarrow u \text{ small in } \Omega.$$

This can sometimes be expressed by a stability estimate of the form

$$\|u\|_\Omega \leq \varphi\big(\|u\|_U , \|Pu\|_{\tilde{\Omega}} , \|u\|_{\tilde{\Omega}} \big), \quad \text{with } \varphi(a,b,c) \to 0 \text{ when } (a,b) \to 0 \text{ with } c \text{ bounded}, \tag{1.1.2}$$

where $U \subset \Omega \subset \tilde{\Omega}$ are nonempty, and for appropriate norms. As we will see, both qualitative and quantitative unique continuation properties have several applications in control theory.

A more tractable problem than (1.1.1) is the so called *local unique continuation* problem: given $x^0 \in \mathbb{R}^n$ and S an oriented local hypersurface containing x^0, do we have the following implication:

$$\text{There is a neighborhood } \Omega \text{ of } x^0, \text{ such that } Pu = 0 \text{ in } \Omega, u|_{\Omega \cap S^-} = 0 \Longrightarrow x^0 \notin \operatorname{supp}(u), \tag{1.1.3}$$

where S^- denotes one side of S. It turns out that proving (1.1.3) for a suitable class of hypersurfaces (with regard to the operator P) is in general a key step in the proof of properties of the type (1.1.1). The first general unique continuation result of the form (1.1.3) is the Holmgren-John Theorem [87, 100], stating that, for operators with analytic coefficients, unique continuation holds across any noncharacteristic hypersurface S (see e.g. [91, Theorem 8.6.5] for a precise statement).

When focusing on operators with (only) smooth (C^∞) coefficients, the most general result was proved by Hörmander [88] (see also [92, Chapter XXVIII]). Uniqueness across a hypersurface holds assuming a strong pseudoconvexity condition (see e.g. Definition 1.2.1 below). This result uses as a key tool Carleman estimates, which were introduced in [36] at first for elliptic operators and developed in [35] for operators with simple characteristics.

Starting from the 70's, and motivated by applications to the unique continuation of the wave equation with non analytic coefficients, people tried to lower the geometric conditions (with respect to "strong pseudoconvexity") without imposing analyticity of all coefficients of the operator involved. The first results of this kind were [150] later generalized in [123], which provide with unique continuation for the wave operator with (only) C^∞ coefficients from subsets of the form $(-\infty, +\infty) \times \omega$. This work was then followed by the seminal [154] proving a similar result but from a set $(-T, T) \times \omega$ with T finite yet non optimal. Then, it was understood in [165] that the right framework was that of **partial analyticity**, that is analyticity with only a subset of variables (e.g. the time variable when dealing with wave equation). This led to successive improvements [94, 156, 167] that interpolate in a satisfying way between Holmgren and Hörmander Theorem. So far, the local unique continuation results are summarized in the first two lines of the Table 1.1. The specific setting of partial analyticity and the related pseudoconvexity property is described more precisely in the Section 1.2.

Theorem	Holmgren, John [87, 100]	Tataru, Robbiano-Zuily, Hörmander [94, 156, 165, 167]	Hörmander [88], [92, Chapter XXVIII]
Regularity of the coefficients needed	analytic coefficients	partially analytic coefficients in some variable x_a	C^∞ (even C^1) coefficients
Geometrical assumption of the hypersurface	Φ non characteristic for P: $p(x, \nabla\Phi) \neq 0$	Φ pseudoconvex in $\{\xi_a = 0\}$	Φ pseudoconvex ($\{p, \{p, \Phi\}\} > 0$ sufficient if real order 2)
Stability estimates	Logarithmic type, see John [101] and [A10] (reviewed in Section 1.2)	Logarithmic type, see [A10] (reviewed in Section 1.2)	Hölder type, see [14, 96, 120], see also [A10] (reviewed in Section 1.2)

Table 1.1: Panorama of the different unique continuation results for a differential operator P with principal symbol p.

Roughly speaking, our work [A10] (see [27] for a related result with loss for the wave operator, obtained simultaneously) is concerned with the last line of Table 1.1. It consists in giving *local and global* quantification (i.e. stability estimates) related to this unique continuation result under partial analyticity. Logarithmic or Hölder type dependence refers to the form of the function φ in (1.1.2). This allowed us to apply these results to the classical wave equation [A7] , [A10] and equations involving hypoelliptic (sum of squares) operators [A19] , [A17] . To summarize, the results we obtain are as follows:

1) A general global quantification of the unique continuation property under partial analyticity assumptions. The main result we present is Theorem 1.2.5, which gives a global quantification of the unique continuation along a foliation of hypersufaces satisfying the appropriate conditions. The general estimates obtained in this setting are described in Section 1.2 and were obtained in [A10] .

2) A logarithmic stability estimate for the observation of the waves from any non empty open subset (that is, without geometric assumptions). This is mainly Theorem 1.3.1 below. The results are described in Section 1.3 and were obtained in [A10] .

3) A constructive proof of the Bardos-Lebeau-Rauch observability result of the wave equation under the Geometric Control Condition (GCC) [16]. We describe how the estimate described in Item 2 is useful to obtain a constructive proof of the observability of the wave equation

under the GCC. This is useful when one wants to have estimates of the observability constants in some regimes. These results are described in Section 1.3 and rely on **[A7]** .

4) Some stability estimates for the observation of hypoelliptic (sum of squares) operators and their evolution counterparts. Namely we obtain quantitative unique continuation for eigenfunctions, wave-type, heat-type equations related to a sum of squares operator. As usual, such estimates can be transfered to (approximate) control results, but also to stabilization of damped equations. The results and some ideas of the proof are presented in Section 1.4. They were obtained in **[A19]** , **[A17]** .

1.2 Quantitative unique continuation under partial analyticity

General results of quantitative unique continuation

In this section, we describe the setting of the general stability result we obtained and present the class of partial differential operators we deal with. We consider domains $\Omega \subset \mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, where $n_a + n_b = n$. We denote by $x = (x_a, x_b)$ the global variables and $\xi = (\xi_a, \xi_b)$ the associated dual variables. The variables x_a will denote the set of variables in which the considered operator is analytic. In the examples studied below, this will be the time variable t when we consider the classical (Riemannian) wave equation, while it will be the full set of variables when hypoelliptic operators are considered.

Given a bounded domain $\Omega \subset \mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$, we say that a smooth function $f : \Omega \to \mathbb{C}$ is analytic with respect to x_a if, for any point $x^0 = (x_a^0, x_b^0) \in \Omega$, there is $\varepsilon > 0$ such that f extends as a holomorphic function in the variable x_a for $x = (x_a, x_b) \in \big(B(x_a^0, \varepsilon) + iB(0, \varepsilon)\big) \times B(x_b^0, \varepsilon)$.

The unique continuation and the associated quantification is proved true under some very general assumptions for the operators, named analytically principally normal operators by Tataru [167, Definition 2.2]. Yet, for simplification, we will give a particular case of this condition that will be sufficient for most of the applications we have in mind.

Assumption 1.2.1. Let P be a differential operator on an open set $\Omega \subset \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$ of order $m \in \mathbb{N}^*$ with smooth coefficients and **real-valued** principal symbol $p(x_a, x_b, \xi_a, \xi_b)$. We will assume that all coefficients of P are **real-analytic in the variable** x_a and that

$$\partial_{x_a} p(x_a, x_b, 0, \xi_b) = 0, \qquad \text{for } (x_a, x_b) \in \Omega, \xi_b \in \mathbb{R}^{n_b}. \tag{1.2.1}$$

We now formulate the definition of strongly pseudoconvex surfaces for an operator P.

Definition 1.2.1 (Strongly pseudoconvex oriented surface)**.** Let $\Omega \subset \mathbb{R}^n$, Γ be a closed conic subset of $T^*\Omega$, and let P be a differential operator with principal symbol p. Let S be a C^2 oriented hypersurface of Ω and $x^0 \in S \cap \Omega$. We say that S is strongly pseudoconvex in Γ at x^0 for P if there exists $\phi \in C^2(\Omega; \mathbb{R})$ such that $S = \{\phi = 0\}$, $\nabla \phi(x^0) \neq 0$, satisfying:

$$\operatorname{Re}\{\overline{p}, \{p, \phi\}\}(x^0, \xi) > 0, \qquad \text{if } p(x^0, \xi) = \{p, \phi\}(x^0, \xi) = 0 \text{ and } \xi \in \Gamma, \xi \neq 0; \tag{1.2.2}$$

$$\frac{1}{i\tau}\{\overline{p}_\phi, p_\phi\}(x^0, \xi) > 0, \qquad \text{if } p_\phi(x^0, \xi) = \{p_\phi, \phi\}(x^0, \xi) = 0 \text{ and } \xi \in \Gamma, \tau > 0, \tag{1.2.3}$$

where $p_\phi(x, \xi) = p(x, \xi + i\tau\nabla\phi)$.

Note that this is a property of the *oriented* surface S solely, and not of the defining function ϕ (see [92], beginning of Section 28.3). If $\Gamma = T^*\Omega$, it is the usual condition of the Hörmander Theorem (see [92, Section 28.3]), that is, under which uniqueness holds for P at x^0 across the hypersurface S, i.e. from $\phi > 0$ to $\phi < 0$.

Below, this condition will always be used for $\Gamma = \{\xi_a = 0\}$. In this case, and using the homogeneity of p in ξ, Assumption (1.2.3) may be rephrased as:

$$\frac{1}{i}\{\overline{p}(x, \xi - i\nabla\phi), p(x, \xi + i\nabla\phi)\}(x^0, 0, \xi_b) > 0, \qquad \text{if } p(\zeta) = \{p, \phi\}(\zeta) = 0, \quad \xi_b \in \mathbb{R}^{n_b},$$

where $\zeta = (x^0, i\nabla_a\phi(x^0), \xi_b + i\nabla_b\phi(x^0))$. An important feature of this definition is that it is invariant by changes of coordinates.

Before stating our main result, let us discuss some cases of operators of particular interest.

Remark 1.2.2 (Hörmander case)**.** If $n_a = 0$, there is no analytic variable. In this case, Definition 1.2.1 coincides with the definition of principally normal operators [92, Chapter XXVIII] and Definition 1.2.1 with $\Gamma = T^*\Omega$ that of strictly pseudoconvex functions. The unique continuation result under consideration is the classical Hörmander theorem [92, Chapter XXVIII].

Remark 1.2.3 (Holmgren-John case)**.** If $n_a = n$, that is the operator is analytic in all the variables, we have $x_a = x, \xi_a = \xi$, and hence $\Gamma = \Omega \times \{\xi_a = 0\} = \Omega \times \{\xi = 0\}$. In this situation, Condition (1.2.1) is empty ($\partial_{x_a} p(x_a, \xi_a)$ is a homogeneous polynomial of degree $m \geq 1$ in ξ_a, where m is the order of P; hence it vanishes at $\xi_a = 0$).

Next, concerning the conditions on the surface $\{\phi = 0\}$, notice that (1.2.2) is also empty since $\Gamma \cap \{\xi \neq 0\} = \emptyset$. For (1.2.3), if $\xi \in \Gamma$, that is $\xi = 0$, we have $p_\phi(x^0, \xi) = p(x^0, i\tau\nabla\phi(x^0)) = (i\tau)^m p(x^0, \nabla\phi(x^0))$: any noncharacteristic surface at x^0 (i.e. satisfying $p(x^0, \nabla\phi(x^0)) \neq 0$) is a strongly pseudoconvex oriented surface. The unique continuation result under consideration is the classical Holmgren-John theorem.

Note that, in the case $n_a = n$, the results presented here hold under Condition (1.2.3), namely:

$$p(x^0, \nabla\phi(x^0)) = 0 \quad \Longrightarrow \frac{1}{i}\{\overline{p}(x, \xi - i\nabla\phi), p(x, \xi + i\nabla\phi)\}(x^0, 0) > 0,$$

which is weaker than the noncharactericity condition $p(x^0, \nabla\phi(x^0)) \neq 0$ of the Holmgren-John theorem. We have used the fact that $p(x^0, \nabla\phi(x^0)) = 0$ implies $\{p, \phi\}(x^0, \nabla\phi(x^0)) = 0$ by the homogeneity in ξ of p.

Remark 1.2.4 (Wave type and Schrödinger type operators)**.** Let us now consider the case of operators P of principal symbol of the form $p_2(x, \xi) = Q_x(\xi)$, where Q_x is a smooth x-family of real quadratic forms in ξ, such that $Q_x(0, \xi_b)$ is positive (or negative) definite on $\mathbb{R}^{n_b}$. This is the case of the wave operator or Schrödinger type operators when x_a is the time variable. Then, Assumption (1.2.2) holds (uniformly with respect to $x \in \Omega$) again according to the positive definiteness of $Q_x(0, \xi_b)$. It is indeed empty since $p_2(x, (0, \xi_b))$ does not vanish for $\xi_b \neq 0$. Moreover, we have $\{p_2, \phi\}(x, \xi) = 2\tilde{Q}_x(\xi, \nabla\phi)$, where $\tilde{Q}_x$ is the polar form of Q_x, and

$$\{p_2, \phi\}(x, \xi + i\nabla\phi) = 2\tilde{Q}_x(\xi, \nabla\phi) + 2iQ_x(\nabla\phi).$$

As a consequence (Q being real), $\text{Im}\{p_2, \phi\}(x, \xi + i\nabla\phi) = 2Q_x(\nabla\phi)$ so that (1.2.3) is also empty (and thus satisfied) for any noncharacteristic hypersurface.

In conclusion, for real quadratic forms which are positive (or negative) definite on $\mathbb{R}^{n_b}$ at $\xi_a = 0$, any noncharacteristic hypersurface is strongly pseudoconvex in the sense of Definition 1.2.1. In the case $n_a = 1$, this includes the following operators of particular interest:

- $P = D^2_{x_a} - \sum_{i,j=1}^{n-1} \alpha_{ij}(x) D_{x_b^j} D_{x_b^i} + \ell.o.t.$ (wave operator) with $p = \xi_a^2 - \sum_{i,j=1}^{n-1} \alpha_{ij}(x)\xi_b^j \xi_b^i$,
- $P = D_{x_a} - \sum_{i,j=1}^{n-1} \alpha_{ij}(x) D_{x_b^j} D_{x_b^i} + \ell.o.t.$ (Schrödinger operator) with $p = -\sum_{i,j=1}^{n-1} \alpha_{ij}(x)\xi_b^j \xi_b^i$,

where the quadratic form with coefficients $\alpha_{i,j}$ is positive definite.

We are now prepared to formulate our main result in the general framework. We first describe the geometric context and then state the Theorem.

Geometric setting: (see Figure 1.1) We first fix two splittings of $\mathbb{R}^n$ as $\mathbb{R}^n = \mathbb{R}^{n-1}_{x'} \times \mathbb{R}_{x_n}$ and $\mathbb{R}^n = \mathbb{R}^{n_a}_{x_a} \times \mathbb{R}^{n_b}_{x_b}$, possibly in two different bases. We let D be a bounded open subset of $\mathbb{R}^{n-1}$ with smooth boundary and $G = G(x', \varepsilon)$ a C^2 function defined in a neighborhood of $\overline{D} \times [0, 1]$, such that

- For all $\varepsilon \in (0, 1]$, we have $\{x' \in \mathbb{R}^{n-1}, G(x', \varepsilon) \geq 0\} = \overline{D}$;
- for all $x' \in D$, the function $\varepsilon \mapsto G(x', \varepsilon)$ is strictly increasing;
- for all $\varepsilon \in (0, 1]$, we have $\{x' \in \mathbb{R}^{n-1}, G(x', \varepsilon) = 0\} = \partial D$.

We set $G(x',0) = 0$, $S_0 = \overline{D} \times \{0\}$ and, for $\varepsilon \in (0,1]$,

$$\begin{aligned} S_\varepsilon &= \{(x', x_n) \in \mathbb{R}^n, x_n \geq 0 \text{ and } G(x',\varepsilon) = x_n\} \\ &= (\overline{D} \times \mathbb{R}) \cap \{(x', x_n) \in \mathbb{R}^n, G(x',\varepsilon) = x_n\}; \\ K &= \{x \in \mathbb{R}^n, 0 \leq x_n \leq G(x',1)\}. \end{aligned}$$

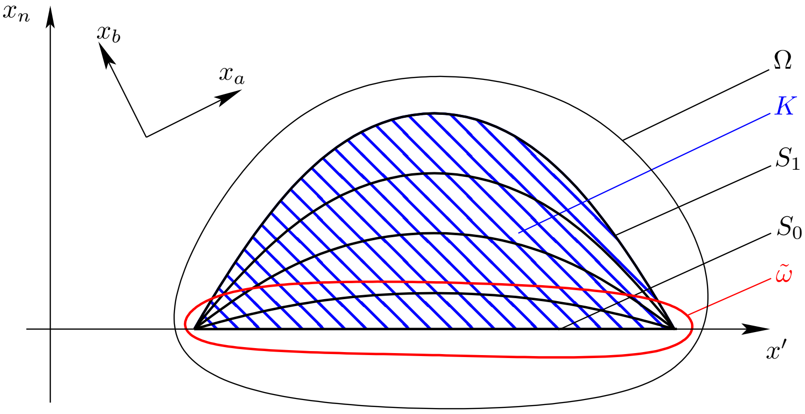


Figure 1.1: Geometric setting of Theorem 1.2.5

Theorem 1.2.5. *In the above geometric setting, we moreover let Ω be a bounded open neighborhood of K, and P be a differential operator of order m, satisfying Assumption 1.2.1 on Ω in $\{\xi_a = 0\}$.*

Assume also that, for any $\varepsilon \in [0, 1+\eta)$, the oriented surfaces $S_\varepsilon = \{\phi_\varepsilon = 0\}$ with $\phi_\varepsilon(x', x_n) := G(x',\varepsilon) - x_n$ are strictly pseudoconvex in $\{\xi_a = 0\}$ for P on the whole S_ε, in the sense of Definition 1.2.1.

Then, for any open neighborhood $\tilde{\omega} \subset \Omega$ of S_0, there exists a neighborhood U of K, and constants $\kappa, C, \mu_0 > 0$ such that for all $\mu \geq \mu_0$ and $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|u\|_{L^2(U)} \leq Ce^{\kappa\mu}\left(\|u\|_{H_b^{m-1}(\tilde{\omega})} + \|Pu\|_{L^2(\Omega)}\right) + \frac{C}{\mu^{m-1}}\|u\|_{H^{m-1}(\Omega)},$$

where we have denoted $\|u\|_{H_b^{m-1}(\tilde{\omega})} = \sum_{|\beta|\leq m-1}\left\|D_b^\beta u\right\|_{L^2(\tilde{\omega})}$.

Note that in the framework of the Hörmander theorem ($n_a = 0$), we can obtain the stronger polynomial-type dependence :

$$\|u\|_{H^{m-1}(U)} \leq C\left(\|u\|_{H^{m-1}(\tilde{\omega})} + \|Pu\|_{L^2(\Omega)}\right)^\delta \|u\|_{H^{m-1}(\Omega)}^{1-\delta}$$

for some $\delta \in (0,1)$. This result was more or less already known even if not written explicitely in this geometric framework for general operators (see [14, 96, 115, 120, 155]).

The formulation of the above result using a foliation by hypersurfaces is inspired by that of [100, Theorem p. 224] in the context of the Holmgren-John theorem. Most of the global Theorems for the wave equations and hypoelliptic equations presented below are proved in the setting of Theorem 1.2.5, after some suitable change of coordinates.

Idea of the proof

As already mentioned, unique continuation theorems (e.g. the Hörmander theorem) are often proved with Carleman estimates, that is, weighted L^2 estimates of the form

$$\left\|e^{\tau\psi}u\right\|_{L^2} \leq C\left\|e^{\tau\psi}Pu\right\|_{L^2}, \tag{1.2.4}$$

where τ is a large parameter and ψ a weight function having levelsets appropriately situated with respect to the surface S. Such inequalities are already quantitative, and hence furnish a good starting point towards local quantitative unique continuation results. This strategy has already been followed in [120, 155] in the case of elliptic operators, see also [14]. Starting from the Carleman inequality (1.2.4), the idea is to apply the estimates to some function $\chi(x)u$ where χ is a cutoff function according to the levelsets of ψ. The exponential weight $e^{\tau\psi(x)}$ in (1.2.4) (giving an exponentially large/small strength to the large/small values of ψ) naturally leads to inequalities of the form

$$\|u\|_{V_2} \leq e^{\kappa\mu}\big(\|u\|_{V_1} + \|Pu\|_{V_3} \big) + e^{-\kappa\mu} \|u\|_{V_3}\,, \tag{1.2.5}$$

uniformly for $\mu \geq \mu_0$ and for small open sets $V_1 \subset V_2 \subset V_3$ depending on the local geometry (namely, on the cutoff function χ, the support of $[P,\chi]$, and hence on the levelsets of ψ). Optimizing in μ (see [155] or [115, Lemma 5.2]) this can then be written as an interpolation estimate

$$\|u\|_{V_2} \leq \big(\|u\|_{V_1} + \|Pu\|_{V_3} \big)^{\delta} \|u\|_{V_3}^{1-\delta}\,,$$

for some $\delta \in (0,1)$. The interest of these interpolation estimates (or directly of estimates like (1.2.5)) is that they can be easily iterated, leading to some global ones. This procedure ends up with a Hölder type dependence. We refer for instance to the survey article [115] for a description of these estimates in the elliptic case, with application to spectral estimates and control results for the heat equation.

Yet, in the context of the unique continuation theorem for partially analytic operators, the Carleman estimates proved in [94, 156, 165, 167] contain a "microlocal" weight of the form $e^{-\frac{\varepsilon}{2\tau}|D_a|^2}e^{\tau\psi(x)}$ instead of $e^{\tau\psi(x)}$. Whereas the usual $e^{\tau\psi}$ is still here to give strength to the levelsets of ψ, the additional term $e^{-\frac{\varepsilon}{2\tau}|D_a|^2}$ is now aimed at localizing in the low frequencies in the variable x_a. In this context, the proof of unique continuation proceeds with a (qualitative) complex analytic argument (maximum principle). Here, this additional argument in the proof of unique continuation also requires to be quantified. As in [155], this procedure naturally leads to local logarithmic (instead of Hölder) stability estimates. The main issue one then has to face when quantifying unique continuation is that such estimate cannot be iterated (or would yield dependence estimates of the type (1.1.2) with a function φ being a composition of as many "log" as steps needed in the iteration).

One idea to overcome this difficulty, proposed by Tataru in his unpublished lecture notes [166], was to propagate some low frequency estimates of the form

$$\left\{ \begin{array}{rcl} \|u\|_{H^{m-1}} & = & 1 \\ \left\|m\left(\frac{D_a}{\mu}\right)\sigma(\frac{x}{R})Pu\right\|_{L^2} & \leq & e^{-\mu^\alpha} \end{array} \right. \Longrightarrow \left\|m\left(\frac{D_a}{\tau}\right)\sigma(\frac{x}{r})u\right\|_{H^{m-1}} \leq e^{-\tau}, \quad \forall \tau < c\,\mu^\alpha \tag{1.2.6}$$

for functions u supported in $\{\phi < \phi(x_0)\}$, for some appropriate compactly supported cutoff functions σ and $m(\xi)$ in the Gevrey class $1/\alpha$, $\alpha < 1$, and for some $r < R$. Such estimates could be propagated and would lead to some global stability estimates of the form (1.1.2) with $\varphi_\varepsilon(a,b,c) = c\left(\log(1+\frac{c}{a+b})\right)^{-(1-\varepsilon)}$.

The loss $1-\varepsilon$ in the power of log is due to the use of functions of class Gevrey α with compact support. The optimal case $\alpha = 1$ would correspond to analytic functions. Yet, analytic functions cannot have compact support, which is a key ingredient in the usual application of Carleman estimates.

Let us now explain our strategy to solve this problem.

Obtaining local information at low frequency

Part of the proof of the quantitative unique continuation is inspired by this idea of propagating only low frequency (in the analytic variable x_a) estimates. However, we replace the Gevrey cutoff functions σ and m in (1.2.6) by some analytic "almost cutoff" functions of the generic form

$$\chi_\lambda := e^{-\frac{|D_a|^2}{\lambda}}\chi, \tag{1.2.7}$$

where χ is smooth with the expected compact support, being convolved/regularized with a heat kernel in the variable x_a (or ξ_a for the "cutoff" in Fourier m), hence analytic in this variable. It turns out that the right choice of the regularization parameter λ is $\lambda = C\mu$ where μ is the frequency where we want to measure our solution. That such functions are not compactly supported makes all commutator estimates (e.g. when applying the Carleman estimate to functions like $\chi_\lambda u$ instead of χu, as explained above) much more intricate and requires a careful study of the dependence with respect the regularisation parameter λ, the local frequency μ and the parameter τ in the Carleman inequality. All estimates are carried out up to an exponentially small remainder (in terms of these parameters).

Following this procedure, the local estimate we prove (which we are in addition able to propagate) is a generalization of (1.2.5), but truncated at low frequencies in the analytic variable x_a. In a neighborhood of a point x^0, it is of the form

$$\left\|m_\mu\left(\frac{D_a}{\beta\mu}\right)\chi_{2,\mu}u\right\|_{H^{m-1}} \leq Ce^{\kappa\mu}\left(\left\|m_\mu\left(\frac{D_a}{\mu}\right)\chi_{1,\mu}u\right\|_{H^{m-1}} + \|Pu\|_{L^2(B(x^0,R))}\right) + Ce^{-\kappa'\mu}\|u\|_{H^{m-1}} \tag{1.2.8}$$

uniformly for $\mu \geq \mu_0$. Here, χ_1 and χ_2 are some cutoff functions in the physical space that localize respectively to the place where the information is taken (locally in $\{\phi > 0\}$) and to where it is propagated (a small neighborhood of x^0). These functions respectively correspond to $\mathbb{1}_{V_1}$ and $\mathbb{1}_{V_2}$ in (1.2.5). The Fourier multipliers m_μ cuts off (analytically) the ξ_a frequencies (m has to be though of as $\mathbb{1}_{B_{\mathbb{R}^{n_a}}(0,1)}$). All these cutoff functions are used only with their analytic regularization according to (1.2.7) with $\lambda = \mu$. They never localize exactly. Using such regularized cutoff functions and Fourier multipliers follows the spirit of analytic semiclassical analysis. However, we do not make use of that theory and rather construct by hand the appropriate mollifiers, making the proof selfcontained in this respect.

The proof of estimates like (1.2.8) mainly proceeds in three steps.

First, as in the usual proofs of unique continuation results, starting from the hypersurface $\{\phi = 0\}$, one needs to construct a weight function ψ with both properties:

- to satisfy the assumptions required to apply the Carleman estimate (ψ should be a strictly pseudoconvex *function*);
- to have level sets appropriately located with respect to those of ϕ (so that propagating uniqueness across levelsets of ψ still corresponds to propagating zero locally from $\phi > 0$ to $\phi < 0$).

This corresponds to the so called "convexification process", see [92, Chapter XXVIII].

Second, we apply as a black box the Carleman estimates of [94, 156, 165, 167] (or some similar ones that we prove in the presence of boundary) to χu, where χ is a particular cutoff function (localizing near the point of interest, and according to levelsets of ψ), containing both rough cutoffs and mollified ones. We then need to estimate all terms arising from the commutator $e^{-\frac{\varepsilon}{2\tau}|D_a|^2}e^{\tau\psi}[P,\chi]$, that are either well localized or yield an exponentially small contribution.

Finally, we need to transfer the information given by the Carleman inequality to some estimate like (1.2.8) on the low frequencies of the function. This is done through a complex analysis argument, the Carleman parameter τ playing the role of complex variable, as in [165]. If ζ is the complex variable, the Carleman estimates corresponds to an estimate on $\zeta = i\tau \in i\mathbb{R}_+$. Combined with *a priori* estimates, a Phragmén-Lindelöf type theorem allows to extend this estimate to part of the real domain, where it corresponds to estimating $\left\|m\left(\frac{D_a}{\beta\mu}\right)\chi u\right\|$. To obtain estimates that are uniform with respect to the frequency (and regularization) parameter μ, we also need, following [166], to use a scaling argument, replacing τ by τ/μ.

Propagating local informations to global ones

Once the local estimates are proved, we need to iterate them to obtain a global estimate. At first, we define some tools that will allow later in an abstract way to propagate easily our local estimate (1.2.8). Estimate (1.2.8) says essentially that, for a solution of $Pu = 0$, information can be transfered from the support of χ_1 to the support of χ_2. We formalize that with the notion of *zone* *of dependence*. Roughly speaking, we say that on open set O_2 depends on O_1 if (1.2.8) holds for every χ_1 equals to 1 on O_1 and any χ_2 supported in O_2. This part allows to formulate the proof of Theorem 1.2.5 as a complete geometric one. Even if quite different in definition, it is close in spirit to the interpolation theory developped in Lebeau [118] to propagate globally the local information obtained by the Cauchy-Kowalevski theorem. Moreover, it should adapt to some more general kinds of foliations. Note that at each step of this propagation argument, we have a loss in the range of frequency: from an information on frequencies $\leq \mu$, we obtain from (1.2.8) an information on frequencies $\leq \beta\mu$, with β small. This is overcome by the fact that we only have a finite number of steps in this iterative procedure.

Once this propagation result is done, we are left with a low frequency information of the solution u. Since we have no information about the high frequency part, the only thing to do is to use some trivial bound of the type

$$\left\| \left(1 - m\left(\frac{D_a}{\mu}\right)\right) u \right\|_{L^2} \leq \frac{C}{\mu^{m-1}} \|u\|_{m-1}.$$

This is actually much worse than the negative exponential that we already had. But it turns out to be the best we can do without any more information. It gives the final estimate of Theorem 1.2.5.

1.3 Applications to the observability and control of the wave equation

Logarithmic stability without geometric assumption

In this section, we describe the motivating applications of the results presented in Section 1.2, i.e. to the wave equation with Dirichlet boundary conditions. In this very particular setting, we are also able to tackle the boundary value problem.

When dealing with a manifold with boundary $\mathcal{M}$, we will always assume that the manifold, the boundary and the metric are smooth. Moreover, $\mathrm{Int}(\mathcal{M})$ will denote the set of points in $\mathcal{M}$ which have a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. The boundary of $\mathcal{M}$, denoted by $\partial\mathcal{M}$, is the complement of $\mathrm{Int}(\mathcal{M})$ in $\mathcal{M}$. All manifolds considered will be assumed to be connected. For a subset $\omega \subset \mathcal{M}$, we will define $\mathcal{L}(\mathcal{M}, \omega) := \sup_{x\in\mathcal{M}} \mathrm{dist}(x, \omega)$ which is finite since $\mathcal{M}$ is compact and connected.

Theorem 1.3.1 (Quantitative unique continuation for waves)**.** *Let $\mathcal{M}$ be a compact Riemannian manifold with (or without) boundary. Let $V \in L^\infty([0,T] \times \mathcal{M})$ analytic in time. For any nonempty open subset ω of $\mathcal{M}$ and assuming $T > 2\mathcal{L}(\mathcal{M}, \omega)$, there exist $C, \kappa, \mu_0 > 0$ such that for any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and associated solution u of*

$$\begin{cases} \partial_t^2 u - \Delta_g u = Vu & \text{in } (0,T) \times \mathrm{Int}(\mathcal{M}), \\ u_{|\partial\mathcal{M}} = 0 & \text{in } (0,T) \times \partial\mathcal{M}, \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) & \text{in } \mathrm{Int}(\mathcal{M}), \end{cases} \tag{1.3.1}$$

we have, for any $\mu \geq \mu_0$,

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq Ce^{\kappa\mu} \|u\|_{L^2((0,T);H^1(\omega))} + \frac{1}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}. \tag{1.3.2}$$

If $\partial\mathcal{M} \neq \emptyset$ and Γ is a non empty open subset of $\partial\mathcal{M}$, for any $T > 2\mathcal{L}(\mathcal{M}, \Gamma)$, there exist $C, \kappa, \mu_0 > 0$ such that for any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and associated solution u of (1.3.1), we have

$$\|(u_0, u_1)\|_{L^2 \times H^{-1}} \leq Ce^{\kappa\mu} \|\partial_\nu u\|_{L^2((0,T)\times\Gamma)} + \frac{1}{\mu} \|(u_0, u_1)\|_{H^1 \times L^2}.$$

Theorem 1.3.1 remains valid if Δ_g is perturbated by lower order terms that are analytic in time but may have low regularity in space. In the special case where they are time independent, the constants in the previous estimates may be chosen uniformly with respect to these perturbations (in the appropiate norms). Note that in (1.3.2), the $L^2(0,T; H^1(\omega))$ norm can actually be replaced by a $L^2(0,T; L^2(\omega))$ norm according to **[A19]** , Section 5.3. This result can also be formulated in an equivalent way, that looks more like a stability estimate. We only give the boundary observation case, the internal observation case being similar.

Corollary 1.3.2. *Assume $\partial\mathcal{M} \neq \emptyset$ and Γ is a non empty open subset of $\partial\mathcal{M}$. Then, for any $T > 2\mathcal{L}(\mathcal{M},\Gamma)$, there exists $C > 0$ such that for any $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M}) \setminus \{(0,0)\}$ and associated solution u of (1.3.1), we have*

$$\|(u_0,u_1)\|_{L^2\times H^{-1}} \leq C \frac{\|(u_0,u_1)\|_{H^1\times L^2}}{\log\left(1+\frac{\|(u_0,u_1)\|_{H^1\times L^2}}{\|\partial_\nu u\|_{L^2(]0,T[\times\Gamma)}}\right)},$$

$$\|(u_0,u_1)\|_{H^1\times L^2} \leq Ce^{C\Lambda}\|\partial_\nu u\|_{L^2((0,T)\times\Gamma)}, \qquad \textit{with } \Lambda = \frac{\|(u_0,u_1)\|_{H^1\times L^2}}{\|(u_0,u_1)\|_{L^2\times H^{-1}}}.$$

In the first estimate, the function on the right hand-side is to be understood as being $\left(\log(1+\frac{1}{x})\right)^{-1}$ for $x > 0$ and 0 for $x = 0$.

In the second estimate, Λ has to be considered as the typical frequency of the initial data. So, the estimate states a cost of observability of the order of an exponential of the typical frequency. As an illustration, taking for initial data $(u_0, u_1) = (\psi_\lambda, 0)$ with ψ_λ a normalized eigenfunction of the Laplace-Dirichlet operator on $\mathcal{M}$, associated to the eigenvalue λ, one has $\Lambda \sim \sqrt{\lambda}$ and Corollary 1.3.2 recovers the tunneling estimate $\|\partial_\nu\psi_\lambda\|_{L^2(\Gamma)} \geq C^{-1}e^{-C\sqrt{\lambda}}$ (see [120]).

The minimal time $2\mathcal{L}(\mathcal{M},\omega)$ or $2\mathcal{L}(\mathcal{M},\Gamma)$ in these two theorems is optimal (even for qualitative unique continuation) in view of the finite speed of propagation for the wave equation. Moreover, as proved by Lebeau [118] in the analytic context, this exponential dependence is sharp in general. More precisely, the form of the estimates of Theorem 1.3.1 and Corollary 1.3.2 is optimal as soon as there is a ray of geometric optics (travelling at speed one) which does not intersect the region $\overline{\Gamma}$ (resp. $\overline{\omega}$ in the internal observation case) in the time interval $[0,T]$ (and only has transverse intersection with the boundary). See [118, Section 2, pages 5 and 6].

The proof of Theorem 1.3.1 (and the variant Theorem 1.3.4) relies on several applications of our main quantitative unique continuation Theorem 1.2.5 with some well chosen foliation of non characteristic hypersurfaces.

As a consequence of Theorem 1.3.1, we obtain the following approximate controllability results, with (optimal in general) estimate on the cost. For the sake of brevity, we only state the case of a boundary control.

Theorem 1.3.3 (Cost of boundary approximate control)**.** *For any $T > 2\mathcal{L}(\mathcal{M},\Gamma)$, there exist $C, c > 0$ such that for any $\varepsilon > 0$ and any $(u_0,u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$, there exists $g \in L^2((0,T)\times\Gamma)$ with*

$$\|g\|_{L^2((0,T)\times\Gamma)} \leq Ce^{\frac{c}{\varepsilon}}\|(u_0,u_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})},$$

such that the solution of

$$\begin{cases} (\partial_t^2 - \Delta)u = 0 & \textit{in } (0,T)\times \mathrm{Int}(\mathcal{M}), \\ u_{|\partial\mathcal{M}} = \mathbb{1}_\Gamma g & \textit{in } (0,T)\times\partial\mathcal{M}, \\ (u,\partial_t u)_{|t=0} = (u_0,u_1), & \textit{in } \mathrm{Int}(\mathcal{M}), \end{cases}$$

satisfies $\left\|(u,\partial_t u)_{|t=T}\right\|_{L^2(\mathcal{M})\times H^{-1}(\mathcal{M})} \leq \varepsilon\|(u_0,u_1)\|_{H_0^1(\mathcal{M})\times L^2(\mathcal{M})}$.

That this result is a consequence of Theorem 1.3.1 is proved in [155, Proof of Theorem 2, Section 3]. The solution of the nonhomogeneous boundary value problem is defined in the sense of transposition, see [128].

The estimates of Theorem 1.3.1 and Corollary 1.3.2 can actually be stated more locally, and interpreted in a different physical context (motivated by [150]). The following Theorem shows that they are independent on the global geometry, and, in particular, do not require that $\mathcal{M}$ is compact if one only wants to recover data supported in a given compact set.

Theorem 1.3.4 (Penetration into shadow for waves)**.** *Let $\mathcal{M}$ be a complete Riemannian manifold with (possibly empty) compact boundary $\partial\mathcal{M}$. Let ω_0 be an open set of $\mathcal{M}$ and ω_1 a compact set of $\mathcal{M}$. Then, for any*

$$T > \mathcal{L}(\omega_1,\omega_0) := \sup_{x\in\omega_1} \mathrm{dist}(x,\omega_0),$$

there exist $C > 0$ such that for any $(u_0, u_1) \in H^1_0(\mathcal{M}) \times L^2(\mathcal{M}) \setminus \{(0,0)\}$ supported in ω_1 and associated solution u of (1.3.1) *(taken on the time interval $(-T,T)$ instead of $(0,T)$), we have,*

$$\|(u_0,u_1)\|_{H^1\times L^2} \le Ce^{C\Lambda} \|u\|_{L^2((-T,T);H^1(\omega_0))}, \qquad \text{with } \Lambda = \frac{\|(u_0,u_1)\|_{H^1\times L^2}}{\|(u_0,u_1)\|_{L^2\times H^{-1}}}.$$

Roughly speaking, the theorem describes the following physical situation: take a noise creating an initial data compactly supported in ω_1, and take an observer located in a zone ω_0. Then, by observing during the time interval $(-\mathcal{L}(\omega_1,\omega_0)-\varepsilon, \mathcal{L}(\omega_1,\omega_0)+\varepsilon)$, $\varepsilon > 0$, the observer will be able to recover at least a proportion of the initial energy of the order $e^{-C\Lambda}$ where Λ is the typical frequency of the data. This result is particularly interesting if the zone ω_1 is in the "shadow" of an obstacle when seen from ω_0, that is if no rays of geometric optic starting from ω_1 ever reach ω_0. In that case, the classical geometric optic approximation would predict that the observer does not receive any information. We refer to [150] for a qualitative result in infinite time; here, Theorem 1.3.4 provides a quantitative result in finite time, which is optimal with respect to the time and the form of the estimate if ω_1 is indeed in the "shadow" region when observed from ω_0. More precisely, [118, Section 2] implies that the $e^{C\Lambda}$ is optimal as soon as there is a ray of geometric optics (having only transverse intersections with $\partial\mathcal{M}$) starting from the interior of ω_1 at time zero and not intersecting $\overline{\omega_0}$ during the time interval $[-T,T]$. Such an estimate in the shadow region is reminiscent to the tunneling effect for waves and in semiclassical analysis.

We also obtain related results for the Schrödinger equation, see **[A10]** , Theorem 1.5. The latter formulate in a similar way but hold in arbitrary small time. Yet, we believe that it is not optimal and could be improved. We refer to the section about perspective below.

Idea of proof Theorem 1.3.1 is proved using several applications of Theorem 1.2.5. According to Remark 1.2.4, it is applicable under the (quite weak) assumption that the hypersurfaces are non characteristic. Then, for each point $x \in \mathcal{M}$, we construct a path that links x to a point $x_1 \in \omega$ of length less than $T > \mathcal{L}(\mathcal{M},\omega)$. Then, it is possible to construct a foliation of non characteristic hypersurfaces along which the application of Theorem 1.2.5 is possible. The construction of such hypersurfaces was inspired by that of [118]. This allows then to transfer the information from a neighborhood of x_1 included in ω to a neighborhood of x. Since we can do that for any $x \in \mathcal{M}$, a compactness argument gives a global estimate quite close to the expected one.

$$\|u\|_{L^2(]-\varepsilon,\varepsilon[\times\mathcal{M})} \le Ce^{\kappa\mu} \|u\|_{L^2(]-T,T[\times\omega)} + \frac{C}{\mu} \|u\|_{H^1(]-T,T[\times\mathcal{M})}.$$

Energy estimates finally allow to estimate by below and above the global space time norms with related norms of the initial data.

Note also that in the case of manifold with boundary, we needed to write new Carleman estimates with boundary for operators of order two with real valued principal symbol.

A constructive proof of the Bardos-Lebeau-Rauch theorem

Another application of Theorem 1.3.1, given in **[A7]** and which was at the origin of the present work, is concerned with the *exact* observability/controllability problem. This property was completely characterized (with optimal geometric conditions) in the seminal paper [16].

The purpose is to prove that if (ω,T) satisfies the Geometric Control Condition (GCC), then, any solution of (1.3.1) satisfies the observability estimate

$$\|(u_0,u_1)\|_{H^1\times L^2} \le C_0 \|u\|_{L^2(0,T;H^1(\omega))}. \tag{1.3.3}$$

The proof in [16] is non constructive, with the drawback that it does not give any information about the constant C_0 involved. The latter is of primary importance in applications since it may be interpreted as the cost of exact controls for the controlled wave equation (via the Hilbert Uniqueness Method, see [128]). It is made in two steps, both of them being non constructive in the previously known literature:

1) high frequency part: proof of a weaker estimate

$$\|(u_0, u_1)\|_{H^1\times L^2} \le C \|u\|_{L^2((0,T);H^1(\omega))} + C \|(u_0, u_1)\|_{L^2\times H^{-1}} \tag{1.3.4}$$

2) low frequency part: getting rid of the lower order term by reducing to a unique continuation type argument.

Step 1 concerning high frequency is proved with microlocal analysis and is therefore always performed "up to lower order terms", which explains the presence of the remainder term in $L^2 \times H^{-1}$-norm. It uses techniques like propagation of regularity, microlocal defect measures or Egorov theorem that are out of the scope of this review article. We implement in **[A7]** a constructive proof inspired by [53] and relying on the Egorov Theorem, in case $\partial\mathcal{M} = \emptyset$.

Step 2 was usually performed in the literature with an argument by contradiction combined with a unique continuation theorem. Theorem 1.3.1 allows to give a completely constructive and direct proof of this step as follows. Combining the high frequency estimate (1.3.4) with our quantitative unique continuation result (1.3.2) gives directly uniformly for $\mu \ge \mu_0$,

$$\|(u_0, u_1)\|_{H^1\times L^2} \le C \|u\|_{L^2((0,T);H^1(\omega))} + Ce^{\kappa\mu} \|u\|_{L^2((0,T);H^1(\omega))} + \frac{1}{\mu} \|(u_0, u_1)\|_{H^1\times L^2}.$$

It implies (1.3.3) when μ is taken large enough so that to absorb the last term of the right-hand side.

This method allows us to obtain estimates of the observability constant (and therefore the cost of the control) in two regimes. We obtain

- the dependence of the control cost (the constant C_0 in (1.3.3)) with respect to the the addition of a potential $V(x)$ in the wave operator
- the dependence of the control cost when the observation time T approaches the critical time of the Geometric Control Condition. In particular, in the case without boundary, we obtain an observability estimate of the form

  $$\|(v_0, v_1)\|_{H^1\times L^2} \le Ce^{\kappa\mathfrak{K}(T)^{-1}} \int_0^T \|b_\omega v(t)\|^2_{H^1(\mathcal{M})} dt,$$

  where $\mathfrak{K}(T) = \min_{\rho\in S^*\mathcal{M}} \int_0^T b_\omega^2 \circ \pi \circ \varphi_t(\rho) dt$ is the average of the observation along the geodesic flow φ_t on S^*M. π is the canonical projection $S^*M \to M$.

We refer to **[A7]** for precise statements.

Some perspectives

1) Schrödinger operator: Our article **[A10]** contains an equivalent of Theorem 1.3.1 where the remainder term is of the form $\|u_0\|_{H^2}$, which is not natural. We expect that the natural term would be $\|u_0\|_{H^1}$. This is a consequence of the fact that it ignores the term $i\partial_t$ which is considered as a lower order term. We should take into account the anisotropy of the Schrödinger operator. In collaboration with Matthieu Léautaud and our common student Spyridon Filippas, we made a first step in that direction by a unique continuation result (that is only qualitative) that allows some coefficients Gevrey in time. This is contained in the thesis of Spyridon Filippas [69] and in [70] recently submitted. We hope to continue in this direction and use the anisotropy of the Schrödinger operator to refine the quantitative estimate.

2) Another possible improvement of the results presented in this section is the regularity of the coefficients. The coefficients of the operators are assumed to be analytic in the variable n_a, but smooth overall. For instance, in the context of the wave equation, we could expect to extend Theorem 1.3.1 to some coefficients of the metric that are only Lipschitz, which is the minimal regularity in the elliptic case. This is work in progress with Matthieu Léautaud.

Another natural extension is the case of metrics with jump along an interface as it is natural in the applications to geoscience. A natural equivalent of Theorem 1.3.1 has been proved by Spyridon Filippas [68] in the context of his PhD thesis [69].

3) The results of unique continuation across a noncharacteristic hypersurface of Tataru, Zuily-Robbiano, Hörmander that we quantified in this section, in the partially analytic framework, have been used extensively in the context of hyperbolic inverse problems. The problem is to recover a metric (i.e. the data of a heterogeneous medium) from measurements at the boundary of the solutions of the wave equation (the Dirichlet-to-Neumann operator). A very important method in this field, the Boundary Control Method, shows that the knowledge of the hyperbolic Dirichlet-to-Neumann operator operator uniquely determines the metric (modulo invariance). This method is based on the unique continuation. Theorem 1.3.1 gives a quantitative estimate of this unique continuation and we hope that it could provide stability estimates for the inverse problem. If the measurements at the edge are close, how does this imply that the two metrics are close? We have already started a project on this problem with Matti Lassas (Helsinki, Finland), Matthieu Léautaud and Lauri Oksanen (Helsinki).

1.4 Applications to the observability and control of hypoelliptic equations

The general result stated in Theorem 1.2.5 actually contains a quantitative version of the classical Holmgren-John theorem, see Remark 1.2.3. A classical result of Bony [25], relying on the Holmgren-John Theorem, proves unique continuation for solutions to $\mathcal{L}u = Vu$ where $\mathcal{L}$ is a hypoelliptic operators with analytic coefficients (and V and analytic potential). In this Section, we propose a quantitative version of this unique continuation result, together with generalizations to eigenfunctions, solutions to wave and heat equations associated to such operators $\mathcal{L}$. Most of the results are taken from **[A19]** , and a few of them from **[A17]** .

In Section 1.4, we first present generalities about hypoelliptic operators and their analysis.

In Section 1.4, we detail our main results concerning eigenfunctions, wave-type operators, heat-type operators and damped equations.

In Section 1.4 we give some ideas of the proofs. The main technical part is the proof for the sub-Riemannian wave operator in Section 1.4, where we apply our quantitative version of the Holmgren-John theorem (Theorem 1.2.5) combined with hypoelliptic estimates. Sections 1.4, 1.4 and 1.4 describe the abstract functional analytic framework to deduce the results from the wave equation to eigenfunctions, heat-like and damped equations.

Generalities about sub-Riemannian geometry and analysis in this context

Let $\mathcal{M}$ be a smooth compact connected manifold without boundary. We denote by $\mathcal{X}^\infty$ the space of smooth vector fields on $\mathcal{M}$ (with real coefficients), which we identify to derivations on $\mathcal{M}$. We assume $\mathcal{M}$ is endowed with a smooth positive density measure ds, so that we may integrate functions on $\mathcal{M}$. We may then define the space $L^2(\mathcal{M}) = L^2(\mathcal{M}, ds)$ of square integrable functions with respect to this measure. For $X \in \mathcal{X}^\infty$, we define by X^* its formal dual operator for the duality of $L^2(\mathcal{M})$, that is,

$$\int_{\mathcal{M}} X^*(u)(x)v(x)ds(x) = \int_{\mathcal{M}} u(x)X(v)(x)ds(x), \qquad \text{for any } u, v \in C^\infty(\mathcal{M}).$$

Given $\mathfrak{m} \in \mathbb{N}$ and $\mathfrak{m}$ vector fields. $X_1, \cdots, X_{\mathfrak{m}} \in \mathcal{X}^\infty$, we are interested in properties of the following (non-positive) second order operator, associated to the X_i's (namely the so-called type I Hörmander operator).

$$\mathcal{L} = \sum_{i=1}^{\mathfrak{m}} X_i^* X_i. \tag{1.4.1}$$

Note that this operator is formally symmetric nonnegative, when defined on functions in $C^\infty(\mathcal{M})$, since we have

$$(\mathcal{L}u, u)_{L^2(\mathcal{M})} = \sum_{i=1}^{\mathfrak{m}} \|X_i u\|^2_{L^2(\mathcal{M})} .$$

Both from the geometric control and the operator theoretic points of view, it is in this context natural to consider iterated Lie brackets of the vector fields X_i. We refer for instance to [1].

Definition 1.4.1. For any family $\mathcal{F}$ of smooth vector fields on $\mathcal{M}$ and $\ell \in \mathbb{N}^*$, we define the subspaces $\mathrm{Lie}^\ell(\mathcal{F})$ of $\mathcal{X}^\infty$ by iteration as follows:

- $\mathrm{Lie}^1(\mathcal{F})$ is the space spanned by $\mathcal{F}$ in $\mathcal{X}^\infty$,
- $\mathrm{Lie}^{\ell+1}(\mathcal{F}) = \mathrm{span}\left(\mathrm{Lie}^\ell(\mathcal{F}) \cup \left\{[X,Y]; X \in \mathcal{F}, Y \in \mathrm{Lie}^\ell(\mathcal{F})\right\}\right)$.

For any point $x \in \mathcal{M}$, $\ell \in \mathbb{N}^*$, we denote by $\mathrm{Lie}^\ell(\mathcal{F})(x)$ the set of all tangent vectors $X(x)$ with $X \in \mathrm{Lie}^\ell(\mathcal{F})$.

We shall always assume that the family (X_i) satisfies the Chow-Rashevski-Hörmander condition (or is "bracket generating").

Assumption 1.4.1. There exists $\ell \geq 1$ so that for any $x \in \mathcal{M}$, $\mathrm{Lie}^\ell(X_1, \cdots, X_{\mathfrak{m}})(x) = T_x\mathcal{M}$. Denote then by $k \in \mathbb{N}^*$ the minimal ℓ for which this holds.

The integer k will sometimes be refered to as the *hypoellipticity index* of $\mathcal{L}$. Assumption 1.4.1 is central in control theory and operator theory, for it characterizes both the controllability of the controlled ODE driven by the vector fields (X_i) and the Hypoellipticity of the operator $\mathcal{L}$. Let us now recall these two seminal results, namely the Chow-Rashevski theorem and the Hörmander theorem, which we both use in the sequel.

Theorem 1.4.2 (Chow-Rashevski)**.** *Under Assumption 1.4.1, the following statement holds: for any $x_0, x_1 \in \mathcal{M}$, any $T > 0$, there exist $u_i \in L^1(0,T)$ for $i \in \{1, \cdots, \mathfrak{m}\}$ such that the unique solution of*

$$\dot{\gamma}(t) = \sum_{i=1}^{\mathfrak{m}} u_i(t) X_i(\gamma(t)), \quad \gamma(0) = x_0$$

satisfies $\gamma(T) = x_1$.

This theorem motivates the following definition.

Definition 1.4.3 (Horizontal path)**.** We say that an absolutely continuous function $\gamma : [0,T] \to \mathcal{M}$ is a horizontal path if there exist $u_i \in L^1(0,T;\mathbb{R})$ for $i = 1, \cdots, \mathfrak{m}$ such that for almost every $t \in [0,T]$, we have $\dot{\gamma}(t) = \sum_{i=1}^{\mathfrak{m}} u_i(t) X_i(\gamma(t))$.

Such a trajectory is in particular absolutely continuous and almost everywhere tangent to the so-called horizontal distribution $\mathrm{span}(X_1, \cdots, X_{\mathfrak{m}})$. The second key role played by Assumption 1.4.1 in analysis is summarized in the following result.

Theorem 1.4.4 (Hörmander [89], Rothschild-Stein [157])**.** *Under Assumption 1.4.1, the operator $\mathcal{L}$ in (1.4.1) is hypoelliptic, that is, for all $u \in \mathcal{D}'(\mathcal{M})$ and $x_0 \in \mathcal{M}$, if $\mathcal{L}u \in C^\infty$ near x_0 then $u \in C^\infty$ near x_0.*

Moreover, it is subelliptic of order $\frac{1}{k}$, that is, the following estimates hold: there is $C > 0$ such that for any $u \in C^\infty(\mathcal{M})$, we have

$$\|u\|^2_{H^{\frac{1}{k}}(\mathcal{M})} \leq C \sum_{i=1}^{\mathfrak{m}} \|X_i u\|^2_{L^2(\mathcal{M})} + C \|u\|^2_{L^2(\mathcal{M})} , \tag{1.4.2}$$

$$\|u\|^2_{H^{\frac{2}{k}}(\mathcal{M})} \leq C \|\mathcal{L}u\|^2_{L^2(\mathcal{M})} + C \|u\|^2_{L^2(\mathcal{M})} . \tag{1.4.3}$$

The hypoellipticity was shown by Hörmander [89], who also provided with a subelliptic estimate with loss. The optimal subelliptic estimate (1.4.2) with gain of $1/k$ derivatives is proved by [157].

Since the operator $\mathcal{L}$ is symmetric non-negative, the hypoellipticity of $\mathcal{L}+1$ and the compactness of $\mathcal{M}$ directly imply that $\mathcal{L}$ is essentially selfadjoint (see e.g. Reed-Simon [152, Theorem X.26]). Hence, it extends uniquely as a selfadjoint operator (its Friedrich extension)

$$\mathcal{L} : D(\mathcal{L}) \subset L^2(\mathcal{M}) \to L^2(\mathcal{M}),$$

with, according to (1.4.3), $H^2(\mathcal{M}) \subset D(\mathcal{L}) \subset H^{\frac{2}{k}}(\mathcal{M})$ (still under Assumption 1.4.1). The operator $\mathcal{L}$ is hence selfadjoint on $L^2(\mathcal{M})$, with compact resolvent: it admits a Hilbert basis of eigenfunctions $(\varphi_j)_{j\in\mathbb{N}}$, associated with the real eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$, sorted increasingly, that is

$$\mathcal{L}\varphi_i = \lambda_i\varphi_i, \qquad (\varphi_i, \varphi_j)_{L^2(\mathcal{M})} = \delta_{ij}, \qquad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to +\infty.$$

Note that a bootstrap argument in (1.4.3) shows that $\varphi_j \in C^\infty(\mathcal{M})$. Also, the spectral decomposition allows to define solutions of the hypoelliptic wave and heat equations (respectively $(\partial_t^2 + \mathcal{L})v = f$ and $(\partial_t + \mathcal{L})u = f$), which we shall consider in this paper.

In addition to Assumption 1.4.1, we will also assume in the main part of the article that everything is real-analytic.

Assumption 1.4.2. The manifold $\mathcal{M}$, the density ds, and the vector fields X_i are **real-analytic**.

In particular, it implies that the operator $\mathcal{L}$ has analytic coefficients in any analytic coordinate set compatible with the manifold $\mathcal{M}$.

This assumption can be lowered (and we give some examples in **[A19]**), but seems hard to avoid totally, due to some counterexamples to unique continuation [13].

Finally, let us mention that hypoelliptic operators appear naturally in several physical and mathematical contexts such as stochastic processes and the theory of functions of several complex variables. We refer to [30, Chapter 2] for a presentation of some of these applications. Classical examples of operators $\mathcal{L}$ encompassed by this frameworks is also provided in **[A19]** , Section 1.1: elliptic operators ($k = 1$, Grushin operators ($k \in N^*$), Heisenberg ($k = 2$), Lie Groups...

Main results for hypoelliptic equations

Our main results under Assumptions 1.4.1 and 1.4.2 are of four different types:

1) Tunneling estimates for eigenfunctions φ_j of $\mathcal{L}$ (Section 1.4);

2) Quantitative approximate observability (and associated controllability) of the hypoelliptic wave equation $(\partial_t^2 + \mathcal{L})v = 0$ from a subset $\omega \subset \mathcal{M}$ (Section 1.4);

3) Quantitative approximate observability (and associated controllability) of the hypoelliptic heat equation $(\partial_t + \mathcal{L})u = 0$ from ω (Section 1.4).

4) Decay for damped hypoelliptic wave equations $(\partial_t^2 + \mathcal{L} + 1_\omega\partial_t)v = 0$, Schrödinger $(i\partial_t + \mathcal{L} + i1_\omega\partial_t)v = 0$ or damped plates $(\partial_t^2 + \mathcal{L}^2 + 1_\omega\partial_t)v = 0$ (Section 1.4)

All of these results depend explicitely on the hypoellipticity index k of the operator considered, i.e. the minimal number of iterated brackets necessary to span the whole tangent space, given by Assumption 1.4.1. We finally prove with an example that the results are optimal in general.

Eigenfunction tunneling

Our first result is the following.

Theorem 1.4.5. *Let ω be a nonempty open subset of $\mathcal{M}$. Then, there is $C, c > 0$ such that for all $(\lambda, \varphi) \in \mathbb{R}_+ \times L^2(\mathcal{M})$ satisfying $\mathcal{L}\varphi = \lambda\varphi$, we have*

$$\|\varphi\|_{L^2(\mathcal{M})} \leq Ce^{c\lambda^{k/2}} \|\varphi\|_{L^2(\omega)}. \tag{1.4.4}$$

This estimate may be read as $\|\varphi\|_{L^2(\omega)} \geq \frac{1}{C}e^{-c\lambda^{k/2}}$ for all normalized eigenfunctions φ, and hence quantizes the possible vanishing rate of eigenfunctions on any subdomain ω.

In the case $k = 1$, i.e. when $\mathcal{L}$ is an elliptic operator, the analyticity assumption 1.4.2 is not needed and the result follows from the Donnelly-Fefferman paper [57]. In this situation, it also holds on a manifold with boundary for Dirichlet eigenfunctions [58, 120] (see also [121] for other boundary conditions).

We shall also deduce from estimates of [20, Section 2.3] that the tunneling estimate (1.4.4) is optimal in the following particular setting.

Example 1.4.6 (Higher order Grushin operators on the rectangle)**.** Consider the manifold *with boundary* $\mathcal{M} = [-1, 1] \times [0, 1]$ or $\mathcal{M} = [-1, 1] \times (\mathbb{R}/\mathbb{Z})$, endowed with the Lebesgue measure dx, and for $\gamma > 0$, define the operator $\mathcal{L}_\gamma = -(\partial_{x_1}^2 + x_1^{2\gamma}\partial_{x_2}^2)$ with Dirichlet conditions on $\partial\mathcal{M}$. If $\gamma \in \mathbb{N}$, then the operator $\mathcal{L}_\gamma$ is hypoelliptic of order $k = \gamma + 1$ (i.e. Assumption 1.4.1 is fulfilled with $k = \gamma + 1$).

Proposition 1.4.7. *Consider, for $\gamma > 0$ the situation of Example 1.4.6. Assume that $\overline{\omega} \cap \{x_1 = 0\} = \emptyset$. Then there exists $C, c_0 > 0$ and a sequence (λ_j, φ_j) of eigenvalues and associated eigenfunctions of $\mathcal{L}_\gamma$ with $\lambda_j \to +\infty$ such that*

$$\|\varphi_j\|_{L^2(\omega)} \leq Ce^{-c_0\lambda_j^{\frac{\gamma+1}{2}}} \|\varphi_j\|_{L^2(\mathcal{M})}.$$

We recall that if $\gamma \in \mathbb{N}^*$, then $\mathcal{L}_\gamma$ is hypoelliptic of order $k = \gamma + 1$, so that Proposition 1.4.7 shows that, in general, one cannot expect a better estimate than that of Theorem 1.4.5.

Note that in the analytic context, the *qualitative* uniqueness:

$$\Big(\mathcal{L}\varphi = \lambda\varphi \text{ on } \mathcal{M}, \quad \varphi = 0 \text{ on } \omega\Big) \implies \varphi \equiv 0 \text{ on } \mathcal{M},$$

was proved by Bony [25], as a consequence of the Holmgren-John theorem. Removing the analyticity assumption, even for such a qualitative unique continuation property, remains a very subtle issue (see [13]).

Quantitative approximate observability of the hypoelliptic wave equation

We will need to define the Sobolev spaces related to the operators $\mathcal{L}$.

$$\mathcal{H}^s_{\mathcal{L}} = \{u \in \mathcal{D}'(\mathcal{M}),\ (1 + \mathcal{L})^{\frac{s}{2}}u \in L^2(\mathcal{M})\}, \quad s \in \mathbb{R},$$

and associated norms

$$\|u\|_{\mathcal{H}^s_{\mathcal{L}}} = \left\|(1 + \mathcal{L})^{\frac{s}{2}}u\right\|_{L^2(\mathcal{M})}, \quad s \in \mathbb{R}.$$

Let us now also introduce basic notions of sub-Riemannian geometry needed to formulate our main result. We refer to [1] for a comprehensive introduction to sub-Riemannian geometry, as well as for further developments. The so-called sub-Riemannian metric associated to the vector fields $(X_1, \cdots, X_\mathfrak{m})$ is defined, for $x \in \mathcal{M}$ and $v \in T_x\mathcal{M}$, by

$$g(x, v) := \begin{cases} \inf\left\{\sum_{i=1}^{\mathfrak{m}} u_i^2 \,\middle|\, (u_1, \cdots, u_\mathfrak{m}) \in \mathbb{R}^\mathfrak{m}, \sum_{i=1}^{\mathfrak{m}} u_i X_i(x) = v\right\} & \text{if } v \in \operatorname{span}(X_i(x), i \in \{1, \cdots, \mathfrak{m}\}), \\ +\infty & \text{if not.} \end{cases}$$

This defines for any $x \in \mathcal{M}$ a positive definite quadratic form $g(x,\cdot)$ on the horizontal space

$$\mathrm{span}(X_1(x), \cdots, X_{\mathfrak{m}}(x)).$$

Remark that, if finite, the infimum is in fact a minimum, and is realized by a vector $(u_1, \cdots, u_{\mathfrak{m}}) \in \mathbb{R}^{\mathfrak{m}}$. Given $\gamma : [0,1] \to \mathcal{M}$ an absolutely continuous path, we define its length accordingly by

$$\mathrm{length}(\gamma) := \int_0^1 \sqrt{g(\gamma(t), \dot{\gamma}(t))}\, dt.$$

The fact that this quantity is finite implies that $\dot{\gamma}(t) \in \mathrm{span}\left(X_1(\gamma(t)), \cdots, X_{\mathfrak{m}}(\gamma(t))\right)$ for almost all $t \in [0,1]$. Note also that when the vectors are linearly independent, the infinimum is among one unique u realizing the decomposition. Also, it is always finite if γ is a horizontal path (in the sense of Definition 1.4.3).

Then, this allows to define a sub-Riemannian (also called Carnot-Carathéodory) distance on $\mathcal{M}$ by

$$d_{\mathcal{L}}(x_0, x_1) = \inf\left\{\mathrm{length}(\gamma)\,|\gamma \text{ horizontal path}, \gamma(0) = x_0, \gamma(1) = x_1\right\}, \quad x_0, x_1 \in \mathcal{M}.$$

The Chow-Rashevski Theorem 1.4.2 implies that, under Assumption 1.4.1, the distance $d_{\mathcal{L}}$ is always finite on $\mathcal{M} \times \mathcal{M}$. We also define accordingly $d_{\mathcal{L}}(x_0, E) = \inf_{x_1 \in E} d_{\mathcal{L}}(x_0, x_1)$ for a point $x_0 \in \mathcal{M}$ and a subset $E \subset \mathcal{M}$.

With these definitions in hand, we may now state our main result, which concerns the quantitative unique continuation (or quantitative approximate observability) for the Hypoelliptic wave equation

$$\left\{\begin{array}{rcl} \partial_t^2 u + \mathcal{L} u & = & 0 \\ (u, \partial_t u)|_{t=0} & = & (u_0, u_1). \end{array}\right. \tag{1.4.5}$$

Theorem 1.4.8. *Let $\mathcal{L}$ as above satisfying Assumptions 1.4.1 and 1.4.2. Assume that ω is a non empty open set of $\mathcal{M}$ and let $T > \sup_{x\in\mathcal{M}} d_{\mathcal{L}}(x,\omega)$. Then, there exist $\kappa, C, \mu_0 > 0$ such that we have*

$$\|(u_0,u_1)\|_{L^2 \times \mathcal{H}_{\mathcal{L}}^{-1}} \leq Ce^{\kappa\mu^k} \|u\|_{L^2(]-T,T[\times\omega)} + \frac{1}{\mu} \|(u_0,u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2} \tag{1.4.6}$$

for all $\mu \geq \mu_0$, for any $(u_0, u_1) \in \mathcal{H}_{\mathcal{L}}^1 \times L^2$, and associated u solution of (1.4.5) on $]-T,T[$.

As before, this estimate could be stated equivalently rewritten under one of the following two formulations: for all $(u_0, u_1) \in \mathcal{H}_{\mathcal{L}}^1 \times L^2 \setminus \{(0,0)\}$, one has

$$\|(u_0,u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2} \leq Ce^{c\Lambda^k} \|u\|_{L^2(]-T,T[\times\omega)}, \quad \text{with } \Lambda = \frac{\|(u_0,u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2}}{\|(u_0,u_1)\|_{L^2 \times \mathcal{H}_{\mathcal{L}}^{-1}}}, \tag{1.4.7}$$

or

$$\|(u_0,u_1)\|_{L^2 \times \mathcal{H}_{\mathcal{L}}^{-1}} \leq C \frac{\|(u_0,u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2}}{\log\left(\frac{\|(u_0,u_1)\|_{\mathcal{H}_{\mathcal{L}}^1 \times L^2}}{\|u\|_{L^2(]-T,T[\times\omega)}} + 1\right)^{\frac{1}{k}}}, \tag{1.4.8}$$

where, in the last expression, the function $x \mapsto \left(\log(1+\frac{1}{x})\right)^{-\frac{1}{k}}$ has to be extended by zero at $x = 0^+$.

Again, in the particular situation of Example 1.4.6, the sequence of eigenfunctions of Proposition 1.4.7 shows that the exponent $e^{\kappa\mu^k}$ in (1.4.6) (resp. $e^{c\Lambda^k}$ in (1.4.7) and $\log^{-\frac{1}{k}}$ in (1.4.8)) cannot be improved in general.

Moreover, the assumption on the time $T > \sup_{x\in\mathcal{M}} d_{\mathcal{L}}(x,\omega)$ is optimal. Indeed, the hypoelliptic wave equation (1.4.5) also satisfies finite speed of propagation. The formulation of this result is similar to the one associated to the classical wave equation, but with the Riemannian distance replaced by the sub-Riemannian distance $d_{\mathcal{L}}$.

As a corollary of this result (see [155] or [112, Appendix]), we obtain the approximate controllability of the Hypoelliptic wave equation, as well as an estimate of the cost of approximate controls. Here, we only state approximate controllability to zero, which is equivalent to approximate controllability to the whole state space $\mathcal{H}_{\mathcal{L}}^1 \times L^2$ on account to the reversibility of the equation.

Corollary 1.4.9 (Cost of approximate control)**.** *For any $T > 2\sup_{x\in\mathcal{M}} d_{\mathcal{L}}(x,\omega)$, there exist $C, c > 0$ such that for any $\varepsilon > 0$ and any $(u_0, u_1) \in \mathcal{H}^1_{\mathcal{L}} \times L^2$, there exists $g \in L^2((0,T)\times\omega)$ with*

$$\|g\|_{L^2((0,T)\times\omega)} \leq Ce^{\frac{c}{\varepsilon^k}} \|(u_0,u_1)\|_{\mathcal{H}^1_{\mathcal{L}}\times L^2},$$

such that the solution of

$$\begin{cases} (\partial_t^2 + \mathcal{L})u = \mathbb{1}_\omega g & \text{in } (0,T)\times\mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0,u_1), & \text{in } \mathcal{M}, \end{cases}$$

satisfies $\|(u,\partial_t u)|_{t=T}\|_{L^2\times\mathcal{H}^{-1}_{\mathcal{L}}} \leq \varepsilon \|(u_0,u_1)\|_{\mathcal{H}^1_{\mathcal{L}}\times L^2}$.

To the authors' knowledge, these results are the first ones concerning the approximate observability/controllability of hypoelliptic waves. They furnish not only the approximate observability/controllability but also an (optimal in general) estimate of the cost. Moreover, in this context, even qualitative unique continuation did not seem to be known.

In the elliptic case $k = 1$, these can be obtained by the theory developed by Lebeau in [118] (even on a manifold with boundary). However, in this (elliptic) case, the analyticity assumption can be removed, as explained in Section 1.3.

Finally, we shall see that we prove actually a more general statement in which the term $\|(u_0,u_1)\|_{\mathcal{H}^1_{\mathcal{L}}\times L^2}$ in the right-handside of Estimate (1.4.6) can be changed into $\|(u_0,u_1)\|_{\mathcal{H}^s_{\mathcal{L}}\times\mathcal{H}^{s-1}_{\mathcal{L}}}$ for any $s > 0$, if changing the power of μ accordingly.

Quantitative approximate observability of the hypoelliptic heat equation

We now turn to the study of observability properties for solutions of the hypoelliptic heat equation

$$\begin{cases} \partial_t y + \mathcal{L}y = 0, & \text{in } (0,T)\times\mathcal{M}, \\ y(0) = y_0 & \text{in } \mathcal{M}, \end{cases} \tag{1.4.9}$$

from a subdomain $\omega \subset \mathcal{M}$. By duality, we are equivalently concerned here with different controllability properties of the following system

$$\begin{cases} (\partial_t + \mathcal{L})u = \mathbb{1}_\omega g, & \text{in } (0,T)\times\mathcal{M}, \\ u(0) = u_0, & \text{in } \mathcal{M}. \end{cases} \tag{1.4.10}$$

We provide with two main results, still under Assumptions 1.4.1 and 1.4.2:

1) For any $k \in \mathbb{N}^*$, we prove an approximate observability result in any time $T > 0$ with a frequency-depending constant of order $Ce^{c\Lambda^k}$, where $\Lambda = \frac{\|y_0\|_{\mathcal{H}^1_{\mathcal{L}}}}{\|y_0\|_{L^2}}$, or, equivalently, approximate controllability with cost $e^{\frac{c}{\varepsilon^k}}$. This is Theorem 1.4.10 below which is the analogues of Theorem 1.4.8 for parabolic equations.

2) Finally, in the very particular case $k = 2$ (including Grushin and Heisenberg operators), we prove an approximate observability/controllability property *to trajectories* in large time with a polynomial cost. This is Theorem 1.4.12 below and may be interpreted as a counterpart of the exact controllability to trajectories for the heat equation [73, 120] (case $k = 1$). There is no similar result if $k > 2$, except if we restric to more regular (Gevrey-type) data.

The first result we obtain provides the cost of approximate observability of the whole state space $L^2(\mathcal{M})$. There is no restriction for the hypoellipticity index k, but the (exponential) cost depends on this parameter.

Theorem 1.4.10. *For all $T > 0$, there exist $C, c > 0$ such that for any $y_0 \in \mathcal{H}^1_{\mathcal{L}}$ and associated solution y of (1.4.9), we have*

$$\|y_0\|^2_{L^2} \leq Ce^{c\Lambda^k} \int_0^T \int_\omega |y(t,x)|^2 \, dx\, dt, \qquad \Lambda = \frac{\|y_0\|_{\mathcal{H}^1_{\mathcal{L}}}}{\|y_0\|_{L^2}}, \tag{1.4.11}$$

and, for any $\mu > 0$,

$$\|y_0\|^2_{L^2} \leq Ce^{c\mu^k} \int_0^T \int_\omega |y(t,x)|^2 \, dx\, dt + \frac{1}{\mu^2} \|y_0\|^2_{\mathcal{H}^1_{\mathcal{L}}}. \tag{1.4.12}$$

Again, in the particular situation of Example 1.4.6, the sequence of eigenfunctions of Proposition 1.4.7 shows that the exponent $e^{\kappa\mu^k}$ in (1.4.12) (resp. $e^{c\Lambda^k}$ in (1.4.11)) cannot be improved in general.

This theorem generalizes the results of Fernandez-Cara-Zuazua and Phung [67, 146] in the elliptic case $k = 1$. Yet, in this framework, the analyticity was not necessary (as in all above stated results in the case $k = 1$) and the setting can be relaxed (uniform dependence of the constants with respect to lower order terms and to the time T, boundary value problems...).

As a corollary (see [112, Appendix]), we obtain, given an initial state and a target state both belonging to the space $L^2(\mathcal{M})$, and given a precision ε, the existence of a control function bringing the initial state in an ε-neighborhood of the target (in appropriate topology). We obtain as well an estimate of the cost of the control.

Corollary 1.4.11 (Cost of approximate control to the state space)**.** *For any $T > 0$, there exist $C, c > 0$ such that for any $\varepsilon > 0$ and any $u_0 \in L^2(\mathcal{M}), u_1 \in L^2(\mathcal{M})$, there exists $g \in L^2((0,T) \times \omega)$ with*

$$\|g\|_{L^2((0,T)\times\omega)} \le Ce^{\frac{c}{\varepsilon^k}} \left\|e^{-T\mathcal{L}}u_0 - u_1\right\|_{L^2(\mathcal{M})},$$

such that the solution of (1.4.10) issued from u_0 satisfies

$$\|u(T) - u_1\|_{\mathcal{H}_{\mathcal{L}}^{-1}} \le \varepsilon \left\|e^{-T\mathcal{L}}u_0 - u_1\right\|_{L^2(\mathcal{M})}.$$

In this statement, $e^{-T\mathcal{L}}u_0$ stands for the solution at time T to Equation (1.4.10) with $g = 0$.

Our second result concerning the hypoelliptic heat equation is, as opposed to the previous one, concerned with final state approximate observability (or equivalently an approximate controllability to trajectories) with a *polynomial* cost, and is restricted to the case $k = 2$.

Theorem 1.4.12. *Assume that $k = 2$. There exist T_0 so that for $T > T_0$, there exists $C > 0$ and $\beta > 0$ such that for all $\eta > 0$ and $y_0 \in L^2(\mathcal{M})$ and associated solution y to (1.4.9),*

$$\|y(T)\|_{L^2}^2 \le \frac{C}{\varepsilon^\beta} \int_0^T \int_\omega |y(t,x)|^2 \, dt \, dx + \varepsilon \|y_0\|_{L^2}^2.$$

This result gives directly the following corollary concerning approximate controllability to trajectories (or, equivalently, to zero) at a polynomial cost (see again [112, Appendix]).

Corollary 1.4.13 (Cost of approximate control to trajectories if $k = 2$)**.** *Assume that $k = 2$, and let $T_0 > 0$ as in Theorem 1.4.12. For any $T > T_0$, there exists $C > 0$, and $\beta > 0$ so that for all $\varepsilon > 0$, we have the following statement: for any $u_0, \tilde{u}_0 \in L^2$, there exists $g \in L^2((0,T) \times \omega)$ with*

$$\|g\|_{L^2((0,T)\times\omega)} \le \frac{C}{\varepsilon^\beta} \|u_0 - \tilde{u}_0\|_{L^2},$$

such that the associated solution u of (1.4.10) satisfies

$$\left\|u(T) - e^{-T\mathcal{L}}\tilde{u}_0\right\|_{L^2(\mathcal{M})} \le \varepsilon \|u_0 - \tilde{u}_0\|_{L^2}.$$

Decay of damped hypoelliptic equations

We finally present a result concerning the damped hypoelliptic wave equation

$$\begin{cases} (\partial_t^2 + \mathcal{L} + 1_\omega \partial_t)u = 0, & \text{on } (0, +\infty) \times \mathcal{M}, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), & \text{on } \mathcal{M}, \end{cases} \tag{1.4.13}$$

where $\omega \subset \mathcal{M}$ is a non-empty open set. Solutions to (1.4.13) enjoy formally the following dissipation identity (obtained by taking the inner product of (1.4.13) with $\partial_t u$ and integrating on $(0, T)$):

$$E(u(T)) - E(u(0)) = -\int_0^T \int_\omega |\partial_t u(t,x)|^2 ds(x)\, dt, \quad E(u) = \frac{1}{2}\left(\sum_{i=1}^{\mathfrak{m}} \|X_i u\|_{L^2(\mathcal{M})}^2 + \|\partial_t u\|_{L^2(\mathcal{M})}^2\right).$$

An important question is then to understand at which rate the energy decays. We denote the damped wave operator $\mathcal{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathcal{L} & -1_\omega(x) \end{pmatrix}$. In this context, we obtain a logarithmic decay rate with power k, which can be proved to be optimal in general.

Theorem 1.4.14 (Decay rates for damped hypoelliptic waves)**.** *Assume Assumptions 1.4.1 and 1.4.2. Then, for all* $(u_0, u_1) \in \mathcal{H}^1_{\mathcal{L}} \times L^2$*, the associated solution to* (1.4.13) *satisfies* $E(u(t)) \to 0$*. Moreover, for all* $j \in \mathbb{N}^*$*, there exists* $C_j > 0$ *such that for all* $(u_0, u_1) \in D(\mathcal{A}^j)$*, the associated solution to* (1.4.13) *satisfies*

$$E(u(t))^{\frac{1}{2}} \leq \frac{C_j}{\log(t+2)^{j/k}} \left\| \mathcal{A}^j(u_0,u_1) \right\|_{\mathcal{H}^1_{\mathcal{L}} \times L^2}, \qquad \text{for all } t \geq 0.$$

This result is the analogue in the hypoelliptic setting of the Lebeau theorem [119] (case $k = 1$). Note that the proof is also gives the same result for $k = 1$ without analyticity and with Dirichlet boundary condition since it gives an abstract implication from estimates of the form of Theorem 1.3.1 or 1.4.8 to logarithmic decay of the damped equation.

We obtain similar results for damped Schrödinger equation $(i\partial_t + \mathcal{L} + i1_\omega \partial_t)v = 0$ and the damped plate equation $(\partial_t^2 + \mathcal{L}^2 + 1_\omega \partial_t)v = 0$. In order to keep the article reasonably short, we refer to **[A17]** for these results.

Idea of the proofs

Quantitative unique continuation for the hypoelliptic wave equation

The proof of Theorem 1.4.8 is based on the general strategy described in Section 1.2 for quantifying and propagating unique continuation properties. We only use here the "Holmgren-John" case, i.e. when the operator has analytic coefficients as described in Remark 1.2.3.

Here, when compared to the case of the classical wave equation described in Section 1.3, two additional difficulties arise: one being of geometric nature, and the other one related to the compatibility between the energy spaces associated to $\mathcal{L}$ and those dealt with in **[A10]** .

Let us first describe the geometric difficulty. The proof is inspired by the case of the classical wave equation explained in Section 1.3: the idea is, given a point $x_0 \in \mathcal{M}$, to take any path $\gamma : [0, 1] \to \mathcal{M}$ with $\gamma(0) = x_0$ and $\gamma(1) \in \omega$ (observation set), of length sufficiently small, and then to construct a family of appropriate non characteristic hypersurfaces in these coordinates near $[-T, T] \times \gamma$. There, we apply the general Theorem 1.2.5, which allows to bound the solution u to $(\partial_t^2 - \Delta)u = 0$ in a neighborhood of $(t, x) = (0, x_0)$ by u in $[-T, T] \times \omega$.

Here, due to the non definiteness/ellipticity of the operator $\mathcal{L}$, we are not able to construct *global* coordinates near *any* path γ together with appropriate noncharacteristic hypersurfaces, in which to apply the results of **[A10]** . To overcome this difficulty, we do not consider *any* path between x_0 and ω, but rather only so called *normal geodesics*, that is, projections on $\mathcal{M}$ of hamiltonian curves of the principal symbol of the operator $\mathcal{L}$. The existence of such paths γ (minimizing the sub-Riemannian distance) from any point x_0 to ω is a well-known result in sub-Riemannian geometry, proved by Rifford and Trélat [153]. Then, locally near a point of γ, the introduction of normal geodesic coordinates allows us to define local coordinates in which to apply a *local* version of a slight variant of Theorem 1.2.5.

Note that this single geometric construction, combined with the usual Holmgren-John theorem would be enough to prove the qualitative uniqueness statement.

These arguments eventually allows to prove an estimate of the form

$$\|u\|_{L^2(]-\varepsilon,\varepsilon[\times\mathcal{M})} \leq Ce^{\kappa\mu} \|u\|_{L^2(]-T,T[\times\omega)} + \frac{C}{\mu} \|u\|_{H^1(]-T,T[\times\mathcal{M})}, \tag{1.4.14}$$

for μ large and u solution to $(\partial_t^2 + \mathcal{L})u = 0$. This estimate is the same as (1.3.2) (after energy estimates) for the wave equation.

This leads us to the second main difficulty we have to face in the proof of Theorem 1.4.8. Whereas the left hand-side of (1.4.14) is bounded from below by the natural $L^2 \times \mathcal{H}^{-1}_{\mathcal{L}}$ norm of the data, the right hand-side is not directly linked to their $\mathcal{H}^1_{\mathcal{L}} \times L^2$ norm. More precisely, the hypoelliptic estimates of Rothschild and Stein [157] of Theorem 1.4.4 imply that $\|u\|_{H^1(]-T,T[\times\mathcal{M})} \leq C\,\|(u_0,u_1)\|_{\mathcal{H}^k_{\mathcal{L}}\times\mathcal{H}^{k-1}_{\mathcal{L}}}$. This provides a weaker version of Theorem 1.4.8 which has exactly the same form as in the case of the wave equation (cost $e^{\kappa\mu}$), but with the norm $\|(u_0,u_1)\|_{\mathcal{H}^k_{\mathcal{L}}\times\mathcal{H}^{k-1}_{\mathcal{L}}}$ in the right hand-side. This weaker version is however interesting for itself since the proof is much less involved.

To obtain the estimate of Theorem 1.4.8 (and in fact a family of such estimates with any $\mathcal{H}^s_{\mathcal{L}}\times\mathcal{H}^{s-1}_{\mathcal{L}}$, $s>0$, in the right hand-side), we thus need to work with a version of (1.4.14) still containing frequency cutoff localization and an $e^{-c\mu}$ small remainder (instead of the $1/\mu$ one). These low-frequency-with-exponentially-small-remainder estimates are then combined with the spectral representation of solutions to $(\partial_t^2+\mathcal{L})u=0$ in order to gain back derivatives in the remainder term.

From waves to eigenfunction tunneling

Theorems 1.4.5 is simply deduced from Theorem 1.4.8 (under the equivalent form of estimate (1.4.7)) by using a particular solution to the wave equation (1.4.5), namely $u(t,x)=\cos(\sqrt{\lambda}t)\varphi(x)$. It only remains to notice that the frequency functions $\Lambda = \frac{\|(u_0,u_1)\|_{\mathcal{H}^1_{\mathcal{L}}\times L^2}}{\|(u_0,u_1)\|_{L^2\times\mathcal{H}^{-1}_{\mathcal{L}}}}$ is of order $\sqrt{\lambda}$ for $(u_0,u_1)=(\varphi,0)$, where $\mathcal{L}\varphi=\lambda\varphi$.

From wave-like to heat-like equations

The proofs of Theorems 1.4.10 and 1.4.12 follow the general idea that the controllability/observability properties for hyperbolic equations implies controllability/observability properties for their parabolic counterpart, see [65, 66, 140, 158] (see also [120]).

This has been named as "transmutation methods" by Luc Miller [140]. Here, we use the method developed in [65] (itself relying on a Lebeau-Robbiano strategy [120]). In that paper, Ervedoza and Zuazua deduced the (exact final time) observability of the heat equation (known from [73, 120]) from the approximate observability estimate for waves (namely the analogue of Theorem 1.4.8) as proved in [148] (with loss) or above Theorem 1.3.1 (without loss). They prove the following result.

Proposition 1.4.15 ([65, 66]). *Let $T,S>0$ and $\alpha>2S^2$. Then, there exists some kernel function $k_T(t,s)$ such that if y is solution of the heat equation (1.4.9), then $u(s)=\int_0^T k_T(t,s)y(t)dt$ is solution of*

$$\begin{cases} \partial_s^2 u+\mathcal{L}u &= 0, \quad \text{for } s\in]-S,S[,\\ (u,\partial_s u)|_{s=0} &= \left(0,\int_0^T \partial_s k_T(t,0)y(t)dt\right)=\left(0,\int_0^T e^{-\alpha\left(\frac{1}{t}+\frac{1}{T-t}\right)}y(t)dt\right); \end{cases}$$

The authors also provide with useful estimates on this kernel.

Both proofs of Theorems 1.4.10 and 1.4.12, follow a similar strategy and apply Theorem 1.4.8 to a solution u of the wave equation constructed with Proposition 1.4.15. Yet, one important difference between Theorems 1.4.10 and 1.4.12 is that the first one is an estimate of the solution at initial time while the second one is at final time. The proofs therefore differ slightly depending on whether or not they use the natural decay of the heat equation. In the first case, Proposition 1.4.15 is applied directly to the solution of the heat equation that we want to observe. In the second case, Proposition 1.4.15 is applied to the low frequency part of the solution, more in the spirit of the "Lebeau-Robbiano" strategy [120]. Let us now give a few more details in each situation.

The proof of Theorem 1.4.10 relies on the transmutation technique of Proposition 1.4.15 applying directly the transmutation kernel $k_T(t,s)$ to the full solution y to the heat equation: $u(t)=\int_0^T k_T(t,s)y(s)ds$ is a solution to the wave equation. We then prove a fine asymptotic analysis of $\int_0^T k_T(0,s)e^{-\lambda s}ds$ for high frequencies together with convexity estimates to bound the "frequency function" of $u(0)$, namely $\frac{\|u(0)\|_{\mathcal{H}^1_{\mathcal{L}}}}{\|u(0)\|_{L^2}}$ by the frequency function of $y(0)$. The proof of this result via a direct transmutation method seems to be new, even for the classical heat equation. The usual proofs [67, 146] rather rely on the exact final time observability estimate, which does not hold here in general.

The proof of Theorems 1.4.12 is very close to that of [65]. Using again the transmutation result together with our estimate (1.4.7) for the wave equation, we obtain a “Lebeau-Robbiano-like” estimate, that is a low frequency observability with a good estimates of the cost with respect to the frequency. More precisely, defining the low frequency spaces $E_\lambda = \operatorname{span}\{\varphi_j, \lambda_j \le \lambda\}$, we obtain the following low-frequency observability estimate, with a precise estimation of the observability constant with respect to the cutoff frequency.

Lemma 1.4.16 (“Lebeau-Robbiano-like” estimates). *There exist $C, \gamma > 0$ such that for any $T > 0, \lambda \ge 0$, for every $y_0 \in E_\lambda$ and associated solution y to (1.4.9), we have*

$$\|y(T)\|_{L^2}^2 \le \frac{C}{T} e^{\left(2\gamma\lambda^{k/2} + \frac{C}{T}\right)} \int_0^T \int_\omega |y(t,x)|^2 \; dt \; dx. \tag{1.4.15}$$

Moreover, there exists $c_0 > 0$ such that for any $T > 0$ there exists $C = C_T > 0$ such that for any $\lambda \ge 0$, any $y_0 \in E_\lambda$ and associated solution y to (1.4.9), we have

$$\|y_0\|_{L^2}^2 \le C e^{2c_0\lambda^{k/2}} \int_0^T \int_\omega |y(t,x)|^2 \; dt \; dx. \tag{1.4.16}$$

Note the difference between (1.4.15) and (1.4.16): the former estimates the *final* data at time T (with explicit dependence with respect to time) whereas the latter estimates the *initial* data. Estimate (1.4.16) is used in the proof of Theorem 1.4.10 whereas Estimate (1.4.15) is used in the proof of Theorem 1.4.12.

The cost $e^{2c_0\lambda^{k/2}}$ in these low-frequency observability estimates has to be compared to the dissipation for high frequencies $\sqrt{\lambda_j} \ge \sqrt{\lambda}$, namely $e^{-t\lambda}$. Hence, we see that the cases $k = 1$ (classical heat equation, already discussed), $k = 2$ (Grushin, Heisenberg...), and $k > 2$ display very different features:

1) In case $k = 2$, the cost of observation of low frequencies $e^{c\lambda}$ and the parabolic dissipation for high frequencies $e^{-t\lambda}$ have the same strength: in this case, we need to wait a time long enough so that the dissipation “beats” the cost of the observability (essentially $t > c$). Moreover, the iterative procedure devised in [120] in order to control/observe *all* frequencies in finite time cannot converge here: each step would need a time $t > c$. Yet, it allows to obtain the approximate controllability result of Theorem 1.4.12.

2) In case $k > 2$, the dissipation for high frequencies $e^{-t\lambda}$ has no chance to compete with the cost of observation of low frequencies $e^{c\lambda^{k/2}}$. The only chance to obtain some positive result is to assume that the initial data are in some Gevrey-type space that allows to compensate for the cost of low frequencies $e^{c\lambda^{k/2}}$. This leads to approximate control results in Gevrey-type spaces (with polynomial cost) that we have chosen not to state here for simplicity. We refer to **[A19]** , Theorem 1.18, for more details.

From waves to resolvent estimates and decay for damped equations

The proof of Theorem 1.4.14 consists in several reductions to resolvent type estimates. First, we relate in an abstract setting decay of damped equations to resolvent estimates according to [19]. More precisely, in the proof of Theorem 1.4.14, it suffices show an estimate of the form $\left\|(is - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H}^1_{\mathcal{L}} \times L^2)} \le C e^{\kappa |s|^k}$ for all $|s| \ge 1$. We then prove that the latter is a consequence of the following estimate:

$$\|v\|_{L^2(\mathcal{M})} \le C e^{\kappa\lambda^k} \left(\|v\|_{L^2(\omega)} + \left\|(\mathcal{L} - \lambda^2)v\right\|_{L^2(\mathcal{M})} \right), \quad \text{for all } v \in \mathcal{H}^2_{\mathcal{L}}, \lambda \ge \lambda_0. \tag{1.4.17}$$

To obtain (1.4.17), for $v \in \mathcal{H}^2_{\mathcal{L}}$ with $(\mathcal{L} - \lambda^2)v = f$, we construct, as in Section 1.4 a particular solution to the wave equation with source term, namely $u(t,x) = \cos(\sqrt{\lambda}t)\varphi(x)$ solution of $(\partial_t^2 + \mathcal{L})u = \cos(\sqrt{\lambda}t)f$. A slight variant of Theorem 1.4.8 with source term applied to u allows us to obtain (1.4.17).

Some perspectives

1) Lemma 1.4.16 are pretty much like estimates called "spectral estimates" (or also Lebeau-Jerison, Lebeau-Robbiano types estimates). They would take certainly take the form

$$\|y_0\|_{L^2} \leq Ce^{c\lambda^{k/2}} \|y_0\|_{L^2(\omega)}$$

for any $y_0 \in E_\lambda$. It would be very interesting to obtain such estimates and seems doable with the technics we developed. Following the Lebeau-Robbiano methods, they would require some estimates with a boundary, but for operators of the form $\partial_s^2 - \mathcal{L}$.

2) All the result presented in this part are stated without boundary. A very natural question is then to obtain similar results in the case with boundary. Yet, we would need to solve several problems. First, the theory of boundary value problems for hypoelliptic operators is much more complicated. Second, our estimates involve an operator $e^{-\frac{D_a^2}{2\tau}}$ where x_a is the analytic variable which was all the variables in our cases. The link between the regularizing operator $e^{-\frac{D_a^2}{2\tau}}$ and the trace estimates could be very complicated. Maybe a first step could be when the variable x_a is a partial variable and is tangential to the boundary.

3) The case $k = 2$, as the Grushin case clearly appears as a limit case. It has been further explored (see [20, 21, 110]) and some geometric conditions are necessary to obtain the controllability of the heat equation. It seems that our method does not see too much this kind of geometry, so further ideas are certainly missing to see the difference between the approximate controllability at polynomial cost that we obtain in general and the exact controllability that is obtained in some particular geometric configurations.

CHAPITRE 2

Some dynamical system methods for unique continuation and stabilization of nonlinear wave equations

In this chapter, we intend to present the general method we developed with Romain Joly for proving the unique continuation property for some nonlinear systems and state its consequences for stabilization and control. We applied it to nonlinear waves, but we expect it to apply to more equations. We use some ideas coming from Dynamical Systems that aimed at proving some regularity properties of the compact attractor. It was developed in **[A3]** , **[A4]** and **[A14]** .

2.1 Introduction

Most of this chapter will be focused on the following semilinear damped wave equation or some variant

$$\begin{cases} \Box u + \gamma(x)\partial_t u + f(u) = 0 & (t,x) \in \mathbb{R}_+ \times \Omega \ , \\ u(t,x) = 0 & (t,x) \in \mathbb{R}_+ \times \partial\Omega \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) & \in H^1_0(\Omega) \times L^2(\Omega) \end{cases} \tag{2.1.1}$$

where $\Box = \partial^2_{tt} - \Delta$ with Δ being the Laplace-Beltrami operator with Dirichlet boundary conditions. The domain Ω is a compact connected C^∞ Riemannian manifold with boundaries of dimension $d \leq 3$. Let $X = H^1_0(\Omega) \times L^2(\Omega)$ be the energy space.

We could also consider many variants: the domain could be infinite (like a compact perturbation of the Euclidian domain), the dimension could be different than 3, we could consider different boundary conditions. Yet, we have chosen this context to simplify the discussion. We refer to **[A3]** where some other cases are treated.

The nonlinearity $f \in C^1(\mathbb{R}, \mathbb{R})$ is assumed to be defocusing, energy subcritical and such that 0 is an equilibrium point. More precisely, we assume that there exists $C > 0$ such that

$$\begin{aligned} f(0) &= 0, && \text{0 is an equilibrium point} \\ sf(s) &\geq 0, && \text{defocusing} \\ |f(s)| \leq C(1+|s|)^p \quad \text{and} \quad |f'(s)| &\leq C(1+|s|)^{p-1}, && \text{energy subcritical} \end{aligned} \tag{2.1.2}$$

with $1 \leq p < 5$ when $d = 3$ and $1 \leq p < +\infty$ if $d = 2$.

The assumption "energy subcritical" is the natural one to obtain local well-posedness theory once sufficient Strichartz estimates are known, which is the case here, see [24, 33]. The defocusing assumption allows us to obtain global solutions. The assumption $f(0) = 0$ is more important for a dynamical point of view. It will be sometimes relaxed when we want to allow some more complicated dynamics.

The damping $\gamma \in L^\infty(\Omega)$ is a non-negative function. We assume γ is effective on ω, that is, there exist an open set $\omega \subset \Omega$, $\alpha > 0$ such that

$$\forall x \in \omega \ , \ \gamma(x) \geq \alpha > 0 \tag{2.1.3}$$

We will sometimes (but not always, especially in Section 2.3) assume that ω satisfies the geometric control condition introduced in [150] and [16]

- (GCC) There exists $L > 0$ such that any generalized geodesic of Ω of length L meets the set ω where the damping is effective.

Note that we assume implicitly that the geodesics of $\overline{\Omega}$ do not have contact of infinite order with $\partial\Omega$ which ensures that the generalized geodesics are well defined in a unique way and for all time, see [135]. We refer to the comments in Section 0.2 of the Introduction.

The associated energy of the system $E \in C^0(X, \mathbb{R}_+)$ is given by

$$E(u) := E(u, \partial_t u) = \frac{1}{2}\int_\Omega (|\partial_t u|^2 + |\nabla u|^2) + \int_\Omega V(u) \ , \tag{2.1.4}$$

where $V(u) = \int_0^u f(s)ds$. Due to Assumption (2.1.2) and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, this energy is well defined and moreover, if u solves (2.1.1), we have, at least formally,

$$\partial_t E(u(t)) = -\int_\Omega \gamma(x)|\partial_t u(x,t)|^2 \, dx \ \leq \ 0 \ . \tag{2.1.5}$$

As explained in Section 0.1 of the introduction, some easy identities, see (0.1.10) imply that the system is dissipative. We are interested in the exponential decay (ED) of the energy of the nonlinear damped wave equation (2.1.1), as described in Section 0.1

Let us now present the previous literature about this question and what was the missing argument to get the expected result, that is the stabilization under the geometric control condition or even weaker assumptions.

The stabilization property (ED) for Equation (2.1.1) has been studied in [83], [176], [177] and [50] for $p < 3$. For $p \in [3, 5)$, our main reference is the work of Dehman, Lebeau and Zuazua [54]. This work is mainly concerned with the stabilization problem previously described, on the Euclidean space $\mathbb{R}^3$ with flat metric and stabilization active outside of a ball, or on a bounded domain, but with stabilization close to the full boundary.

The main purpose of this chapter is to extend their result to more general geometries where multiplier methods cannot be used or do not give the optimal result with respect to the geometry, and to give further applications. Other stabilization results for the nonlinear wave equation can be found in [8] and the references therein. Some works have been done in the difficult critical case $p = 5$, we refer to [51] and [T5] .

Let us briefly present the scheme of the proof of the result of [54]. They prove the exponential decay of solutions of (2.1.1) on Ω bounded domain with the Euclidian metric and where ω a neighborhood of the full boundary.

As presented in Section 0.1 of the introduction, the main point is to prove an observability estimate

$$E(u(0)) \leq C \int_0^T \int_\Omega \gamma_\omega(x)|\partial_t u(x,t)|^2 \, dx \, dt, \tag{2.1.6}$$

for any solution with initial datum $E(U_0) \leq E_0$. It was proved by an argument by contradiction, following the compactness uniqueness methods described in Section 0.2. It uses two main ingredients:

1) **Propagation of compactness:** u_n converges strongly on ω implies that it converges strongly in Ω

2) **Unique continuation:** $u \equiv 0$ is the unique solution of

$$\begin{cases} \Box u + f(u) &= 0 \\ \partial_t u &= 0 \quad \text{on } [0,T] \times \omega \ . \end{cases} \tag{2.1.7}$$

For the propagation of compactness 1), the authors of [54] use the microlocal defect measure which is very efficient and allows to obtain the compactness in a more general context, assuming the geometric control condition. Using propagation of regularity, they also manage to prove that the solutions of (2.1.7) are actually smooth if ω satisfies the geometric control condition. .

In particular, even if the results of [54] concern a specific geometry, following carefully, the proof, we can extract a rough statement of the type:

"geometric control condition" + "unique continuation" $\Longrightarrow$ "exponential decay".

for defocusing subcritical nonlinearities. This type of implication is even stated explicitly in some related works for the nonlinear Schrödinger equation [52] and **[T2]** .

So, the remaining part is the unique continuation step 2).

Why the unique continuation is a problem?

For proving the unique continuation, the usual argument is to notice that $w = \partial_t u$ is a solution of

$$\left\{ \begin{array}{rcll} \Box w + Vw & = & 0 & \text{on} \quad [0,T]\times\Omega \\ w & = & 0 & \text{on} \quad [0,T]\times\omega. \end{array} \right. \tag{2.1.8}$$

with $V = f'(u)$.

At this point, in the geometric configuration of [54], that is ω a neighborhood of the full boundary, the authors could use unique continuation results based on classical Carleman estimates (that is the Hörmander theorem corresponding to $n_a = 0$ in the notations of Section 1.2). Indeed, it is not so complicated to construct some foliation of pseudoconvex hypersurfaces that allow to obtain the unique continuation. For instance, for any $c \in [0,1)$, $(t_1, x_1) \in \mathbb{R}^{1+d}$, the function $\Psi(t,x) = |x-x_1|^2 - c^2(t-t_1)^2$ can be checked to satisfy the pseudoconvexity condition of Definition (1.2.1) with $n_a = 0$ for the wave operator $\Box$. Using a compactness argument and letting c close to one, one may deduce the following global result.

Proposition 2.1.1. *Let Ω be a bounded domain of $\mathbb{R}^d$ and ω be an open neighborhood of $\partial\Omega$ in Ω, and fix any $x_1 \in \mathbb{R}^d$. Then, assuming*

$$T > \sup\{|x - x_1|, x \in \Omega \setminus \omega\},$$

and letting $V \in L^\infty((0,T)\times\Omega;\mathbb{C})$, any solution to (2.1.8) satisfies

$$(u, \partial_t u)|_{t=0} = (0,0) \text{ in } \Omega.$$

We refer to **[P1]** (Theorem 2.8, Section 2) of the Survey we did for a proof of this result (see also Theorem 1.2 of **[A14]** under the assumption that there exists a pseudo-convex foliation of Ω.)

This allow to conclude the proof of the unique continuation and therefore of the exponential decay in the case of [54], that is ω a neighborhood of $\partial\Omega$. But, we would liked to consider more general geometric situations.

Yet, for other configuration, the geometric assumptions resulting from the use of Hörmander theorem for the unique continuation are not very natural and are stronger than (GCC). For instance for a flat metric, the pseudoconvexity condition of an hypersurface $\{\Psi = 0\}$ for the wave operator writes

$$X_t^2 = |X_x|^2 \quad \text{and} \quad d\Psi(x_0)(X) = 0 \Longrightarrow \text{Hess}\,\Psi(x_0)(X,X) > 0 \quad \text{for all } X = (X_t, X_x) \in \mathbb{R}^{1+d}\setminus\{0\}.$$

and imply a kind of convexity of the hypersurface. The usual geometric assumption that appear are often of "multiplier type" (or Morawetz type) that is ω is a neighbourhood of $\{x \in \partial\Omega \,|\, (x - x_0)\cdot n(x) > 0\}$ which are known to be stronger than the geometric control condition (see [137] for a discussion about the links between these assumptions). Moreover, on curved spaces, this type of condition often needs to be checked by hand in each situation, which is mostly impossible in general. Moreover, concerning the unique continuation (2.1.8) with V even smooth, the classical counterexamples of Alinhac-Bahouendi [2, 5, 6] refined by Hörmander [95] are quite striking and prove that the Hörmander condition of pseudoconvexity is not far from being optimal for local unique continuation.

For any $s > 1$ and $d \geq 2$, they construct some u and $V \in \mathcal{G}^s(B_{\mathbb{R}^{1+d}}(0,1);\mathbb{C})$ (Gevrey functions) so that

$$\partial_t^2 u - \Delta u = Vu,$$
$$\operatorname{supp}(u) = \{(t, x_1, \dots, x_d); x_1 \geq 0\}.$$

It seems to indicate that in geometrical situations where the strong pseudoconvexity of the hypersurface is not satisfied, we can not expect local unique continuation for potential V that does not have analyticity.

Another counterexample that kills one possible strategy to prove unique continuation for (2.1.7) is the one of Métivier in [136]. He proved that a nonlinear version of the Holmgren theorem fails in general.

So, at that point, a natural idea for proving unique continuation for (2.1.7) would be to try to apply the unique continuation theory with partial analyticity developed by Tataru [165, 167]-Robbiano-Zuily [156] - Hörmander [94], described in Section 1.2. In particular, a globalization of the local unique continuation property given by this theory implies

Theorem 2.1.2. *Let Ω be a smooth bounded domain. Let $V \in L^\infty([0,T]\times\Omega)$ analytic in time. For any nonempty open subset ω of Ω and assuming $T > 2\mathcal{L}(\mathcal{M},\omega)$, the unique solution u in $H^1([0,T]\times\Omega)$ of*

$$\begin{cases} \Box u + Vu &= 0 \quad \text{on} \quad [0,T]\times\Omega \\ u &= 0 \quad \text{on} \quad [0,T]\times\partial\Omega, \\ u &= 0 \quad \text{on} \quad [0,T]\times\omega. \end{cases} \tag{2.1.9}$$

is $u = 0$.

But the main problem is that for a solution of the nonlinear wave equation, $V = f'(u)$ has no reason to be analytic in time. The semigroup of the nonlinear wave equation is not smoothing.

It turns out that under some very general geometric assumptions, we can actually obtain this analyticity for the pathological solutions of 2.1.7 in infinite time $T = +\infty$. That will use some smoothing in infinite time proved by Hale and Raugel in [81] for compact trajectories and some refinment with weaker assumptions. These arguments of Dynamical systems were originally performed to prove the regularity of the global attractor. We will also get results in this direction. This will be sufficient to obtain the decay of the nonlinear damped wave equation.

More precisely, the results we obtain are as follows:

1) Section 2.2 presents the results we obtain in the case where the geometric control condition is satisfied. We obtain exponential decay when zero is the only equilibrium and the existence of a compact global attractor otherwise.

2) In Section 2.3, we presents some results where some weaker assumptions are required. We had to present a new proof of "asymptotic smoothness" in this context. We also obtain the stabilization to zero.

3) In Section 2.4, we present applications to global controllability. The idea is to travel through the attractor to get to the expected target.

2.2 Damped nonlinear wave equations: the case with geometric control condition

In all this Section, for simplicity, we only consider the case where Ω is of dimension $d = 3$.

The main theorem we obtain in **[A3]** is the following.

Theorem 2.2.1. *Assume that the damping γ satisfies (2.1.3) and the geometric control condition (GCC). If f is real analytic and satisfies (2.1.2) with $1 \leq p < 5$. , then the exponential decay property (ED).*

Theorem 2.2.1 applies for nonlinearities f which are globally analytic. Of course, the nonlinearities $f(u) = |u|^{p-1}u$ are not analytic if $p \notin \{1, 3\}$, but we can replace these usual nonlinearities by similar ones as $f(u) = (u/\text{th}(u))^{p-1}u$, which are analytic for all $p \in [1, 5)$. Note that the estimates (2.1.2) are only required for $s \in \mathbb{R}$, so that it does not imply that f is polynomial.
The statements of the theorem lead to some remarks.

- Of course, our results and their proofs should easily extend to any space dimension $d \geq 3$ if the exponent p of the nonlinearity satisfies $p < (d+2)/(d-2)$.
- Actually, it may be possible to get $\lambda > 0$ in (ED) uniform with respect to the size of the data. We can take for instance $\lambda = \tilde{\lambda} - \varepsilon$ where $\tilde{\lambda}$ is the decay rate of the linear equation. The idea is that once we know the existence of a decay rate, we know that the solution is close to zero for a large time. Then, for small solutions, the nonlinear term can be neglected to get almost the same decay rate as the linear equation. We refer for instance to **[T3]** in the context of KdV equation. Notice that the possibility to get the same result with a constant K independent on E_0 is an open problem.
- The geometric control condition is known to be not only sufficient but also necessary for the exponential decay of the linear damped equation. The proof of the optimality uses some sequences of solutions which are asymptotically concentrated outside of the damping region. We can use the same idea in our nonlinear stabilization context. First, the observability for a certain time eventually large is known to be equivalent to the exponential decay of the energy. This was for instance noticed in [51] Proposition 2, in a similar context. Then, we take as initial data the same sequence that would give a counterexample for the linear observability. The linearizability property (see [74]) allows to obtain that the nonlinear solution is asymptotically close to the linear one. This contradicts the observability property for the nonlinear solution as it does for the linear case. Hence, the geometric control condition is also necessary for the exponential decay of the nonlinear equation.
- It can be proved that the property of Theorem 2.2.1 holds not only for all f analytic with the desired growth, but also for a generic set (in the sense of Baire) for a well-chosen topology of C^1 functions satisfying (2.1.2)

Yet, in the next Section 2.3, we will see that if we allow a smoother initial datum and sometimes a weaker decay, it is possible to prove the stabilization to zero with some weaker geometric assumptions.

Moreover, we give two applications of our results in both contexts of control theory and dynamical systems. First, as is usual in control theory, some results of stabilization can be coupled with local control theorems to provide global controllability in large time.

Theorem 2.2.2. *Assume that f satisfies the conditions of Theorem 2.2.1. Let $R_0 > 0$ and ω satisfying the geometric control condition (GCC). Then, there exists $T > 0$ such that for any (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1)$ in $H_0^1(\Omega) \times L^2(\Omega)$, with*

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leq R_0; \qquad \|(\tilde{u}_0, \tilde{u}_1)\|_{H^1 \times L^2} \leq R_0$$

there exists $g \in L^\infty([0,T], L^2(\Omega))$ supported in $[0,T] \times \omega$ such that the unique strong solution of

$$\begin{cases} \Box u + f(u) &= g \quad \text{on} \quad [0,T] \times \Omega \\ (u(0), \partial_t u(0)) &= (u_0, u_1). \end{cases}$$

satisfies $(u(T), \partial_t u(T)) = (\tilde{u}_0, \tilde{u}_1)$.

The proof of the local controllability close to zero was done in [54] and their proof applies without modification. Therefore, once the stabilization property is obtained, the proof of the global controllability is actually the same as in [54] and is quite usual, see Figure 2.1. It goes as follows: the stabilization Theorem 2.2.1 allows to get a control of the form $g = \gamma \partial_t u$ that bring any initial state as close as we want to zero, in large time. The local controllability allows to get exactly to zero once we are small enough. That means that we have proved that we can steer any state (u_0, u_1)

to zero in large time. We can also therefore find a control that drives $(\tilde{u}_0, -\tilde{u}_1)$ to zero in large time. Since the equation is invariant t to $-t$, which changes the initial state $(\tilde{u}_0, -\tilde{u}_1)$ to $(\tilde{u}_0, \tilde{u}_1)$ for any we can find a control from zero to $(\tilde{u}_0, \tilde{u}_1)$. By concatenating the two solutions, we get the expected control. The second application of our results concerns the existence of a compact global

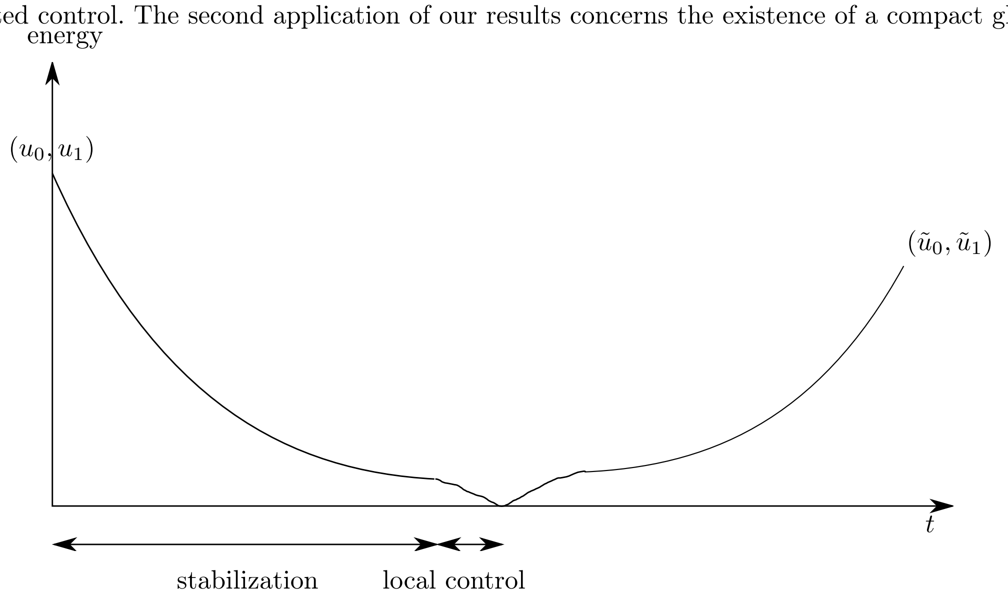


Figure 2.1: Global strategy by stabilization

attractor. We refer to Section 0.2 in the introduction for the proper definitions.

Theorems 2.2.1 shows that $\{0\}$ is a global attractor for the damped wave equation (2.1.1). Of course, it is possible to obtain a more complex attractor by considering an equation of the type

$$\begin{cases} \Box u + \gamma(x)\partial_t u + f(x,u) = 0 & (t,x) \in \mathbb{R}_+ \times \Omega \ , \\ u(x,t) = 0 & (t,x) \in \mathbb{R}_+ \times \partial\Omega \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) \end{cases} \tag{2.2.1}$$

where $f \in C^\infty(\Omega \times \mathbb{R}, \mathbb{R})$ is real analytic with respect to u and satisfies the following variant of (2.1.2). There exist $C > 0$, $R > 0$, $p \in [1, 5)$ such that for all $(x, u) \in \Omega \times \mathbb{R}$,

$$\begin{gathered} |f(x,u)| \leq C(1+|u|)^p \ , \quad |f'_x(x,u)| \leq C(1+|u|)^p \ , \quad |f'_u(x,u)| \leq C(1+|u|)^{p-1}, \\ uf(x,u) \geq 0, \qquad \text{for all } u \text{ with } |u| \geq R, \\ f(x,u) = 0, \qquad \text{for all } x \in \partial\Omega. \end{gathered} \tag{2.2.2}$$

Theorem 2.2.3. *Assume that f is as above. Then, the dynamical system generated by (2.2.1) in $H^1_0(\Omega) \times L^2(\Omega)$ is gradient and admits a compact global attractor $\mathcal{A}$.*

This type of result was obtained in [103] with weaker nonlinearity.

In this context, the scheme of proof for Theorem 2.2.2, roughly described by Figure 2.1 does not apply. But a similar result will also be possible by "travelling through the attractor", see Section 2.4.

Idea of the proof

The proof of the stabilization follows the same line described in Section 0.2 of the introduction and in the introduction of this chapter. One first difference is that we prove the existence of a time $T > 0$ so that the observability (2.1.6) holds, without control of this time. This is sufficient for the stabilization. One advantage, is that with the argument by contradiction of the compactness-uniqueness argument, we end up with a sequence of solutions u_n and some times $T_n \to +\infty$ so that

$$\int_0^{T_n} \int_\Omega \gamma(x) |\partial_t u_n(x,t)|^2 \, dx \, dt \leq \frac{1}{n} E(u_n(0)). \tag{2.2.3}$$

which is slightly stronger than the situation in the usual case (0.2.8) since the limit situation is in infinite time. Denoting $U_n = (u_n, \partial_t u_n)$ for simplicity, we are left to the following two tasks

1) **Asymptotic compactness:** $U_n(t+t_n) \underset{t_n\to+\infty}{\to} U_\infty(t)$

2) **Unique Continuation** Prove that $U_\infty = 0$, that is the attractor set is $\{0\}$.

The Asymptotic compactness step 1) is very similar to Propagation of compactness step 1) (page 40) and could also be proved as in [54]. Yet, we will give below a simpler proof taking advantage of the fact that $T_n \to +\infty$.

The Unique Continuation 2) is also very similar to the Unique continuation step 2) (page 40) except that the time is infinite. Indeed, we easily get $E(u_\infty)(t_1) = E(u_\infty)(t_2)$ for all $t_1, t_2 \in \mathbb{R}$ and the unique continuation write as follows

$u_\infty \equiv 0$ is the unique strong solution in the energy space of

$$\left\{ \begin{array}{rcll} \Box u + f(u)(+\gamma\partial_t u) & = & 0 & \text{on} \quad \mathbb{R}\times\Omega \\ \partial_t u & = & 0 & \text{on} \quad \mathbb{R}\times\omega. \end{array} \right. \tag{2.2.4}$$

One crucial point for the proof here is that we need to check the unique continuation in **infinite time**.

One of the advantages of our method, which could make it more easily applicable to other situations, is that we use as a black box the exponential decay of the linear damped equation. This is a direct consequence of the assumption (GCC) from the classical result of [16] (see also [119]).

We first describe how we obtain the first point, that is the asymptotic compactness. It could actually be proved following the lines of [54] using microlocal defect measure. Yet, we proposed an easier abstract way of proving it, relying only on the exponential decay of the semi-group. The idea is to write the Duhamel formula and use the (relative) compactness of the nonlinearity.

Proposition 2.1. *Let $f \in \mathcal{C}^1(\mathbb{R})$ satisfying (2.1.2), let (u_0^n, u_1^n) be a sequence of initial data which is bounded in $H_0^1(\Omega)\times L^2(\Omega)$ and let (u_n) be the corresponding solutions of the damped wave equation (2.1.1). Let $(t_n) \in \mathbb{R}$ be a sequence of times such that $t_n \to +\infty$ when n goes to $+\infty$.*

Then, there exist subsequences $(u_{\phi(n)})$ and $(t_{\phi(n)})$ and a global solution u_∞ of (2.1.1) such that

$$\forall T > 0\ , \quad (u_{\phi(n)}, \partial_t u_{\phi(n)})(t_{\phi(n)} + .) \longrightarrow (u_\infty, \partial_t u_\infty)(.) \quad \text{in } \mathcal{C}^0([-T,T],X)\ .$$

Since the proof is quite elementary, we give a sketch of the proof in the easier case $f(u) = O(|u|^p)$ with $p < 3$ where the Sobolev embedding allows to treat the nonlinearity as a compact term in dimension 3. Denoting $U = (u, \partial_t u)$, we can write the equation $\partial_t U = AU + F(u)$ with A the infinitesimal generator of the linear damped wave equation, that is

$$D(A) = (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega), \quad A = \begin{pmatrix} 0 & Id \\ \Delta & -\gamma(x) \end{pmatrix}, \tag{2.2.5}$$

and $F(U) = \begin{pmatrix} 0 \\ f(u) \end{pmatrix}$. The Duhamel formulation writes, assuming for simplicity that $t_n \in \mathbb{N}$,

$$\begin{aligned} U_n(t_n) &= e^{At_n}U_n(0) + \int_0^{t_n} e^{sA}F(U_n(t_n - s))\,ds \\ &= e^{At_n}U_n(0) + \sum_{k=0}^{t_n} e^{kA}\int_0^1 e^{sA}F(U_n(t_n-k-s))\,ds \\ &= e^{At_n}U_n(0) + \sum_{k=0}^{t_n} e^{kA}I_{k,n}. \end{aligned} \tag{2.2.6}$$

The main idea is the following

- the first term (free term) converges to zero since $\|e^{At}\| \leq Ce^{-\lambda t}$ and $U_n(0)$ bounded in $H_0^1(\Omega) \times L^2(\Omega)$
- the terms corresponding to the nonlinear term will be compact since the nonlinearity is compact

More precisely, since $f(u) = O(|u|^p)$ with $p < 3$ in dimension 3, f maps a bounded set of $H^1(\Omega)$ into a bounded set of $H^\varepsilon(\Omega)$ for some $\varepsilon > 0$. In particular, the terms $I_{n,k} = \int_0^1 e^{sA}F(U_n(t_n - k - s))\, ds$ are bounded by some constant M in $H^{1+\varepsilon}(\Omega) \times H^\varepsilon(\Omega)$ uniformly in n and k. The last terms of (2.2.6) can be estimated by

$$\left\| \sum_{k=0}^{t_n} e^{kA} I_{k,n} \right\|_{H^{1+\varepsilon}(\Omega)\times H^\varepsilon(\Omega)} \leq \sum_{k=0}^{t_n} Ce^{-\lambda k}M \leq C_\lambda M.$$

Note that we have used that the damped wave operator e^{tA} is also exponentially decreasing in other Sobolev spaces, which can be easily verified.

So, we have written U_n as : one term converging to zero + a compact term. Therefore, it is compact. It finishes the sketch of proof of Proposition 2.1 in the case $p < 3$. Under the more general subcritical assumption $p < 5$ for the nonlinearity, we need to use Strichartz type estimates. In particular, the following gain of regularity for composition can be used top complete the compactness argument in this general case.

Theorem 2.2.4 ([54]). *Let $s \in [0, 1)$, $R > 0$ and $T > 0$.*
There exist $\varepsilon > 0$ and (q, r) satisfying $\frac{1}{q} + \frac{3}{r} = \frac{1}{2}$, $q \in [7/2, +\infty]$ and $C > 0$ such that, if $v \in L^\infty([0, T], H^{1+s}(\Omega))$ has a Strichartz $L^q([0, T], L^r(\Omega))$ norm bounded by R, then

$$\|f(v)\|_{L^1([0,T],H^{s+\varepsilon}(\Omega))} \leq C\|v\|_{L^\infty([0,T],H^{1+s}(\Omega))} \ .$$

Now, we get back to the second step in the proof of the observability estimate, which is the unique continuation problem. As we explained earlier, thanks to Theorem 2.1.2, the remaining point is the proof of the analyticity in time of the pathological solution of (2.2.4)

The main tool comes from the following abstract result which was initially designed in order to prove analyticity of some compact attractor.

Theorem 2.2.5 (Hale-Raugel [81]). *Let $U(t)$ be a global solution in a Banach space X of*

$$\partial_t U(t) = AU(t) + G(U(t)) \quad \forall t \in \mathbb{R} \ .$$

We further assume that

(i) $\|e^{At}\|_{\mathcal{L}(X)} \leq Me^{-\lambda t}$ for all $t \geq 0$
(ii) $\{U(t), t \in \mathbb{R}\}$ is contained in a compact set $K \subset B(0, r)$ of X
(iii) $\{DG(U(t))U_2 \,|t \in \mathbb{R}, \|U_2\|_X \leq 1\}$ is a relatively compact set of X
(iv) G is analytic in $B(0, 4r) + iB(0, \rho)$ with $\rho > 0$
(v) there exist projectors P_n converging to the identity and commuting with the unbounded part of A.
Then, the solution $U(t)$ is analytic from $t \in \mathbb{R}$ into X.

Note that for a solution U, the fact to be compact in positive time is not surprising for positive times t, but we are asking this also in negative time, which is more pathological if we expect the backward semigroup e^{-sA} to be growing in time. Indeed, the solutions of (2.2.4) are quite pathological in this sense.

Let us first comment on how the assumptions are proved to be satisfied in the easier case $p < 3$:
(i) is the exponential decay of the linear damped equation and is a consequence of the geometric control condition (GCC),
(ii) and (iii) require some kind of compactness of the trajectory. They are consequences of the asymptotic compactness proved previously in Proposition 2.1,
(iv) is a consequence of the fact that we have chosen f analytic,
(v) are fulfilled choosing the projectors $P_n = \mathbb{1}_{[0,n]}(-\Delta)$.

In the more general case $p \in [3,5)$, F is not a compact application on X and the well-posedness is only possible thanks to Strichartz estimates. So, the idea is to use arguments similar to the proof of the propagation of compactness to some propagation of regularity. We obtain that U is bounded in a higher Sobolev space where the nonlinearity F is compact and allows to apply Theorem 2.2.5 in a different space.

In the next Section 2.3, we will extend this type of result to some nonuniform decay. This will be the occasion to introduce the ideas of the proof of Theorem 2.2.5 and how we could extend it to some nonuniform decay.

Let us now give some words about the proof of Theorem 2.2.3, that is the case where we weakened the sign assumption to allow some more complicated dynamics and a nontrivial global attractor. We refer to Section 0.2 of the introduction. The purpose is therefore to prove that the assumptions of Theorem 0.2.9 are fulfilled for S the flow of the damped equation (2.2.1) in X.

- S is gradient: the nonlinear energy E decays thanks to (2.1.5). To verify that it is a strict Lyapunov functional, we need to check that a solution that has conserved energy satisfies $\partial_t u = 0$ on $\mathbb{R} \times \omega$. The same ideas will prove that u is analytic and then that $\partial_t u = 0$, that is u is an equilibrium point.
- S is asymptotically smooth: this is exactly the first step "Asymptotic compactness"
- for any bounded set $B \subset X$, there exists $t_0 \geq 0$ such that $\bigcup_{t \geq t_0} S(t)B$ is bounded: we can take $t_0 = 0$ and it is a consequence of the energy estimates using the sign assumption (2.2.2) of $f(x,u)$ for large u.
- S is pointwise dissipative thanks to the energy decay.

Some perspectives

1) Finite time unique continuation: Along the proof, we have proved the unique continuation for the nonlinear equation (2.2.4), that is in infinite time. It is sufficient for the proof of stabilization. The contradiction argument actually yields the unique continuation in a large time T depending on the side of the data (E_0 in (ED)). Yet, it would be natural to obtain this unique continuation in a natural geometric time independent of the size of the data. This is a work in progress with Cristóbal Loyola.

2) Problems with stronger nonlinearity: In the proof of the unique continuation, or the stabilization, it was very useful that the nonlinearity is compact at a suitable choice of regularity. That would be interesting to see how far we can go with this method to obtain some results of unique continuation, maybe assuming some a priori regularity of the solution. Is it possible with a nonlinearity of the form $f(u, \nabla_{t,x} u)$? Or even with a quasilinear equation?

3) Is the analyticity of f necessary? Of course, it would be nice, to remove the assumption that f is analytic, but it seems that it would require more ideas...

4) Critical nonlinearity: We proved the stabilization for some subcritical defocusing nonlinearities $p < 5$ in dimension 3. For the energy critical problem ($p = 5$ in dimension 3), there are some known results in domains where some Morawetz type multiplier can be applied, which, as already commented, is more restrictive than (GCC). In my work of thesis [T5] , I proved the compactness part of the proof by compactness uniqueness. By the critical nature of the equation, it required some concentration compactness argument as described in the end of Section 0.2 and some profile decomposition. This allowed to prove stabilization for energy bounded solutions, but with an additional smallness in L^2. It would be nice to see if an extension of the method presented here could allow to obtain the unique continuation and get the stabilization under (GCC) (or slightly stronger assumption).

5) Other equations: It is quite clear in the proof that the method we developed is not specific to the wave equation. We hope to apply it to other equations like the nonlinear Schrödinger equation. This is a work in progress by Cristóbal Loyola.

Note also that the method we presented has already been used afterwards by other authors for other problems. Perrin [145] applied it for the focusing nonlinear wave below the ground state, Jellouli-Khenissi [97] for a system KdV-BBM.

2.3 Damped nonlinear wave equations: some cases without geometric control condition

This section is a continuation of the previous section. It corresponds to **[A14]** . We still consider the defocusing nonlinear wave equation (2.1.1) with the same notations. Yet, we want to consider geometric situations where ω does not any more satisfies (GCC).

When some of the geodesic rays are trapped, the stabilization of the linear semigroup is semi-uniform in the sense that $\|e^{At}A^{-1}\| \leq h(t)$ for some function h with $h(t) \to 0$ when $t \to +\infty$. We provide general tools to deal with the semilinear stabilization problem in the case where $h(t)$ has a sufficiently fast decay.

In the general case, as soon as $\gamma \not\equiv 0$, the decay rate can be taken as $h(t) = \mathcal{O}(1/\ln t)$ as shown in [119, 121] with a small loss or Theorem 1.4.14 for $k = 1$. We will consider some more favorable situations where γ misses the geometric control condition, but very closely: typically there are only very few geodesics that do not meet the support of the damping and these geodesics are unstable. In this case, we may hope for a better decay, see for example the other references of Figure 2.2.

We will only assume a semi-uniform decay, but sufficiently fast in the following sense.

Assumption 2.3.1. There exists a function $h(t)$ such that

$$\forall U_0 \in D(A) \ , \ \|e^{At}U_0\|_X \ \leq \ h(t) \, \|U_0\|_{D(A)} \tag{2.3.1}$$

and there is $\sigma_h \in (0, 1]$ such that

$$\lim_{t\to\infty} h(t) = 0 \quad \text{and} \ \ \forall \sigma \in [0, \sigma_h] \ , \ \int_0^\infty h(t)^{1-\sigma} \, \mathrm{d}t \ < \ \infty \ . \tag{2.3.2}$$

Condition (2.3.2) requires a decay rate fast enough to be (slightly better than) integrable. The main idea behind this assumption is to be able to give a sense to $\int_0^{+\infty} e^{As}F(s)ds$ where the source term F is bounded in some suitable Sobolev space.

Roughly speaking, we will show in this section that this condition (or a slight reinforcement) is sufficient to obtain a stabilization of the semilinear equation. We obtain the first (up to our knowledge) results concerning the stabilization of this semilinear equation in cases where γ does not satisfy the geometric control condition.

Theorem 2.3.1. *Assume $d = 2$. Consider the damped wave equation (2.1.1) assuming f real analytic and satisfies (2.1.2) with $1 \leq p < +\infty$. Assume in addition that:*

- *the damping γ is of class $\mathcal{C}^1$ and Assumption 2.3.1 holds,*
- *the decay rate $h(t)$ of the semigroup in (2.3.1) satisfies $h(t) = \mathcal{O}(t^{-\beta})$ with $\beta > 2p$.*

Then, any solution u of (2.1.1) satisfies

$$\|(u, \partial_t u)(t)\|_{H^1_0 \times L^2} \underset{t\to+\infty}{\longrightarrow} 0 \ .$$

Moreover, for any R and $\sigma \in (0, 1]$, there exists $h_{R,\sigma}(t)$ which goes to zero when t goes to $+\infty$ such that the following stabilization hold. For any $U_0 \in H_0^{1+\sigma}(\Omega) \times H^\sigma(\Omega)$, if u is the solution of (2.1.1), then

$$\|(u_0, u_1)\|_{H^{1+\sigma}\times H^\sigma} \leq R \quad \Longrightarrow \quad \|(u, \partial_t u)(t)\|_{H^1_0 \times L^2} \leq h_{R,\sigma}(t) \underset{t\to+\infty}{\longrightarrow} 0 \ .$$

Our assumptions on the decay of the linear semigroup may seem strong. They are satisfied in the cases where the set of trapped geodesics, the ones which do not meet the support of the damping, is small and hyperbolic in some sense. Several geometries satisfying the hypothesis have been studied in the literature, see the concrete examples of Figure 2.2 and the references therein. Notice in particular that the example of a domain with holes is particularly relevant for applications where we want to stabilize a nonlinear material with holes by adding a damping or a control in the external part. There is a huge literature about the damped wave equation and the purpose of the examples presented here is mainly to illustrate our theorem to nonspecialists. Moreover, the subject is growing fast, giving more and more examples of geometries where we understand the effect of the damping and where we may be able to apply our results. Other related examples are [147] or [117]. We do not pretend to exhaustivity and refer to the bibliography of the more recent [99] for instance.

We expect that the decay rate $h_{R,\sigma}(t)$ is related to the linear decay rate $h(t)$ of Assumption 2.3.1. We are able to obtain this link for the typical decays of the examples of Figure 2.2.

Proposition 2.2. *Consider a situation where the stabilization stated in Theorems 2.3.1 holds. Then,*

- *if the decay rate of Assumption 2.3.1 satisfies $h(t) = \mathcal{O}(t^{-\alpha})$ with $\alpha > 1$, then the nonlinear equation admits a decay of the type $h_{R,\sigma}(t) = \mathcal{O}(t^{-\sigma\alpha})$.*
- *if the decay rate of Assumption 2.3.1 satisfies $h(t) = \mathcal{O}(e^{-at^{1/\beta}})$ with $a > 0$ and $\beta > 0$, then the nonlinear equation admits a decay of the type $h_{R,\sigma}(t) = \mathcal{O}(e^{-b\sigma t^{1/(\beta+1)}})$ for some $b > 0$.*

Notice that this result is purely local in the sense that the decay rate is obtained when the solution is close enough to 0. Our proofs do not provide an explicit estimate of the time needed to enter this small neighborhood of 0. Also notice that the loss in the power of the second case of Proposition 2.2 is due to an abstract setting (see idea of the proof below). In the concrete examples, we may avoid this loss.

In the cases where the geometric control condition fails, the decay of the linear semigroup is not uniform. At least, if γ does not vanish everywhere, it is proved in [47] (see also [84]) that the trajectories of the linear semigroup goes to zero. In fact, the decay can be estimated with a loss of derivative as

$$\|e^{At}U\|_{H^1\times L^2} \leq h(t)\|U\|_{H^2\times H^1} \qquad \text{with} \quad h(t) \xrightarrow[t\to+\infty]{} 0 \,. \tag{2.3.3}$$

Notice that our results deeply rely on the fact that the decay rate of (2.3.3) is integrable. Typically, for the situations of Figure 2.2, it is of the type $h(t) = \mathcal{O}(e^{-\lambda t^\alpha})$ or $h(t) = \mathcal{O}(1/t^\beta)$ with sufficiently large $\beta > 0$.

Theorems 2.3.1 concern the stabilization of the solutions of (2.1.1) in the sense that their $H^1 \times L^2$-norm goes to zero. Notice that, since the energy of the damped wave equation is non-increasing (see Section 0.1), we know that this $H^1 \times L^2$-norm is at least bounded. Such a uniform bound is not clear a priori for the $H^2 \times H^1$-norm. However, basic arguments provide this bound as a corollary of the theorem if the decay is fast enough, which is the case of the "disk with holes", the "peanut of rotation" and the "hyperbolic surfaces" of Figure 2.2.

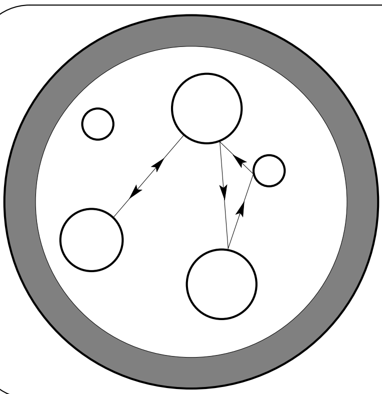
Theorem 2.3.2. *Consider the damped wave equation (2.1.1) in the framework of Theorems 2.3.1. Assume that for all $R > 0$, the decay rate $h_{R,1}(t)$ is faster than polynomial, i.e. $h_{R,1}(t) = o(t^{-k})$ for any $k \in \mathbb{N}$. Also assume that γ is of class $\mathcal{C}^1$ and f is of class $\mathcal{C}^2(\overline{\Omega}\times\mathbb{R},\mathbb{R})$. Then the $H^2(\Omega)\times H^1(\Omega)$-norm of the solutions are bounded in the following sense. For any $R > 0$, there exists $C(R) > 0$ such that, for any $U_0 \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$ such that $\|U_0\|_{H^2\times H^1} \leq R$, the solution u of (2.1.1) satisfies*

$$\sup_{t\geq 0} \|(u, \partial_t u)(t)\|_{H^2\times H^1} \leq C(R) \,.$$

Note that, in the case without damping, this result is sometimes expected to be false. It is related to the weak turbulence, described as a transport from low frequencies to high frequencies, see [39, 75, 82].

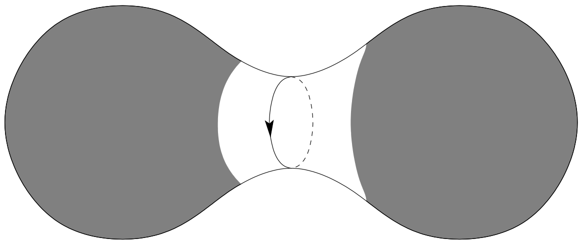
The main purpose of [A14] was to obtain new examples of stabilization for the semilinear damped wave equation and to introduce the corresponding methods and tools. We do not pretend to be exhaustive and the method may be easily used to obtain further or more precise results. For example:

- the boundary condition may be modified, typically in the case of the disk with holes, Neumann boundary condition may be chosen at the exterior boundary.



Disk with holes
We set Ω to be a convex flat surface with a damping γ efficient near the boundary. Typically, we may take the flat disk $B(0,1)$ of $\mathbb{R}^2$ and assume that there exist $r \in (0,1)$ and $\underline{\gamma} > 0$ such that $\gamma(x) \geq \underline{\gamma} > 0$ for $|x| > r$. Inside the interior zone without damping, assume that at least two holes exist; to simplify, we also assume that these holes are disks and are small in a sense specified later. Notice that there exist some periodic geodesics which do not meet the support of the damping. This example has been studied in [31, 34] for resolvent estimates. We proved that their result imply that we can take $h(t) = e^{-\lambda t^{1/3}}$.

The peanut of rotation
We consider a compact two-dimensional manifold without boundary. We assume that the damping γ is effective, that is uniformly positive, everywhere except in the central part of the manifold. This part is a manifold of negative curvature and



invariant by rotation along the $y-$axis. More precisely, let us set this part to be equivalent to the cylinder endowed with the metric $g(y,\theta) = \mathrm{d}y^2 + \cosh^2(y)\mathrm{d}\theta^2$. This central part admits a unique (up to change of orientation) periodic geodesic which is unstable; any other geodesic meets the support of the damping. This example has been studied for example in [38], [160] and obtained stabilization with $h(t) = e^{-\lambda\sqrt{t}}$.

Hyperbolic surfaces
We consider a compact connected hyperbolic surface with negative curvature (for example a surface of genus 2 cut out from Poincaré disk). The damping is any non zero smooth function $\gamma(x) \geq 0$. This example has been studied in [99] and then [63] following the fractal uncertainty principle of [28] who proved the result with $h(t) = e^{-\lambda t}$. We also refer to other results in any dimension with pressure conditions [159] following ideas of [9].

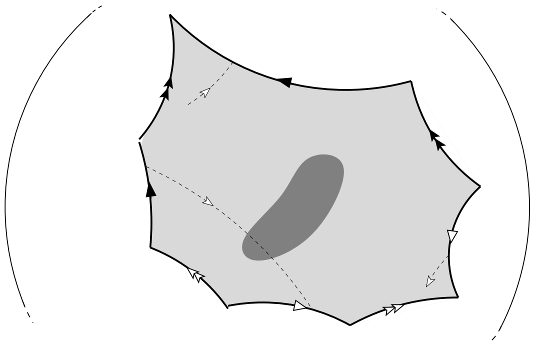


Figure 2.2: *the main applications of Theorems 2.3.1. The gray parts show the localization of the damping (white=no damping).*

- for simplicity, the examples of Figure 2.2 and the main results of this article concern two-dimensional manifolds. However, the same arguments can be used to deal with higher-dimensional manifolds. There are some technical complications, mainly due to the Sobolev embeddings. For example, in dimension $d = 3$, the degree of f in (2.1.2) should satisfy $p < 3$ ($p < 5$ if we use Strichartz estimates as done in Section 2.2). To simplify, we choose to state our results in dimension $d = 2$. However, several intermediate results in this article are stated for dimensions $d = 2$ or $d = 3$.

- It is also possible to combine the strategy of this paper with other tricks and technical arguments. For example, we may consider unbounded manifolds or manifolds of dimension $d = 3$ with nonlinearity of degree $p \in [3, 5)$, which are supercritical in the Sobolev sense. This would require to use Strichartz estimates in addition to Sobolev embeddings as done in [50], [54] or [A3] described in Section 2.2.

- As in Section 2.2, we can use a more general form of nonlinearity as in (2.2.1) and replace the sign condition by an asymptotic sign condition as in (2.2.2) in order to allow nontrivial equilibrium points. However, the arguments of [A14] show that the energy E is a strict Lyapunov functional and that any solution converges to the set of equilibrium points. We can also show the existence of a *weak compact attractor* in the sense that there is an invariant compact set $\mathcal{A} \subset H^1_0(\Omega) \times L^2(\Omega)$, which consists of all the bounded trajectories and such that any regular set $\mathcal{B}$ bounded in $(H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$ is attracted by $\mathcal{A}$ in the topology of $H^1_0(\Omega) \times L^2(\Omega)$. Notice that this concept of weak attractor is the one of Babin and Vishik in [12]. At this time, the asymptotic compactness property of the semilinear damped wave equation was not discovered and people thought that a strong attractor (attracting bounded sets of $H^1_0(\Omega) \times L^2(\Omega)$) was impossible due to the lack of regularization property for the damped wave equation. Few years later, Hale [79] and Haraux [84] obtain this asymptotic compactness property and the existence of a strong attractor. Thus this notion of weak attractor has been forgotten. It is noteworthy that it appears again here. Notice that we cannot hope a better attraction property since even in the linear case, $\{0\}$ is not an attractor in the strong sense.

Idea of the proof

The proof follows the same line as in the case where (GCC) was satisfied: prove the asymptotic compactness 1 and the unique continuation 2. But one of the notable differences is the loss of regularity when we apply the decay estimate (2.3.1). Note also that we do not prove an observability estimate. The contradiction argument directly comes the negation of $U(t_n) \to 0$. The compactness will give the uniform convergence to a limit W. It is easy to see that this W has to be bounded in positive and negative time and with conserved energy. In particular, it should solve (2.2.4). The uniqueness of the possible limits will give the result, that is $W = 0$. So, we sketch the proof of the compactness and the analytic regularity in this case.

For the asymptotic compactness, we want to prove an analog of the compactness result Proposition 2.1, when (GCC) was satisfied. For simplicity, we prove the compactness of a single trajectory, that is $U_n = U$. We first replace (2.3.1) by a decay in other spaces

$$\forall U_0 \in X^{\sigma_1} \ , \ \ \|e^{At}U_0\|_{X^{\sigma_2}} \leq h(t)^{\sigma_1-\sigma_2}\|U_0\|_{X^{\sigma_1}} \ .$$

for all $0 \leq \sigma_2 < \sigma_1 \leq 1$ and where we have defined $X^0 = X = H^1_0 \times L^2$, $X^1 = D(A)$ and X^σ is the interpolation space between X^0 and X^1. We know that the solution is bounded in energy. In dimension 2, any polynomial nonlinearity f will send the energy space to $H^{1-\sigma}$ for $\sigma \in (0, 1]$. Then, the integrability property (2.3.2) allows to estimate the nonlinear term that appeared in the proof of Proposition 2.1.

$$\left\| \int_0^{t_n} e^{As} F(U(t_n - s)) \, \mathrm{d}s \right\|_{X^{\sigma_2}} \leq \int_0^{+\infty} h(t)^{1-\sigma-\sigma_2} \, \|F(U)(t_n - s)\|_{X^{1-\sigma}} \leq C \int_0^{+\infty} h(t)^{1-\sigma-\sigma_2}. \tag{2.3.4}$$

Assumption 2.3.1 gives that the integral is bounded for some $0 < \sigma + \sigma_2 \leq \sigma_h$. So, we can make a choice with $\sigma_2 > 0$, which will give the compactness in the energy space of this term. The term $e^{t_n A}U_0$ also converges to zero in the energy space with eventually a nonuniform rate. If we want a uniform rate, we need to ask for a bit more regularity, which gives a compactness result weaker than Proposition 2.1 but sufficient for our later purpose.

Let us now give a rough idea of the proof of analyticity of the "pathological" trajectory. That will come from an extension of Theorem 2.2.5 to the case where the uniform decay of the linear damped equation is not satisfied.

Theorem 2.3.3. *Assume that the assumptions of Theorem 2.3.1 are fulfilled. Let $U(t) \in \mathcal{C}^0(\mathbb{R}, X)$ be a mild solution of (2.1.1) and assume moreover that there exists $\sigma_0 \in (0,1)$ and $C > 0$ such that*

$$\forall t \in \mathbb{R} \ , \quad \|U(t)\|_{X^{\sigma_0}} \leq C \ .$$

Then the mapping $t \in \mathbb{R} \mapsto U(t) \in X^{\sigma_0}$ is analytic with respect to t.

We briefly present the idea of the proof as it came in Theorem 2.2.5 and detail how it should be adapted for nonuniform decay.

The goal is to prove that U can be extended as a holomorphic function in a complex strip $\mathbb{R} + i(-\varepsilon, \varepsilon)$. We would like to use the fixed point theorem for contracting maps as in the proof of the Cauchy-Lipschitz theorem for the following map (defined formally)

$$\begin{aligned} U(t) \longmapsto \qquad & e^{A(t-t_0)}U(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}F(U(\tau))\, d\tau \\ \underset{s=t-\tau}{=} \quad & e^{A(t-t_0)}U(t_0) + \int_0^{t-t_0} e^{As}F(U(t-s))\, ds \\ \underset{t_0\to-\infty}{=} \quad & \int_0^{\infty} e^{As}F(U(t-s))\, ds \ . \end{aligned} \tag{2.3.5}$$

The interest of this last formulation is that we killed the free term $e^{At}U_0$ that is not analytic for a generic U_0. Since F is analytic, if we could justify the infinite integration, we could hope to apply a fixed point for functions holomorphic in $\mathbb{R} + i(-\varepsilon, \varepsilon)$. By some uniqueness, it would imply the analyticity of the solution. Yet, U is not assumed small and it seems difficult to perform a fixed point argument. The idea is to prove a finite determining modes argument, that is to prove that once some low frequencies of my solutions are fixed, all the high frequencies are determined. Of course, this can only be true in some class of "pathological solutions", that is here bounded in positive and negative times. In dynamical systems, this idea expresses the property that the compact attractor is almost finite-dimensional.

More precisely, we decompose $U = U_L + U_H = \mathcal{P}_n^L U + \mathcal{P}_n^H U$ the Low and High frequency projections of U. Here n is a large parameter to be chosen and $\mathcal{P}_n^L = \mathbb{1}_{[0,n]}(-\Delta)$ and $\mathcal{P}_n^H = \mathbb{1}_{(n,+\infty)}(-\Delta)$.

The expectation is that, for n large enough, we can define an analytic application G so that $U_H = G(U_L)$ for a "pathological solution".

Applying the projectors to the equation, we get

$$\left\{ \begin{array}{l} \partial_t U_L = (\mathcal{P}_n^L A \mathcal{P}_n^L)U_L + (\mathcal{P}_n^L A \mathcal{P}_n^H)U_H + \mathcal{P}_n^L F(U_L + U_H) \\ \partial_t U_H = (\mathcal{P}_n^H A \mathcal{P}_n^H)U_H + (\mathcal{P}_n^H A \mathcal{P}_n^L)U_L + \mathcal{P}_n^H F(U_L + U_H) \end{array} \right. \tag{2.3.6}$$

Since $U_H(t)$ is also bounded uniformly in $t \in \mathbb{R}$, we can expect to write the high frequencies U_H with a fixed point depending only on U_L, analog to the expression (2.3.5).

$$U_H(t) = \int_0^{\infty} e^{\mathcal{P}_n^H A \mathcal{P}_n^H s}\Big(\underbrace{\mathcal{P}_n^H A \mathcal{P}_n^L}_{\text{small for large } n} U_L(t-s) + \underbrace{\mathcal{P}_n^H F(U_L + U_H)(t-s)\Big)}_{\text{small for large n by compactness of } F} \, \mathrm{d}s \ .$$

Note that A almost commute with $\mathcal{P}_n^H$ (up to the damping γ), so that $\mathcal{P}_n^H A \mathcal{P}_n^H s$ is expected to become regularizing and small for large n. The second term becomes small for large n because the nonlinearity F is "compact" when we chose the correct spaces.

This can actually be made more precise with the following difficulties when we have non uniform decay

- we need to prove a decay of the high frequency semigroup $e^{\mathcal{P}_n^H A \mathcal{P}_n^H s}$ that would be uniform in n. To prove this, we had to use the equivalence of the decay of the semigroup to some resolvent estimates for the damped operator (see [19, 26]) and prove some links with resolvents of the free Laplace operator, inspired by [10]. This step implies some loss and stronger assumption on h.

- in a similar way as in estimate (2.3.4), we have to face the nonuniform decay of the semigroup in the sense that there is a loss of regularity.

The details are written in **[A14]** . Once this is done, we can write $U_H = G(U_L)$ for an operator G. Note that G is a nonlocal operator that needs the whole "story" of U_L for all $t \in \mathbb{R}$. Said in a more mathematical way, we solve the fixed point in some Banach space $C(\mathbb{R}, \mathcal{P}_n^H X^\sigma)$ (high frequencies) for a well chosen σ (and with similar holomorphic space in a complex strip $\mathbb{R} + i(-\varepsilon, \varepsilon)$). Replacing this into the equation of U_L in (2.3.6), we see that the equation of $U_L(\cdot + s)$ is an ODE in the Banach space $C(\mathbb{R}, \mathcal{P}_n^L X^\sigma)$ (low frequencies) since the operator A is bounded on the space . So U_L is analytic in time. Therefore, it is also the case for $U_H = G(U_L)$.

Some perspectives

The same perspectives evocated in Section 2.2 would of course still be very interesting in this more general context: it would be nice to extend this kind of results to other equations and stronger nonlinearities: with derivatives, quasilinear... Besides these questions, a natural question specific to this part is the role of the decay h.

Is the limitation on the decay of h necessary? In this section, we have clearly not tried to optimize the assumptions on the decay of h. Yet, beyond this certainly unnecessary losses, there is an assumption that seems crucial to the argument, which is the integrability of the decay (2.3.2). It seems very important for justifying some expressions like (2.3.5)

We accepted some losses at several points:

- as was briefly explained in the proof, we actually needed a decay for the semigroup $e^{\mathcal{P}_n^H A \mathcal{P}_n^H}$ projected on the high frequency. In order to obtain it in an abstract framework, we used some implications between resolvent estimates and the decay of the semigroup. But there is some loss in this path. In each example, we could certainly avoid this loss by proving directly the decay of the projected

- We did not use Strichartz estimates to simplify the proof. It would certainly avoid some losses at several points.

2.4 Travelling through the attractor

In Theorem 2.2.2 of the previous Section 2.2, we used the stabilization to zero to obtain some global controllability result. In this section, we intend to obtain similar results in some more general cases as the situation in Theorem 2.2.3. In that situation, the stabilized solution, instead of converging to zero, "converges" to some compact attractor that is a priori unknown. The idea will be to travel inside of the attractor from one equilibrium to another one. The results are taken from **[A4]** .

We consider the following equation with control where satisfies the weaker assumptions 2.2.2

$$\begin{cases} \Box u + f(x,u) = \mathbb{1}_\omega g & \text{in } \mathbb{R}_+ \times \Omega \ , \\ u = 0 & \text{in } \mathbb{R}_+ \times \partial\Omega \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) & \text{in } \Omega. \end{cases} \tag{2.4.1}$$

In this more general context, our main result writes in a similar way as Theorem 2.2.2, that is controllability in large time.

Theorem 2.4.1. *Assume f satisfies the same assumptions of Theorem 2.2.3. Let $R_0 > 0$ and ω satisfying the geometric control condition (GCC). Then, there exists $T > 0$ such that for any (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1)$ in $H_0^1(\Omega) \times L^2(\Omega)$, with*

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leq R_0; \qquad \|(\tilde{u}_0, \tilde{u}_1)\|_{H^1 \times L^2} \leq R_0,$$

there exists $g \in L^\infty([0,T], L^2(\omega))$ *such that the unique strong solution of* (2.4.1) *satisfies* $(u(T), \partial_t u(T)) = (\tilde{u}_0, \tilde{u}_1)$.

As seen before, Theorem 2.4.1 was already known when (2.2.2) is replaced by the more restricted sign condition $f(x,s)s > 0$ for all $s \in \mathbb{R}^*$. In this simpler case, choosing one γ supported in ω so that we have $\gamma(x) \geq \varepsilon > 0$ for some $\widetilde{\omega} \Subset \omega$ and still satisfying (GCC), we could use that all the solutions of the associated damped wave equation

$$\Box v + \gamma \partial_t v + f(x,v) = 0 \tag{2.4.2}$$

converge to zero. When releasing the sign assumption to the asymptotic one (2.2.2), the dynamics of (2.4.2) become more complicated and described by Theorem 2.2.3. The main idea will be to show how one can use the heteroclinic connections of (2.4.2) to travel from one equilibrium to another and still obtain global controllability.

Before getting into the idea of proof, let us present some already-known results of controllability of nonlinear wave equations (see [41] and [174]). To our knowledge, the control problem is solved mainly in three different cases:

- *Local control.* We assume $f(x,0) = 0$, that is that $v \equiv 0$ is a steady state of (2.1.1). The problem of local controllability states as property (SGC), except that $\mathcal{B}$ is not any bounded set but is a small neighborhood of 0. The local controllability is known for both internal and boundary control, see [37] and [175].

  The result of Coron and Trélat [42] is quite different but could be described roughly as local controllability near a whole connected component of the set of steady states. The authors study local controllability by boundary control near a path of steady states, in dimension $d = 1$. Notice that their method using quasi-static deformations, enables them to compute effectively the control. Remark also that an important part of our proof consists in proving that we can go from one equilibrium to another using the global compact attractor and is therefore similar, in spirit, to their result.

- *Quasilinear nonlinearity.* Another class of problems for which (SGC) holds, is the case where f is almost linear, for instance globally lipschitzian or super-linear but with a growth of the type $s \ln^\beta s$, see [72] [111], [125], [178], [41] and the references therein. Notice that these papers prove in fact the global controllability, that is the time T in (SGC) can be chosen independently of $\mathcal{B}$.

- *Sign condition* $f(x,s)s \geq 0$. As described in the previous Section 2.2 [176], [54] and [A3] (Theorem 2.2.1 described in Section 2.2), stabilization results are proved, that is that, if $f(x,s)s \geq 0$ and if f satisfies some additional conditions, all the solutions of the damped wave equation

$$v_{tt}(x,t) - \Delta v(x,t) + f(x, v(x,t)) = -\mathbb{1}_\omega v_t(x,t) \tag{2.4.3}$$

  goes to 0 when t goes to $+\infty$. Adding a local control result near 0, one easily gets a global control result by using $u = -v_t$ as control in (2.1.1). Therefore, [176], [54] [62] and [A3] also yield the semi-global controllability of semilinear wave equations, under a sign assumption for f. Notice that this sign condition cannot be removed carelessly if f is super-linear, due to the counter-example of [178].

Idea of the proof

We first give an idea of the proof without caring about the uniformity of the time of control. We refer also to Figure 2.3 and 2.4 for an idea of the strategy.

It will be the consequence of a few statements that we list:

1) the damped equation is gradient: the solutions converge to the equilibrium set $\mathcal{E}$ as $t \to +\infty$.
2) local controllability near equilibrium points
3) the compact global attractor set is connected.

The strategy will be as follows, writing "we go from V_0 to V_1" instead of " there exists $T \geq 0$ and $g \in L^1([0,T], L^2)$ such that the unique solution v of (2.4.1) satisfies $(v, \partial_t v)(T) = V_1$".

We define the symmetrization operator $R(u_0, u_1) = (u_0, -u_1)$. The main basic facts are the following when $(u(t), \partial u(t))$ is a solution of the damped equation (2.4.2) and $t_1 \leq t_2$:

Basic fact 1: lastly, for an equilibrium e which is indeed of the form $(u_0, 0)$, we have local controllability in a neighborhood $\mathcal{N}(e)$. Up to reducing $\mathcal{N}(e)$, we can assume $\mathcal{N}(e) = R\mathcal{N}(e)$. In particular, for any $V_0 \in \mathcal{N}(e)$, there is a control to RV_0.

Basic fact 2: we can go from $(u(t_1), \partial_t u(t_1))$ to $(u(t_2), \partial_t u(t_2))$ by taking the control $g = -\gamma \partial_t u$.

Basic fact 3: using the backward equation, we can go from $R(u(t_2), \partial_t u(t_2))$ to $R(u(t_1), \partial_t u(t_1))$.

First, using the damped equation, we can go from any point to a neighborhood of an equilibrium point. Using the backward equation, we show that for any fixed arrival point V_1, there is a point close to an equilibrium that we can link to V_1. So, the main point will be to go from any equilibrium to another, that is to travel through the attractor.

Then, we show the following statement: Let $U(t) = (u(t), \partial_t u(t))$ be a globally bounded trajectory, that is a trajectory inside $\mathcal{A}$ (see Definition 0.2.4), then, for any t_0, $t_1 \in \mathbb{R}$, we can go from $U(t_0)$ to $U(t_1)$.

If $t_0 \leq t_1$, this is easy by the basic fact 2. If $t_0 > t_1$, we use a "double u-turn argument": Thanks to Theorem 0.2.8, we know that there is a subsequence so that $U(t_n)$ converges as $t_n \to \pm\infty$ to some equilibrium $e_\pm$. Let $t_\pm$ with $t_- \leq t_1 \leq t_0 \leq t_+$ so that $U(t_\pm) \in \mathcal{N}(e_\pm)$, the neighborhood where it is locally controlable. We use Basic fact 2 to go from $U(t_0)$ to $U(t_+)$. We use Basic fact 1 to go from $U(t_+)$ to $R(U(t_+))$. We use Basic fact 3 to go from $R(U(t_+)$ to $R(U(t_-))$. We use again Basic fact 1 to go from $R(U(t_-))$ to $U(t_-)$. We use again basic fact 2 to go from $U(t_-)$ to $U(t_1)$ as expected.

If the compact global attractor was made of a finite number of isolated equilibrium points connected by heteroclinic orbits (which is the more general situation), this statement would allow to conclude. In the general case, we conclude using the fact that $\mathcal{A}$ is connected and prove that the reachable set in $\mathcal{A}$ from an equilibrium is open-closed.

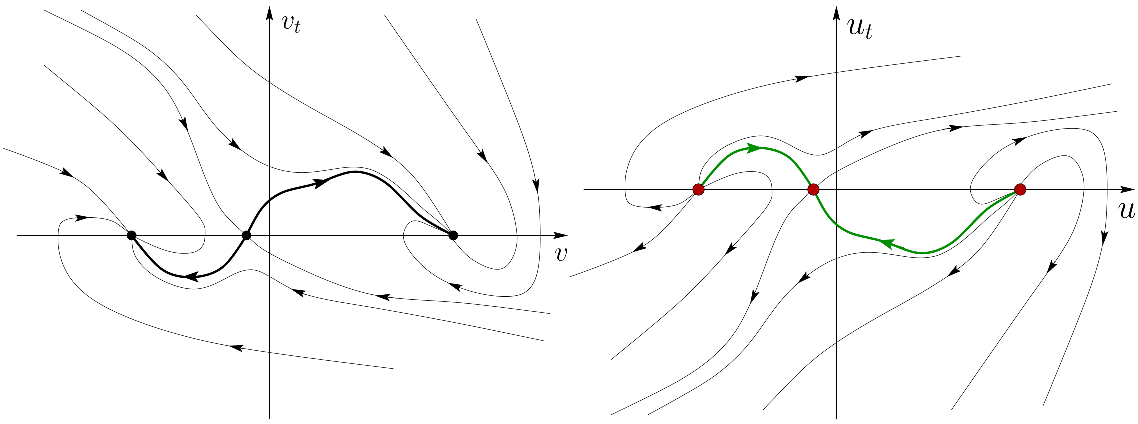


Figure 2.3: *Right: the flow of the dynamical system $S(t)$ generated by the feedback control $u = -\gamma(x)v_t$. The flow is represented in the phase plane (v, v_t). The compact global attractor $\mathcal{A}$ (in bold) consists in three equilibrium points and two heteroclinic orbits connecting them. Left: the flow generated by the feedback control $u = \gamma(x)v_t$. It is deduced from the flow of $S(t)$ by reversing time and orientation of the second coordinate.*

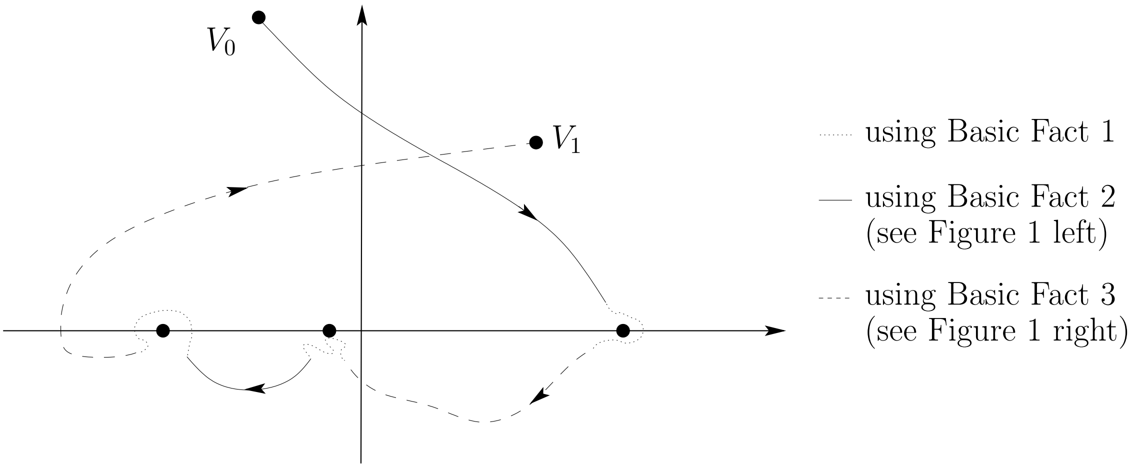


Figure 2.4: *An example of global control using successively the Basic Facts 1, 2 and 3, introduced at the beginning of the idea of proof of Theorem 2.4.1. The resulting trajectory consists of switching from one of the flows of Figure 2.3 to the other, using in between the local controls near the equilibrium points.*

CHAPITRE 3

Cost of the control in asymptotic regimes

This Chapter is concerned with the series of work about two related questions about the cost of parabolic equations that we can summarize as follows

- Concerning the cost of the control of the heat equation $\partial_t u - \Delta_g u = \mathbb{1}_\omega g$, it was conjectured by Miller that the cost of the control for short time T was blowing up as $e^{\frac{\mathfrak{K}}{T}}$ with $\mathfrak{K} = \frac{\sup_{x\in\mathcal{M}} \operatorname{dist}(x,\omega)^2}{4}$. It was based on the idea that the heat kernel provides the most concentrated solutions of the heat equation.
- Concerning the cost of the control of the transport equation in the vanishing viscosity limit $\partial_t u - \varepsilon\Delta_g u + X \cdot \nabla u = \mathbb{1}_\omega g$ where X is a vector field, it was believed that the minimal time T to get a cost of control uniform in $\varepsilon > 0$ was the time of controllability of the transport equation. It was based on the idea that this time of uniform controllability should be the one of the limit equation.

Quite surprisingly, in both of these examples, we provide a similar answer

- the natural conjecture is false because of some counterexamples that have strong oscillations
- the conjecture is true if we consider the related observability estimate, but restricted to positive solutions.

Note that besides the results themselves, the informal conclusion of this series of results is that the study of observability of parabolic equations for positive solutions or in the general case are quite different topics. Indeed, by the physical interpretation of the heat equation as Brownian motion, it would be tempting to try to give a proof using probabilistic methods that are, a priori, more applicable to positive function. It seems that it has not been done so far. Our study shows that if it was possible to obtain the observability in the general case from the knowledge of this result for positive functions, at least the constants involved would be very different.

Before turning into the results we obtain, let us just precise that, very often, results concerning optimal constants can sometimes become quite technical, so, we have chosen not to present all the results that we obtained in this subject. In particular, the results in [A11] has implications for other constants important in control theory. We refer to the articles [A11] for the cost of the control in small time and [A16] [A21] [A20] for the control in the vanishing viscosity limit, for the detail of all the results.

3.1 Cost of the control of the heat equation in small time

In the whole presentation, we are given a connected compact Riemannian manifold $(\mathcal{M}, g)$ with or without boundary $\partial\mathcal{M}$, we denote by Δ_g the (negative) Laplace-Beltrami operator on $\mathcal{M}$. For readability, we first focus in the next section on results concerning the observability constant for the heat equation.

Here, we study the so-called *cost of controllability* of the heat equation. It is well known since the seminal papers of Lebeau-Robbiano [120] and Fursikov-Imanuvilov [73] that for any time $T > 0$, the heat equation is controllable to zero. More precisely, by duality, the controllability problem is equivalent to the observability problem for solutions of the free heat equation (see e.g. [41, Section 2.5.2]): For any non-empty open set ω and $T > 0$, there exist $C_0(T, \omega)$ such that we have

$$\left\|e^{T\Delta_g}u\right\|^2_{L^2(\mathcal{M})} \leq C_0(T,\omega)^2 \int_0^T \left\|e^{t\Delta_g}u\right\|^2_{L^2(\omega)} dt, \qquad \text{for all } T > 0 \text{ and all } u \in L^2(\mathcal{M}). \tag{3.1.1}$$

Here, $(e^{t\Delta_g})_{t>0}$ denotes the semigroup generated by the *Dirichlet* Laplace operator on $\mathcal{M}$ (otherwise explicitely stated). The observability constant $C_{T,\omega}$ is then directly related to the cost of the control to zero and has been the object of several studies.

It has been proved by Seidman [161] in dimension one (in the closely related case of a boundary observation) and by Fursikov-Imanuvilov [73] in general (see also [142] for obtaining this result via the Lebeau-Robbiano method), that the cost in small time blows up at most exponentially:

$$\omega \neq \emptyset \quad \Longrightarrow \quad \text{there is } C, \mathfrak{K} > 0 \text{ such that } C_0(T,\omega) \leq Ce^{\frac{\mathfrak{K}}{T}} \quad \text{for all } T > 0. \tag{3.1.2}$$

Guïchal [78] in one dimension and Miller [138] in the general case proved that exponential blowup indeed occurs:

$$\overline{\omega} \neq \mathcal{M} \quad \Longrightarrow \quad \text{there is } c > 0 \text{ such that } C_0(T,\omega) \geq ce^{\frac{c}{T}} \quad \text{for all } T > 0.$$

This suggest to define

$$\mathfrak{K}_{heat}(\omega) = \inf\left\{\mathfrak{K} > 0, \exists C > 0 \text{ s.t. (3.1.1) holds with } C_0(T,\omega) = Ce^{\frac{\mathfrak{K}}{T}}\right\}, \tag{3.1.3}$$

which, according to the abovementionned results satisfies $\mathfrak{K}_{heat}(\omega) < \infty$ as soon as $\omega \neq \emptyset$ and $\mathfrak{K}_{heat}(\omega) > 0$, as soon as $\overline{\omega} \neq \mathcal{M}$. This constant depends only on the geometry of the manifold $(\mathcal{M}, g)$ and the subset ω. It is expected to contain geometric features of short time heat propagation, and has thus received a lot of attention in the past fifteen years [17, 48, 64, 66, 138, 139, 141–143, 149, 169, 170].

In this direction, the result of Miller [138] is actually more precise and provides a geometric lower bound: for all $(\mathcal{M}, g), \omega$, we have

$$\mathfrak{K}_{heat}(\omega) \geq \frac{\mathcal{L}(\mathcal{M},\omega)^2}{4},$$

where, for $E \subset \mathcal{M}$, we write

$$\mathcal{L}(\mathcal{M}, E) = \sup_{x\in\mathcal{M}} \operatorname{dist}_g(x, E). \tag{3.1.4}$$

The proof relies on heat kernel estimates. In [138, 140], Luc Miller also proved that in case ω satisfies the Geometric Control Condition in $(\mathcal{M}, g)$ (see [16]) we have

$$\mathfrak{K}_{heat}(\omega) \leq \alpha_* L^2_\omega, \tag{3.1.5}$$

where L_ω is the maximal length of a "ray of geometric optics" (i.e. geodesic curve in case $\partial\mathcal{M} = \emptyset$) not intersecting ω, and $\alpha_* \leq 2$ is an absolute constant (independent of the geometry). Based on these results and the idea that the heat kernel provides the most concentrated solutions of the heat equation, he formulated the following conjecture [138, Section 2.1]-[141, Section 3.1].

Conjecture 3.1.1 (Luc Miller)**.** *For all $(\mathcal{M}, g)$ and $\omega \subset \mathcal{M}$ such that $\overline{\omega} \neq \mathcal{M}$, we have $\mathfrak{K}_{heat}(\omega) = \frac{\mathcal{L}(\mathcal{M},\omega)^2}{4}$.*

Note that it has been proved in [131] that, in the related context of the 1D heat equation with a boundary observation, the factor $\frac{1}{4}$ might not be correct (and should be replaced by $\frac{1}{2}$, see Section 3.1 below). Our first result disproves Conjecture 3.1.1 in a stronger sense.

Theorem 3.1.2 (Counterexamples)**.** *Assume* $(\mathcal{M}, g)$ *is one of the following*

1) $\mathcal{M} = \mathbb{S}^n \subset \mathbb{R}^{n+1}$ *and* g *is the canonical metric;*
2) $\mathcal{M} = \mathbb{D} = \left\{(x_1, x_2) \in \mathbb{R}^2 \,\middle|\, x_1^2 + x_2^2 \leq 1\right\} \subset \mathbb{R}^2$ *is the unit disk,* g *the Euclidean metric and Dirichlet conditions are taken on* $\partial\mathcal{M}$*;*
3) $\mathcal{M} = \mathcal{S} \subset \mathbb{R}^3$ *is a surface of revolution diffeomorphic to the sphere* $\mathbb{S}^2$*, and* g *is the metric induced by the Euclidean metric on* $\mathbb{R}^3$ *(with additional non degeneracy conditions.*

Then, for any $C > 0$*, there exists* $\omega \subset \mathcal{M}$ *so that* $\mathfrak{K}_{heat}(\omega) \geq C\mathcal{L}(\mathcal{M}, \omega)^2$ *and* $\mathfrak{K}_{heat}(\omega) \geq C$*.*
More precisely, assume that x_0 *is either*

1) *any point in* $\mathbb{S}^n$*,*
2) *the center of* $\mathbb{D}$*.*
3) *one of the two points that intersect the axis of revolution of* $\mathcal{S} \subset \mathbb{R}^3$*,*

Then, there exists $C > 0$ *and* $r_0 > 0$ *so that we have*

$$\mathfrak{K}_{heat}(B_g(x_0, r)) \geq C|\log(r)|^2, \quad \text{for any } 0 < r \leq r_0. \tag{3.1.6}$$

Here, $B_g(x_0, r)$ denotes the geodesic ball of $\mathcal{M}$ centered at x_0 of radius r. The results we obtain are slightly more precise. In particular, the constant C is an explicit geometric constant. The lower bounds are related to an appropriate Agmon distance associated to the problem. We refer to estimates 3.1.18 below and the sketch of proof for more precise estimates.

Note that the common geometric factor of all these examples is that there exists a stable closed geodesic and the point of concentration x_0 is far from this geodesic. We should also notice that this blowup of $\mathfrak{K}_{heat}(B(x_0, r))$ for small r does not always happen and is due here to a particular (de)concentration phenomenon. For instance on $\mathcal{M} = \mathbb{T}^1$, the set $\omega = B(x_0, r)$ always satisfies the Geometric Control Condition for any time $T > 1-2r$. Abstract results give $\mathfrak{K}_{heat}(B(x_0, r)) \leq \alpha_* \leq 2$ for any $r > 0$ and blowup does not occur.

Then, we showed that the blowup given by (3.1.6) is optimal as far as the asymptotic of $\mathfrak{K}_{heat}$ for small balls is concerned. We prove the following observability result from small balls (closely related to previous results of Jerison-Lebeau).

Theorem 3.1.3. *For all* $x_0 \in \mathcal{M}$*, there exist* $C > 0$ *such that for all* $r > 0$ *we have*

$$\mathfrak{K}_{heat}(B(x_0, r)) \leq C|\log(r)|^2 + C.$$

Note that Bardos and Phung [17, 149] recently proved independently that $\mathfrak{K}_{heat}(B(x_0, r)) \leq \frac{C_\epsilon}{r^\epsilon} + C_\epsilon$ for all $\epsilon > 0$ in case $\mathcal{M} \subset \mathbb{R}^n$ is star-shaped w.r.t. x_0.

These results seem to suggest that $\mathcal{L}(\mathcal{M}, \omega)$ is not the only appropriate parameter needed for estimating $\mathfrak{K}_{heat}(\omega)$. There are indeed some solutions of the heat equation concentrating more than the heat kernel for small times.

Our last result concerning the heat equation goes actually in the opposite direction. It provides with a large class of solutions of the heat equation, namely *positive* solutions, that do not concentrate more than the heat kernel, thus proving Conjecture 3.1.1 when restricted to this class of solutions. Recall that $\mathcal{L}(\mathcal{M}, E)$ is defined in (3.1.4).

Theorem 3.1.4. *Assume that* $(\mathcal{M}, g)$ *has geodesically convex boundary* $\partial\mathcal{M}$*. Then, for any nonempty open set* $\omega \subset \mathcal{M}$ *and* $z_0 \in \mathcal{M}$*, for any* $\varepsilon > 0$*, there exist* $C, D > 0$ *so that for any* $0 < T \leq D$*, we have*

$$\|u(T)\|^2_{L^2(\mathcal{M})} \leq \frac{C}{T} e^{\frac{(1+\varepsilon)(\mathcal{L}(M,\omega)+\varepsilon)^2}{2T}} \int_0^T \|u(t, \cdot)\|^2_{L^2(\omega)} \, dt, \tag{3.1.7}$$

$$\|u(T)\|^2_{L^2(\mathcal{M})} \leq \frac{C}{T} e^{\frac{(1+\varepsilon)(\mathcal{L}(M,z_0)+\varepsilon)^2}{2T}} \int_0^T u(t, z_0)^2 \, dt, \tag{3.1.8}$$

for all $u_0 \in L^2(\mathcal{M})$ *such that* $u_0 \geq 0$ *a.e. on* $\mathcal{M}$ *and associated solution* u *to*

$$(\partial_t - \Delta_g)u = 0 \text{ on } \mathbb{R}^+_* \times \mathrm{Int}(\mathcal{M}), \quad u|_{t=0} = u_0 \text{ in } \mathrm{Int}(\mathcal{M}), \quad \partial_\nu u = 0 \text{ on } \mathbb{R}^+ \times \partial\mathcal{M}.$$

Theorem 3.1.4 follows from classical Li-Yau estimates [126] described in Theorem 0.2.11 in the introduction. Notice that here, Neumann boundary conditions are taken (ν denotes a unit vector field normal to $\partial\mathcal{M}$), and an additional geometric assumption is made (convexity of $\partial\mathcal{M}$). The result still holds without the convexity assumption up to replacing $(1+\varepsilon)$ in the exponent by a geometric constant. We also recall that for nonnegative initial data $u_0 \geq 0$, the solution of the heat equation remains nonnegative for all times. Of course, the counterexamples of Theorem 3.1.2 prevent these estimates from holding in general. Estimate (3.1.8) is particularly surprising (even without considering the value of the constants) and of course only true for positive solutions (otherwise just taking z_0 in a nodal set of an eigenfunction of Δ_g invalidates (3.1.8)).

Previous results

We are not aware of other situations in which the constants described in the previous paragraph are known exactly. We collect in this section previous results on the constants $\mathfrak{K}_{heat}$, which received a lot of attention in the past fifteen years.

Parabolic equations in dimension one The most studied case concerns the constant $\mathfrak{K}_{heat}$, with observation/control at the boundary in the one-dimensional case, say $\mathcal{M} = [-1, 1]$. Yet, it seems that the constant $\mathfrak{K}_{heat}(\{-1, 1\})$ is still unknown. Note that the latter has particular importance since it has applications to higher dimensions (with geometric conditions) via the transmutation method of Luc Miller [140].

Here, we list previous results on $\mathcal{M} = [-1, 1]$ with Neumann trace observation (Dirichlet control) on both sides of the interval. Note also that each improvement of the constant was also the occasion of finding new techniques of proofs.

- $\mathfrak{K}_{heat}(\{-1, 1\}) \leq 2\left(\frac{36}{37}\right)^2$ Miller [140], using the transmutation method;
- $\mathfrak{K}_{heat}(\{-1, 1\}) \leq \frac{3}{4}$ Tenenbaum-Tucsnak [169], using results of analytic number theory;
- $\mathfrak{K}_{heat}(\{-1, 1\}) \geq \frac{1}{2}$, Lissy [131], using complex analysis arguments;
- $\mathfrak{K}_{heat}(\{-1, 1\}) \leq 0,7$, Dardé-Ervedoza [48], combining Carleman estimates and complex analysis.

Note that in this setting, the analogue of Conjecture 3.1.1 would be $\mathfrak{K}_{heat}(\{-1, 1\}) = \frac{1}{4}$, which [131] disproved in this context (by a factor 2). However, this result does not in general prevent the existence of a universal constant $C > 0$ so that $\mathfrak{K}_{heat}(\omega) = C\mathcal{L}(\mathcal{M}, \omega)^2$.

Parabolic equations in higher dimensions There are many papers concerning the controllability properties of the heat equation. We only mention those providing estimates on the constants studied in this section.

The first computable estimates were obtained using the transmutation method to give estimates similar to (3.1.5). We can find several references improving the universal constant involved, see [48, 138, 140, 169].

Some other estimates were obtained using the Lebeau-Robbiano strategy. It consists in proving the controllability of the heat equation from some estimates of non-vanishing of sums of eigenfunctions (so-called Lebeau-Robbiano spectral inequality) [98, 120, 122]: there exist $C, \mathfrak{K}$ such that we have

$$\|u\|_{L^2(\mathcal{M})} \leq Ce^{\mathfrak{K}\sqrt{\lambda}} \|u\|_{L^2(\omega)}, \quad \text{for all } \lambda > 0 \text{ and all } u \in E_{\leq\lambda}, \tag{3.1.9}$$

where $E_{\leq\lambda} := \text{span}\{E_{\lambda_j}, \lambda_j \in \text{Sp}(-\Delta_g), \lambda_j \leq \lambda\}$ is the space of linear combinations of eigenfunctions associated to eigenvalues less than λ. It is then natural to define $\mathfrak{K}_\Sigma$ as the best possible $\mathfrak{K}$ in the previous inequality, that is more precisely

$$\mathfrak{K}_\Sigma(\omega) = \inf\{\mathfrak{K} > 0, \exists C > 0 \text{ s.t. (3.1.9) holds}\}. \tag{3.1.10}$$

Some authors made this link quantitative and obtained $\mathfrak{K}_{heat}(\omega) \leq 4\mathfrak{K}_\Sigma(\omega)^2$, see [142, Corollary 1 and Section 2.4] (see also [162] for a proof of $\mathfrak{K}_{heat}(\omega) \leq 8\mathfrak{K}_\Sigma(\omega)^2$). Many authors then obtained estimates for $\mathfrak{K}_{heat}$ by estimates on $\mathfrak{K}_\Sigma$.

In [169], the authors prove $\mathfrak{K}_\Sigma(\omega^*) \leq 3\log(\frac{(4\pi e)^N}{|\omega^*|})$ where $\mathcal{M} = (0,\pi)^N$ is a cubic domain and $|\omega^*|$ is the volume of the biggest rectangle included in ω. The proof of this result uses a number theoretic argument of Turán concerning families of the complex exponential $(e^{ikx})_{k\in\mathbb{Z}}$ (which can be interpreted as an estimate of $\mathfrak{K}_\Sigma(I)$ for I a subinterval of $\mathbb{T}$). Remark that in this particular flat-torus geometry, we have no idea of what the right constant should be.

In [17], the authors prove $\mathfrak{K}_\Sigma(B(0,r)) \leq \frac{C_\varepsilon}{r^\varepsilon}$ for all $\varepsilon > 0$ in convex geometries. This has just been extended by Phung [149]. Our Theorem 3.1.3 improves this result. Note also that [143] gave results related to this in a periodic setting, tracking uniformity with respect to several parameters.

In the Euclidian space $\mathbb{R}^n$ where Δ is the usual flat Laplacian, spectral estimates as (3.1.9) can be interpreted as a manifestation of the uncertainty principle. Several results relying on this fact have been recently stated. We refer for instance to the recent articles [64] and [172] and the references therein.

Idea of the proof

Lower bound for $\mathfrak{K}_{heat}$: exponentially small eigenfunctions

The lower bound (3.1.6) for $\mathfrak{K}_{heat}$ will be simply obtained by plugging a well chosen sequence of eigenfunctions of the Laplacian as an initial datum for the heat equation. It will be natural to introduce, the best rate of decay for eigenfunctions. In particular, by [57], there exist $C, \mathfrak{K}$ such that we have for $E_\lambda := \text{span}\{\psi \in L^2(\mathcal{M}), -\Delta_g\psi = \lambda\psi\}$,

$$\|\psi\|_{L^2(\mathcal{M})} \leq Ce^{\mathfrak{K}\sqrt{\lambda}} \|\psi\|_{L^2(\omega)}, \quad \text{for all } \lambda \in \text{Sp}(-\Delta_g) \text{ and } \psi \in E_\lambda. \tag{3.1.11}$$

It is then natural to define $\mathfrak{K}_{eig}$ as we did for a sum of eigenfunctions, the best possible $\mathfrak{K}$ in the previous inequality, that is more precisely

$$\mathfrak{K}_{eig}(\omega) = \inf\{\mathfrak{K} > 0, \exists C > 0 \text{ s.t. (3.1.11) holds}\}. \tag{3.1.12}$$

Assume that the observability estimate (3.1.1) is satisfied with $C_0(T,\omega) = Ce^{\frac{\mathfrak{K}_{heat}}{T}}$ uniformly for $0 < T \leq 1$. We apply it to $u(t,x) = e^{-t\lambda}\psi(x)$ with $\lambda \in \text{Sp}(-\Delta_g) \setminus \{0\}$ and $\psi \in E_\lambda$, and choosing $T = \sqrt{\frac{\mathfrak{K}}{\lambda}}$, we have

$$\|\psi\|^2_{L^2(\mathcal{M})} \leq \frac{C}{2\lambda} e^{4\sqrt{\mathfrak{K}_{heat}\lambda}} \|\psi\|^2_{L^2(\omega)}.$$

In particular,

$$\frac{\mathfrak{K}_{eig}(\omega)^2}{4} \leq \mathfrak{K}_{heat}(\omega). \tag{3.1.13}$$

So, if we are able to construct a sequence $\lambda_k \to +\infty$ of eigenfunctions exponentially small on an open set, that is with

$$\|\psi_k\|_{L^2(\omega)} \leq Ce^{-\mathfrak{K}\sqrt{\lambda_k}} \|\psi_k\|_{L^2(\mathcal{M})}, \tag{3.1.14}$$

this will give the lower bound $\mathfrak{K} \leq \mathfrak{K}_{eig}(\omega)$ and then $\frac{\mathfrak{K}^2}{4} \leq \mathfrak{K}_{heat}(\omega)$.

At that point, to contradict the conjecture, it is enough to make estimates on some well-known classical eigenfunctions on the sphere

- on the sphere $\mathbb{S}^2$, parametrized by $(s,\theta) \in (0,\pi) \times \mathbb{S}^1$, we can take the zonal eigenfunctions $\psi_k(s,\theta) = c_k \sin(s)^k e^{ik\theta}$ with the polynomial normalization constant $c_k = O(k^{1/4})$ for $\lambda_k = k(k+1)$. If we take $\omega = B(N,r)$ a small ball around the north pole, we have $\|\psi_k\|^2_{L^2(\omega)} = O(\sin(r)^k)$, that gives $\mathfrak{K}_{eig} \geq |\log(\sin(r))|$.

- the disk $\mathbb{D}$, the whispering galleries on the disk are eigenfunctions that concentrate on the boundary $\partial\mathbb{D}$. Using asymptotics of Bessel functions, it is possible to obtain a sequence satisfying (3.1.14) for $\omega = B(N,r)$ with $\mathfrak{K}_{eig}(\omega) \geq \mathfrak{K} \geq -\tanh(\alpha(r)) - \alpha(r)$ with $\alpha(r) = \cosh^{-1}(1/r)$.

For r small, we recognize the bound (3.1.6) in both cases which already contradicts Conjecture 3.1.1 when r is small enough.

Let us now put these results in a broader context, and introduce several related geometric constants appearing in tunneling estimates. We consider surfaces of revolutions of the form $\mathcal{M} = \mathcal{S} \subset \mathbb{R}^3$ a smooth compact surface diffeomorphic to the sphere $\mathbb{S}^2$. We assume moreover that it has revolution invariance around an axis, that intersects $\mathcal{S}$ in two points, the north and the south poles, respectively $N, S \in \mathcal{S}$. Then, we describe (almost all) the surface by two coordinates, namely $s = \text{dist}_g(\cdot, N)$, the geodesic distance to the north pole and $\theta \in \mathbb{S}^1$, the angle of rotation. The variable s is in $(0, L)$ where $L = \text{dist}_g(N, S)$. The surface is characterized by the function $R(s)$ associating to s the Euclidean distance in $\mathbb{R}^3$ to the symmetry axis, which, by definition, is rotationally invariant, and satisfies $R(0) = 0 = R(L)$. This function R is the "profile" of the revolution surface $\mathcal{S}$. In the coordinates $(s, \theta) \in (0, L) \times \mathbb{S}^1$ valid on $\mathcal{S} \setminus \{N, S\}$, the Laplace-Beltrami operator is given by

$$\Delta_{s,\theta} = \frac{1}{R(s)} \partial_s R(s) \partial_s + \frac{1}{R(s)^2} \partial_\theta^2. \tag{3.1.15}$$

We are looking for eigenfunction very oscillating in the θ variable, of the form $e^{ik\theta} f(s)$ with $k \in \mathbb{Z}$, $f \in C^\infty(0, L)$ with suitable conditions at 0 and L, solution of

$$-\frac{1}{R(s)} \partial_s \left(R(s) \partial_s f \right) + \frac{k^2}{R(s)^2} f = \lambda f. \tag{3.1.16}$$

If $\lambda = k^2 \mu$ for a constant μ to be chosen, this is a (variation of a) semiclassical 1 D Schrödinger equation with $h = k^{-2}$ and the potential $V(s) = \frac{1}{R(s)^2} - \mu$ that goes to $+\infty$ at 0 and L. We shall now assume that R reaches at s_0 a global maximum. This kind of problem has been very well studied [85, 86]. The rough idea is to consider the symbol $p = \xi^2 + V$ and observe that most of the energy is concentrated in the classically allowed region: $K = \{s \in (0, L); V(s) \geq 0\}$. It is possible to construct some eigenfunction with $\mu \approx \frac{1}{R(s_0)^2}$ so that the minimum of V is 0 and is reached at s_0.

We introduce the relevant Agmon distance to the "equator" $s = s_0$, defined by

$$d_A(s) = \left| \int_{s_0}^{s} \sqrt{V(s) - V(s_0)} \, dy \right|. \tag{3.1.17}$$

with $V = \frac{1}{R(y)^2}$.

The proofs rely on classical semiclassical decay estimates for eigenfunctions [86, 163]. We refer to the monographs [56, 85] for the historical background and more references. An additional difficulty here is linked to the degeneracy of the function R close to the north and south poles. Note also that, to our knowledge, the idea of constructing such examples on surfaces of revolution is due to Lebeau [119] and Allibert [7].

We proved in this context the following Agmon type result

Theorem 3.1.5. *Assume that $s \mapsto R(s)$ admits a* non-degenerate strict global *maximum at $s_0 \in (0, L)$. Then, for all $k \in \mathbb{N}$, there exists $\psi_k \in C^\infty(\mathcal{S})$, and $\lambda_k \geq 0$ such that*

$$\lambda_k \sim \frac{k^2}{R(s_0)^2}, \qquad \|\psi_k\|_{L^2(\mathcal{S})} = 1, \qquad -\Delta_g \psi_k = \lambda_k \psi_k.$$

Moreover, there exist $C, k_0 > 0$ such that, for all $k \in \mathbb{N}$, $k \geq k_0$ and all $0 \leq r \leq s_0$, we have the estimate

$$\|\psi_k\|_{L^2(B(N,r))} \leq C \lambda_k^C e^{-d_A(r)\left(R(s_0)\sqrt{\lambda_k} - C\right)}.$$

In particular, we get

$$\mathfrak{K}_{eig}(B_g(N, r)) \geq d_A(r) R(s_0). \tag{3.1.18}$$

Then, the asymptotic behavior of d_A when $s \to 0$, namely

$$d_A(s) = -\log(s) + O(1), \quad \text{as } s \to 0^+. \tag{3.1.19}$$

is the key to contradict Conjecture 3.1.1 since $d_A(s)$ can be made as large as we want close to 0 (corresponding to the north pole) while $\mathcal{L}(\mathcal{M}, \omega)$, the bigger distance to $\omega = B(N, r)$, remains bounded. Note that we recover the value computed explicitly in $\mathbb{S}^2$ and $\mathbb{D}$. Another interesting remark is that in the limit of small r, (3.1.19) allows to rewrite as

$$\|\psi_k\|_{L^2(B(N,r))} \leq Ce^{C\sqrt{\lambda_k}} r^{R(s_0)\sqrt{\lambda_k}-C}, \tag{3.1.20}$$

and, in any local chart centered at N, we have $\partial^\alpha \psi_k(N) = 0$ for all $|\alpha| < R(s_0)\sqrt{\lambda_k} - C - n/2$. The same estimates actually hold for the eigenfunctions we constructed before on the sphere and the disk. These eigenfunctions saturate the maximal vanishing rate predicted by the Donnelly-Fefferman Theorem [57].

Upper bound for $\mathfrak{K}_{heat}$: uniform Lebeau-Robbiano spectral inequalities

We now give some ideas for proving the upper bound counterpart to the previous counterexamples, that is Theorem 3.1.3 which aim at proving some observability estimates of the form (3.1.1) with $C_0(T, \omega) = Ce^{\frac{\mathfrak{K}}{T}}$ uniformly for $0 < T \leq 1$.

As already mentioned before, the Lebeau-Robbiano strategy reduces the proof of the controllability of the heat equation to the proof of some spectral inequalities. More precisely, with $E_{\leq\lambda} := \text{span}\{E_{\lambda_j}, \lambda_j \in \text{Sp}(-\Delta_g), \lambda_j \leq \lambda\},$, we can obtain the following uniform estimate.

Theorem 3.1.6 (Uniform Lebeau-Robbiano spectral inequality with observation on small balls)**.** *Let $(\mathcal{M}, g)$ be a compact Riemannian manifold with (or without) boundary $\partial\mathcal{M}$. For all $x_0 \in \mathcal{M}$, there exist constants $C_1, C_2 > 0$ such that for all $r > 0$, $\lambda \geq 0$ and $\psi \in E_{\leq\lambda}$, we have*

$$\|\psi\|_{L^2(\mathcal{M})} \leq e^{\left(C_1\sqrt{\lambda}+C_2\right)(1+|\log(r)|)} \|\psi\|_{L^2(B(x_0,r))}.$$

Note that a careful inspection of the proofs (of all Carleman estimates used, that are stable by small perturbations) shows that the constant C_1, C_2 can actually be taken independent of the point x_0. This is an estimate on $\mathfrak{K}_\Sigma$ as discussed earlier in (3.1.10).

When restricted to eigenfunctions, this result is the counterpart of (3.1.20) and was obtained by Donnelly-Fefferman [57], and roughly states that eigenfunctions vanish at most like $r^{C\sqrt{\lambda}+C}$ on balls of radius r (λ is the eigenvalue). It has been generalized in some sense to sums of eigenfunctions by Jerison and Lebeau [98]. We prove here a variant of this result under the form of a uniform Lebeau-Robbiano spectral inequality with observation on small balls. $\leq \lambda$. Note that we prove the result in the case of Neumann boundary conditions as well. This uniform Lebeau-Robbiano spectral inequality directly implies Theorem 3.1.3 using [142, Corollary 1].

The usual trick to prove Theorem 3.1.6, introduced by Lebeau-Robbiano [120], is to obtain some interpolation for the augmented manifold $\mathbb{R}_s \times \mathcal{M}$ for the operator $\partial_s^2 + \Delta_g$. Interpolation inequalities are some inequalities as presented in Lemma 3.1.7 below that express stability estimates of Hölder type. As seen in the Introduction of Chapter 1, this is the usual result of quantification of smallness for the usual Carleman estimates, that is $n_a = 0$ in the notation of this chapter.

The scheme of the proof consists of several steps for propagating the information (that is iterating interpolation inequalities) from $\{s = 0\} \cap B(0, r)$ to $\mathcal{M}$:

- from a small part of the boundary $\{s = 0\} \cap B(0, 2r)$ to $B(0, r)$: after a suitable scaling, we use a boundary Carleman estimates for metric almost flat,
- from small balls to unit balls: this is in this step that the $\log(r)$ cost appears from a Carleman-Aronszajn estimates,
- from unit balls to the full manifold: we have to iterate the interpolation estimates until we cover the full space.
- conclude using the Lebeau-Robbiano trick: apply the global interpolation result with observation at $\{s = 0\} \cap B(0, 2r)$, we obtained to the function (in case $\partial\mathcal{M} = \emptyset$ where there is no zero eigenvalues)
$$v(s) = \frac{\sinh(s\sqrt{-\Delta_{\mathfrak{g}}})}{\sqrt{-\Delta_{\mathfrak{g}}}} w,$$

where $\Delta_{\mathfrak{g}}$ is the Dirichlet Laplacian. v solution to

$$(-\partial_s^2 - \Delta_{\mathfrak{g}})v = 0, \quad v|_{(0,S_0)\times\partial M} = 0, \quad (v, \partial_s v)|_{s=0} = (0, w).$$

In all these steps, especially the first two ones, we have to keep track of the dependence to r.

We only detail the second step which is the one where the cost $\log(r)$ appears. The corresponding interpolation estimate is the following.

Lemma 3.1.7 (Local interpolation inequality from small balls to unit balls)**.** *Let $P = -\partial_s^2 - \Delta_g$ and let B_r denote balls centered at $(s_0, x_0) \in (-T, T) \times \mathcal{M}$, away from the boundary. Then, there exists $r_1 > 0$ such that for all $0 < r_0 \le r_1$, there is $C > 0$ such that for all $r \in (0, \frac{r_0}{10})$, and $F \in H^2(B_{r_0})$, we have*

$$\|F\|_{H^1(B_{\frac{r_0}{4}})} \le C \left(\|PF\|_{L^2(B_{r_0})} + \|F\|_{H^1_r(B_r)} \right)^{\alpha_r} \|F\|^{1-\alpha_r}_{H^1(B_{r_0})}, \quad \alpha_r = \frac{\log 2}{\log\left(\frac{2r_0}{r}\right) + \log 2}.$$

The proof of this Lemma relies on a Carleman estimate (with singular weight) due to Aronszajn [11]. We use the version stated in [57, Proposition 2.10] and [58, Proposition 2.10] (and slightly modified according to the remarks in [98, Beginning of Section 14.3]), which we now state. An alternative proof of a closely related estimate is given by Hörmander in [90, Inequality (17.2.11), Chapter XVII.2].

Proposition 3.1.8. *[11][57, Proposition 2.10] Let $P = -\partial_s^2 - \Delta_g$ and let $(\rho, t) \in (0, r_1) \times \mathbb{S}^n$ be geodesic polar coordinates around a point $(s_0, x_0) \in X_S$ away from the boundary. Then, there exists a function $\bar{\rho}(\rho)$ with*

$$\bar{\rho} = \rho + O(\rho^2), \qquad \text{as } \rho \to 0^+, \tag{3.1.21}$$

and constants $\tau_0, C, r_0 > 0$, such that we have

$$C \int |\bar{\rho}^{-\tau} P u|^2 \rho^{-1} d\rho dt \ge \int \left(|\bar{\rho}^{-\tau} \nabla u|^2 + |\bar{\rho}^{-\tau} u|^2 \right) \rho^{-1} d\rho dt, \quad \text{for all } \tau \ge \tau_0, \quad u \in C_0^\infty(B_{r_0} \setminus \{0\}).$$

We will apply the Carleman-Aronszajn estimate to $\chi_r(\rho)F$ where χ is a radial cutoff with gradient supported around $\rho = r_0$ and around $\rho = r_0 \approx 1$. $P\chi_r(\rho)F$ will create one term supported around $\rho = r$ where the weight has the form $r^{-\tau} = e^{-\log(r)\tau}$ and another one supported around $\rho = r_0$ where the weight has the form $r_0^{-\tau}$. This will give the $\log(r)$ term in Lemma 3.1.7 after optimization.

Finally, note that in all these steps, we were careful about making proofs valid and uniform in a class of Lipschitz metrics. It is of course interesting by itself to lower the regularity, but it also allows to use the classical trick of doubling a regular manifold with boundary by symmetry (Neumann boundary conditions) or antisymmetry (Dirichlet boundary conditions) of the solution to be able to treat the Dirichlet or Neumann boundary conditions from the result on a case without boundary.

In particular, the 3rd step contains some uniform Lipschitz Carleman estimates (see [A11]) that have their own interest and were actually used later in the context of the unique continuation for Schrödinger equation.

The observability constant for positive solutions

Now, we aim to give an idea for proving Theorem 3.1.4 concerning the observability of positive solutions to the heat equation. The main tool will be the Li-Yau estimates described in Theorem 0.2.11 in Section 0.2. We refer to this section for comments on this kind of estimate.

The idea is that for any $x \in \mathcal{M}$ and $\eta > 0$, there exists one $y \in \omega$ with $d(x, y) \le \mathcal{L}(\mathcal{M}, \omega) + \eta$ for which Theorem 0.2.11 gives For a large constant r to be chosen, and for all $t \in [0, T]$, Theorem 0.2.11 with $t_1 = t/r$ and $t_2 = t$ yields for $\alpha > 1$

$$u(t/r, x) \le C e^{\frac{\alpha(\mathcal{L}(\mathcal{M},\omega)+\eta)^2}{4(1-1/r)t}} u(t, y). \tag{3.1.22}$$

where C is a constant depending on all the parameters but which will not be important. After a compactness argument, the idea is then to integrate in y in a small ball inside ω and integrate in $x \in \mathcal{M}$ and $t \in [\lambda T, T]$ for $\lambda \in (0,1)$ close to 1. The real computation is actually a bit more technical (y depends on x actually), but the main idea is to make the observation in ω at a time much larger than where we want the estimate and only close to the final time. For λ and α close 1 and small η, the exponential term in (3.1.22) is close to $e^{\frac{\mathcal{L}(\mathcal{M},\omega)^2}{4T}}$ which is the expected term.

Finally, it is worth saying that it was proved by Le Balc'h that the observability of positive solutions implies that we can control to a positive state, with the corresponding cost. So our observability cost implies a "cost to positive function".

Lemma 3.1.9. *[114, Theorem 4.1] Let $L^2(\mathcal{M};\mathbb{R}^+)$ be the set of positive functions. Assume moreover that there exists $C_{0,+}(T,\omega) > 0$ so that*

$$\left\|e^{T\Delta_g} u_0\right\|_{L^2(\mathcal{M})} \leq C_{0,+}(T,\omega)\left\|e^{t\Delta_g} u_0\right\|_{L^2([0,T]\times\omega)}, \qquad \text{for all } u_0 \in L^2(\mathcal{M};\mathbb{R}^+).$$

Then, for any $y_0 \in L^2(\mathcal{M})$ and $0 < \varepsilon \leq \varepsilon_0$, there exists a control $h \in L^2([0,T], L^2(\mathcal{M}))$ with

$$\|h\|_{L^2([0,T],L^2(\mathcal{M}))} \leq C_{0,+}(T,\omega)\,\|y_0\|_{L^2(\mathcal{M})}$$

so that the solution of

$$\begin{cases} \partial_t y - \Delta_g y = \mathbb{1}_\omega h & \text{in } (0,T)\times\Omega, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{3.1.23}$$

(with boundary condition if necessary) satisfies $y(T) \in L^2(\mathcal{M};\mathbb{R}^+)$.

We refer also to **[A16]** , Lemma 5.9., for a more general version as observability for data in conic subspace.

3.2 Transport equations in the vanishing viscosity limit

In this part, we consider the controllability of transport equations and their viscous approximations. In order to simplify the problem, we first consider smooth connected compact manifold $\mathcal{M}$ without boundary, a smooth real-valued vector field X on $\mathcal{M}$ and a real-valued potential $q(x)$, we consider the question of observability/detectability for the autonomous transport equation

$$\begin{cases} (\partial_t - X)u = 0, & \text{in } \mathbb{R}\times\mathcal{M}, \\ u|_{t=0} = u_0, & \text{on } \mathcal{M}, \end{cases} \tag{3.2.1}$$

from an observation (open) set $\omega \subset \mathcal{M}$ through the time interval $(0,T)$. More precisely, the question is whether there exists a constant $C_0 = C_0(T,\omega) > 0$ such that

$$C_0^2 \int_0^T \int_\omega |u(t,x)|^2 ds(x)dt \geq \|u(T)\|^2_{L^2(\mathcal{M})}, \qquad \text{for all } u_0 \in L^2(\mathcal{M}) \text{ and } u \text{ solution of (3.2.1)}. \tag{3.2.2}$$

Here, $ds(x)$ denotes any positive density measure on $\mathcal{M}$, and the L^2 norm is defined accordingly. The observability question (3.2.2) is naturally solved by introducing an appropriate Geometric Control Condition (recall $\partial\mathcal{M} = \emptyset$): we say that $(\mathcal{M}, X, \omega, T)$ satisfies (GCC)[1] if for all $x \in \mathcal{M}$, there is $t \in (0,T)$ such that $\phi_{-t}(x) \in \omega$, where $(\phi_t)_{t\in\mathbb{R}}$ denotes the flow of X. We also say that $(\mathcal{M}, X, \omega)$ satisfies (GCC) if $(\mathcal{M}, X, \omega, T)$ does for some $T > 0$; and if so, we denote by $T_{GCC}(\mathcal{M}, X, \omega)$ the infimum of times for which $(\mathcal{M}, X, \omega, T)$ satisfies (GCC).

[1]Note that we keep the notation (GCC) which was already used before in the context of the wave equation. Indeed, the geometric control condition (GCC) of Bardos-Lebeau-Rauch states that the set $S^*\omega \subset S^*\mathcal{M}$ satisfies the Geometric Control Condition in the previous sense for the vector field H_p where H_p is the Hamiltonian of p, the principal symbol of $-\Delta$.

On the other hand, endowing $\mathcal{M}$ with a Riemannian metric g, one may want to investigate the observability question for the viscously damped transport equation:

$$\begin{cases} (\partial_t - X - \varepsilon\Delta_g)u = 0, & \text{in } \mathbb{R}_*^+ \times \mathcal{M}, \\ u|_{t=0} = u_0, & \text{on } \mathcal{M}, \end{cases} \tag{3.2.3}$$

from the same observation set $(0,T)\times\omega$. The question is whether there exists a constant $C_0(T,\varepsilon) > 0$ such that

$$C_0(T,\varepsilon)^2 \int_0^T \int_\omega |u(t,x)|^2 ds(x) dt \geq \|u(T)\|^2_{L^2(\mathcal{M})}, \quad \text{for all } u_0 \in L^2(\mathcal{M}) \text{ and } u \text{ solution of (3.2.3)}, \tag{3.2.4}$$

(and one may then choose the Riemannian volume density $ds(x) = d\,\mathrm{Vol}_g(x)$ without changing the problem). For fixed $\varepsilon > 0$, Equation (3.2.3) is similar to the heat equation described in the previous Section and the observability inequality (3.2.4) is also known to hold for any open set $\omega \neq \emptyset$ and $T > 0$ by the results of [73] (see also [120] and its variant in [116]). Of course, in such results, the observability constant $C_0(T,\varepsilon)$ in (3.2.4) depends *a priori* on ε (we drop the dependence in ω here). For many different reasons (the theory of conservation laws, control of Navier-Stokes from Euler...), it is interesting to investigate the behavior of the observability constant $C_0(T,\varepsilon)$ in the vanishing viscosity limit $\varepsilon \to 0^+$. This problem was first studied in the one-dimensional setting by Coron and Guerrero in [40], and later extended to any dimension by Guerrero and Lebeau [77]. Their main result in this direction can be formulated as follows.

Theorem 3.2.1 (Guerrero-Lebeau [77]). *Given an open set $\omega \subset \mathcal{M}$, the following two results hold.*

- *[77, Theorem 1] Assume $(\mathcal{M}, X, \overline{\omega}, T)$ does not satisfy (GCC). Then there is $C, \varepsilon_0 > 0$ such that any constant $C_0(T,\varepsilon)$ in (3.2.4) satisfies $C_0(T,\varepsilon) \geq \exp(C/\varepsilon)$ for $\varepsilon \in (0,\varepsilon_0)$.*
- *[77, Theorem 3] Assume $(\mathcal{M}, X, \omega)$ satisfies (GCC). Then there is $T_{unif}(\omega) \geq T_{GCC}(\mathcal{M}, X, \omega)$ and $K_0 > 0$ such that for all $T \geq T_{unif}(\omega)$, (3.2.4) holds with $C_0(T,\varepsilon) \leq K_0$ for all $\varepsilon \leq 1$.*

Note that the results in [77] are even more general since time-dependent vector fields are allowed and the boundary-value problem is also considered (with Dirichlet boundary conditions).

Note that if (3.2.4) holds for some T_0 and constant $C_0(T_0,\varepsilon)$, then it also holds for all times $T \geq T_0$ with the same constant $C_0(T_0,\varepsilon)$. In [77], the question of the minimal time $T_{unif}(\omega)$, more precisely defined by

$$T_{unif}(\omega) = \inf\{T > 0 \text{ for which there exist } K_0, \varepsilon_0 > 0 \\ \text{such that (3.2.4) holds with } C_0(T,\varepsilon) \leq K_0 \text{ for all } \varepsilon \in (0,\varepsilon_0)\},$$

and its link with the minimal observation time $T_{GCC}(\mathcal{M}, X, \omega)$ associated to the limit problem (3.2.1) is left open. In particular, the formulation of the results in [77] (see e.g. Theorem 2 and the discussion thereafter in that reference) suggests the possible existence of a universal constant $\mathfrak{K} \geq 1$ such that

$$T_{unif}(\omega) \leq \mathfrak{K}\, T_{GCC}(\mathcal{M}, X, \omega). \tag{3.2.5}$$

We investigate this question in a very particular case, namely assuming the vector field X is a gradient vector field, i.e. $X = \nabla_g \mathfrak{f}$ for a function $\mathfrak{f} \in W^{2,\infty}(\mathcal{M};\mathbb{R})$ (note that the gradient is taken with respect to the Riemannian metric g). Hence, Equation (3.2.3) becomes

$$\begin{cases} (\partial_t - \nabla_g \mathfrak{f} \cdot \nabla_g - \varepsilon\Delta_g)u = 0, & \text{in } \mathbb{R}_*^+ \times \mathcal{M}, \\ u|_{t=0} = u_0, & \text{on } \mathcal{M}, \end{cases} \tag{3.2.6}$$

In this context, as already said, our results have the same form as in the previous section concerning the cost of the control in small time: the natural conjecture that we could expect is false but true for positive solutions.

The negative results can be (loosely) stated as follows.

Theorem 3.2.2. *1) There are geometries* $(\mathcal{M}, g)$ *such that for all* $\Lambda > 0$*, one can find* $\mathfrak{f} \in C^\infty(\mathcal{M})$ *and* ω *open such that* $(\mathcal{M}, \nabla_g \mathfrak{f}, \omega)$ *satisfies (GCC) and* $T_{unif}(\omega) \geq \Lambda\, T_{GCC}(\mathcal{M}, \nabla_g \mathfrak{f}, \omega)$.

2) There are $(\mathcal{M}, \mathfrak{f}, X, \omega)$ *such that for all* $\Lambda > 0$*, one can find a metric* g_Λ *on* $\mathcal{M}$ *such that*

- $X = \nabla_{g_\Lambda} \mathfrak{f}$,
- $(\mathcal{M}, X, \omega)$ *satisfies (GCC),*
- $T_{unif}(\omega) \geq \Lambda\, T_{GCC}(\mathcal{M}, X, \omega)$.

In particular, Theorem 3.2.2 states that there is no $\mathfrak{K}$ such that (3.2.5) holds for all $(\mathcal{M}, X, \omega)$. The second item in Theorem 3.2.2 stresses the importance of the viscosity one chooses. Namely, with the same vector field X, changing the metric g, that is the viscous perturbation, may change the minimal uniform observability time. We also obtain related results for domains of $\mathbb{R}^n$.

Our second main result in this setting concerns the uniform observability of *positive* solutions to (3.2.6) and is parallel to Theorem 3.1.4. Recall that nonnegative data $u_0 \geq 0$ give rise to positive solutions to (3.2.6). We define $C_0^+(T, \varepsilon)$ the observability constant for positive solutions, that is for which (3.2.2) holds for all $u_0 \geq 0$, and accordingly set

$$T_{unif}^+(\omega) = \inf \{T > 0 \text{ for which there exist } K_0, \varepsilon_0 > 0 \text{ such that (3.2.4) holds} \\ \text{for all } u_0 \geq 0, \text{ with } C_0^+(T, \varepsilon) \leq K_0 \text{ for all } \varepsilon \in (0, \varepsilon_0)\big\}. \quad (3.2.7)$$

Theorem 3.2.3 (Positive solutions)**.** *For all* $\mathfrak{f} \in C^3(\mathcal{M}; \mathbb{R})$*, and* $\omega \subset \mathcal{M}$ *such that* $(\mathcal{M}, \nabla_g \mathfrak{f}, \omega)$ *satisfies (GCC), we have* $T_{unif}^+(\omega) = T_{GCC}(\mathcal{M}, \nabla_g \mathfrak{f}, \omega)$.

As usual, these uniform observability/non-observability results can be reformulated in terms of uniform controllability/non-controllability statements for an adjoint controlled equation. We also obtain in this context that an observation of positive solutions allows us to get a control to a positive final state as in Lemma 3.1.9.

After the counterexample is done, the situation seems even more complicated than it was before. The natural question was what could be T_{unif}. So, we decided to try to see what could be done in the 1D problem and if it was possible to see what are the parameters that come into the problem.

It turns out that an important function is the potential $V(x) = \frac{|\mathfrak{f}'(x)|^2}{4}$. In the case that V has the shape of a potential well with additional technical assumptions, we can make some very precise upper and lower bounds on T_{unif}. The results become quite technical so, I will just present the important terms and ideas that come out and refer to [A21] for the result and [A20] for the related study of the localization of the eigenfunction of the semiclassical problem.

In both cases, the upper and lower bounds on T_{unif} have the form: "geometric quantity"+"spectral quantity".

The geometric quantity comes from the localization of the eigenfunctions and involves some Agmon distance related to V as in (3.1.17).

The spectral quantity comes from the behavior of the eigenvalues. It involves the function $\Phi(E) = \frac{1}{2} \operatorname{Vol}\left(\left\{(x, \xi) \in [0, L] \times \mathbb{R}, \quad \xi^2 + V(x) \leq E\right\}\right)$ that appears in the Weyl Law of the natural semiclassical operator $-\varepsilon^2 \partial_x^2 + V$. The terms appearing in our lower and upper bounds are related to the quantity $\Phi'(E)$ that measures how much the eigenvalues are separated.

Idea of the proof

The first step of the proof is actually the one that explains why we have chosen X as a gradient of a function. It consists of a conjugation argument that changes our operator to a "semiclassical selfadjoint operator"

The first basic computation is the following:

$$e^{-\frac{\mathfrak{f}}{2\varepsilon}} \Delta_g e^{\frac{\mathfrak{f}}{2\varepsilon}} = \Delta_g + \frac{1}{\varepsilon} \nabla_g \mathfrak{f} \cdot \nabla_g + \frac{|\nabla_g \mathfrak{f}|_g^2}{4\varepsilon^2} + \frac{\Delta_g \mathfrak{f}}{2\varepsilon}.$$

We denote by

$$\frac{1}{\varepsilon^2}P_\varepsilon := -\Delta_g + \frac{|\nabla_g \mathfrak{f}|_g^2}{4\varepsilon^2} + \frac{\Delta_g \mathfrak{f}}{2\varepsilon}, \quad \text{that is} \quad P_\varepsilon := -\varepsilon^2\Delta_g + \frac{|\nabla_g \mathfrak{f}|_g^2}{4} + \varepsilon q_f, \tag{3.2.8}$$

where $q_f = \frac{\Delta_g \mathfrak{f}}{2}$. P_ε is mostly the semiclassical operator $-\varepsilon^2\Delta_g + V$ with $V = \frac{|\nabla_g \mathfrak{f}|_g^2}{4}$. We obtain then

Lemma 3.2.4. *The following statements are equivalent.*

1) The function u solves

$$(\partial_t - \nabla_g \mathfrak{f} \cdot \nabla_g - \varepsilon\Delta_g)u = 0, \ \text{in } (0, T_0) \times \mathrm{Int}(\mathcal{M}), \tag{3.2.9}$$

$$\text{with} \quad \|u(T_0)\|^2_{L^2(\mathcal{M})} \le C_0^2 \int_0^{T_0} \|u\|^2_{L^2(\omega)}\, dt, \tag{3.2.10}$$

2) The function $v(t,x) = e^{\mathfrak{f}(x)/2\varepsilon}u(t,x)$ solves

$$\varepsilon\partial_t v + P_\varepsilon v = 0, \ \text{in } (0, T_0) \times \mathrm{Int}(\mathcal{M}), \tag{3.2.11}$$

$$\text{with} \quad \left\| e^{-\frac{\mathfrak{f}}{2\varepsilon}} v(T_0)\right\|^2_{L^2(\mathcal{M})} \le C_0^2 \int_0^{T_0} \left\| e^{-\frac{\mathfrak{f}}{2\varepsilon}} v\right\|^2_{L^2(\omega)} dt, \tag{3.2.12}$$

Note that the constant coefficient one dimensional problem introduced in [40] enters the "gradient flow" setting with $\mathcal{M} = (0, L) \subset \mathbb{R}$, $g = 1$, $\Delta_g = \partial_x^2$, $\mathfrak{f} = Mx$ for $M \in \mathbb{R}$, and thus $\nabla_g \mathfrak{f} \cdot \nabla_g = M\partial_x$. In that context, this form together with its formulation (3.2.8) have already been used in [40, 76, 130, 131].

So, we are led to consider the observation with weight of a semiclassical heat equation. Once we have made this observation, the proof will follow a similar scheme as in the previous section concerning the cost of the heat equation in short time: the lower bounds are made using some well-chosen eigenfunctions on surfaces of revolution while the bound for positive solutions will use some Harnack type inequalities (also taken form Li-Yau).

Lower bound for T_{unif} : exponentially small eigenfunctions

We consider the same type of surfaces of revolution $\mathcal{S}$ described in the previous section that are isomorphic (sometimes if we avoid some pole) to $(0, L) \times \mathbb{S}^1$ parametrized by (s, θ). In a similar way that we did in Section 3.1, we are looking for eigenfunctions of P_ε that have the form $e^{ik\theta}w(s)$. The operator P_ε applying to functions under this form becomes then

$$P_\varepsilon^{(k)} w = -\frac{\varepsilon^2}{R(s)}\partial_s\left(R(s)\partial_s w\right) + \left(\frac{\varepsilon^2 k^2}{R(s)^2} + \frac{|\mathfrak{f}'(s)|^2}{4} + \varepsilon q_f\right) w. \tag{3.2.13}$$

We make the following choice of the parameter ε:

$$\varepsilon = \varepsilon_k = ck^{-1} \tag{3.2.14}$$

considered as a semiclassical parameter, where $c > 0$ is a fixed parameter that will be chosen but fixed. In particular, $P_\varepsilon^{(k)}$ becomes roughly a semiclassical operator modeled by $-\varepsilon^2\partial_s^2 + V_c$ with the new potential

$$V_c(s) := \frac{c^2}{R(s)^2} + \frac{|\mathfrak{f}'(s)|^2}{4}. \tag{3.2.15}$$

The first part $\frac{c^2}{R(s)^2}$ comes from the Riemannian geometry, that is from Δ_g while the second part come from the vector field. At this point, we guess that it will be possible to choose some situations where the term $1/R(s)^2$ will have a strong effect even if the transport equation is controllable. We see here that the choice of the metric, that is the choice of the small viscosity term Δ_g has some effect. To end the proof, we proceed as in the previous section constructing some eigenfunctions exponentially decreasing. They satisfy some similar Agmon type estimates similar to Theorem 3.1.5 but with the Agmon distance defined with respect to the new potential V_c. The details are made in **[A16]** .

Uniform observability constant for positive solutions

Again, the main tool for proving Theorem 3.2.3 about the uniform observability for positive solutions will be some Li-Yau estimates. Thanks to Lemma 3.2.4, we can use the formulation as a semiclassical Schrödinger type operator with potential $V = \frac{|\nabla_g \mathfrak{f}|_g^2}{4}$.

We will use some estimate of the heat kernel in the semiclassical limit taken from Li-Yau [126].

Theorem 3.2.5 (Theorem 6.1 of [126]). *Let $\mathcal{M}$ be a compact manifold without boundary. Let $V \in C^2(\mathcal{M})$. For any $\varepsilon > 0$, we consider H_ε, the fundamental solution of*

$$\partial_t w - \Delta_g w + \frac{1}{\varepsilon^2} V(x) w = 0, \qquad \text{on } (0,+\infty) \times \mathcal{M}.$$

Then, we have

$$\lim_{\varepsilon \to 0} \varepsilon \log H_\varepsilon(x,y,\varepsilon t) = -\rho(x,y,t) \tag{3.2.16}$$

where

$$\rho(x,y,t) = \inf\left\{ \int_0^t \frac{1}{4} |\dot\gamma(s)|_g^2 + V\big(\gamma(s)\big) ds, \gamma \in W^{1,\infty}([0,t];\mathcal{M}), \gamma(0) = x, \gamma(t) = y \right\}. \tag{3.2.17}$$

We recall that $H_\varepsilon(x,y,t)$ is defined to be the unique solution to

$$\begin{cases} \Big(\partial_t - \Delta_g + \frac{1}{\varepsilon^2} V(x)\Big) H_\varepsilon(x,y,t) = 0, & \text{for } (t,x) \in \mathbb{R}_*^+ \times \mathcal{M}, \\ H_\varepsilon(x,y,t)|_{t=0} = \delta_{x=y}, & \text{for } x \in \mathcal{M}, \end{cases} \tag{3.2.18}$$

where $y \in \mathcal{M}$ is fixed, and the differential operator $-\Delta_g + \frac{1}{\varepsilon^2} V_\varepsilon(x)$ acts in the x-variable.

So, the proof of Theorem 3.2.3 are done in the following steps

- Prove that an "observability for the heat kernel" is equivalent to an observability estimates in L^1 for positive solutions. This step of independent interest is actually valid for any positive kernel and only involves measure theory and positivity.
- prove that some L^1 observability estimates for positive solutions will imply some estimate in L^2. Here, we need to get some uniform decay of the solutions that are consequences of estimates in [77].
- make the link between the decay rate ρ defined by (3.2.17) for $V = \frac{|\nabla_g \mathfrak{f}|_g^2}{4}$ and the transport equation. Having in mind that we want to have weighted estimates with weight $e^{\mathfrak{f}(x)/2}$, it is natural to define instead the kernel $d_{\nabla_g \mathfrak{f}}(x,y,t) := \rho(x,y,t) + \frac{\mathfrak{f}(x) - \mathfrak{f}(y)}{2}$. We prove that we have $d_{\nabla_g \mathfrak{f}}(x,y,t) \geq 0$ and $d_{\nabla_g \mathfrak{f}}(x,y,t) = 0$ if and only if there exists a trajectory of $\dot\gamma(s) = \nabla_g \mathfrak{f}(\gamma(s))$ with $\gamma(0) = x, \gamma(t) = y$.

Some perspectives

The question of determining T_{unif} is wide open and our study actually shows that it is certainly more complicated than expected.

As the general problem seems very difficult, it seems reasonable for the time being to try to build classes of solutions to intuit what the right constant might be. One could begin by constructing classes of solutions, for instance using complex geometric optics to try to understand the quantities that may be involved.

CHAPITRE 4

Classifications of solutions from their asymptotic behavior

In this chapter, we present two different results where we classify solutions of certain PDE from their asymptotic behavior. Both of them are in collaboration with Raphaël Côte.

- in the first result **[P2]** , we prove the existence, in a suitable space, of a scattering operator for nonlinear elliptic equations at $r \to +\infty$. That is we prove that each nonlinear solution is asymptotically close to a linear solution while every linear solution can be approximated by a nonlinear one in a neighborhood of ∞. It is a way of classifying all solutions of the nonlinear elliptic equation close to ∞
- in the second result **[A18]** , we classify linear solutions of the wave equations that, asymptotically as $t \to \pm\infty$, have no energy outside of a truncated cone. We also derive several asymptotic formulas for the solution and the asymptotic energy.

In both cases, the results have their own interest, but the initial motivation came from the recent program of Duyckaerts-Kenig-Merle and the channel of energy method (see Section 0.1).

4.1 A scattering operator for some nonlinear elliptic equation

The purpose of this section is to present the results of **[P2]** that gives a classification of solutions of certain nonlinear elliptic equations, by their behavior at infinity. We consider equations of the form

$$\Delta u = f(u, \nabla u), \tag{4.1.1}$$

where the nonlinearity f is analytic, and with an extra emphasis on the elliptic nonlinear equation with $\dot{H}^1$ critical power nonlinearity, conformal equations in dimension 2, and smooth harmonic maps. Roughly speaking, we will construct, in these considered examples, a scattering operator: we prove that when considering the vicinity of (spatial) infinity, there is a one-to-one correspondence between linear solutions of

$$\Delta u_L = 0 \quad \text{on} \quad \mathbb{R}^d \setminus B(0,1), \quad u_L|_{\mathbb{S}^{d-1}} = u_0, \tag{4.1.2}$$

where u_0 is a given function on $\mathbb{S}^{d-1}$ and nonlinear solutions of (4.1.1) defined for sufficiently large x; furthermore nonlinear solutions behave as a linear one in an appropriate space.

This space is strong enough to distinguish each linear solution from another one only from their asymptotic behavior. For instance, since all linear solutions converge to 0 at infinity, the space we consider should be much finer than L^∞ or $\dot{H}^1$. The space we use specifically translates the behavior of the linear elliptic solutions. In particular, it implies some analyticity in the angular variable.

Note also that the full classification is obtained in some general examples that are critical or with additional assumptions. Yet, the construction of nonlinear solutions from their behavior at infinity (that is one part of the scattering operator) is made in great generality.

The problem we consider is natural and has its own interest; we believe it will also prove useful for related evolution problems. Indeed, one extra motivation comes from the evidence that the asymptotic behavior of nonlinear objects like the well-known soliton plays a fundamental role in dynamical contexts, as it drives the interactions: for example, the construction of blow-up solutions, the construction of multi-solitons, the analysis of collision of solitons, the soliton resolution conjecture, etc.

One of the key roadblocks in generalizing the above results to the nonradial setting is the lack of understanding of the nonradial nonlinear objects, such as spectral properties when linearizing around them, or their asymptotic behavior.

Our work here provides a first description in the nonradial context, within a framework that encompasses semilinear elliptic equations together with harmonic maps or the H-system.

We address the question by recasting the elliptic equation in terms of an evolution equation on the sphere, where time is played by the radial variable. After performing a conformal change of variable, the equation is obviously (strongly) ill-posed, as discussed in Section 0.2 of the Introduction, but is amenable to resolution from infinity, for data *without growing modes*. As discussed in Section 0.2, the idea is to define some spaces that depend on some parameter t seen as an evolution parameter (called X_t in Section 0.2). Here, the evolution parameter t in some conformal variables will be related to the radius $r = e^t$ in the original variables.

The definition of the spaces depending on a variable parameter will be crucial in the analysis and is one of the novelty of our result. So, we will begin in the next paragraph by defining the functional setting, in order to state our results with the following ones. We advise the reader to make the parallel with the elementary discussion on the explicit resolution of $\partial_t^2 u + \partial_\theta^2 u = 0$ in (0.2) giving rise to the bound (0.2.5) to have an intuition on these spaces, keeping in mind that we will make a change of variable $r = e^t$ so that $r^\ell = e^{\ell t}$.

The functional setting

The proof will be performed after a conformal change of coordinates from $\mathbb{R}^d$ to $\mathbb{R} \times \mathbb{S}^{d-1}$ using spherical coordinates. The harmonical analysis on $\mathbb{S}^{d-1}$ will play a crucial role. We begin with a few generalities.

Let $d \geq 2$. We denote $\Delta_{\mathbb{S}^{d-1}}$ the Laplace-Beltrami operator on the sphere $\mathbb{S}^{d-1}$, and let $(\phi_{\ell,m})_{\ell\in\mathbb{N}, m\leq N_\ell}$ be an L^2 orthogonal basis of normalized spherical harmonics, so that $\phi_{\ell,m}$ is the restriction of a harmonic homogeneous polynomial of degree $\ell \in \mathbb{N}$. Recall that $\phi_{\ell,m}$ are eigenfunctions for $-\Delta_{\mathbb{S}^{d-1}}$

$$-\Delta_{\mathbb{S}^{d-1}} \phi_{\ell,m} = \ell(\ell + d - 2)\phi_{\ell,m},$$

and so N_ℓ is the dimension of the eigenspace of $-\Delta_{\mathbb{S}^{d-1}}$ for the eigenvalue $\ell(\ell + d - 2)$. Let P_ℓ be the orthogonal projection onto this eigenspace: if v is a function defined on $\mathbb{S}^{d-1}$,

$$P_\ell v = \sum_{m=1}^{N_\ell} \langle f, \phi_{\ell,m}\rangle \phi_{\ell,m}, \quad \|P_\ell v\|^2_{L^2(\mathbb{S}^{d-1})} = \sum_{m=1}^{N_\ell} |\langle v, \phi_{\ell,m}\rangle|^2. \tag{4.1.3}$$

We consider the positive elliptic operator on the sphere $\mathbb{S}^{d-1}$

$$\mathfrak{D} = \sqrt{-\Delta_{\mathbb{S}^{d-1}} + \left(\frac{d-2}{2}\right)^2},$$

so that for all $\ell \in \mathbb{N}$ and $m \leq N_\ell$,

$$\mathfrak{D}\phi_{\ell,m} = \left(\ell + \frac{d-2}{2}\right)\phi_{\ell,m}.$$

We denote $L_0^2(\mathbb{S}^{d-1}) = \mathrm{Span}(\phi_{\ell,m}, \ell \in \mathbb{N}, m \leq N_\ell\}$ the space of (finite) linear combinations of eigenfunctions of $\mathfrak{D}$; all normed spaces below will be meant as completion of L_0^2 for the underlying norm.

The space $H^s(\mathbb{S}^{d-1})$ is the completion of $L_0^2(\mathbb{S}^{d-1})$ for the H^s norm defined as

$$\|v\|_{H^s(\mathbb{S}^{d-1})}^2 := \sum_{\ell=0}^{+\infty} \langle \ell \rangle^{2s} \|P_\ell v\|_{L^2(\mathbb{S}^{d-1})}^2, \tag{4.1.4}$$

where $\langle \ell \rangle = \sqrt{1+|\ell|^2}$ is the japanese bracket (here ℓ is an integer, but we also use the notation for a vector or a multi-index).

We will state our results using the spaces $Z_{s,r}^\infty$ and $Z_{s,r}^0$, made of functions on $\mathbb{S}^{d-1}$, and which are the completion of $L_0^2(\mathbb{S}^{d-1})$ for the respective norms:

$$\|v\|_{Z_{s,r}^\infty} := r^{\frac{d-2}{2}} \left\| r^{\mathfrak{D}} v \right\|_{H^s(\mathbb{S}^{d-1})} = \left(\sum_{\ell=0}^{+\infty} \langle \ell \rangle^{2s} r^{2(\ell+d-2)} \|P_\ell v\|_{L^2(\mathbb{S}^{d-1})}^2 \right)^{1/2}$$

Notice that for $1 \leq r < r'$, we have the continuous embedding $Z_{s,r'}^\infty \subset Z_{s,r}^\infty \subset H^s$ together with $\|v\|_{Z_{s,r}^\infty} \leq \|v\|_{Z_{s,r'}^\infty}$.

For functions defined on $\mathbb{R}^d \setminus B(0,r_0)$, we will be interested in the $Z_{s,r}$ regularity on rescaled restrictions $u(r\cdot) : (y \mapsto u(ry)$ (defined on the sphere $\mathbb{S}^{d-1}$) (for $r \leq r_0$: this dependence of the space on the radius prompt us to the following definitions, which are meaningful due to the continuous embedding mentioned above.

We say that $u \in \mathcal{Z}_{s,r_0}^\infty$ if u defined on $\mathbb{R}^d \setminus B(0,r_0)$ and such that, for all $r \geq r_0$, $u(r\cdot) \in Z_{s,r/r_0}^\infty$,

$$(\rho \mapsto u(\rho\cdot)) \in \mathcal{C}([r,+\infty), Z_{s,r/r_0}^\infty)$$

and such that the following norm is finite:

$$\|u\|_{\mathcal{Z}_{s,r_0}^\infty} := r_0^{\frac{d-2}{2}} \sup_{r \geq r_0} \|u(r\cdot)\|_{Z_{s,r/r_0}^\infty}.$$

The exponent ∞ reminds us that we will be interested in the behavior for $|x| \to +\infty$. Note also that in the other applications involving some derivatives in the nonlinearity, the space involved is more complicated and involves some radial derivative.

The regularity index s appears as a fine tuning parameter: it plays an important role in the product laws, and so in the multi-linear estimates; one could simply fix

$$s > \frac{d}{2} + \frac{3}{2}.$$

For some purposes, we stated some intermediary results with more precision on s: except if a specific weaker bound is precised, all the following results assume s as above.

$u \in \mathcal{Z}_{s,r_0}^\infty$ implies that u has the same decay as a linear solution of $\Delta u_L = 0$, that is $|u(x)| = \mathcal{O}(|x|^{-(d-2)})$. It is actually more precise: when decomposing in spherical harmonics, each component decays as a linear solution, that is $\|P_\ell u(r\cdot)\|_{L^\infty(\mathbb{S}^{d-1})} = \mathcal{O}_\ell(r^{-(d-2)-\ell})$ as $r \to +\infty$.

Finally, we drop the index r when $r=1$, for example $\mathcal{Z}_s^\infty := \mathcal{Z}_{s,1}^\infty$.

Main results on the semilinear equation

In this section, we state those of our results that are concerned with the case of the (non derivative) nonlinearity

$$f(y) = \sum_{p \in \mathbb{N}} a_p y^p, \tag{4.1.5}$$

with a positive radius of convergence.

The first statement constructs, for a general nonlinearity, nonlinear solutions having a prescribed linear behavior at infinity. The second one, restricted to H^1-critical analytic nonlinearities, realizes the converse, that is, establishes that a finite energy solution behaves as a linear solution at infinity. The combination of both results leads to a kind of scattering operator identifying linear and nonlinear germs of solutions.

Theorem 4.1. *Let $d \geq 3$. Assume that f as in (4.1.5) satisfies the supercriticality assumption for some $\nu_0 > 0$,*

$$\forall p \in \mathbb{N}, \quad [a_p \neq 0 \Longrightarrow (d-2)p - d \geq \nu_0 > 0.] \tag{4.1.6}$$

Let $u_0 \in H^s(\mathbb{S}^{d-1})$, and $u_L \in \mathcal{Z}_s^\infty$ the linear solution given in (4.1.2).

Then, there exist $r_0 \geq 1$ and a unique small solution $u \in \mathcal{Z}_{s,r_0}^\infty$ of (4.1.1) on $\{|x| \geq r_0\}$ and such that

$$\|(u - u_L)(r\cdot)\|_{\mathcal{Z}_{s,r/r_0}^\infty} \lesssim r^{-\nu_0} \underset{r\to+\infty}{\longrightarrow} 0. \tag{4.1.7}$$

Moreover, the map $u_0 \mapsto u$ is injective; and if $\|u_0\|_{H^s(\mathbb{S}^{d-1})}$ is small enough, we can take $r_0 = 1$.

(Here and below, we say that there is a unique small solution u in a Banach space Z if there exists $\varepsilon > 0$ such that u is the unique solution in the ball centered at 0 and of radius ε of Z.) (4.1.6) means that we have $(d-2)p - d \geq \nu_0 > 0$ for the smallest p in series (4.1.5) defining f.

When we restrict to $\dot{H}^1$ critical exponents with analytic nonlinearities (which actually leaves the three possibilities mentioned below), we obtain a full classification of all possible solutions close to infinity.

Theorem 4.2 (Semilinear energy critical equation)**.** *We consider the equation*

$$\Delta u = \kappa u^p. \tag{4.1.8}$$

where $\kappa \in \mathbb{R}$, and we assume to be in one of the following situations:

$$(d,p) \in \{(3,5),(4,3),(6,2)\}.$$

1) Let $u \in \dot{H}^1(\{|x| \geq 1\})$ be a solution of (4.1.8) in the weak sense. Then, there exists $r_0 \geq 1$ so that $u \in \mathcal{Z}_{s,r_0}^\infty$ and a unique $u_L \in \mathcal{Z}_{s,r_0}^\infty$ solution of $\Delta u_L = 0$ on $\{|x| \geq r_0\}$ so that

$$\|u(r\cdot) - u_L(r\cdot)\|_{\mathcal{Z}_{s,r/r_0}^\infty} \leq Cr^{-2} \underset{r\to+\infty}{\longrightarrow} 0. \tag{4.1.9}$$

2) Reciprocally, given $u_0 \in H^s(\mathbb{S}^{d-1})$, and $u_L \in \mathcal{Z}_s^\infty$ as in (4.1.2), there exists $r_0 \geq 1$ and a unique small solution $u \in \mathcal{Z}_{s,r_0}^\infty$ of (4.1.8) on $\{|x| \geq r_0\}$ satisfying (4.1.9).

To our knowledge, such classification did not appear elsewhere in the literature, for any elliptic type equation. It gives both a complete rigidity and a fine description for nonlinear solutions, concerning their behavior at infinity.

In particular, the previous theorem also implies a result of unique continuation at infinity.

Corollaire 4.1. *In the situation of the previous Theorem 4.2 1), if $u \in \dot{H}^1(\{|x| \geq 1\})$ is a solution of (4.1.8) so that*

$$\forall \ell \in \mathbb{N}, \quad r^{d-2+\ell} \|P_\ell u(r\cdot)\|_{H^s(\mathbb{S}^{d-1})} \to 0 \quad as \quad r \to +\infty,$$

then $u = 0$ on $\mathbb{R}^d \setminus B(0,1)$. In particular, if $u(x) = \mathcal{O}(|x|^{-\beta})$ for any $\beta \in \mathbb{R}$, then $u = 0$.

The results in this direction we are aware of (see for instance [29, 49]) would be obtained considering u^p as Vu for some potential $V = u^{p-1}$. They require exponential decay without distinction between the spherical harmonics.

For power nonlinearities u^p, $p > \frac{d}{d-2}$ (p integer), Theorem 4.1 constructs a lot of solutions with prescribed asymptotic linear behavior. We can perform a classification under further decay assumption.

Theorem 4.3 (Semilinear equation with decay)**.** *Let $d \geq 3$, $\kappa \in \mathbb{R}$ and $p \in \mathbb{N}^*$ with $p > \frac{d}{d-2}$ and consider the equation*

$$\Delta u = \kappa u^p \tag{4.1.10}$$

1) Let $u \in \dot{H}^1(\{|x| \geq 1\})$ solution of (4.1.10) in the weak sense so that we have $|u(x)| \leq C|x|^{-\frac{2}{p-1}-\eta}$ for some $\eta > 0$ and $C > 0$. Then, there exists $r_0 \geq 1$ so that $u \in \mathcal{Z}^\infty_{s,r_0}$ and there exists a unique $u_L \in \mathcal{Z}^\infty_{s,r_0}$ solution of $\Delta u_L = 0$ on $\{|x| \geq r_0\}$ so that

$$\|u(r\cdot) - u_L(r\cdot)\|_{\mathcal{Z}^\infty_{s,r/r_0}} \leq Cr^{-((d-2)p-d)} \underset{r\to+\infty}{\longrightarrow} 0. \tag{4.1.11}$$

2) Reciprocally, given $u_0 \in H^s(\mathbb{S}^{d-1})$, and $u_L \in \mathcal{Z}^\infty_s$ as in (4.1.2), there exists $r_0 \geq 1$ and a unique small solution $u \in \mathcal{Z}^\infty_{s,r_0}$ of (4.1.10) on $\{|x| \geq r_0\}$ satisfying (4.1.11).

In the defocusing case, the decay can be obtained using results of Véron [171] for solutions constructed by Benilan-Brézis-Crandall in [22].

Corollaire 4.2. *Let $d \geq 3$, $p \in 2\mathbb{N}+1$ with $p > \frac{d}{d-2}$. Let $f \in L^1(\mathbb{R}^d)$ be real valued with compact support. Due to [22], there exist a unique real valued solution $u \in L^{\frac{d}{d-2},\infty}(\mathbb{R}^d)$ with $\Delta u \in L^1(\mathbb{R}^d)$ of $\Delta u = u^p + f$.*

Then the conclusion of Theorem 4.3 1) holds for u.

Here $L^{q,\infty}(\mathbb{R}^d)$ are the weak-L^q spaces, for $1 < q < +\infty$, and are called spaces of Marcinkiewicz $M^q(\mathbb{R}^d)$ in the above reference, see [22, Appendix] or [164, Chap V.3].

In particular, in the defocusing case, the assumption of additional decay in Theorem 4.3 is not necessary and can be obtained under reasonable assumptions on the solution. Yet, this assumption is sometimes necessary with the power $\frac{2}{p-1}$ being optimal. For instance, for a $\dot{H}^1$-supercritical nonlinearity $p > \frac{d+2}{d-2}$, in the focusing case $\kappa < 0$, it is known that there exist radial positive solutions that behave like $C|x|^{-\frac{2}{p-1}}$ at infinity. These solutions have a slower decay than the solutions we construct in $\mathcal{Z}^\infty_s$ which decay as the linear solutions, that is $C|x|^{-(d-2)}$. We refer to [104, Theorem 5.2] for a nice summary. We refer also to [23, Theorem 3.3] for a dichotomy result in the case of positive solutions in the defocusing case and $p \neq \frac{d+2}{d-2}$, $p > 1$.

The class of equations covered by our theorems for constructing solutions are quite general. Yet, the regularity results and uniqueness of the Dirichlet boundary value problem have to be adapted to each equation. This is the reason why we only treated some examples for the classification; we nonetheless believe that the strategy can be applied in many more cases. Assuming that we are able to construct a solution with prescribed behavior at infinity, the road map for the classification in Theorem 4.2 and 4.3 goes as follows:

- prove by scaling and regularity arguments that, for a rescaled version of the solution, the trace on $\mathbb{S}^{d-1}$ is small in $H^s(\mathbb{S}^{d-1})$ with s large enough.
- construct a solution in the space $\mathcal{Z}^\infty_s$ with the same Dirichlet data on $\mathbb{S}^{d-1}$. By construction, this solution has the correct decay and will "scatter" to a linear solution.
- prove a uniqueness result for the Dirichlet value problem in some appropriate space containing the original solution and the solution we constructed.

The full classification as in Theorem 4.2 is not always true, but we believe that some modifications of the methods we introduce in this paper might lead to similar results. It would be natural to try to construct, by a modification of the space $\mathcal{Z}^\infty_s$, other sets of nonlinear solutions with different asymptotic behavior.

We completed this classification in the following cases:

- energy solution of conformal equations in dimension 2,

$$\Delta u = -A(u)(\nabla u, \nabla u) - H(u)(\partial_x u, \partial_y u) \tag{Conf-E}$$

where A is the second fundamental form[1] of the embedding $\mathcal{N} \subset \mathbb{R}^M$ and for $z \in qN$ and $H(z)$ an antisymmetric form with additional form. This includes the Harmonic maps and the H system

- energy solution of Harmonic maps in dimension $d \geq 3$ with enough regularity ($\mathcal{C}^2$)

Note that the case of Harmonic maps (or more generally conformal equations) in dimension 2 presents several additional difficulties that we have chosen not to describe precisely in this manuscript. First, the nonlinearities with derivatives are quite more difficult to handle. Moreover, in dimension 2, quadratic nonlinearities (with respect to the derivatives) are critical cases and we had to use some better properties of the first iterate of the Picard iteration. This improved behavior is observed in the case of conformal equations in dimension 2 where a "null" structure is observed.

We also obtain some results close to 0.

Theorem 4.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function with a positive radius of convergence and such that $f(0) = 0$.*

1) For any smooth solution u of $\Delta u = f(u)$ on $B(0,1)$, there exist a solution u_L of $\Delta u_L = 0$ and g analytic on $B(0, r_0)$ for some $0 \leq r_0 < 1$ so that u can be written

$$u = u_L + |x|^2 g. \tag{4.1.12}$$

2) Reciprocally, for any u_L bounded solution of $\Delta u_L = 0$ on $B(0,1)$ with $u_L(0) = 0$, there exist $0 < r_0 \leq 1$ and a unique small analytic solution u of $\Delta u = f(u)$ on $B(0, r_0)$ so that (4.1.12) holds for one g analytic on $B(0, r_0)$.

Moreover, the application $u_L \mapsto u$ is injective.

The decomposition (4.1.12) is known in the literature as the Fischer decomposition of the function u. This decomposition is already known to hold for any analytic function, so the first part is not really new. The main part of our proof is the construction of the nonlinear solution. This result can be seen as a local solvability result for a semi-linear elliptic equation with a prescribed behavior at a point. It seems that the available results in this context only prescribe the first 2 derivatives at one point, see for instance [168, Section 14.3, Proposition 3.3]. So, our result constructs many more local solutions, and actually all of them.

Idea of the proof

In all the results of scattering, there are two parts:

- Given a linear solution u_L, find a nonlinear solution u so that $u \approx u_L$ as $t \to +\infty$.
- Given a nonlinear solution u, prove that there is a linear solution u_L so that $u_L \approx u$ as $t \to +\infty$

So, our proof naturally contains both parts. The first one is very general under some supercriticality assumption while for the second, we need to adapt more to each equation and choose the correct space that ensures enough decay.

Solving the nonlinear equation with prescribed linear asymptotic behavior

To simplify the exposition, we will assume that the nonlinearity is of the form $f(u, \nabla u) = u^p$, that is we only prove Theorem 4.1 in the case f is a monomial, while our result is more general and allows some derivative and non-polynomial nonlinearities.

The first step is to make the conformal change of variable $v(t,y) = e^{\frac{(d-2)t}{2}} u(e^t y)$ and to notice that u defined on $x \in \mathbb{R}^d \setminus B(0,R)$ will satisfy

$$\Delta u = u^p \tag{4.1.13}$$

[1] we denote $A(u)(\nabla u, \nabla u) = \sum_{i=1}^d A(u)(\partial_{x_i} u, \partial_{x_i} u)$

if and only if v defined on $[\log(R), +\infty) \times \mathbb{S}^{d-1}$ satisfies

$$\partial_{tt} v - \mathfrak{D}^2 v = g(t, v) =: e^{-\kappa_0 t} v^p, \tag{4.1.14}$$

with $\kappa_0 = p\frac{d-2}{2} - \frac{d+2}{2}$ which is essentially dictated by the scaling. Note that the supercriticality assumption (4.1.6) gives $\kappa_0 > -1$. So, the exponential factor in $e^{-\kappa_0 t}$ can be slightly positive, but not too much.

So, our task is now to solve the elliptic problem (4.1.14) with a solution v is asymptotically like a linear solution v_L as $t \to +\infty$.

But in these coordinates, the solution v_L is very easy to compute (see also Section 0.2 formula (0.2.4) for $d = 2$ where $-\mathfrak{D}^2 = \partial_\theta^2$). But we selected the solutions v_L that are decaying at $+\infty$. They all have the form

$$v_L(t) = e^{-t\mathfrak{D}} v_0 \tag{4.1.15}$$

for one v_0 defined on $\mathbb{S}^{d-1}$. When written in these new coordinates, the natural equivalent of the space $Z_{s,t}$ defined in the introduction is the space $Y_{s,t}$ with the norm

$$\|g\|_{Y_{s,t}} = \|e^{t\mathfrak{D}} g\|_{H^s(\mathbb{S}^{d-1})},$$

so that we have $\|g\|_{Z^\infty_{s,r}} = r^{\frac{d-2}{2}} \|g\|_{Y_{s,\log(r)}}$ and $\|u(r\cdot)\|_{Z^\infty_{s,r}} = \|v(t)\|_{Y_{s,t}}$ with $t = \log(r)$.

This space is exactly suited for measuring linear solutions since we have

$$\|v_L(t)\|_{Y_{s,t}} = \|v_0\|_{H^s} \text{ for any } t \geq 0. \tag{4.1.16}$$

So, now, our new problem is to solve (4.1.14) with the asymptotic property

$$\|v(t) - v_L(t)\|_{Y_{s,t}} \underset{t\to+\infty}{\longrightarrow} 0. \tag{4.1.17}$$

Note that (4.1.16) makes it obvious that the map $u_0 \mapsto u$ in Theorem 4.1, if it exists, is injective. since if v_L and $\widetilde{v_L}$ are different linear solutions, then $\|\widetilde{v_L}(t) - v_L(t)\|_{Y_{s,t}} \underset{t\to+\infty}{\not\longrightarrow} 0$.

The idea is to see (4.1.14) as an evolution equation (wave type) with linear approximation at infinity, as it is done in scattering theory. Formally, if we forget, for a moment, about the convergence, the natural formula that we would expect in scattering theory would be the following Duhamel type formula

$$v(t) = v_L(t) - \int_t^{+\infty} \frac{\sinh\left((\tau - \sigma)\mathfrak{D}\right)}{\mathfrak{D}} g(\sigma, v(\sigma)) d\sigma.$$

So, the main point in the proof will be to give a meaning to the right-hand side and perform a fixed point argument. For a nonlinearity without derivative (otherwise, we need to add the time derivative in the space), we will work in the following space

$$\|v\|_{\mathcal{Y}_s} = \sup_{\tau\geq 0} \|v(\tau)\|_{Y_{s,\tau}}. \tag{4.1.18}$$

for v, defined on $[0, +\infty) \times \mathbb{S}^{d-1}$. It is quite clear by formula (4.1.16) that the linear solution belongs to this space.

We begin by studying the Duhamel formula in relation to these spaces. For a source term F defined on $[0, +\infty) \times \mathbb{S}^{d-1}$, we define the Duhamel formula

$$\Phi(F)(t) = \int_t^{+\infty} \frac{\sinh\left((t - \sigma)\mathfrak{D}\right)}{\mathfrak{D}} F(\sigma) d\sigma \tag{4.1.19}$$

We can prove the following estimate (if $d \geq 3$ otherwise, we have to pay attention to the frequence 0)

$$\|\Phi(F)(t)\|_{Y_{s,t}} \lesssim \int_t^{+\infty} \|F(\sigma)\|_{Y_{s-1,\sigma}} d\sigma. \tag{4.1.20}$$

In order to understand more the estimate (4.1.20) and simplify the computations, let us treat the case $F(t) = P_\ell F(t)$ for some $\ell \in \mathbb{N}$, so that $\mathfrak{D}F(t) = \lambda_\ell F(t)$ with $\lambda_\ell = \ell + \frac{d-2}{2}$. We have, using $\sinh(s) \leq e^{|s|}$

$$\begin{aligned}\|\Phi(F)(t)\|_{Y_{s,t}} &= e^{\lambda_\ell t}\left\|\int_t^{+\infty} \frac{\sinh((t-\sigma)\lambda_\ell)}{\lambda_\ell} F(\sigma)d\sigma\right\|_{H^s} \\ &\leq e^{\lambda_\ell t}\int_t^{+\infty} \frac{e^{-(t-\sigma)\lambda_\ell}}{\lambda_\ell}\|F(\sigma)\|_{H^s} d\sigma \leq e^{\lambda_\ell t}\int_t^{+\infty} \frac{e^{-(t-\sigma)\lambda_\ell}}{\lambda_\ell} e^{-\lambda_\ell \sigma}\|F(\sigma)\|_{Y_{s,\sigma}} d\sigma \\ &\leq e^{\lambda_\ell t}\int_t^{+\infty} \|F(\sigma)\|_{Y_{s-1,\sigma}} d\sigma.\end{aligned}$$

Which is the Duhamel type estimate (4.1.20) in this specific case. The general case is made by summing up similar estimates for $\ell \in \mathbb{N}$.

With such an estimate at hand, if we want to perform a fixed point argument, we need to be able to estimate the nonlinear term in some space $Y_{s-1,t}$. The crucial point will be to prove that the spaces $Y_{s-1,t}$ form actually an algebra. We can even prove something better in the sense that it is an algebra with norm less than $Ce^{-\frac{d-2}{2}t}$ uniformly in $t \geq 0$. More precisely, we can prove the following crucial product estimate.

Proposition 4.1 (Product law in the $Y_{s,t}$ spaces)**.** *Let $d \geq 2$ and $s > \frac{d}{2} + \frac{1}{2}$. There exists $C > 0$ so that*

$$\forall t \geq 0,\ \forall u, v \in Y_{s,t}, \quad \|uv\|_{Y_{s,t}} \leq Ce^{-\frac{d-2}{2}t}\|u\|_{Y_{s,t}}\|v\|_{Y_{s,t}}. \tag{4.1.21}$$

The proof of Proposition 4.1 is quite technical and we will just give a rough idea of where the gain comes from. Note that it is valid until $t = 0$, which is a fact that large Sobolev spaces are algebra (we have a small loss for the optimal s here, but it is not important for what follows).

The two important classical facts will be

- The Sogge estimates applied to the sphere

$$\|P_\ell u\|_{L^\infty(\mathbb{S}^{d-1})} \leq C\langle \ell\rangle^{\frac{d}{2}-1}\|P_\ell u\|_{L^2(\mathbb{S}^{d-1})}. \tag{4.1.22}$$

- Let ℓ_1 and $\ell_2 \in \mathbb{N}$ and ϕ_{ℓ_i} be two spherical harmonics of degree ℓ_i. The product $\phi_{\ell_1}\phi_{\ell_2}$ can be written as a sum of spherical harmonics of degree ℓ with $|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2$.

Again, in order to get an intuition on why Proposition 4.1 can be true, let us specify to the simpler case $u = P_{\ell_1}u$ and $v = P_{\ell_2}v$ assuming $\ell_2 \geq \ell_1$. We compute, using the previous second fact about the product of spherical harmonics

$$\begin{aligned}\|uv\|^2_{Y_{s,t}} &= \sum_{|\ell_1-\ell_2|\leq\ell\leq\ell_1+\ell_2} \langle\ell\rangle^{2s}e^{2(\ell+\frac{d-2}{2})t}\|P_\ell(uv)\|^2_{L^2} \\ &\leq \sum_{\ell_2-\ell_1\leq\ell\leq\ell_1+\ell_2} \langle\ell\rangle^{2s}e^{2(\ell+\frac{d-2}{2})t}\|u\|^2_{L^\infty}\|v\|^2_{L^2} \\ &\leq \langle\ell_1\rangle^{d-2}\sum_{\ell_2-\ell_1\leq\ell\leq\ell_1+\ell_2} \langle\ell\rangle^{2s}e^{2(\ell+\frac{d-2}{2})t}\|u\|^2_{L^2}\|v\|^2_{L^2}\end{aligned}$$

where we have used Sogge estimate. Now, since $u = P_{\ell_1}u$ and $v = P_{\ell_2}v$, we have $\|u\|_{L^2} = \langle\ell_1\rangle^{-s}e^{-(\ell_1+\frac{d-2}{2})t}\|u\|_{Y_{s,t}}$ and the same for v, so we arrive to

$$\|uv\|^2_{Y_{s,t}} \leq \langle\ell_1\rangle^{\frac{d}{2}-1}\langle\ell_1\rangle^{-2s}\langle\ell_2\rangle^{-2s}\sum_{|\ell_1-\ell_2|\leq\ell\leq\ell_1+\ell_2} \langle\ell\rangle^{2s}e^{2(\ell+\frac{d-2}{2})t}e^{-2(\ell_1+\frac{d-2}{2})t}e^{-2(\ell_2+\frac{d-2}{2})t}\|u\|^2_{Y_{s,t}}\|v\|^2_{Y_{s,t}}.$$

At that point, the most important part is to count the exponents in the exponential and to notice that $\ell \leq \ell_1 + \ell_2$ (coming from the product property) implies

$$(\ell + \frac{d-2}{2}) - (\ell_1 + \frac{d-2}{2}) - (\ell_2 + \frac{d-2}{2}) \leq -\frac{d-2}{2}.$$

So, the final estimate in this particular case is

$$\|uv\|_{Y_{s,t}}^2 \leq e^{-(d-2)t} \|u\|_{Y_{s,t}}^2 \|v\|_{Y_{s,t}}^2 \langle \ell_1 \rangle^{d-2} \langle \ell_1 \rangle^{-2s} \langle \ell_2 \rangle^{-2s} \sum_{|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2} \langle \ell \rangle^{2s}.$$

Now, estimating $\sum_{\ell_2 - \ell_1 \leq \ell \leq \ell_1 + \ell_2} \langle \ell \rangle^{2s} \lesssim \langle \ell_2 \rangle^{2s} \langle \ell_1 \rangle$, the remaining powers are

$$\langle \ell_1 \rangle^{d-2-2s+1}.$$

This requires $d - 2 - 2s + 1 < 0$, that is $s > \frac{d-1}{2}$. This would be the optimal exponent so that $H^s(\mathbb{S}^{d-1})$ is an algebra. In the full proof, we need to make some summation and we require s slightly bigger. We don't know if it is optimal, but the value of s will not be of great importance anyway.

Now that the crucial Duhamel estimates (4.1.20) and the product Proposition 4.1 are proved, we are ready to perform the fixed point argument.We are looking for a solution $v = w + v_L$ with w solution of

$$\partial_{tt} w - \mathfrak{D}^2 w = g(t, w + v_L) \tag{4.1.23}$$

with g as in (4.1.14). w is required to be decaying as $\|w(t)\|_{Y_{s,t}} \underset{t\to+\infty}{\longrightarrow} 0$ so as (4.1.17) is satisfied. We are looking for a fixed point of

$$\Psi : w \mapsto \Phi(g(t, w + v_L)). \tag{4.1.24}$$

We will consider the simplest case where v_L is small and we can look for a fixed point in some small balls of the space $\mathcal{Y}_s$ defined in (4.1.18). After a bit of numerology the following exponential contributions to the decay appear when we estimate $g(t, w + v_L))$ in the space $Y_{s,t}$

- κ_p is the exponential decay appearing in the definition of g in (4.1.14).
- We have a multiplication of p functions in $Y_{s,t}$. So, it creates an exponential gain of factor $(p-1)\frac{d-2}{2}$ due to Lemma 4.1.

Adding all the above yield the rate $e^{-\nu_0 t}$ with

$$\nu_0 = \kappa_0 + (p-1)\frac{d-2}{2} = p(d-2) - d. \tag{4.1.25}$$

That is we will have the estimate

$$\|g(t, w + v_L)\|_{Y_{s,t}} \leq Ce^{-\nu_0 t} \|w + v_L\|_{Y_{s,t}}^p. \tag{4.1.26}$$

The assumption of Theorem 4.1 ensures $\nu_0 > 0$. Therefore, the integral (4.1.20) when estimating $\Phi(g(t, w+v_L))$ is convergent and with norm $Y_{s,t}$ converging to zero. This allows after some estimates to perform a fixed point for Ψ and therefore construct a solution with v_L as asymptotic behavior, as expected.

Proving that a nonlinear solution scatters

The class of equations covered by our theorems for constructing solutions is much more general than the one described just before. We refer to [P2] for more precisions. Yet, for the converse, that is proving that a nonlinear solution scatters, it has to be adapted to each situation and assume some decay in spaces related to the equation.

Yet, the regularity results and uniqueness of the Dirichlet boundary value problem have to be adapted to each equation. This is the reason why we only treated some examples for the classification; we nonetheless believe that the strategy can be applied in many more cases. Assuming that we are able to construct a solution with prescribed behavior at infinity, the road map for the classification in Theorem 4.2 and 4.3 goes as follows:

- prove by scaling and regularity arguments that, for a rescaled version of the solution, the trace on $\mathbb{S}^{d-1}$ is small in $H^s(\mathbb{S}^{d-1})$ with s large enough.
- construct a solution in the space $\mathcal{Z}_s^\infty$ with the same Dirichlet data on $\mathbb{S}^{d-1}$. By construction, this solution has the correct decay and will "scatter" to a linear solution.
- prove a uniqueness result for the Dirichlet value problem in some appropriate space containing the original solution and the solution we constructed.

We give more precision in the specific case presented in Theorem 4.2, that is the case of H^1 critical equation. That means that the equation is invariant by some scaling that keeps the H^1 norm invariant (at least when considered on the full space $\mathbb{R}^d$).

So, we consider one solution in $H^1(\mathbb{R}^d \setminus B(0,1))$. Let $\varepsilon > 0$ small to be chosen later. Since $u \in \dot{H}^1(\{|x| \geq 1\})$, there exists r_0 so that $\|\nabla u\|_{L^2(|x| \geq r_0/2)} \leq \varepsilon$. Denoting $u^{r_0}(x) = (r_0)^{d/2-1} u(r_0 x)$, the function u^{r_0} satisfies $\|\nabla u^{r_0}\|_{L^2(|x|\geq 1/2)} \leq \varepsilon$ and is solution of the same elliptic equation (this is because the equation is H^1 critical). Denote $2^* = \frac{2d}{d-1}$ the Sobolev exponent so that $\dot{H}^1 \subset L^{2^*}$. By Sobolev estimate, we have also $\|u^{r_0}\|_{L^{2^*}(|x|\geq 1/2)} \lesssim \varepsilon$. We will use the fact that solutions of these H^1 critical equations are known to be smooth. In particular, following the classical method of Trudinger, we prove the quantitative version of trace estimates.

Proposition 4.2 (Trace regularity)**.** *Assume $p = 2^* - 1 \in \mathbb{N}$ and $s > 0$. There exists $C_s > 0$ and $\varepsilon_0 > 0$ and $C > 0$ so that for any real valued $u \in \dot{H}^1(\{|x| \geq 1/2\})$ solution of $\Delta u = \kappa u^p$ on $\{|x| \geq 1/2\}$, with $|\kappa| \leq 1$ and so that*

$$\varepsilon := \|u\|_{L^{2^*}(\{|x|\geq 1/2\})} \leq \varepsilon_0,$$

then, $u|_{\mathbb{S}^{d-1}} \in H^s(\mathbb{S}^{d-1})$ and $\|u|_{\mathbb{S}^{d-1}}\|_{H^s(\mathbb{S}^{d-1})} \leq C_s \varepsilon$.

In particular, if r_0 is chosen large enough, it gives $\left\| u^{r_0}_{|\mathbb{S}^{d-1}} \right\|_{H^s(\mathbb{S}^{d-1})} \lesssim \varepsilon$.

We notice that for the critical exponent $p = 2^* - 1 = \frac{d+2}{d-2}$, Theorem 4.1 holds with $\nu_0 = (d-2)p - d = 2$.

Using a variant of the Duhamel formula adapted to the Dirichlet problem, it is possible to prove the following existence theorem for the Dirichlet problem in our spaces. It also implies, with similar techniques the existence of an asymptotic "linear behavior".

Theorem 4.5 (The Dirichlet problem in our spaces)**.** *Under the same assumptions as Theorem 4.1, let $u_0 \in H^s(\mathbb{S}^{d-1})$ with $\|u_0\|_{H^s(\mathbb{S}^{d-1})}$ small enough.*

Then, there exists a unique small solution $\widetilde{u} \in \mathcal{Z}_s^\infty$ of (4.1.1) on $\{|x| \geq 1\}$ and such that $\widetilde{u}_{|\mathbb{S}^{d-1}} = u_0$.

Moreover, there exists a unique $u_{+,L} \in \mathcal{Z}_s^\infty$ solution of $\Delta u_{+,L} = 0$ so that

$$\|(\widetilde{u} - u_{+,L})(r\cdot)\|_{\mathcal{Z}_{s,r}^\infty} \to 0 \quad as \quad r \to +\infty.$$

We apply this theorem with $u_0 = u^{r_0}{}_{|\mathbb{S}^{d-1}}$ and obtain a solution $\widetilde{u} \in \mathcal{Z}_s^\infty$ so that $\widetilde{u}_{|\mathbb{S}^{d-1}} = u^{r_0}_{|\mathbb{S}^{d-1}}$.

In particular, u and $\widetilde{u}$ are two small solutions of the same nonlinear elliptic equation (4.2) with the same Dirichlet boundary condition. Using some uniqueness results for small solutions of the Dirichlet problem, it is then possible to verify that we actually have $\widetilde{u} = u^{r_0}$ and therefore, $u^{r_0} \in \mathcal{Z}_s^\infty$. Hence $u \in \mathcal{Z}_{s,r_0}^\infty$.

The scattering result (4.1.9) is obtained from the similar statement in Theorem 4.5.

4.2 Classification of solutions of linear waves from their energy outside of cones

In this part, whose results are contained in [A18] , we consider solutions to the linear wave equation in any dimension $d \geq 1$.

$$\begin{cases} \partial_{tt} u - \Delta u = 0, \\ (u, \partial_t u)_{|t=0} = (u_0, u_1), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{4.2.1}$$

We are interested in the following question: does the information outside the (translated) light cone give you information about the full solution?

We are particularly interested in understanding how the energy of w concentrates around the light cone for large times, that is, provide some formulas for quantities which are typically

$$\lim_{t\to+\infty} \|\nabla_{x,t} u\|_{\dot{H}^1\times L^2(|x|\geq t+R)}$$

where $R\in\mathbb{R}$ is fixed, in terms of the initial data (u_0,u_1). These kinds of quantities are very natural when thinking of finite speed of propagation for solutions to the linear wave equation, but are also useful in nonlinear contexts, for example for the channels of energy method, we refer for example to [59] for one of the first time it was used in the context of the energy critical nonlinear wave equation. Such formulas were given in the radial setting, notably in [45] and [107], and we aim at generalizing the result therein to nonradial linear waves.

Our main result in this section is Theorem 4.6 below concerning the exterior energy outside of the cone for $R>0$ and odd dimension. Before stating it, we will first present several other results that are sometimes not completely new, but in a unified presentation that we find interesting.

We can formulate our first results on solutions of the half-wave equation, that is, consider $e^{t|D|}f$, where $|D|$ is the operator defined as a multiplier in Fourier space

$$\widehat{|D|f}(\xi) = |\xi|\hat{f}(|\xi|),$$

where $\hat{f}$ is the d-dimensional Fourier transform of f:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx.$$

For functions of several variables (say s and other ones), we will consider in an analogous way $|D_s|$, where the Fourier transform is restricted to the s variable.

Our results on the half-wave equation will transfer to the wave equation as its solutions can be written (at least formally)

$$u = e^{it|D|}f + e^{-it|D|}g \quad \text{where} \quad f := \frac{1}{2}\left[u_0 + \frac{1}{i|D|}u_1\right] \text{ and } g := \frac{1}{2}\left[u_0 - \frac{1}{i|D|}u_1\right]. \tag{4.2.2}$$

We now introduce some notation. Given a function f on $\mathbb{R}^d$ and $\omega\in\mathbb{S}^{d-1}$, let $f_\omega^\pm:\mathbb{R}\to\mathbb{C}$ be such that its 1-dimensional Fourier transform (as a function of $\nu\in\mathbb{R}$) is

$$\mathcal{F}_{\mathbb{R}}(f_\omega^\pm)(\nu) = \mathbb{1}_{\pm\nu\geq 0}|\nu|^{\frac{d-1}{2}}\hat{f}(\nu\omega). \tag{4.2.3}$$

We also use the notation

$$\tau := \frac{d-1}{4}\pi \quad \text{and} \quad c_0 = \frac{1}{\sqrt{2(2\pi)^{d-1}}}. \tag{4.2.4}$$

Finally, we define the operator $\mathcal{T}$ as follows: for a function v defined on $\mathbb{R}^d$, $\mathcal{T}v$ is a function of two variables (s,ω), defined on $\mathbb{R}\times\mathbb{S}^{d-1}$ by its (partial) Fourier transform in the first variable s:

$$\mathcal{F}_{s\to\nu}(\mathcal{T}v)(\nu,\omega) = c_0|\nu|^{\frac{d-1}{2}}(e^{i\tau}\mathbb{1}_{\nu<0} + e^{-i\tau}\mathbb{1}_{\nu\geq 0})\hat{v}(\nu\omega). \tag{4.2.5}$$

that is,

$$(\mathcal{T}v)(s,\omega) = c_0\left(e^{i\tau}v_\omega^-(s) + e^{-i\tau}v_\omega^+(s)\right).$$

Our first result is the description of the asymptotic for large times of solutions of the half-wave equation, and then of the wave equation.

Proposition 4.3 (Radiation field and concentration of energy on the light cone)**.**
1) (Half-wave equation) Let $f\in L^2(\mathbb{R}^d)$. Then as $t\to+\infty$, the convergence holds

$$(e^{it|D|}f)(x) - \frac{e^{i\tau}}{(2\pi|x|)^{\frac{d-1}{2}}} f^-_{x/|x|}(|x|-t) \to 0 \quad \textit{in} \quad L^2(\mathbb{R}^d). \tag{4.2.6}$$

Furthermore, one has

$$\limsup_{t\to\pm\infty} \|e^{it|D|}f\|_{L^2(||x|-|t||\geq R)} \to 0 \quad as \quad R \to +\infty. \tag{4.2.7}$$

2) (Wave equation) Let $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$*, and* u *be the solution to* (4.2.1)*. Then as* $t \to +\infty$*, the convergence holds*

$$\nabla_{t,x}u(t,x) - \frac{1}{\sqrt{2}|x|^{\frac{d-1}{2}}}(\partial_s \mathcal{T}u_0 - \mathcal{T}u_1)\left(|x| - t, \frac{x}{|x|}\right) \times \begin{pmatrix} -1 \\ x/|x| \end{pmatrix} \to 0 \quad in \quad L^2(\mathbb{R}^d)^{1+d}. \tag{4.2.8}$$

Furthermore, one has

$$\limsup_{t\to\pm\infty} \|\nabla_{t,x}u(t)\|_{L^2(||x|-|t||\geq R)} \to 0 \quad as \quad R \to +\infty. \tag{4.2.9}$$

Of course, for $g \in L^2(\mathbb{R}^d)$, one obtains the corresponding expression for $e^{-it|D|}g$ by considering the complex conjugate in (4.2.6):

$$(e^{-it|D|}g)(x) - \frac{e^{-i\tau}}{(2\pi|x|)^{\frac{d-1}{2}}} g^+_{x/|x|}(|x| - t) \underset{t\to+\infty}{\longrightarrow} 0 \quad \text{in} \quad L^2(\mathbb{R}^d). \tag{4.2.10}$$

This also gives an expansion for $t \to -\infty$. Also, 2) is a rather direct consequence of 1), as we will prove the following equality which has its own interest:

$$(\partial_s \mathcal{T}u_0 - \mathcal{T}u_1)(s,\omega) = 2c_0\partial_s(e^{i\tau} f^-_\omega + e^{-i\tau} g^+_\omega)(s). \tag{4.2.11}$$

This result is therefore a computation of the radiation field of Friedlander [71]. We refer to [18]; in odd dimension, it can be classically written thanks to the Radon transform (see [134], [113]), to which the operator $\mathcal{T}$ is related. For smooth functions, these computations already appeared in several previous papers concerning nonlinear wave equations [3, 4, 93, 102] where the asymptotic profile appears in some criterium for blow-up or existence in large time.

The map defined on $\mathcal{S}(\mathbb{R}^d)$ by $\mathcal{F}^{-1}_{\nu\to s}\left[\hat{f}(\nu\omega)\right]$ is the Radon transform in the direction ω. That is

$$\mathcal{R}f(s,\omega) := \int_{\omega\cdot y = s} f(y)dy = \mathcal{F}^{-1}_{\nu\to s}\left[\hat{f}(\nu\omega)\right](s). \tag{4.2.12}$$

As a consequence, one has the equality as operators $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R} \times \mathbb{S}^{d-1})$:

$$\mathcal{T} = m_d(D_s)\mathcal{R} \quad \text{where} \quad m_d(\nu) := c_0|\nu|^{\frac{d-1}{2}}\left(e^{i\tau}\mathbb{1}_{\nu<0} + e^{-i\tau}\mathbb{1}_{\nu\geq 0}\right).$$

In particular, if the space dimension d is odd, then $\mathcal{T}$ is a local operator:

$$\mathcal{T} = c_0(-1)^{\frac{d-1}{2}}\partial_s^{\frac{d-1}{2}}\mathcal{R}.$$

However, as far as we can tell, the correct computation of the convergence in L^2 seems to be not classic, especially in even dimension, although an analogous formula to (4.2.8) can be found in [106] by Katayama (relying on the Radon transform). Our proof follows from a rather elementary and short stationary phase analysis.

Note also, that the formula makes a clear difference between the odd and even dimensions. Indeed, the operator has some nicer properties of locality with respect to the support.

Our formula is amenable to further computations. For example, as an easy consequence, we can also compute the energy outside a (shifted) light cone, or the asymptotic energy at $+\infty$ and $-\infty$ in the following sense.

Definition 4.2.1. Given $R \in \mathbb{R}$ and a space time function v, we denote

$$E_{ext,R}(v) := \frac{1}{2}\Bigg(\lim_{t\to+\infty}(\|\nabla v\|^2_{L^2(|x|\geq t+R)} + \|\partial_t v\|^2_{L^2(|x|\geq t+R)}) + \lim_{t\to-\infty}(\|\nabla v\|^2_{L^2(|x|\geq|t|+R)} + \|\partial_t v\|^2_{L^2(|x|\geq|t|+R)})\Bigg)$$

assuming that the limits exist.

Then there hold

Corollary 4.2.2 (Mass outside the light cone)**.** *Let $R \in \mathbb{R}$. We have the formula*

$$\begin{aligned}\lim_{t\to+\infty} &\|u\|^2_{L^2(|x|\geq t+R)} \\ &= \frac{1}{(2\pi)^{d-1}} \int_{\omega\in\mathbb{S}^{d-1}} \left\| f^-_\omega(s) + e^{-i\frac{\pi}{2}(d-1)} g^+_\omega(s) \right\|^2_{L^2([R,+\infty))} d\omega. \end{aligned} \tag{4.2.13}$$

Also, in the case of an initial datum $(u_0, u_1) \in \dot{H}^1 \times L^2$ in the energy space, we have the formula

$$\lim_{t\to+\infty} \|\nabla u\|^2_{L^2(|x|\geq t+R)} = \lim_{t\to+\infty} \|\partial_t u\|^2_{L^2(|x|\geq t+R)} \tag{4.2.14}$$

$$= \frac{1}{(2\pi)^{d-1}} \int_{\omega\in\mathbb{S}^{d-1}} \left\| e^{i\tau}\partial_s f^-_\omega(s) + e^{-i\tau}\partial_s g^+_\omega(s) \right\|^2_{L^2([R,+\infty))} d\omega.$$

$$= \frac{1}{2} \|\partial_s \mathcal{T} u_0 - \mathcal{T} u_1\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}. \tag{4.2.15}$$

(The first equality in (4.2.14) is equipartition). As a consequence, there is asymptotic orthogonality in the sense that

$$E_{ext,R}(u) = \|\partial_s \mathcal{T} u_0\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})} + \|\mathcal{T} u_1\|^2_{L^2([R,+\infty)\times\mathbb{S}^{d-1})}. \tag{4.2.16}$$

(Here and below, $\mathbb{R} \times \mathbb{S}^{d-1}$ is equipped with the standard product measure). The last two formulas (4.2.15) and (4.2.16) involving $\mathcal{T}$ reveal the important role of this operator in our analysis. We can reformulate (4.2.16) in the following way: denoting u^e [resp. u^o] the solution to (4.2.1) with initial data $(u_0, 0)$ [resp. $(0, u_1)$] then

$$E_{ext,R}(u) = E_{ext,R}(u^e) + E_{ext,R}(u^o).$$

Odd dimension

In odd dimension, we are able to precise the previous results and the asymptotic energy outside truncated cones $|x| \geq t + R$ with $R \geq 0$.

We first consider the easier case $R = 0$. From our computations, we can easily recover the following result, which goes back at least to Duyckaerts, Kenig and Merle [59].

Proposition 4.4. *Assume d odd and u be a a solution to (4.2.1) with initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$. Then, we have*

$$E_{ext,0}(u) = \frac{1}{2} \|(u_0, u_1)\|^2_{\dot{H}^1\times L^2(\mathbb{R}^d)}. \tag{4.2.17}$$

Then, we consider the case $R > 0$ where the previous result cannot hold. We are nonetheless able to determine the solutions u that have vanishing asymptotic energy on the exterior light cone $|x| \geq t + R$ with $R > 0$, that is

$$E_{ext,R}(u) = 0.$$

By finite speed of propagation, initial data which are compactly supported in $|x| \leq R$ obviously satisfy this condition. We will call this space

$$\mathcal{K}_{R,comp} = \left\{ (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d) : (u_0, u_1)|_{\{|x|>R\}} = 0 \right\}.$$

where the equality is in the distributional sense.

It turns out that these are not the only examples.

We will now need some further notation. We follow the same notations as in Section 4.1 and denote $(\phi_{\ell,m})_{\ell\in\mathbb{N}, m\leq N_\ell}$ a countable orthonormal basis of spherical harmonics of $\mathbb{S}^{d-1}$. $\phi_{\ell,m}$ is the restriction to $\mathbb{S}^{d-1}$ of a harmonic (homogeneous) polynomial. ℓ is the degree of this polynomial.

The non-radiative functions will be the following. Denote for $k \in \mathbb{N}$,

$$\alpha_k := -\ell - d + 2k + 2.$$

α_k also depends on ℓ, but here and below, we silence this dependence to keep notations light. Then let

$$g_k(x) = \mathbb{1}_{\{|x|>R\}} |x|^{\alpha_k} \phi_{\ell,m}\left(\frac{x}{|x|}\right) \tag{4.2.18}$$

Note that $g_k \in L^2 \iff \alpha_k < -d/2$. We introduce

$$\mathcal{N}^0_{R,\ell,m} = \operatorname{Span}\left(g_k;\ \text{for } k \in \mathbb{N} \text{ such that } \alpha_k < -d/2\right)$$

Similarly, let

$$f_k(x) = \begin{cases} \left(\frac{|x|}{R}\right)^{\alpha_k} \phi_{\ell,m}\left(\frac{x}{|x|}\right) & \text{for } |x| > R \\ \left(\frac{|x|}{R}\right)^{\ell} \phi_{\ell,m}\left(\frac{x}{|x|}\right) & \text{for } |x| \leq R. \end{cases} \tag{4.2.19}$$

Note that $f_k \in \dot{H}^1 \iff \alpha_k < -d/2 + 1$. Also, the value of f_k in $|x| \leq R$ is not very important; our choice permits to keep continuity and that the restriction $f_k|_{\{|x|<R\}}$ is a harmonic polynomial, so that f_k is orthogonal to (in $\dot{H}^1$) to functions with compact support in $B(0, R)$. Let

$$\mathcal{N}^1_{R,\ell} = \operatorname{Span}\left(f_k;\ \text{for } k \in \mathbb{N} \text{ such that } \alpha_k < -d/2 + 1\right).$$

For any $\ell, m \in \mathcal{M}$, we note the space

$$P_{\ell,m}(R) = \mathcal{N}^0_{R,\ell,m} \times \mathcal{N}^1_{R,\ell,m}, \quad \text{and} \quad P(R) = \mathcal{K}_{R,comp} \overset{\perp}{\oplus} \overset{\perp}{\bigoplus_{\substack{\ell \in \mathbb{N} \\ m \leq N_\ell}}} P_{\ell,m}(R),$$

(the orthogonality is related to the natural scalar product of $\dot{H}^1 \times L^2$). Then we will prove that if u is a linear wave solution that is non-radiative, that is such that $E_{ext,R}(u) = 0$, then $(u, \partial_t u)|_{t=0} \in P(R)$ (and the converse is true as well). We actually have a quantitative version of this fact: this is our second main result.

Theorem 4.6. *Assume d is odd, $d \geq 3$, and let $R > 0$. Let u be the solution to the linear wave equation with initial data $(u, \partial_t u)_{|t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2$. Then, we have*

$$\|(u_0, u_1)\|^2_{\dot{H}^1 \times L^2(\mathbb{R}^d)} = 2E_{ext,R}(u) + \|\pi_R(u_0, u_1)\|^2_{\dot{H}^1 \times L^2(\mathbb{R}^d)}. \tag{4.2.20}$$

where π_R is the orthogonal projection (in $\dot{H}^1 \times L^2$) onto the space $P(R)$.

Moreover, if $(u_0, u_1) \in P(R)$, then the equality

$$u(t, x) = \sum_{\substack{\ell \in \mathbb{N} \\ m \leq N_\ell}} v_{\ell,m}(t, r) \phi_{\ell,m}(w)$$

holds for all (t, x) in the (outer) truncated cone $\mathcal{C}_R = \left\{(t, x) \in \mathbb{R}^d; |x| - |t| \geq R\right\}$, where

$$v_{\ell,m}(t, r) = \sum_{j=1}^{B} \frac{1}{r^{d+l-j}} \sum_{i=0}^{B-j} d_{i,j} t^i. \tag{4.2.21}$$

for some $d_{i,j} \in \mathbb{C}$, and where $B := \frac{d+1}{2} + \ell$.

The theorem above is the generalization to nonradial data of the main result in [107]. Upon completion of [A18] , Liu-Shen-Wei [124] gave a description of non-radiative solutions u to the wave equation (that is, such that $E_{ext,R}(u) = 0$), in odd and even dimensions, but still in the radial case.

Moreover, some time after the publication of our result, it has been extended by Li-Shen-Wang-Wei [127] (without the quantitative statement (4.2.20) which is false) to even dimensions.

Even dimension

In even dimension, we are able to give a more tractable formula for $E_{ext,0}(u)$.

Proposition 4.5. *Assume that d is even and let u be a solution to (4.2.1) with initial data $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$. Then, we have*

$$\begin{aligned} E_{ext,0}(u) &= \frac{1}{2}\|(u_0,u_1)\|^2_{\dot{H}^1\times L^2(\mathbb{R}^d)} \\ &+ \frac{(-1)^{\frac{d}{2}}}{(2\pi)^{d+1}}\mathrm{Re}\int_{\omega\in\mathbb{S}^{d-1}}\int_0^\infty\int_0^\infty (rs)^{\frac{d-1}{2}}\frac{\overline{s\widehat{u_0}(s\omega)}r\widehat{u_0}(-r\omega)-\overline{\widehat{u_1}(s\omega)}\widehat{u_1}(-r\omega)}{r+s}drdsd\omega. \end{aligned}$$

More precisely, there hold

$$\begin{aligned} 2\lim_{t\to+\infty}\|\nabla u\|^2_{L^2(|x|\geq t)} &= E_{ext,0}(u) \\ &+ \frac{2}{(2\pi)^{d+1}}\mathrm{Re}\int_{\omega\in\mathbb{S}^{d-1}}\int_0^\infty\int_0^\infty \frac{r^{\frac{d+1}{2}}\widehat{u_0}(r\omega)s^{\frac{d-1}{2}}\overline{\widehat{u_1}(s\omega)}}{r-s}drdsd\omega. \end{aligned}$$

This is therefore an equivalent of Proposition 4.4. It extends the results of [45] where this formula first appeared for radial data (to recover this formula, notice that $\hat{u}_0(-r) = \hat{u}_0(r)$ when u_0 is radial). Also, short time before our result, Delort derived a similar formula in [55].

Idea of the proof

We give a rough idea of the proof of Proposition 4.3 and one of its consequences, Theorem 4.6. The other results are more computational and we refer to **[A18]** for more precisions.

For the proof of Proposition 4.3, it is sufficient to prove the asymptotic (4.2.6) for the half wave equation $e^{it|D|}$. The case of the full wave equation comes from the decomposition of a solution of the wave equation as a sum of a solution of the half-wave equation and of the anti-half-wave equation (that is $e^{-it|D|}$).

We want to find an asymptotic formula for $v = e^{it|D|}f$. The scheme of proof is the following:

- assuming first that $f \in \mathcal{S}(\mathbb{R}^d)$ is smooth and decaying, and that $\hat{f} \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$ is smooth and has compact support away from 0, we compute an asymptotic of the solution by two successive applications of (non)stationary phase lemma: first for $|x|$ and then for t as large parameter.
- we use energy estimates to verify that the convergence also holds in L^2.

For the first step, the Fourier inversion formula gives

$$\begin{aligned} v(t,x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}e^{it|\xi|}\hat{f}(\xi)d\xi \\ &= \frac{1}{(2\pi)^d}\int_0^{+\infty} r^{d-1}e^{itr}\underbrace{\int_{\mathbb{S}^{d-1}} e^{irx\cdot\omega}\hat{f}(r\omega)d\omega}_{I(r,x,t)}\,dr. \end{aligned}$$

For r fixed, we use stationary phase for large $|x|$: the two stationary points are $\omega = \pm\frac{x}{|x|}$ and we obtain the approximation (of course with remainder to be treated)

$$\begin{aligned} I(r,x,t) \sim &\left(\frac{2\pi}{r|x|}\right)^{\frac{d-1}{2}} e^{-i\tau}e^{ir|x|}\hat{f}(r\frac{x}{|x|}) \\ &+ \left(\frac{2\pi}{r|x|}\right)^{\frac{d-1}{2}} e^{i\tau}e^{-ir|x|}\hat{f}(-r\frac{x}{|x|}), \end{aligned} \tag{4.2.22}$$

So, we are left with

$$v(t,x) \sim \left(\frac{1}{|x|}\right)^{\frac{d-1}{2}} \frac{1}{(2\pi)^{\frac{d+1}{2}}}$$
$$\times \int_0^{+\infty} r^{\frac{d-1}{2}} e^{itr} \left[e^{-i\tau} e^{ir|x|} \hat{f}(r\frac{x}{|x|}) + e^{i\tau} e^{-ir|x|} \hat{f}(-r\frac{x}{|x|}) \right] dr$$

The first term is small again by (non)stationary phase with large parameter $t+|x|$ thanks to the oscillating term $e^{ir(t+|x|)}$. For the second term, we recognize the inverse Fourier transform at $|x|-t$ of $e^{i\tau}\mathbb{1}_{r\leq 0}|ru|^{\frac{d-1}{2}}\hat{f}(r\omega)$ as expected.

Then, with L^2 Fourier theory, we need to prove that the approximation of v for large t "did not lose any mass", that is has the same L^2 norm. This will be enough to prove the convergence in the general case.

This describes mostly the arguments for the proof of Proposition 4.3. We now get concerned by how to obtain Theorem 4.6.

First, we notice by a simple functional analytic argument in Hilbert space that estimate (4.2.20) is a consequence of Pythagore theorem once we have computed the space $P(R)$ of non-radiative solutions. After a few manipulations, we see that the key step is to compute $\mathcal{N}_R$ the Kernel of $\mathbb{1}_{|s|\geq R}\mathcal{T} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}\times\mathbb{S}^{d-1})$, that is the function so that the solution u with initial datum $(0,f)$ is non-radiative: $E_{ext,1}(u)=0$

The computation of $Ker\mathbb{1}_{|s|\geq R}\mathcal{T}$ could certainly be done by direct computation, but we tried to avoid complicated computations by using scaling arguments and the stability of the Kernel by Δ. It is actually inspired by a typical trick in control theory (that we saw in [16]) when we want to compute the set of invisible solutions (often reduced to zero) in a problem of observation. If we already know a weak observability inequality (that means up to more regular term), we know by Fredholm's theory that this set of invisible solutions is of finite dimension and stable by application. We tried to execute a similar trick in this context. The proof relies on the following steps.

1) first, using the decomposition by spherical harmonics, we reduce to a $1D$ problem for fixed spherical harmonics ℓ: we need to compute the kernel $\mathcal{N}_{R,\ell,m}$ reduced to functions of the form $f(x) = w(|x|)\phi_{\ell,m}\left(\frac{x}{|x|}\right)$ where $\phi_{\ell,m}$ is a spherical harmonics

2) $\mathcal{N}^0_{1,\ell}$ is finite-dimensional: for this, we use a formula of Ludwig [133] writing the Radon transform of functions $f(x) = w(|x|)\phi_{\ell,m}\left(\frac{x}{|x|}\right)$. It is a convolution operator involving polynomials and it is not too hard to see that the set of functions in the Kernel are such that the Radon transform is polynomial.

3) $\mathcal{N}_{R,\ell,m}$ is spanned by functions of the type $\ln(|r|)^p r^\alpha$, $\alpha\in\mathbb{C}$, $p\in\mathbb{N}$: the main idea here is to use scaling: "if a function is non-radiative, then it is the same for its dilation", that is roughly speaking that our space is stable by dilation. After a logarithmic change of variable that our space is stable, we use a classical ODE result stating that a finite-dimensional functional space stable by translation is made of solutions of a linear ODE with constant coefficients and contains functions of the form $t^p e^{\alpha t}$.

4) we need to get rid of the ln term. The main idea is to use the stability by Δ (correctly localized): "if a function f is non-radiative, then so is Δf". This will imply that $\mathcal{N}_{R,\ell,m}$ is spanned by the g_k.

Some perspectives

Our classification result Theorem 4.6 has been extended later by Li-Shen-Wang-Wei [127] (without the quantitative statement (4.2.20) which is false) to even dimensions. So, it seems that the linear case is now completely clarified. The natural perspective would be to have a classification of the nonlinear non-radiative solution, at least for small solutions. This is a work in progress with Raphaël Côte.

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